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ABSTRACT

Already in 1926, Schrödinger constructed "minimum uncertainty wave packets" for the harmonic oscillator, later popularized as "coherent states", which today are widely used, for instance in quantum optics (laser theory) and in quantum field theory (infrared behaviour in QED and QCD). It is argued that Schrödinger's discovery of coherent states played a crucial role in Heisenberg's discovery of the uncertainty principle in 1927.

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1. Introduction*)

I would like to draw attention to an almost forgotten short note of six pages by Erwin Schrödinger /1/ published on July 9, 1926. The paper belongs to a series of six papers written within the first six months of this year. The other five papers /2/ contain Schrödinger's monumental contribution to wave mechanics, four of which carrying the programmatic title "Quantization as an eigenvalue problem". At the beginning of 1927 the six papers appeared in book form /3/, which may be regarded as the earliest textbook on quantum mechanics. In his obituary for Schrödinger, Max Born said: "Was gibt es Großartigeres in der theoretischen Physik als seine ersten sechs Arbeiten zur Wellenmechanik?" /4/.
Schrödinger's paper /1/ was criticized by Werner Heisenberg /5/ in his famous paper on the uncertainty relation, received March 23, 1927. This seems to be the reason why later authors did not consider the paper worth to be read or to be cited, respectively.

I would like to point out that Schrödinger's paper /1/ contains several important contributions to quantum mechanics in general and, in particular, introduces "coherent states" for the first time. Today, coherent states play an important role in many branches of physics, for instance in quantum optics (laser theory), nuclear physics, high energy physics, superstring theory and in the formulation of Feynman path integrals. Actually, Schrödinger's paper is completely correct with the exception of the very last sentence where Schrödinger makes an incorrect extrapolation from the harmonic oscillator to the "wave groups" for "Kepler ellipses" in the H-atom.

In his paper /1/, Schrödinger wanted to illustrate by the example of "Planck's linear oscillator" that it is always possible to find solutions of his "undulatory mechanics" in the form of well-localized wave packets whose center of gravity oscillates <u>without change of shape</u> with the period of the corresponding classical motion and thus describe the classical trajectory of a point particle like, for example, the Kepler orbits of the electron in the H-atom. <u>Wave packets</u> were introduced for the first time in quantum mechanics by Schrödinger in his second historical paper (see the second paper cited in /2/). The first explicit construction of a wave packet as an exact solution of the time-dependent Schrödinger equation was achieved in /1/. At this occasion Schrödinger introduced and applied for the first time the <u>superposition principle</u>. ("Ähnlich wie bei der Differentialgleichung der schwingenden Saite oder irgendeines anderen schwingenden Systems ergibt sich ψ als Superposition ..." /1/.)

*) The present text is a slightly enlarged version of the original contribution to the Schrödinger Workshop, to be published in the Proceedings.

2. Schrödinger's construction of wave packets for the harmonic oscillator

For the harmonic oscillator with classical energy $(m/2)\dot{q}^2 + (m\omega^2/2)q^2$, Schrödinger /1/ gives the complete solution

$$\begin{cases} \psi_n = e^{-\frac{x^2}{2}} H_n(x)\, e^{i\frac{E_n}{\hbar}t} \\ E_n = \hbar\omega\left(n+\frac{1}{2}\right) \quad ; \; n = 0, 1, 2, \ldots \end{cases} \tag{1}$$

($x = q/\sqrt{\hbar/m\omega}$ = dimensionless elongation, $H_n(x)$ = Hermite polynomials; Schrödinger uses $2\pi\nu_0$ instead of ω and $2\pi\nu_n$ instead of $E_n/\hbar$; the time-independent eigenfunctions were already derived in Schrödinger's second paper /2/). The ψ_n's are orthonormal if multiplied by $(2^n n!)^{-1/2}$ ("Normierungsfaktor").

The unusual sign in the time-dependent phase factor in (1) is explained by a footnote which appeared to the above equation and reads: "i means $\sqrt{-1}$. On the right-hand side the real part is to be taken, as usual". The important question, when did Schrödinger realize that his wave function unavoidably has to be complex, has been addressed by C.N. Yang in a beautiful talk entitled "Square root of minus one, complex phases and Erwin Schrödinger" /6/. Concerning the above mentioned footnote in Schrödinger's paper /1/, Yang states: "revealing his general attitude on this matter, which was the same as in the usual linear circuit theory: ψ may be complex, but one always takes the real part in the end". And somewhat later Yang continues: "What one can be certain is that between June 11 and June 23 Schrödinger was finally convinced that ψ is complex" /6/.*) I disagree with the last conclusion, because it becomes quite obvious from reading the last sentences of Schrödinger's sixth paper /2/, that he disliked the idea of a complex wave function and still preferred "the much more sympathetic interpretation, that the state of the system is given by a real function and its derivative with respect to time". Schrödinger writes at the end of his sixth paper /2/: "Eine gewisse Härte liegt ohne Zweifel zurzeit noch in der Verwendung einer komplexen Wellenfunktion. Würde sie grundsätzlich unvermeidlich und nicht eine bloße Rechenerleichterung sein, so würde das heißen, daß grundsätzlich zwei Wellenfunktionen existieren, die erst zusammen Aufschluß über den Zustand des Systems geben. Diese etwas unsympathische Folgerung läßt, wie ich glaube, die sehr viel sympathischere Deutung zu, daß der Zustand des Systems durch eine reelle Funktion und ihre Ableitung nach der Zeit gegeben ist."

*) Actually, June 23 should read June 21, since Yang /6/ erroneously assumes that Schrödinger's sixth paper (the last paper cited in /2/) was received on June 23; the correct date is June 21, 1926.

Looking into Schrödinger's subsequent papers on wave mechanics, one finds - exactly one year later! - in a paper /7/ received on June 10, 1927 a footnote (after equation (2)) where Schrödinger states for the first time: "Die Wellenfunktion ψ hat als wesentlich komplex zu gelten."

Using his superposition principle, Schrödinger constructs from the eigenfunctions (1) the following wave packet ($A \gg 1$)

$$\psi = \sum_{n=0}^{\infty} \left(\frac{A}{2}\right)^n \frac{\psi_n}{n!} \quad . \tag{2}$$

("Ich wähle eine Zahl $A \gg 1$ ... und bilde folgendes Aggregat von Eigenschwingungen ..., was, wie man leicht überlegt, darauf hinauskommt, eine relativ schmale Gruppe in der Umgebung des n-Wertes $n = A^2/2$ herauszugreifen." /1/). With the help of an identity from Courant-Hilbert, Schrödinger carries out the summation in (2) and (after having taken the real part!) arrives at

$$\psi = e^{\frac{A^2}{4} - \frac{1}{2}(x - A\cos\omega t)^2} \cdot \cos\left[\frac{\omega t}{2} + (A\sin\omega t)\left(x - \frac{A}{2}\cos\omega t\right)\right] \quad . \tag{3}$$

The most striking property of this wave packet derives from the first factor in (3), which has the form of a "Gaussian error curve" with a sharp maximum ($A \gg 1$) at $x = A\cos\omega t$, i.e. at the classical trajectory. Since the main contribution in (2) comes from eigenfunctions around $n = A^2/2 \gg 1$, the energy of the wave packet is given by

$$E \sim \hbar\omega \cdot \frac{A^2}{2} = \frac{m}{2}\omega^2 a^2 \tag{4}$$

which is exactly the classical energy of an oscillator with frequency ω and classical amplitude $a = A\sqrt{\hbar/m\omega}$. Thus the wave packet (3) is the earliest example of an exact result derived from quantum mechanics which satisfies Bohr's "correspondence principle". "Die 'Korrespondenz' ist also auch in dieser Hinsicht eine vollkommene" /1/.

Before we continue with a discussion of Schrödinger's paper, let us add some remarks on the unique role played by the harmonic oscillator. Already in February 1927, Dirac wrote his famous paper on "The quantum theory of the emission and absorption of radiation" /8/ which marks the beginning of quantum electrodynamics. In order to quantize the radiation field, Dirac introduced an infinite set of non-Hermitian operators (annihilation and creation operators a, a^+) satisfying the commutation relation $[a,a^+] = 1$ for Bose operators.*) A few months later, this was generalized by Jordan /9/ to Fermi operators obeying anticommutation relations. Based on these results, Jordan derived in collaboration with Klein and Wigner /10/ the canonical commutation relations for field operators. Many years later, the forced harmonic oscillator played an important role in Feynman's treatment of quantum electrodynamics /11/. The systematic use for Bose systems of states based on non-Hermitian operators is due to Schwinger /12/. Today, states described by wave packets of the form (2) are called "coherent states" and are used in many different fields of physics /13/. Recently, the concept of coherent states was generalized to so-called "squeezed states" ("two-photon states") /14/, which should lead to improved phase sensitivity in interferometers.

3. Schrödinger's missed opportunity and Heisenberg's discovery of the uncertainty principle

Schrödinger writes in /1/: "Unsere Wellengruppe hält dauernd zusammen, breitet sich nicht im Laufe der Zeit auf ein immer größeres Gebiet aus, wie man es sonst, z. B. in der Optik, gewohnt ist." And a few sentences later: " ... eine solche Wellengruppe bleibt dauernd zusammen, im Gegensatz zu einem Wellenpaket der klassischen Optik, das sich im Laufe der Zeit zerstreut. Der Unterschied dürfte davon herrühren, daß unsere Gruppe aus einzelnen diskreten harmonischen Komponenten aufgebaut ist, nicht aus einem Kontinuum von solchen". Thus Schrödinger is completely aware of the fact that a well-localized wave packet in optics spreads so much in one period that the very notion of a classical trajectory looses its meaning, yet he believes that the peculiar behaviour of the wave packet (2) is a general feature of quantum mechanical wave packets and therefore allows the construction of wave groups corresponding to Kepler ellipses in the H-atom. "Es läßt sich mit Bestimmtheit voraussehen, daß man auf ganz ähnliche Weise auch die Wellengruppen konstruieren kann, welche auf hochquantigen Keplerellipsen umlaufen und das undulationsmechanische Bild des Wasserstoffelektrons sind; nur sind da die rechentechnischen Schwierigkeiten größer als in dem hier behandelten, ganz besonders einfachen Schulbeispiel" /1/.

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*) Note added in proof: On October 22, 1987 I learned from Professor S. Schweber [talk given at the "Pascual Jordan Memorial Colloquium" at the University of Hamburg] that the quantization of a system of infinitely many harmonic oscillators was carried out already in 1925 in the famous paper by Born-Heisenberg-Jordan [Z. Physik 35 (1925) 606]. It is found in the last paragraph of that paper, which is entirely due to Jordan, where the energy fluctuations in a field of black-body radiation are calculated.

Schrödinger did not realize that the peculiar properties of his "minimum uncertainty wave packet" are a direct consequence of the fact that the energy levels of the harmonic oscillator are equidistant. But this is a special case, which is only true for the harmonic oscillator, as Heisenberg /5/ observed in his famous paper on the uncertainty relation. Heisenberg writes: "Der Übergang von der Mikro- zur Makromechanik ist schon von Schrödinger behandelt worden, aber ich glaube nicht, daß die Schrödingersche Überlegung das Wesen des Problems trifft ...". Using the transformation theory, independently developed by Dirac and Jordan /15/ in December 1926, Heisenberg derives for a free particle his uncertainty relation.
At this point it is instructive to look a few months back. Already on May 24, 1926 Pauli sent a letter from Hamburg to Schrödinger in which he writes /16/: "Was nun das schwierige Problem der 'Geburt der Elektronenbahnen' bei großen Quantenzahlen betrifft ... . Ich glaube, man macht sich die Sache am einfachsten klar am Fall des kräftefrei translatorisch bewegten Massenpunktes." This is exactly what Heisenberg did! It is almost certain that Pauli and Schrödinger discussed these matters in Zürich during the week June 21 - 26, 1926. (Langevin, Pauli, Sommerfeld, Stern ... came to Zürich on an invitation by Debye). On October 4, 1926 Schrödinger gave a talk in Copenhagen, where Bohr had invited him together with Heisenberg. The almost fanatic discussions between Bohr and Schrödinger have been vividly described by Heisenberg in his autobiography /17/. It is quite obvious, that it was mainly Schrödinger's philosophical point of view which prevented him from forming the idea of quantum mechanical wave packets spreading in time in complete analogy to the optical case well-known to him.

That Heisenberg's discovery of the uncertainty principle has been triggered by Schrödinger's paper /1/, can also clearly be infered from a letter, which Heisenberg /18/ wrote to Pauli on February 23, 1927 - one month before he submitted his paper /5/. Heisenberg writes: "Der Übergang von Mikro- zu Makromechanik. Die folgenden Überlegungen sind mir eigentlich das wichtigste am ganzen: Erst möcht' ich sagen, wie dieser Übergang (von Mikro- zu Makromech.) nicht geht. Schrödinger nimmt an, es sei möglich, Eigenfunktionen des Atoms so zu addieren, daß Wellenpakete resultieren, die für beliebige Zeiten beisammen bleiben und die periodische Bewegung des Elektrons liefern ... . Das tun aber die Eigenfunktionen nie - ausgenommen eben den Spezialfall des Oszillators. Sie tun es aber näherungsweise, aber das heißt nur, daß das Wellenpaket erst nach hinreichend langer Zeit zerläuft ... . Dies zeigt, daß Schrödingers Vorschlag nicht geht, und daß es ganz hoffnungslos scheint, zu einer 'Bahn' zu kommen ... . So werden wir also eine Lösung der quantenmechanischen Gleichungen konstruieren, die zur Zeit t_o für den Ort des Elektrons den Platz q_o mit der betreffenden Ungenauigkeit ergibt; z. B. ein Wellenpaket à la Schrödinger" /18/.

Thus the "wave packets à la Schrödinger" were an important ingredient for Heisenberg's discovery of the uncertainty principle!

4. Epilogue

The purpose of this contribution was two-fold. First, I draw attention to the important contributions which Schrödinger made to quantum mechanics in his nearly unknown paper /1/. For the first time, we find in that paper the formulation of the superposition principle and, as an application of it, the explicit construction of a very special wave packet, known today as a coherent state. Secondly, I discussed the historical fact that Schrödinger had all the necessary ingredients in his hands for discovering the uncertainty principle, but that his philosophical motives acted against such a revolutionary step.

Before closing, I would like to recall that Schrödinger realized very soon the fundamental significance of Heisenberg's discovery, and that he himself derived the most general form of the uncertainty principle /19/ valid for an arbitrary pair of observables.

I would like to close with a quotation from a much later publication by Schrödinger entitled "The meaning of wave mechanics - La signification de la méchanique ondulatoire" which was published in 1953 simultaneously in English and French (on opposite pages!) in the volume "Louis de Broglie, Physicien et Penseur" /20/. Schrödinger writes: "Louis de Broglie's great theoretical discovery of the wave phenomenon associated with the electron ... embraces the entire domain of physics and chemistry, and may be said to hold the field to-day along the whole line, albeit not precisely in the way de Broglie and his early followers had intended. - For it must have given to de Broglie the same shock and disappointment as it gave to me, when we learnt that a sort of transcendental, almost psychical interpretation of the wave phenomenon had been put forward, which was very soon hailed by the majority of leading theorists as the only one reconcilable with experiments, and which has now become the orthodox creed, accepted by almost everybody, with a few notable exceptions. ... M. de Broglie, so I believe, disliked the probability interpretation of wave mechanics as much as I did. But very soon and for a long period one had to give up opposing it, and to accept it as an expedient interim solution. I shall point out some of the reasons why the originally contemplated alternative seemed deceptive and, after

all, too naive . ... As long as a particle, an electron or proton etc., was still believed to be a permanent, individually identifiable entity, it could not adequately be pictured in our mind as a wave parcel. For as a rule, apart from artificially constructed and therefore irrelevant exceptions, no wave parcel can be indicated which does not eventually disperse into an ever increasing volume of space. ... All this was known 25 years ago, and abated the hopes of 'naive' wave-mechanists. The now orthodox view about the wave function as 'probability amplitude' was put forward and was worked out into a scheme of admirable logical consistency."

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