**DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY**

DESY 86-044
~~April~~ 1986



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ISSN 0418-9833

NOTKESTRASSE 85 · 2 HAMBURG 52

DESY 86-044
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ISSN 0418-9833

# ON GEOMETRY AND PHYSICS OF STAGGERED LATTICE FERMIONS

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Talk given at the NATO Workshop on Clifford Algebra, Canterbury 1985

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ABSTRACT. This report contains: 1.Introduction. 2.The Problem of Lattice Fermions. 3.Relation between Dirac Fields and Differential Forms. 4.Dirac Kaehler Equation and Clifford Product on a Cubic Lattice. 5.The Symmetry of the Dirac Kaehler Equation on the Lattice. 6.A Physics Problem. 7.Summary and Outlook.

## 1. INTRODUCTION

In a series of lectures E.Kaehler [1] discussed the possibility of interpreting the equation ('Dirac Kaehler Equation': DKE)

$$(d - \delta + m)\Phi = 0 \tag{1}$$

as a generalization of the Dirac equation. Why have people showed a renewed interest in the DKE at the present time? It was shown [2] [3] [4] that a systematic lattice approximation of Equ. (1) is equivalent to the Kogut Susskind description [5] of lattice fermions! This approximation was done in the spirit of the DeRham isomorphy which maps the differential form $\Phi$, the exterior differentiation $d$ , and the codifferential operator $\delta$ on cochains,boundary operator, and coboundary operator on the lattice. How is this fact related to our topic of *CLIFFORD ALGEBRA*? The connection between the DKE and the Dirac equation relies heavily on the introduction of a Clifford product for differential forms.Thus the many important physics questions which one wants to solve with the help of the lattice approximation of the Quantum Chromodynamics(QCD) lead to interesting new problems related to Clifford algebras. It is the aim of my lecture to illustrate this assertion by examples taken from actual research in lattice QCD.

## 2. THE PROBLEM OF LATTICE FERMIONS

The fundamental formula for the calculation of the expectation value of a physical observable $\Omega$ is [6] [7]

$$\langle\Omega\rangle = \frac{1}{Z}\int \mathcal{D}[\mathbf{A}]\mathcal{D}[\psi,\overline{\psi}]\Omega[\mathbf{A},\overline{\psi},\psi]e^{-S(\mathbf{A},\overline{\psi},\psi)}. \tag{2}$$

$\mathbf{A}(x)$ denotes the colour gauge potential describing the gluons, $\psi(x), \overline{\psi}(x)$ the quark fields. The action $S = S_g + S_q$ consists of a gluon part

$$S_g = \frac{1}{2g^2} \int dx \;\; trace(\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu}), \tag{3}$$

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + i[\mathbf{A}_\mu, \mathbf{A}_\nu], \tag{4}$$

and a part describing quark fields interacting with gluon fields

$$S_q = \int dx \overline{\psi}(x)(D_\mu \gamma^\mu + m)\psi(x), \tag{5}$$

$$D_\mu \psi(x) = (\partial_\mu + i\mathbf{A}_\mu)\psi(x). \tag{6}$$

The expression (2) is supposed to describe averaging $\Omega[...]$ with a normalized measure $\frac{1}{Z}\mathcal{D}[\mathbf{A}]\mathcal{D}[\psi, \overline{\psi}]$ over all field configurations. This procedure is not well defined mathematically. The significance of formula (2) for physics is based mainly on its formal perturbative evaluation, which leads to an understanding of many experimental facts in QCD, and it leads in QED to experimentally confirmed results of highest precision [6]. In order to make Eq. (2) more rigorous mathematically, one considers it for a field theory in Euclidean space time, using imaginary time coordinates. Real time field theory follows by analytic continuation [8].This first step we have already performed by the definition of an 'Euclidean' action in (3), (5).The real time action would have been imaginary. The next step is to approximate Euclidean space time by a finite lattice. Then the 'path integral' in (2) becomes a finite dimensional integral. It is hoped that the 'thermodynamic limit' to an infinite volume of space time, and a 'renormalized continuum limit' leads to an interpretation of (2) which is acceptable for physics.

A guiding principle for the formulation of the lattice approximation of gauge theories is given by the geometrical meaning of the basic quantities. This leads to the 'Wilson action' for gauge fields on the lattice. Let us denote in a cubic lattice $\overline{\Gamma}$, by $\overline{x} = \overline{b}(\overline{n}^1, ..., \overline{n}^d)$ the lattice points, by $[\overline{x}, \mu_1, ..., \mu_h]$ the lattice h-cells, by $\overline{e}_\mu$ the unit lattice vector (see Fig.1.);we put the lattice constant $\overline{b} = 1$ most of the time. Then the geometric meaning of the gauge potential $\mathbf{A}_\mu$ as connection in the principle bundle of the colour group$\mathcal{P}[R^d, SU(3)]$ leads to the associated lattice description of $\mathbf{A}_\mu$ by finite parallel transports $U(\overline{x})$ of a colour vector along links (1-cells) $[\overline{x}, \mu]$:

$$U(\overline{x}, \mu) \sim \exp(i \int_{[\overline{x}, \mu]} \mathbf{A}_\mu(x) dx^\mu), \qquad\qquad U(\overline{x} + \overline{e}_\mu, -\mu) = U^{-1}(\overline{x}, \mu). \tag{7}$$

The lattice analog of the field strength is the parallel transport around a plaquette $\mathbf{P} = [\overline{x}, \mu\nu]$

$$U(\mathbf{P}) = U(\overline{x}, \mu\nu) = U^{-1}(\overline{x}, \nu)U^{-1}(\overline{x} + \overline{e}_\nu, \mu)U(\overline{x} + \overline{e}_\mu, \nu)U(\overline{x}, \mu) \sim \exp(i\mathbf{F}_{\mu\nu}(\overline{x})). \tag{8}$$

With these quantities we can define the Wilson action for the SU(n)-gauge theory [9]

$$S_g = -\frac{\beta}{2n} \sum_{\mathbf{P} \in \overline{\Gamma}} (U(\mathbf{P}) + U^{-1}(\mathbf{P}) - 2), \tag{9}$$

which in a formal continuum limit $\overline{b} \to 0$ approaches the gauge action $S_g$ of Eq. (3). Of course, these classical limits mean little for the existence of the renormalized quantum mechanical limit.

Next we have to find the lattice approximation of the quark fields. In a first naive attempt one considers fermion fields as defined on lattice points, and one substitutes differentiation by a difference approximation which respects the Hermite symmetry of the Dirac operator. This 'naive' free lattice fermion action reads then to

$$S_q = \sum_{\overline{x}}(\sum_{\mu} \frac{1}{2\overline{b}}[\overline{\psi}(\overline{x})\gamma_\mu\psi(\overline{x}+\overline{e}_\mu) - \overline{\psi}(\overline{x}+\overline{e}_\mu)\gamma_\mu\psi(\overline{x})] + \overline{m}\overline{\psi}(\overline{x})\psi(\overline{x})). \tag{10}$$

The propagator related to $S_q$ is in momentum representation

$$G(\overline{p}) = (\sum_{\mu} \gamma_\mu \frac{1}{\overline{b}} \sin \overline{p}_\mu \overline{b} + \overline{m})^{-1}, \tag{11}$$

with $\overline{p}$ varying in the first Brillouin zone: $-\frac{\pi}{\overline{b}} < \overline{p} \leq \frac{\pi}{\overline{b}}$. However,this propagator poses some serious problems for physics! For lattice spacing $\overline{b} \to 0, G(\overline{p})$ is non-vanishing in 16 different regions, namely for $\overline{p}_\mu \approx 0, \pi, \quad \mu = 1,2,3,4$, where it approaches the continuum expression $G_{cont}(p) = (p_\mu\gamma^\mu + m)^{-1}$. Thus the action (10) describes not one but 16 Dirac particles [10].

This spectrum degeneracy is related to the symmetry [11] of the action $S_q$ under the transformations

$$(\hat{M}^H\psi)(\overline{x}) = e^{i\pi(\overline{e}_H,\overline{x})}M^H\psi(\overline{x}), \qquad (\hat{M}^H\overline{\psi})(\overline{x}) = e^{i\pi(\overline{e}_H,\overline{x})}\overline{\psi}(\overline{x})M^{H\dagger},$$

$$M^H = \prod_{\mu\in H} M^\mu, \qquad M^\mu = i\gamma^5\gamma^\mu, \qquad \overline{e}_H = \sum_{\mu\in H} \overline{e}_\mu. \tag{12}$$

Here we used a multi-index notation $H = (\mu_1, ...\mu_h), \quad \mu_1 < \mu_2 < ... < \mu_h$. The $\hat{M}$ satisfy the defining relations of the $\gamma$- matrices: $\hat{M}^\mu\hat{M}^\nu + \hat{M}^\nu\hat{M}^\mu = 2\delta^{\mu\nu}$. They transform the $\gamma$-matrices like

$$(\hat{M}^H)^{-1}\gamma^K\hat{M}^H = e^{i\pi(\overline{e}_H,\overline{e}_K)}\gamma^K, \tag{13}$$

from which the invariance of the action (10) follows immediately. The phase factor $\exp i\pi(\overline{e}_H,\overline{x})$ acts on the Fourier transform $\tilde{\psi}(\overline{p}) = (2\pi)^{d/2}\sum_{\overline{x}} e^{i\overline{p}\overline{x}}\psi(\overline{x})$ by translating the argument
$(\hat{M}^H\tilde{\psi})(\overline{p}) = M^H\tilde{\psi}(\overline{p}+\pi\overline{e}_H)$ The group $\{\hat{M}^H\}$ acts as a permutation group of the 16 regions of degeneracy. Therefore it is called sometimes 'spectrum doubling group': SDG.

In order to reduce the unwanted inflation of fields, it was suggested [5] that one use the symmetry of the SDG to separate the action (10) into parts with lower degeneracy. For this we follow a procedure which prepares our later discussion. We imbed the lattice $\overline{\Gamma}$ in a lattice $\Gamma$ with the double lattice spacing (see Fig.1.):

$$x \in \Gamma : x = b(n^1, ..., n^d) = 2\overline{b}(\overline{n}^1, ..., \overline{n}^d), \qquad e_\mu = 2\overline{e}_\mu;$$

$$\overline{x} \in \overline{\Gamma} : \overline{x} = x + \frac{1}{2}e_{H(x)}, \qquad \phi(x, H) \equiv \chi(\overline{x}) \tag{14}$$

With this notation we transform the naive Dirac fields

$$\psi_a(\overline{x}) = \gamma_{ai}^{H(\overline{x})}\chi_i(\overline{x}), \qquad \chi_i(\overline{x}) = \sum_a (\gamma^{H(\overline{x})})^{-1}_{ia}\psi_a(\overline{x}). \tag{15}$$

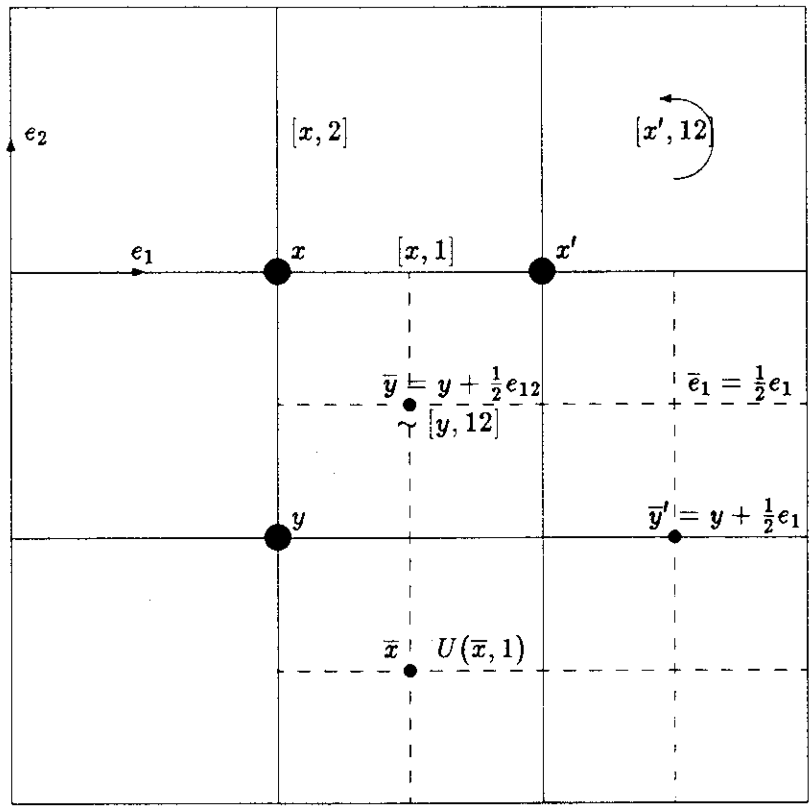


Figure 1: Illustration of the Lattice Notions

It follows from (13) that $\chi_i(\overline{x})$ transforms without an $\overline{x}-$dependent phase $(\hat{M}^H\chi)_i(\overline{x}) = M^H_{ik}\chi_k(\overline{x})$. Since the Dirac-operator $\gamma^\mu\overline{\triangle}^\mu$ commutes with the transformations $\hat{M} \in SDG$, the action separates in the irreducible components $\chi_i(\overline{x})$. Indeed we get

$$S_q = \sum_{i=1}^{4}\sum_{\overline{x}}\{\sum_{\mu}\frac{\check{\rho}_{\mu H}}{2\overline{b}}[\overline{\chi}_i(\overline{x})\chi_i(\overline{x}+\overline{e}_\mu) - \overline{\chi}_i(\overline{x}+\overline{e}_\mu)\chi_i(\overline{x})] + \overline{m}\overline{\chi}_i(\overline{x})\chi_i(\overline{x})\}, \qquad (16)$$

with a sign function $\check{\rho}_{\mu H}$ explicitely defined in Eq. (19). The lattice fields $\chi_i(\overline{x})$ are called the 'staggered fermion components' of the naive Dirac fields. The restriction to one component $\chi(\overline{x})$ reduces the degeneracy by a factor 4 in 4-dimensional Euclidean space time. In the following sections we are mainly concerned with the geometry and physics of staggered fermion fields.

Our presentation of the problem of lattice fermions might give the impression that spectrum doubling is a more or less superficial problem. This is not correct. Mathematically it was shown by H.B.Nielson and M.Ninomiya [12] and others [13] that it originates deeply in the topological structure of the 'kinetic energy' terms defined on the periodic Brillouin zone. It can be explained by the theorems on 'spectral flow' [14]. The methods applied in physics to deal with the problem are rather arbitrary. One procedure [15] is to add an arbitrary term which breaks the degeneracy in such a way that only one Dirac field survives in the continuum limit. However, this destroys important symmetries of the lattice theory. The point of view we present in this paper is that staggered fermions represent in an appropriate form the spectrum problem [16]. The underlying geometry may give some hints on interesting structure in elementary particle physics.

## 3. RELATION BETWEEN DIRAC FIELDS AND DIFFERENTIAL FORMS

Before we give a geometric interpretation of staggered lattice fermions by a lattice approximation of the DKE, we want to clarify the relation between the DKE and the Dirac equation in the Euclidean space time continuum. The essential point in this consideration is the introduction of a Clifford product for differential forms[1]. Let us first introduce our notation. $(x^\mu)$ are coordinates with respect to an orthogonal basis of Euclidean $R^d$. We use for the expansion of a complex differential form $\Phi$ in the basis generated by the wedge products of $dx^\mu$ ('Cartesian basis') the multi-index notation of Eq. (12)

$$\phi = \sum_{h=1,\mu_i}^{d} \frac{1}{h!}\phi_{\mu_1\ldots\mu_h}(x)dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_h} \equiv \sum_H \phi(x,H)dx^H. \tag{17}$$

The bilinear, associative Clifford algebra $\mathcal{CL}\{\Phi\}$ is generated by the products of $dx^\mu$

$$dx^\mu \vee dx^\nu = dx^\mu \wedge dx^\nu + \delta^{\mu\nu}. \tag{18}$$

It follows for the products of the basis elements $dx^H$

$$dx^H \wedge dx^K = \hat{\rho}_{H,K}dx^{H\cup K}, \qquad dx^H \vee dx^K = \check{\rho}_{H,K}dx^{H\triangle K},$$
$$\hat{\rho}_{H,K}=0 \quad for H\cap K\neq\emptyset, \quad \hat{\rho}_{H,K}=(-1)^\nu \quad for \quad H\cap K=\emptyset, \qquad \check{\rho}_{H,K}=(-1)^\nu,$$
$$\nu \quad number \quad of \quad pairs \quad (\mu_i^1,\mu_j^2), \quad \mu_i^1\in H, \quad \mu_j^2\in K, \quad \mu_i^1>\mu_j^2. \tag{19}$$

The main automorphism $\mathcal{A}: \mathcal{A}(\Phi\wedge\Psi)=\mathcal{A}\phi\wedge\mathcal{A}\Psi, \quad \mathcal{A}(\Phi\vee\Psi)=\mathcal{A}\phi\vee\mathcal{A}\Psi$
and the main antimorphism $\mathcal{B}: \mathcal{B}(\Phi\wedge\Psi)=\mathcal{B}\Psi\wedge\mathcal{B}\Phi, \quad \mathcal{B}(\Phi\vee\Psi)=\mathcal{B}\Psi\vee\mathcal{B}\Phi$
transform the multi-indexed basis elements as

$$\mathcal{A}dx^H = (-1)^h dx^H, \qquad \mathcal{B}dx^H = (-1)^{h(h-1)/2}dx^H. \tag{20}$$

The contraction operator $e_\mu\neg$, defined linearly as antiderivation

$$e_\mu\neg(\Phi\wedge\Psi) = (e_\mu\neg\Phi)\wedge\Psi + \mathcal{A}\Phi\wedge e_\mu\neg\Psi, \qquad e^\mu\neg dx^\nu = \delta^{\mu\nu}, \quad e^\mu\neg 1 = 0.$$

relates Clifford product and wedge product

$$dx^\mu \vee \Phi = dx^\mu\wedge\Phi + e^\mu\neg\Phi. \tag{21}$$

With this notation the differential operators $d$ and $\delta$ can be expressed conveniently expressed by

$$d\Phi = dx^\mu\wedge\partial_\mu\Phi, \qquad \delta\Phi = -e^\mu\neg\partial_\mu\Phi,$$

and because of Eq. (21)

$$(d-\delta)\Phi = dx^\mu\vee\partial_\mu\Phi. \tag{22}$$

$\partial_\mu$ denoting differentiation of the components $\phi(x,H)$.

The relation between the DKE (1) and the Dirac equation in flat space is a consequence of the isomorphism $dx^\mu \leftrightarrow \gamma^\mu$ between the Clifford algebra $\mathcal{CL}\{\Phi\}$ and the algebra of the Dirac matrices $\gamma^\mu$ which follows from the equivalence of Eq. (18) with the defining relations of the $\gamma^\mu$ : $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta^{\mu\nu}$. In order to describe this relation, we use an explicite construction of the decomposition of $\mathcal{CL}\{\Phi\}$ into primitive left ideals [2][17] [18]. For this we use in even dimension d the $2^{d/2}$–dimensional, irreducible representation of the $\gamma^H = \gamma^{\mu_1} \cdot\cdot \gamma^{\mu_h}$, to define a new basis $Z = (Z_a^b)$:

$$\Phi = \sum_H \phi(x, H) dx^H = \sum_{a,b} \psi_a^b(x) Z_a^b,$$

$$Z = 2^{-d/2} \sum_H (\gamma^H)^T \mathcal{B} d^H. \tag{23}$$

This basis has the property

$$dx^\mu \vee Z = \gamma^{\mu T} Z, \qquad Z \vee dx^\mu = Z\gamma^{\mu T}, \qquad Z_a^b \vee Z_c^d = Z_a^d \delta_c^b. \tag{24}$$

The transformation between the 'Cartesian'-components $\phi(x, H)$ and the 'Dirac' components $\psi_a^b$ of $\Phi$:

$$\phi(x, H) = 2^{-d/2} trace(\gamma^{H\dagger}\psi(x, H)), \qquad \psi(x) = \sum_H \phi(x, H)\gamma^H, \tag{25}$$

results from the completeness and orthogonality relations of the $\gamma$-matrices. Now it follows from Eqs. (22), (24), (13) that the Dirac components of a solution of the DKE satisfy the Dirac equation

$$(\gamma^\mu \partial_\mu + m)\psi^b(x) = 0. \tag{26}$$

In this sense, the DKE in flat space is equivalent to a set of $2^{d/2}$ Dirac equations.

This degeneracy of the DKE is descibed by the following symmetry. Let

$$c(u) = \sum_H u(H) dx^H = \sum_{a,b} u_a^b Z_a^b \tag{27}$$

be a constant differential form: $\partial_\mu c = 0$. Due to the associativity of the $\vee$-multiplication , $\Phi \rightarrow \Phi \vee c$, transforms a solution of the the DKE into a solution,

$$0 = ((d - \delta + m)\Phi) \vee c = (dx^\mu \vee \partial_\mu \Phi + m\Phi) \vee c$$

$$= (dx^\mu \partial_\mu (\Phi \vee c) + m(\Phi \vee c). \tag{28}$$

It follows immediately from Eq. (24) that $c(u)$ transforms the Dirac components by the matrix $u_c^b$:

$$(\Phi \vee c(u))_a^b = \sum_d \psi_a^d u_d^b. \tag{29}$$

In case $(u_a^b) \in SU(4)$, these transformations are called flavour transformations. Eqs (27), (29) describe the flavour transformations by Clifford right multiplication, i.e.by an element of the Clifford group [17].

c(u) may also represent a projection operator. Thus $\mathbf{P}_a = \frac{1}{4}(1 \pm i dx^{12})(1 \pm dx^{1234})$ projects on the 4 different flavour Dirac components in a representation of the $\gamma$-matrices where $\gamma^5, \gamma^{12}$ are diagonal.

Finally we want to express the scalar product of differential forms by Dirac components:

$$(\Phi, \Psi)(x) = \sum_H \phi^*(x, H)\psi(x, H) = 2^{-d/2} \sum_{a,b} \phi_a^{*b}(x)\psi_a^b(x),$$

with

$$\psi_a^b(x) = (Z_a^b, \Psi), \qquad \phi_a^{*b}(x) = (\Phi, Z_a^b). \tag{30}$$

The Clifford product allows an algebraic definition of

$$(\Phi, \Psi)_0 = (\Phi, \Psi)dx^{1234}: \qquad (\Phi, \Psi)_0 = (\mathcal{B}\Phi^* \vee \Psi) \wedge dx^{1234}.$$

In terms of these expressions, the free action might be written as

$$S_q = \frac{1}{4}\int (\overline{\Phi}, (d - \delta + m)\Phi)_0 = \sum_b \int dx^{1234}\overline{\psi}^b(x)(\gamma^\mu\partial_\mu + m)\psi^b(x), \tag{31}$$

which should be compared with Eq. (5).

## 4. DKE AND CLIFFORD PRODUCT ON A CUBIC LATTICE

Now we consider the lattice approximation of the DKE with the aim to show that this leads to a geometric interpretation of staggered lattice fermions. Imitating the mapping between DeRham complexes and simplicial complexes [19], we get as lattice correspondence of the differential form $\Phi$

$$\Phi(C) = \int_C \Phi, \tag{32}$$

where $C$ is a sum of lattice cells. $\Phi(C)$ , a linear functional on linear combinations of cells, can be expressed in a basis $\{d^{x,H}\}$:

$$\Phi = \sum_{x,H} \Phi(x, H)dx^{x,H}, \qquad dx^{x,H}([x', H']) = \delta^x_{x'}\delta^H_{H'}. \tag{33}$$

This is the lattice analog of (17). Because of Stokes' theorem, the mapping Eq. (32) implies

$$d \to \check{\triangle}, \qquad \check{\triangle}\Phi(C) = \Phi(\triangle C), \qquad \delta \to \check{\nabla}, \qquad \check{\nabla}\Phi(C) = \Phi(\nabla C).$$

Here $\triangle$ and $\nabla$ are the boundary and the coboundary operator applied to lattice cells. Thus the DKE on the lattice becomes

$$(\check{\triangle} - \check{\nabla} + m)\Phi = 0. \tag{34}$$

In the continuum theory, it was important for the further treatment of the DKE to introduce a Clifford product for differential forms. Now we want to discuss the possibilities of a Clifford product on the lattice. We follow the procedure of algebraic topology and introduce the cup product and the cap product corresponding to the $\wedge$- product and $e^\mu\neg$ -operator

$$\begin{aligned} d^{x,H} \wedge d^{y,K} &= \hat{\rho}_{H,K}\delta^{x+e_H,y}d^{x,H\cup K}, \\ e^\mu \neg d^{y,K} &= \tilde{\rho}_{\mu,K}d^{x,K-\mu}, \end{aligned} \tag{35}$$

($\tilde{\rho}_{\mu,K}$ as in the continuum), and combine both like in Eq. (21) for a Clifford product

$$d^{x,H} \vee d^{y,K} = \check{\rho}_{H,K}\delta^{x+e_H,y}d^{x+e_\Lambda,H\triangle K}; \qquad \Lambda = H \cap K, \qquad H\triangle K = H \cup K - \Lambda. \tag{36}$$

Some examples are illustrated in Fig.(2). The wedge product is associative and satisfies

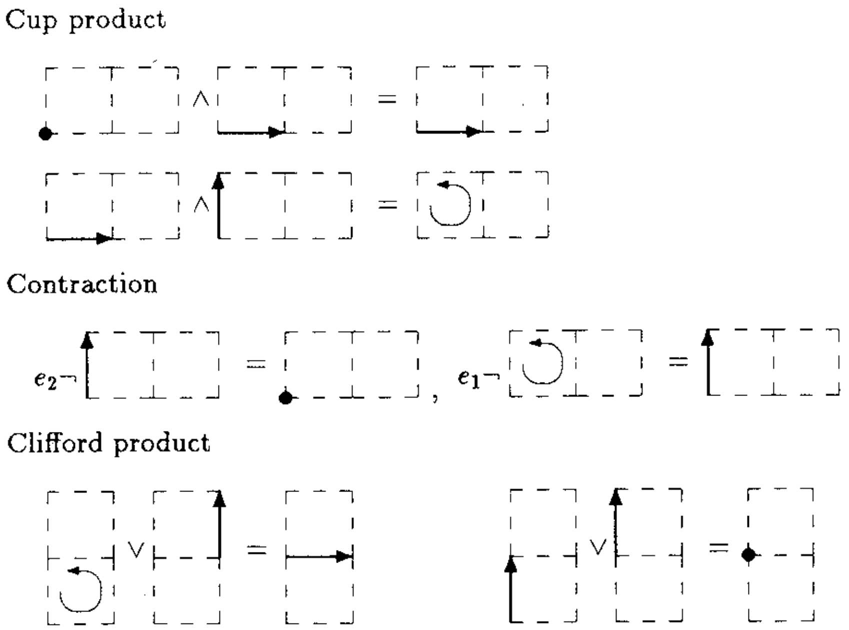


Figure 2: Examples of of operations with elementary cochains of a 2-dimensional lattice: contraction, $\wedge$ product and $\vee$ product. These illustrate the 'matching' conditions.

$$\check{\Delta}(\Phi \wedge \Psi) = \check{\Delta}\Phi \wedge \Psi + \not{A}\Phi \wedge \check{\Delta}\Psi .$$

The important rules for the lattice contraction operator $e_\mu \neg$ are

$$e_\mu \neg (d^{x,H} \wedge d^{y,K}) = (e_\mu \neg d^{x,H}) \wedge T_{e_\mu} d^{x,K} + \not{A} d^{x,H} \wedge (e_\mu \neg d^{x,K}),$$

$$T_{e_\mu} d^{x,H} = d^{x-e_\mu,H}, \qquad\qquad e^\mu \neg e^\nu \neg + e^\nu \neg e^\mu \neg = 0.$$

However,the Clifford product is not associative, but satisfies

$$(d^{x-e_{H\cap L},H} \vee d^{y,K}) \vee d^{z-e_{H\cap K},L} = d^{x,H} \vee (d^{y,K} \vee d^{z,L}). \tag{37}$$

This is a consequence of the 'matching conditions' which must be satisfied by the $\vee$ -product of elementary cochains $d^{x,H}$ for giving a non-vanishing result.

Is it worthwhile to consider such a Clifford product on the lattice? Formally it helps us to procede with our analogy to the continuum. Defining, with help of the translation operator $T_{e_\mu}$, difference operators $\partial^+_\mu = T_{e_\mu} - 1, \partial^-_\mu = 1 - T_{-e_\mu}$ allows a representation of $\check{\Delta}, \check{\nabla}$ very similar to (22):

$$\check{\Delta} = d^\mu \wedge \partial^-_\mu, \qquad \check{\nabla} = -e^\mu \neg \partial^-_\mu, \qquad \check{\Delta} - \check{\nabla} = d^\mu \vee \partial^-_\mu. \tag{38}$$

with $d^H = \sum_x d^{x,H}$. Summing over x,y in Eq. (37) leads to an 'associative' law: $(d^H \vee d^{x,K}) \vee d^L = d^H \vee (d^{x,K}) \vee d^L)$. From this follows, as in Eq. (28),that $\Phi \to \Phi \vee d^H$ transforms a solution of the lattice DKE into a solution. Thus the Clifford product on the lattice describes an important symmetry of staggered fermions. We shall discuss this in more detail in the following sections.

Now we show that the lattice Dirac Kaehler field $\phi(x, H)$ becomes a staggered fermion field $\chi(\overline{x})$ if we make the identification

$$\phi(x, H) = \phi(\overline{x}, H(\overline{x})) = \chi(\overline{x}), \tag{39}$$

with notations explained in Eq. (14). By this mapping the r-cochains of the 'coarse' lattice $\Gamma : \phi(x, H)$ get identified with lattice fields defined at the points of the 'fine' lattice $\overline{x} \in \overline{\Gamma}$ which are central points of the cells $[x, H]$ (see Fig.1). The proof of the equivalence of Dirac Kaehler fermions with staggered fermions consists by the demonstration that a Dirac field defined according to Eq. (15):

$$\psi_a(\overline{x}) = \gamma_{ai}^{H(\overline{x})}\phi(\overline{x}, H(\overline{x}))$$

i arbitrarily fixed, satisfies the naive Dirac equation, iff $\phi(x, H)$ satisfies the DKE. This is shown by the following calculation. We have the following formula

$$\begin{aligned}
(\overline{\triangle}_\mu\psi)(x,H) &= (\partial_\mu^+\psi)(x, H\setminus\{\mu\}) & if\ \mu\ \in H,\\
&= (\partial_\mu^-\psi)(x, H\cup\{\mu\}) & if\ \mu\ \notin H.\\
(\overline{\triangle}_\mu\psi)(\overline{x}) &= \tfrac{1}{2}(\psi(\overline{x}+\overline{e}_\mu) - \psi(\overline{x}-\overline{e}_\mu)).
\end{aligned}$$

and hence

$$\gamma^{H\dagger}(\gamma^\mu\overline{\triangle}_\mu\psi)(x,H) = \sum_{\mu\in H}(\gamma^{H\dagger}\gamma^\mu\partial_\mu^+\psi(x, H\setminus\{\mu\}) + \sum_{\mu\notin H}(\gamma^{H\dagger}\gamma^\mu\partial_\mu^-\psi(x, H\cup\{\mu\})$$

$$= \sum_{\mu\in H}\hat{\rho}_{\mu,H}\partial_\mu^+\Phi(x, H\setminus\{\mu\}) + \sum_{\mu\notin H}\tilde{\rho}_{\mu,H}\partial_\mu^-\Phi(x, H\cup\{\mu\}) \quad = \quad ((\check{\triangle} - \check{\nabla})\Phi)(x,H).$$

By a similar calculation the equalitity of the action for staggered fermions, Eq. (16), with the Dirac Kaehler action

$$S_q = \frac{1}{4}\sum_{x,H}\{\overline{\Phi}(x,H)((\check{\triangle} - \check{\nabla} + m)\Phi)(x,H)\} \tag{40}$$

follows.

The DKE is defined geometrically on the coarse lattice. We want to check the energy momentum spectrum. For this we multiply the DKE by the adjoint operator

$$\begin{aligned}
&(-(\check{\triangle} - \check{\nabla} + m)(\check{\triangle} - \check{\nabla} + m)\Phi\\
&= (\check{\triangle}\check{\nabla} + \check{\nabla}\check{\triangle} + m^2)\Phi = (-\partial_\mu^+\partial^{\mu-} + m^2)\Phi = 0.
\end{aligned} \tag{41}$$

The iterated DKE is indeed the correct Klein Gordon equation on the lattice. We consider plane wave solutions

$$\phi(x, H) = u(p, H)e^{-ipx}.$$

With this ansatz Eq. (41) becomes

$$(\sum_\mu(\frac{2}{b}\sin\frac{p_\mu b}{2})^2 + m^2)u(p,H) = 0.$$

Because $(\frac{2}{b}\sin\frac{p_\mu b}{2})$ is monotonous in the cut-off momentum range $-\frac{\pi}{b} < p_\mu \leq \frac{\pi}{b}$,the energy momentum spectrum has the same multiplicity as that of the DKE in the continuum. This is in contrast to the spectrum problem which arises from the naive lattice approximation of the Diac equation. J.M.Rabin [20] has given a topological argument why the mapping Eq. (32) does not introduce new zero solutions of the kinetic energy: The zero solutions present harmonic forms, which are in 1-1 correspondence to the homology classes in the continuum and on the lattice according to DeRham's Theorem.

## 5. THE SYMMETRY OF THE DKE ON THE LATTICE

The transformations of the spinorial Euclidean group $S\mathcal{E}$ and of the flavour group $\mathcal{F}$ generate a symmetry group $\mathcal{G}$ of the DKE in the continuum: $\mathcal{G} \simeq \mathcal{F} \times S\mathcal{E}$ . What is left of this symmetry by the lattice approximation?

Before we answer this question, we want to describe the continuum symmetry in an appropriate way [3]. The infinitesimal translations $\delta_\mu \phi_a^b(x) = \partial_\mu \phi_a^b(x)$ and rotations $\delta_{\mu\nu}\phi_a^b(x) = (x_\mu\partial_\nu - x_\nu\partial_\mu)\phi_a^b(x) + \frac{1}{4}([\gamma_\mu, \gamma_\nu]\phi)_a^b(x)$ of the spinor components can be expressed directly as operations on the forms

$$\delta_\mu \Phi = \partial_\mu \Phi,$$

$$\delta_{\mu\nu}\Phi = (x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi + \frac{1}{2}S_{\mu\nu} \vee \Phi, \qquad S_{\mu\nu} = dx_\mu \wedge dx_\nu. \tag{42}$$

These transformations generate the unit component of the group $S\mathcal{E}$ which should be supplemented by space reflections. On the other hand, if the Cartesian components transform as O(4) tensors, we get the transformation law

$$\delta^G_{\mu\nu}\Phi = (x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi + \frac{1}{2}(S_{\mu\nu} \vee \Phi - \Phi \vee S_{\mu\nu}). \tag{43}$$

These 'geometric' rotations differ from spinor rotations by a flavour transformation of the form Eq. (29). Thus we may express an element g of the symmetry group of the DKE either as a direct product $g = (f) \otimes (a, s) \equiv (f, a, s)$, with $(f) \in \mathcal{F}, (a, s) \in S\mathcal{E}$, or we compose it by a geometric transformation $(s, a, s) \in G\mathcal{E}$ and another flavour transformation $(\overline{f})$

$$g = (\overline{f}) \circ (s, a, s) = (\overline{f}s, a, s) = [\overline{f}, a, R(s)]. \tag{44}$$

The group multiplication in these different equivalent forms is:

$$(f, a, s) \circ (f', a', s') = (ff', R(s)a' + a, ss'),$$

$$[f, a, R(s)] \circ [f', a', R(s')] = [fsf's^{-1}, R(s)a' + a, R(s)R(s')]. \tag{45}$$

Restricting space time to a lattice implies restrictions of the symmetry group of the DKE. These are most obvious for the geometrical transformations generating the geometrical Euclidean group $G\mathcal{E}$. The contineous translation group gets restricted to the group of lattice translations:
$\mathcal{T}_L = \{[a]\}, a = b(n^1 \dots n^d), n^i \in Z$. The cubic lattice allows only rotations belonging to the symmetry group $W_d$ of the d-dimensional cube. In d = 4 dimensions, $W_4$ is a group with 384 elements which is generated by rotations $R_{\mu\nu}$ in the $(\mu, \nu)$ -plane by $\frac{\pi}{2}$ , and by a reflection $\Pi : (x^1, x^2, x^3, x^4) \to (-x^1, -x^2, -x^3, x^4)$. These rotation-reflections map r-cells, $r = 1 \dots 4$, onto r-cells, and commute with the boundary and coboundary operations. Therefore the DKE is invariant under the group $\mathcal{T}_L \propto W_4 \simeq G\mathcal{E}_L \subset G\mathcal{E}$ of these transformations;($\propto$ denotes a semi-direct product). The transformation law of staggered fermion fields is

$$([a]\Phi)(\overline{x}, H(\overline{x})) = \Phi(\overline{x} - a^\mu e_\mu, H(\overline{x})), \qquad H(\overline{x} - a^\mu e_\mu) = H(\overline{x}),$$

$$([R]\Phi)(\overline{x}, H(\overline{x})) = \rho(R, H(\overline{x}))\Phi(R^{-1}\overline{x}, H(R^{-1}\overline{x})), \qquad R \in W_4, \tag{46}$$

where the sign $\rho(R, H)$ is the same as in the transformation of the basis differential of the continuum : $Rdx^H = \rho(R, H)dx^{R^{-1}\circ H}$.

The lattice restriction of the flavour transformations is determined by our definition of the Clifford product on the lattice. The symmetry transformation of the DKE

$$d^K\Phi : \Phi \rightarrow \epsilon\Phi \vee (d^K)^{-1}, \ \ \epsilon = \pm 1,$$

implies for staggered fermion fields

$$(d^K\Phi)(\overline{x}, H(\overline{x})) = \epsilon\check{\rho}_{H(\overline{x}),K}\Phi(\overline{x} + \frac{1}{2}e_K, H(\overline{x} + \frac{1}{2}e_K)). \tag{47}$$

It follows that $(d^K)^2 = [-e_K]$, i.e. the flavour transformations generate translations. These transformations on the lattice appear as the following restrictions of the continuum group $\mathcal{G} : [F, a, R] = [\epsilon\gamma^K, -\frac{1}{2}e_K, 1]$. The group $\mathcal{K}_4 = \{[\epsilon\gamma^K, -\frac{1}{2}e_K + a, 1]\}/\mathcal{T}_L$ is isomorphic to the finite group of 32 elements generated by the $\gamma-$matrices. The group $\mathcal{K}_4$ plays an important role in the discussion of Clifford algebras [21]. Now we may state in a crystallographic language [24]: Flavour transformations form a generalized point group isomorphic to the group $\mathcal{K}_4$ which is a non-symmorphic extension of the translation group.

Putting these considerations together, we find the following sub-group $\mathcal{G}_L \subset \mathcal{G}$ which is a symmetry group of the lattice DKE : $\mathcal{G}_L = \{[\epsilon d_k, -\frac{1}{2}e_K + a, R] \mid a \in \mathcal{T}_L, R \in W_4\}$ with the composition law following from (46)

$$[\epsilon d_K, -\frac{1}{2}e_K + a, R] \circ [\epsilon' d_L, -\frac{1}{2}e_L + a', R']$$

$$= [\epsilon\epsilon' \rho(R, R \circ L)\check{\rho}_{K,R\circ L} d^{K \triangle R\circ L}, -\frac{1}{2}(e_K + Re_L) + Ra' + a, RR']. \tag{48}$$

There are further symmetries of the DKE. The complex DKE is invariant with respect to phase transformations. In the massless case,there are independent phase transformations of the even and odd forms. In the following we restrict ourselves to the discussion of $\mathcal{G}_L$.

Now we try to give a short summary of the representation theory of $\mathcal{G}_L$ [22], [23]. Because $\mathcal{T}_L$ is a normal sub-group of $\mathcal{G}_L$ with $\mathcal{G}_L/\mathcal{T}_L \simeq \mathcal{K}_4 \otimes W_4$, the induction procedure by E.P.Wigner - G.W. Mackey [25] allows the construction of the irreducible, unitary representations ('irreps') of $\mathcal{G}_L$ from the representations of $\mathcal{T}_L$, of the finite group $\mathcal{K}_4$ , and of the sub-groups of $W_4$. We shall give a short glossary of this Wigner- Mackey procedure. Because we apply it to a relatively simple discrete group, we can omit all mathematical sophistication.

Take a group G with a normal sub-group N and the factor group F. An irrep $g \rightarrow U(g)$ of G restricted to $N = \{n\}$ decomposes in irreps $L^p$. Here $p \in \hat{N}$ is a label of the irreps of N. It is the first step of the procedure to analyze this restriction $U(G)|_N$ . For this we construct an orthonormal basis of the representation space in which $U(G)|_N$ decomposes explicitly:

$$\{\Phi_{p,a;\eta}\} : \qquad U(n)\Phi_{p,a;\eta} = \sum_{\overline{a}} \Phi_{p,\overline{a};\eta} L^p_{\overline{a},a}(n), \tag{49}$$

$\eta$ being a degeneracy label. ( A direct integral decomposition we treat formally in the same way).

Since N is a normal sub-group of $G$, $L^p(g^{-1}ng)$ is equivalent to some $L^{p'}(n)$ ,i.e. G acts as a transformation group of $\hat{N} : p' = gp$ . Acting with $U(g^{-1}ng)$ on the basis (49) , we see that p varies over G -orbits $\Theta_j = \{gp_j | g \in G\} \subset \hat{N}$, where $p_j$ denotes an arbitrary but fixed reference point. Indeed, an irrep U(g) is characterized by a single orbit $\Theta_j$. The group $S_j^{(1)}$ leaving $p_j$ invariant: $S_j^{(1)} = \{s | sp_j = p_j, s \in \mathcal{G}\}$ is called the little group of the first kind. N is a normal sub-group of $S_j^{(1)}$. The little group of second kind is defined as the factor group $S_j^{(2)} = S_j^{(1)}/N$.

For the further procedure it is important that one can extend always $(L^{p_j}_{a',a}(n))$, up to equivalence, uniquely to a projective representation $\overline{L}^{p_j}(s), \quad s \in S_j^{(1)}$ :

$$\overline{L}^{p_j}(s)\overline{L}^{p_j}(t) = \sigma_j(s,t)\overline{L}^{p_j}(st), \qquad s,t \in S_j^{(1)}, \tag{50}$$

such that the multiplier $\sigma_j(s,t)$ only depends on the N-cosets, i.e. on the elements of $S_j^{(2)}$. For the proof we refer to [25]. Since $S_j^{(1)}$ is the stability group of $p_j$, the sub-space spanned by $\{\Phi_{p_j}\}$ is invariant under the transformations of $S_j^{(1)}$:

$$U(s)\Phi^{j;\chi}_{p_j,a,r} = \sum_{a',r'} = \Phi^{j;\chi}_{p_j,a',r'}\overline{L}^{p_j}_{a',a}(s)D^{\chi}_{r',r}(s). \tag{51}$$

Here $D^{\chi}(s)$ is a projective representation of $S_j^{(2)} = S_j^{(1)}/N$, with the multiplier $\sigma_j^{-1}(s,t)$ .It may be considered as an unfaithful projective representation of $S_j^{(1)}$.

We can now formulate the MAIN THEOREM. [26] All irreducible unitary representations of a group G with a normal sub-group N are characterized by the G-orbits $\Theta^j \subset \hat{N}$ , and the irreducible projective representations of the related little groups of second kind $S_j^{(2)} \to D^{\chi}(s)$ with multiplier of the equivalence class $\sigma_j^{-1}(s,t)$.

For the explicit construction we define for the points p of the orbit $\Theta^j$ a 'boost' transformation: $p = g(p)p_j, g(p) \in G$ and standardize by equivalence transformations the representation $L^p(n)$ and the basis $\{\Phi^{j;\chi}_{p,a,r}\}$ :

$$\begin{aligned} L^p_{m,m'}(n) &= L^{p_j}_{m,m'}(g^{-1}(p)ng(p)), \\ U(g(p))\Phi^{j;\chi}_{p_j,a,r} &= \Phi^{j;\chi}_{p,a,r}. \end{aligned} \tag{52}$$

Because we can write an arbitrary $g \in G$ as a product of boost transformations and an element of $S_j^{(1)}$:

$$g = g(gp)s(g,p)g^{-1}(p), \qquad i.e. \qquad s(g,p) = g^{-1}(gp)gg(p) \qquad \in S_j^{(1)}, \tag{53}$$

it follows from Eq. (51) and (52) that

$$U(g)\Phi^{j;\chi}_{p,a,r} = \sum_{a',r'} \Phi^{j;\chi}_{gp,a',r'}\overline{L}^{p_j}_{a',a}(s(g,p))D^{\chi}_{r',r}(s(g,p)). \tag{54}$$

One may verify by direct calculation that Eq. (54) defines a representation of G . This representation is irreducible if p is restricted to a single orbit $\Theta_j$,and if $D^{\chi}_{r',r}(s)$ is irreducible. In the construction above the representation of an orbit $\Theta_j$ by a reference point $p_j$ and boost transformations g(p) induces some arbitrariness. One convinces oneself easily that the choice of different reference points and boost transformations lead to equivalent representations of G. Similarly, equivalence transformations of $D^{\chi}(s)$ and $L^{p_j}$ lead to equivalent representations.

The iterated application of the Wigner-Mackey procedure leads to a complete classification of the irreps of $\mathcal{G}_L$ by a ' MOMENTUM STAR $St_j$', a ' FLAVOUR ORBIT $\Theta_{j,k}$ ', and a ' REDUCED SPIN $\sigma$'.

In order to show this,we consider in a first application of the Wigner-Mackey procedure the translation group $\mathcal{T}_L$ as normal sub-group of $\mathcal{G}_L$. The 1-dimensional irreps of $\mathcal{T}_L : [a] \rightarrow e^{ipa}$ are labelled by 'momenta' $p = (p_1, \dots p_4)$ varying in the Brillouin zone: $-\frac{\pi}{b} < p_\mu \leq \frac{\pi}{b}$. We denote by the star $St_j$ the orbit of the rotations $R \in W_4$ applied to the momenta $p \rightarrow Rp$. Depending on the orientation of $p$ there are 13 qualitatively different stars. For each $St_j$ one may choose a reference point $p_j \in St_j$, boost operators $\Lambda(p)p_j = p, \ \ \Lambda(p) \in W_4$ , and determine the stability group $\overline{S}_j = \{R|Rp_j = p_j\}$.

The little group of the first kind $S_j^{(1)}$ in this application of the Wigner-Mackey procedure is generated by the translations, flavour transformations, and the rotations of $\overline{S}_j$ .The little group of the second kind $S_j^{(2)} \simeq S_j^{(1)}/\mathcal{T}_L$ is generated by $\overline{S}_j$ and the elements of $\mathcal{K}_4$ . The representations of $\mathcal{G}_L$, corresponding to Eqs. (52), (53), (54) have the form

$$U[\epsilon d^K, -\frac{1}{2}e_K + a, R]\Phi^{j,\chi}_{p,n} = e^{i(Rp,a-\frac{1}{2}e_K)} \sum_{n'} \Phi^{j,\chi}_{Rp,n'} D^{\chi}_{n'n}(s(g,p)), \tag{55}$$

with

$$p \in St_j, \qquad s(g,p) = [1,0,\Lambda^{-1}(Rp)] \circ [\epsilon d^K, -\frac{1}{2}e_K + a, R] \circ [1,0,\Lambda(p)] \in S^1_j.$$

The extension of the representation of $\mathcal{T}_L : \quad [a] \rightarrow e^{ip_j a}$ to a representation $\overline{L}(s)$ of $S_j^{(1)}$ is given trivially by $[\epsilon d^K, -\frac{1}{2}e_K + a, R] \rightarrow e^{-i(p,a-1/2e_K)}, \quad R \in \overline{S}_j$. $D^{\chi}(s)$ is an irrep of $S_j^{(2)}$ which we have to construct now.

This little group of the second kind $S_j^{(2)}$ contains $\mathcal{K}_4$ as a normal sub-group. We apply for the construction of the irreps of $S_j^{(2)}$ the Wigner-Mackey procedure for a second time. For this we have to consider first the irreps of $\mathcal{K}_4$. These are the well known 4-dimensional representation $\epsilon d^K \rightarrow \epsilon\gamma^K$ which we give the label L = 0, and the 16 one dimensional representations of the factor group

$$\mathcal{K}_4 : \qquad \epsilon d^K \rightarrow e^{i\pi(e_L,e_K)} \equiv \Gamma^L(\epsilon d^K), \qquad e_L = \sum_{\mu\in L} e_\mu.$$

L is a multi-index like in Eq. (12). For the construction of the representations of $S_j^{(2)}$, we have to consider further the transformations of the irreps of $\mathcal{K}_4$ under the rotations of $\overline{S}_j$: $\Gamma^L(R^{-1}(\epsilon d^K)R) \simeq \Gamma^{R\circ L}$. In the case L=0, this is an equivalence transformation for all R, therefore
$R \circ (L = 0) = (L = 0)$. The set of 1-dimensional representations decompose under the rotations of $\overline{S}_j$ in 'flavour orbits' $\Theta_{j,k}, k = 1, \dots N_j$. For flavour orbits we again may fix a refence point $\overline{L}_k$, choose boost operators $f(L)\overline{L}_k = L, f(l) \in \overline{S}_j$, and determine the stability group $S_{j,k} = \{R|R\overline{L}_k = \overline{L}_k; R \in S_j\} \subset S_j$ The little group of the first kind $S_{j,k}^{(1)}$ is a semi-direct product of $S_{j,k}$ with $\mathcal{K}_4$, $\mathcal{K}_4$ normal sub-group. The little group of the second kind is $S_{j,k}^{(1)}/\mathcal{K}_4 \simeq S_{j,k}$

With these concepts we can construct the irreps of $S_j^{(2)}$ according to Eqs. (52), (53), (54),

$$\mathcal{U}(\epsilon d^K, R)\Phi^{k,\sigma}_{L,a,n} = \sum_{a',n'} \Phi^{k,\sigma}_{L,a',n'} \tilde{\Gamma}^{\overline{L}_k}_{a',a}(\xi(s,L))\mathcal{D}^{\sigma}_{n',n}(\xi(s,L)), \tag{56}$$

with
$$L \in \Theta_{j,k}, \qquad s = [(\epsilon d^K, R)] \in S_j^{(2)}, \qquad \xi(s,L) = \in S_{j,k}^{(1)}.$$

$\tilde{\Gamma}^0_{a',a}(\xi), \xi \in S^{(1)}_{j,k}$ denotes the extension of the 4-dimensional representation of $K_4$ to a projective representation of $S^{(1)}_{j,k}$. Here the representations of the rotations $R \in S_{j,k}$ are given by the projective representation of $W_4$ generated by $R_{\mu\nu} \to \frac{1}{2}(1+\gamma_\mu\gamma_\nu)$, $\Pi \to \gamma^4$, restricted to $S_{j,k}$. In the extensions of the 1-dimensional representations: $\tilde{\Gamma}^{\overline{L}_k}_{\overline{a},a}(\xi)$,these rotations,reflections are represented trivially: $R_{\mu\nu} \to 1$, $\Pi \to 1$.

The groups $S_{j,k}$ are direct products of the symmetric groups Sym(n) ,n = 2,3,4, the dihedral group $D_4$, and the hyper-cubic group $W_4$. Their projective irreps are wellknown, or can be constructed easily. The irreps of $S_{j,k} \ni \xi \to \mathcal{D}^\sigma_{r',r}(\xi)$, which determine the reduced spin $\sigma$ , may then be labelled by combinations of the primitive characters of these elementary groups.

The discussion of the irreps of $S_j^{(1)}$ together with Eq. (55) completes the construction of the irreps of the symmetry group $\mathcal{G}_L$ of the lattice DKE. The substitution of $D^X_{n',n}$ in Eq. (55) by $\mathcal{U}(s)$ in Eq. (56) is straightforward. However, it leads to clumsy formulas which we do not want to reproduce here. Instead we want to illustrate the different concepts introduced here by an important example [10]. For a more complete discussion of the representation theory of $\mathcal{G}_L$ we refer to the original publication.

We consider the momentum star of 8 points
$$St_4: \qquad (p_\mu) = (0,0,0,\pm E), (0,0,\pm E, 0), ..$$

The stability group of the reference point $p_j = (0,0,0,E)$, $E > 0$, of this star is $\overline{S}_6 \simeq W_3$ . In this case there are three types of flavour orbits

(a) The 1-point orbit $\Gamma^0$

(b) 1-point orbits $\Theta_{6,k} = \{(e_L)\}, k = 1\ldots4$ with $S_{4,k} \simeq W_4$:
$\{(0,0,0,0)\}, \{(0,0,0,1)\}, \{(1,1,1,0)\}, \{(1,1,1,1)\}$.

(c) 3-point orbits $\Theta_{6,k}, k = 5\ldots8$, with $S_{4,j} \simeq D_4 \times Z_2$ :
$\{(1,0,0,0),(0,1,0,0),(0,0,1,0)\}$, $\{(0,1,1,0),\ldots\}$, $\{(1,0,0,1),\ldots\}$, $\{(0,1,1,1),\ldots\}$.

In this example we have $S_{4,k} \simeq Sym(4) \times Z_2 \simeq W_3$ for L = k = 0,..4, and $S_{4,k} \simeq D_4 \times Z_2$ for k = 5,..8.
For k=0 we have to consider projective representations of $W_3$ with the multipliers of the spinor representations. There are two 2-dimensional and one 4-dimensional of such representations for Sym(4). This means altogether six reduced spin-parity combinations for this class of representations characterized by $St_4$ and $\Theta_{4,0}$. We denote them by $(\tilde{2}^\pm), (\tilde{2}'^\pm), (\tilde{4}^\pm)$. The symmetric group Sym(4), i.e. the proper cubic group, has two 1-dimensional, one 2-dimensional, and two 3-dimensional representations. Together with the parities $\Pi = \pm1$, this gives 10 spin parity combinations related to the reduced spin group of the orbits $\Theta_{4,k}$ k = 1,..4. We denote them by $(1^\pm)_{W_3}, (1'^\pm)_{W_3}, (2^\pm)_{W_3}, (3^\pm)_{W_3}, (3'^\pm)_{W_3}$.
The dihedral group $D_4$ has four 1-dimensional representations and one 2-dimensional representaion. This gives again 10 spin parity combinations, however of a different type which belong to the flavour orbits $\Theta_{j,k}$, k = 5,..8. These are denoted by $(1^\pm)_{D_4}, (1'^\pm)_{D_4}, (1''^\pm)_{D_4}, (1'''^\pm)_{D_4}, (2^\pm)_{D_4}$.
Thus we have a complete classification of the 86 irreps of $\mathcal{G}_L$ with momentum star $St_4$.

The group theoretical analysis of the DKE is one of the main tools for the investigation of its physical significance. We shall illustrate this in the next section.

## 6. A PHYSICS PROBLEM

How do these different geometric considerations relate to actual calculations of physical quantities? In this section I shall make an attempt to describe in a cursory way the calculation of the meson spectrum. Because of the very crude approximations involved, the results shall not have any real significance. But they should illustrate more directly the physics related to the solution of the lattice fermion problem offered by the Dirac-Kaehler approach.

The aim is to calculate the 'meson propagator' $\langle\overline{\psi}\psi(x)\overline{\psi}\psi(x')\rangle$ with help of the lattice approximation of formula (2). We use the Wilson action, (9), for the gluon field, and of course we use an action of staggered fermion fields for the description of the quarks:

$$S_q = \sum_{\overline{x}}\{\sum_{\mu}\check{\rho}_{\mu H(\overline{x})}\frac{1}{2}[\overline{\chi}(\overline{x})U(\overline{x},\mu)\chi(\overline{x}+\overline{e}_\mu) - \overline{\chi}(\overline{x}+\overline{e}_\mu)U^{-1}(\overline{x},\mu)\chi(\overline{x})] + \overline{m}\overline{\chi}(\overline{x})\chi(\overline{x})\}$$

$$\equiv \overline{m}\sum_{\overline{x},\overline{x}'}\overline{\chi}(\overline{x})(\delta_{\overline{x},\overline{x}'} + (\overline{x}|Q[U]|\overline{x}')\chi(\overline{x}). \qquad (57)$$

This is the gauge invariant version of the free action Eq. (16), or (40),$\bar{b}=1$.

The total action $S = S_g + S_q$ is symmetric under the group $\mathcal{G}_L$ if $\chi(\overline{x}) = \Phi(\overline{x}, H(\overline{x}))$ transforms according to Eqs. (46), (47), and $U(\overline{x})$ transforms geometrically under [ a,R ] and according to

$$U(\overline{x},\mu) \xrightarrow{d^K} U(\overline{x}+\overline{e}_K)$$

under lattice flavour transformations. We shall use this symmetry below for the classification of the different meson states.

There is some arbitrariness in the introduction of a gauge invariant interaction into the Dirac Kaehler action (40)[3]. Different from the 'Susskind' -coupling defined by the action (57), one could consider also a 'coarse' coupling [2] [27]. In this case , gauge fields are defined on the links of the coarse lattice only. The local gauge transformations $\phi(x,H) \rightarrow g(x)\phi(x,H)$ act at the points of the coarse lattice. However, an action based on such a gauge invariant coupling is not invariant under the transformations of $\mathcal{G}_L$. This has serious consequences for the renormalized continuum limit [28].

In the evaluation of the path integral

$$\langle\overline{\chi}\chi(\overline{x})\overline{\chi}\chi(\overline{x}')\rangle = \frac{1}{Z}\int\int \mathcal{D}[U]\mathcal{D}[\chi,\overline{\chi}]\overline{\chi}(\overline{x})\chi(\overline{x})\overline{\chi}(\overline{x}')\chi(\overline{x}')\exp -(S_g + \overline{m}\overline{\chi}(1+Q[U])\chi), \quad (58)$$

we perform first the 'Gaussian integration'[29] with respect to the 'Grassmann' variables $\overline{\chi}(\overline{x}), \chi(\overline{x})$.
This results in

$$\langle\overline{\chi}\chi(\overline{x})\overline{\chi}\chi(\overline{x}')\rangle = \frac{1}{Z}\int \mathcal{D}[U]\frac{1}{\overline{m}^2}(\overline{x}|(1+Q)^{-1}|\overline{x}')(\overline{x}'|(1+Q)^{-1}|\overline{x})\det\overline{m}(1+Q)e^{-S_g}. \quad (59)$$

For the calculation of the quark determinant $\det\overline{m}(1+Q)$ and the quark propagator $S(\overline{x},\overline{y};U) = \frac{1}{\overline{m}}(\overline{x}|(1+Q)^{-1}|\overline{y})$, we use as an intermediate step the so-called hopping parameter expansion:

$$S(\overline{x},\overline{y};U) = \frac{1}{\overline{m}}\sum_{L=0}^{\infty}(-1)^L(\overline{x}|(Q)^L|\overline{y}) = \frac{1}{\overline{m}}\sum_{L=0}^{\infty}(\frac{-1}{2\overline{m}})^L\sum_{C_L(\overline{x},\overline{y})}\prod_{l_\mu\in C_L}\rho(l_\mu)U(l_\mu),$$

$$\det(1+Q) = \exp\{\sum_{L=0}^{\infty}\sum_{x}\frac{1}{L}(\frac{-1}{2\overline{m}})^L \sum_{C_L(\overline{x},\overline{x})}\prod_{l_\mu \in C_L} \rho(l_\mu)U(l_\mu)\}. \tag{60}$$

Here $C_L(x,y)$ are paths of length L from $\overline{y}$ to $\overline{x}$ composed of the links $l_\mu = [\overline{z},\mu]$, $\rho(l_\mu) = \check{\rho}_{\mu,H(\overline{z})}, \rho(-l_\mu) = -\rho(l_\mu)$. The product of the $U(l_\mu)$ form the parallel transport of the quark colour vector along $C_L(x,y)$ . The expression of $(\overline{x}|Q^L|\overline{y})$ by sums of parallel transports along paths depends crucially on the fact that the matrix $(\overline{x}|Q|\overline{y})$ is only different from zero for neighbouring points $\overline{x}, \overline{y}$.

After inserting this expression (60) in Eq. (59) we may perform the integration with respect to the gauge field variables $U(\overline{x},\mu)$ by expanding the Boltzmann factor $e^{-\beta S_g'}$ in $\beta$ ('strong coupling approximation'). The orthogonality of matrix elements of irreducible group representations with respect to averaging with the Haar measure

$$\int_{SU(3)} d\mu(g) U_{\alpha,\beta}(g) U^*_{\alpha',\beta'} = \frac{1}{\dim U}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'} \tag{61}$$

allows a systematic evaluation of gauge field averages. Already in zero order we get a non-trivial result. The meson propagator is given by the sum over all strictly parallel quark lines, see Fig.(3).

$$\overline{\chi}\chi(\overline{x})\overline{\chi}\chi(0) = \sum_L (\frac{-1}{4\overline{m}^2})^L B_{\overline{x}}(L). \tag{62}$$

$B_{\overline{x}}(L)$ is the number of double paths from 0 to $\overline{x}$.In this approximation the quark and anti-quark are strongly bound; they can not separate by even one lattice link. The next order in $\beta$ contains paths as in Fig.(3). This order describes 'internal relative motion' over the distance of one lattice link.

Now we approximate even further the zero order expression (62) by restricting the summation to tree graphs, and omitting paths with closed loops [30]. In order to perform the summation over all tree graphs in (62), we use a combinatorial method introduced by O.Martin [31] for this type of calculation. He remarked that a general tree graph might be composed by a trunk and branches. Branches are $q\overline{q}$ lines which branch off a single quark or anti-quark line. Trunks are tree graphs without branches. In (62) one has to sum only over all trunks , if one renormalizes the mass $\overline{m} \to \frac{1}{2}(\overline{m}+\sqrt{\overline{m}^2+7})$ . Setting $\alpha = \overline{m}+\sqrt{\overline{m}^2+7}$ , we get instead of Eq. (62)

$$\overline{\chi}\chi(\overline{x})\overline{\chi}\chi(0) = \sum_L (\frac{-1}{\alpha^2})^L T_{\overline{x}}(L). \tag{63}$$

$T_{\overline{x}}(L)$ is the number of trunks from 0 to $\overline{x}$. Trunks are $\overline{q}q-$lines which are not back tracking, otherwise they would have branches. Therefore it is useful to classify trunks according to their last step. $T_{\mu,\overline{x}}(L)$ is the number of trunks from 0 to $\overline{x}$ with last step in $\mu$-direction. We have the recursion relation

$$T_{\mu,\overline{x}}(L+1) = \sum_{\nu\neq-\mu} T_{\nu,\overline{x}-\overline{e}_\nu}(L).$$

Fourier transformation with respect to $\overline{x}$ gives

$$\tilde{T}_\mu(L+1) = \sum_\nu e^{i\overline{p}_\mu}(1-\delta_{\mu,-\nu}\tilde{T}_\nu(L)) = \sum_\nu M_{\mu,\nu}(\overline{p})\tilde{T}_\nu(L). \tag{64}$$

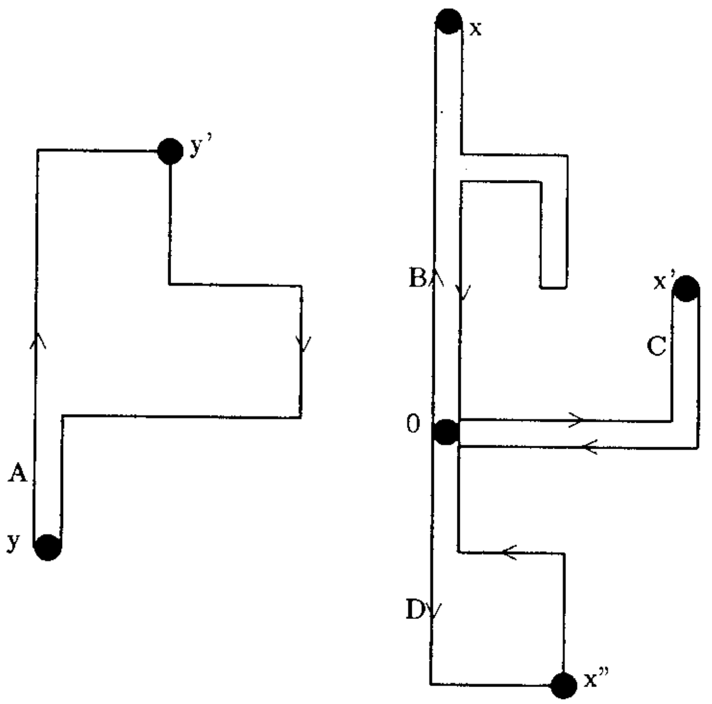


Figure 3: $q\bar{q}$-paths contributing to the meson propagator: A.'General' path from y to y'. B.Trunk with branch from 0 to x. C.Trunk from 0 to x'. D.1-order contribution.

This formula allows the summation over the length L in Eq. (63) for the Fourier transform of the meson propagator $G(\overline{p})$:

$$G(\overline{p}) = \sum_{\overline{x}} e^{i\overline{p}\overline{x}} \langle \overline{\chi}\chi(\overline{x})\overline{\chi}\chi(\overline{0})\rangle = \xi^{\dagger}(1 + \frac{1}{\alpha^2}\mathcal{M}(\overline{p}))^{-1}\xi. \tag{65}$$

$\xi^{\dagger}$, $\xi$ denote the vectors of the final and initial orientations of the paths, which are given by the details of the meson field $\overline{\chi}\chi(x)$. The poles of the propagator , the eigenvalues $\lambda(\overline{p})$ of $\mathcal{M}(\overline{p})$ with $\lambda(\overline{p}) = -\alpha^2$, determine the masses of the mesons. In a way, this eigenvalue problem plays a role similar to the energy eigen-value problem of the Schroedinger equation in non-relativistic quantum mechanics, or the Bethe Salpeter equation in perturbative relativistic field theory. It allows us to illustrate the use of the group theoretical methods developed in the last section.

The symmetry of the lattice DKE can be used to classify the meson states. It follows from Eq. (46) that mesons described by $\overline{\chi}\chi(\overline{x})$ transform by 1-dimensional flavour transformations. In particular the fields $\mu^L(\overline{x}) = \overline{\chi}(\overline{x})e^{i\pi e_L \overline{x}}\chi(\overline{x})$ transform like

$$([\epsilon d^K]\mu^L)(\overline{x}) = e^{i\pi(e_L, e_k)}\mu^L(\overline{x} + \overline{e}_K). \tag{66}$$

The $\mu^L$–propagator matrix $\mathcal{M}^L(p)$ may be determined from Eq. (64) for meson configurations with 'zero space-like momenta', i.e for meson states with quantum numbers belonging to the class of irreps with momentum star $St_j$, as described at the and of the last section. We get in 0-order

$$\mathcal{M}^L(p) = e^{ip_\mu/2} e^{i\pi(e_L,e_\mu)}(1-\delta_{\mu,-\nu}). \tag{67}$$

(Reminder: $\bar{p}$ in (64) refers to the Fourier transformation in the fine lattice, whereas the momenta of $St_j$ refer to the coarse lattice.)
$\mathcal{M}^L$ is a $8\times 8$ matrix, $\mu,\nu = \pm 1, \ldots \pm 4$, transforming the space $\mathcal{H} = \{\xi\}$. The 8-dimensional representation D(s) of $S_j^{(1)}$ in $\mathcal{H}$ decomposes in the following irreps (notation of Sect. 5):

$$\begin{array}{ll} D(s) \sim 3(1^+)_{W_3} + (2^+)_{W_3} + (3'^+)_{W_3} & \text{for} \quad L \sim k = 1, ..4, \\ D(s) \sim 4(1^+)_{D_4} + (1^-)_{D_4} + (1''^+)_{D4} + (2^-)_{D_4} & \text{for} \quad L \sim k = 5, ..8. \end{array}$$

As a consequence of the $\mathcal{G}_L$ -invariance of the theory $\mathcal{M}^L$ commutes with D(s). Schur's lemma implies that $\mathcal{M}^L$ decomposes in sub-matrices which leave the irreducible sub-spaces invariant. The eigen-value condition $\lambda(iE) = -\alpha^2$ can only be satisfied in sub-spaces belonging to the quantum numbers: $(1^+)_{W_3}$, k = 2,4 and $(1^+)_{D_4}$, k = 7,8.
They lead to the result

$$\cosh E = \frac{\alpha^4 - 6\alpha^2 + 7}{2\alpha^2} + 2j, \qquad j = 0,1,2,3 \quad \text{for} \quad k = 4,8,7,2.$$

The next problem would be to associate these states with physical particles. There is the following necessary condition which can be solved by the group theoretical method. The irreps of the continuum group $\mathcal{G}$ determine spin parity, and SU(4)-flavour of physical particles. A state of a lattice meson can be associated to such a particle, if the irrep of $\mathcal{G}$ restricted to $\mathcal{G}_L$ contains the lattice representation of the meson. With help of Mackey's sub-group theorem [26] it is a straight forward calculation to get such 'branching' rules. For details we refer to [23].

A realistic physical discussion requires higher order calculations supplemented by Monte Carlo calculations[34]. In 1-order in $\beta$, the matrices $\mathcal{M}^L$ become 80 dimensional. They can be treated successfully with help of the group theoretical methods sketched here[33]. However, a critical physical discussion is beyond the scope of this report. It was only my intention to give a realistic impression of the long way one has to go if one wants to analyze the physics of the DKE and its symmetry in the context of Quantum Chromodynamics.

## 7. SUMMARY AND OUTLOOK

The consistent quantum mechanical treatment of the standard gauge theoretical models of elementary particles by the method of lattice approximation suggests a description of the fermions by the Dirac Kaehler equation. The reason for this suggestion is the spectrum doubling problem of lattice fermions: The maximal reduction of the number of Dirac particles described by the naive lattice approximation of the Dirac equation leads to staggered fermion fields, and these fields are equivalent to a systematic lattice approximation of the Dirac Kaehler forms.

However, this natural looking solution of the lattice fermion problem leads to some open physical problems. Since complex Dirac Kaehler forms are equivalent to four Dirac fields, there is an additional degree of freedom, called (Susskind) flavour hidden in the DKE. The physical meaning of the Susskind flavour is completely unclear. One may have the following opinions:

-In spite of the possible solution of the lattice fermion problem by the DKE, the question of the 'Susskind flavour' is a lattice artifact. It should be eliminated (artificially ?!) in the contiuum limit.

-Spectrum doubling is a quantum effect of (chiral) fermion fields, related to the problem of anomalies [13] The Susskind flavour of complex (or real) Dirac Kaehler forms has some physical meaning. It might be related to the quark flavour in QCD [35], to weak isospin [36], to the family structure of elementary particles, etc. However there is not yet a physically relevant theory of such a realistic interpretation of Susskind flavour.

In order to judge on the importance of these aspects of the Dirac Kaehler formalism for elementary particle physics, one has to treat specific problems. Symmetry considerations form always an important link between a dynamical equation and physical states. For this reason we gave a description of the symmetry group of Dirac Kaehler fermions together with a sketch of its representation theory. An application to the strong coupling scheme for the calculation of the meson spectrum in the frame work of QCD was used to illustrate these type of considerations. In the studies of the DKE on the lattice the structure of the Clifford algebra and its extension appeared in a new context: as a non-asssociative algebra of cochains on a cubic lattice, the group $\mathcal{K}_4$[21] as a generalized point group of a non-symmorphic space group, $\mathcal{K}_4$ as a lattice flavour group etc.

There is a general interest in the study of lattice aspects of different physical ideas relevant for elementary particle physics. If fermions are involved, one has to rely on the Dirac Kaehler formalism. Therefore it is a first step in a lattice approximation to formulate such a problem in the continuum in terms of differential forms instead of Dirac spinors. We mention shortly a few examples for such a procedure. Some of these problems are treated more extensively in another context in other contributions to this proceedings. [37]

(i) CLASSICAL SOLUTIONS OF THE DKE IN AN EXTERNAL INSTANTON FIELD. It is generally believed that certain configurations of gauge fields, which are close to solutions of the classical field equations with a topological charge -'instantons'- , play an important role in the dynamics of QCD. These gauge field configurations allow zero mass solutions of the Dirac equation of a given chirality, as it is stated by the Atiyah Singer index theorem.[38] There are attempts to define topological charges also for lattice approximations of gauge fields [39] and to investigate these by numerical calculations. Recent calculations include the effect of these gauge field configurations on the spectrum of fermion systems. From this we derive an interest in the structure of solutions of the DKE

$$(d - \delta)\Phi - i\mathbf{A} \vee \Phi = 0 \tag{68}$$

in a classical SU(2) - instanton field

$$\mathbf{A}(x) = \frac{x^2}{x^2 + \lambda^2}\omega \tag{69}$$

with

$$\omega = \frac{1}{x^2}(x^i dx^4 - x^4 dx^i + \epsilon^i_{kl} x^k dx^l)\tau_i \tag{70}$$

The $\tau_i, i = 1,2,3$ are the Pauli matrices. $\omega$ is closely related to the Lie algebra valued Maurer Cartan forms of the group SU(2). With help of the Maurer Cartan relations, one gets after a short calculation the solution

$$\Phi = (r^2 + \lambda^2)^{-\frac{3}{2}}(1 - r dr \wedge \omega) \vee (1 + dx^{1234}), \qquad r = \sqrt{x^2}. \tag{71}$$

Of course this solution corresponds to the solution of the Dirac equation with positive chirality which is postulated by the index theorem. It would be intersting to see if something of the correlation between the gauge field and the fermion form expressed by Eq. (71) survives the lattice approximation.

(ii)REMARK ON GRAVITATION There are also attempts to investigate the theory of gravitation in lattice approximation [40]. The natural way to include fermions in these considerations leads to the DKE. Since the DKE is defined for a general ('pseudo'-) Riemannian manifold, it describes in a straightforward manner the interaction the interaction with a classical gravitational field. However, it was pointed out by several authors [41], that in curved space there is no longer an equivalence between the Dirac equation and the DKE. The general relativistic DKE postulates an interaction of the flavour degrees of freedom with the gravitational field. In a weak gravitational field described by a metric tensor:
$g_{00} = -(1 + V(x)) = -g(x), g_{\mu\nu} = \delta_{\mu\nu} \quad for \quad \mu \neq 0, \nu \neq 0,$
$V(x)$ a static potential, the DKE might be written [42]:

$$-\frac{1}{g}\partial_0\Phi + \sum_{i=1}^{3} dx^{0i} \vee \partial_i\Phi + m dx^0 \vee \Phi$$

$$-\frac{1}{g^2}\partial_i V dx^i \vee e_0 \neg \Phi - \frac{1}{2g}\partial_i V \cdot \Phi \vee dx^{0i} = 0. \tag{72}$$

The first line corresponds to the Dirac equation in a gravitational field.The second line expresses the deviations introduced by the Dirac Kaehler approach. These are proportional to the gradient of V, and hence are small in slowly varying fields. However, in fundamental theories which include gravitation, e.g. like super-symmetric graviatation [43], the gravitational interaction of flavour might open interesting aspects for its physical understanding.

(iii) SUPER-SYMMETRY In the recent attempts of the formulation of fundamental theories, the consideration of super-symmetry plays an important role The Dirac Kaehler formalism represents Dirac fields by a coherent superposition of inhomogeneous differential forms. On the other hand, forms describe in a natural way tensor fields which are bosonic. This similarity strongly suggests the investigation of super-symmetry between bosons and fermions in terms of differential forms. In order to be more specific, one may consider a bosonic form $\Phi$ and a fermionic form $\Psi$ which satisfy the DKE. Then the expressions

$$j^f_\mu = (\Psi \vee f, dx_\mu \wedge \overline{\Phi}), \qquad \overline{\Phi} = \mathcal{A}\Phi \tag{73}$$

describe conserved currents depending on the flavour transformations f. It is possible to show [45], that in an explicit 2-dimensional model the charges of this mixed boson fermion currents: $Q = \int j^f_0 dx^{123}$ are closely related to the generators of super-symmetry. Other examples are discussed in the contribution by R.W.Tucker. Here we want only to emphasize that the formulation of super-symmetry in differential forms opens the interesting possibility of treating super-symmetric models in lattice approximation with help of the methods developped in Sections 3 -5. [46]

The investigation of geometry and physics of lattice fermions is still a large ,unsettled program.

ACKNOWLEDGEMENT.I would like to thank Dr.A.K.Common for his comments with respect to the final form of the manuscript.

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