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Path Integral on a Group Manifold and the Lattice Gauge Theory Hamiltonian

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Abstract:

For a quantum mechanical system living on the manifold of a compact simple Lie group we present explicit formulae for the quantum corrections, both in the Hamiltonian and, for the most common time discretization, in the path integral. As a special application of this rather general procedure, we compare, for lattice gauge theories, the path integral corresponding to the Kogut-Süsskind Hamiltonian and the Wilson action. The latter is shown to correspond to a very special but elegant way of discretizing the time variable.

I. Introduction

Recently these has been some renewed interest in the path integral formulation of quantum mechanical systems and quantum field theories on curved spaces, or, more general, on topologically nontrivial manifolds. In quantum field theory Christ and Lee /1/ several years ago pointed out that, when formulating Yang-Mills theories in the Coulomb gauge, new terms appear in the action integral. They result from a nontrivial metric in the space of the gauge-fixed field variables /2-4/, and they lead, in perturbation theory, to new interaction vertices. These terms had been overlooked before /5/. In the context of quantum mechanical problems there has been some recent progress in calculating path integrals, which now allows to handle quite a few problems in the path integral formulation which had been untractable before /6, 7 and refs. therein/. In most of these cases symmetries are playing an essential rôle. This motivates a strong interest in formulating path integrals on group manifolds /8, 9/. In string theory one faces the task of doing quantum mechanics on topologically nontrivial manifolds (e.g. Riemann manifolds with nonzero genus). Although it may be too much to expect that one might be able to write down a closed expression for the action integral on the whole manifold, one should be able to formulate the theory, at least, on coordinate patches. The basic task then is the same as in the other examples: to handle the path integral of a quantum mechanical system on a manifold with a nonflat metric, e.g. on group manifolds.

There exists a well-established procedure for both canonical quantization /10 - 12/ and writing down the path integral for quantum mechanics on curved spaces /13 - 21/. In refs. /1-4/ this formalism has partially been used to determine the correct Yang-Mills Hamiltonian in the Coulomb gauge. In the context of quantum mechanical problems,

however, not much use has been made yet of this procedure. In refs. /6, 7/ special cases of group manifolds have been studied, such as SU(2), SO(n), SU(1,1) and SO(n,m). All these group manifolds have in common that they can easily be embedded in euclidian (or pseudo-euclidian) flat space, and this special property has been made use of for deriving the path integral. It is clear that this method does not work for other groups of interest /8/. There is also a potential danger in this way of deriving the path integral /3/. It therefore seems very much preferable to directly use the standard procedure, which is always applicable /9/. In the first part of this paper we perform, in a rather explicit manner, both canonical quantization and the derivation of the path integral for a general compact simple Lie group, following the standard routine of refs. /13-21/. In particular, we explicitly calculate the quantum corrections which are necessary for the correct formulation of the quantum theory.

There is an interesting application in lattice gauge theories, namely the interrelation between the Wilson action and the corresponding lattice Hamiltonian. Usually a lattice gauge theory is defined through a partition function on a 4-dimensional euclidian lattice, using the Wilson action in the Boltzmann-factor. In order to derive the corresponding Hamiltonian, one singles out the time direction of the lattice, defines the transfer matrix and finally takes the lattice spacing in time direction to zero /22, 23/. For the simplest nonabelian case of SU(2) the Hamiltonian has been found by Kogut and Susskind /24/

$$H = -\frac{g_t^2}{2a_s} \sum_{\{i,j\}} \Delta_{LB}\{i,j\} + \frac{2}{g_s^2 a_s} V . \tag{1.1}$$

Here the kinetic term (electric energy) comes as a sum over all links, and for each link we have the Laplace-Beltrami operator on the SU(2) group manifold S_3. The second term (magnetic energy) plays the rôle of the potential which depends on all the link variables. g_t is the coupling constant in time direction and a_s is the spacial lattice spacing. The (asymmetric) Wilson action in the temporal gauge (a_t is the lattice spacing in time direction)

$$S = \sum_k \left\{ \frac{2a_s}{g_t^2 a_t} \sum_{\{i,j\}} Re\, Tr\left[1 - U^{-1}_{ij}(k+1)\, U_{ij}(k)\right] - \frac{2a_t}{g_s^2 a_s} V \right\} \tag{1.2}$$

can be viewed as the action of the quantum mechanical path integral of eq. (1.1): each link variable U_{ij} is like a particle with mass $m = \frac{a_s}{g_t^2}$ living on the SU(2) manifold. If we now apply to eq. (1.1) the standard procedure for deriving the path integral, the resulting quantum action will, in general, look much more complicated than the Wilson action (eq. (1.2)). We shall show, however, that the path integral based upon eq. (1.2) matches the Hamiltonian (1.1): the Wilson action "has chosen" a very clever way of disretizing time. From the point of view of the standard routine this discretization scheme may look peculiar, but the simplicitly of eq. (1.2) is certainly striking. We shall show that the equivalence of eqs. (1.1) and (1.2) generalizes to any compact simple Lie group. We thus end with the conclusion that, with a time discretization scheme which at first sight looks complicated, the path integral can always be cast into the elegant "Wilson form". As a by-product, we present a device for finding rather easily the lattice Hamiltonian for Lie groups other than SU(2).

This paper is organized as follows. We first (section II) review, for a general compact simple Lie group, how canonical quantization is done and how the path integral is derived. In section III we then turn to lattice gauge theories and study the interrelation between the Wilson action and the lattice Hamiltonian. In an appendix we briefly outline how the (mostly well-known) results for SU(2) are reproduced.

II. Quantum Theory on a Group Manifold

a) Metrical Quantities of the Group

In the following we consider a compact simple Lie group G . It may be construed as a differentiable manifold M furnished with a group structure. Elements of G correspond to points on M and may be parametrized in terms of the real coordinates ω^1. As usual $\omega = 0$ determines the unity element of the group. The dimension n of the manifold is identical to that of the group (i.e. the dimension of the associated Lie algebra as a vector space). The group structure is fixed if we know the composition function ϕ which determines the group multiplication. Let $\omega_1^{\,1}$ and $\omega_2^{\,1}$ be the parameters of two elements g_1 and g_2 of G . Then

$$\omega^\ell = \phi^\ell(\omega_1, \omega_2), \quad \ell = 1, \dots, n \tag{2.1}$$

corresponds to the product $g = g_1 g_2$ of these two elements. The left auxiliary functions are defined as /25/,

$$\eta^{\ell\ell_1}(\omega) = \left.\frac{\partial \phi^\ell(\omega, \omega_1)}{\partial \omega_1^{\ell_1}}\right|_{\omega_1 = 0} \quad . \tag{2.2}$$

The inverse of η are the components of the Maurer Cartan form σ :

$$\eta^{\ell_1\ell}\, \sigma^{\ell\ell_2} = \delta^{\ell_1\ell_2} \quad . \tag{2.3}$$

The associativity of the group multiplication

$$\phi(\omega_3, \phi(\omega_2, \omega_1)) = \phi(\phi(\omega_3, \omega_2), \omega_1) \tag{2.4}$$

leads via differentiation to Lie's $2^{\underline{nd}}$ theorem

$$\eta^{\ell\ell_1}\partial_\ell\, \eta^{\ell_3\ell_2} - \eta^{\ell\ell_2}\partial_\ell\, \eta^{\ell_3\ell_1} = f^{\ell_1\ell_2\ell_4}\, \eta^{\ell_3\ell_4} \tag{2.5}$$

where ∂_ℓ denotes $\frac{\partial}{\partial \omega^\ell}$. In terms of the σ-fields this relation is also called the Maurer-Cartan equation:

$$\partial_{\ell_3} \sigma^{\ell_1 \ell_2} - \partial_{\ell_2} \sigma^{\ell_1 \ell_3} = f^{\ell_1 \ell_4 \ell_5} \sigma^{\ell_4 \ell_2} \sigma^{\ell_5 \ell_3} . \tag{2.6}$$

In eqs. (2.5) and (2.6), $f^{l_1 l_2 l_3}$ are the structure constants of the group. Since we are considering compact simple groups, the structure constants can be chosen to be totally antisymmetric.

Now we consider a matrix representation $U(\omega)$ of G, satisfying (without loss of generality)

$$\partial_\ell \mathcal{U}\Big|_{\omega=0} = -i\, T^\ell \tag{2.7}$$

where T^l are the generators of the group in the chosen representation. The T^l satisfy the commutation relation

$$[T^{\ell_1}, T^{\ell_2}] = i f^{\ell_1 \ell_2 \ell_3} T^{\ell_3} , \tag{2.8}$$

and they are normalized to

$$\mathrm{Tr}\, T^{\ell_1} T^{\ell_2} = \frac{1}{2} \delta^{\ell_1 \ell_2} . \tag{2.9}$$

The left auxiliary functions are related to derivatives of U. This can be seen by differentiating

$$\mathcal{U}(\phi(\omega_1, \omega_2)) = \mathcal{U}(\omega_1)\, \mathcal{U}(\omega_2) \tag{2.10}$$

with respect to ω_2 at $\omega_2 = 0$. This yields

$$\eta^{\ell_2 \ell_1} \partial_{\ell_2} \mathcal{U} = -i\, \mathcal{U}\, T^{\ell_1} , \tag{2.11}$$

or, in terms of σ (*)

$$\sigma^{\ell_1 \ell} T^{\ell_1} = i\, \mathcal{U}^{-1} \partial_\ell \mathcal{U} . \tag{2.12}$$

Eqs. (2.11) and (2.12) imply that

$$L^\ell = \eta^{\ell_1 \ell} \partial_{\ell_1} \tag{2.13}$$

is the infinitesimal generator of transformations via group multiplication from the right: $U \to U\,U'$. Note that σ (and therefore also η and L) are invariant under global transformations from the left: $U \longrightarrow U'U$. The role of the left and right is reversed, when in eq. (2.2) the derivative of ϕ is taken with respect to the first argument. This would lead to right invariant $\bar\sigma$'s and $\bar\eta$'s and to the right invariant generators of left multiplications. In this sense left and right multiplications are completely on an equal footing. However, it is sufficient to use only one set of functions, either η and σ or $\bar\eta$ and $\bar\sigma$, since they are not independent: in a representation U, $\bar\sigma$ is given by

$$(\partial_\ell \mathcal{U})\,\mathcal{U}^{-1} = -i\, \bar\sigma^{\ell_1 \ell} T^{\ell_1} \Rightarrow \bar\sigma^{\ell_1 \ell} = R^{\ell_1 \ell_2} \sigma^{\ell_2 \ell} , \tag{2.14}$$

where the orthogonal matrix $R^{l_1 l_2}$ is defined by:

$$\mathcal{U}^{-1} T^{\ell_1} \mathcal{U} = R^{\ell_1 \ell_2}\, T^{\ell_2} \tag{2.15}$$

Note that this matrix R does not depend on the representation U. It is common to use η and σ instead of $\bar\eta$ and $\bar\sigma$, which gives rise to an artificial asymmetry.

The "natural" metric on M is expected to be invariant under global both left and right multiplication. If G is a simple compact group, this bi-invariant metric is unique up to multiplication by a positive constant /26/:

(*) Eq. (2.13) shows that $\sigma^{l_1 l} d\omega^l$ indeed is the Maurer Cartan 1-form: $(\sigma^{\ell_1 \ell} d\omega^\ell) T^{\ell_1} = i\, \mathcal{U}^{-1} d\, \mathcal{U}$ where d is the exterior derivative on M /25/.

$$g_{\ell_1\ell_2} = \sigma^{\ell\ell_1}\sigma^{\ell\ell_2} = \bar{\sigma}^{\ell\ell_1}\bar{\sigma}^{\ell\ell_2} \quad , \tag{2.16}$$

Our choice of the multiplicative constant is motivated by the desire that σ (and not a multiple of σ) can be interpreted as a vielbein field. Then the inverse of the metric reads:

$$g^{\ell_1\ell_2} = \eta^{\ell_1\ell}\eta^{\ell_2\ell} \quad . \tag{2.17}$$

Note that this definition (eq. (2.16)) of the metric is independent of any representation. For a given representation U, satisfying eqs. (2-7) - (2.9), the metric can also be written as:

$$g_{\ell_1\ell_2} = -2\,\mathrm{Tr}\,[U^{-1}(\partial_{\ell_1}U)U^{-1}(\partial_{\ell_2}U)] \; . \tag{2.18}$$

For the purpose of quantization we also need the Christoffel symbols and the scalar curvature. We use the conventions

$$\Gamma_{\ell_1}{}^{\ell_3}{}_{\ell_2} = \frac{1}{2}g^{\ell_3\ell_4}(\partial_{\ell_1}g_{\ell_2\ell_4} + \partial_{\ell_2}g_{\ell_1\ell_4} - \partial_{\ell_4}g_{\ell_1\ell_2}) \; , \tag{2.19}$$

$$R = g^{\ell_1\ell_2}(\partial_{\ell_3}\Gamma_{\ell_1}{}^{\ell_3}{}_{\ell_2} - \partial_{\ell_1}\Gamma_{\ell_2}{}^{\ell_3}{}_{\ell_3} + \Gamma_{\ell_1}{}^{\ell_3}{}_{\ell_2}\Gamma_{\ell_3}{}^{\ell_4}{}_{\ell_4} - \Gamma_{\ell_1}{}^{\ell_3}{}_{\ell_4}\Gamma_{\ell_2}{}^{\ell_4}{}_{\ell_3}) \; . \tag{2.20}$$

Making use of eqs. (2.5) and (2.6), these quantities can be calculated to be

$$\Gamma_{\ell_1}{}^{\ell_3}{}_{\ell_2} = \frac{1}{2}\eta^{\ell_3\ell_4}(\partial_{\ell_1}\sigma^{\ell_4\ell_2} + \partial_{\ell_2}\sigma^{\ell_4\ell_1}) = \eta^{\ell_3\ell_4}(\partial_{\ell_1}\sigma^{\ell_4\ell_2} + \frac{1}{2}f^{\ell_4\ell_5\ell_6}\sigma^{\ell_5\ell_1}\sigma^{\ell_6\ell_2}) \tag{2.21}$$

$$R = \frac{1}{4}f^{\ell_1\ell_2\ell_3}f^{\ell_1\ell_2\ell_3} \tag{2.22}$$

Now we have all the tools at our disposal which are needed for the quantization procedure. We finally note that many of these calculations greatly simplify if one makes use of the calculus of differential forms /25, 27/. For our purposes, however, it will be more appropriate to stay in the component formulation: this applies, in particular, to the calculation of the quantum corrections in the path integral.

Since the left auxiliary functions η and correspondingly all metrical quantities are not explicitly known except for U(1) and SU(2), we now want to give a Taylor expansion of η . It can be obtained from the Campbell-Baker-Hausdorff formula (for a recent account, see /28/)

$$\ln e^{A}e^{B} = A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] - \frac{1}{720}[A,[A,[A,[A,B]]]] + \ldots \quad , \tag{2.23}$$

where the dots indicate higher orders in B and higher commutators. Identifying $A = -i\omega_1{}^{\ell}T^{\ell}$, $B = -i\omega_2{}^{\ell}T^{\ell}$ and differentiating with respect to $\omega_2{}^{1}$ we obtain

$$\eta^{\ell_1\ell}(\omega) = \delta^{\ell_1\ell} + \frac{1}{2}f^{\ell_1\ell_2\ell}\omega^{\ell_2} + \frac{1}{12}f^{\ell_1\ell_2\ell_3}f^{\ell_3\ell_4\ell}\omega^{\ell_2}\omega^{\ell_4} - \frac{1}{720}f^{\ell_1\ell_2\ell_3}f^{\ell_3\ell_4\ell_5}f^{\ell_5\ell_6\ell_7}f^{\ell_7\ell_8\ell}\omega^{\ell_2}\omega^{\ell_4}\omega^{\ell_6}\omega^{\ell_8} + O(\omega^6) \quad . \tag{2.24}$$

This leads to the approximate expression for the metric:

$$g^{\ell_1\ell} = \delta^{\ell_1\ell} - \frac{1}{12}f^{\ell_1\ell_2\ell_3}f^{\ell_3\ell_4\ell}\omega^{\ell_2}\omega^{\ell_4} + \frac{1}{240}f^{\ell_1\ell_2\ell_3}f^{\ell_3\ell_4\ell_5}f^{\ell_5\ell_6\ell_7}f^{\ell_7\ell_8\ell}\omega^{\ell_2}\omega^{\ell_4}\omega^{\ell_6}\omega^{\ell_8} + O(\omega^6) \quad , \tag{2.25}$$

which may be used in order to calculate approximations to all metrical quantities. Clearly, this form of $\eta^{\ell_1 \ell}$ and $g^{\ell_1 \ell}$ is only valid for the parametrization $U = e^{-i\omega^\ell T^\ell}$. Any other parametrization would imply different composition functions and therefore different metrical quantities.

b) Canonical Quantization

Since the correct quantization procedure in spite of being well established for a long time /10, 12/, is not yet sufficiently well-known, it may be useful to recapitulate the basic ideas. First let us consider canonical quantization. The classical kinetic Hamiltonian is of the form $\frac{1}{2}\tilde{g}^{lm} p_l p_m$ where p_l are the canonical momenta and $\tilde{g}^{lm}$ is the metric (rescaled by the mass), which in general depends on the coordinates q^l. Upon quantization, q and p become operators obeying the canonical commutation relations. This leads to an operator ordering ambiguity in the kinetic Hamiltonian. It can be resolved by requiring that the Hamilton operator has the correct classical limit, that it is invariant with respect to arbitrary coordinate transformations and, finally, that it is hermitean with respect to the canonical integration measure $d^n q\sqrt{g}$. The unique result is[(*)]

$$H = \frac{1}{2} g^{-\frac{1}{4}} p_{\ell_1} \tilde{g}^{\ell_1 \ell_2} g^{\frac{1}{2}} p_{\ell_2} g^{-\frac{1}{4}} + V \tag{2.26}$$

where
$$\tilde{g}_{\ell_1 \ell_2} = m\, g_{\ell_1 \ell_2} \quad , \quad g = \det(g_{\ell_1 \ell_2}) \tag{2.27}$$

and V is assumed to depend only on q. In the coordinate representation the momenta are

(*) We ignore a possible curvature term, which is present in ref. /10/, but forbidden in ref. /11/, since, for our case, the curvature is a constant anyway.

$$p_\ell = -i\, g^{-\frac{1}{4}} \partial_\ell\, g^{\frac{1}{4}} = -i\,\partial_\ell - \frac{i}{4} g^{\ell_1 \ell_2}(\partial_\ell g_{\ell_2 \ell_1}) . \tag{2.28}$$

This again follows from the hermiticity requirement with respect to the canonical integration measure. In this representation the kinetic Hamiltonian is the Laplace-Beltrami operator

$$\begin{aligned} H_{kin} &= -\frac{1}{2} g^{-\frac{1}{2}} \partial_{\ell_1} \tilde{g}^{\ell_1 \ell_2} g^{\frac{1}{2}} \partial_{\ell_2} \\ &= -\frac{1}{2} \tilde{g}^{\ell_1 \ell_2} \partial_{\ell_1} \partial_{\ell_2} + \frac{1}{2} \tilde{g}^{\ell_1 \ell_2} \Gamma_{\ell_1}{}^{\ell_3}{}_{\ell_2} \partial_{\ell_3} . \end{aligned} \tag{2.29}$$

The form of H given in eq. (2.26) is not suitable for practical purposes. Hence it must be reordered according to an appropriate scheme /19/. We shall make use of the standard and the Weyl ordered form of H_{kin}:

standard:

$$H_{kin} = \frac{1}{2} \tilde{g}^{\ell_1 \ell_2} p_{\ell_1} p_{\ell_2} - \frac{i}{2}(\partial_{\ell_1} \tilde{g}^{\ell_1 \ell_2}) p_{\ell_2} - \frac{1}{8}(\partial_{\ell_1} \partial_{\ell_2} \tilde{g}^{\ell_1 \ell_2}) + \Delta V_1 , \tag{2.30}$$

Weyl:

$$H_{kin} = \frac{1}{8}(\tilde{g}^{\ell_1 \ell_2} p_{\ell_1} p_{\ell_2} + 2 p_{\ell_1} \tilde{g}^{\ell_1 \ell_2} p_{\ell_2} + p_{\ell_1} p_{\ell_2} \tilde{g}^{\ell_1 \ell_2}) + \Delta V_1 . \tag{2.31}$$

In both cases:

$$\Delta V_1 = \frac{1}{8}(\tilde{g}^{\ell_1 \ell_2} \Gamma_{\ell_1}{}^{\ell_3}{}_{\ell_4} \Gamma_{\ell_2}{}^{\ell_4}{}_{\ell_3} - \tilde{R}) . \tag{2.32}$$

On the group manifold, g, Γ and R are given by eqs. (2.17), (2.21) and (2.22), respectively. Now ΔV_1 can be calculated to be:

$$\Delta V_1 = \frac{1}{8m} (\partial_{\ell_1} \eta^{\ell_2 \ell})(\partial_{\ell_2} \eta^{\ell_1 \ell}) . \tag{2.33}$$

Our final item in the canonical context are the generators of group transformations.

The hermitian left-invariant generators of right multiplications and the right-invariant generators of left multiplications are

$$\hat{L}^{\ell_1} = -\frac{1}{2}(\eta^{\ell_2\ell_1} p_{\ell_2} + p_{\ell_2}\eta^{\ell_2\ell_1})$$

$$= -\eta^{\ell_2\ell_1} p_{\ell_2} + \frac{i}{2}(\partial_{\ell_2}\eta^{\ell_2\ell_1}) \quad , \tag{2.34}$$

$$\hat{R}^{\ell_1} = -\frac{1}{2}(\bar{\eta}^{\ell_2\ell_1} p_{\ell_2} + p_{\ell_2}\bar{\eta}^{\ell_2\ell_1})$$

$$= R^{\ell_1\ell_2}\hat{L}^{\ell_2} \quad , \tag{2.35}$$

respectively. In the coordinate representation $\hat{L}$ agrees, up to a factor of i, with eq. (2.13). The generators satisfy the algebra

$$[\hat{L}^{\ell_1}, \hat{L}^{\ell_2}] = i f^{\ell_1\ell_2\ell_3}\hat{L}^{\ell_3} \quad ,$$

$$[\hat{R}^{\ell_1}, \hat{R}^{\ell_2}] = -i f^{\ell_1\ell_2\ell_3}\hat{R}^{\ell_3} \quad ,$$

$$[\hat{L}^{\ell}, \hat{L}^{2}] = [\hat{L}^{\ell_1}, \hat{R}^{\ell_2}] = [\hat{R}^{\ell}, \hat{R}^{2}] = 0 \quad . \tag{2.36}$$

For a representation U with eqs. (2.7) - (2.9) application of the generators yields:

$$[\hat{L}^{\ell}, \mathcal{U}] = \mathcal{U} T^{\ell} \quad , \quad [\hat{R}^{\ell}, \mathcal{U}] = T^{\ell}\mathcal{U} \quad . \tag{2.37}$$

The quadratic Casimir operators of $\hat{R}$ and $\hat{L}$ coincide:

$$\hat{R}^2 = \hat{L}^2$$

$$= g^{\ell_1\ell_2} p_{\ell_1} p_{\ell_2} - i(\eta^{\ell_2\ell_1}\partial_{\ell_2}\eta^{\ell_3\ell_1} + \eta^{\ell_3\ell_1}\partial_{\ell_2}\eta^{\ell_2\ell_1}) p_{\ell_3}$$

$$-\frac{1}{2}\eta^{\ell_2\ell_1}(\partial_{\ell_2}\partial_{\ell_3}\eta^{\ell_3\ell_1}) - \frac{1}{4}(\partial_{\ell_2}\eta^{\ell_2\ell_1})(\partial_{\ell_3}\eta^{\ell_3\ell_1}) \quad . \tag{2.38}$$

Comparing this to eqs. (2.27) and (2.30), we find

$$H_{kin} = \frac{1}{2m}\hat{L}^2 \quad , \tag{2.39}$$

i.e. the kinetic part of the Hamilton operator is just the (unique) quadratic Casimir operator of the group /25/.

c) <u>Path Integral Quantization</u>

Let us now turn to the path integral description of the quantum system. The path integral approach to the quantization on curved spaces (or on flat spaces in non cartesian coordinates) has also been known for many years /13-21/. Although it is well understood by now, it may - especially for the community of particle physicists - be helpful to present the procedure in some detail(*).

The path integral is a device for the calculation of the probability that a state $|q'\rangle$ at a time t' evolves into a state $|q''\rangle$ at a later time t":

$$PI = \langle q'', t'' | q', t' \rangle = \langle q'' | e^{-iH(t''-t')} | q' \rangle \quad . \tag{2.40}$$

Usually the first step for the evaluation of PI is the insertion of intermediate states. For this purpose we need the completeness-relation for the q-eigenstates. Using the natural (geometrical) integration measure, it reads

$$1 = \int d^n q \; g^{\frac{1}{2}}(q) \, |q\rangle\langle q| \quad , \tag{2.41}$$

(*) Most textbooks describe the path integral approach only in flat systems using cartesian coordinates. One of the exceptions, the textbook of T.D. Lee /29/, uses a normalization which we consider to be somewhat unnatural: instead of our eq. (2.42) he uses

$$\langle \psi | \chi \rangle = \int d^n q \; \psi^*(q)\chi(q) \; ,$$

i.e. the integration measure is not invarant under general coordinates transformations.

which implies

$$\langle \psi | \chi \rangle = \int d^n q \; g^{\frac{1}{2}}(q) \; \psi^*(q) \, \chi(q) \; , \tag{2.42}$$

i.e. the scalar product of two wave functions contains the coordinate invariant integration measure. The coordinate representation of the momentum operator (eq. (2.28)) determines the p eigenfunction in the coordinate representation:

$$\langle q | p \rangle = g^{-\frac{1}{4}}(q) \; e^{i p_\ell q^\ell} \; , \tag{2.43}$$

which, in turn, fixes the completeness for the p eigenstates and the normalization of all states:

$$1 = \int \frac{d^n p}{(2\pi)^n} \; |p\rangle\langle p| \; , \tag{2.44}$$

$$\langle q_1 | q_2 \rangle = g^{-\frac{1}{4}}(q_1) g^{-\frac{1}{4}}(q_2) \int \frac{d^n p}{(2\pi)^n} \; e^{i p_\ell (q_1 - q_2)^\ell} \; , \tag{2.45}$$

$$\langle p_1 | p_2 \rangle = \int d^n q \; e^{i q^\ell (p_2 - p_1)_\ell} \; . \tag{2.46}$$

Dividing the time interal t"-t' into N equal parts of size ε, the path integral may be written as $\left(e^a = \lim\limits_{N\to\infty} (1 + \frac{a}{N})^N \right)$

$$PI = \lim_{N\to\infty} g^{-\frac{1}{4}}(q'') g^{-\frac{1}{4}}(q') \cdot \int \left(\prod_{k=1}^{N-1} d^n q_k \right) \cdot$$

$$\cdot \prod_{k=0}^{N-1} \left[g^{\frac{1}{4}}(q_{k+1}) \, g^{\frac{1}{4}}(q_k) \, \langle q_{k+1} | 1 - i \varepsilon H | q_k \rangle \right] \; , \tag{2.47}$$

where $q_0 = q'$, $q_N = q''$. Hence we have to evaluate the short time kernel

$$K(k+1,k) := g^{\frac{1}{4}}(q_{k+1}) \, g^{\frac{1}{4}}(q_k) \, \langle q_{k+1} | 1 - i\varepsilon H | q_k \rangle \tag{2.48}$$

$$= \int \frac{d^n p}{(2\pi)^n} \; e^{i p_\ell (q_{k+1} - q_k)^\ell}$$

$$- i \varepsilon \, g^{\frac{1}{4}}(q_{k+1}) \, g^{\frac{1}{4}}(q_k) \, \langle q_{k+1} | H | q_k \rangle \; . \tag{2.49}$$

The potential contained in H does not provide any difficulties, since it depends only on q. The kinetic part, however, cannot be used in the form of eq. (2.26). Instead, it has to be reordered into a form belonging to a "λ -ordering" /19/. The most convenient (and most commonly used) one is the Weyl-ordering scheme, defined by:

$$\{ p^r q^m \}_W = \frac{1}{2^m} \sum_{\ell=0}^{m} \frac{m!}{\ell!\,(m-\ell)!} \; q^{m-\ell} p^r q^\ell \; . \tag{2.50}$$

It can be shown that for r = 2 this reduces to eq. (2.31) (this is posed as an excercise in Lee's book /29/). Therefore we have:

$$\langle q_{k+1} | H_{kin} | q_k \rangle = \langle q_{k+1} | \tfrac{1}{2} \{ \tilde{g}^{\ell_1 \ell_2} p_{\ell_1} p_{\ell_2} \}_W + \Delta V_1(q) | q_k \rangle \tag{2.51}$$

$\tilde{g}^{l_1 l_2}$ is expanded in a power series in q, and we use

$$\langle q_{k+1} | \{ q^m p^r \}_W | q_k \rangle$$

$$= \bar{q}_k^{\,m} \cdot g^{-\frac{1}{4}}(q_{k+1}) \, g^{-\frac{1}{4}}(q_k) \int \frac{d^n p}{(2\pi)^n} \; p^r \, e^{i p \cdot \Delta_k} \tag{2.52}$$

where $\bar{q}_k := \frac{1}{2}(q_{k+1} + q_k)$ and $\Delta_k := q_{k+1} - q_k$. This gives for the short time kernel:

$$K(k+1,k) = \int \frac{d^n p}{(2\pi)^n} \; e^{i p_\ell \Delta_k^\ell} \cdot$$

$$\left[1 - i\varepsilon \left(\tfrac{1}{2} \tilde{g}^{\ell_1 \ell_2}(\bar{q}_k) p_{\ell_1} p_{\ell_2} + \Delta V_1 + V \right) \right] \; , \tag{2.53}$$

where ΔV_1 and the potential V may be taken at arbitrary q, for instance at $\bar{q}$. From here it is obvious that Weyl-ordering of the Hamiltonian corresponds to a "midpoint discretization" of the metric. Hence there is a one-to-one correspondence between discretization and the quantum correction ΔV_1. As an example for another discretization scheme, standard ordering would have given

$$K(k+1,k) = \int \frac{d^n p}{(2\pi)^n}\, e^{i p_\ell \Delta_k^{\ell}} \cdot$$

$$\Big[1 - i\varepsilon\Big(\tfrac{1}{2}\tilde{g}^{\ell_1\ell_2}(q_{k+1})\, p_{\ell_1} p_{\ell_2} - \tfrac{i}{2}(\partial_{\ell_1}\tilde{g}^{\ell_1\ell_2})(q_{k+1})\, p_{\ell_2}$$

$$- \tfrac{1}{8}\partial_{\ell_1}\partial_{\ell_2}\tilde{g}^{\ell_1\ell_2} + \Delta V_1 + V\Big)\Big]\,. \tag{2.54}$$

Since in the path integral we only need to be precise up to $O(\varepsilon)$, the integrand of the kernel is exponentiated and the canonical (phase-space) form of the path integral reads:

$$PI = g^{-\frac{1}{4}}(q'')\, g^{-\frac{1}{4}}(q') \lim_{N\to\infty} \int \Big(\prod_{k=1}^{N-1} d^n q_k\Big)\Big(\prod_{k=0}^{N-1} \frac{d^n p^k}{(2\pi)^n}\Big)\cdot$$

$$\cdot \exp\Big\{ i \sum_{k=0}^{N-1} \Big[p^k_\ell \Delta^\ell_k - \frac{\varepsilon}{2}\tilde{g}^{\ell_1\ell_2}(\bar{q}_k)\, p^k_{\ell_1} p^k_{\ell_2} - \varepsilon \Delta V_1 - \varepsilon V\Big]\Big\}\,. \tag{2.55}$$

The p-integration is gaussian and can be performed

$$PI = g^{-\frac{1}{4}}(q'')\, g^{-\frac{1}{4}}(q') \lim_{N\to\infty} \int \Big(\prod_{k=1}^{N-1} d^n q_k\Big) \cdot \prod_{k=0}^{N-1} \Big\{ g^{\frac{1}{2}}(\bar{q}_k)\,\cdot$$

$$\cdot \exp i \Big[\frac{1}{2\varepsilon}\tilde{g}_{\ell_1\ell_2}(\bar{q}_k)\Delta_k^{\ell_1}\Delta_k^{\ell_2} - \varepsilon\Delta V_1 - \varepsilon V\Big]\Big\}\,. \tag{2.56}$$

where a constant normalization factor has been omitted. Eq. (2.56) shows that Δ is

of order $\sqrt{\varepsilon}$, since $d^n q_{k+1}\, d^n q_k = d^n \bar{q}_k\, d^n \Delta_k$ and

$$\int d^n\Delta\; \Delta^{\ell_1}\Delta^{\ell_2}\, e^{\frac{i}{2\varepsilon}\tilde{g}_{\ell_3\ell_4}\Delta^{\ell_3}\Delta^{\ell_4}}$$

$$= i\varepsilon\, \tilde{g}^{\ell_1\ell_2} \int d^n\Delta\; e^{\frac{i}{2\varepsilon}\tilde{g}_{\ell_3\ell_4}\Delta^{\ell_3}\Delta^{\ell_4}}\,. \tag{2.57}$$

In the context of path integrals this equation is often abbreviated by the symbol $\doteq$ /13/:

$$\Delta^{\ell_1}\Delta^{\ell_2} \doteq i\varepsilon\, \tilde{g}^{\ell_1\ell_2}\,, \tag{2.58}$$

where the inverse metric is taken at $\bar{q}$ of the corresponding time slice.

In order to compare eq. (2.56) to the usual covariant (configuration space) path integral, which has the integration measure $\prod_k d^n q_k \sqrt{g(q_k)}$, we still have to manipulate the measure. By Taylor expansion of $g(q_k)$ and $g(q_{k+1})$ around $\bar{q}_k$ up to order ε (i.e. Δ^2), we obtain:

$$g^{\frac{1}{2}}(\bar{q}_k) = g^{\frac{1}{4}}(q_{k+1})\, g^{\frac{1}{4}}(q_k)\Big(1 - \tfrac{1}{8}\Delta_k^{\ell_1}\Delta_k^{\ell_2}\,\partial_{\ell_1}\Gamma_{\ell_2}{}^{\ell_3}{}_{\ell_3}\Big)$$

$$\doteq g^{\frac{1}{4}}(q_{k+1})\, g^{\frac{1}{4}}(q_k)\Big(1 - \frac{i\varepsilon}{8}\,\tilde{g}^{\ell_1\ell_2}\,\partial_{\ell_1}\Gamma_{\ell_2}{}^{\ell_3}{}_{\ell_3}\Big)\,. \tag{2.59}$$

This is exponentiated and inserted into eq. (2.56):

$$PI = \lim_{N\to\infty} \int \prod_{k=1}^{N-1} \Big(d^n q_k\; g^{\frac{1}{2}}(q_k)\Big)\cdot$$

$$\cdot \exp\Big\{ i \sum_{k=0}^{N-1} \Big[\frac{1}{2\varepsilon}\tilde{g}_{\ell_1\ell_2}(\bar{q}_k)\Delta_k^{\ell_1}\Delta_k^{\ell_2} - \varepsilon\Delta V - \varepsilon V\Big]\Big\}\,, \tag{2.60}$$

where

$$\Delta V = \frac{1}{8}\,\tilde{g}^{\ell_1\ell_2}\,\partial_{\ell_1}\Gamma_{\ell_2}{}^{\ell_3}{}_{\ell_3} + \Delta V_1 \tag{2.61}$$

and again we have omitted the constant normalization factor.

Eq. (2.60) requires some discussion. Most important, the path integral is <u>not</u> simply the integral over the field variables of the exponential of the classical action. An additional term ΔV has appeared which has to be interpreted as a quantum correction, since it has its origin in the noncommutativity of the operators p and q. Our discussion shows that this new term strongly depends upon the way in which time is discretized. We have started from the Weyl-ordered Hamiltonian, and in our result g is taken at the midpoint value $\bar{q}$. Another ordering could have led, for example, to $g(q_{k+1})$ ("standard ordering") or $g(q_k)$ ("antistandard ordering"), and in each case we would have found a different result for ΔV. Therefore, writing the exponent of eq. (2.60) as an integral (without further specification)

$$S = \int dt \left(\frac{1}{2} \tilde{g}_{l_1 l_2} \dot{q}^{l_1} \dot{q}^{l_2} - V - \Delta V \right) \tag{2.62}$$

is extremely misleading: the kinetic term seems to be the same in any scheme of time discretization. But ΔV will differ from scheme to scheme, and eq. (2.62) then suggests different answers for different schemes. This cannot be correct, since our starting point, the matrix element (2.40) does not depend upon our choice of discretization of time. The resolution to this lies in the fact that the kinetic term in eq. (2.60) <u>does</u> depend upon the scheme. In order to be unambigous one should therefore, either avoid taking in eq. (2.60) the limit $\varepsilon \to 0$ or supply the expression (2.62) with the additional specification of the discretization scheme.

In the literature eq. (2.60) is often written in a different form:

$$\begin{aligned} PI = \lim_{N\to\infty} \int \prod_{k=1}^{N-1} \left(d^n q_k \, g^{\frac{1}{2}}(q_k) \right) \cdot \\ \cdot \exp\Big\{ i \sum_{k=0}^{N-1} \Big[\frac{1}{2\varepsilon} \tilde{g}_{l_1 l_2}(\bar{q}_k) \Delta_k^{l_1} \Delta_k^{l_2} + \frac{1}{6} \varepsilon \tilde{R} \\ + \frac{1}{48\varepsilon} \left(\partial_{l_1} \partial_{l_2} \tilde{g}_{l_3 l_4} - 2 \tilde{g}_{l_5 l_6} T_{l_1}{}^{l_5}{}_{l_2} T_{l_3}{}^{l_6}{}_{l_4} \right) \Delta_k^{l_1} \Delta_k^{l_2} \Delta_k^{l_3} \Delta_k^{l_4} \Big] \Big\} \end{aligned} \tag{2.63}$$

The equivalence can be checked using a generalization of eq. (2.58):

$$\Delta^{l_1} \Delta^{l_2} \Delta^{l_3} \Delta^{l_4} \doteq -\varepsilon^2 \left(\tilde{g}^{l_1 l_2} \tilde{g}^{l_3 l_4} + \tilde{g}^{l_1 l_3} \tilde{g}^{l_2 l_4} + \tilde{g}^{l_1 l_4} \tilde{g}^{l_2 l_3} \right) . \tag{2.64}$$

Finally we want to write down the path integral for our quantum theory on a group manifold. ΔV_1 has already been calculated in eq. (2.33), the complete ΔV is given by

$$\begin{aligned} \Delta V = \frac{1}{8m} \Big\{ (\partial_{l_1} \eta^{l_2 l})(\partial_{l_2} \eta^{l_1 l}) - \eta^{l_1 l} \partial_{l_1} \partial_{l_2} \eta^{l_2 l} \\ - \eta^{l_1 l} (\partial_{l_1} \eta^{l_2 l}) \eta^{l_3 l_4} \partial_{l_2} \sigma^{l_4 l_3} \Big\} . \end{aligned} \tag{2.65}$$

Hence eq. (2.60) becomes:

$$\begin{aligned} PI = \lim_{N\to\infty} \int \prod_{k=1}^{N-1} \left(d^n \omega_k \, \det(\sigma(\omega_k)) \right) \cdot \\ \cdot \exp\Big\{ i \sum_{k=0}^{N-1} \Big[\frac{m}{2\varepsilon} (\sigma^{l l_1} \sigma^{l l_2})(\bar{\omega}_k) \Delta_k^{l_1} \Delta_k^{l_2} - \varepsilon V \\ - \frac{\varepsilon}{8m} \Big((\partial_{l_1} \eta^{l_2 l})(\partial_{l_2} \eta^{l_1 l}) - \eta^{l_1 l} \partial_{l_1} \partial_{l_2} \eta^{l_2 l} \\ - \eta^{l_1 l} (\partial_{l_1} \eta^{l_2 l}) \eta^{l_3 l_4} \partial_{l_2} \sigma^{l_4 l_3} \Big) \Big] \Big\} . \end{aligned} \tag{2.66}$$

In order to cast this equation into the form of eq. (2.63) we have to calculate

$$\begin{aligned} \left(\partial_{l_1} \partial_{l_2} \tilde{g}_{l_3 l_4} - 2 \tilde{g}_{l_5 l_6} T_{l_1}{}^{l_5}{}_{l_2} T_{l_3}{}^{l_6}{}_{l_4} \right) \Delta^{l_1} \Delta^{l_2} \Delta^{l_3} \Delta^{l_4} \\ = 2m \, \sigma^{l l_1} \partial_{l_2} \partial_{l_3} \sigma^{l l_4} \, \Delta^{l_1} \Delta^{l_2} \Delta^{l_3} \Delta^{l_4} , \end{aligned} \tag{2.67}$$

which leads to:

$$PI = \lim_{N\to\infty} \int \prod_{k=1}^{N-1} \left(d^n\omega_k \, \det\left(\sigma(\omega_k)\right)\right) \cdot$$
$$\cdot \exp\Big\{ i \sum_{k=0}^{N-1} \Big[\frac{m}{2\varepsilon} \left(\sigma^{\ell\ell_1}\sigma^{\ell\ell_2}\right)(\bar{\omega}_k)\, \Delta_k^{\ell_1}\Delta_k^{\ell_2} + \frac{\varepsilon}{24m} f^{\ell_1\ell_2\ell_3} f^{\ell_1\ell_2\ell_3}$$
$$+ \frac{m}{24\varepsilon}\, \sigma^{\ell\ell_1}\partial_{\ell_2}\partial_{\ell_3}\sigma^{\ell\ell_4}\Delta^{\ell_1}\Delta^{\ell_2}\Delta^{\ell_3}\Delta^{\ell_4} - \varepsilon V \Big]\Big\} . \qquad (2.68)$$

This form of the path integral and, more general, eq. (2.63) is particularly well-suited for comparing eq. (2.68) with any other path integral formulation which uses a different discretization. One expands the exponent of the latter path integral around the midpoint $\bar{\omega}$. If the resulting power series in Δ (up to order ε) agrees with eq. (2.68), both path integrals describe the same physics, namely a quantum mechanical system on the group manifold. In the following section such a comparison will be carried out for the Wilson action of a lattice gauge theory.

III. Lattice Gauge Theory

Lattice gauge theories serve as an interesting application of the results of the preceeding section. As we shall see, they provide an instructive example, how a very peculiar way of discretizing time may lead to a particularly appealing form of the path integral. For simplicity, we limit ourselves to unitary representations of the gauge group , which is thought to be a compact and simple Lie group.

Usually, a lattice gauge theory is defined through the partition function on the 4-dimensional euclidian lattice, using the Wilson action in the Boltzmann factor. In order to derive the lattice Hamiltonian, one goes into the temporal gauge $A_0^{\,l} = 0$ (l is an algebra-index) and singles out the time direction:

$$Z = \int \prod_k \Big(\prod_{\{i,j\}} dU_{ij}(k) \Big) \cdot$$
$$\cdot \exp\Big\{ -\sum_k \Big[\frac{2a_s}{g_t^2 a_t} \sum_{\{i,j\}} \mathrm{Tr}\left(1 - \mathrm{Re}\, U_{ij}^{\dagger}(k+1)\, U_{ij}(k)\right) - \frac{2a_t}{g_s^2 a_s} V_k \Big]\Big\} \qquad (3.1)$$

where

$$V_k = \sum_{\substack{\text{plaquettes} \\ P}} \mathrm{Tr}\left(1 - \mathrm{Re}\, U(\partial P)\right) , \qquad (3.2)$$

Here a_s and a_t are the lattice spacings in spacelike and timelike directions, respectively. g_s and g_t are the two lattice coupling constants which in this asymmetric lattice have to be distinguished from each other. In eqs. (3.1) and (3.2) the lattice has been sliced: k refers to the time slice, and $\{i,j\}$ labels spacelike links. The sum in eq. (3.2) then extends over all spacelike plaquettes belonging to the time-slice k. Finally the Hamiltonian is derived by writing eq. (3.1) in terms of the transfer-matrix: in the limit $a_t \longrightarrow 0$ one the obtains the lattice Hamiltonian (Kogut-Susskind-Hamiltonian):

$$H = \frac{g_t^2}{2a_s} \sum_{\{i,j\}} L_{ij}^2 + \frac{2}{g_s^2 a_s} V \quad . \tag{3.3}$$

The hermitian generators L^l_{ij} satisfy the algebra

$$[L_{ij}^{l_1} , L_{ij}^{l_2}] = i f^{l_1 l_2 l_3} L_{ij}^{l_3} \quad , \tag{3.4}$$

$$[L_{ij}^{l} , U_{ji}] = U_{ji} T^{l} \quad . \tag{3.5}$$

All these operators act onto states which are normalized according to

$$\int (dU) \, |U\rangle\langle U| = 1 \quad , \tag{3.6}$$

where (dU) is the Haar-measure of the gauge group.

Let us return for a moment to the integral of eq. (3.1). In the limit $a_t \longrightarrow 0$ (with fixed length of the lattice in the time direction) it can be viewed as a path integral, where each link is like a quantum mechanical "particle" which lives on the group manifold. It is then clear that the derivation of the Hamiltonian from eq. (3.1), which we have just reviewed. is the "inverse" of the procedure described in the previous section(*). There one starts from the Hamiltonian and then derives the path integral. Since the "path integral" (3.1) does not have quite the form that we would expect from the considerations of the previous section, we shall apply the standard procedure to the Hamiltonian (eq. (3.3)) and then compare the resulting path integral with eq. (3.1).

(*) Throughout this section we shall use euclidian time. Contact with the previous section has therefore to be made through the usual Wick rotation.

Let us first show that, for each link, the kinetic part of eq. (3.3) agrees with eq. (2.39). To this end we observe that the algebra of eqs. (3.4) and (3.5) coincides with eqs. (2.36) and (2.37), if we identify $\hat{L} = L_{ij}$, $U = U_{ji}$ and $m = \frac{a_s}{g_t^2}$. Furthermore the integration measures of the normalization condition of the states of eqs. (3.6) and (2.41), when applied to the group manifold, agree. Hence, for each link, the kinetic part of the Kogut-Susskind-Hamiltonian describes a quantum theory of a particle with mass $\frac{a_s}{g_t^2}$ constrained to move on the gauge group manifold. Therefore we can write the Hamiltonian as

$$H = \frac{g_t^2}{2a_s} \sum_{\{i,j\}} \left[\{ g^{l_1 l_2}(\omega) P_{l_1} P_{l_2} \}_W (i,j) + \Delta V_1 (i,j) \right] + \frac{2}{g_s^2 a_s} V \tag{3.7}$$

where ω^l_{ij} are the parameters of the group at the link $\{i,j\}$, and $g^{l_1 l_2}(\omega)$ and ΔV_1 are given in eq. (2.18) and (2.30), respectively. In terms of the canonical operators the hermititan generators of right group transformations at each link are given by: (cf. eq. (2.31))

$$L_{ij}^{l} = -\frac{1}{2} \left(\eta^{l_1 l} P_{l_1} + P_{l_1} \eta^{l_1 l} \right) (i,j) \quad . \tag{3.8}$$

As a by-product, this comparison provides an easy method for finding, for a general group, the lattice Hamiltonian: each link variables looks like "a particle which lives on its own group manifold". The magnetic potential term the couples these particles together.

If we would now apply the standard method of the previous section to the kinetic part of the Kogut-Susskind-Hamiltonian, eq. (3.3), we would, of course, end up with eq. (2.68) for each link. It therefore remains to be shown that Wilson's form of the path integral, eq. (3.1), is identical to eq. (2.69): the only reason why, at least at first sight, eq. (3.1) looks quite different from eq. (2.69), lies in the use of a very special scheme of time discretization. In order to see this we shall rewrite eq. (3.1) into the discretization scheme of eq. (2.68). Then the only difference between eqs. (3.1) and (2.69) will be recognized to be an overall normalization constant. Let us begin with the exponent of eq. (3.1) (in the following we shall disregard the potential which is irrelevant for our discussion; we also suppress the summation over the links, and we write ε instead of a_t):

$$S^{kin}_{Wilson} = \sum_k \frac{2a_s}{g_t^2\varepsilon}\, \mathrm{Tr}\,[1 - \mathrm{Re}\, \mathcal{U}^+(k+1)\,\mathcal{U}(k)]$$

$$= \sum_k \frac{a_s}{g_t^2\varepsilon}\, \mathrm{Re\,Tr}\,[(\mathcal{U}(k+1)-\mathcal{U}(k))^+(\mathcal{U}(k+1)-\mathcal{U}(k))]\,. \tag{3.9}$$

Obviously, the time discretization in eq. (3.9) does not correspond to the midpoint rule used in eq. (2.63). For comparison, we have to expand the kinetic part of the Wilson action around the "midpoint" of all time intervalls, keeping all terms up to order ε (terms of higher order than ε are irrelevant in the path integral). Since the kinetic term has a factor ε^{-1} and $\Delta_k = \omega_{k+1} - \omega_k$ is of order $\sqrt{\varepsilon}$, we have to expand up to fourth order in Δ (*). This gives:

$$\mathcal{U}(k+1) - \mathcal{U}(k) = \Delta_k^{\ell_1}\partial_{\ell_1}\mathcal{U} + \frac{1}{24}\,\Delta_k^{\ell_1}\Delta_k^{\ell_2}\Delta_k^{\ell_3}\,\partial_{\ell_1}\partial_{\ell_2}\partial_{\ell_3}\,\mathcal{U}\,, \tag{3.10}$$

where the derivatives of U have to be taken at the midpoint $\bar\omega_k = \frac{1}{2}(\omega_{k+1}+\omega_k)$

The quadratic term gives the classical kinetic term:

$$\frac{a_s}{g_t^2\varepsilon}\,\Delta^{\ell_1}\Delta^{\ell_2}\,\mathrm{Re\,Tr}\,[(\partial_{\ell_1}\mathcal{U})^+(\partial_{\ell_2}\mathcal{U})]$$

$$= \frac{a_s}{2g_t^2\varepsilon}\,\sigma^{\ell\ell_1}\sigma^{\ell\ell_2}\Delta^{\ell_1}\Delta^{\ell_2} = \frac{m}{2\varepsilon}\,g_{\ell_1\ell_2}\,\Delta^{\ell_1}\Delta^{\ell_2}\,, \tag{3.11}$$

(*) It has already been shown in ref. /30/, that an expansion only up to second order leads to erroneous results, as it had to be expected.

where we have defined the mass parameter precisely as in the Hamiltonian approach. The quartic term of S^{kin}_{Wilson} reads (in an obvious matrix notation):

$$\frac{m}{12\varepsilon}\,\Delta^{\ell_1}\Delta^{\ell_2}\Delta^{\ell_3}\Delta^{\ell_4}\,\mathrm{Re\,Tr}\,[(\partial_{\ell_1}\mathcal{U}^+)(\partial_{\ell_2}\partial_{\ell_3}\partial_{\ell_4}\mathcal{U})]$$

$$= \frac{-m}{12\varepsilon}\Big\{(\sigma\cdot\Delta)^{\ell_1}(\sigma\cdot\Delta)^{\ell_2}(\sigma\cdot\Delta)^{\ell_3}(\sigma\cdot\Delta)^{\ell_4}\;\mathrm{Re\,Tr}\;T^{\ell_1}T^{\ell_2}T^{\ell_3}T^{\ell_4}$$

$$+3(\sigma\cdot\Delta)^{\ell_1}(\sigma\cdot\Delta^{\ell_2})\Delta^{\ell_3}\Delta^{\ell_4}\partial_{\ell_3}\sigma^{\ell\ell_4}\;\mathrm{Re\,Tr}\;i\,T^{\ell_1}T^{\ell_2}T^{\ell}$$

$$-(\sigma\cdot\Delta)^{\ell_1}\Delta^{\ell_2}\Delta^{\ell_3}\Delta^{\ell_4}\partial_{\ell_2}\partial_{\ell_3}\sigma^{\ell\ell_4}\;\mathrm{Re\,Tr}\;T^{\ell_1}T^{\ell}\Big\}\,. \tag{3.12}$$

We now use eq. (2.9) and the identity

$$X^{\ell_1}X^{\ell_2}\;\mathrm{Re\,Tr}\;i\,T^{\ell_1}T^{\ell_2}T^{\ell} = 0\,, \tag{3.13}$$

valid for any X because of the hermiticity of the generators. This simplifies the Wilson action to

$$S^{kin}_{Wilson} = \sum_k \Big\{\frac{m}{2\varepsilon}\,g_{\ell_1\ell_2}(\bar\omega_k)\,\Delta_k^{\ell_1}\Delta_k^{\ell_2}$$

$$+\Big[\frac{m}{24\varepsilon}\,\sigma^{\ell\ell_1}\partial_{\ell_2}\partial_{\ell_3}\sigma^{\ell\ell_4}$$

$$-\frac{m}{12\varepsilon}\,\sigma^{\ell_5\ell_1}\sigma^{\ell_6\ell_2}\sigma^{\ell_7\ell_3}\sigma^{\ell_8\ell_4}\;\mathrm{Re\,Tr}\;T^{\ell_5}T^{\ell_6}T^{\ell_7}T^{\ell_8}\Big]\cdot$$

$$\cdot\,\Delta_k^{\ell_1}\Delta_k^{\ell_2}\Delta_k^{\ell_3}\Delta_k^{\ell_4}\Big\}\,. \tag{3.14}$$

If we compare this result with the exponent of the path integral expression of the gauge group of eq. (2.68), there is a difference:

$$\Delta S \doteq \sum_k \Big(\frac{\varepsilon}{24m} f^{l_1 l_2 l_3} f^{l_1 l_2 l_3} + \frac{m}{12\varepsilon} g^{l_5 l_1} g^{l_6 l_2} g^{l_7 l_3} g^{l_8 l_4}\, \mathrm{Re}\,\mathrm{Tr}\, T^{l_5} T^{l_6} T^{l_7} T^{l_8} \cdot \Delta_k^{l_1} \Delta_k^{l_2} \Delta_k^{l_3} \Delta_k^{l_4} \Big). \tag{3.15}$$

In the path integral, however, ΔS is equivalent to a constant (use eq. (2.64)):

$$\Delta S \doteq \sum_k \Big[\frac{\varepsilon}{24m} f^{l_1 l_2 l_3} f^{l_1 l_2 l_3} - \frac{\varepsilon}{12m} \mathrm{Re}\,\mathrm{Tr} \left(2 T^{l_1} T^{l_1} T^{l_2} T^{l_2} + T^{l_1} T^{l_2} T^{l_1} T^{l_2} \right) \Big] \tag{3.16}$$

$$= \sum_k \frac{\varepsilon}{8m} \Big(\frac{1}{2} f^{l_1 l_2 l_3} f^{l_1 l_2 l_3} - 2\, \mathrm{Re}\,\mathrm{Tr}\, T^{l_1} T^{l_1} T^{l_2} T^{l_2} \Big). \tag{3.17}$$

Obviously, this is a constant, hence the only difference between the path integrals of eqs. (3.1) and (2.69) may safely be absorbed into the normalization of the path integral. In this way it is explicitly verified that the standard procedure for the derivation of a path integral and the transfer matrix formalism are indeed inverse operations.

Our result also implies that the path integral on a group manifold can be cast into an especially simple form, namely (we now include the correct normalization factor):

$$\langle q'', t'' | q', t' \rangle$$

$$= \exp\Big\{ i \frac{t''-t'}{16m} \big[f^{l_1 l_2 l_3} f^{l_1 l_2 l_3} - 4\, \mathrm{Re}\,\mathrm{Tr}((T^l T^l)^2) \big] \Big\}$$

$$\cdot \lim_{N\to\infty} \left(\frac{m}{2\pi i \varepsilon} \right)^{N\cdot\frac{n}{2}} \int \prod_{k=1}^{N-1} \left(d^n \omega_k\, g^{\frac{1}{2}}(\omega_k) \right)$$

$$\cdot \exp\Big\{ i \sum_{k=0}^{N-1} \frac{2m}{\varepsilon} \mathrm{Tr} \left[1 - \mathrm{Re}\, \mathcal{U}^+(\omega_{k+1})\, \mathcal{U}(\omega_k) \right] \Big\} . \tag{3.18}$$

Here the U's form a unitary representation of the group, parametrized by ω and satisfying eq. (2.7)(*).

Finally, we want to state the result for ΔS for the case of SU(N). Here the structure constants satisfy /31/

$$f^{l_1 l_2 l_3} f^{l_1 l_2 l_4} = N\, \delta^{l_3 l_4} , \tag{3.19}$$

and we have $\delta^{ll} = N^2-1$. If the U's are chosen in the fundamental representation, we can use the normalization of eq. (2.9) and Schur's lemma to calculate

$$T^l T^l = \frac{1}{2N} (N^2-1) \cdot \mathbb{1} . \tag{3.20}$$

This implies for ΔS:

$$\Delta S = \sum_k \frac{\varepsilon}{16m} \frac{N^2-1}{N} . \tag{3.21}$$

(*) Note that all expressions for the quantum mechanics on the group manifold, eqs. (2.33), (2.61), (2.68), and (3.18), also apply to the case of nonlinear sigma models, which are the field theoretic extensions of our quantum mechanical system.

IV. Conclusion

In this paper we first have reviewed the basic formulae for quantum mechanics on the manifold of a compact simple Lie group, both for canonical quantization and for the derivation of the path integral. Particular attention has been given to the non-cartesian nature of any parametrization of the group, and explicit expressions have been presented for the quantum corrections which are associated with this feature.

In the second part we have used these formulae in order to gain, in lattice gauge theories, further insight into the relationship between the Wilson action and the lattice Hamiltonian (Kogut-Susskind-Hamiltonian). The latter is shown to be the canonical Hamiltonian on the group manifold (up to the potential part). This identification is an agreement with the picture that attached to each (spacial) link there is a group manifold, and the link variables U_{ij} behave like "quantum mechanical particles" living on these group spaces. This may be considered as the lattice counterpart of the continuum fibre bundle picture. The identification of the Hamiltonians also allows to write the lattice Hamiltonian in terms of canonical coordinate and momentum operators and of the left-axuxiliary functions η of the Lie group. In the parametrization $U = \exp(-i\omega^{\ell}T^{\ell})$ the Campbell-Baker-Hausdorff formula provides a tool to calculate η (and hence the Hamiltonian) to any given order of accuracy.

The usual four dimensional euclidian partition function with the Wilson action, on the other hand, becomes a path integral when (in temporal gauge) the timelike lattice spacing is taken to zero (keeping the spacelike lattice spacing fixed). The action integral of this path integral ("Wilson form"), however, does not look at all like the one that follows from applying the standard rules (with a simple time

discretization) to the lattice Hamiltonian ("standard form"). We have explicitly demonstrated that both forms are equivalent (up to an irrelevant normalization constant): the (superficial) difference lies in the way in which time is descretized. The "Wilson form" corresponds to a very special discretization scheme, whereas the "standard form" e.g. uses the midpoint rule.

We finally like to stress that this equivalence of two seemingly different forms of the path integral not only applies to the context of lattice gauge theories. For any quantum system on a compact simple Lie group the path integral can be written in the elegant "Wilson form".

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Appendix A: Path Integral on the Group Manifold of SU(2):

In SU(2) the generators are $T^{l} = \frac{1}{2}\tau^{l}$ with τ^{l} being the Pauli-matrices. They satisfy

$$T^{l_1}T^{l_2} = \frac{1}{4}\delta^{l_1 l_2} + \frac{i}{2}\varepsilon^{l_1 l_2 l_3}T^{l_3} , \qquad (A.1)$$

where $\varepsilon^{l_1 l_2 l_3}$ is the totally antisymmetric Levi-Civita tensor in three dimensions. We want to present the path integral for SU(2) in two different wide-spread representations, namely $U = \exp(-iB^{l}T^{l})$ /32/ and $U = x^{o} + ix^{l}\tau^{l}$ /33/.

a) First we want to use the familiar parametrization: /32/

$$\mathcal{U} = e^{-iT^{l}B^{l}} = c - 2i\,\frac{B^{l}T^{l}}{B}\,s , \qquad (A.2)$$

where

$$B = \sqrt{B^{l}B^{l}} ,\quad c = \cos\frac{B}{2} ,\quad s = \sin\frac{B}{2} . \qquad (A.3)$$

In this case, the composition functions are rather complicated, but we do not need to know them. The σ 's can calculated using eq. (2.12), which is applicable since eq. (A.2) satisfies eq. (2.7):

$$\sigma^{l_1 l_2} = \frac{2cs}{B}P^{l_1 l_2} + \frac{B^{l_1}B^{l_2}}{B^2} + 2\,\frac{s^2}{B^2}\,\varepsilon^{l_1 l_2 l_3}B^{l_3} , \qquad (A.4)$$

where the projector P is defined by

$$P^{l_1 l_2} = \delta^{l_1 l_2} - \frac{B^{l_1}B^{l_2}}{B^2} . \qquad (A.5)$$

This implies the left-auxiliary functions

$$\eta^{l_1 l_2} = \frac{1}{2}\,\frac{c}{s}\,B\,P^{l_1 l_2} + \frac{B^{l_1}B^{l_2}}{B^2} - \frac{1}{2}\varepsilon^{l_1 l_2 l_3}B^{l_3} . \qquad (A.6)$$

Eqs. (A.18) and (A.20) can be used to calculate the metric and its inverse:

$$g_{l_1 l_2} = \frac{4s^2}{B^2}P^{l_1 l_2} + \frac{B^{l_1}B^{l_2}}{B^2} , \qquad (A.7)$$

$$g^{l_1 l_2} = \frac{B^2}{4s^2}P^{l_1 l_2} + \frac{B^{l_1}B^{l_2}}{B^2} . \qquad (A.8)$$

Note that the Taylor expansion of eq. (A.8) coincides with eq. (2.25), as it should be. The Christoffel symbols can be calculated by using eq. (2.19) or eq. (2.21):

$$\Gamma^{l_3}_{l_1 l_2} = \left(\frac{1}{B} - \frac{2cs}{B^2}\right)P^{l_1 l_2}\frac{B^{l_3}}{B} + \left(\frac{c}{2s} - \frac{1}{B}\right)\left(P^{l_1 l_3}\frac{B^{l_2}}{B} + P^{l_2 l_3}\frac{B^{l_1}}{B}\right) , \qquad (A.9)$$

and the curvature is given by eq. (2.22):

$$R = \frac{1}{4}\varepsilon^{l_1 l_2 l_3}\varepsilon^{l_1 l_2 l_3} = \frac{3}{2} . \qquad (A.10)$$

The quantum correction $\Delta \cdot V_1$ in the Weyl-ordered Hamiltonian is (cf. eqs. (2.32) and (2.33))

$$\Delta V_1 = \frac{1}{8m}\left(-\frac{3}{2} - \frac{1}{2}\frac{c^2}{s^2} - \frac{1}{s^2} + \frac{c}{2s^3}B + \frac{2}{B^2}\right) . \qquad (A.11)$$

The quantum corrections in the path integral, which uses the midpoint discretization, may be written in two alternative forms, either as a q-dependent potential or as a power series in Δ . In the form of a potential it reads (cf. eqs. (2.60), (2.61) and (2.65)):

$$\Delta V = \frac{1}{8m}\left(-\frac{3}{2} - \frac{1}{2}\frac{c^2}{s^2} - \frac{5}{2}\frac{1}{s^2} + \frac{c}{s^3}B + \frac{4}{B^2}\right) . \qquad (A.12)$$

For the power series in Δ we have, for each time slice (cf. eqs. (2.63) and (2.68)):

$$\Delta L = \frac{\varepsilon}{4m} + \frac{m}{6\varepsilon}\Big\{\Big[\frac{3}{2}\frac{cs}{B^3} - \frac{s^2}{B^4}(2+c^2)\Big]\, P^{\ell_3\ell_4}$$

$$+\Big[-\frac{1}{B^2} - \frac{c^2}{2B^2} + 6\,\frac{s^2}{B^4}\Big]\frac{B^{\ell_3}B^{\ell_4}}{B^2}\Big\} P^{\ell_1\ell_2}\Delta^{\ell_1}\Delta^{\ell_2}\Delta^{\ell_3}\Delta^{\ell_4}, \tag{A.13}$$

which is, of course, equivalent to $-\varepsilon\cdot\Delta V$. Hence the correct path integral on the SU(2) group manifold takes the form:

$$PI = \lim_{N\to\infty}\int\prod_{k=1}^{N-1}\left(d^n B_k\; g^{\frac{1}{2}}(B_k)\right)\cdot$$

$$\cdot\exp\Big\{i\sum_{k=0}^{N-1}\Big[\frac{m}{2\varepsilon}\, g_{\ell_1\ell_2}(\bar{B}_k)\,\Delta_k^{\ell_1}\Delta_k^{\ell_2} - \varepsilon\cdot V(\bar{B}_k) \tag{A.14}$$

$$+\frac{\varepsilon}{8m}\Big(\frac{3}{2} + \frac{1}{2}\frac{c^2}{s^2} + \frac{5}{2}\frac{1}{s^2} - \frac{c}{s^3}B - \frac{4}{B^2}\Big)\Big]\Big\}$$

or

$$PI = \lim_{N\to\infty}\int\prod_{k=1}^{N-1}\left(d^n B_k\; g^{\frac{1}{2}}(B_k)\right)\cdot$$

$$\cdot\exp\Big\{i\sum_{k=0}^{N-1}\Big[\frac{m}{2\varepsilon}\, g_{\ell_1\ell_2}(\bar{B}_k)\,\Delta_k^{\ell_1}\Delta_k^{\ell_2} - \varepsilon\cdot V(\bar{B}_k) + \frac{\varepsilon}{4m}$$

$$+\frac{m}{6\varepsilon}\Big(\Big(\frac{3}{2}\,\frac{cs}{B^3} - \frac{s^2}{B^4}(2+c^2)\Big) P^{\ell_3\ell_4}$$

$$+\Big(-\frac{1}{B^2} - \frac{1}{2}\frac{c^2}{B^2} + 6\,\frac{s^2}{B^4}\Big)\frac{B^{\ell_3}B^{\ell_4}}{B^2}\Big) P^{\ell_1\ell_2}\Delta_k^{\ell_1}\Delta_k^{\ell_2}\Delta_k^{\ell_3}\Delta_k^{\ell_4}\Big]\Big\}, \tag{A.15}$$

where we have omitted the common normalization factor.

b) In this part of the appendix we consider the parametrization /33/:

$$\mathcal{U} = x^0 + i\, x^\ell \tau^\ell \,, \quad x^0 = \sqrt{1 - x^\ell x^\ell}\,. \tag{A.16}$$

Note that this parametrization does not fulfil eq. (2.7), hence the results of section II should be applied with some care. Therefore we start from the very

beginning: the composition function reads:

$$\phi^\ell(x_1, x_2) = x_1^{\,0} x_2^{\,\ell} + x_1^{\,\ell} x_2^{\,0} - \varepsilon^{\ell\ell_1\ell_2} x_1^{\,\ell_1} x_2^{\,\ell_2}\,, \tag{A.17}$$

which implies for the left auxiliary functions

$$\eta^{\ell\ell_1} = x^0\,\delta^{\ell\ell_1} + \varepsilon^{\ell\ell_1\ell_2} x^{\ell_2}\,. \tag{A.18}$$

Let us mention that eq. (A.18) cannot be obtained from eq. (A.6) by simply applying the coordinate transformation $B^\ell \to x^\ell(B) = -s\cdot\frac{B^\ell}{B}$ and using the proper transformation behaviour of the vielbein η. This would yield

$$\tilde{\eta}^{\ell\ell_1}(x) = \eta^{\ell\ell_2}(B(x))\,\frac{\partial x^{\ell_1}}{\partial B^{\ell_2}} = -\frac{1}{2}\,\eta^{\ell\ell_1}(x)\,. \tag{A.19}$$

The reason for this lies in the fact that when changing the parametrization from eq. (A.2) to (A.16) we also make a change of the orthogonal basis of the Lie algebra according to $T \to -\tau$. As a result, under this combined transformation, the vielbein recieves an additional factor of -2, compared to the simple coordinate transformation. The redefinition of the basis also changes the structure constants: eqs. (2.5) and (A.18) lead to

$$f^{\ell_1\ell_2\ell_3} = -2\,\varepsilon^{\ell_1\ell_2\ell_3}\,. \tag{A.20}$$

For the σ's we find

$$\sigma^{\ell\ell_1} = x^0\,\delta^{\ell\ell_1} + \frac{x^\ell x^{\ell_1}}{x^0} - \varepsilon^{\ell\ell_1\ell_2} x^{\ell_2}\,. \tag{A.21}$$

Instead of eq. (2.12) they satisfy

$$\mathcal{U}^{-1}\partial_\ell\,\mathcal{U} = i\,\tau^\ell\,\sigma^{\ell\ell_1}\,, \tag{A.22}$$

which again indicates the change in the basis of the algebra. σ and η give the metric and its inverse:

$$g_{\ell_1\ell_2} = \delta^{\ell_1\ell_2} + \frac{x^{\ell_1}x^{\ell_2}}{x^{0\,2}} \quad , \tag{A.23}$$

$$g^{\ell_1\ell_2} = \delta^{\ell_1\ell_2} - x^{\ell_1}x^{\ell_2} \quad . \tag{A.24}$$

The Christoffel symbols and the scalar curvature are calculated to be:

$$\Gamma_{\ell_1\ \ \ell_2}^{\ \ \ell_3} = x^{\ell_3} g_{\ell_1\ell_2} \quad , \tag{A.25}$$

$$R = \frac{1}{4} f^{\ell_1\ell_2\ell_3} f^{\ell_1\ell_2\ell_3} = \varepsilon^{\ell_1\ell_2\ell_3}\varepsilon^{\ell_1\ell_2\ell_3} = 6 \quad . \tag{A.26}$$

The difference between eqs. (A.26) and (A.10) can be understood by the following transformation: first perform the coordinate transformation $B \longrightarrow x$, which leaves R invariant and then rescale the metric by a factor of 4 (due to the change of the basis in the algebra), which enlarges R by the same factor.

The quantum correction ΔV_1 in the Weyl-ordered Hamiltonian is given by (see eqs. (2.32) and (2.33))

$$\Delta V_1 = \frac{1}{8m} \left(\frac{x^\ell x^\ell}{x^{0\,2}} - 6 \right) \tag{A.27}$$

The quantum corrections in the midpoint rule discretized path integral can be stated as (cf. eqs. (2.60), (2.61), (2.63) and (2.69)):

$$\Delta V = \frac{1}{8m} \left(5 \frac{x^\ell x^\ell}{x^{0\,2}} - 3 \right) \quad , \tag{A.28}$$

$$\Delta L = \frac{\varepsilon}{m} + \frac{m}{24\varepsilon} g_{\ell_1\ell_2} \left(g_{\ell_3\ell_4} + 3 g_{\ell_3\ell_5} g_{\ell_4\ell_6} x^{\ell_5} x^{\ell_6} \right) \cdot \Delta^{\ell_1}\Delta^{\ell_2}\Delta^{\ell_3}\Delta^{\ell_4} \tag{A.29}$$

Therefore the correct path integral reads:

$$PI = \lim_{N\to\infty} \int \prod_{k=1}^{N-1} \left(d^n x_k \; g^{\frac{1}{2}}(x_k) \right) \cdot \exp\left\{ i \sum_{k=0}^{N-1} \left[\frac{m}{2\varepsilon} g_{\ell_1\ell_2}(\bar{x}_k) \Delta_k^{\ell_1} \Delta_k^{\ell_2} - \frac{\varepsilon}{8m} \left(5 \frac{x^\ell x^\ell}{x^{0\,2}} - 3 \right) - \varepsilon V(\bar{x}_k) \right] \right\} , \tag{A.30}$$

or

$$PI = \lim_{N\to\infty} \int \prod_{k=1}^{N-1} \left(d^n x_k \; g^{\frac{1}{2}}(x_k) \right) \cdot \exp\left\{ i \sum_{k=0}^{N-1} \left[\frac{m}{2\varepsilon} g_{\ell_1\ell_2}(\bar{x}_k) \Delta_k^{\ell_1} \Delta_k^{\ell_2} - \varepsilon V(\bar{x}_k) + \frac{\varepsilon}{m} + \frac{m}{24\varepsilon} g_{\ell_1\ell_2} \left(g_{\ell_3\ell_4} + 3 g_{\ell_3\ell_5} g_{\ell_4\ell_6} x^{\ell_5} x^{\ell_6} \right) \Delta_k^{\ell_1} \Delta_k^{\ell_2} \Delta_k^{\ell_3} \Delta_k^{\ell_4} \right] \right\} , \tag{A.31}$$

where, again, we have suppressed the common normalization factor.

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