A Central Limit Theorem for Modified Massive Arratia Flow

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Abstract

The modified massive Arratia flow is a model for the dynamics of passive particle clusters moving in a random fluid that accounts for the effects of mass aggregation. We show a central limit theorem for the point process associated to the cluster positions when the system is started from a uniform configuration. The critical mixing estimate is obtained by coupling the system to countably many independent Brownian motions.

1 Introduction

The modified massive Arratia flow (MMAF) introduced in [Kon10a, Kon17b] is a model for the evolution of an ensemble of passive particle clusters in a random 1D-fluid. It describes the positions and sizes of a family of clusters moving as independent scaled Brownian motions on the real line as long as no collisions between clusters occur, the diffusion rate of each cluster being inverse proportional to its mass. In case of a collision clusters coalesce and form a new cluster of aggregate size (cf. Definition 2.1 below). This model is an overdamped intertial version of the well-studied Arratia flow model of coalescing Brownian motions that has no masses and no diffusivity rescaling [Arr79, Dor04, DO10, SSS14, BGS15, DKG17, SSS17, DV18, GF18, Ria18, DV20, Dor24]. Variants of

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the MMAF for different initial conditions, asymptotic properties of the trajectories and its large deviations were investigated in [Kon10a, Kon10b, Kon14, Kon17a, Kon17b, Mar18, KvR19, Mar21, KM23]. A reversible extension of the MMAF model featuring a splitting mechanism was introduced and studied in [Kon23, KvR24].

In this note we study the point process induced from the cluster positions when the MMAF system is started from the uniform configuration on the integer lattice $\mathbb{Z}$. Unlike for the standard Arratia flow [MRTZ06, TZ11] precise formulas or the structure of the associated point process of the cluster ensemble are yet unknown, but one can ask for asymptotic properties. The aim of this work is to show a central limit theorem for the occupation measure in Theorem 2.2 below. The statement is analogous to the corresponding result in [DH23] for the Arratia flow, however in the MMAF-case the correlation structure is more involved, leading to a different approach for the critical mixing estimate in Section 3. The main idea is based on coupling the MMAF to a countable family of independent Brownian motions, which is the main technical part of this work.

2 Statement of Main Result

For a rigorous statement of our result we recall the definition of the MMAF from [Kon10a], where we consider the special case of a uniform starting configuration on $\mathbb{Z} \subset \mathbb{R}$.

Definition 2.1. A family of continuous processes $\{x_k(t),\ t \geq 0,\ k \in \mathbb{Z}\}$ is called a *modified massive Arratia flow started from* $\mathbb{Z}$ if it satisfies the following properties

(F1) for each $k \in \mathbb{Z}$ the process x_k is a continuous square-integrable martingale with respect to the filtration

$$\mathcal{F}_t = \sigma(x_k(s),\ s \leq t,\ k \in \mathbb{Z}), \quad t \geq 0;$$

(F2) for each $k \in \mathbb{Z}$, $x_k(0) = k$;

(F3) for each $k < l$ from $\mathbb{Z}$ and $t \geq 0$, $x_k(t) \leq x_l(t)$;

(F4) for each $k \in \mathbb{Z}$ the quadratic variation of x_k equals

$$\langle x_k \rangle_t = \int_0^t \frac{ds}{m_k(s)}, \quad t \geq 0,$$

where $m_k(t) = \#\{l : \ \exists s \leq t \ \ x_k(s) = x_l(s)\}$ and $\#A$ denotes a number of elements in A;

(F5) for each $k, l \in \mathbb{Z}$

$$\langle x_k, x_l \rangle_{t \wedge \tau_{k,l}} = 0, \quad t \geq 0,$$

where $\tau_{k,l} = \inf\{t : \ x_k(t) = x_l(t)\}$.

The existence of such a family of processes is shown in [Kon10a, Theorem 2]. Moreover, conditions (F1)-(F5) uniquely determine the distributions of $(x_k)_{k\in\mathbb{Z}}$ in the space $\mathrm{C}\left([0,\infty)\right)^{\mathbb{Z}}$.

We can introduce the associated *occupation measure* induced on the real line by

$$\mu_t(A) = \#(A \cap \{x_k(t),\ k \in \mathbb{Z}\}), \quad A \in \mathcal{B}(\mathbb{R}).$$

Let $\mathcal{P}$ denote the set of bounded measurable functions $f : \mathbb{R} \to \mathbb{R}$ with period one, i.e. $f(x) = f(x+1)$ for all $x \in \mathbb{R}$. For every $f \in \mathcal{P}$ and $k \in \mathbb{Z}$ denote

$$A_{k,t}f := \int_{k-1}^{k} f(u)\mu_t(du).$$

In Proposition 4.1 below, we show that the random variables $A_{k,t}f$, $k \in \mathbb{Z}$, have finite moments of every order. Moreover, it is easy to see that the sequence $(A_{k,t}f)_{k\in\mathbb{Z}}$ is stationary. This directly follows from the fact that the distributions of $(x_k)_{k\in\mathbb{Z}}$ and $(\tilde{x}_k^l)_{k\in\mathbb{Z}}$ coincides in $\mathrm{C}([0,\infty))^{\mathbb{Z}}$ for every $l \in \mathbb{Z}$, where $\tilde{x}_k^l(t) = x_{k+l}(t) - l$, $t \geq 0$.

With this our main result reads as follows.

Theorem 2.2. *For every $f \in \mathcal{P}$ and $t > 0$ the sequence*

$$Y_t^n(f) = \frac{\sum_{k=1}^n (A_{k,t}f - \mathbb{E}\left[A_{k,t}f\right])}{\sqrt{n}}, \quad n \geq 1,$$

converges in distribution to a Gaussian random variable with mean 0 and variance

$$\sigma_t^2(f) = \mathrm{Var}\, A_{0,t}f + 2\sum_{k=1}^{\infty} \mathrm{Cov}\left(A_{0,t}f, A_{k,t}f\right).$$

The proof of the statement is based on the classical central limit theorem for stationary sequences, c.f [IL71, Theorem 18.5.3]. Hence, the main task is to establish quantitative mixing for the MMAF model which we achieve below by coupling to independent Brownian motions.

3 Mixing estimate

Fixing $f \in \mathcal{P}$ and $t > 0$ and we introduce the mixing coefficient for $i \in \mathbb{Z}$

$$\alpha_i(j) = \sup\left\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|,\ A \in \mathfrak{M}_{-\infty}^i,\ B \in \mathfrak{M}_j^{\infty}\right\}, \quad j \in \mathbb{Z},\ j > i,$$

where $\mathfrak{M}_a^b = \sigma\{A_{k,t}f,\ a \leq k \leq b\}$ for $-\infty \leq a < b \leq \infty$. Our goal is to prove the following proposition.

Proposition 3.1. *There exist constants $C > 0$ and $\beta > 0$ depending only on f and t such that*

$$\alpha_i(j) \leq Ce^{-\beta\sqrt{j-i}}$$

for all $i < j$ from $\mathbb{Z}$.

The idea of proof of the proposition is to construct a modified massive Arratia flow in such a way that particles which came to $(-\infty, l]$ and $[k, +\infty)$ at time t are "almost" independent. An important role is played by the following construction from [Kon10a].

Let $T > 0$ be fixed and $\mathrm{C}_{\leq}([0,T],\mathbb{R}^n)$ denote the Banach space of continuous functions $f : [0,T] \to \mathbb{R}^n$ satisfying $f_1(0) \leq \dots \leq f_n(0)$. We equip $\mathrm{C}_{\leq}([0,T],\mathbb{R}^n)$ with the uniform norm. We next define a map $F_n : \mathrm{C}_{\leq}([0,T],\mathbb{R}^{2n+1}) \to \mathrm{C}_{\leq}([0,T],\mathbb{R}^{2n+1})$ as follows. For $f \in \mathrm{C}_{\leq}([0,T],\mathbb{R}^{2n+1})$ we set $\tau^{(0)} = 0$ and

$$\Theta^{(0)} := \{\pi \subseteq \{-n,\dots,n\} : \; k,l \in \pi \iff f_k(0) = f_l(0)\}.$$

Note that $\Theta^{(0)}$ is a partition of the set $\{-n,\dots,n\}$. Let $\pi_k^{(0)}$ denote the set from $\Theta^{(0)}$ which contains $k \in \{-n,\dots,n\}$. Define

$$f_k^{(0)}(t) = f_{\tilde{k}}(0) + \frac{1}{\sqrt{m_k^{(0)}}}\left(f_{\tilde{k}}^{(0)}(t) - f_{\tilde{k}}^{(0)}(0)\right), \quad t \in [0,T],$$

where $\tilde{k}$ is the element from $\pi_k^{(0)}$ with the minimal absolute value and $m_k^{(0)} = \#\pi_k^{(0)}$. By induction, we construct $f^{(p)} \in \mathrm{C}_{\leq}([0,T],\mathbb{R}^{2n+1})$ for all $p \in \{1,\dots,2n\}$ as follows. For $k \in \{-n,\dots,n-1\}$, set

$$\tau_k^{(p)} = \inf\left\{t \geq 0 : \; f_k^{(p-1)}(t) = f_{k+1}^{(p-1)}(t)\right\} \wedge T,$$

and

$$\tau^{(p)} = \inf\left\{\tau_k^{(p)} > \tau^{(p-1)} : \; k \in \{-n,\dots,n-1\}\right\},$$

where as usual $\inf\emptyset = +\infty$. Let $\Theta^{(p)}$ be the partition of $\{-n,\dots,n\}$ defined by

$$\Theta^{(p)} := \{\pi \subseteq \{-n,\dots,n\} : \; k,l \in \pi \iff f_k^{(p-1)}(\tau^{(p)}) = f_l^{(p-1)}(\tau^{(p)})\}.$$

Let $\pi_k^{(p)}$ be the element of $\Theta^{(p)}$ which contains $k \in \{-n,\dots,n\}$. If $\tau^{(p)} = T$, we set $f_k^{(p)}(t) = f_k^{(p-1)}(t)$, $t \in [0,T]$. Otherwise,

$$f_k^{(p)}(t) = \begin{cases} f_k^{(p-1)}(t), & \text{if } 0 \leq t < \tau^{(p)}, \\ f_{\tilde{k}}^{(p-1)}(\tau^{(p)}) + \frac{\sqrt{m_k^{(p-1)}}}{\sqrt{m_k^{(p)}}}\left(f_{\tilde{k}}^{(p-1)}(t) - f_{\tilde{k}}^{(p-1)}(\tau^{(p)})\right), & \text{if } \tau^{(p)} \leq t \leq T, \end{cases}$$

where $\tilde{k}$ is the element from $\pi_k^{(p)}$ with the minimal absolute value and $m_k^{(p)} = \#\pi_k^{(p)}$. We now define

$$F_n(f) = f^{(2n)}.$$

We also define maps

$$F_n^+ : \mathrm{C}_{\leq}([0,T],\mathbb{R}^{n+1}) \to \mathrm{C}_{\leq}([0,T],\mathbb{R}^{n+1})$$

and

$$F_n^- : \mathrm{C}_\leq([0,T],\mathbb{R}^{n+1}) \to \mathrm{C}_\leq([0,T],\mathbb{R}^{n+1})$$

similarly to F_n, where the set $\{-n,\dots,n\}$ is replaced by $\{0,\dots,n\}$ and $\{-n,\dots,0\}$, respectively. By the construction, it is clear that maps F_n, F_n^+ and F_n^- are measurable.

For $l \in \mathbb{Z}, n \in \mathbb{N}$ and a fixed family of independent Brownian motions $w_k(t)$, $t \in [0,T]$, $k \in \mathbb{Z}$, on $\mathbb{R}$ with diffusion rate 1 and $w_k(0) = k$, we define continuous stochastic processes $X_k^{l,n}$, $k \in \{l-n,\dots,l+n\}$, $X_k^{l,n,+}$, $k \in \{l,\dots,l+n\}$, and $X_k^{l,n,-}$, $k \in \{l-n,\dots,l\}$, by

$$(X_{l-n}^{l,n},\dots,X_{l+n}^{l,n}) := F_n(w_{l-n},\dots,w_{l+n}), \quad (X_l^{l,n,+},\dots,X_{l+n}^{l,n,+}) := F_n^+(w_l,\dots,w_{l+n})$$

and

$$(X_{n-l}^{l,n,-},\dots,X_l^{l,n,-}) := F_n^-(w_{n-l},\dots,w_l).$$

We next recall the following lemma proved in [Kon10a, Lemma 5].

Lemma 3.2. *Let $w_k(t)$, $t \in [0,T]$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, be a family of independent Brownian motions on $\mathbb{R}$ with diffusion rate 1 and $w_k(0) = k$. Then for every $\varepsilon \in \left(0, \frac{1}{2}\right)$ the equality*

$$\mathbb{P}\left\{\max_{k\in\{0,\dots,n\}} \max_{t\in[0,T]} w_k(t) \leq n + \frac{1}{2}, \quad \min_{t\in[0,T]} w_{n+1}(t) > n + \frac{1}{2} + \varepsilon \text{ i.o.}\right\} = 1$$

holds.

Let $\mathbb{Z}_{\leq l} = \{l, l-1, \dots\}$ and $\mathbb{Z}_{\geq l} = \{l, l+1, \dots\}$. Using the lemma above and the construction of F_n, F_n^+ and F_n^-, similarly to the proof of [Kon10a, Theorem 2] we get the following statement.

Proposition 3.3. *Let $X_k^{l,n}$, $X_k^{l,n,+}$, $X_k^{l,n,-}$ be the stochastic processes constructed above. Then for every $l \in \mathbb{Z}$ the following statement holds.*

(i) For every $k \in \mathbb{Z}$ the sequence $\{X_k^{l,n},\ n \geq 1\}$ converges a.s. in the discrete topology of $\mathrm{C}[0,T]$ to a process X_k^l. Moreover, the family X_k^l, $k \in \mathbb{Z}$, is a modified massive Arratia flow started from $\mathbb{Z}$, i.e. it satisfies properties (F1)-(F4).

(ii) For every $k \in \mathbb{Z}_{\geq l}$ the sequence $\{X_k^{l,n,+},\ n \geq 1\}$ converges a.s. in the discrete topology of $\mathrm{C}[0,T]$ to a process $X_k^{l,+}$. Moreover, the family $X_k^{l,+}$, $k \in \mathbb{Z}_{\geq l}$, satisfies properties (F1)-(F4) with $\mathbb{Z}$ replaced by $\mathbb{Z}_{\geq l}$.

(iii) For every $k \in \mathbb{Z}_{\leq l}$ the sequence $\{X_k^{l,n,-},\ n \geq 1\}$ converges a.s. in the discrete topology of $\mathrm{C}[0,T]$ to a process $X_k^{l,-}$. Moreover, the family $X_k^{l,-}$, $k \in \mathbb{Z}_{\leq l}$, satisfies properties (F1)-(F4) with $\mathbb{Z}$ replaced by $\mathbb{Z}_{\leq l}$.

We further define the events

$$A^+_{l,j}(t) := \left\{ \max_{k\in\{l,\dots,l+j\}} \max_{s\in[0,t]} w_k(s) \le l+j+\frac{1}{2}, \quad \min_{s\in[0,t]} w_{l+j+1}(s) > l+j+\frac{1}{2} \right\}$$

and

$$A^-_{l,j}(t) := \left\{ \min_{k\in\{l-j,\dots,l\}} \min_{s\in[0,t]} w_k(s) \ge l-j-\frac{1}{2}, \quad \max_{s\in[0,t]} w_{l-j-1}(s) < l-j-\frac{1}{2} \right\}$$

for all $l \in \mathbb{Z}$, $j \in \mathbb{N}_0$ and $t \in [0,T]$.

Lemma 3.4. *Let $l \in \mathbb{Z}$ and $j \in \mathbb{N}_0$ and let the families $\{X^l_k,\ k \in \mathbb{Z}\}$, $\{X^{l,+}_k,\ k \in \mathbb{Z}_{\ge l}\}$, $\{X^{l,-}_k,\ k \in \mathbb{Z}_{\le l}\}$ be constructed above. Then $X^l_k = X^{l+p,+}_k$ on $A^+_{l,j}(T)$ for each $k > l+j$, $p \in \{0,\dots,j\}$, and $X^l_k = X^{l-p,-}_k$ on $A^-_{l,j}(T)$ for each $k < l-j$, $p \in \{0,\dots,j\}$.*

Proof. The statement of the lemma directly follows from the construction of the families of random processes X^l_k, $X^{l,+}_k$, $X^{l,-}_k$ and the events $A^+_{l,j}(T)$, $A^-_{l,j}(T)$. □

We denote

$$B^+_{l,N}(t) = \bigcup_{j=1}^{N} A^+_{l,j}(t), \quad B^-_{l,N}(t) = \bigcup_{j=1}^{N} A^-_{l,j}(t), \quad l \in \mathbb{Z},\ \ N \in \mathbb{N},\ \ t \in [0,T].$$

Lemma 3.5. *For each $T > 0$ there exist a constant $C = C_T > 0$ and a function $\beta_T(t) : (0,T] \to (0,\infty)$ depending only on T such that $t\beta_T(t) \to \frac{1}{8\sqrt{2}}$ as $t \to 0+$ and for every $l \in \mathbb{Z}$ and $N \in \mathbb{N}$*

$$\mathbb{P}\left(B^\pm_{l,N}(t)\right) \ge 1 - Ce^{-\beta_T(t)\left[(\sqrt{N}-\sqrt{2})\vee 1\right]}$$

for all $l \in \mathbb{Z}$, $N \in \mathbb{N}$ and $t \in [0,T]$.

Proof. Note that $\mathbb{P}\left(B^+_{l,N}(t)\right) = \mathbb{P}\left(B^-_{l,N}(t)\right) = \mathbb{P}\left(B^+_{0,N}(t)\right)$. Therefore, it is enough to estimate $\mathbb{P}\left(B^+_{0,N}(t)\right)$ for each $N \in \mathbb{N}$. We denote

$$M_{j,n}(t) = \left\{ \max_{k\in\{0,\dots,j\}} \max_{s\in[0,t]} w_k(s) = \max_{k\in\{j-n+2,\dots,j\}} \max_{s\in[0,t]} w_k(s) \right\}$$

and

$$R_{j,n}(t) = \left\{ \max_{k\in\{j-n+2,\dots,j\}} \max_{s\in[0,t]} w_k(s) \le j+\frac{1}{2}, \min_{s\in[0,t]} w_{j+1}(s) > j+\frac{1}{2} \right\}$$

for all $2 \le n \le j$ and $t \in [0,T]$. Then, by Funibi's theorem,

$$\begin{aligned}
\mathbb{P}\left(M_{j,n}(t)^c\right) &\le \mathbb{P}\left\{ \max_{k\in\{0,\dots,j-n+1\}} \max_{s\in[0,t]} w_k(s) > j \right\} \le \sum_{k=0}^{j-n+1} \mathbb{P}\left\{ \max_{s\in[0,t]} w_k(s) > j \right\} \\
&\le \sum_{k=0}^{j-n+1} \frac{\sqrt{2}}{\sqrt{\pi t}} \int_{j-k}^{+\infty} e^{-\frac{u^2}{2t}} du \le \sum_{k=n-1}^{\infty} \frac{\sqrt{2}}{\sqrt{\pi t}} \int_{k}^{+\infty} e^{-\frac{u^2}{2t}} du \\
&\le \frac{\sqrt{2}}{\sqrt{\pi t}} \int_{n-2}^{+\infty} \left(\int_x^{+\infty} e^{-\frac{u^2}{2t}} du \right) dx \\
&\le \frac{\sqrt{2}}{\sqrt{\pi t}} \int_{n-2}^{+\infty} u e^{-\frac{u^2}{2t}} du \le \frac{\sqrt{2t}}{\sqrt{\pi}} e^{-\frac{(n-2)^2}{2t}}.
\end{aligned} \tag{3.1}$$

Using the independence of w_k, $k \in \mathbb{Z}$, we obtain

$$\begin{aligned} \mathbb{P}\left(R_{j,n}(t)\right) &= \frac{\sqrt{2}}{\sqrt{\pi t}} \int_0^{\frac{1}{2}} e^{-\frac{u^2}{2t}} du \prod_{k=j-n+2}^{j} \frac{\sqrt{2}}{\sqrt{\pi t}} \int_0^{j+\frac{1}{2}-k} e^{-\frac{u^2}{2t}} du \\ &= \left(1 - \frac{\sqrt{2}}{\sqrt{\pi t}} \int_{\frac{1}{2}}^{\infty} e^{-\frac{u^2}{2t}} du\right) \prod_{k=0}^{n-2} \left(1 - \frac{\sqrt{2}}{\sqrt{\pi t}} \int_{k+\frac{1}{2}}^{+\infty} e^{-\frac{u^2}{2t}} du\right) \end{aligned} \tag{3.2}$$

for all $2 \leq n \leq j$ and $t \in [0, T]$. Taking a constant $a_T > 0$ such that $\ln(1-x) \geq -a_T x$ for all $0 \leq x \leq \frac{\sqrt{2}}{\sqrt{\pi T}} \int_{\frac{1}{2}}^{\infty} e^{-\frac{u^2}{2T}} du = \frac{\sqrt{2}}{\sqrt{\pi}} \int_{\frac{1}{2T}}^{\infty} e^{-\frac{u^2}{2}} du$, we next estimate

$$\begin{aligned} \ln \mathbb{P}\left(R_{j,n}(t)\right) &= \ln \left(1 - \frac{\sqrt{2}}{\sqrt{\pi t}} \int_{\frac{1}{2}}^{\infty} e^{-\frac{u^2}{2t}} du\right) + \sum_{k=0}^{n-2} \ln \left(1 - \frac{\sqrt{2}}{\sqrt{\pi t}} \int_{k+\frac{1}{2}}^{+\infty} e^{-\frac{u^2}{2t}} du\right) \\ &\geq -\frac{\sqrt{2} a_T}{\sqrt{\pi t}} \int_{\frac{1}{2}}^{\infty} e^{-\frac{u^2}{2t}} du - \sum_{k=0}^{\infty} \frac{\sqrt{2} a_T}{\sqrt{\pi t}} \int_{k+\frac{1}{2}}^{+\infty} e^{-\frac{u^2}{2t}} du \\ &\geq -\frac{2\sqrt{2} a_T}{\sqrt{\pi t}} \int_{\frac{1}{2}}^{\infty} e^{-\frac{u^2}{2t}} du - \frac{a_T \sqrt{2}}{\sqrt{\pi t}} \int_{\frac{1}{2}}^{\infty} \int_x^{+\infty} e^{-\frac{u^2}{2t}} du dx \geq -\tilde{C}_T e^{-\frac{1}{8t}}, \end{aligned}$$

where $\tilde{C}_T := \frac{10\sqrt{2T} a_T}{\sqrt{\pi}}$. Thus, for each $2 \leq n \leq j$ and $t \in [0, T]$, we get

$$\mathbb{P}\left(R_{j,n}(t)\right) \geq e^{-\tilde{C}_T e^{-1/8t}}.$$

Fix $p \in \mathbb{Z}_{\geq 3}$ and set $N_k = p + k$, $k \in \{1, \ldots, p\}$,

$$G_k(t) = R_{\tilde{N}_k, N_k}(t) \cap M_{\tilde{N}_k, N_k}(t),$$

where $\tilde{N}_k = \sum_{i=1}^k N_i$. Since $\bigcup_{k=1}^p G_k(t) \subseteq B_{0,N}^+(t)$ for $N = \tilde{N}_p = \frac{(3p+1)p}{2}$, one can estimate

$$\begin{aligned} \mathbb{P}\left(B_{0,N}^+(t)\right) &\geq \mathbb{P}\left(\bigcup_{k=1}^p G_k(t)\right) = \mathbb{P}\left(\bigcup_{k=1}^p \left(R_{\tilde{N}_k, N_k}(t) \setminus M_{\tilde{N}_k, N_k}(t)^c\right)\right) \\ &\geq \mathbb{P}\left(\left(\bigcup_{k=1}^p R_{\tilde{N}_k, N_k}(t)\right) \setminus \left(\bigcup_{k=1}^p M_{\tilde{N}_k, N_k}(t)^c\right)\right) \\ &\geq \mathbb{P}\left(\bigcup_{k=1}^p R_{\tilde{N}_k, N_k}(t)\right) - \mathbb{P}\left(\bigcup_{k=1}^p M_{\tilde{N}_k, N_k}(t)^c\right) \\ &\geq 1 - \mathbb{P}\left(\bigcap_{k=1}^p R_{\tilde{N}_k, N_k}(t)^c\right) - C_T \sum_{k=1}^p e^{-\frac{(N_k-2)^2}{2t}}, \end{aligned}$$

where we used (3.1) in the last step. Note that

$$\sum_{k=1}^p e^{-\frac{(N_k-2)^2}{2t}} = \sum_{k=1}^p e^{-\frac{(p+k-2)^2}{2t}} \leq \int_{p-2}^{+\infty} e^{-\frac{u^2}{2t}} du \leq 2T e^{-\frac{(p-2)^2}{2t}}, \tag{3.3}$$

since $p \geq 3$. Set $\gamma_T := -\ln\left(1 - e^{-\tilde{C}_T e^{-1/8T}}\right) > 0$. Using the inequality $1 - e^{-x} \leq x$ for $x \geq 0$, we get

$$1 - e^{-\tilde{C}_T e^{-1/8T}} \leq \tilde{C}_T e^{-\frac{1}{8T}} = e^{\ln \tilde{C}_T - \frac{1}{8T}}$$

Hence, by the independence of $R_{\tilde{N}_k, N_k}$, $k \in \{1, \dots, p\}$, and estimate (3.3), there exists a constant $C > 0$ depending on T such that

$$\mathbb{P}\left(B^+_{0,N}(t)\right) \geq 1 - \prod_{k=1}^{p} \mathbb{P}\left(R_{\tilde{N}_k, N_k}(t)^c\right) - Ce^{-\frac{(p-2)^2}{2t}}.$$

Using the observation (3.2) and the inequality $1 - e^{-x} \leq x$ for $x \geq 0$, we continue

$$\begin{aligned}\mathbb{P}\left(B^+_{0,N}(t)\right) &\geq 1 - (1 - e^{-\tilde{C}_T e^{-1/8t}})^p - Ce^{-\frac{(p-2)^2}{2t}} \\ &\geq 1 - e^{-\left[\left(\frac{1}{8t} - \ln \tilde{C}_T\right) \vee \gamma_T\right] p} - Ce^{-\frac{(p-2)^2}{2t}} \geq 1 - C_T e^{-\beta_T(t) p}\end{aligned}$$

for all $p \in \mathbb{Z}_{\geq 3}$ and $N = \frac{(3p+1)p}{2}$, where $\beta_T(t) = \left[\left(\frac{1}{8t} - \ln \tilde{C}_T\right) \vee \gamma_T\right] \wedge \frac{1}{t}$ and the constant C_T depends only on T. Next, for every $N \in \mathbb{Z}_{\geq 18}$ and $p = \left\lfloor \sqrt{N/2} \right\rfloor$ we have $N \geq \frac{(3p+1)p}{2}$. Hence,

$$\begin{aligned}\mathbb{P}\left(B^+_{0,N}(t)\right) &\geq \mathbb{P}\left(B^+_{0,\frac{(3p+1)p}{2}}(t)\right) \geq 1 - C_T e^{-\beta_T(t) p} = 1 - C_T e^{-\beta_T(t) \left\lfloor \sqrt{\frac{N}{2}} \right\rfloor} \\ &\geq 1 - C_T e^{-\beta_T(t)\left(\frac{\sqrt{N}}{\sqrt{2}} - 1\right)}\end{aligned}$$

for every $N \geq 18$.

It only remains to get the estimate for $N < 18$. We first note that

$$\begin{aligned}\mathbb{P}\left(A^+_{l,j}(t)^c\right) &\leq (j+2)\mathbb{P}\left\{\max_{s \in [0,t]} w_0(s) \geq \frac{1}{2}\right\} \\ &\leq \frac{\sqrt{2}(j+2)}{\sqrt{\pi t}} \int_{\frac{1}{2}}^{\infty} e^{-\frac{u^2}{2t}} du \leq \frac{4\sqrt{2T}(j+2)}{\sqrt{\pi}} e^{-\frac{1}{8t}}.\end{aligned}$$

Thus, for all integer numbers $N < 18$

$$\mathbb{P}\left(B^+_{0,N}(t)^c\right) \geq 1 - \sum_{j=1}^{N} \frac{4\sqrt{2T}(j+2)}{\sqrt{\pi}} e^{-\frac{1}{8t}} \geq 1 - C_T e^{-\frac{1}{8t}}.$$

This completes the proof of the lemma. □

Proof of Proposition 3.1. Let X^l_k, $k \in \mathbb{Z}$, $X^{l,+}_k$, $k \in \mathbb{Z}_{\geq l}$ and $X^{l,-}_k$, $k \in \mathbb{Z}_{\leq l}$, be the families of processes constructed in Proposition 3.3 for each $l \in \mathbb{Z}$. We fix $i < j$ from $\mathbb{Z}$, $t > 0$ and define for $l = \left\lfloor \frac{i+j}{2} \right\rfloor$ the measures

$$\mu^l_t(A) = \#\left(A \cap \{X^l_k(t),\ k \in \mathbb{Z}\}\right),$$

and

$$\mu_t^{l,+}(A) = \#\left(A \cap \{X_k^{l,+},\ k \in \mathbb{Z}_{\geq l}\}\right), \quad \mu_t^{l,-}(A) = \#\left(A \cap \{X_k^{l,-},\ k \in \mathbb{Z}_{\leq l}\}\right),$$

for all $A \in \mathcal{B}(\mathbb{R})$. Set also

$$A_{k,t}^l f = \int_{k-1}^k f(u)\mu_t^l(du) \quad \text{and} \quad A_{k,t}^{\pm} f = \int_{k-1}^k f(u)\mu_t^{l,\pm}(du), \quad k \in \mathbb{Z}.$$

By Proposition 3.3, the distributions of $(A_{k,t}^l f)_{k\in\mathbb{Z}}$ and $(A_{k,t}f)_{k\in\mathbb{Z}}$ coincide in $\mathbb{R}^{\mathbb{Z}}$.

We next assume that $j - i \geq 3$ and take arbitrary sets $A \in \mathfrak{M}_{-\infty}^i$, $B \in \mathfrak{M}_j^{+\infty}$. Then there exist Borel measurable sets $\tilde{A} \subseteq \mathbb{R}^{\mathbb{Z}_{\leq i}}$ and $\tilde{B} \subseteq \mathbb{R}^{\mathbb{Z}_{\geq j}}$ such that

$$A = \{(A_{k,t}f)_{k\in\mathbb{Z}_{\leq i}} \in \tilde{A}\} \quad \text{and} \quad B = \{(A_{k,t}f)_{k\in\mathbb{Z}_{\geq j}} \in \tilde{B}\}.$$

Then, using Lemma 3.5, we can estimate

$$\begin{aligned}
|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| &= \Big|\mathbb{P}\left(\left\{(A_{k,t}f)_{k\in\mathbb{Z}_{\leq i}} \in \tilde{A}\right\} \cap \left\{(A_{k,t}f)_{k\in\mathbb{Z}_{\geq j}} \in \tilde{B}\right\}\right) \\
&\qquad - \mathbb{P}\left\{(A_{k,t}f)_{k\in\mathbb{Z}_{\leq i}} \in \tilde{A}\right\}\mathbb{P}\left\{(A_{k,t}f)_{k\in\mathbb{Z}_{\geq j}} \in \tilde{B}\right\}\Big| \\
&= \Big|\mathbb{P}\left(\left\{(A_{k,t}^l f)_{k\in\mathbb{Z}_{\leq i}} \in \tilde{A}\right\} \cap \left\{(A_{k,t}^l f)_{k\in\mathbb{Z}_{\geq j}} \in \tilde{B}\right\}\right) \\
&\qquad - \mathbb{P}\left\{(A_{k,t}^l f)_{k\in\mathbb{Z}_{\leq i}} \in \tilde{A}\right\}\mathbb{P}\left\{(A_{k,t}^l f)_{k\in\mathbb{Z}_{\geq j}} \in \tilde{B}\right\}\Big| \\
&\leq \Big|\mathbb{P}\left(\left\{(A_{k,t}^l f)_{k\in\mathbb{Z}_{\leq i}} \in \tilde{A}\right\} \cap \left\{(A_{k,t}^l f)_{k\in\mathbb{Z}_{\geq j}} \in \tilde{B}\right\} \cap B_{l,j-l}^+ \cap B_{l,l-i}^-\right) \\
&\qquad - \mathbb{P}\left(\left\{(A_{k,t}^l f)_{k\in\mathbb{Z}_{\leq i}} \in \tilde{A}\right\} \cap B_{l,l-i}^-\right)\mathbb{P}\left(\left\{(A_{k,t}^l f)_{k\in\mathbb{Z}_{\geq j}} \in \tilde{B}\right\} \cap B_{l,j-l}^+\right)\Big| \\
&\quad + Ce^{-\beta\sqrt{j-i}}
\end{aligned}$$

for some positive constants C, β independent on i, j and A, B. By Lemmas 3.4 and 3.5, we get

$$\begin{aligned}
&|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \\
&\leq \Big|\mathbb{P}\left(\left\{(A_{k,t}^{l,-} f)_{k\in\mathbb{Z}_{\leq i}} \in \tilde{A}\right\} \cap \left\{(A_{k,t}^{l+1,+} f)_{k\in\mathbb{Z}_{\geq j}} \in \tilde{B}\right\} \cap B_{l,j-l}^+ \cap B_{l,l-i}^-\right) \\
&\qquad - \mathbb{P}\left(\left\{(A_{k,t}^{l,-} f)_{k\in\mathbb{Z}_{\leq i}} \in \tilde{A}\right\} \cap B_{l,l-i}^-\right)\mathbb{P}\left(\left\{(A_{k,t}^{l+1,+} f)_{k\in\mathbb{Z}_{\geq j}} \in \tilde{B}\right\} \cap B_{l,j-l}^+\right)\Big| \\
&\quad + Ce^{-\beta\sqrt{j-i}} \\
&\leq \Big|\mathbb{P}\left(\left\{(A_{k,t}^{l,-} f)_{k\in\mathbb{Z}_{\leq i}} \in \tilde{A}\right\} \cap \left\{(A_{k,t}^{l+1,+} f)_{k\in\mathbb{Z}_{\geq j}} \in \tilde{B}\right\}\right) \\
&\qquad - \mathbb{P}\left\{(A_{k,t}^{l,-} f)_{k\in\mathbb{Z}_{\leq i}} \in \tilde{A}\right\}\mathbb{P}\left\{(A_{k,t}^{l+1,+} f)_{k\in\mathbb{Z}_{\geq j}} \in \tilde{B}\right\}\Big| \\
&\quad + C_1 e^{-\beta\sqrt{j-i}},
\end{aligned}$$

where $C_1 > 0$ is a constant independent on i, j and A, B. Hence, using independence of $(A_{k,t}^{l,-})_{k\in\mathbb{Z}\leq i}$ and $(A_{k,l}^{l+1,+})_{k\in\mathbb{Z}\geq j}$, we can conclude that

$$|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \leq C_1 e^{-\beta\sqrt{j-i}}.$$

Now, taking the supremum over $A \in \mathfrak{M}_{-\infty}^{i}$ and $B \in \mathfrak{M}_{j}^{+\infty}$, we obtain the statement of the proposition. □

4 Proof of Theorem 2.2

In this section, we will prove Theorem 2.2. According to [IL71, Theorem 18.5.3] and Proposition 3.1, the statement of Theorem 2.2 follows from the fact that $\mathbb{E}\left[(A_{k,t}f)^{2+\delta}\right] < \infty$ for some $\delta > 0$. We will show that $A_{k,t}f$ have finite moments of all orders.

Proposition 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. Then for every $a < b$ from $\mathbb{R}$, $T > 0$ and $p \geq 1$ there exists a constant $C > 0$ such that*

$$\mathbb{E}\left[\left|\int_a^b f(u)\mu_t(du)\right|^p\right] < C$$

for all $t \in [0, T]$.

Proof. Set $\|f\|_\infty := \sup_{u\in\mathbb{R}} |f(u)|$. Using the definition of μ_t, we estimate

$$\begin{aligned}
\mathbb{E}\left[\left|\int_a^b f(u)\mu_t(du)\right|^p\right] &\leq \|f\|_\infty^p \mathbb{E}\left[\mu_t([a,b])^p\right] \\
&= \|f\|_\infty^p \mathbb{E}\left[\left(\sum_{l\in\mathbb{Z}} \mathbb{I}_{\{x_l(t)\in[a,b]\}}\right)^p\right] \\
&= \|f\|_\infty^p \sum_{k<n} \mathbb{E}\left[\left(\sum_{l=k}^{n} \mathbb{I}_{\{x_l(t)\in[a,b]\}}\right)^p \mathbb{I}_{B_{k,n}^{a,b}(t)}\right]
\end{aligned}$$

where $B_{k,n}^{a,b}(t) := \{x_{k-1}(t) < a, x_k(t) \geq a, x_n(t) \leq b, x_{n+1}(t) > b\}$. By the Cauchy-Schwartz inequality, we get

$$\mathbb{E}\left[\left|\int_a^b f(u)\mu_t(du)\right|^p\right] \leq \|f\|_\infty^p \sum_{k<n} \mathbb{E}\left[\left(\sum_{l=k}^{n} \mathbb{I}_{\{x_l(t)\in[a,b]\}}\right)^{2p}\right]^{\frac{1}{2}} \mathbb{P}(B_{k,n}^{a,b}(t))^{\frac{1}{2}}.$$

Next, using Hölder's inequality, we obtain

$$\mathbb{E}\left[\left|\int_a^b f(u)\mu_t(du)\right|^p\right] \leq \|f\|_\infty^p \sum_{k<n} (n-k+1)^{p-\frac{1}{2}} \mathbb{E}\left[\sum_{l=k}^{n} \mathbb{I}_{\{x_l(t)\in[a,b]\}}\right]^{\frac{1}{2}} \mathbb{P}(B_{k,n}^{a,b}(t))^{\frac{1}{2}}.$$

Since $(x_{k+l}(t)-l)_{k\in\mathbb{Z}}$ and $(x_k(t))_{k\in\mathbb{Z}}$ have the same distributions, we conclude

$$\mathbb{E}\left[\sum_{l=k}^{n}\mathbb{I}_{\{x_l(t)\in[a,b]\}}\right]=\sum_{l=k}^{n}\mathbb{E}\left[\mathbb{I}_{\{x_l(t)-l\in[-l+a,-l+b]\}}\right]$$
$$=\sum_{l=k}^{n}\mathbb{E}\left[\mathbb{I}_{\{x_0(t)\in[-l+a,-l+b]\}}\right]\leq b-a+1.$$

Consequently, we have

$$\mathbb{E}\left[\left|\int_a^b f(u)\mu_t(du)\right|^p\right]\leq\|f\|_\infty^p(b-a+1)^{\frac{1}{2}}\sum_{k<n}(n-k+1)^{p-\frac{1}{2}}\mathbb{P}(B_{k,n}^{a,b}(t))^{\frac{1}{2}}.$$

Now it remains to show that the series $\sum_{k<n}(n-k+1)^{p-\frac{1}{2}}\mathbb{P}(B_{k,n}^{a,b}(t))^{\frac{1}{2}}$ converges and is uniformly bounded on $[0,T]$. We first estimate $\mathbb{P}\{x_0(t)\geq c\}$ for every $c\geq 1$. According to [IW89, Theorem II.7.2'], there exists a Brownian motion $w(t)$, $t\geq 0$, probably on an extended probability space, such that $x_0(t)=w\left(\langle x_0\rangle_t\right)$, $t\geq 0$. Since the quadratic variation

$$\langle x_0\rangle_t=\int_0^t\frac{ds}{m_0(s)}\leq t$$

for all $t\geq 0$, we get

$$\begin{aligned}\mathbb{P}\{x_0(t)\geq c\}&\leq\mathbb{P}\left\{\max_{s\in[0,t]}x_0(s)\geq c\right\}=\mathbb{P}\left\{\max_{s\in[0,t]}w\left(\langle x_0\rangle_s\right)\geq c\right\}\\&\leq\mathbb{P}\left\{\max_{s\in[0,t]}w(s)\geq c\right\}=\frac{\sqrt{2}}{\sqrt{\pi t}}\int_c^\infty e^{-\frac{u^2}{2t}}du\leq\frac{2\sqrt{2t}}{\sqrt{\pi}}e^{-\frac{c^2}{2t}}.\end{aligned}\tag{4.1}$$

Similarly,

$$\mathbb{P}\{x_0(t)\leq -c\}\leq\frac{2\sqrt{2t}}{\sqrt{\pi}}e^{-\frac{c^2}{2t}}\tag{4.2}$$

for all $c\geq 1$. We next rewrite

$$\begin{aligned}\sum_{k<n}(n-k+1)^{p-\frac{1}{2}}\mathbb{P}(B_{k,n}^{a,b}(t))^{\frac{1}{2}}&=\sum_{k=0}^{+\infty}\sum_{n=k+1}^{+\infty}(n-k+1)^{p-\frac{1}{2}}\mathbb{P}(B_{k,n}^{a,b}(t))^{\frac{1}{2}}\\&+\sum_{k=-\infty}^{-1}\sum_{n=k+1}^{0}(n-k+1)^{p-\frac{1}{2}}\mathbb{P}(B_{k,n}^{a,b}(t))^{\frac{1}{2}}\\&+\sum_{k=-\infty}^{-1}\sum_{n=1}^{+\infty}(n-k+1)^{p-\frac{1}{2}}\mathbb{P}(B_{k,n}^{a,b}(t))^{\frac{1}{2}}.\end{aligned}\tag{4.3}$$

In the first term of the right hand side of (4.3), we estimate

$$\mathbb{P}\left(B_{k,n}^{a,b}(t)\right)\leq\mathbb{P}\{x_{k-1}(t)\leq a,x_n(t)\leq b\}=\mathbb{E}\left[\mathbb{I}_{\{x_{k-1}(t)\leq a\}}\mathbb{I}_{\{x_n(t)\leq b\}}\right]$$

$$\leq \sqrt{\mathbb{E}\left[\mathbb{I}^2_{\{x_{k-1}(t)\leq a\}}\right]}\sqrt{\mathbb{E}\left[\mathbb{I}^2_{\{x_n(t)\leq b\}}\right]}$$
$$= \sqrt{\mathbb{P}\left\{x_{k-1}(t)\leq a\right\}}\sqrt{\mathbb{P}\left\{x_n(t)\leq b\right\}}.$$

Since the distributions of the random variables $x_l(t) - l$ and $x_0(t)$ coincide, we get

$$\mathbb{P}\left\{x_{k-1}(t)\leq a\right\} = \mathbb{P}\left\{x_0(t)\leq -(k-1)+a\right\} \leq \frac{2\sqrt{2t}}{\sqrt{\pi}}e^{-\frac{(k-1-a)^2}{2t}}$$

for all $k \geq a+2$, and

$$\mathbb{P}\left\{x_n(t)\leq b\right\} = \mathbb{P}\left\{x_0(t)\leq -(n-b)\right\} \leq \frac{2\sqrt{2t}}{\sqrt{\pi}}e^{-\frac{(n-b)^2}{2t}}$$

for all $n \geq b+1$, by (4.2). This implies that

$$\sum_{k=0}^{+\infty}\sum_{n=k+1}^{+\infty}(n-k+1)^{p-\frac{1}{2}}\mathbb{P}(B^{a,b}_{k,n}(t))^{\frac{1}{2}} \leq C$$

for all $t \in [0,T]$ and some constant $C > 0$. Similarly, estimating $\mathbb{P}(B^{a,b}_{k,n}(t))$ by $\mathbb{P}\big\{x_k(t) \geq a, x_{n+1}(t) \geq b\big\}$ and $\mathbb{P}\left\{x_k(t) \geq a, x_n(t) \leq b\right\}$ in the second and third terms of the right hand side of (4.3), respectively, and using (4.1), (4.2), we get

$$\sum_{k=-\infty}^{-1}\sum_{n=k+1}^{0}(n-k+1)^{p-\frac{1}{2}}\mathbb{P}(B^{a,b}_{k,n}(t))^{\frac{1}{2}} + \sum_{k=-\infty}^{-1}\sum_{n=1}^{+\infty}(n-k+1)^{p-\frac{1}{2}}\mathbb{P}(B^{a,b}_{k,n}(t))^{\frac{1}{2}} \leq C$$

for all $t \in [0,T]$. This completes the proof of the lemma. $\square$

We will next show that $\sigma^2_t(f)$ is strictly positive for some time $t > 0$ and function f. The following lemma is true.

Lemma 4.2. *Let $f \in \mathrm{C}^3_b(\mathbb{R})$ be an odd, 1-periodic function. Then $\frac{\sigma^2_t(f)}{t} \to (f'(0))^2$ as $t \to 0+$. In particular, there exists $t > 0$ such that $\sigma^2_t(f) > 0$ if $f'(0) \neq 0$.*

Proof. For the proof of the lemma, we will use the fact that the particles in the modified massive Arratia flow become more independent for small times t.

Let

$$\tilde{A}_{k,t}f := \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} f(u)\mu_t(du)$$

for each $k \in \mathbb{Z}$ and $t \geq 0$. Using the fact that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$ and the periodicity of f, we conclude that $\mathbb{E}\big[\tilde{A}_{k,t}f\big] = 0$. Let

$$\tilde{Y}^n_t(f) := \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\tilde{A}_{k,t}f = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\left(\tilde{A}_{k,t}f - \mathbb{E}\left[\tilde{A}_{k,t}f\right]\right), \quad n \geq 1.$$

Similarly to the proof of Proposition 3.1, one can show that the family $\left(\tilde{A}_{k,t}f\right)_{k\in\mathbb{Z}}$ satisfies the mixing condition with the same bound for the mixing coefficients. Hence, $\left\{\tilde{Y}^n_t(f)\right\}_{n\geq 0}$ converges to a Gaussian random variable with mean 0 and variance

$$\begin{aligned}\tilde{\sigma}^2_t(f) &:= \operatorname{Var}\tilde{A}_{0,t}f + 2\sum_{k=1}^{\infty}\operatorname{Cov}\left(\tilde{A}_{0,t}f, \tilde{A}_{k,t}f\right)\\ &= \mathbb{E}\left[\left(\tilde{A}_{0,t}f\right)^2\right] + 2\sum_{k=1}^{\infty}\mathbb{E}\left[\tilde{A}_{0,t}f\tilde{A}_{k,t}f\right].\end{aligned}$$

On the other side, one can estimate

$$\mathbb{E}\left[\left(Y^n_t(f) - \tilde{Y}^n_t(f)\right)^2\right] = \frac{1}{n}\mathbb{E}\left[(\xi_n - \mathbb{E}\left[\xi_n\right])^2\right] \leq \frac{1}{n}\mathbb{E}\left[\xi_n^2\right],$$

where

$$\xi_n := \int_0^{\frac{1}{2}} f(u)\mu_t(du) - \int_n^{n+\frac{1}{2}} f(u)\mu_t(du).$$

Due to Lemma 4.1 and the stationarity of the modified massive Arratia flow, we estimate

$$\mathbb{E}\left[\xi_n^2\right] \leq 2\mathbb{E}\left[\left(\int_0^{\frac{1}{2}} f(u)\mu_t(du)\right)^2\right] + 2\mathbb{E}\left[\left(\int_{\frac{1}{2}}^{1} f(u)\mu_t(du)\right)^2\right] < \infty.$$

Thus, $\mathbb{E}\left[\left(Y^n_t(f) - \tilde{Y}^n_t(f)\right)^2\right] \to 0$ as $n \to \infty$. Therefore, according to Theorem 2.2, $\left\{\tilde{Y}^n_t(f)\right\}_{n\geq 0}$ converges in distribution to the same limit as $\left\{Y^n_t(f)\right\}_{n\geq 0}$. This implies that $\sigma^2_t(f) = \tilde{\sigma}^2_t(f)$.

Let $(w_k)_{k\in\mathbb{Z}}$ be a family of Brownian motions that were used for the definition $X^{l,n}_k$ in Proposition 3.3. Using Proposition 3.3, we get

$$\begin{aligned}\mathbb{E}\left[\left(\tilde{A}_{0,t}f\right)^2\right] &= \mathbb{E}\left[f^2(w_0(t))\right]\\ &\quad + \mathbb{E}\left[\left(\left(\tilde{A}_{0,t}f\right)^2 - f^2(w_0(t))\right)\mathbb{I}_A\right],\end{aligned} \tag{4.4}$$

where $A := \left\{|X_0(t)| \geq \frac{1}{2}\right\} \cup \left\{X_{-1}(t) \geq -\frac{1}{2}\right\} \cup \left\{X_1(t) \leq \frac{1}{2}\right\}$. By Taylor's formula and the equality $f(0) = 0$, that follows from the fact that f is odd function, the first term of the right hand side of the equality above can be rewritten as

$$\begin{aligned}\mathbb{E}\left[f^2(w_0(t))\right] &= f^2(0) + \frac{1}{2}\frac{d^2f^2}{dx^2}(0)\mathbb{E}\left[w_0^2(t)\right] + o(t)\\ &= \left(f'(0)\right)^2 t + o(t).\end{aligned}$$

Using Hölder's inequality, the square of the second term of (4.4) can be estimated by

$$I_t := \left(\mathbb{E}\left[\left(\tilde{A}_{0,t}f\right)^4\right] + \mathbb{E}\left[f^4(w_0(t))\right]\right)\mathbb{P}\left(A\right).$$

Now, by the boundedness of f and Proposition 4.1, there exists $C > 0$ such that

$$I_t \leq C\left(\mathbb{P}\left\{|X_0(t)| \geq \frac{1}{2}\right\} + \mathbb{P}\left\{X_{-1}(t) \geq -\frac{1}{2}\right\} + \mathbb{P}\left\{X_1(t) \leq \frac{1}{2}\right\}\right)$$
$$\leq 3C\mathbb{P}\left\{|X_0(t)| \geq \frac{1}{2}\right\} \leq 3C\mathbb{P}\left\{|w_0(t)| \geq \frac{1}{2}\right\} \leq Ce^{-\frac{1}{8t}}.$$

Thus,

$$\frac{1}{t}\mathbb{E}\left[\left(\tilde{A}_{0,t}f\right)^2\right] \to (f'(0))^2$$

as $t \to 0$.

Similarly, we estimate $\mathbb{E}\big[\tilde{A}_{0,t}f\tilde{A}_{k,t}f\big]$ for each $k \in \mathbb{N}$. Using the notation from the proof of Proposition 3.1 and denoting

$$\tilde{A}^l_{k,t}f := \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} f(u)\mu^l_t(du) \quad \text{and} \quad \tilde{A}^{\pm}_{k,t}f := \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} f(u)\mu^{l,\pm}_t(du), \quad k \in \mathbb{Z},$$

we estimate for $k \geq 1$ and $l := \left\lfloor \frac{k}{2} \right\rfloor$

$$\mathbb{E}\left[\tilde{A}_{0,t}f\tilde{A}_{k,t}f\right] = \mathbb{E}\left[\tilde{A}^l_{0,t}f\tilde{A}^l_{k,t}f\right]$$
$$= \mathbb{E}\left[\tilde{A}^l_{0,t}f\tilde{A}^l_{k,t}f\mathbb{I}_{B^+_{l,k-l}\cap B^-_{l,l}}\right] + \mathbb{E}\left[\tilde{A}^l_{0,t}f\tilde{A}^l_{k,t}f\mathbb{I}_{\left(B^+_{l,k-l}\cap B^-_{l,l}\right)^c}\right].$$

By Lemma 3.5, Proposition 4.1, and Hölder's inequality, the square of the second term of the equality above can be estimated by

$$\mathbb{E}\left[\left(\tilde{A}^l_{0,t}f\right)^2\right]\mathbb{E}\left[\left(\tilde{A}^l_{k,t}f\right)^2\right]\left(\mathbb{P}\left(\left(B^+_{l,k-l}\right)^c\right) + \mathbb{P}\left(\left(B^-_{l,l}\right)^c\right)\right) \leq C_T e^{-\beta_T(t)\left[\left(\sqrt{k}-\sqrt{2}\right)\vee 1\right]}$$

for all $t \in [0, T]$, where the function $\beta_T : (0, T] \to (0, \infty)$ and the constant C_T depend only on T and $t\beta_T(t) \to \frac{1}{8\sqrt{2}}$ as $t \to 0$. Using now Lemma 3.4, we get

$$\mathbb{E}\left[\tilde{A}^l_{0,t}f\tilde{A}^l_{k,t}f\mathbb{I}_{B^+_{l,k-l}\cap B^-_{l,l}}\right] = \mathbb{E}\left[\tilde{A}^{l,-}_{0,t}f\tilde{A}^{l+1,+}_{k,t}f\mathbb{I}_{B^+_{l,k-l}\cap B^-_{l,l}}\right]$$
$$= \mathbb{E}\left[\tilde{A}^{l,-}_{0,t}f\tilde{A}^{l+1,+}_{k,t}f\right] + \mathbb{E}\left[\tilde{A}^{l,-}_{0,t}f\tilde{A}^{l+1,+}_{k,t}f\mathbb{I}_{\left(B^+_{l,k-l}\cap B^-_{l,l}\right)^c}\right].$$

We can similarly estimate the square of the second term of the equality above by the expression $C_T e^{-\beta_T(t)\left[\left(\sqrt{k}-\sqrt{2}\right)\vee 1\right]}$. By the independence of $\tilde{A}^{l,-}_{0,t}f$ and $\tilde{A}^{l+1,+}_{k,t}f$, the first term of the equality above equals $\mathbb{E}\big[\tilde{A}^{l,-}_{0,t}f\big]\mathbb{E}\big[\tilde{A}^{l+1,+}_{k,t}f\big]$. Furthermore,

$$0 = \mathbb{E}\left[\tilde{A}_{0,t}f\right]\mathbb{E}\left[\tilde{A}_{k,t}f\right] = \mathbb{E}\left[\tilde{A}^{l,-}_{0,t}f\right]\mathbb{E}\left[\tilde{A}^{l+1,+}_{k,t}f\right] + R_{k,t},$$

where

$$|R_{k,t}|^2 \leq C_T e^{-\beta_T(t)\left[\left(\sqrt{k}-\sqrt{2}\right)\vee 1\right]}$$

for all $t \in [0, T]$ and $k \geq 1$. Combining the estimates above, we have shown that

$$\begin{aligned}\left|\mathrm{Cov}\left(\tilde{A}_{0,t}f\tilde{A}_{k,t}f\right)\right| &\leq \left|\mathbb{E}\left[\tilde{A}_{0,t}f\tilde{A}_{k,t}f\right] - \mathbb{E}\left[\tilde{A}_{0,t}f\right]\mathbb{E}\left[\tilde{A}_{k,t}f\right]\right| \\ &\leq C_T e^{-\frac{\beta_T(t)}{2}\left[\left(\sqrt{k}-\sqrt{2}\right)\vee 1\right]}.\end{aligned}$$

Using the dominated convergence theorem and the fact that $t\beta_T(t) \to \frac{1}{8\sqrt{2}}$, we conclude that

$$\frac{1}{t}\sum_{k=1}^{\infty}\left|\mathrm{Cov}\left(\tilde{A}_{0,t}f, \tilde{A}_{k,t}f\right)\right| \leq \frac{1}{t}\sum_{k=1}^{\infty} C_T e^{-\frac{\beta_T(t)}{2}\left[\left(\sqrt{k}-\sqrt{2}\right)\vee 1\right]} \to 0$$

as $t \to 0+$. This completes the proof of the statement. □

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