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by

H. Mais, G. Ripken

Deutsches Elektronen-Synchrotron DESY, Hamburg

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Spin-Orbit Motion in a Storage Ring in the Presence of Synchrotron Radiation using a Dispersion Formalism

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Abstract

In this report we investigate the spin-orbit motion of particles in storage rings. Having summarized the fully six-dimensional description of the orbital motion with coordinates $(x, p_x, z, p_z, \sigma, \eta = \frac{\Delta E}{E_o})$ we introduce the dispersion. Since the dispersion function is introduced via a canonical transformation, the symplectic structure of the equations of motion (and thus of all the transfer matrices) in the absence of radiation effects is completely preserved. In this formulation the coupling between transverse and longitudinal motion only appears in the cavities.

As physical applications of this approach we calculate the damping constants, the beam emittance matrix and the depolarization time.

Contents

1. Introduction and statement of the problem

In the widely used polarization program SLIM[1] coupled linear spin-orbit motion in the presence of synchrotron radiation is treated using a six-dimensional formalism with the variables x, p_x, z, p_z, σ, $\eta = \delta E/E_0$. The aim of the present work is to develop a six dimensional formalism which explicitly involves dispersion and which as a result is simpler and is better suited for diagnostic purposes. Then, using the same mathematical methods as developed earlier[2] the damping constants, beam emittance matrix and depolarization time τ_D are calculated.

This formalism can then serve as a basis for rewriting the numerical code SLIM using the dispersion approximation.

2. The equations of motion

We begin the investigation of spin-orbit motion in a storage ring with the statement of the equations of motion whereby in our quasi linear framework motion in sextupoles can also be included.

2.1 Orbital motion

The central equation for the orbital motion is written in the form[2]

$$\vec{y}\,' = (\underline{A} + \delta\underline{A})\vec{y} + \vec{c}_{sex} + \vec{c}_0 + \vec{c}_1 + \delta\vec{c} \quad . \qquad (2.1)$$

$\vec{y}$ is the six-dimensional orbit vector describing the transverse and longitudinal particle motion. In coordinates $\vec{y}$, is given by:

$$\vec{y}^T = (x,\ p_x,\ z,\ p_z,\ \sigma,\ \eta) \qquad (2.2)$$

$$p_x = x' - Hz$$
$$p_z = z' + Hx$$

The matrix $\underline{A}$

$$\vec{A} = \begin{pmatrix} \underline{B} & & & & \vec{0}_4 & \vec{K} \\ -K_x & 0 & -K_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{e\hat{V}}{E_0}\cdot k\cdot\frac{2\pi}{L}\cos\Phi\cdot\sum_\mu \delta(s-s_\mu) & 0 \end{pmatrix} \qquad (2.3)$$

with

$$\underline{B} = \begin{pmatrix} 0 & 1 & H & 0 \\ -(K_x^2+g+H^2) & 0 & N & H \\ -H & 0 & 0 & 1 \\ N & -H & -(K_z^2-g+H^2) & 0 \end{pmatrix} \tag{2.3a}$$

$$\vec{K}^T = (0,\ K_x,\ 0,\ K_z) \quad ; \tag{2.3b}$$

$$g = \frac{e}{E_o}\left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \quad ; \tag{2.4a}$$

$$N = \frac{1}{2}\frac{e}{E_o}\left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z}\right)_{x=z=0} \quad ; \tag{2.4b}$$

$$H = \frac{1}{2}\frac{e}{E_o} B_\tau^{(o)} \tag{2.4c}$$

and the matrix $\delta\underline{A}$

$$\delta\underline{A} = ((\delta A_{mn})) \quad ;$$

$$\begin{cases} \delta A_{22} = -\dfrac{e\hat{V}}{E_o}\sin\Phi \cdot \sum\limits_{\mu} \delta(s-s_\mu) \quad ; \\ \delta A_{44} = \delta A_{22} \quad ; \\ \delta A_{61} = -C_1 \cdot [(K_x^2+K_z^2)\cdot K_x + 2K_x \cdot g] \quad ; \\ \delta A_{63} = -C_1 \cdot [(K_x^2+K_z^2)\cdot K_z - 2K_z \cdot g] \quad ; \\ \delta A_{66} = -2C_1 \cdot (K_x^2+K_z^2) \quad ; \\ \delta A_{mn} = 0 \qquad \text{otherwise} \end{cases} \tag{2.5}$$

describe the influence of the various beam line elements.

The vector $\vec{c}_{sex}$

$$\vec{c}^{\,T}_{sex} = \frac{1}{2}\lambda(s) \quad (0,\ z^2 - x^2,\ 0,\ 2xz,\ 0,\ 0) \quad ; \tag{2.6}$$

$$\lambda(s) = \frac{e}{E_0}\cdot\left(\frac{\partial^2 B_z}{\partial x^2}\right)_{x=z=0}$$

describes the influence of sextupoles and $\vec{c}_0$

$$\vec{c}^{\,T}_0 = (0,0,0,0,0,\ \frac{e\hat{V}}{E_0}\sin\Phi\cdot\sum_{\mu}\delta(s - s_\mu) - C_1\cdot(K_x^2 + K_z^2)) \tag{2.7}$$

describes the energy variation due to photon emission in the bending magnets and due to particle acceleration in the cavities, where

$$C_1 = \frac{2}{3}e^2\frac{\gamma_0^4}{E_0} \tag{2.7a}$$

The vector $\vec{c}_1$

$$\vec{c}^{\,T}_1 = (0,\ -\Delta B_z,\ 0,\ \Delta B_x,\ 0,\ 0) \tag{2.8}$$

describes the influence of field errors caused by misalignments of the magnets. The quantity $\delta\vec{c}$

$$\delta\vec{c}^{\,T} = (0,\ 0,\ 0,\ 0,\ 0,\ \delta c) \tag{2.9}$$

describes the influence of the quantum fluctuations with

$$<\delta c(s)\cdot\delta c(s')> = \omega(s)\cdot\delta(s - s') \quad ; \tag{2.9a}$$

$$<\delta c(s)> = 0 \quad ; \tag{2.9b}$$

$$\omega(s) = |K_{x,z}(s)|^3\cdot C_2 \quad ; \tag{2.10}$$

$$C_2 = \frac{55\cdot\sqrt{3}}{48}\cdot C_1\cdot\Delta\cdot\gamma_0^2 \quad ; \tag{2.10a}$$

$$\Delta = \frac{\hbar}{m_0 c} \quad . \tag{2.10b}$$

2.2 Spin motion

The equation of motion for the spin is given by [2]

$$\frac{d}{ds}\vec{\xi} = \underline{\Omega}\cdot\vec{\xi} \tag{2.11}$$

with

$$\vec{\xi} = \begin{pmatrix} \xi_\tau \\ \xi_x \\ \xi_z \end{pmatrix} \quad ; \tag{2.11a}$$

$$\underline{\Omega} = \begin{pmatrix} 0 & -\Omega_z & \Omega_x \\ \Omega_z & 0 & -\Omega_\tau \\ -\Omega_x & \Omega_\tau & 0 \end{pmatrix} \quad ; \quad \vec{\Omega} = \begin{pmatrix} \Omega_\tau \\ \Omega_x \\ \Omega_z \end{pmatrix} \quad ; \tag{2.11b}$$

$$\begin{aligned}
\vec{\Omega} = -\,(1 + K_x\cdot x + K_z\cdot z)\ \frac{e}{E_0(1+\eta)}\cdot\{[1 + a\gamma_0(1+\eta)]\cdot\vec{B}\ - \\
- \frac{a\gamma_0^2}{1+\gamma_0}\ (1+2\eta)\cdot[1 - \frac{\gamma_0}{1+\gamma_0}\cdot\eta]\cdot[\vec{e}_x\cdot x'B_\tau + \vec{e}_z\cdot z'B_\tau + \\
+ \vec{\tau}\cdot(B_\tau + x'\cdot B_x + z'\cdot B_z)]\ + \\
+ \gamma_0(1+\eta)\cdot[a + \frac{1}{1+\gamma_0}\cdot(1 - \frac{\gamma_0}{1+\gamma_0}\cdot\eta)\cdot\varepsilon_\tau[x'\cdot\vec{e}_z - z'\cdot\vec{e}_x]\}\ - \\
- K_z\cdot\vec{e}_x + K_x\cdot\vec{e}_z \quad ;
\end{aligned} \tag{2.11c}$$

(ε_τ = cavity field) .

The components Ω_τ, Ω_x, Ω_z of $\underline{\Omega}$ given in (2.11b) can then be extracted from (2.11c). For each of the lens types one finds:

1) For a sextupole

$$\left\{ \begin{aligned} \frac{e}{E_0} B_x &= \lambda(s)\cdot xz \quad ; \\ \frac{e}{E_0} B_z &= \frac{1}{2}\lambda(s)\cdot(x^2 - z^2) \quad ; \\ K_x &= K_z = 0 \quad ; \end{aligned} \right.$$

$$\begin{cases} \Omega_\tau = 0 \quad ; \\ \Omega_x = -\lambda(s) \cdot (1 + a\gamma_0) \cdot xz \quad ; \\ \Omega_z = -\frac{1}{2}\lambda(s) \cdot (1 + a\gamma_0) \cdot (x^2 - z^2) \quad ; \end{cases} \tag{2.12a}$$

2) For a quadrupole

$$\begin{cases} \frac{e}{E_0} B_x = g \cdot z \quad ; \\ \frac{e}{E_0} B_z = g \cdot x \quad ; \\ K_x = K_z = 0 \quad ; \end{cases}$$

$$\begin{cases} \Omega_\tau = 0 \quad ; \\ \Omega_x = -g(s) \cdot (1 + a\gamma_0) \cdot z \quad ; \\ \Omega_z = -g(s) \cdot (1 + a\gamma_0) \cdot x \quad ; \end{cases} \tag{2.12b}$$

3) For a skew quadrupole

$$\begin{cases} \frac{e}{E_0} B_x = N \cdot x \quad ; \\ \frac{e}{E_0} B_z = -N \cdot z \quad ; \\ K_x = K_z = 0 \quad ; \end{cases}$$

$$\begin{cases} \Omega_\tau = 0 \quad ; \\ \Omega_x = -N(s) \cdot (1 + a\gamma_0) \cdot x \quad ; \\ \Omega_z = +N(s) \cdot (1 + a\gamma_0) \cdot z \quad ; \end{cases} \tag{2.12c}$$

4) For a bending magnet

a) $K_x \neq 0 \quad ; \quad K_z = 0 \quad ;$

$$\begin{cases} \frac{e}{E_0} B_x = 0 \quad ; \\ \frac{e}{E_0} B_z = K_z \quad ; \end{cases}$$

$$\begin{cases} \Omega_\tau = +\dfrac{a\gamma_0^2}{1+\gamma_0} \cdot z' \cdot K_x \quad ; \\ \Omega_x = 0 \quad ; \\ \Omega_z = -K_x \cdot a\gamma_0 - (1 + a\gamma_0) \cdot K_x^2 \cdot x + K_x \cdot \eta \quad ; \end{cases} \tag{2.12d}$$

b) $K_x = 0$; $K_z \neq 0$;

$$\begin{cases} \dfrac{e}{E_0} B_x = -K_z \quad ; \\ \dfrac{e}{E_0} B_z = 0 \quad ; \end{cases}$$

$$\begin{cases} \Omega_\tau = -\dfrac{a\gamma_0^2}{1+\gamma_0} \cdot x' \cdot K_z \quad ; \\ \Omega_x = +K_z \cdot a\gamma_0 + (1 + a\gamma_0) \cdot K_z^2 \cdot z - K_z \cdot \eta \quad ; \\ \Omega_z = 0 \quad ; \end{cases} \tag{2.12e}$$

5) For a solenoid

$$\begin{cases} \dfrac{e}{E_0} B_\tau = 2 \cdot H \quad ; \\ \dfrac{e}{E_0} B_x = -H' \cdot x \quad ; \\ \dfrac{e}{E_0} B_z = -H' \cdot z \quad ; \end{cases}$$

$$K_x = K_z = 0 \quad ;$$

$$\begin{cases} \Omega_\tau = -2H \cdot \left[1 + a\gamma_0 \cdot \dfrac{1}{1+\gamma_0}\right] + \eta \cdot 2H \cdot \left[1 + \dfrac{a\gamma_0^2}{(1+\gamma_0)^2}\right] \quad ; \\ \Omega_x = (1 + a\gamma_0) \cdot H' \cdot x + a\gamma_0 \cdot \dfrac{\gamma_0}{1+\gamma_0} \cdot 2H \cdot x' \quad ; \\ \Omega_z = (1 + a\gamma_0) \cdot H' \cdot z + a\gamma_0 \cdot \dfrac{\gamma_0}{1+\gamma_0} \cdot 2H \cdot z' \quad ; \end{cases} \tag{2.12f}$$

6) For a cavity

$$\frac{e}{E_0} \varepsilon(s) = \frac{e\hat{V}}{E_0} \{\sin\Phi + \sigma(s) \cdot k \cdot \frac{2\pi}{L} \cos\Phi\} \cdot \sum_\mu \delta(s - s_\mu) \quad ;$$

$$K_x = K_z = 0 \quad ;$$

$$\begin{cases} \Omega_\tau = 0 \quad ; \\ \Omega_x = (a\gamma_0 + \dfrac{\gamma_0}{1+\gamma_0}) \cdot z' \cdot \dfrac{e\hat{V}}{E_0} \sin\Phi \cdot \sum\limits_\mu \delta(s - s_\mu) \quad ; \\ \Omega_z = -(a\gamma_0 + \dfrac{\gamma_0}{1+\gamma_0}) \cdot x' \cdot \dfrac{e\hat{V}}{E_0} \sin\Phi \cdot \sum\limits_\mu \delta(s - s_\mu) \quad ; \end{cases} \qquad (2.12g)$$

7) For a dipole kicker magnet

a) x-direction

$$\vec{B} = \begin{pmatrix} 0 \\ \Delta B_x \\ 0 \end{pmatrix} \quad ; \quad \Delta B_x = \Delta\hat{B}_x \cdot \delta(s - s_0) \quad ;$$

$$K_x = K_z = 0 \; ;$$

$$\begin{cases} \Omega_\tau = 0 \quad ; \\ \Omega_x = -(1 + a\gamma_0) \cdot \dfrac{e}{E_0} \Delta\hat{B}_x \cdot \delta(s - s_0) \quad ; \\ \Omega_z = 0 \quad ; \end{cases} \qquad (2.12k)$$

b) z-direction

$$\vec{B} = \begin{pmatrix} 0 \\ 0 \\ \Delta B_z \end{pmatrix} \quad ; \quad \Delta B_z = \Delta\hat{B}_z \cdot \delta(s - s_0) \quad ;$$

$$K_x = K_z = 0 \quad ;$$

$$\begin{cases} \Omega_\tau = 0 \quad ; \\ \Omega_x = 0 \quad ; \\ \Omega_z = -(1 + a\gamma_0) \cdot \dfrac{e}{E_0} \Delta\hat{B}_z \cdot \delta(s - s_0) \quad . \end{cases} \qquad (2.12i)$$

3. Introduction of a new reference trajectory (closed orbit)

Linear spin-orbit motion in a storage ring can be completely described by eqs. (2.1) and (2.11) but the equations must be solved in several steps. First of all it is necessary to eliminate the inhomogeneous terms $\vec{c}_0$ and $\vec{c}_1$ in (2.1). This is achieved in the usual way by finding the (only) periodic solution $\vec{y}_0$ of eq. (2.1). Without $\delta\vec{c}$ we have:

$$\begin{aligned} \vec{y}_0' &= (\underline{A} + \delta\underline{A})\vec{y}_0 + \vec{c}_0 + \vec{c}_1 + \vec{c}_{sex} \quad ; \\ \vec{y}_0(s_0 + L) &= \vec{y}_0(s_0) \quad . \end{aligned} \tag{3.1}$$

and the general solution of (2.1) can be written:

$$\vec{y} = \vec{y}_0 + \vec{\tilde{y}} \tag{3.2}$$

Putting (3.2) into (2.1) and taking into account (3.1) one obtains in linear approximation for $\vec{\tilde{y}}$:

$$\frac{d}{ds}\vec{\tilde{y}} = (\underline{\tilde{A}} + \delta\underline{A}) \cdot \vec{\tilde{y}} + \delta\vec{c} \tag{3.3}$$

with

$$\underline{\tilde{A}} = \begin{pmatrix} \underline{\tilde{B}} & & & & \vec{0}_4 & \vec{K} \\ -K_x & 0 & -K_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{e\hat{V}}{E_0} k \cdot \frac{2\pi}{L} \cos\Phi \cdot \sum\limits_{\mu} \delta(s-s_\mu) & 0 \end{pmatrix} ; \tag{3.4}$$

$$\underline{\tilde{B}} = \underline{B} + \underline{B}_{sex} \quad ; \tag{3.5a}$$

$$\underline{B}_{sex} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -x_0 & 0 & z_0 & 0 \\ 0 & 0 & 0 & 0 \\ z_0 & 0 & x_0 & 0 \end{pmatrix} \cdot \lambda(s) \quad . \tag{3.5b}$$

The vector $\vec{\tilde{y}}$ describes the synchro-betatron oscillations around the new equilibrium orbit $\vec{y}_0$ (the "closed orbit").

By subdividing $\vec{\tilde{y}}$ into transverse and longitudinal pieces in the form

$$\vec{\tilde{y}} = \begin{pmatrix} \vec{\tilde{y}}^{\perp} \\ \tilde{y}_5 \\ \tilde{y}_6 \end{pmatrix} \tag{3.6}$$

one obtains from (3.3) and (3.4)

$$\frac{d}{ds}\,\vec{\tilde{y}}^{\perp} = (\underline{\tilde{B}} + \delta\underline{B}) \cdot \vec{\tilde{y}}^{\perp} + \tilde{y}_6 \cdot \vec{K} \quad ; \tag{3.7a}$$

$$\frac{d}{ds}\,\tilde{y}_5 = - K_x \cdot \tilde{y}_1 - K_z \cdot \tilde{y}_3 \quad ; \tag{3.7b}$$

$$\frac{d}{ds}\,\tilde{y}_6 = \frac{e\hat{V}}{E_o} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\Phi \cdot \sum_{\mu} \delta(s-s_\mu) \cdot \tilde{y}_5 + \\ + \sum_{\mu=1}^{4} \delta A_{6\mu} \cdot \tilde{y}_\mu + \delta A_{66} \cdot \tilde{y}_6 + \delta c \tag{3.7c}$$

with

$$\delta B_{ik} = \delta A_{ik} \quad (i,k = 1,2,3,4) \tag{3.8}$$

4. Introduction of dispersion

For our further investigation of eq. (3.1) we introduce the dispersion $\vec{D}$:

$$\begin{cases} \frac{d}{ds}\vec{D} = (\underline{\tilde{B}} + \delta\underline{B}) \cdot \vec{D} + \vec{K} \ ; \\ \vec{D}(s_0 + L) = \vec{D}(s_0) \end{cases} \tag{4.1}$$

with

$$\vec{D}^T = (D_1, D_2, D_3, D_4)$$

If we ignore the (small) radiation damping terms, $\delta\underline{B}$, we can write

$$\frac{d}{ds}\vec{D} = \underline{\tilde{B}} \cdot \vec{D} + \vec{K} \ . \tag{4.1a}$$

Using the ansatz

$$\vec{\tilde{y}}^{\perp} = \vec{\overline{y}}^{\perp} + \overline{y}_6 \cdot \vec{D} \ ; \tag{4.2a}$$

$$\overline{y}_6 = \tilde{y}_6 \tag{4.2b}$$

eq. (3.7c) takes the form

$$\frac{d}{ds}\overline{y}_6 = \frac{e\hat{V}}{E_0} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\Phi \cdot \sum_{\mu} \delta(s-s_\mu) \cdot \tilde{y}_5 + \sum_{\mu=1}^{4} \delta A_{6\mu} \cdot \overline{y}_\mu + \\ + [\delta A_{66} + \sum_{\mu=1}^{4} \delta A_{6\mu} \cdot D_\mu] \cdot \overline{y}_6 + \delta c \ . \tag{4.3}$$

Taking into account (4.1) and (4.3) eq. (3.7a) can be written as

$$\frac{d}{ds}\vec{\overline{y}}^{\perp} = (\underline{\tilde{B}} + \delta\underline{B}) \cdot \vec{\overline{y}}^{\perp} - \overline{y}_6' \cdot \vec{D} \\ = (\underline{\tilde{B}} + \delta\underline{B}) \cdot \vec{\overline{y}}^{\perp} - \vec{D} \cdot \frac{e\hat{V}}{E_0} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\Phi \cdot \sum_{\mu} \delta(s-s_\mu) \cdot \tilde{y}_5 - \\ - \vec{D} \cdot \sum_{\mu=1}^{4} \delta A_{6\mu} \cdot \overline{y}_\mu - \vec{D} \cdot [\delta A_{66} + \sum_{\mu=1}^{4} \delta A_{6\mu} \cdot D_\mu] \cdot \overline{y}_6 - \vec{D} \cdot \delta c \tag{4.4}$$

with

$$(\vec{\overline{y}}^{\perp})^T = (\overline{y}_1, \overline{y}_2, \overline{y}_3, \overline{y}_4) \ .$$

Furthermore we put

$$\bar{y}_5 = \tilde{y}_5 - \tilde{y}_2 \cdot D_1 - \tilde{y}_4 \cdot D_3 + \tilde{y}_1 \cdot D_2 + \tilde{y}_3 \cdot D_4$$
$$= \tilde{y}_5 - \bar{y}_2 \cdot D_1 - \bar{y}_4 \cdot D_3 + \bar{y}_1 \cdot D_2 + \bar{y}_3 \cdot D_4 \qquad (4.5)$$

and using the relations

$$\tilde{y}_1' = \tilde{y}_2 + H \cdot \tilde{y}_3 \quad ; \quad D_1' = D_2 + HD_3$$
$$\tilde{y}_3' = \tilde{y}_4 - H \cdot \tilde{y}_1 \quad ; \quad D_3' = D_4 - HD_1$$

one obtains:

$$\begin{aligned}
\bar{y}_5' = & - (K_X \cdot \tilde{y}_1 + K_Z \cdot \tilde{y}_3) \\
& - \{ \sum_{\mu=1}^{4} (\tilde{B}_{2\mu} + \delta A_{2\mu}) \cdot \tilde{y}_\mu + \tilde{y}_6 \cdot K_X \} \cdot D_1 \\
& - \{ \sum_{\mu=1}^{4} (\tilde{B}_{4\mu} + \delta A_{4\mu}) \cdot \tilde{y}_\mu + \tilde{y}_6 \cdot K_Z \} \cdot D_3 \\
& + (\tilde{y}_2 + H \cdot \tilde{y}_3) \cdot D_2 + (\tilde{y}_4 - H \cdot \tilde{y}_1) \cdot D_4 \\
& - \tilde{y}_2 \cdot (D_2 + H \cdot D_3) - \tilde{y}_4 \cdot (D_4 - H \cdot D_1) \\
& + \tilde{y}_1 \{ \sum_{\mu=1}^{4} (\tilde{B}_{2\mu} + \delta A_{2\mu}) \cdot D_\mu + K_X \} \\
& + \tilde{y}_3 \{ \sum_{\mu=1}^{4} (\tilde{B}_{4\mu} + \delta A_{4\mu}) \cdot D_\mu + K_Z \} \\
= & - \bar{y}_6 \cdot (K_X \cdot D_1 + K_Z \cdot D_3) \\
& + \delta A_{22} \cdot (\bar{y}_1 \cdot D_2 - \bar{y}_2 \cdot D_1) + \delta A_{44} \cdot (\bar{y}_3 \cdot D_4 - \bar{y}_4 \cdot D_3) \ . \qquad (4.6)
\end{aligned}$$

The eqs. (4.3), (4.4) and (4.6) can be put in matrix form

$$\frac{d}{ds}\vec{\bar{y}} = (\underline{\bar{A}} + \delta\underline{\bar{A}}) \cdot \vec{\bar{y}} + \delta c \cdot \begin{pmatrix} -\vec{D} \\ 0 \\ 1 \end{pmatrix} \tag{4.7}$$

with

$$\vec{\bar{y}} = \begin{pmatrix} \vec{\bar{y}}^{1} \\ \bar{y}_5 \\ \bar{y}_6 \end{pmatrix} \tag{4.8}$$

and

$$\underline{\bar{A}} = \underline{A}_o + \underline{A}_c \quad ; \tag{4.9}$$

$$\underline{A}_o = \begin{pmatrix} \underline{\tilde{B}} & & & & \vec{0}_4 & \vec{0}_4 \\ 0 & 0 & 0 & 0 & 0 & -[K_x \cdot D_1 + K_z \cdot D_3] \\ 0 & 0 & 0 & 0 & \frac{e\hat{V}}{E_o} k \cdot \frac{2\pi}{L} \cdot \cos\Phi \cdot \sum_\mu \delta(s-s_\mu) & 0 \end{pmatrix} \tag{4.9a}$$

$$\underline{A}_c = \frac{e\hat{V}}{E_o} k \cdot \frac{2\pi}{L} \cdot \cos\Phi \cdot \sum_\mu \delta(s-s_\mu) \begin{pmatrix} \vec{D} \cdot \vec{D}^T \cdot \underline{S} & -\vec{D} & \vec{0}_4 \\ \vec{0}_4^T & 0 & 0 \\ -\vec{D}^T \cdot \underline{S} & 0 & 0 \end{pmatrix} \tag{4.9b}$$

$$\underline{S} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad ; \tag{4.10}$$

as well as

$$\delta\underline{\bar{A}} = ((\delta\bar{A}_{nm})) \quad ;$$

$$\delta\bar{A}_{mn} = \delta A_{mn} - D_m \cdot \delta A_{6n} \qquad \text{für } m,n = 1, 2, 3, 4 \quad ;$$

$$\delta\bar{A}_{m6} = -D_m \cdot \{\delta A_{66} + \sum_{\mu=1}^{4} \delta A_{6\mu} \cdot D_\mu\} \qquad \text{für } m = 1, 2, 3, 4 \quad ;$$

$$\delta\bar{A}_{51} = D_2 \cdot \delta A_{22}$$

$$\delta\bar{A}_{52} = -D_1 \cdot \delta A_{22}$$

$$\delta\bar{A}_{53} = D_4 \cdot \delta A_{44}$$

$$\delta\bar{A}_{54} = -D_3 \cdot \delta A_{44}$$

$$\delta\bar{A}_{66} = \delta A_{66} + \sum_{\mu=1}^{4} \delta A_{6\mu} \cdot D_\mu \quad ;$$

$$\delta\bar{A}_{mn} = \delta A_{mn} \qquad \text{otherwise} . \tag{4.11}$$

The matrix $\underline{A}_C$ describes the coupling between synchrotron and betatron oscillations. From eq. (4.9b) it is obvious that the coupling terms disappear if the dispersion vector is zero in the cavity regions[3]:

$$V(s) \cdot \vec{D} = 0 \tag{4.12}$$

with $\quad V(s) = \hat{V} \cdot \sum_{\mu} \delta(s - s_\mu)$.

It is worthwhile noting that eq. (4.7) without the perturbing terms $\delta\underline{\bar{A}}$ and $\delta\vec{\bar{c}}$

$$\frac{d}{ds}\vec{\bar{y}} = \underline{\bar{A}} \cdot \vec{\bar{y}} \tag{4.13}$$

can be written in canonical form with the Hamiltonian

$$\begin{aligned} K = {} & \frac{1}{2}(\bar{p}_x + H \cdot \bar{z})^2 + \frac{1}{2}(\bar{p}_z - H \cdot \bar{x})^2 + \frac{1}{2} G_1 \cdot \bar{x}^2 + \frac{1}{2} G_2 \cdot \bar{z}^2 - \\ & - [N + \lambda \cdot z_0] \cdot \bar{x}\bar{z} - \frac{1}{2}[K_x \cdot D_1 + K_z \cdot D_3] \cdot \bar{\eta}^2 - \\ & - \frac{1}{2} \frac{e\hat{V}}{E_0} k \cdot \frac{2\pi}{L} \cdot \cos\Phi \cdot \sum_{\mu} \delta(s - s_\mu) \quad \times \\ & \times \quad [\bar{\sigma} + D_1 \cdot \bar{p}_x + D_3 \cdot \bar{p}_z - D_2 \cdot \bar{x} - D_4 \cdot \bar{z}]^2 \end{aligned} \tag{4.14}$$

with

$$\begin{cases} G_1 = K_x^2 + g + H^2 + \lambda \cdot x_0 \quad ; \\ G_2 = K_z^2 - g + H^2 - \lambda \cdot x_0 \quad ; \end{cases} \tag{4.15}$$

$$\vec{\bar{y}}^T = (\bar{x}, \bar{p}_x, \bar{z}, \bar{p}_z, \bar{\sigma}, \bar{\eta}) \quad . \tag{4.16}$$

This can be checked easily by putting (4.14) into the canonical equations of motion

$$\begin{cases} \bar{x}' = \dfrac{\partial K}{\partial \bar{p}_x} \quad ; \quad \bar{p}_x' = -\dfrac{\partial K}{\partial \bar{x}} \quad ; \\ \bar{z}' = \dfrac{\partial K}{\partial \bar{p}_z} \quad ; \quad \bar{p}_z' = -\dfrac{\partial K}{\partial \bar{z}} \quad ; \\ \bar{\sigma}' = \dfrac{\partial K}{\partial \bar{\eta}} \quad ; \quad \bar{\eta}' = -\dfrac{\partial K}{\partial \bar{\sigma}} \end{cases} \tag{4.17}$$

Remark:

By neglecting the perturbing terms $\delta\underline{A}$ and $\delta\vec{c}$ the starting equation (3.6) for the orbital vector $\vec{\tilde{y}}$ can also be derived from a Hamiltonian

$$\mathcal{H} = \frac{1}{2}(\tilde{p}_x + H\cdot\tilde{z})^2 + \frac{1}{2}(\tilde{p}_z - H\cdot\tilde{x})^2 + \frac{1}{2}G_1\cdot\tilde{x}^2 + \frac{1}{2}G_2\cdot\tilde{z}^2 -$$

$$- [N + \lambda\cdot z_0]\cdot\tilde{x}\tilde{z} - (K_x\cdot\tilde{x} + K_z\cdot\tilde{z})\cdot\tilde{\eta} -$$

$$- \frac{1}{2}\tilde{\sigma}^2\cdot\frac{e\hat{V}}{E_0}k\cdot\frac{2\pi}{L}\cdot\cos\Phi\cdot\sum_{\mu}\delta(s - s_\mu) \quad . \tag{4.18}$$

The Hamiltonian K given in (4.14) can be obtained from $\mathcal{H}$ by applying to (4.18) a canonical transformation of the form[4,5]

$$F_2(\tilde{x}, \tilde{z}, \tilde{\sigma}, \bar{p}_x, \bar{p}_z, \bar{\eta}, s) = \bar{p}_x\cdot(\tilde{x} - \bar{\eta}\cdot D_1) + \bar{\eta}\cdot D_2\cdot\tilde{x} +$$

$$+ \bar{p}_z\cdot(\tilde{z} - \bar{\eta}\cdot D_3) + \bar{\eta}\cdot D_4\cdot\tilde{z} -$$

$$- \frac{1}{2}[D_1D_2 + D_3D_4]\cdot\bar{\eta}^2 + \bar{\eta}\cdot\tilde{\sigma} \tag{4.19}$$

The corresponding transformation equations describe the transition from $\vec{\tilde{y}}$ to $\vec{\bar{y}}$ and they agree with eqs. (4.2a), (4.2b) and (4.5).

5. Description of the spin motion

5.1 Perturbation theory

Using eqs. (3.2), 4.2) and (4.5) the orbit vector $\vec{y}$ has the form

$$\vec{y} = \vec{y}_0 + \vec{\tilde{y}} \tag{5.1a}$$

with

$$\begin{cases} \vec{\tilde{y}}^{\perp} = \vec{\bar{y}}^{\perp} + \bar{y}_6 \cdot \vec{D} \quad ; \\ \tilde{y}_5 = \bar{y}_5 + \bar{y}_2 \cdot D_1 + \bar{y}_4 \cdot D_3 - \bar{y}_1 \cdot D_2 - \bar{y}_3 \cdot D_4 \quad ; \\ \tilde{y}_6 = \bar{y}_6 \ . \end{cases} \tag{5.1b}$$

In order to utilize the equation of motion (2.8) for the spin it is now necessary to divide the "spin matrix" $\underline{\Omega}$ into two components and to do this in a way corresponding to the division of the vector $\vec{y}$ in eq. (5.1)

$$\underline{\Omega} = \underline{\Omega}(\vec{y}) = \underline{\Omega}^{(o)} + \underline{\omega} \tag{5.2}$$

with

$$\underline{\Omega}^{(o)} = \underline{\Omega}(\vec{y}_0) \ . \tag{5.3}$$

For the matrix $\underline{\omega}$

$$\underline{\omega} = \underline{\Omega}(\vec{y}) - \underline{\Omega}^{(o)} \equiv \begin{pmatrix} 0 & -\omega_Z & \omega_X \\ \omega_Z & 0 & -\omega_T \\ -\omega_X & \omega_T & 0 \end{pmatrix} \tag{5.4}$$

one obtains using (2.9) and (5.1b):

1) Sextupole:

$$\begin{cases} \omega_T = 0 \quad ; \\ \omega_X = -\lambda(s) \cdot (1 + a\gamma_0) \cdot \{x_0 \cdot [\bar{z} + \bar{\eta} \cdot D_3] + z_0 \cdot [\bar{x} + \bar{\eta} \cdot D_1]\} \quad ; \\ \omega_Z = -\lambda(s) \cdot (1 + a\gamma_0) \cdot \{x_0 \cdot [\bar{x} + \bar{\eta} \cdot D_1] - z_0 \cdot [\bar{z} + \bar{\eta} \cdot D_3]\} \quad ; \end{cases} \tag{5.5a}$$

2) Quadrupole:

$$\begin{cases} \omega_\tau = 0 \ ; \\ \omega_x = g(s) \cdot (1 + a\gamma_0) \cdot (\bar{z} + \bar{\eta} \cdot D_3) \ ; \\ \omega_z = g(s) \cdot (1 + a\gamma_0) \cdot (\bar{x} + \bar{\eta} \cdot D_1) \ ; \end{cases} \tag{5.5b}$$

3) Skew quadrupole:

$$\begin{cases} \omega_\tau = 0 \ ; \\ \omega_x = - N(s) \cdot (1 + a\gamma_0) \cdot (\bar{x} + \bar{\eta} \cdot D_1) \ ; \\ \omega_z = + N(s) \cdot (1 + a\gamma_0) \cdot (\bar{z} + \bar{\eta} \cdot D_3) \ ; \end{cases} \tag{5.5c}$$

4) Bending magnet:

a) $K_x \neq 0 \ ; \ K_z = 0 \ ;$

$$\begin{cases} \omega_\tau = + \dfrac{a\gamma_0^2}{1 + \gamma_0} \cdot (\bar{z}' + \bar{\eta} \cdot D_4) \cdot K_x \ ; \\ \omega_x = 0 \ ; \\ \omega_z = - (1 + a\gamma_0) \cdot K_x^2 \cdot (\bar{x} + \bar{\eta} \cdot D_1) + K_x \cdot \bar{\eta} \ ; \end{cases} \tag{5.5d}$$

b) $K_x = 0 \ ; \ K_z \neq 0 \ ;$

$$\begin{cases} \omega_\tau = - \dfrac{a\gamma_0^2}{1 + \gamma_0} \cdot (\bar{x}' + \bar{\eta} \cdot D_2) \cdot K_z \ ; \\ \omega_x = + (1 + a\gamma_0) \cdot K_z^2 \cdot (\bar{z} + \bar{\eta} \cdot D_3) - K_z \cdot \bar{\eta} \ ; \\ \omega_z = 0 \ ; \end{cases} \tag{5.5e}$$

5) Solenoid:

$$\begin{cases} \omega_\tau = \bar{\eta} \cdot 2H \cdot [1 + \dfrac{a\gamma_0^2}{(1 + \gamma_0)^2}] \ ; \\ \omega_x = (1 + a\gamma_0) \cdot H' \cdot (\bar{x} + \bar{\eta} \cdot D_1) + \\ \qquad + a\gamma_0 \cdot \dfrac{\gamma_0}{1 + \gamma_0} \cdot 2H \cdot [(\bar{p}_x + \bar{\eta} \cdot D_2) + H \cdot (\bar{z} + \bar{\eta} \cdot D_3)] \ ; \\ \omega_z = (1 + a\gamma_0) \cdot H' \cdot (\bar{z} + \bar{\eta} \cdot D_3) + \\ \qquad + a\gamma_0 \cdot \dfrac{\gamma_0}{1 + \gamma_0} \cdot 2H \cdot [(\bar{p}_z + \bar{\eta} \cdot D_4) - H \cdot (\bar{x} + \bar{\eta} \cdot D_1)] \ ; \end{cases} \tag{5.5f}$$

6) Cavity

$$\omega_\tau = 0 \quad ;$$

$$\omega_x = (a\gamma_0 + \frac{\gamma_0}{1+\gamma_0}) \cdot (\overline{z}' + \overline{\eta} \cdot D_4) \cdot \frac{e\hat{V}}{E_0} \sin\Phi \cdot \sum_\mu \delta(s - s_\mu) \quad ;$$

$$\omega_z = -(a\gamma_0 + \frac{\gamma_0}{1+\gamma_0}) \cdot (\overline{x}' + \overline{\eta} \cdot D_2) \cdot \frac{e\hat{V}}{E_0} \sin\Phi \cdot \sum_\mu \delta(s - s_\mu) \quad . \qquad (5.5g)$$

In the following we consider $\underline{\omega}$ to be a small perturbation in linear approximation. Then using the ansatz

$$\begin{pmatrix} \xi_\tau \\ \xi_x \\ \xi_z \end{pmatrix} = \begin{pmatrix} \xi_\tau^{(o)} \\ \xi_x^{(o)} \\ \xi_z^{(o)} \end{pmatrix} + \begin{pmatrix} \xi_\tau^{(1)} \\ \xi_x^{(1)} \\ \xi_z^{(1)} \end{pmatrix} \quad ; \qquad (5.6)$$

$$\frac{d}{ds} \begin{pmatrix} \xi_\tau^{(o)} \\ \xi_x^{(o)} \\ \xi_z^{(o)} \end{pmatrix} = \underline{\Omega}^{(o)} \begin{pmatrix} \xi_\tau^{(o)} \\ \xi_x^{(o)} \\ \xi_z^{(o)} \end{pmatrix} \quad . \qquad (5.7)$$

we obtain for the vector $\vec{\xi}^{(1)}$

$$\frac{d}{ds} \vec{\xi}^{(1)} = \underline{\Omega}^{(o)} \cdot \vec{\xi}^{(1)} + \underline{\omega} \cdot \vec{\xi}^{(o)} \quad . \qquad (5.8)$$

5.2 The $(\vec{n}, \vec{m}, \vec{\ell})$-orthonormal system of the spin motion

In order to further simplify the description of the spin motion we introduce, in the usual way[2], a new orthonormal system $\vec{n}(s)$, $\vec{m}(s)$, $\vec{\ell}(s)$ defined in terms of the one turn matrix $\underline{N}(s_0+L, s_0)$ resulting from (5.7).

$$\vec{\xi}^{(o)}(s_0+L) = \underline{N}(s_0+L, s_0)\,\vec{\xi}^{(o)}(s_0) \tag{5.9}$$

With this aim in mind we investigate the eigenvalue spectrum of the matrix $\underline{N}$

$$\underline{N}(s_0+L, s_0)\,\vec{r}_\mu(s_0) = \alpha_\mu \cdot \vec{r}_\mu(s_0) \tag{5.10}$$

with

$$\begin{cases} \alpha_1 = 1 & ; \quad \vec{r}_1(s_0) = \vec{n}_0(s_0) \quad ; \\ \alpha_2 = e^{+i\cdot 2\pi\nu} & ; \quad \vec{r}_2(s_0) = \vec{m}_0(s_0) + i\cdot\vec{\ell}_0(s_0) \quad ; \\ \alpha_3 = e^{-i\cdot 2\pi\nu} & ; \quad \vec{r}_3(s_0) = \vec{m}_0(s_0) - i\cdot\vec{\ell}_0(s_0) \quad ; \end{cases} \tag{5.11}$$

$$\begin{cases} \vec{n}_0(s_0) = \vec{m}_0(s_0) \times \vec{\ell}_0(s_0) \quad ; \\ \vec{m}_0(s_0) \perp \vec{\ell}_0(s_0) \quad ; \\ |\vec{n}_0(s_0)| = |\vec{m}_0(s_0)| = |\vec{\ell}_0(s_0)| = 1 \end{cases} \tag{5.12}$$

and put

$$\vec{n}(s) = \underline{N}(s,s_0)\,\vec{n}_0(s_0) \tag{5.13}$$

and

$$\vec{m}(s) + i\cdot\vec{\ell}(s) = e^{-i[\Psi(s)-\Psi(s_0)]}\cdot\underline{N}(s,s_0)[\vec{m}_0(s_0) + i\cdot\vec{\ell}_0(s_0)]. \tag{5.14}$$

If we require for the phase function Ψ the relation

$$\Psi(s_0+L) - \Psi(s_0) = 2\pi\nu \tag{5.15}$$

then

$$\begin{cases} \vec{n}(s) = \vec{m}(s) \times \vec{\ell}(s) \quad ; \\ \vec{m}(s) \perp \vec{\ell}(s) \quad ; \\ |\vec{n}(s)| = |\vec{m}(s)| = |\vec{\ell}(s)| = 1 \quad ; \end{cases} \tag{5.16}$$

$$(\vec{n},\vec{m},\vec{\ell})_{s=s_0+L} = (\vec{n},\vec{m},\vec{\ell})_{s=s_0} \tag{5.17}$$

i.e. the vectors $\vec{n}$, $\vec{m}$, $\vec{\ell}$ are actually an orthonormal right-handed vector basis which transforms into itself after one circuit around the ring.

By differentiating the vectors $\vec{m}$ and $\vec{\ell}$ and considering (5.7) and (5.14):

$$\begin{cases} \frac{d}{ds}\vec{m}(s) = \underline{\Omega}^{(o)} \cdot \vec{m}(s) + \Psi'(s) \cdot \vec{\ell}(s) \quad ; \\ \frac{d}{ds}\vec{\ell}(s) = \underline{\Omega}^{(o)} \cdot \vec{\ell}(s) - \Psi'(s) \cdot \vec{m}(s) \quad . \end{cases} \tag{5.18a}$$

At the same time using (5.13)

$$\frac{d}{ds}\vec{n}(s) = \underline{\Omega}^{(o)} \cdot \vec{n}(s) . \tag{5.18b}$$

Thus $\vec{m}$ and $\vec{\ell}$ depend on the behaviour of the phase function Ψ which can be arbitrary except that the phase advance per circuit must be $2\pi\nu$ as given in eq. (5.15).

6. The complete equations of coupled spin-orbit motion

With the help of eqs. (4.7) and (5.7), (5.8), (5.18a,b) we are now in the position to construct the complete defining equations of the linearized spin-orbit motion. With this aim in mind, following A. Chao we solve the eqs. (5.7) and (5.8) by introducing the ansatz

$$\begin{cases} \vec{\xi}^{(o)} = \xi_0 \cdot \vec{n} \; ; & (6.1a) \\ \vec{\xi}^{(1)} = \xi_0 \cdot [\alpha(s) \cdot \vec{m} + \beta(s) \cdot \vec{\ell}] \; ; & (6.1b) \end{cases}$$

$$(\alpha^2 + \beta^2 \ll 1).$$

Using eqs. (6.1a) and (5.18b) eq. (5.7) is already fulfilled and by substituting (6.1a,b) in (5.8) and using (5.18) one obtains

$$\alpha' = (\ell_\tau, \ell_x, \ell_z) \begin{pmatrix} \omega_\tau \\ \omega_x \\ \omega_z \end{pmatrix} + \beta \cdot \Psi' \; ; \qquad (6.2a)$$

$$\beta' = -(m_\tau, m_x, m_z) \begin{pmatrix} \omega_\tau \\ \omega_x \\ \omega_z \end{pmatrix} - \alpha \cdot \Psi' \; ; \qquad (6.2b)$$

Furthermore, with eq. (5.5) one can put

$$\begin{pmatrix} \omega_\tau \\ \omega_x \\ \omega_z \end{pmatrix} = \underline{F}_{(3\times6)} \cdot \vec{y} \qquad (6.3)$$

where $\underline{F}$ is extracted from (5.5a-g) for the various lenses.

In detail one has:

1) Sextupole:

$$F_{21} = -\lambda(s) \cdot (1 + a\gamma_0) \cdot z_0 \quad ;$$

$$F_{23} = -\lambda(s) \cdot (1 + a\gamma_0) \cdot x_0 \quad ;$$

$$F_{26} = F_{21} \cdot D_1 + F_{23} \cdot D_3 \quad ;$$

$$F_{31} = F_{23} \quad ;$$

$$F_{33} = - F_{21} \quad ;$$

$$F_{36} = F_{31} \cdot D_1 + F_{33} \cdot D_3 \quad ;$$

$$F_{ik} = 0 \quad \text{otherwise} . \tag{6.4a}$$

2) Quadrupole:

$$F_{23} = - g(s) \cdot (1 + a\gamma_0) \quad ;$$

$$F_{26} = F_{23} \cdot D_3 \quad ;$$

$$F_{31} = F_{23} \quad ;$$

$$F_{36} = F_{23} \cdot D_1 \quad ;$$

$$F_{ik} = 0 \quad \text{otherwise} . \tag{6.4b}$$

3) Skew quadrupole:

$$F_{21} = -N(s) \cdot (1 + a\gamma_0) \quad ;$$

$$F_{26} = F_{21} \cdot D_1 \quad ;$$

$$F_{33} = - F_{21} \quad ;$$

$$F_{36} = F_{33} \cdot D_3 \quad ;$$

$$F_{ik} = 0 \quad \text{otherwise} . \tag{6.4c}$$

4) Bending magnet:

a) $K_X \neq 0$; $K_Z = 0$;

$$F_{14} = \frac{a\gamma_0^2}{1+\gamma_0} \cdot K_X \; ;$$

$$F_{16} = F_{14} \cdot D_4 \; ;$$

$$F_{31} = -(1 + a\gamma_0) \cdot K_X^2 \; ;$$

$$F_{36} = F_{31} \cdot D_1 + K_X \; ;$$

$$F_{ik} = 0 \quad \text{otherwise} . \tag{6.4d}$$

b) $K_X = 0$; $K_Z \neq 0$;

$$F_{12} = -\frac{a\gamma_0^2}{1+\gamma_0} \cdot K_Z \; ;$$

$$F_{16} = F_{12} \cdot D_2 \; ;$$

$$F_{23} = (1 + a\gamma_0) \cdot K_Z^2 \; ;$$

$$F_{26} = F_{23} \cdot D_3 - K_Z \; ;$$

$$F_{ik} = 0 \quad \text{otherwise} . \tag{6.4e}$$

5) Solenoid:

$$F_{16} = 2H \cdot [1 + \frac{a\gamma_0^2}{(1+\gamma_0)^2}] \; ;$$

$$F_{21} = (1 + a\gamma_0) \cdot H' \; ;$$

$$F_{22} = \frac{a\gamma_0^2}{1+\gamma_0} \cdot 2H \; ;$$

$$F_{23} = F_{22} \cdot H \; ;$$

$$F_{26} = F_{21} \cdot D_1 + F_{22} \cdot D_2 + F_{23} \cdot D_3 \; ;$$

$$F_{31} = -F_{23} \; ;$$

$$F_{33} = F_{21} \; ;$$

$$F_{34} = F_{22} \; ;$$

$$F_{36} = F_{31} \cdot D_1 + F_{33} \cdot D_3 + F_{34} \cdot D_4 \; ;$$

$$F_{ik} = 0 \quad \text{otherwise} . \tag{6.4f}$$

6) Cavity:

$$F_{24} = (a\gamma_0 + \frac{\gamma_0}{1+\gamma_0}) \cdot \frac{e\hat{V}}{E_0} \sin\Phi \cdot \sum_{\mu} \delta(s - s_\mu) \quad ;$$

$$F_{26} = F_{24} \cdot D_4 \quad ;$$

$$F_{32} = - F_{24} \quad ;$$

$$F_{36} = F_{32} \cdot D_2 \quad ;$$

$$F_{ik} = 0 \quad \text{otherwise} . \tag{6.4g}$$

Taking into account (6.3), the spin equation (6.2) takes the form

$$\frac{d}{ds}\vec{\xi} = \underline{G}_0 \cdot \vec{y} + \underline{D}_0 \cdot \vec{\xi} \quad ; \tag{6.5}$$

$$\vec{\xi} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad ; \tag{6.6a}$$

$$\underline{G}_0 = \underline{R} \cdot \underline{F} \tag{6.6b}$$

with
$$\underline{R} = \begin{pmatrix} \ell_\tau & \ell_x & \ell_z \\ -m_\tau & -m_x & -m_z \end{pmatrix} \quad ; \tag{6.6c}$$

$$\underline{D}_0 = \begin{pmatrix} 0 & \Psi' \\ -\Psi' & 0 \end{pmatrix} \quad ; \tag{6.6d}$$

where the orbit vector in (6.5) is defined by (4.7).

By combining the orbit vector $\vec{y}$ and the spin vector $\vec{\xi}$ we can construct an 8-dimensional vector of the form

$$\vec{u} = \begin{pmatrix} \vec{y} \\ \vec{\xi} \end{pmatrix} \tag{6.7}$$

and then finally the spin eq. (6.5) and the orbit eq. (4.7) can be combined into a matrix equation representing the complete equations of motion

$$\frac{d}{ds}\vec{u} = (\underline{\hat{A}} + \delta\underline{\hat{A}}) \cdot \vec{u} + \delta\vec{\hat{c}} \qquad (6.8)$$

with

$$\underline{\hat{A}} = \begin{pmatrix} \underline{\bar{A}} & \underline{0} \\ \underline{G}_o & \underline{D}_o \end{pmatrix} \; ; \quad \delta\underline{\hat{A}} = \begin{pmatrix} \delta\underline{\bar{A}} & \underline{0} \\ \underline{0} & \underline{0} \end{pmatrix} \; ; \qquad (6.8a)$$

$$\delta\vec{\hat{c}} = \delta c \begin{pmatrix} -\vec{D} \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} . \qquad (6.8b)$$

Since the spin basis $(\vec{n},\vec{m},\vec{\ell})$ is periodic (eq. (5.17)), $\underline{\hat{A}}$ is also periodic. This equation serves as the starting point for further developments.

7. The unperturbed problem

7.1 The eigenvalue spectrum of the one turn transfer matrix

We now restrict ourselves to the unperturbed spin-orbit problem

$$\frac{d}{ds}\vec{u} = \underline{\hat{A}}\,\vec{u} \tag{7.1}$$

i.e.

$$\begin{cases} \dfrac{d}{ds}\vec{y} = \underline{\bar{A}}\,\vec{y} & (7.1a) \\[2ex] \dfrac{d}{ds}\vec{\xi} = \underline{G}_0\,\vec{y} + \underline{D}_0\,\vec{\xi} & (7.1b) \end{cases}$$

and investigate the eigenvalue spectrum of the one turn matrix $M_{8\times8}(s_0+L,s_0)$ which solves for eq. (7.1):

$$\underline{M}_{(8\times8)}(s_0+L,s_0)\;\vec{q}_\mu(s_0) = \lambda_\mu\cdot\vec{q}_\mu(s_0)\;. \tag{7.2}$$

We then separate the components of the eigenvector $\vec{q}_\mu$ given in (7.2) into a 6 component orbit part, $\vec{v}_\mu$, and a 2 component spin part, $\vec{w}_\mu$:

$$\vec{q}_\mu = \begin{pmatrix}\vec{v}_\mu\\ \vec{w}_\mu\end{pmatrix}\;. \tag{7.3}$$

In addition, we require that the stability condition

$$|\lambda_\mu| \leq 1\;. \tag{7.4}$$

be satisfied.

The solution to the spin equation (7.1b) is then given as

$$\vec{\xi}(s) = \underline{D}(s,s_0)\,\vec{\xi}(s_0) + \underline{G}(s,s_0)\,\vec{y}(s_0) \tag{7.5}$$

with

$$\underline{G}(s,s_0) = \int_{s_0}^{s} ds'\cdot\underline{D}(s,s')\,\underline{G}_0(s')\,\underline{M}(s',s_0)\;; \tag{7.5a}$$

$$\underline{D}(s'',s') = \begin{pmatrix}\cos[\Psi(s'')-\Psi(s')] & \sin[\Psi(s'')-\Psi(s')]\\ -\sin[\Psi(s'')-\Psi(s')] & \cos[\Psi(s'')-\Psi(s')]\end{pmatrix} \tag{7.5b}$$

where $\underline{M}(s,s_0)$ is the transfer matrix which solves for the orbit eq. (7.1a):

$$\vec{y}(s) = \underline{M}(s,s_0)\,\vec{y}(s_0)\;.$$

The matrix $M_{8\times 8}(s_0+L,s_0)$ can now be written as:

$$\underline{M}_{(8\times 8)}(s_0+L,s_0) = \begin{pmatrix} \underline{M}(s_0+L,s_0) & \underline{0} \\ \underline{G}(s_0+L,s_0) & \underline{D}(s_0+L,s_0) \end{pmatrix} . \qquad (7.6)$$

where by (7.5b) and (5.20), $D(s_0+L,s_0)$ is given by

$$\underline{D}(s_0+L,s_0) = \begin{pmatrix} \cos 2\pi\nu & \sin 2\pi\nu \\ -\sin 2\pi\nu & \cos 2\pi\nu \end{pmatrix} . \qquad (7.7)$$

With (7.3) and (7.6) the eigenvalue eq. (7.2) transforms into

$$\underline{M}(s_0+L,s_0)\,\vec{v}_\mu(s_0) = \lambda_\mu \cdot \vec{v}_\mu(s_0) \quad ; \qquad (7.8a)$$

$$\underline{G}(s_0+L,s_0)\,\vec{v}_\mu(s_0) + \underline{D}(s_0+L,s_0)\,\vec{w}_\mu(s_0) = \lambda_\mu \cdot \vec{w}_\mu(s_0) \; . \qquad (7.8b)$$

From (7.8a) it is immediately clear that the vectors $\vec{v}_\mu(s_0)$ are simply the eigenvectors of the one turn matrix $\underline{M}(s_0+L,s_0)$ for the orbit motion. This matrix is symplectic:

$$\underline{M}^T(s_0+L,s_0) \cdot \underline{S} \cdot \underline{M}(s_0+L,s_0) = \underline{S} \qquad (7.9)$$

with

$$\underline{S} = \begin{pmatrix} \underline{S}_2 & & 0 \\ & \underline{S}_2 & \\ 0 & & \underline{S}_2 \end{pmatrix} \qquad (7.10a)$$

$$\underline{S}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad (7.10b)$$

since the matrix $\underline{A}$ of coefficients given in (4.9) satisfies the condition

$$\underline{\bar{A}}^T \cdot \underline{S} + \underline{S} \cdot \underline{\bar{A}} = 0 \; . \qquad (7.11)$$

(The symplecticity of the matrix $\underline{M}$ follows also from the fact that the equation of motion (7.1a) can be written in canonical form using the Hamiltonian (4.14))[6].

As a result, the eigenvectors $\vec{v}_\mu(s_0)$ occur in pairs

$$(\vec{v}_k(s_0),\ \vec{v}_{-k}(s_0)) \quad ; \quad k = I,\ II,\ III \tag{7.12a}$$

with the reciprocal eigenvalues

$$\lambda_k \cdot \lambda_{-k} = 1 \tag{7.12b}$$

Thus, the stability condition (7.4) can be written as

$$|\lambda_k| = 1 \qquad ; \quad \lambda_{-k} = \lambda_k^* \tag{7.13}$$

$$\Longrightarrow \quad \vec{v}_{-k}(s_0) = \vec{v}_k^*(s_0) \quad ; \quad \lambda_k = e^{-i\cdot 2\pi Q_k}\ ,\ Q_k \text{ real} \tag{7.14}$$

so that all eigenvalues must lie on a unit circle and with the normalization condition for the $\vec{v}_k(s_0)$

$$\vec{v}_k^+(s_0)\, \underline{S}\, \vec{v}_k(s_0) = i \qquad (k = I,\ II,\ III).$$

we obtain the relations

$$\vec{v}_k^+(s_0) \cdot \underline{S} \cdot \vec{v}_\ell(s_0) = - \vec{v}_{-k}^+(s_0) \cdot \underline{S} \cdot \vec{v}_{-\ell}(s_0) = i \cdot \delta_{k\ell} \quad ; \tag{7.15a}$$

$$\vec{v}_k^+(s_0) \cdot \underline{S} \cdot \vec{v}_{-\ell}(s_0) = \vec{v}_{-k}^+(s_0) \cdot \underline{S} \cdot \vec{v}_\ell(s_0) = 0 \quad ; \tag{7.15b}$$

$$(k, \ell = I,\ II,\ III)$$

Once the vectors $\vec{v}_{\pm k}(s_0)$ have been found by solving (7.8a), then the spin parts $\vec{w}_{\pm k}(s_0)$ of the complete eigenvectors

$$\vec{q}_{\pm k}(s_0) = \begin{pmatrix} \vec{v}_{\pm k}(s_0) \\ \vec{w}_{\pm k}(s_0) \end{pmatrix}$$

may be obtained (using eq. (7.8b)) as

$$\vec{w}_k(s_0) = - [\underline{D}(s_0+L, s_0) - \lambda_k \cdot \underline{1}]^{-1} \cdot \underline{G}(s_0+L, s_0)\ \vec{v}_k(s_0) \quad ; \tag{7.16a}$$

$$\vec{w}_{-k}(s_0) = \vec{w}_k^*(s_0) \tag{7.16b}$$

so that in general we can put

$$\vec{q}_k(s_0) = \begin{pmatrix} \vec{v}_k(s_0) \\ \vec{w}_k(s_0) \end{pmatrix} \quad ; \tag{7.17a}$$

$$\vec{q}_{-k}(s_0) = \vec{q}_k^*(s_0) \qquad ; \qquad (k = I,\ II,\ III) \tag{7.17b}$$

where $\vec{w}_k(s_0)$ is given by (7.16a).

From (7.17) we already have 6 of the eigenvectors of the 8-dimensional matrix $\underline{M}_{8\times 8}(s_0+L,s_0)$. In order to find the two remaining eigenvectors we use the ansatz

$$\vec{q}_{\pm IV}(s_0) = \begin{pmatrix} \vec{0}_6 \\ \vec{w}_{\pm IV}(s_0) \end{pmatrix} \tag{7.18}$$

and obtain from (7.7) and (7.8)

$$\vec{w}_{IV}(s_0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i\Psi(s_0)} \quad ; \tag{7.19a}$$

$$\vec{w}_{-IV}(s_0) = \vec{w}^{*}_{IV}(s_0) \tag{7.19b}$$

so that

$$\lambda_{IV} = e^{-i\cdot 2\pi Q_{IV}} \quad ; \quad \lambda_{-IV} = \lambda^{*}_{IV} \tag{7.20}$$

with

$$Q_{IV} = \nu \quad . \tag{7.21}$$

7.2 Floquet's Theorem

The vectors $\vec{q}_{\pm k}(s_0)$ are the eigenvectors of the one turn matrix $\underline{M}_{8\times 8}(s_0+L,s_0)$ with the starting point s_0. Since the matrix $\underline{\hat{A}}$ (eq. (6.8a)) is periodic, the matrix $\underline{M}_{8\times 8}$ is also periodic so that the eigenvectors of the matrix $\underline{M}_{8\times 8}(s+L,s)$ with starting point s can be obtained by operating with $\underline{M}_{8\times 8}(s,s_0)$:

$$\vec{q}_{\pm k}(s) \equiv \begin{pmatrix} \vec{v}_{\pm k}(s) \\ \vec{w}_{\pm k}(s) \end{pmatrix} = \underline{M}_{(8\times 8)}(s,s_0)\, \vec{q}_{\pm k}(s_0) \quad ; \tag{7.22}$$

$$\underline{M}_{(8\times 8)}(s+L,s)\, \vec{q}_{\pm k}(s) = \lambda_{\pm k}\cdot \vec{q}_{\pm k}(s) \quad . \tag{7.23}$$

where the eigenvalues remain unchanged.

$$\lambda_{\pm k}(s) = \lambda_{\pm k}(s_0) \equiv \lambda_{\pm k} \quad . \tag{7.24}$$

In particular one finds

$$\vec{v}_{\pm k}(s) = \underline{M}(s,s_0)\,\vec{v}_{\pm k}(s_0) \quad ; \tag{7.25}$$

$$\begin{cases} \vec{w}_k(s) = -\,[\underline{D}(s+L,s) - \lambda_k \cdot \underline{1}]^{-1} \cdot \underline{G}(s+L,s)\,\vec{v}_k(s) \; ; \\ \vec{w}_{-k}(s) = \vec{w}_k^{\,*}(s) \end{cases} \tag{7.26}$$

with

$$\underline{D}(s+L,s) = \begin{pmatrix} \cos 2\pi\nu & \sin 2\pi\nu \\ -\sin 2\pi\nu & \cos 2\pi\nu \end{pmatrix} = \underline{D}(s_0+L,s_0) \; ; \tag{7.27}$$

$$\begin{cases} \vec{w}_{IV}(s) = \underline{D}(s,s_0)\,\vec{w}_{IV}(s_0) = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i\cdot\Psi(s)} \; ; \\ \vec{w}_{-IV}(s) = \vec{w}_{IV}^{\,*}(s) \; . \end{cases} \tag{7.28}$$

The vectors $\vec{v}_{\pm k}(s)$ defined by (7.25) also fulfill the same orthogonality relations (7.15) as $\vec{v}_{\pm k}(s_0)$:

$$\begin{cases} \vec{v}_k^{\,+}(s)\cdot\underline{S}\cdot\vec{v}_\ell(s) = -\,\vec{v}_{-k}^{\,+}(s)\cdot\underline{S}\cdot\vec{v}_{-\ell}(s) = i\cdot\delta_{k\ell} \; ; \\ \vec{v}_k^{\,+}(s)\cdot\underline{S}\cdot\vec{v}_{-\ell}(s) = \vec{v}_{-k}^{\,+}(s)\cdot\underline{S}\cdot\vec{v}_\ell(s) = 0 \; . \end{cases} \tag{7.29}$$

Similar relations are also valid for the vectors $\vec{w}_{\pm IV}$:

$$\begin{cases} \vec{w}_{IV}(s)\cdot\underline{S}_2\cdot\vec{w}_{IV}(s) = -\,\vec{w}_{-IV}(s)\cdot\underline{S}_2\cdot\vec{w}_{-IV}(s) = i \; ; \\ \vec{w}_{IV}(s)\cdot\underline{S}_2\cdot\vec{w}_{-IV}(s) = \vec{w}_{-IV}(s)\cdot\underline{S}_2\cdot\vec{w}_{IV}(s) = 0 \; ; \end{cases} \tag{7.30}$$

as one can see by substitution of (7.8) into the left side of (7.30).
Putting

$$\vec{q}_\mu(s) = \hat{\vec{q}}_\mu(s)\cdot e^{-i\cdot 2\pi Q_\mu\cdot\frac{s}{L}} \; ; \tag{7.31}$$

$$\begin{cases} \vec{v}_\mu(s) = \hat{\vec{v}}_\mu(s)\cdot e^{-i\cdot 2\pi Q_\mu\cdot\frac{s}{L}} \; ; \\ \vec{w}_\mu(s) = \hat{\vec{w}}_\mu(s)\cdot e^{-i\cdot 2\pi Q_\mu\cdot\frac{s}{L}} \; ; \quad (\mu = \pm k \text{ with } k = I,II,III) \end{cases}$$

the factor $\hat{\vec{q}}_\mu(s)$

$$\hat{\vec{q}}_\mu(s) = \vec{q}_\mu(s)\cdot e^{+i\cdot 2\pi Q_\mu\cdot\frac{s}{L}}$$

is seen to be a periodic function with period L:

$$\hat{q}_\mu(s+L) = \underbrace{\vec{q}_\mu(s+L)}_{e^{-i\cdot 2\pi Q_\mu}\cdot\vec{q}_\mu(s)}\cdot e^{+i\cdot 2\pi Q_\mu\cdot\frac{s+L}{L}} = \vec{q}_\mu(s)\cdot e^{+i\cdot 2\pi Q_\mu\cdot\frac{s}{L}} = \hat{\vec{q}}_\mu(s). \tag{7.32}$$

The representation of the eigenvectors $\vec{q}_\mu(s)$ as a product of a periodic function $\hat{\vec{q}}_\mu(s)$ and a harmonic function $e^{-i\cdot 2\pi Q_\mu \frac{s}{L}}$ is an example of "Floquet's Theorem".

With the derivation of this theorem and the orthogonality relations (7.29) and (7.30) we now have a connection with the spin-orbit formalism of Report DESY 83-062. The remaining work on the complete spin-orbit eq. (6.8) can now be carried through in direct analogy to the methods of DESY 83-062 and thus, in the following the methods need only be sketched.

For later considerations we mention here that the matrix $\underline{M}(s+L,s)$ has the simple block diagonal form

$$\underline{M}(s+L,s) = \begin{pmatrix} M^{(\beta)}_{(4\times 4)}(s+L,s) & \underline{0}_{(4\times 2)} \\ \underline{0}_{(2\times 4)} & \underline{M}^{(\sigma)}_{(2\times 2)}(s+L,s) \end{pmatrix} \tag{7.33}$$

if the matrix $\underline{A}_c$ (eq. (4.9b)), describing the coupling between synchrotron and betatron motion, vanishes. Furthermore, the 2-dimensional one turn matrix $\underline{M}^{(\sigma)}_{(2\times 2)}(s+L,s)$ which is defined by the equation of synchrotron motion

$$\begin{cases} \dfrac{d}{ds}\,\bar{\sigma} = -\,[K_x\cdot D_x + K_z\cdot D_z]\cdot\bar{\eta}\ ; \\[2ex] \dfrac{d}{ds}\,\bar{\eta} = \dfrac{e\hat{V}}{E_0}\cdot k\cdot\dfrac{2\pi}{L}\cdot\cos\phi\cdot\sum\limits_{\mu}\ \delta(s-s_\mu)\cdot\bar{\sigma} \end{cases} \tag{7.34}$$

(see eq. (7.1a) and (4.9)) can be represented in the form

$$\underline{M}^{(\sigma)}_{(2\times 2)}(s+L,s) = \begin{pmatrix} \cos 2\pi Q_\sigma + \alpha_\sigma(s)\cdot\sin 2\pi Q_\sigma & \beta_\sigma\cdot\sin 2\pi Q_\sigma \\ -\,\gamma_\sigma\cdot\sin 2\pi Q_\sigma & \cos 2\pi Q_\sigma - \alpha_\sigma(s)\cdot\sin 2\pi Q_\sigma \end{pmatrix} \tag{7.35}$$

with $\beta_\sigma\gamma_\sigma = \alpha_\sigma^2 + 1$.

From eq. (7.33) and (7.35) one then sees that for the orbit eigenvectors $\vec{v}_k(s)$ one can write

$$\vec{v}_k = \begin{pmatrix} \vec{v}_k^{(\beta)} \\ \vec{\underline{0}}_2 \end{pmatrix} \quad ; \quad k = I,\ II \quad ; \qquad (7.36a)$$

$$\vec{v}_{III} = \begin{pmatrix} \vec{0}_4 \\ \vec{v}^{(\sigma)} \end{pmatrix} \quad ; \quad \vec{v}^{(\sigma)} = \frac{1}{\sqrt{2\beta_\sigma}} \begin{pmatrix} \beta_\sigma(s) \\ -[\alpha_\sigma(s)+i] \end{pmatrix} \cdot e^{-i\cdot\Psi_\sigma(s)} \qquad (7.36b)$$

where, in the case that the betatron oscillations are decoupled:

$$\underline{M}^{(\beta)}_{(4\times4)}(s+L,s) = \begin{pmatrix} M^{(x)}_{(2\times2)}(s+L,s) & \underline{0}_{(2\times2)} \\ \underline{0}_{(2\times2)} & \underline{M}^{(z)}_{(2\times2)}(s+L,s) \end{pmatrix};$$

$$\underline{M}^{(y)}_{(2\times2)}(s+L,s) = \begin{pmatrix} \cos 2\pi Q_y + \alpha_y(s)\cdot\sin 2\pi Q_y & \beta_y(s)\cdot\sin 2\pi Q_y \\ -\gamma_y(s)\cdot\sin 2\pi Q_y & \cos 2\pi Q_y - \alpha_y(s)\cdot\sin 2\pi Q_y \end{pmatrix}$$

$$\beta_y\cdot\gamma_y = \alpha_y^2 + 1 \qquad (y \equiv x,\ z) \qquad (7.37)$$

the vectors $\vec{v}_I$ and $\vec{v}_{II}$ take a form similar to $\vec{v}_{III}$:

$$\vec{v}_I^{(\beta)} = \begin{pmatrix} \vec{v}^{(x)} \\ \vec{0}_2 \end{pmatrix} \quad ; \quad \vec{v}_{II}^{(\beta)} = \begin{pmatrix} \vec{0}_2 \\ \vec{v}^{(z)} \end{pmatrix} \quad ; \qquad (7.38a)$$

$$\vec{v}^{(y)} = \frac{1}{\sqrt{2\beta_y(s)}} \begin{pmatrix} \beta_y(s) \\ -[\alpha_y(s)+i] \end{pmatrix} \cdot e^{-i\cdot\Psi_y(s)} \quad ; \qquad (7.38b)$$

$$(y \equiv x,\ z)$$

Remark:

An approximate form for the matrix $\underline{M}^{(\sigma)}(s+L,s)$ can be established in which in the equation of motion (7.34) the coefficients of $\bar{\eta}$ and $\bar{\sigma}$ are averaged over one turn:

$$[K_x \cdot D_x + K_z \cdot D_z] \longrightarrow \varkappa = \frac{1}{L} \int_s^{s+L} d\tilde{s} \cdot [K_x(\tilde{s}) \cdot D_x(\tilde{s}) + K_z(\tilde{s}) \cdot D_z(\tilde{s})] \quad (7.39a)$$

(momentum compaction factor) ;

$$\frac{e\hat{V}}{E_0} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\Phi \cdot \sum_\mu \delta(s-s_\mu) \longrightarrow \frac{1}{L} \int_s^{s+L} d\tilde{s} \cdot \frac{e\hat{V}}{E_0} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\Phi \cdot \sum_\mu \delta(\tilde{s}-s_\mu)$$

$$= \frac{1}{L} \cdot k \cdot \frac{2\pi}{L} \cdot \frac{\cos\Phi}{\sin\Phi} \cdot \int_s^{s+L} d\tilde{s} \cdot \frac{e\hat{V}}{E_0} \sin\Phi \cdot \sum_\mu \delta(\tilde{s}-s_\mu)$$

$$= \frac{\Omega^2}{\varkappa}$$

with

$$\Omega^2 = \frac{\varkappa}{L} \cdot k \cdot \frac{2\pi}{L} \cdot \mathrm{ctg}\Phi \cdot \frac{U_0}{E_0} \quad ; \quad (7.39b)$$

$$U_0 = \int_s^{s+L} d\tilde{s} \cdot e\hat{V} \cdot \sin\Phi \cdot \sum_\mu \delta(\tilde{s} - s_\mu) \quad (7.40)$$

$\equiv$ average energy per particle per turn picked up from the cavities

$$= E_0 \cdot \int_s^{s+L} d\tilde{s} \cdot C_1 \cdot [K_x^2(\tilde{s}) + K_z^2(\tilde{s})]$$

(average energy lost per particle per turn)

Thus, eq. (7.34) transforms into the differential equation system

$$\begin{cases} \frac{d}{ds}\bar{\sigma} = -\varkappa \cdot \bar{\eta} \quad ; \\ \frac{d}{ds}\bar{\eta} = \frac{\Omega^2}{\varkappa} \cdot \bar{\sigma} \end{cases} \quad (7.41)$$

with the solution

$$\begin{pmatrix} \bar{\sigma}(s) \\ \bar{\eta}(s) \end{pmatrix} = \begin{pmatrix} \cos\Omega(s-s_0) & -\frac{\varkappa}{\Omega}\sin\Omega(s-s_0) \\ \frac{\Omega}{\varkappa}\sin\Omega(s-s_0) & \cos\Omega(s-s_0) \end{pmatrix} \begin{pmatrix} \bar{\sigma}(s_0) \\ \bar{\eta}(s_0) \end{pmatrix} . \quad (7.42)$$

Thus, one obtains for the one turn matrix

$$\underline{M}^{(\sigma)}_{(2\times2)}(s+L,s) = \begin{pmatrix} \cos\Omega L & -\frac{\varkappa}{\Omega}\sin\Omega L \\ \frac{\Omega}{\varkappa}\sin\Omega L & \cos\Omega L \end{pmatrix} \qquad (7.43)$$

and by comparison of (7.43) with (7.35) we find ($\beta_\sigma > 0$)

$$2\pi Q_\sigma = -\Omega \cdot L \quad ; \qquad (7.43a)$$

$$\beta_\sigma = \frac{\varkappa}{\Omega} \quad ; \qquad (7.43b)$$

$$\alpha_\sigma = 0 \quad ; \qquad (7.43c)$$

$$\gamma_\sigma = \frac{\Omega}{\varkappa} \quad . \qquad (7.43d)$$

where the quantities Ω and $\varkappa$ are taken from (7.39a,b).In particular by substituting (7.39) and (7.40) in (7.43b) one obtains

$$\beta_\sigma^2 = \frac{L}{2\pi}\,\frac{1}{k}\,\frac{\varkappa \cdot \mathrm{tg}\,\Phi}{\frac{1}{L}\int_{s_0}^{s_0+L} d\tilde{s}\cdot C_1\cdot[K_x^2(\tilde{s}) + K_z^2(\tilde{s})]} \qquad (7.44)$$

Since $\vec{v}^{(\sigma)}(s)$ must be a solution to (7.41), one sees that the phase function $\Psi_\sigma(s)$ introduced in (7.36b) is given by

$$\Psi_\sigma(s) = -\Omega\cdot(s - s_0) \quad . \qquad (7.45)$$

so that with (7.43a), the condition

$$\Psi_\sigma(s+L) - \Psi_\sigma(s) = 2\pi Q_\sigma \qquad (7.46)$$

is fulfilled.

For our further investigations we assume that the one turn matrix $\underline{M}(s_0+L,s_0)$ takes the block diagonal form given by eq. (7.33), i.e. that the coupling of synchro-betatron motion (approximately) vanishes. By suppression of the dispersion $\vec{D}$ in the cavities this can always be achieved exactly as can be seen from eqs. (4.9b) and (4.12). Of course, one can retain the coupling terms and one will obtain exactly the same numerical results as given by the normal fully coupled 6x6-formalism of Refs. (1) and (2). As we shall see later, by ignoring the coupling terms the formalism becomes simpler and more physically transparent.

8. Solution ansatz for the perturbed problem. Bogoliubov averaging

The general solution of the unperturbed equation of motion (7.1) can be represented (see eq. (7.22)) by

$$\vec{u} = \sum_{\substack{k=I,II,\\ III,IV}} \{A_k \cdot \vec{q}_k + A_{-k} \cdot \vec{q}_{-k}\} \tag{8.1}$$

with the integration constants A_k, A_{-k} (k = I, II, III, IV).

To solve the perturbed problem (7.8) we make the ansatz (Variation of Constants)

$$\vec{u} = \sum_{\substack{k=I,II,\\ III,IV}} \{A_k(s) \cdot \vec{q}_k + A_{-k}(s) \cdot \vec{q}_{-k}\} \tag{8.2}$$

and obtain by substituting (8.2) into (7.8)

$$\sum_{\substack{k=I,II,\\ III}} \{A_k'(s) \cdot \vec{v}_k + A_{-k}'(s) \cdot \vec{v}_{-k}\} = \delta \underline{\overline{A}} \sum_{\substack{k=I,II,\\ III}} \{A_k \cdot \vec{v}_k + A_{-k} \cdot \vec{v}_{-k}\} + \delta c \cdot \begin{pmatrix} -\vec{D} \\ 0 \\ 1 \end{pmatrix} ; \tag{8.3a}$$

$$A_{IV}'(s) \cdot \vec{w}_{IV} + A_{-IV}'(s) \cdot \vec{w}_{-IV} = - \sum_{\substack{k=I,II\\ III}} \{A_k'(s) \cdot \vec{w}_k + A_{-k}'(s) \cdot \vec{w}_{-k}\} . \tag{8.3b}$$

With the help of the orthogonality relations (7.29) and (7.30) these equations can be solved for $A_k'(s)$ (k=I,II,III,IV).

If one then uses the Bogoliubov averaging technique one obtains, using (7.36)

$$A_k(s) = e^{-i \cdot \frac{2\pi}{L} \cdot \delta Q_k \cdot (s-s_0)} \cdot \{A_k(s_0) - i \cdot \int_{s_0}^{s} ds' \cdot e^{i \cdot \frac{2\pi}{L} \cdot \delta Q_k \cdot (s'-s_0)} \cdot f_k^*(s') \cdot \delta c(s')\} ; \tag{8.4a}$$

$$A_{-k}(s) = [A_k^*(s)] \quad \text{for } k = I, II, III ; \tag{8.4b}$$

$$A_{IV}(s) = A_{IV}(s_0) + \sum_{\substack{k=I,II,\\III}} \int_{s_0}^{s} ds' \cdot \{f_k^*(s') \cdot \delta c(s') \cdot \vec{w}_{IV}^+(s') \; \underline{S}_2 \; \vec{w}_k(s') -$$

$$- f_k(s') \cdot \delta c(s') \cdot \vec{w}_{IV}^+(s') \; \underline{S}_2 \; \vec{w}_{-k}(s')\} \quad ; \qquad (8.5a)$$

$$A_{-IV}(s) = A_{IV}^*(s) \qquad (8.5b)$$

where

$$f_k^* = \vec{v}_k^+ \; \underline{S} \cdot \begin{pmatrix} -\vec{D} \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{cases} (v_{k1}^{(\beta)})^* \cdot D_2 - (v_{k2}^{(\beta)})^* \cdot D_1 + (v_{k3}^{(\beta)})^* \cdot D_4 - (v_{k4}^{(\beta)})^* \cdot D_3 & \text{für } k=I,II \; ; \\ -\sqrt{\dfrac{\beta_\sigma}{2}} \, e^{+i \cdot \Psi_\sigma} & \text{für } k = III \; ; \end{cases} \qquad (8.6)$$

$$\delta Q_k = \frac{1}{2\pi} \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{v}_k^+(\tilde{s}) \cdot \underline{S} \cdot \delta\underline{\bar{A}}(\tilde{s}) \cdot \vec{v}_k(\tilde{s}) \; . \qquad (8.7)$$

As already shown in (7), the quantities δQ_k immediately give the complex Q-shift of the kth oscillation mode resulting from the perturbation matrix $\delta\underline{A}$.

Together with eq. (8.2), which is built from the orbit part

$$\vec{y} = \sum_{\substack{k=I,II,\\III}} \{A_k(s) \cdot \vec{v}_k(s) + A_{-k}(s) \cdot \vec{v}_{-k}(s)\} \qquad (8.8a)$$

and the spin part

$$\vec{\xi} = \sum_{\substack{k=I,II,\\III}} \{A_k(s) \cdot \vec{w}_k(s) + A_{-k}(s) \cdot \vec{w}_{-k}(s)\} +$$

$$+ A_{IV}(s) \cdot \vec{w}_{IV}(s) + A_{-IV}(s) \cdot \vec{w}_{-IV}(s) \qquad (8.8b)$$

these equations describe the spin-orbit motion in the presence of the synchrotron radiation damping terms in $\delta\underline{\bar{A}}$.

9. Influence of synchrotron radiation on orbit motion

9.1 The beam emittance matrix

As a result of the stochastic excitation term $\delta\vec{\hat{c}}$ in the equation of motion (6.8) the only meaningful quantities that can be evaluated are averages of the form

$$< f(\vec{u}(s)) \cdot g(\vec{u}(s)) >_{\delta c}$$

which depend on the statistical properties of the quantum fluctuations δc.

We first consider only the orbital part of the motion which by eqs. (8.4) and (8.8a) is independent of the spin behaviour and we determine the moments

$$\begin{aligned} < \bar{y}_m(s) \cdot \bar{y}_n(s) >_{\delta c} = \sum_{k,\ell=I,II,III} \{ & A_{(k,\ell)}(s) \cdot v_{km} v_{\ell n} + \\ & + A_{(k,-\ell)}(s) \cdot v_{km} v_{-\ell,n} + A_{(-k,\ell)}(s) \cdot v_{-k,m} v_{\ell n} + \\ & + A_{(-k,-\ell)}(s) \cdot v_{-k,m} v_{-\ell,n} \} \ ; \end{aligned} \qquad (9.1)$$

$$(m,\ n = 1,\ 2,\ 3,\ 4,\ 5,\ 6)$$

with

$$A_{(k,\ell)}(s) = < A_k(s) \cdot A_\ell(s) >_{\delta c} = [A_{(-k,-\ell)}(s)]^* \qquad (9.2a)$$

$$A_{(k,-\ell)}(s) = < A_k(s) \cdot A_{-\ell}(s) >_{\delta c} = [A_{(-k,\ell)}(s)]^* \qquad (9.2b)$$

which give directly the width of the beam distribution.

Since we are mainly interested in the stationary (or equilibrium) values of the beam dimensions at an arbitrary position s in the storage ring we shall calculate

$$A^{stat}_{(k,\pm\ell)}(s) = \lim_{N\to\infty} A_{(k,\pm\ell)}(s + N \cdot L) \ . \qquad (9.3)$$

From eq. (8.4), taking into account

$$< \delta c(s) > \;=\; 0 \quad ; \tag{9.4a}$$

$$< \delta c(s) \cdot \delta c(s') > \;=\; \omega(s) \cdot \delta(s - s') \tag{9.4b}$$

and using the abbreviations

$$\delta Q_k \;=\; \delta\hat{Q}_k - \frac{i}{2\pi} \cdot \alpha_k \quad ; \tag{9.5}$$

$$\begin{cases} \delta\hat{Q}_k = \mathrm{Re}\,\{\delta Q_k\} & ; \\ \alpha_k = -\,2\pi \cdot \mathrm{Im}\,\{\delta Q_k\} & ; \end{cases}$$

and from now on ignoring integrals over oscillating functions[2)], we obtain:

$$\begin{aligned} A_{(k,-k)}(s+N\cdot L) &\equiv\; < |A_k(s+N\cdot L)|^2 >_{\delta c} \\ &=\; < |A_k(s)|^2 >_{\delta c} \cdot e^{-2\alpha_k \cdot N} \;+ \\ &+\; \frac{1-e^{-2\alpha_k\cdot N}}{e^{2\alpha_k}-1} \cdot \int_s^{s+L} d\tilde{s} \cdot e^{-2\alpha_k \cdot \frac{1}{L}(s-\tilde{s})} \cdot \omega(\tilde{s}) \cdot |f_k(\tilde{s})|^2 \;; \end{aligned} \tag{9.6a}$$

$$\begin{aligned} A_{(k,\pm\ell)}(s+N\cdot L) &\equiv\; < A_k(s+N\cdot L) \cdot A_{\pm\ell}(s+N\cdot L) >_{\delta c} \\ &=\; < A_k(s) \cdot A_{\pm\ell}(s) >_{\delta c} \cdot e^{-i\cdot 2\pi[\delta\hat{Q}_k + \delta\hat{Q}_\ell]\cdot N} \cdot e^{-(\alpha_k+\alpha_\ell)\cdot N} \end{aligned}$$

(otherwise) (9.6b)

From (9.6a,b) it is however clear, that stationary equilibrium values of the $A_{(k,\pm\ell)}$ are only possible if α_k in (9.5) satisfy the condition

$$\alpha_k > 0 \qquad (k = \mathrm{I},\, \mathrm{II},\, \mathrm{III}) \;. \tag{9.7}$$

If

$$\alpha_k < 0$$

the particle motion is, according to (8.8a) and (9.6), unstable.

In the following we assume that the stability condition (9.7) is valid.

Thus, finally one obtains for the equilibrium case the stationary values ($N\alpha_k \gg 1$):

$$A^{stat}_{(k,-k)}(s) \equiv \langle |A_k(s)|^2 \rangle^{stat}_{\delta c}$$

$$= \frac{1}{e^{2\alpha_k} - 1} \cdot \int_s^{s+L} d\tilde{s} \cdot e^{2\alpha_k \frac{1}{L}(s-\tilde{s})} \cdot \omega(\tilde{s}) \cdot |f_k(\tilde{s})|^2 \; ; \qquad (9.8a)$$

$$A^{stat}_{(k,\pm\ell)}(s) = 0 \qquad \text{otherwise} \qquad (9.8b)$$

and from (9.1) and (9.2) we get:

$$\langle \bar{y}_m(s) \cdot \bar{y}_n(s) \rangle^{stat}_{\delta c} =$$

$$= 2 \cdot \sum_{k=I,II,III} \langle |A_k(s)|^2 \rangle^{stat}_{\delta c} \cdot \mathrm{Re}\{ v_{km}(s) \cdot v_{kn}^*(s) \} \qquad (9.9)$$

which describes the "Beam Emittance Matrix".

Normally, $\alpha_k \ll 1$, so that one can replace (9.8a) in good approximation by

$$\langle |A_k(s)|^2 \rangle^{stat}_{\delta c} = \frac{1}{2\alpha_k} \cdot \int_s^{s+L} d\tilde{s} \cdot \omega(\tilde{s}) \cdot |f_k(\tilde{s})|^2$$

or using (8.6) by

$$\langle |A_k(s)|^2 \rangle^{stat}_{\delta c} = \begin{cases} \frac{1}{2\alpha_k} \cdot \int_s^{s+L} d\tilde{s} \cdot \omega(\tilde{s}) \cdot |v_{k1} \cdot D_2 - v_{k2} \cdot D_1 + v_{k3} \cdot D_4 - v_{k4} \cdot D_3|^2 & \text{for } k=I,II\,; \\ \frac{1}{4\alpha_k} \cdot \int_s^{s+L} d\tilde{s} \cdot \omega(\tilde{s}) \cdot \beta_\sigma(\tilde{s}) & \text{for } k = III\,. \end{cases} \qquad (9.10)$$

In this case, the integral is periodic and $\langle |A_k(s)|^2 \rangle^{stat}_{\delta c}$ is independent of the position of the start points:

$$\langle |A_k(s)|^2 \rangle^{stat}_{\delta c} = \text{const.} \qquad (9.11)$$

Using the same approximation in (9.6a) one obtains (for N = 1)

$$\frac{d}{ds} < |A_k(s)|^2 >_{\delta c} \approx \frac{< |A_k(s+L)|^2 >_{\delta c} - < |A_k(s)|^2 >_{\delta c}}{L}$$

$$= - \frac{2\alpha_k}{L} \cdot < |A_k(s)|^2 >_{\delta c} + \frac{1}{L} \int_s^{s+L} d\tilde{s} \cdot \omega(\tilde{s}) \cdot |f_k(\tilde{s})|^2 . \quad (9.12)$$

From this it is clear that the stationary values given by (9.8a) and (9.10) arise from an equilibrium between the stochastic excitation caused by quantum fluctuations in the synchrotron radiation (function $\omega(s)$ in (9.6a)) and a damping of the synchro-betatron oscillations caused by the continuous emission of synchrotron light, where the quantity α_k is clearly the damping constant.

From (8.7) and (9.5) the damping constants can be written as

$$\alpha_k = \frac{i}{2} \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{v}_k^{\,+}(\tilde{s}) \cdot [\underline{S} \cdot \delta\underline{\bar{A}}(\tilde{s}) + \delta\underline{\bar{A}}^T(\tilde{s}) \cdot \underline{S}] \cdot \vec{v}_k(\tilde{s}) \quad (9.13)$$

and by using (2.3) as well as (7.15) and (7.38) we get

$$\alpha_k = \frac{1}{2} \frac{U_0}{E_0} + \text{Im} \int_{s_0}^{s_0+L} d\tilde{s} \quad \times$$

$$\times \; [-v_{k1}^* \cdot D_2 + v_{k2}^* \cdot D_1 - v_{k3}^* \cdot D_4 + v_{k4}^* \cdot D_3] \cdot \sum_{\mu=1}^{4} \delta A_{6\mu} \cdot v_{k\mu} \quad \text{for } k=I,II ; \quad (9.14a)$$

$$\alpha_{III} = \frac{U_0}{E_0} - \frac{1}{2} \int_{s_0}^{s_0+L} d\tilde{s} \cdot \sum_{\mu=1}^{4} \delta A_{6\mu}(\tilde{s}) \cdot D_\mu(\tilde{s}) \quad (9.14b)$$

where the quantity U_0 comes from eq. (7.40).

These results have already been obtained by G. Leleux[8] and A. Piwinski[9] for the case where $H \equiv 0$ (no solenoids).

For the sum

$$\alpha_I + \alpha_{II} + \alpha_{III}$$

one obtains from eq. (9.14)

$$\alpha_I + \alpha_{II} + \alpha_{III} = 2\frac{U_0}{E_0} + \frac{1}{2}\sum_{\mu=1}^{4}\int_{s_0}^{s_0+L} d\tilde{s}\cdot\delta A_{6\mu}(\tilde{s})\cdot\{-D_\mu +$$

$$+ \sum_{k=I,II} [- i\cdot(\vec{v}_k^{(\beta)+}\cdot\underline{S}_4\cdot\vec{D})\cdot v_{k\mu}^{(\beta)} +$$

$$+ i\cdot(\vec{v}_k^{(\beta)+}\cdot\underline{S}_4\cdot\vec{D})\cdot v_{k\mu}^{(\beta)*}] \tag{9.15}$$

with

$$\underline{S}_4 = \begin{pmatrix} \underline{S}_2 & \underline{0}_{(2\times 2)} \\ \underline{0}_{(2\times 2)} & \underline{S}_2 \end{pmatrix} .$$

The dispersion vector $\vec{D}$ may now be expanded in terms of the eigenvectors

$$\vec{v}_k^{(\beta)} \quad \text{and} \quad \vec{v}_{-k}^{(\beta)} \equiv (\vec{v}_k^{(\beta)})^* \qquad (k = I, II) \quad :$$

$$\vec{D} = \sum_{k=I,II} \{c_k\cdot\vec{v}_k^{(\beta)} + c_{-k}\cdot\vec{v}_{-k}^{(\beta)}\} \tag{9.16}$$

where the coefficients c_k and c_{-k} are given according to (7.15) and (7.38a) by

$$c_k = - i\cdot(\vec{v}_k^{(\beta)+}\cdot\underline{S}_4\cdot\vec{D}) \quad ;$$

$$c_{-k} = c_k^* . \tag{9.17}$$

It is then clear from (9.16) and (9.17) that the second summand on the right side of (9.15) vanishes so that finally

$$\alpha_I + \alpha_{II} + \alpha_{III} = 2\frac{U_0}{E_0} \tag{9.18}$$

results.

This relation is known as the Robinson Theorem and allows one of the damping constants to be defined in terms of the other two.

9.2 Special case: decoupled machine

With the help of eqs. (9.9), (9.10) and (9.11) we are now in the position to calculate the damping constants and beam size of an arbitrarily coupled machine but we continue for now with consideration of the uncoupled case

$$N = H = 0 \quad . \tag{9.19}$$

In this case, using the relations

$$K_x = \frac{e}{E_0} \cdot B_z^{(0)} \quad ; \tag{9.20a}$$

$$K_z = -\frac{e}{E_0} \cdot B_x^{(0)} \tag{9.20b}$$

one gets for the quantities $\delta A_{6\mu}$ (μ = 1, 2, 3, 4) appearing in (9.14)

$$\begin{aligned} \delta A_{61} &= -C_1 \cdot (K_x^2 + K_z^2) \cdot C_x \quad ; \\ \delta A_{63} &= -C_1 \cdot (K_x^2 + K_z^2) \cdot C_z \quad ; \\ \delta A_{62} &= \delta A_{64} = 0 \; . \end{aligned} \tag{9.21}$$

with

$$C_x = K_x + \frac{2K_x^2}{K_x^2 + K_z^2} \cdot \frac{1}{B_z^{(0)}} \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \quad ; \tag{9.22a}$$

$$C_z = K_z + \frac{2K_z^2}{K_x^2 + K_z^2} \cdot \frac{1}{B_x^{(0)}} \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \quad . \tag{9.22b}$$

Using (7.38a,b) together with (9.21), the damping constants as given by (9.14a,b) become

$$\alpha_I = \frac{1}{2}\frac{U_0}{E_0} - \frac{1}{2}\int_{s_0}^{s_0+L} d\tilde{s} \cdot C_1 \cdot [K_x^2 + K_z^2] \cdot C_x(\tilde{s}) \cdot D_x(\tilde{s}) \quad ; \tag{9.23a}$$

$$\alpha_{II} = \frac{1}{2}\frac{U_0}{E_0} - \frac{1}{2}\int_{s_0}^{s_0+L} d\tilde{s} \cdot C_1 \cdot [K_x^2 + K_z^2] \cdot C_z(\tilde{s}) \cdot D_z(\tilde{s}) \quad ; \tag{9.23b}$$

$$\alpha_{III} = \frac{U_0}{E_0} + \frac{1}{2}\int_{s_0}^{s_0+L} d\tilde{s} \cdot C_1 \cdot [K_x^2 + K_z^2] \cdot [C_x(\tilde{s})D_x(\tilde{s}) + C_z(\tilde{s})D_z(\tilde{s})] . \tag{9.23c}$$

Furthermore, using these α_k, the excitation strengths $<|A_k|^2>^{stat}_{\delta c}$ for the synchro-betatron oscillations given by (9.10) become

$$<|A_I|^2>^{stat}_{\delta c} = \frac{1}{4\alpha_I}\cdot\int_s^{s+L} d\tilde{s}\cdot\omega(\tilde{s})\cdot\{\beta_x(\tilde{s})\cdot D_x'^2(\tilde{s}) + 2\alpha_x(\tilde{s})\cdot D_x(\tilde{s})\cdot D_x'(\tilde{s}) + \gamma_x(\tilde{s})\cdot D_x^2(\tilde{s})\} \quad ; \tag{9.24a}$$

$$<|A_{II}|^2>^{stat}_{\delta c} = \frac{1}{4\alpha_{II}}\cdot\int_s^{s+L} d\tilde{s}\cdot\omega(\tilde{s})\cdot\{\beta_z(\tilde{s})\cdot D_z'^2(\tilde{s}) + 2\alpha_z(\tilde{s})\cdot D_z(\tilde{s})\cdot D_z'(\tilde{s}) + \gamma_z(\tilde{s})\cdot D_z^2(\tilde{s})\} \quad ; \tag{9.24b}$$

$$<|A_{III}|^2>^{stat}_{\delta c} = \frac{\varkappa}{\Omega}\cdot\frac{1}{4\alpha_{III}}\cdot\int_s^{s+L} d\tilde{s}\cdot\omega(\tilde{s}) \quad . \tag{9.24c}$$

where the expression for β_σ given in (7.43b) has been used.

With (9.9) and (9.14), the beam emittance matrix may also be calculated. In particular one finds for the mean square energy spread:

$$\begin{aligned} <\bar{\eta}^2>^{stat}_{\delta c} &= 2\cdot<|A_{III}|^2>^{stat}_{\delta c}\cdot|v_{III,6}|^2 && \text{according to (9.9)} \\ &= <|A_{III}|^2>^{stat}_{\delta c}\cdot\gamma_\sigma(\tilde{s}) && \text{according to (7.36b)} \\ &= \frac{1}{4\alpha_{III}}\cdot\int_s^{s+L} d\tilde{s}\cdot\omega(\tilde{s}) && \text{according to (9.24c) and to (7.43b)} \end{aligned} \tag{9.25}$$

and for the average bunch length:

$$\begin{aligned} <|\bar{\sigma}|^2>_{\delta c} &= 2\cdot<|A_{III}|^2>^{stat}_{\delta c}\cdot|v_{III,5}|^2 && \text{according to (9.9)} \\ &= <|A_{III}|^2>^{stat}_{\delta c}\cdot\beta_\sigma(s) && \text{according to (7.36b)} \\ &= \frac{\varkappa^2}{\Omega^2}\cdot<|\bar{\eta}|^2>^{stat}_{\delta c} && \text{according to (9.24c) and to (7.43d)} \end{aligned} \tag{9.26}$$

with (see (7.39))

$$\frac{\varkappa^2}{\Omega^2} = \varkappa L\cdot\frac{1}{k}\cdot\frac{L}{2\pi}\cdot \mathrm{tg}\,\Phi\cdot\frac{U_0}{E_0} \quad . \tag{9.27}$$

These results (9.24, 9.25, 9.26) will be recognized as being identical to those obtained by more elementary means[10].

10 Spin-depolarization

The expressions for the damping constants, α_k, and the beam emittance matrix $<\overline{y}_m(s)\cdot\overline{y}_n(s)>_{\delta c}^{stat}$ obtained in the previous chapter already provide a general description of the orbit motion.

In order to investigate the spin-orbit motion, in addition to the quantities

$$A_{(k,\pm\ell)}(s) = <A_k(s)\cdot A_{\pm\ell}(s)>_{\delta c} \qquad (k,\ell = I, II, III)$$

we also require the terms with the factor A_{IV}:

$$A_{(IV,-IV)}(s) \equiv <A_{IV}(s)\cdot A_{IV}^*(s)>_{\delta c} = A_{(-IV,IV)}(s) \quad ;$$

$$A_{(IV,IV)}(s) \equiv <A_{IV}(s)\cdot A_{IV}(s)>_{\delta c} = [A_{(-IV,-IV)}(s)]^* \quad ;$$

$$A_{(IV,-k)}(s) \equiv <A_{IV}(s)\cdot A_k^*(s)>_{\delta c} = [A_{(-IV,k)}]^* \quad ;$$

$$A_{(IV,k)}(s) \equiv <A_{IV}(s)\cdot A_k(s)>_{\delta c} = [A_{(-IV,-k)}(s)]^* \quad .$$

For the term $A_{(IV,-IV)}$ one obtains, using (8.5) together with (7.28) and (7.10b)

$$\begin{aligned} A_{(IV,-IV)}(s) &\equiv <A_{IV}(s)\cdot A_{IV}^*(s)>_{\delta c} \\ &= <A_{IV}(s_0)\cdot A_{IV}^*(s_0)>_{\delta c} + \\ &\quad + 2\int_{s_0}^{s} d\tilde{s}\cdot\omega(\tilde{s})\cdot\{[\mathrm{Im}\sum_{k=I,II,III}(f_k^*\cdot w_{k1})]^2 + \\ &\quad + [\mathrm{Im}\sum_{k=I,II,III}(f_k^*\cdot w_{k2})]^2\} \end{aligned} \tag{10.1}$$

and for the remaining terms $A_{(IV,IV)}$ and $A_{(IV,\pm k)}$ one obtains

$$<A_{IV}(s_0+N\cdot L)\cdot A_{IV}(s_0+N\cdot L)>_{\delta c} = <A_{IV}(s_0)\cdot A_{IV}(s_0)>_{\delta c} \; ; \tag{10.2}$$

$$\begin{aligned} &<A_{IV}(s_0+N\cdot L)\cdot A_{\pm k}(s_0+N\cdot L)>_{\delta c} = \\ &= e^{-N\cdot\alpha_k}\cdot e^{-i\cdot 2\pi\cdot\delta\hat{Q}_k\cdot N}\cdot<A_{IV}(s_0)\cdot A_k(s_0)>_{\delta c} \end{aligned} \tag{10.3}$$

(for k = I, II, III)

if one neglects the integrals over the oscillating terms.

From eq. (8.8b) we also obtain for the spin components α and β the expressions

$$<\alpha(s)>_{\delta c} \;=\; \sum_{\substack{k=I,II,\\ III,IV}} \{<A_k(s)>_{\delta c}\cdot w_{k1} + <A_k^*(s)>_{\delta c}\cdot w_{k1}^*\} \;;$$

$$<\beta(s)>_{\delta c} \;=\; \sum_{\substack{k=I,II,\\ III,IV}} \{<A_k(s)>_{\delta c}\cdot w_{k2} + <A_k^*(s)>_{\delta c}\cdot w_{k2}^*\} \;;$$

$$<\alpha^2(s) + \beta^2(s)>_{\delta c} = 2\cdot \mathrm{Re} \sum_{\substack{k,\ell=I,II,\\ III,IV}} \{A_{(k,\ell)}(s)\cdot[w_{k1}\,w_{\ell 1} + w_{k2}\,w_{\ell 2}] + $$
$$+ A_{(k,-\ell)}(s)\cdot[w_{k1}\,w_{\ell 1}^* + w_{k2}\,w_{\ell 2}^*]\} \;. \qquad (10.4)$$

Furthermore, we assume that the initial stochastic averages of the spin components at an arbitrary starting point $s=s_0$, are given by the relations

$$\text{a)}\quad <A_k(s_0)>_{\delta c} \;=\; 0 \qquad \text{for}\quad k = I,II,III,IV \qquad (10.5a)$$

$$\Longrightarrow \qquad <\alpha(s_0)>_{\delta c} \;=\; <\beta(s_0)>_{\delta c} \;=\; 0 \;; \qquad (10.5b)$$

$$\text{b)}\quad <\alpha^2(s_0) + \beta^2(s_0)>_{\delta c} \;=\; 0 \;; \qquad (10.6)$$

$$\text{c)}\quad A_{(IV,IV)}(s_0) \;=\; 0 \;. \qquad (10.7)$$

Equations (10.5) and (10.6) express the assumption that at $s=s_0$ the beam is polarized in the direction of the n-axis so that following (6.1) the existing degree of polarization is given by

$$P(s_0) \;=\; \xi_0 \;. \qquad (10.8)$$

After N circuits we then obtain (using (9.4a) together with (8.4), (8.5) and (10.2)):

$$\left.\begin{array}{l} <A_k(s_0+N\,L)>_{\delta c} = 0 \\ \text{for } k = I,II,III,\ IV \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} <\alpha(s_0+N\,L)>_{\delta c} = 0 \;; \\ <\beta(s_0+N\,L)>_{\delta c} = 0 \;; \end{array}\right. \qquad (10.9)$$

$$<A_{(IV,IV)}(s_0+N\,L)>_{\delta c} = 0 \qquad (10.10)$$

and if in particular we consider the case when the orbital motion is in equilibrium:

$$N \cdot \alpha_k \gg 1$$

$$\begin{cases} A_{(k,\ell)}(s_0+N\cdot L) = 0 \; ; \\ A_{(k,-\ell)}(s_0+N\cdot L) = 0 \quad \text{for } \ell \neq k \; ; \\ A_{(k,-k)}(s_0+N\cdot L) = \langle |A_k(s_0)|^2 \rangle_{\delta c}^{stat} \; ; \quad (k \neq IV) \end{cases}$$

and evaluate (20.4), using (7.28) and (7.32) together with (9.6a,b), (10.3) and (10.10), we obtain

$$\frac{1}{2} \cdot \langle \alpha^2(s_0+N\,L) + \beta^2(s_0+N\,L) \rangle_{\delta c} =$$

$$= \sum_{k=I,II,III} \langle |A_k(s_0)|^2 \rangle_{\delta c}^{stat} \cdot [|w_{k1}(s_0)|^2 + |w_{k2}(s_0)|^2] +$$

$$+ \langle |A_{IV}(s_0+N\cdot L)|^2 \rangle_{\delta c} \qquad (10.11)$$

where N must nevertheless be sufficiently small so that the condition given in (6.1b):

$$\alpha^2(s) + \beta^2(s) \ll 1$$

is fulfilled; this is necessary for the application of the perturbation theory given in section 5.1, i.e.

$$\langle \alpha^2(s_0+N\cdot L) + \beta^2(s_0+N\cdot L) \rangle_{\delta c} \ll 1 \; . \qquad (10.12)$$

Since on average, the spin components α and β in (10.9) at $s=s_0+N\,L$ vanish, the polarization vector continues to point along the direction $\vec{n}$ and the left side of (10.11) immediately gives the relative change in the polarization

$$\frac{P(s_0) - P(s_0+N\cdot L)}{P(s_0)}$$

after N circuits (see fig. 1). Thus we may write:

$$\frac{P(s_0) - P(s_0+N\cdot L)}{P(s_0)} = \sum_{k=I,II,III} \langle |A_k(s_0)|^2 \rangle_{\delta c}^{stat} \cdot [|w_{k1}(s_0)|^2 +$$

$$+ |w_{k2}(s_0)|^2] + \langle |A_{IV}(s_0+N\cdot L)|^2 \rangle_{\delta c} \; . \qquad (10.13)$$

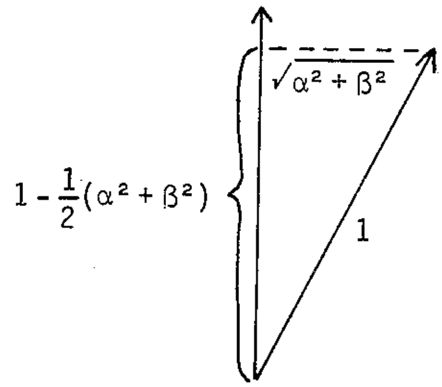

Figure 1

For the depolarization time τ_D:

$$\tau_D^{-1} = -\frac{1}{P}\,\frac{dP}{dt} = -\frac{c}{P}\,\frac{dP}{ds}$$

we then obtain

$$\tau_D^{-1} \approx -\frac{c}{P(s_o+N\cdot L)}\quad\frac{P(s_o+(N+1)\cdot L)-P(s_o+N\cdot L)}{L}$$

$$\approx \frac{c}{L}\cdot\frac{1}{P(s_o)}\cdot\{P(s_o+N\cdot L)-P(s_o+(N+1)\cdot L)\}$$

$$\approx \frac{c}{L}\cdot\frac{1}{P(s_o)}\cdot\{[P(s_o)-P(s_o+(N+1)\cdot L)]-[P(s_o)-P(s_o+N\cdot L)]\}$$

$$\approx \frac{c}{L}\cdot\{<|A_{IV}(s_o+(N+1)\cdot L)|^2>_{\delta c}-<|A_{IV}(s_o+N\cdot L)|^2>_{\delta c}\}$$

and using (20.1) we find

$$\tau_D^{-1} = 2\cdot\frac{c}{L}\int_{s_o}^{s_o+L} d\tilde{s}\cdot w(\tilde{s})\cdot\{[I_m\sum_{k=I,II,III}(f_k^*\cdot w_{k1})]^2 + [I_m\sum_{k=I,II,III}(f_k^*\cdot w_{k2})]^2\}\,. \qquad (10.14)$$

where the components w_{k1} and w_{k2} of the vector $\vec{w}_k$ in (7.26) are to be used. (10.14) is suitable for a numerical calculation of the depolarization time and it can also serve as starting point for a systematic optimization of the degree of polarization in storage rings as shown in (11).

With the help of this equation we are finally in the position to estimate the depolarizing effect of the synchrotron radiation on the spin motion.

In discussion of eq. (10.14) it is also useful to make a principal axis transformation of the factor

$$[\underline{D}(\tilde{s}+L,\tilde{s}) - \lambda_k \cdot \underline{1}]^{-1}$$

of eq. (7.26). Thus:

$$\underline{D}(s+L,s) \equiv \begin{pmatrix} \cos 2\pi\nu & \sin 2\pi\nu \\ -\sin 2\pi\nu & \cos 2\pi\nu \end{pmatrix} = \underline{U}\ \underline{K}\ \underline{U}^{-1}$$

$$\text{with}\quad \underline{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} ; \quad \underline{K} = \begin{pmatrix} e^{i\cdot 2\pi\nu} & 0 \\ 0 & e^{-i\cdot 2\pi\nu} \end{pmatrix}$$

$$\Longrightarrow \quad [\underline{D}(\tilde{s}+L,\tilde{s}) - \lambda_k \cdot \underline{1}]^{-1} = [\underline{U} \cdot (\underline{K} - e^{-i\cdot 2\pi Q_k} \cdot \underline{1}) \cdot \underline{U}^{-1}]^{-1}$$

$$= \underline{U} \cdot (\underline{K} - e^{-i\cdot 2\pi Q_k} \cdot \underline{1})^{-1} \cdot \underline{U}^{-1}$$

so that the vector $\vec{w}_k$ can be written in the form

$$\vec{w}_k(\tilde{s}) = +\frac{i}{2}\ \underline{U} \begin{pmatrix} e^{i\pi(Q_k-\nu)} \dfrac{1}{\sin\pi(Q_k+\nu)} & 0 \\ 0 & e^{i\pi(Q_k+\nu)} \dfrac{1}{\sin\pi(Q_k-\nu)} \end{pmatrix}$$

$$\times\ \underline{U}^{-1}\underline{G}(\tilde{s}+L,\tilde{s})\ \vec{v}_k(\tilde{s}) \qquad (10.15)$$

It is then clear that the components of $\vec{w}_k$ (and thereby according to (10.14) the quantity $\tau_D{}^{-1}$) become infinitely large, when

$$Q_k \pm \nu = \text{integer} . \qquad (10.16)$$

At these resonance positions the polarization is then destroyed.

By substituting the forms for f_k given in (8.6) into (10.14) and using this principal axis transformation we see that when the machine is uncoupled as in section 9.2, τ_D^{-1} has exactly the same form as given by Yokoya[12]. In this case, the depolarizing effect can be written in terms of simple integrals over the β-functions and the dispersion.

11. Summary

We have investigated the spin-orbit motion of particles in storage rings using the dispersion formalism. Starting from the fully six-dimensional description of the orbital motion as described in Ref. (2), the dispersion function was introduced via a canonical transformation so that the symplectic structure of the equations of motion and thus all the transfer matrices are completely preserved in the absence of radiation effects. The coupling in the synchro-betatron oscillations now appears in the cavities and vanishes if the dispersion in the cavities is equal to zero. Neglecting the synchro-betatron coupling, the transfer matrix of the orbit has block diagonal form. In this case, it is no longer necessary to construct the eigenvectors of the full 8x8 one turn matrix. Instead, only 4x4 and 2x2 eigenproblems need to be handled.

Together with the report in Ref. (2), these investigations cover the whole linear theory of spin-orbit motion.

The formalism developed in this work has been used as a basis for rewriting the spin part of SLIM in thick lens form neglecting transverse-longitudinal coupling[13]. Of course, this formalism can also be reduced to a thin lens form[14].

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APPENDIX

Calculation of the eight-dimensional transfer matrix of a cavity in the dispersion formalism taking into account the synchro-betatron coupling.

In this appendix we show how the 8x8 transfer matrix of a cavity can be calculated in the version of the dispersion formalism in which all synchro-betatron coupling terms are retained (see section 4 , especially eq. (4.9)).

For this purpose one has to investigate the solution of the orbital equation of motion

$$\frac{d}{ds}\vec{y} = \underline{\bar{A}}\,\vec{y} \qquad (1)$$

and the spin equation

$$\frac{d}{ds}\vec{\xi} = \underline{G}_0\,\vec{y} + \Psi' \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (2)$$

for a cavity (see eq. (7.1) and (6.6d).

For a (pointlike) cavity at the position $s = s_0$ the matrix $\underline{\bar{A}}$ is given according to eq. (4.9) by

$$\underline{\bar{A}} = \delta(s - s_0)\cdot \underline{K}_1 \qquad (3)$$

with

$$\underline{K}_1 = \begin{pmatrix} \vec{D}\cdot\vec{D}^T\cdot\underline{S} & -\vec{D} & \vec{0}_4 \\ \vec{0}_4^T & 0 & 0 \\ -\vec{D}^T\cdot\underline{S} & 1 & 0 \end{pmatrix} \cdot \frac{e\hat{V}}{E_0}\,k \cdot \frac{2\pi}{L}\cos\Phi \ . \qquad (4)$$

For $\underline{G}_0$ one gets with (6.4g), (6.6b) and (6.6c)

$$\underline{G}_0 = \delta(s - s_0)\cdot \underline{K}_2 \qquad (5)$$

with

$$\underline{K}_2 = (a\gamma_0 + \frac{\gamma_0}{1+\gamma_0})\cdot \frac{e\hat{V}}{E_0}\sin\Phi \begin{pmatrix} \ell_\tau & \ell_x & \ell_z \\ -m_\tau & -m_x & -m_z \end{pmatrix} \cdot \underline{\hat{F}} \qquad (6)$$

and

$$\underline{\hat{F}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & D_4 \\ 0 & -1 & 0 & 0 & 0 & -D_2 \end{pmatrix} \qquad (7)$$

With these definitions eq. (1) and eq. (2) can be rewritten in the form

$$\frac{d}{ds}\vec{y} = \delta(s - s_0) \cdot \underline{K}_1(s)\,\vec{y} \tag{8a}$$

$$\frac{d}{ds}\vec{\zeta} = \delta(s - s_0) \cdot \underline{K}_2(s) \cdot \vec{y} + \psi' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad . \tag{8b}$$

In solving eqs. (8a, 8b) it is important that the terms

$$\underline{K}_1(s)\,\vec{y} \quad \text{and} \quad \underline{K}_2(s)\,\vec{y}$$

which multiply the δ-functions are continuous functions of s at s_0 although $\vec{y}$ changes discontinuously at s_0 as can be seen from (8a).

The continuity can be proven easily if one takes into account the fact that $\vec{\tilde{y}}^{\perp}$ and $\bar{\sigma}$ are continuous at $s = s_0$ (see eq. (3.4), (3.5), (2.3a)) and that

$$\vec{D}^T \cdot \underline{S}\,\vec{y}^{\perp} = \vec{D} \cdot \underline{S} \cdot \vec{\tilde{y}}^{\perp}$$

$$\hat{\underline{F}} \cdot \vec{y} = \begin{pmatrix} 0 \\ \tilde{y}_4 \\ -\tilde{y}_2 \end{pmatrix}$$

and if one rewrites $\underline{K}_1\,\vec{y}$ and $\underline{K}_2\,\vec{y}$ in the form

$$\underline{K}_1\,\vec{y} = \begin{pmatrix} \vec{D} \cdot \vec{D}^T \cdot \underline{S}\,\vec{\tilde{y}}^{\perp} & -\vec{D}\;\bar{\sigma} \\ -\vec{D}^T \cdot \underline{S}\,\vec{\tilde{y}}^{\perp} & +\bar{\sigma} \end{pmatrix} \cdot \frac{e\hat{V}}{E_0} \cdot k \cdot \frac{2\pi}{L}\cos\Phi \tag{9a}$$

$$\underline{K}_2\,\vec{y} = \left(a\gamma_0 + \frac{\gamma_0}{1+\gamma_0}\right) \cdot \frac{e\hat{V}}{E_0} k \cdot \frac{2\pi}{L}\sin\Phi \begin{pmatrix} \ell_\tau & \ell_x & \ell_z \\ -m_\tau & -m_x & -m_z \end{pmatrix} \begin{pmatrix} 0 \\ \tilde{y}_4 \\ -\tilde{y}_2 \end{pmatrix} . \tag{9b}$$

Now, integrating both sides of eq. (8a, b) from $s_0 - \varepsilon$ to $s_0 + \varepsilon$ one immediately obtains ($\varepsilon \to 0$)

$$\vec{y}(s_0 + o) = [\underline{1} + \underline{K}_1(s_0)]\,\vec{y}(s_0 - o) \tag{10a}$$

$$\vec{\zeta}(s_0 + o) = \vec{\zeta}(s_0 - o) + \underline{K}_2(s_0) \cdot \vec{y}(s_0 - o) \tag{10b}$$

From eq. (10a, b) one can then extract the eight-dimensional transfer matrix of a cavity which reads

$$\underline{M}_{(8\times 8)}(s_0+o,\ s_0-o) = \begin{pmatrix} \underline{M}_{(6\times 6)}(s_0+o,\ s_0-o) & \underline{0} \\ \underline{G}(s_0+o,\ s_0-o) & \underline{1}_{(2\times 2)} \end{pmatrix} \tag{11}$$

with

$$\underline{M}_{(6\times 6)}(s_0+o,\ s_0-o) = \underline{1} + \underline{K}_1(s_0) \tag{11a}$$

$$\underline{G}(s_0+o,\ s_0-o) = \underline{K}_2(s_0)\ . \tag{11b}$$

$\underline{K}_1(s_0)$ is defined by eq. (4) and $\underline{K}_2(s_0)$ is given by

$$\underline{K}_2(s_0) = (a\gamma_0 + \frac{\gamma_0}{1+\gamma_0})\cdot\frac{e\hat{V}}{E_0}\sin\Phi\begin{pmatrix} 0 & -\ell_z & 0 & \ell_x & 0 & (D_4\ell_x - D_2\ell_z) \\ 0 & m_z & 0 & -m_x & 0 & -(D_4 m_x - D_2 m_z) \end{pmatrix} \tag{12}$$

(see eq. (6) and (7)).

$\underline{M}_{(6\times 6)}(s_0+o,\ s_0-o)$ in eq. (11) is the transfer matrix of the orbital motion and $\underline{G}(s_0+o,\ s_0-o)$ is the spin-orbit coupling matrix of a cavity.

$\underline{M}_{(6\times 6)}(s_0+o,\ s_0-o)$ is symplectic, i.e. it fulfills the condition

$$\underline{M}^T_{(6\times 6)}(s_0+o,\ s_0-o)\cdot\underline{S}\cdot\underline{M}_{(6\times 6)}(s_0+o,\ s_0-o) = \underline{S} \tag{13}$$

which can be checked easily by putting (11a) into (13).

Eqs. (11a) and (4) also clearly show again that the transverse and longitudinal motions are completely decoupled if the dispersion vector $\vec{D}$ vanishes in the cavity region (see section 4, especially eq. (4.12)).

When this (fully coupled) version of the cavity matrix is used the dispersion formalism will give exactly the same results as the usual six-dimensional formalism of coupled synchro-betatron oscillations[2].

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