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Integrals for Two-Loop Calculations in Massless QCD

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Abstract:

We present tools that have been developed for the extension of the Sterman-Weinberg formula in two-loop order. They are the essential ingredients for any perturbative two-loop calculation in a massless theory. Introducing dimensional regularization we deal with poles up to fourth order in the dimensional parameter. We study virtual two-loop integrations as well as real ones over phase space.

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1. Introduction

In this work we evaluate various integrals that appeared in the course of the calculation of e^+e^- jet cross sections to order α_s^2. This calculation [1] was done in the framework of massless parturbative QCD. Dimensional regularization has been used throughout for the regularization of the ultraviolet and the infrared divergencies 1). In the course of the calculation of the 2-jet cross sections [1], which is the one most involved, poles of up to fourth order in the dimensional parameter 2) appeared. Thus this is a true two-loop calculation because one-loop results contain at most poles ε^{-2}. So this work extends earlier papers on the techniques [6] of dimensionally regularized massless theories, which all stop at the next to leading order level. We hope that the techniques we present will be of help in other two-loop QCD calculations for ep and $p\bar{p}$ processes where at the moment the ε^{-2} level is arrived at [9].

Some of the methods are described only for the scalar integrals. However, they can all be extended to the case where numerators (coming from the Diray traces) are involved. We will present also results for this case, e.g. eq. (33). The calculation of the total hadronic cross section in e^+e^- annihilation can be reduced to the calculation of the photon 2-point function via the optical theorem. Though it is a genuine two-loop problem fortunately only poles up to ε^{-2} appeared here and could be handled with the help of the Gegenbauer technique [7]. For completeness we will review some features of this technique in the beginning of section 2. However, for the 3-, 4- and 5-point functions with one particle off shell, which are needed for jet cross sections, this method is of no advantage as compared to the usual technique of introducing Feynman parameters [8]. Then we give an elegant way for calculating the ladder diagram (fig. 2a) and use Feynman parameters to calculate the crossed diagram (fig. 2b). In section 3 we discuss boxdiagrams, where a one-loop virtual integration has to be done and also one real gluon has to be integrated out.

This means that one integrates over those parts of phase space where the gluon is infrared or collinear with one of the other partons. This adds poles in ε to the poles coming from the virtual integrations.

Sections 4 and 5 give two prescriptions how to handle the tree-level diagrams, where a two-gluon Bremsstrahlung has to be integrated out. Again poles of up to fourth order in ε emerge.

In physical applications [1] these cancel against the infrared singularities from the virtual integrations.

2. Integrals for virtual two-loop diagrams

The Gegenbauer expansion technique in momentum space [11] proves useful for diagrams with only two external legs, e.g. for the vacuum polarization of the photon [7]. In ref. [4, 7] extensive use is made of it. Here we describe some of its features for the sake of completeness.

The Gegenbauer polynomials C_j^λ , j = 0, 1, ..., $\lambda > -\frac{1}{2}$ form a complete set of functions in the interval (-1, 1) [23]. They are a generalization of the Legendre polynomials ($\lambda = \frac{1}{2}$) and the Chebyshev polynomials ($\lambda = 1$) . The method consists in expanding propagators into Gegenbauer polynomials (cf. eq. (2) of ref. [11]). Then one does the angular integrations with the help of the orthogonality relation eq. (A.1) of ref. [4] and is left with radial integrations in the form of powers of the momenta. In simple cases such as the example in [11] this leads to infinite series which can be evaluated with the help of standard summation formulas [14]. The method works for integrals which depend only on one momentum, i.e. self energy graphs. They must fulfill an additional property, namely there must exist a parametrization of integration momenta in such a way that not more than two Gegenbauer polynomials

appear within one angular integration. For integrals of products of three Gegenbauer polynomials no simple closed form exists.

In the case of the vacuum polarization of the photon in order α_s^2 [7] all graphs are calculable with the Gegenbauer technique with the exception of the graph of fig. 1, which is "irreducible".

Next we consider the 2-parton contributions of the 2-jet cross section (fig. 2). These diagrams can also be thought to define the (singular) electromagnetic form factor of massless quarks [12]. After the Dirac traces are done the dotproducts that emerge are transformed into sums of squares of momenta in such a way that as many of them as possible cancel against denominator factors, e.g.

$$kp = \frac{1}{2} k^2 - \frac{1}{2}(k-p)^2 \tag{1}$$

for $p^2 = 0$. In general one is left with relatively complicated scalar integrals, which we will calculate in the following, and simple tensorial integrals. For the tensorial as well as for some of the scalar integrals (fig. 2c – e) the strategy is to introduce Feynman parameters with the help of

$$\frac{1}{k^2(k-p)^2} = \int_0^1 \frac{dx}{[(k-p\,x)^2]^2} \tag{2}$$

which holds for $p^2 = 0$, i.e. one tries to preserve the massless structure of the theory. In general one ends up with standard integrals given in appendix A.

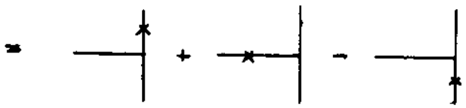
Fig. 2a, b cannot be met with this general strategy. However, planar diagrams like fig. 2a can be solved with the help of "partial integration". This method has been described in [13]. Fig. 2a can be reduced to the scalar integral [3)]

$$L_s = \int \frac{d^n k\, d^n l}{k^2 l^2 (l+p_1)^2 (l-p_2)^2 (k+l+p_1)^2 (k+l-p_2)^2} \tag{3}$$

and some tensorial integrals which can be calculated by using the standard strategy (2). To calculate L_s by the method of ref. [13] we need some notation: Let  mean an additional factor of k^2 in the numerator and  an additional factor of k^2 in the denominator. Define

$$= 2k_1(k_1+k_2) \tag{4}$$

$$= (k_1+k_2)^2 + k_1^2 - k_2^2 \tag{5}$$

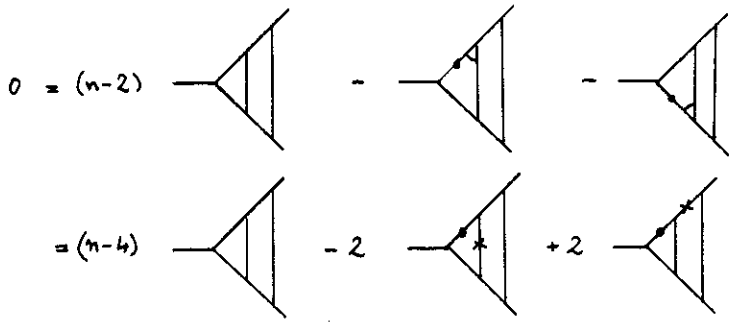


(6)

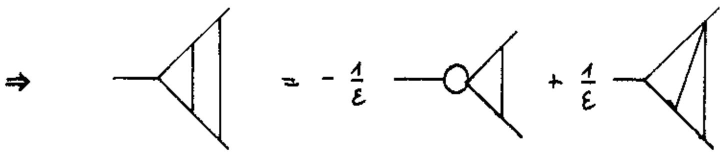
Because of the translational invariance of the integral one has the identity

$$\int d^n k\, d^n l\, \frac{\partial}{\partial k_\mu} \frac{(k+l-p_2)_\mu}{k^2 l^2 (l+p_1)^2 (l-p_2)^2 (k+l+p_1)^2 (k+l-p_2)^2} = 0 \tag{7}$$

After differentiation one receives

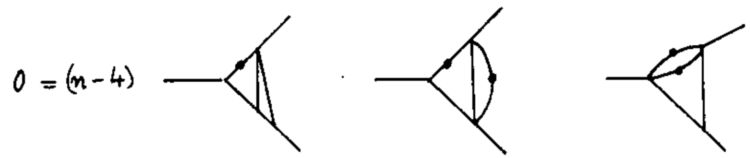


(8)



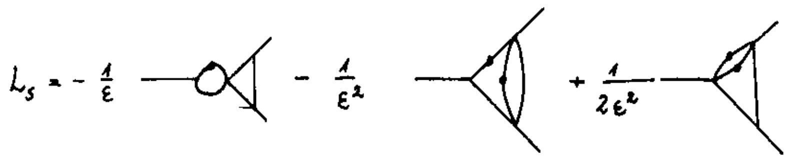
(9)

The first integral on the right hand side is very simple and can be done with the standard formulas of appendix A. The second integral can be simplified in the same way as the original one:



(10)

The result is



(11)

With the help of appendix A one gets

$$L_s = \pi^{4-2\varepsilon} (q^2)^{-2-2\varepsilon}\, \Gamma(1-\varepsilon)\, \Gamma(1+\varepsilon)\, \Gamma(1+2\varepsilon)$$

$$\left\{ \frac{1}{4\varepsilon^2} + \frac{\zeta_2}{2\varepsilon^2} + \frac{11\zeta_3}{2\varepsilon} + \frac{3}{4}\zeta_4 \right\} \tag{12}$$

Using Feynman parameters Gonsalves [12] has also arrived at this result. However, our method is much more elegant. The momentum dependence of L_s could have been derived from a simple dimensional analysis. The appearence of

$$\zeta_s = \sum_{k=1}^{\infty} k^{-s} \tag{13}$$

in connection with poles ε^{s-4} is a characteristic feature of the result (12).

The crossed diagram (fig. 2b) cannot be done with this method because it is not planar. (To see this try to apply (8) to it!) The scalar crossed integral

$$K_s = \int \frac{d^n k \, d^n l}{k^2 l^2 (l-p_1)^2 (k-l)^2 (k-l-p_2)^2 (k-p_1-p_2)^2} \tag{14}$$

can however be calculated with the help of Feynman parameters.

$$K_s = \int \frac{d^n k \, d^n l}{k^2 (l-p_1 x)^4 (k-l-p_2 y)^4 (k-p_1-p_2)^2} \tag{15}$$

Any further Feynman parameter is accompanied by a "mass term" in the denominator. One gets

$$K_s = \pi^{2-\varepsilon} (q^2)^{-1-\varepsilon} (3+\varepsilon)(2+\varepsilon) \int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 du \; u^{1+\varepsilon} (1-u) \int \frac{d^n k}{(k^2+C)^{4+\varepsilon}} \tag{16}$$

$$C = z(1-z) q^2 (1-u) + u(1-u) (qz - x p_1 - y p_2)^2 \tag{17}$$

The k-integration can be done with eq. (A.1).

The y-integration is then straightforward:

$$K_s = -\pi^{4-2\varepsilon} (q^2)^{-2-2\varepsilon} \frac{2}{\varepsilon} \frac{\Gamma^2(1-\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma(1-2\varepsilon)} \int_0^1 du \; u^{\varepsilon} (1-u)^{-1-2\varepsilon}$$

$$\int_0^1 dx \int_0^1 \frac{dz}{z-x} z^{-1-2\varepsilon} (1-z)^{-1-2\varepsilon} \left\{ \left(1-u \frac{z-x}{z}\right)^{-1-2\varepsilon} - \left(1-u \frac{x-z}{1-z}\right)^{-1-2\varepsilon} \right\} \tag{18}$$

The divergence for $x = z$ is an artifact of the y-integration and disappears in the difference. The u-integration leads to hypergeometric functions. If one wants their series representation to converge one has to distinguish the regions $x < z$ and $x > z$. Then

$$K_s = \pi^{4-2\varepsilon} (q^2)^{-2-2\varepsilon} \Gamma(1-\varepsilon)\Gamma(1+2\varepsilon)\Gamma(1+\varepsilon) \, \hat{J} \cdot \varepsilon^{-2} \tag{19}$$

$$\begin{aligned} \hat{J} = & \int_0^1 dz \int_0^z dx \; z^{-1-2\varepsilon} (1-z)^{-1-2\varepsilon} \mathcal{F}\left(1-\frac{x}{z}\right) (z-x)^{-1} \\ & + \int_0^1 dz \int_0^z dx \; x^{-1-2\varepsilon} (1-z)^{-1-2\varepsilon} G\left(1-\frac{z}{x}\right) (z-x)^{-1} \\ & - \int_0^1 dz \int_z^1 dx \; z^{-1-2\varepsilon} (1-z)^{-1-2\varepsilon} \mathcal{F}\left(1-\frac{1-x}{1-z}\right) (z-x)^{-1} \\ & - \int_0^1 dz \int_z^1 dx \; (1-x)^{-1-2\varepsilon} z^{-1-2\varepsilon} G\left(1-\frac{1-z}{1-x}\right) (z-x)^{-1} \end{aligned} \tag{20}$$

with

$$\mathcal{F}(a) = {}_2F_1(1+2\varepsilon, 1+\varepsilon, 1-\varepsilon, a) \tag{21}$$

$$G(a) = {}_2F_1(1+2\varepsilon, -2\varepsilon, 1-\varepsilon, a) \tag{22}$$

We change from x to 1-x and from z to 1-z in the last two terms of J and get

$$\hat{J} = 2(J_0 + J_1 + J_2) \tag{23}$$

$$I_0 = \int_0^1 dz \int_0^z dx\, z^{-1-2\varepsilon} \left[(1-z)^{-1-2\varepsilon} - (1-x)^{-1-2\varepsilon} \right] (z-x)^{-1}$$

$$= \Gamma^{-1}(1+2\varepsilon) \sum_{0} \frac{\Gamma(k+1+2\varepsilon)}{k!\,(k-2\varepsilon)} \left(\psi(k+1) - \psi(1) \right) \tag{24}$$

is the contribution from the first term in the series expansion of the hypergeometric functions. I_1 and I_2 contain the integration of all other terms. Because of absolute convergence of the hypergeometric series one can exchange summation and integration. The sums that finally appear can be done with the help of standard formulas [14]. The final result is

$$K_s = \pi^{4-2\varepsilon} (q^2)^{-2-2\varepsilon} \Gamma(1-\varepsilon)\Gamma(1+\varepsilon)\Gamma(1+2\varepsilon) \left\{ \frac{1}{\varepsilon^4} - \frac{9\zeta_2}{\varepsilon^2} - \frac{25\zeta_3}{\varepsilon} + \frac{15}{2}\zeta_4 \right\} \tag{25}$$

For completeness we give also the results for the scalar integrals of fig. 2c - e.

$$\int \frac{d^n k\, d^n l}{k^2 (k-p_1)^4 (k-p_1-p_2)^2 l^2 (k-p_1-l)^2} = v_0 \left(-\frac{3}{4\varepsilon^2} - 3 + \frac{3}{2}\zeta_2 \right) \tag{26}$$

$$\int \frac{d^n k\, d^n l}{k^2 l^2 (k-l)^2 (l-p_1)^2 (k-p_1)^2 (k-p_1-p_2)^2} = \tag{27}$$

$$= v_0 \left(\frac{3}{4\varepsilon^3} - \frac{3}{2\varepsilon^2} + \frac{3 - 9/2\,\zeta_2}{\varepsilon} - 6 - \frac{15}{2}\zeta_3 + 9\zeta_4 \right)$$

$$\int \frac{d^n k\, d^n l}{k^2 l^2 (k-p_1)^2 (k+l-p_1)^2 (k+p_2)^2} = v_0 \left(\frac{3}{2\varepsilon^2} + \frac{3}{2\varepsilon} + \frac{9}{2} - 6\zeta_2 \right) \tag{28}$$

Here

$$v_0 = \pi^{4-2\varepsilon} (q^2)^{-2-2\varepsilon} \Gamma(1-\varepsilon)\Gamma(1+\varepsilon)\Gamma(1+2\varepsilon) \tag{29}$$

In the case of fig. 2b one cannot reduce all the tensor structure from the Dirac trace to the scalar integral K_s and the standard integrals of appendix A. In addition one needs

$$R_l = \int \frac{d^n k\, d^n l\, (2kp_2)^l}{k^2 l^2 (l-p_1)^2 (k-l)^2 (k-l-p_2)^2 (k-p_1-p_2)^2} \tag{30}$$

for l = 1, 2, 3.

The denominator in eq. (14) is invariant under the two transformations $(p_1 \leftrightarrow p_2, l \leftrightarrow k-l)$ and $(k \leftrightarrow p_1+p_2-k, l \leftrightarrow p_1-l)$. Using the second invariance one can derive

$$R_1 = K_s/2 \tag{31}$$

$$R_3 = -K_s/2 + 3R_2/2 \tag{32}$$

R_2 can be calculated by the same methods as K_s:

$$R_2 = \pi^{4-2\varepsilon} (q^2)^{-2-2\varepsilon} \frac{\Gamma^3(1-\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma(1-3\varepsilon)} \left\{ \frac{1}{2\varepsilon^4} + \frac{1}{\varepsilon^3} + \right.$$

$$\left. \frac{4}{\varepsilon^2} - \frac{5\zeta_2}{2\varepsilon^2} + (14 - 4\zeta_2 - \frac{17}{2}\zeta_3)/\varepsilon + 46 - 12\zeta_2 - \frac{8}{5}\zeta_3 - \frac{85}{4}\zeta_4 \right\} \tag{33}$$

3. Discussion of box diagrams

In section 2 the "two-loop" calculation consisted in doing two virtual integrations. Here we consider diagrams like fig. 3, where one loop integration has to be done and one real particle (namely the gluon) has to be integrated "out" subsequently (i.e. a phase space integration over infrared and collinear regions has to be carried out). As an example we consider the scalar box integral with three particles on mass shell $p_1^2 = p_2^2 = p_3^2 = 0$. The case with all four particles on shell is contained in it. If all incoming and outgoing particles are off shell the integral can be done in 4 dimensions, because no infrared singularity exists. (Box integrals are ultraviolet finite anyhow.) Four dimensional box integrals are calculated in the 3rd paper of [6]. To calculate

$$J_{Box} = \int \frac{d^n k}{k^2(k+p_2)^2(k-p_3)^2(k-p_1-p_3)^2} \tag{34}$$

we follow the general strategy described in section 2. Namely we introduce Feynman parameters avoiding massterms in the denominator

$$J_{Box} = \int_0^1 dx \int_0^1 dy \int \frac{d^n k}{(k+p_2 x)^4 (k-p_3-p_1 y)^4} \tag{35}$$

After a shift of variables one can use the standard formula (A.2) for doing the virtual integration. The Feynman parameter integrations lead to hypergeometric functions

$$J_{Box} = -2(q^2)^{-2-\varepsilon}\pi^{2-\varepsilon}\frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\varepsilon^2\Gamma(1-2\varepsilon)}\frac{1}{y_{13}y_{23}}$$

$$\left\{ {}_2F_1\left(1,-\varepsilon,1-\varepsilon,-\frac{y_{12}}{y_{13}y_{23}}\right) - y_{13}^{-\varepsilon}\,{}_2F_1\left(1,-\varepsilon,1-\varepsilon,-\frac{y_{12}}{y_{23}}\right) - y_{23}^{-\varepsilon}\,{}_2F_1\left(1,-\varepsilon,1-\varepsilon,-\frac{y_{12}}{y_{13}}\right)\right. \tag{36}$$

$y_{ij} = 2p_ip_j/q^2$ are dotproducts normalized to the energy. $y_{12} + y_{23} + y_{13} = 1$ is energy conservation. The ε^{-2}-pole is due to the infrared singularity from the virtual integration. The q^2 dependence could also have been derived from a dimensional analysis of eq. (34). Note also the singularity $y_{13}^{-1}\ y_{23}^{-1}$ which can produce another ε^{-2}-pole after the integration over phase space. The hypergeometric functions in eq. (36) have the power series expansion

$${}_2F_1(1,-\varepsilon,1-\varepsilon,z) = 1-\varepsilon\sum_{j=1}^{\infty}\frac{z^j}{j-\varepsilon} \tag{37}$$

The series in eq. (37) is the expansion of a logarithm generalized to $n = 4-2\varepsilon$ dimensions. If expanded in a power series in ε we obtain

$$\sum_{j=1}^{\infty}\frac{z^j}{j-\varepsilon} = -\ln(1-z) + \sum_{j=1}^{\infty}\varepsilon^j L_{j+1}(z) \tag{38}$$

where L_n is the generalized Euler dilogarithm

$$L_n(z) = \sum_{j=1}^{\infty} z^j/j^n \tag{39}$$

For further integration over phase space the expansion (38) is not useful, because additional singularities may make the inclusion of L_n, $n > 2$ necessary and at the integration boundaries (39) may not be convergent.

So we first transform the hypergeometric functions in eq. (36) according to

$${}_2F_1\left(1,-\varepsilon,1-\varepsilon,-\frac{a}{b}\right) = \left(\frac{b}{a+b}\right)^{\varepsilon} E\left(\frac{a}{a+b}\right) \tag{40}$$

with $E(z) = {}_2F_1(-\varepsilon,-\varepsilon,1-\varepsilon,z)$. The hypergeometric function E has an everywhere convergent series expansion

$$E(z) = 1 + \frac{\varepsilon^2}{\Gamma(1-\varepsilon)}\sum_{k=1}^{\infty}\frac{\Gamma(k-\varepsilon)}{k!(k-\varepsilon)}z^k \tag{41}$$

$$\mathcal{I}_{Box} = -2\,(q^2)^{-2-\varepsilon}\,\pi^{2-\varepsilon}\,\frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\varepsilon^2\,\Gamma(1-2\varepsilon)}\;y_{13}^{-1-\varepsilon}\;y_{23}^{-1-\varepsilon}$$

$$\Big\{(y_{13}y_{23}+y_{12})^{\varepsilon}\,E\Big(\frac{y_{12}}{y_{13}y_{23}+y_{12}}\Big)-(y_{12}+y_{23})^{\varepsilon}\,E\Big(\frac{y_{12}}{y_{12}+y_{23}}\Big) \tag{42}$$

$$-(y_{12}+y_{13})^{\varepsilon}\,E\Big(\frac{y_{12}}{y_{12}+y_{13}}\Big)\Big\}$$

One now wants to integrate eq. (42) over the 3-particle phase space drawn in fig. 4 [15]. One can divide it into a "2-jet" region where (42) has singularties and a "3-jet" region where (42) is finite. In the finite region $E(z) = 1+\varepsilon^2 L_2(z)$ and the integration of the L_2-term can be done in 4 dimensions.

The singular region is defined by $y_{13} < y$ or $y_{23} < y$. (42) has no singulartiy for $y_{12} \to 0$. So this region which physically is 2-jet can be formally added to the finite 3-jet region.

$y \ll 1$ is the invariant mass cut defining a jet [15]. It is both an energy and an angle cut. $\{y_{13}<y,\, y_{23}>y\}$ is the collinear region $1 \| 3$, $\{y_{13}>y,\, y_{23}<y\}$ is the collinear region $2 \| 3$ and $\{y_{13}<y,\, y_{23}<y\}$ is the infrared region $3 \to 0$. Because of the symmetry the 2-jet region of 3-particle phase space can be written as

$$\int_0^y dy_{13}\, y_{13}^{-\varepsilon}\,\Big\{2\int_0^{1-y_{13}} - \int_0^y\Big\}\,dy_{23}\,y_{23}^{-\varepsilon}\,(1-y_{13}-y_{23})^{-\varepsilon} \tag{43}$$

As long as one neglects contributions of order y one can restrict oneself to the y_{13}-pole in (42). So the 2-jet contribution from the scalar box integral is

$$B_y := \int_0^y dy_{13}\, y_{13}^{-1-\varepsilon}\,\Big\{2\int_0^1 - \int_0^y\Big\}\,dy_{23}\,y_{23}^{-\varepsilon}(1-y_{23})^{-\varepsilon}\,\lim_{y_{13}\to 0} y_{13}\,\mathcal{I}_{Box} \tag{44}$$

which can be calculated straightforwardly. The result has an ε^{-4} pole with a structure similar to the two-loop virtual results in (12) or (25). In addition one has terms $\ln^k y/\varepsilon^{4-k}$, k = 0, 1, 2, 3, 4:

$$B_y = \pi^{2+\varepsilon}(q^2)^{-2-2\varepsilon}\,\frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}\,\Big\{\frac{1}{2\varepsilon^4}-\frac{3\zeta_2}{2\varepsilon^2}-\frac{2\zeta_3}{\varepsilon}-\frac{25}{8}\zeta_4 \tag{45}$$

$$+4\zeta_2\ln y/\varepsilon+4\zeta_3\ln y-6\zeta_2\ln^2 y-2\ln^2 y/\varepsilon^2+4\ln^3 y/\varepsilon-\frac{14}{3}\ln^4 y\Big\}$$

4. Tree diagrams

In the case of a diagram like fig. 5 one has to do no virtual integrations. However, two gluons have to be integrated "out" here to calculate its 2-jet contribution - the 3- and 4-jet contributions from such diagrams have been discussed extensively in the literature [2], [3], [16], [17], [18]. In the 4-jet case no ε singularities appear. In the 3-jet case one has to deal with ε^{-2} singularities coming from an integration measure very similar to (43).

There exist two topologically distinct 2-jet configurations here:

(46)

+ permutations + infrared configurations.

Because of the pole structure of the matrix element squared only the permutations $1 \leftrightarrow 2$ and/or $3 \leftrightarrow 4$ are necessary. All other configurations which are physically 2-jet lead to the $O(y)$ contributions. However, because of the appearance of terms $\sim y/\varepsilon$ they cannot be simply added to the 3- or 4-jet contributions, but must in principle be cancelled independently. In section 5 we describe a method where such terms are avoided. That method will also provide us with all finite logarithms necessary for the 2-jet cross section, whereas the procedure described here is only sufficient to cancel the singularities $\sim \mathrm{const}/\varepsilon$ or $\ln^k y/\varepsilon^{4-k}$

We have used the representation of 4-particle phase space given in [3]. In the 2-jet limit (46) the 4-particle phase space reduces to

$$2\int_0^y dy_{24}\, y_{24}^{-\varepsilon} \int_0^y dy_{13}\, y_{13}^{-\varepsilon} \int_y^1 dy_{134}\, y_{134}^{-\varepsilon}(1-y_{134})^{-\varepsilon} \int_0^1 du\, u^{-\varepsilon}(1-u)^{-\varepsilon}$$

$$\int_0^\pi \frac{d\theta'}{N_{\theta}'} \sin^{-2\varepsilon}\theta'$$

$$+ \int_0^y dy_{134}\, y_{134}^{1-2\varepsilon} \int_0^1 dz\, z^{-\varepsilon}(1-z)^{-\varepsilon} \int_0^\pi \frac{d\theta'}{N_{\theta'}} \sin^{-2\varepsilon}\theta' \cdot$$

$$\left\{ 2\int_0^1 dy_{24} \int_0^1 du - \int_y^1 dy_{24} \int_0^y dy_{23} - \int_0^y dy_{24} \int_y^1 dy_{23} - \int_0^y dy_{24} \int_0^y dy_{23} \right\}$$

$$y_{24}^{-\varepsilon}(1-y_{24})^{1-2\varepsilon}\, u^{-\varepsilon}(1-u)^{-\varepsilon} \tag{47}$$

Here $y_{ijk} := y_{ij} + y_{jk} + y_{ik}$, $u := y_{23}/(y_{12}+y_{23})$, $z = y_{13}/y_{134}y_{123}$

and $N_{\theta'} = 2^{2\varepsilon}\pi\, \Gamma(1-2\varepsilon)\, \Gamma^{-2}(1-\varepsilon)$ is the normalization of the θ'-integration. The first term in (47) operates on the $y_{13}\, y_{24}$-poles and the second term on the y_{134}-double poles of the matrix elements squared. These double poles have a structure much more complicated than any pole structure which appeared so far. We will not write down the full expression here but only describe the calculation of two characteristic integrals. These integrals stem from poles in y_{14} in the matrix element squared.

$$\lim_{y_{134}\to 0} y_{14}/y_{134} = u(1-z) + z u y_{24} - 2\cos\theta' \sqrt{u(1-u)z(1-z)y_{24}} \tag{48}$$

has a relatively complicated structure (see appendix B).

We want to calculate

$$J_{\alpha\beta}^{\gamma\delta} := \int_0^1 dy\, y^{\alpha-\varepsilon}(1-y)^{\beta-2\varepsilon} \int_0^1 dz\, z^{-\varepsilon}(1-z)^{-\varepsilon} \int_0^1 du\, u^{\delta-\varepsilon}(1-u)^{\gamma-\varepsilon}\, J/N_{\theta'} \tag{49}$$

to order ε where

$$\mathcal{J} = \int_0^{\pi} \frac{d\theta' \sin^{-2\varepsilon}\theta'}{ay+b-2\sqrt{aby}\,\cos\theta'} \tag{50}$$

with a = uz, b = (1-u)(1-z). The θ'-integration can be done

$$\mathcal{J}/N_{\theta'} = (ay+b)^{-1}\, {}_2F_1\left(\tfrac{1}{2}, 1, 1-\varepsilon, \frac{4aby}{(ay+b)^2}\right) \tag{51}$$

The hypergeometric function can be transformed by a "quadratic transformation" [10] so that

$$\mathcal{J}/N_{\theta'} = \left(\frac{s_+}{2}\right)^{2\varepsilon} r_-^{-1-2\varepsilon}\, {}_2F_1\left(-\varepsilon, -2\varepsilon, 1-\varepsilon, \frac{s_-}{s_+}\right) \tag{52}$$

where $s_\pm = r_+ \pm r_-$, $r_\pm = |ay \pm b|$. We can set ${}_2F_1(-\varepsilon,-2\varepsilon,1-\varepsilon,\frac{s_-}{s_+}) = 1$ because terms of order $\varepsilon^2 \frac{s_-}{s_+}$ give a contribution $O(\varepsilon)$ in the full result. The reason for that is that s_-/s_+ cancels all singularities which appear (u^{-1}, y^{-1}) with the exception of the y_{134}-singularity. So the u-, y- and z-integration are finite. One has a ε^{-1} from the y_{134}-integration but this pole is removed by the ε^2 factor.

Next the z-integration is done. One divides the region of integration into two regions: 1) $z < \gamma$ and b) $z > \gamma$ ($\gamma := v/\beta$, $\beta := 1-u(1-y)$, $v := 1-u$)

Then one has

$$\mathcal{J}^{\gamma\delta}_{\alpha\beta} = \int_0^1 dy\, y^{\alpha-\varepsilon}(1-y)^{\beta-2\varepsilon} \int_0^1 du\, u^{\delta-\varepsilon}(1-u)^{\gamma-\varepsilon}\,(M_a + M_b) \tag{53}$$

where

$$\begin{aligned} \beta M_a &= \beta \int_0^{\gamma} dz\, z^{-\varepsilon}(1-z)^{-\varepsilon}(v-\beta z)^{-1-\varepsilon}\, v^{2\varepsilon}(1-z)^{2\varepsilon} \\ &= \gamma^{-\varepsilon}\,\frac{\Gamma(1-\varepsilon)\Gamma(-2\varepsilon)}{\Gamma(1-3\varepsilon)}\, {}_2F_1(-\varepsilon, 1-\varepsilon, 1-3\varepsilon, \gamma) \\ &= [\beta M_b]_{\gamma \leftrightarrow 1-\gamma} \end{aligned} \tag{54}$$

Expanding the hypergeometric function in eq. (54) in powers of ε and $1-\gamma$ one gets [20]

$$\beta(M_a+M_b) = \left[\frac{\Gamma(1-\varepsilon)\Gamma(-2\varepsilon)}{\Gamma(1-3\varepsilon)} + \frac{\Gamma(\varepsilon)\Gamma(-2\varepsilon)}{\Gamma(-\varepsilon)}\right]\left(\frac{1-\gamma}{\gamma}\right)^{-\varepsilon} + \frac{\Gamma(1-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(1-2\varepsilon)} + O(uy\,\varepsilon^2) \tag{55}$$

The terms of order $uy\varepsilon^2$ do not contribute (see the remarks after eq. (52)). We conclude

$$\mathcal{J}^{\gamma\delta}_{\alpha\beta} = \frac{\Gamma(1-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(1-2\varepsilon)}\, F^{\gamma\delta}_{1\alpha\beta} + \left[\frac{\Gamma(1-\varepsilon)\Gamma(-2\varepsilon)}{\Gamma(1-3\varepsilon)} + \frac{\Gamma(\varepsilon)\Gamma(-2\varepsilon)}{\Gamma(-\varepsilon)}\right] F^{\gamma\delta}_{2\alpha\beta} \tag{56}$$

where

$$\mathcal{F}^{\gamma\delta}_{n\alpha\beta} = \int_0^1 dy \int_0^1 du\, y^{\alpha-n\varepsilon}(1-y)^{\beta-2\varepsilon} u^{\delta-n\varepsilon}(1-u)^{\gamma-(2-n)\varepsilon}\,(1-u(1-y))^{-1} \tag{57}$$

can be calculated by expanding $(1-u(1-y))^{-1}$ near 1:

$$\mathcal{F}^{\gamma\delta}_{n\alpha\beta} = \frac{\Gamma(\beta+1-2\varepsilon)\Gamma(\alpha+\gamma+1-2\varepsilon)}{\Gamma(\alpha+\beta+1-(n+2)\varepsilon)} \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha+1-n\varepsilon)\Gamma(k+\delta+1-n\varepsilon)}{k!\,\Gamma(k+\gamma+\delta+\alpha+2-(n+2)\varepsilon)(k+\alpha+\beta+1-(n+2)\varepsilon)} \tag{58}$$

For all combinations of $\alpha,\beta,\gamma,\delta$ that appear in the problem the sum in eq. (57) is convergent.

As an example we now calculate $\mathcal{J}^{1-1}_{-10}$. We have

$$\mathcal{F}^{1-1}_{2-10} = \Gamma^4(1-2\varepsilon)\,/\,4\varepsilon^2\Gamma^2(1-4\varepsilon) \tag{59}$$

$$\mathcal{F}^{1-1}_{1-10} = \frac{\Gamma^2(1-2\varepsilon)\Gamma^2(-\varepsilon)}{\Gamma^2(1-3\varepsilon)} + \frac{\Gamma^2(1-2\varepsilon)}{\Gamma(-3\varepsilon)} \sum_{k=1}^{\infty} \frac{\Gamma(k-\varepsilon)^2}{k!\,\Gamma(k+1-3\varepsilon)(k-3\varepsilon)} \tag{60}$$

One has

$$\sum_{k=1}^{\infty} \frac{\Gamma^2(k-\varepsilon)}{k!\,\Gamma(k+1-3\varepsilon)(k-3\varepsilon)} = \sum_{k=1}^{\infty} \frac{\Gamma(k+\varepsilon)}{k!\,(k-2\varepsilon)(k-4\varepsilon)} + O(\varepsilon^3) \tag{61}$$

and the sum on the right hand side of (60) can be done with (6.6.2) of ref. [14]. In total one gets

$$\mathcal{J}^{1-1}_{-10} = -\frac{1}{\varepsilon^3} + \frac{11\zeta_2}{2\varepsilon} + 18\zeta_3 + \frac{31}{2}\zeta_4\varepsilon + O(\varepsilon^2) \tag{62}$$

An integral closely related to $\mathcal{J}^{\gamma\delta}_{\alpha\beta}$ is

$$K^{\gamma\delta}_{\alpha\beta} := \int_0^1 dy\, y^{\alpha-\varepsilon}(1-y)^{\beta-2\varepsilon} \int_0^1 dz\, z^{-1-\varepsilon}(1-z)^{-\varepsilon} \int_0^1 du\, u^{\delta-\varepsilon}(1-u)^{\gamma-\varepsilon}$$
$$\int_0^{\pi} \frac{d\theta'}{N_{\theta'}} \frac{-2\sqrt{aby}\,\cos\theta'}{ay+b-2\sqrt{aby}\,\cos\theta'} \tag{63}$$

Because of $\sqrt{aby}$ in the numerator it is less divergent than $\mathcal{J}^{\gamma\delta}_{\alpha\beta}$. One can see this by writing (see (53))

$$K^{\gamma\delta}_{\alpha\beta} = \int_0^1 dy\, y^{\alpha-\varepsilon}(1-y)^{\beta-2\varepsilon} \int_0^1 du\, u^{\delta-\varepsilon}(1-u)^{\gamma-\varepsilon}\,(N_a+N_b) \tag{64}$$

One finds

$$N_a+N_b = -2\ln y\,(N_a+N_b) - 2\ln\gamma + \varepsilon\ln^2\gamma + 2\varepsilon L_2(1-\gamma) + O(\varepsilon^2\ln y) \tag{65}$$

We write

$$-\ln\gamma + \varepsilon\ln^2\gamma + 2\varepsilon L_2(1-\gamma) = 2\,\frac{1-\gamma}{1-\varepsilon}\;{}_2F_1(1,1,2-\varepsilon,1-\gamma) + O(\varepsilon^2 u y) \tag{66}$$

So $K^{\gamma\delta}_{\alpha\beta}$ is reduced to $J^{\gamma\,\delta+1}_{\alpha+1\,\beta}$ modulo a third type of integral

$$L^{\gamma\delta}_{\alpha\beta} = \int_0^1\!\!\int_0^1 \frac{dy\,du}{1-\varepsilon}\, y^{\alpha+1-\varepsilon}(1-y)^{\beta-2\varepsilon} u^{\delta+1-\varepsilon}(1-u)^{\gamma-\varepsilon}\; \frac{{}_2F_1\left(1,1,2-\varepsilon,\frac{uy}{1-u(1-y)}\right)}{1-u(1-y)} \tag{67}$$

which is finite for all combinations of $\alpha,\beta,\gamma,\delta$ that appear. Introducing γ instead of u as integration variable we can do the y-integration. This leads to another hypergeometric function. Expanding the hypergeometric functions into series we can calculate $L^{\gamma\delta}_{\alpha\beta}$ in a standard way.

5. The partial fractioning approach

To get all finite contributions of the tree diagrams to the 2- and 3-jet cross section a representation of the matrix element squared has proved fruitful, which has singularities only when a certain y_{ij}, say y_{13}, is zero [21, 22]. In contrast to eq. (47) the structure of the poles in ε can be derived from considering the region $y_{13}<y$ solely. Finite contributions from other regions can be easily included by numerical integrations [22]. In the 3- and 4-jet case they can even be calculated so as to include contributions of order y [22]. We will give here some typical integrals which appear in the course of the analytical calculation of the singular contributions to the two-jet cross section. (In the three-jet case there

is essentially one complicated integral, see eq. (A.1) of ref. [3].)

One can write the matrix element squared in the form

$$ME = \frac{A}{y_{13}} + (1-2) + (3-4) + (1-2,3-4) \tag{68}$$

Because of symmetry

$$(\text{full phase space})\ \frac{A}{y_{13}} = \frac{1}{4}\cdot(\text{full phase space})\ ME \tag{69}$$

So it is enough to consider the term A/y_{13}. The most interesting region is the region $y_{134}<y$, which is part of the two-jet region. Let us first consider an integral which originates from the pole component $A|_{z=0}$, namely

$$J_p = (\text{phase space})_{y_{134}<y}\ \frac{1}{y_{13}\,y_{134}\,(y_{13}+y_{23})(y_{13}+y_{24})} \tag{70}$$

Here

$$(\text{phase space})_{y_{134}<y} = \int_0^y dy_{134}\, y_{134}^{-2\varepsilon}\int_0^1 dy_{123}\, y_{123}^{1-2\varepsilon}(1-y_{123})^{-\varepsilon}\int_0^1 dz\, z^{-\varepsilon}(1-z)^{-\varepsilon}\int_0^1 dv\, v^{-\varepsilon}(1-v)^{-\varepsilon} \tag{71}$$

(see (47)).

From

$$y_{123} = 1 - y_{24} + O(y_{134}) \tag{72}$$

$$z = y_{13}/y_{134}y_{123} \tag{73}$$

$$u = 1-v = y_{23}/(y_{123}-y_{13}) \tag{74}$$

one gets

$$J_p = \int_0^y dy_{134}\, y_{134}^{-1-2\varepsilon} \int_0^1 dz\, z^{-1-\varepsilon}(1-z)^{-\varepsilon} \int_0^1 dv\, v^{-\varepsilon}(1-v)^{-\varepsilon}\left(1-v\left(1-z y_{134}\right)\right)^{-1} \tag{75}$$

$$\int_0^1 dy_{123}\, y_{123}^{-2\varepsilon}\left(1-y_{123}\right)^{-\varepsilon}\left(1-y_{123}\left(1-z y_{134}\right)\right)^{-1}$$

If one neglects terms of order y one can use the appropriate formula

$$\int_0^1 dx\, x^{k-a\varepsilon}(1-x)^{-\varepsilon}\left(1-x\left(1-z y_{134}\right)\right)^{-1} = \Gamma(k+1-a\varepsilon)\Gamma(-\varepsilon)\Gamma^{-1}(k+1-(a+1)\varepsilon) \tag{76}$$

$$+\Gamma(1-\varepsilon)\Gamma(\varepsilon)\, z^{-\varepsilon} y_{134}^{-\varepsilon} + O(y_{134})$$

which can be derived from 3.197(3) and 9.131 of ref. [20]. The final formula for J_p thus contains terms $\sim y^{-3\varepsilon}, y^{-4\varepsilon}$ which in section 4 appeared only after subtraction of doubly counted infrared regions (see (47)).

$$J_p = \frac{y^{-2\varepsilon}}{-2\varepsilon}\, C \cdot B_1^2 + 2\Gamma(1-\varepsilon)\Gamma(\varepsilon)\left(C+B_1\right)\frac{y^{-3\varepsilon}}{-3\varepsilon}\, B_2 \tag{77}$$

$$+ 4\Gamma^2(1-\varepsilon)\Gamma^2(\varepsilon)\, \frac{y^{-4\varepsilon}}{-4\varepsilon}\, B_3$$

$$B_n = \Gamma(1-\varepsilon)\Gamma(-n\varepsilon)/\Gamma(1-(n+1)\varepsilon) \tag{78}$$

$$C = \Gamma(1-2\varepsilon)\Gamma(-\varepsilon)/\Gamma(1-3\varepsilon) \tag{79}$$

A remark is in order: in the scheme described in section 4 a denominator $\sim y_{13} y_{14} y_{23} y_{24}$ emerged. This made the θ'-integration in the region $y_{134} < y$ so tedious. In the partial fractioning scheme it turns out that one has only three of those four y_{ij} appearing at a time, e.g. $y_{12}(y_{12}+y_{23})(y_{13}+y_{24})$ (I), as in eq. (70), or $y_{13}(y_{13}+y_{14})(y_{14}+y_{24})$ (I). To avoid complicated θ'-dependences in the case II we change the coordinate frame. Exchanging the role of particles 2 and 4 the θ'-integration is trivial again and eq. (76) can be applied. One should note that the structure of invariants is a little different in the two systems. In contrast to eq. (48) one has here

$$\lim_{y_{34}\to 0} y_{12} = \frac{y_{123}}{1-z y_{123}}\left[(1-u)(1-z) + u z y_{24} - 2\cos\theta' \sqrt{u(1-u)z(1-z)y_{24}}\right] \tag{80}$$

with $u = y_{34}/(y_{134}-y_{13})$

Now we turn to some integrals which emerge in the non-pole part $A - A|_{\varepsilon=0}$ of eq. (68). A typical example is

$$J_v := (\text{phase space})_{y_{134}<y}\ \frac{y_{12}}{y_{123}\, y_{134}\left(y_{13}+y_{14}\right)\left(y_{14}+y_{24}\right)} \tag{81}$$

To avoid θ'-dependences in the denominator we work in the system of eq. (80). The θ'-dependence in the numerator is removed by antisymmetry. One gets

$$J_v = \int_0^y dy_{134}\, y_{134}^{-1-2\varepsilon} \int_0^1 dy_{123}\, y_{123}^{1-2\varepsilon}\left(1-y_{123}\right)^{-\varepsilon} \int_0^1 dz\, z^{-\varepsilon}(1-z)^{-\varepsilon} H_v \tag{82}$$

where

$$H_V := \int_0^1 \frac{du\, u^{-\varepsilon}(1-u)^{1-\varepsilon}}{(1-u(1-zy_{123}))(1-y_{123}(1-zy_{134}))} + z\int_0^1 \frac{du\, u^{-\varepsilon}(1-u)^{-\varepsilon}(2u-1)}{(1-zy_{123})(1-u(1-zy_{123}))} \tag{83}$$

The second integral in eq. (83) can be done after partial fractioning

$$\frac{1}{(1-zy_{123})(1-u(1-zy_{123}))} = \frac{1}{1-zy_{123}} + \frac{u}{1-u(1-zy_{123})} \tag{84}$$

One is led to one integral of the type

$$K(k,l,m,n) := \int_0^1 dy\, y^{k-2\varepsilon}(1-y)^{l-\varepsilon}\int_0^1 dz\, z^{m-\varepsilon}(1-z)^{n-\varepsilon}(1-zy)^{-1} \tag{85}$$

and one integral of the type

$$\mathcal{I}(k,l,m,n,a,b) := \frac{\Gamma(a+b-2\varepsilon)}{\Gamma(a-\varepsilon)\Gamma(b-\varepsilon)} \int_0^1 dy\, y^{k-2\varepsilon}(1-y)^{l-\varepsilon}\int_0^1 dz\, z^{m-\varepsilon}(1-z)^{n-\varepsilon} \int_0^1 du\, u^{a-1-\varepsilon}(1-u)^{b-1-\varepsilon}(1-u(1-zy))^{-1} \tag{86}$$

K can be calculated by expanding $(1-zy)^{-1}$ near 1. I can be calculated by doing the n-integration with the help of the same formulas that led to eq. (76). In contrast tp eq. (76) in eq. (86) no approximation can be made. Instead one introduces (absolutely convergent) hypergeometric series and gets

$$\mathcal{I}(k,l,m,n,a,b) = \frac{a+b-1-2\varepsilon}{\Gamma(a-\varepsilon)}\Gamma(1+\varepsilon-b)\Gamma(l+1-\varepsilon)\Gamma(n+1-\varepsilon)\,(\sigma_2-\sigma_1) \tag{87}$$

where

$$\sigma_2 = \sum_{j=0}^{\infty} \frac{\Gamma(j+a+b-1-2\varepsilon)\Gamma(j+k+b-3\varepsilon)\Gamma(j+m+b-2\varepsilon)}{j!\,\Gamma(j+k+l+b+1-4\varepsilon)\Gamma(j+m+n+b+1-3\varepsilon)} \tag{88}$$

and

$$\sigma_1 = \sum_{j=0}^{\infty} \frac{\Gamma(j+a-\varepsilon)\Gamma(j+k+1-2\varepsilon)\Gamma(j+m+1-\varepsilon)}{\Gamma(j+2+\varepsilon-b)\Gamma(j+k+l+2-3\varepsilon)\Gamma(j+m+n+2-2\varepsilon)} \tag{89}$$

$\sigma_2-\sigma_1$ converges for all values of k, l, m, n, a and b that appear and can be calculated by a suitable expansion in ε.

Now we come to the first integral in eq. (83). Call it H_1.

$$H_1 = \frac{\Gamma(2-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(3-2\varepsilon)}\, {}_2F_1(1,1-\varepsilon,3-2\varepsilon,1-zy_{123})(1-y_{123}(1-zy_{134}))^{-1} \tag{90}$$

Define H_l to be the limit of H_1 for $y_{123}\to 0$, so that in H_l eq. (76) can be used and in $H_k = H_1 - H_l$ one can put $(1-y_{123}(1-zy_{134}))^{-1} = (1-y_{123})^{-1}$. Then for H_k one needs $\mathcal{I}(1,-1,0,0,1,2)$ and for H_l

$$h_\ell := \int_0^1 dz\, z^{-2\epsilon}(1-z)^{-\epsilon}\, {}_2F_1(1,1-\epsilon,3-2\epsilon,1-z) \tag{91}$$

Both can be evaluated by standard techniques [14, 24, 25].

In section 4 it was very simple to figure out whether a term contributed O(y) in the region $y_{134} < y$. One simply had to put in the phase space variables z, u and y_{24} and count the power of y_{134} that emerged. Only double poles in y_{134} had to be taken into account (see eq. (71)). Here we have an integrand which cannot be neglected even though formally it is of order y_{134}^{-1}. The integrand is $y_{12}\,y_{123}\,y_{124}/(y_{13}+y_{24})(y_{123}+y_{14})(y_{13}+y_{23})(y_{14}+y_{24})$. The contribution arises for $y_{24} \to 0$ as can be seen by partial fractioning

$$\frac{1}{(y_{13}+y_{24})(y_{14}+y_{24})} = \frac{1}{y_{13}-y_{14}}\left(\frac{1}{y_{24}+y_{14}} - \frac{1}{y_{24}+y_{13}}\right) \tag{92}$$

Working in the system of eq. (80) one has

$$J_N = \int_0^y dy_{134}\, y_{134}^{-1-2\epsilon} \int_0^1 dy_{123}\, y_{123}^{-2\epsilon}(1-y_{123})^{-\epsilon} \int_0^1 dv\, v^{2-\epsilon}(1-v)^{-\epsilon}$$

$$\int_0^1 dz\, z^{-\epsilon}(1-z)^{-\epsilon}\,(z-v(1-z))^{-1}\,(z+v(1-z))^{-1}\,(1-v(1-z\,y_{134}))^{-1} \tag{93}$$

$$\left[\frac{1}{1-y_{123}+v(1-z)y_{134}} - \frac{1}{1-y_{123}+z\,y_{134}}\right]$$

In eq. (93) the limit $y_{24} \to 0$ has been taken whereever it is allowed.

The y_{123}-integration can be done with the help of eq. (76). (One should convince oneself that it is allowed to replace $1-x(1-z\,y_{134})$ by $1-x+z\,y_{134}$ in the denomonator of eq. (76)). One gets

$$J_N = \Gamma(1-\epsilon)\Gamma(\epsilon)\int_0^y dy_{134}\, y_{134}^{-1-2\epsilon}\int_0^1 dz\, z^{-\epsilon}(1-z)^{-\epsilon}\int_0^1 dv\, v^{2-\epsilon}(1-v)^{-\epsilon}$$

$$\left(v^{-\epsilon}(1-z)^{-\epsilon}-z^{-\epsilon}\right)\left(1-v(1-z\,y_{134})\right)^{-1}\left(z+v(1-z)\right)^{-1}\left(z-v(1-z)\right)^{-1} \tag{94}$$

Define J_N^ℓ to be the limit of J_N for $v \to 1$, so that in $J_N^h = J_N - J_N^\ell$ one can put $(1-v(1-z\,y_{134}))^{-1} = (1-v)^{-1}$.

$$J_N^\ell = \Gamma(1-\epsilon)\Gamma(\epsilon)\int_0^y dy_{134}\, y_{134}^{-1-2\epsilon}\int_0^1 dz\, z^{-\epsilon}(1-z)^{-\epsilon}\,(2z-1)^{-1}$$

$$\left((1-z)^{-\epsilon}-z^{-\epsilon}\right)\int_0^1 dv\, v^{-\epsilon}(1-v)^{-\epsilon}\left(1-v(1-z\,y_{134})\right)^{-1} \tag{95}$$

In J_N^ℓ the v-integration can be done with the help of eq. (76). One gets

$$J_N = \Gamma^2(1-\epsilon)\Gamma^2(\epsilon)\,\frac{y^{-4\epsilon}}{-4\epsilon}\,Z_0 + \Gamma(1-\epsilon)\Gamma(\epsilon)\,\frac{y^{-3\epsilon}}{-3\epsilon}\,Z_1 \tag{96}$$

where

$$Z_0 = \int_0^1 dz\, z^{-2\epsilon}(1-z)^{-\epsilon}\,\frac{(1-z)^{-\epsilon}-z^{-\epsilon}}{2z-1} \tag{97}$$

and

$$Z_1 = \int_0^1 dz\, z^{-\varepsilon}(1-z)^{-\varepsilon} \int_0^1 dv\, v^{2-\varepsilon}(1-v)^{-1-\varepsilon} \frac{v^{-\varepsilon}(1-z)^{-\varepsilon} - z^{-\varepsilon}}{\left(1-(1-z)(1+v)\right)\left(1-(1-z)(1-v)\right)} \tag{98}$$

Z_o may be calculated by expanding in ε. This is not possible for Z_1, because Z_1 has a singularity for $v \to 1$. By successive partial fractioning one can isolate this singularity into an elementary integral

$$\frac{1}{\left(1-z_1(1+v)\right)\left(1-z_1(1-v)\right)} = \frac{1}{2z\left(1-z_1(1+v)\right)} + \frac{1}{2z\left(1-z_1(1-v)\right)} \tag{99}$$

$$\frac{1}{(1-v)\left(1-z_1(1+v)\right)} = \frac{1}{(1-2z_1)(1-v)} - \frac{z}{(1-2z_1)\left(1-z_1(1+v)\right)} \tag{100}$$

with $z_1 = 1-z$.

The result is

$$2^{-4\varepsilon} Z_1 = -\frac{3}{2}\zeta_2 + \varepsilon\left(-\frac{21}{8}\zeta_3 - \frac{3}{2}\zeta_2\right) - \frac{183}{16}\zeta_4 - \frac{9}{8}\zeta_3 + \frac{3}{2}\zeta_2 - 6\eta \tag{101}$$

Here

$$\eta = L_4\left(\tfrac{1}{2}\right) + \frac{7}{8}\zeta_3 \ln 2 - \frac{1}{4}\zeta_2 \ln^2 2 + \frac{1}{24}\ln^4 2 \tag{102}$$

is the part of $L_4\left(\frac{1}{2}\right)$ proportional to ζ_4, which however is only known numerically [25]

$$\eta = 1.05857 \tag{103}$$

6. Summary

We have described the calculation of various integrals which appear in two-loop calculations. All integrals can be calculated by analytical means. Properties of the hypergeometric functions and of the gamma function and its derivatives are the essential ingredients of the calculations.

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Footnotes

1) Ultraviolet and infrared singularities can in principle be disentangled by examining the low energy and the high energy behaviour of the integrals. We do not distinguish them in the following, because the ultraviolet singularities are removed by counter terms which are known a priori.

2) If n is the dimension we define $\varepsilon = (4-n)/2$

3) The integrations are done in euclidean space. Continuation to Minkowski space is done in the result of the integration in such a way that the propagators are the usual causal propagators $(k^2+i\mu)^{-1}$.

Figure captions

Fig. 1 contribution to the vacuum polarization of the photon in $O(\alpha_s^2)$, the symbols in brackets always denote the momenta of the particles.

Fig. 2 two parton contributions to the 2-jet cross section in $O(\alpha_s^2)$.
always means a one-loop insertion.

Fig. 3 a typical box diagram

Fig. 4 3 particle phase space for $e^+e^- \to q\bar{q}g$. It is divided into a 2-jet region which contains the singularities and a 3-jet region free of singularities.

Fig. 5 a typical tree level diagram in $O(\alpha_s^2)$.

Appendix A: Standard virtual integrals

With the help of [8]

$$\int \frac{d^n k}{(2\pi)^n} \frac{(k^2)^r}{(k^2-C)^m} = \frac{i(-1)^{r-m}}{(16\pi^2)^{n/4}} C^{r-m+n/2} \tag{A.1}$$

$$\frac{\Gamma(r+\frac{n}{2})\Gamma(m-r-\frac{n}{2})}{\Gamma(n/2)\Gamma(m)}$$

one gets

$$\mathcal{J} := \int \frac{d^n k}{(k^2)^\alpha ((k-q)^2)^\beta} = i(-q^2)^{-\varepsilon} \tag{A.2}$$

$$\pi^{2-\varepsilon} (q^2)^{2-\alpha-\beta} \frac{\Gamma(2-\varepsilon-\beta)\Gamma(2-\varepsilon-\alpha)}{\Gamma(4-2\varepsilon-\alpha-\beta)}$$

$$\frac{\Gamma(\alpha+\beta+\varepsilon-2)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$\mathcal{J}_\mu := \int \frac{d^n k \; k_\mu}{(k^2)^\alpha ((k-q)^2)^\beta} = q_\mu \frac{2-\varepsilon-\alpha}{4-2\varepsilon-\alpha-\beta} \mathcal{J} \tag{A.3}$$

$$\mathcal{J}_{\mu\nu} := \int \frac{d^n k \; k_\mu k_\nu}{(k^2)^\alpha ((k-q)^2)^\beta} \tag{A.4}$$

$$= g_{\mu\nu} (q^2)^{3-\alpha-\beta} V_1(\alpha,\beta) + q_\mu q_\nu (q^2)^{2-\alpha-\beta} V_2(\alpha,\beta)$$

$$\mathcal{J}_{\mu\nu\rho} := \int \frac{d^n k \; k_\mu k_\nu k_\rho}{(k^2)^\alpha ((k-q)^2)^\beta} \tag{A.5}$$

$$= Z_{\mu\nu\rho}(q) (q^2)^{3-\alpha-\beta} W_1(\alpha,\beta) + q_\mu q_\nu q_\rho (q^2)^{2-\alpha-\beta}$$

$$W_2(\alpha,\beta)$$

$$\mathcal{J}_{\mu\nu\rho\sigma} := \int \frac{d^n k \; k_\mu k_\nu k_\rho k_\sigma}{(k^2)^\alpha ((k-q)^2)^\beta}$$

$$= X_{\mu\nu\rho\sigma}(q^2)^{4-\alpha-\beta} W_5(\alpha,\beta) + Y_{\mu\nu\rho\sigma}(q)(q^2)^{3-\alpha-\beta} \tag{A.6}$$

$$W_4(\alpha,\beta) + q_\mu q_\nu q_\rho q_\sigma (q^2)^{2-\alpha-\beta} W_3(\alpha,\beta)$$

$$V_1(\alpha,\beta) = i\pi^{2-\varepsilon}(-q^2)^{-\varepsilon} \frac{\Gamma(3-\alpha-\varepsilon)\Gamma(3-\beta-\varepsilon)}{2\Gamma(6-\alpha-\beta-2\varepsilon)} \tag{A.7}$$

$$\frac{\Gamma(\alpha+\beta+\varepsilon-3)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$V_2(\alpha,\beta) = i\pi^{2-\varepsilon}(-q^2)^{-\varepsilon} \frac{\Gamma(4-\alpha-\varepsilon)\Gamma(2-\beta-\varepsilon)}{\Gamma(6-\alpha-\beta-2\varepsilon)} \tag{A.8}$$

$$\Gamma(\alpha+\beta+\varepsilon-2)/\Gamma(\alpha)\Gamma(\beta)$$

$$W_1(\alpha,\beta) = \frac{3-\alpha-\varepsilon}{6-\alpha-\beta-2\varepsilon} V_1(\alpha,\beta) \tag{A.9}$$

$$W_2(\alpha,\beta) = \frac{4-\alpha-\varepsilon}{6-\alpha-\beta-2\varepsilon} V_2(\alpha,\beta) \tag{A.10}$$

$$W_3(\alpha,\beta) = \frac{5-\alpha-\varepsilon}{7-\alpha-\beta-2\varepsilon} W_2(\alpha,\beta) \tag{A.11}$$

$$W_4(\alpha,\beta) = \frac{4-\alpha-\varepsilon}{7-\alpha-\beta-2\varepsilon} W_1(\alpha,\beta) \tag{A.12}$$

$$W_5(\alpha,\beta) = i\pi^{2-\varepsilon}(-q^2)^{-\varepsilon} \frac{\Gamma(4-\alpha-\varepsilon)\Gamma(4-\beta-\varepsilon)}{4\Gamma(8-\alpha-\beta-2\varepsilon)} \tag{A.13}$$

$$\Gamma(\alpha+\beta+\varepsilon-4)/\Gamma(\alpha)\Gamma(\beta)$$

$$Z_{\mu\nu\rho}(q) = q_\mu g_{\nu\rho} + q_\nu g_{\mu\rho} + q_\rho g_{\mu\nu} \tag{A.14}$$

$$Y_{\mu\nu\rho\sigma}(q) = q_\mu q_\nu g_{\rho\sigma} + \text{permutations} \tag{A.15}$$

$$X_{\mu\nu\rho\sigma} = g_{\mu\nu} g_{\rho\sigma} + \text{permutations} \tag{A.16}$$

Appendix B: Phase space formulas

The phase space for j massless final state particles in n dimensions is

$$PS^{(j)} = (2\pi)^n \int \prod_{i=1}^{j} \frac{d^n p_i}{(2\pi)^{n-1}} \, \delta^+(p_i^2) \, \delta^n\left(q - \sum_{i=1}^{j} p_i\right) \tag{B.1}$$

For j = 3 and q^2-channel processes it can be fully expressed by the invariants y_{13}, y_{23}:

$$PS^{(3)} = \frac{q^2 \, (4\pi/q^2)^{2\varepsilon}}{128\pi^3 \, \Gamma(2-2\varepsilon)} \int_0^1 dy_{13} \, y_{13}^{-\varepsilon} \int_0^{1-y_{23}} dy_{23} \, y_{23}^{-\varepsilon} \, (1-y_{13}-y_{23})^{-\varepsilon} \tag{B.2}$$

For j = 4 two angle variables θ, θ' are needed. They are defined as follows. One chooses a system, where $\vec{p}_1 + \vec{p}_3 = 0$ and where $\vec{p}_2 \parallel \vec{e}_z$ [3]

$$p_1 = \sqrt{y_{13}}/2 \, (1, \ldots, \sin\theta \cos\theta', \cos\theta) \sqrt{q^2} \tag{B.3}$$

$$p_2 = (y_{123} - y_{13})/2\sqrt{y_{13}} \, (1, \ldots, 0, 1) \sqrt{q^2} \tag{B.4}$$

$$p_3 = \sqrt{y_{13}}/2 \, (1, \ldots, -\sin\theta \cos\theta', -\cos\theta) \sqrt{q^2} \tag{B.5}$$

$$p_4 = (y_{134} - y_{13})/2\sqrt{y_{13}} \, (1, \ldots, \sin\beta, \cos\beta) \sqrt{q^2} \tag{B.6}$$

Setting $v = \frac{1}{2}(1 - \cos\theta)$ one gets

$$PS^{(4)} = \frac{q^4 \, (4\pi/q^2)^{3\varepsilon}}{2048\,\pi^5 \, \Gamma(2-2\varepsilon)\Gamma(1-\varepsilon)} \int dy_{123} \, dy_{134} \, dy_{13} \, (y_{134} y_{123} - y_{13})^{-\varepsilon}$$

$$(y_{13} + 1 - y_{123} - y_{134})^{-\varepsilon} \, y_{13}^{-\varepsilon} \, \theta(y_{13}) \, \theta(y_{134} y_{123} - y_{13}) \tag{B.7}$$

$$\theta(y_{13} + 1 - y_{123} - y_{134}) \int_0^1 dv \, v^{-\varepsilon} (1-v)^{-\varepsilon} \int_0^{\pi} \frac{d\theta'}{N_{\theta'}} \sin^{-2\varepsilon}\theta'$$

$N_{\theta'}$ is defined after eq. (47).

For integrations over full of phase space a representation of $PS^{(4)}$ is useful, where all integrations are between 0 and 1:

$$PS^{(4)} = \frac{q^4 \, (4\pi/q^2)^{3\varepsilon}}{2048\pi^5 \, \Gamma(2-2\varepsilon)\Gamma(1-\varepsilon)} \int_0^1 dy_{134} \, y_{134}^{1-2\varepsilon} (1-y_{134})^{2-3\varepsilon} \int_0^1 ds \, s^{1-2\varepsilon} (1-s)^{-\varepsilon}$$

$$\int_0^1 dz \, z^{-\varepsilon} (1-z)^{-\varepsilon} (1 - z\,y_{134})^{-2+2\varepsilon} \int_0^1 dv \, v^{-\varepsilon} (1-v)^{-\varepsilon} \int_0^{\pi} \frac{d\theta'}{N_{\theta'}} \sin^{-2\varepsilon}\theta' \tag{B.8}$$

Here $z = y_{13}/y_{134} y_{123}$ and $s = y_{123}(1 - z\,y_{134})/(1-y_{134})$. The invariants y_{ij} may be expressed with the help of variables appearing in (B.8):

$$y_{12} = (1-y_{134}) \, s \, v \tag{B.9}$$

$$y_{23} = (1-y_{134}) \, s \, (1-v) \tag{B.10}$$

$$y_{14} = y_{134}(1-zy_{123})\left(v(1-\Upsilon) + \Upsilon(1-v) - 2\cos\theta'\sqrt{v(1-v)\Upsilon(1-\Upsilon)}\right) \tag{B.11}$$

$$y_{34} = y_{134}(1-zy_{123})\left((1-v)(1-\Upsilon) + v\Upsilon + 2\cos\theta'\sqrt{v(1-v)\Upsilon(1-\Upsilon)}\right) \tag{B.12}$$

$$y_{24} = y_{134}\, s \tag{B.13}$$

where $\Upsilon := zy_{24}/(1-zy_{123})(1-zy_{134})$.

In the main text we are concentrating on the region $y_{134} < y$. There the invariants may be approximated by

$$y_{12} = y_{123}\, v \tag{B.14}$$

$$y_{23} = y_{123}(1-v) \tag{B.15}$$

$$y_{24} = 1 - y_{123} \tag{B.16}$$

$$y_{14} = y_{134}\left(v(1-z) + zy_{24}(1-v) - 2\cos\theta'\sqrt{v(1-v)z(1-z)y_{24}}\right) \tag{B.17}$$

$$y_{34} = y_{134}\left((1-v)(1-z) + vzy_{24} + 2\cos\theta'\sqrt{v(1-v)z(1-z)y_{24}}\right) \tag{B.18}$$

The phase space in this limit is

$$PS^{(4)}_{y_{134}<y} = \int_0^y dy_{134}\, y_{134}^{1-2\epsilon} \int_0^1 dy_{24}\, y_{24}^{-\epsilon}(1-y_{24})^{1-2\epsilon} \int_0^1 dz\, z^{-\epsilon}(1-z)^{-\epsilon} \int_0^1 dv\, v^{-\epsilon}(1-v)^{-\epsilon} \int_0^\pi \frac{d\theta'}{N_{\theta'}} \sin^{-2\epsilon}\theta' \tag{B.19}$$

If one exchanges the role of particles 2 and 4 in (B.7) and evaluates the limit $y_{134} \to 0$ one gets back (B.19). However, v now has a different meaning and the structure of the invariants differs from (B.14) – (B.18) (apart from $2 \leftrightarrow 4$ interchange):

$$y_{14} = y_{134}(1-zy_{123})\, v \tag{B.20}$$

$$y_{34} = y_{134}(1-zy_{123})(1-v) \tag{B.21}$$

$$y_{24} = 1 - y_{123} \tag{B.22}$$

$$y_{12} = \frac{y_{123}}{1-zy_{123}}\left(v(1-z) + z(1-v)y_{24} - 2\cos\theta'\sqrt{v(1-v)z(1-z)y_{24}}\right) \tag{B.23}$$

$$y_{23} = \frac{y_{123}}{1-zy_{123}}\left((1-v)(1-z) + zvy_{24} + 2\cos\theta'\sqrt{v(1-z)z(1-v)y_{24}}\right) \tag{B.24}$$

Appendix C: Miscellaneous Series

In this appendix we have selected some series which have proven useful in two-loop calculations and which are not standard like, for instance, the hypergeometric series. Most of them are taken from ref. [14]. A few appear in ref. [24] in the form of integrals.

$$\sum_{k=0}^{\infty} \frac{\Gamma(k+c)}{k!\, q(k)} = \Gamma(c)\Gamma(1-c) \sum_{i=1}^{n} \frac{\Gamma(-a_i)}{\Gamma(1-c-a_i)\, q'(a_i)} \;, \quad c < 1 \tag{C.1}$$

where q(k) is a polynimial of degree n in k and q' its derivative. The roots a_i of q(k) must be simple.

$$\sum_{k=1}^{\infty} \frac{\Gamma(k+b)}{\Gamma(k+a)\, k} = \frac{\Gamma(b)}{\Gamma(a)} \left(\psi(a) - \psi(a-b)\right), \qquad a > b \tag{C.2}$$

ψ is the logarithmic derivative of the Γ-function.

$$\sum_{k=0}^{\infty} \frac{\Gamma(k+c)}{k!\,(ka+b)^2} = \frac{\Gamma(c)\Gamma(1-c)\Gamma(b/a)}{a^2\, \Gamma(b/a-c+1)} \left(\psi(b/a-c+1) - \psi(b/a)\right) \tag{C.3}$$

$$\sum_{k=0}^{\infty} \psi^{(n)}(k+1)\, x^k = \frac{(-1)^{n+1}}{n!} \; \frac{L_{n+1}(1) - L_{n+1}(x)}{1-x} \;, \quad -1 < x < 1 \tag{C.4}$$

$$n = 1, 2, \ldots$$

L_n are Eulers n-logarithms.

$$\sum_{k=1}^{\infty} x^k / k \left(\psi(k) - \psi(1)\right) = \ln^2(1-x)/2 \;, \quad -1 \leq x < 1 \tag{C.5}$$

$$\sum_{k=1}^{\infty} \psi''(k) / k = 2\zeta_3 \tag{C.6}$$

$$\sum_{k=1}^{\infty} \left(\psi(k) - \psi(1)\right) / k^2 = \zeta_3 \tag{C.7}$$

$$\sum_{k=1}^{\infty} \left(\psi(k+1) - \psi(1)\right)^2 / k^2 = 17\zeta_4/4 \tag{C.8}$$

$$\sum_{k=1}^{\infty} \left(\psi(k) - \psi(1)\right) / k^3 = \zeta_4/4 \tag{C.9}$$

$$\sum_{j=1}^{\infty} \frac{(-1)^j \psi(j)'}{j^2} = -\frac{85}{16}\zeta_4 + 4L_4\left(\tfrac{1}{2}\right) + \tfrac{7}{2}\zeta_3 \ln 2 \tag{C.10}$$

$$- \zeta_2 \ln^2 2 + \tfrac{1}{6} \ln^4 2 \tag{C.11}$$

$$\sum_{j=1}^{\infty} \frac{(-1)^j}{j^3} \left(\psi(j) - \psi(1)\right) = -\frac{15}{8}\zeta_4 + 2L_4\left(\tfrac{1}{2}\right) + \tfrac{7}{4}\zeta_3 \ln 2 \tag{C.12}$$

$$- \zeta_2 \ln^2 2/2 + \tfrac{1}{12} \ln^4 2$$

Sums with a denominator (k+a)(k+b) ... can be calculated by partial fractioning, e.g. $\sum_1 \psi'_k / k(k+1) = 1$. Further sums can be found in [25].

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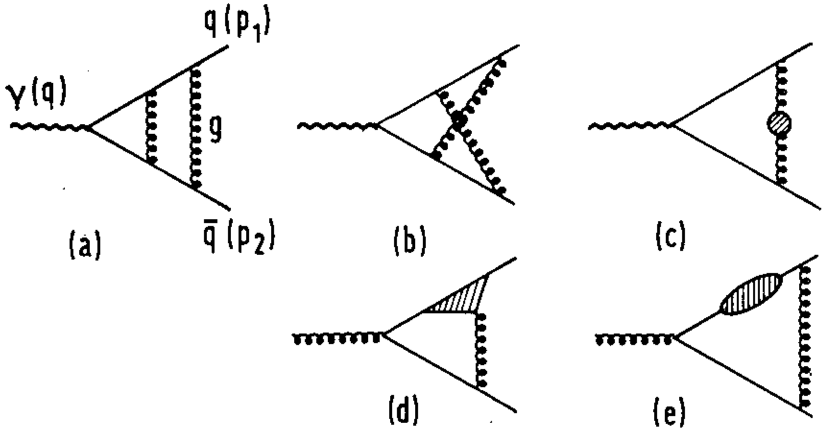
figure 1



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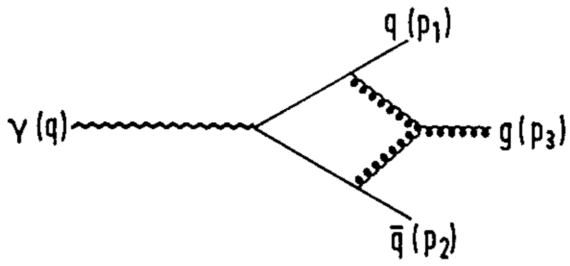

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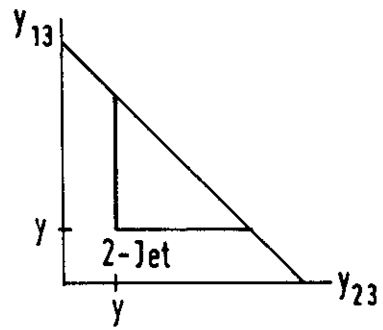

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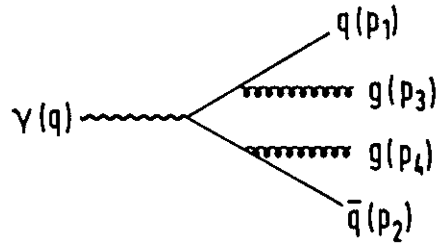

figure 5