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Abstract

It is shown that the chiral and conformal anomalies of fermions in 2n dimensions are related to a statistical mechanics system in 2n+1 dimensions for which a magnetization-like quantity corresponding to the chiral anomaly density has a nontrivial infinite temperature limit.

1. Introduction

In the past few years considerable progress has been made in the understanding of the algebraic and topological properties of chiral and gravitational anomalies /1/. Nevertheless their physical origin remained obscure, since the only "reason" for their existence we know is that for certain quantum field theories there does not exist any regularization scheme which simultaneously respects all symmetries present at the classical level. This is a rather formal statement and it would be desirable to understand the origin of anomalies in more physical terms which are closer to one's intuition /2/. The major obstacle for any attempt in this direction is the fact that anomalies are effects due to regularization which persist when the UV-regulator is removed. In this sense they can be thought of as the collective effect of an infinite number of field modes, all contributing on equal footing, irrespective of their momentum or other quantum numbers. This is particularly obvious in Fujikawa's treatment /3/. For the divergence of the axial vector current of a 2n-dimensional massless Dirac fermion interacting with an external Yang-Mills potential he obtains the formal expression

$$\partial_\mu \langle 0|\bar{\Psi}\gamma^\mu \gamma_{2n+1} \Psi|0\rangle \equiv \mathcal{A}_{2n}(x) = 2\sum_i \varphi_i^+(x)\, \gamma_{2n+1}\, \varphi_i(x) \,. \tag{1.1}$$

The sum runs over a complete set of basis functions φ_i of the space of spinor fields. In more physical terms this means that all fermionic vacuum fluctuations contribute with the same weight to $\mathcal{A}(x)$. The sum, however, is ill-defined. According to the general procedure it is replaced by the well-behaved, gauge invariant expression

$$\mathcal{A}_{2n}(x) = \lim_{M\to\infty} \mathcal{A}_{2n}(x;M) \tag{1.2}$$

where

$$\mathcal{A}_{2n}(x;M) = 2\sum_i \varphi_i^+(x)\,\gamma_{2n+1}\, f\left(\frac{\not{D}^2}{M^2}\right)\varphi_i(x) \tag{1.3}$$

and f is any smooth function, which obeys

$$f(0) = 1\ ,\quad \lim_{s\to\infty} f^{(n)}(s) = 0 \quad \text{for all } n \geq 0\ . \tag{1.4}$$

For example Fujikawa's choice /3/ is

$$f(s) = e^{-s}\ . \tag{1.5}$$

We now assume the φ_i's to be eigenfunctions of the Dirac operator: $\not{D}\varphi_i = \lambda_i\varphi_i$. Effectively only vacuum fluctuations with $\lambda_i \lesssim M$ contribute to the regularized anomaly. Using the well known Seeley-De Witt expansion /4/ of the heat-kernel $\langle x|\exp(-\not{D}^2/M^2)|x\rangle$ it is easily seen that (1.2) reproduces the results of perturbation theory in all even dimensions. In particular, the space integral of $\mathcal{A}(x)$ is twice the index of $\not{D}$. (We assume 2n-dimensional space-time to have euclidean signature.)

Usually the insertion of the cutoff factor $f(\not{D}^2/M^2)$ is considered a purely mathematical tool to render the sum (1.1) finite. The choice (1.5) however is reminiscent of the Boltzmann factor in statistical mechanics and suggests to interpret (1.3) as a thermal average within equilibrium thermodynamics, the rôle of temperature being played by M^2. In a sense which will be made precise below, this amounts to representing the vacuum fluctuations $\varphi_i(x^\mu)$ by an ideal quantum gas of particles living in a (2n+1)-dimensional (x^μ, t) world. The extra dimension is the time coordinate with respect to which the system is translational invariant in thermal equilibrium. In this picture the emergence of the anomaly as the collective effect of all vacuum fluctuations (for $M\to\infty$) can be illustrated by looking

at the high-temperature limit of a certain magnetization-like quantity /5/. Of course, this does not mean that anomalies can be explained completely in terms of statistical mechanics. One still has to use quantum field theory to derive (1.2), (1.3). But it means that there is an auxiliary system which can serve as a physical illustration of the behaviour of the field modes $\varphi_i(x)$ when the regulator is removed.

2. The case of two dimensions: Pauli-electrons

We start the discussion with a seemingly unrelated gedanken-experiment. Assume we have a particle, which can be in two different energetically degenerate states, and which can be converted freely from one state to the other. This degeneracy can be lifted by an external magnetic field $\vec{B}$. Hereby the energy shift experienced in the two states is $-\alpha|\vec{B}|$ and $+\alpha|\vec{B}|$, respectively, where α is some real constant. Stated differently, the energy shift is given by the eigenvalues of $H_{int} = -\alpha\,\vec{\sigma}\cdot\vec{B}$ where $\vec{\sigma}$ are the Pauli matrices. This is the quantum mechanical coupling of a nonrelativistic (Pauli) electron to an external magnetic field via its magnetic moment. Let us consider the case of a time independent, constant magnetic field, which is choosen to lie along the 3-axis. For a single electron the "magnetization" in the canonical ensemble at temperature $T \equiv \beta^{-1}$ is given by

$$\langle\sigma_3\rangle \equiv \frac{\mathrm{tr}[\sigma_3 \exp(\alpha\beta\sigma_3 B)]}{\mathrm{tr}[\exp(\alpha\beta\sigma_3 B)]} = \tanh\left(\alpha\frac{B}{T}\right)\ . \tag{2.1}$$

This expression vanishes in the infinite temperature limit. The same is also true for a fixed number N of non-interacting electrons in the canonical ensemble. If we consider the case of Boltzmann statistics for a moment the total magnetization is

$$M(T,B) = N\langle \sigma_3 \rangle \tag{2.2}$$

and M as well as the susceptibility $\chi \equiv \partial M/\partial B$ become zero as T goes to infinity.

Let us now try to modify the model so as to have a non-zero, but finite value of the magnetization and the susceptibility even for $T \to \infty$. To this end we assume that our system is allowed to exchange particles with a reservoir so that N is no longer fixed, but is a function of temperature. This means that we replace the canonical ensemble by a grand canonical one. The particle number is now given by the derivative of the grand canonical potential

$$\mathcal{J}(T,\mu,V) = -T \ln \Xi(T,\mu,V) \tag{2.3}$$

with respect to the chemical potential μ:

$$\langle N \rangle = -\frac{\partial \mathcal{J}}{\partial \mu} \tag{2.4}$$

Ξ denotes the grand partition function given in terms of the canonical partition functions Z by

$$\Xi(T,\mu,V) = \sum_{N=0}^{\infty} e^{\beta\mu N} Z(T,V,N) \tag{2.5}$$

In the case of Boltzmann statistics for a gas of noninteracting particles we have the well known result

$$\Xi(T,\mu,V) = \exp\left[e^{\beta\mu} Z(T,V,1)\right] \tag{2.6}$$

with the one-particle partition function

$$Z(T,V,1) = \mathrm{Tr}\, e^{-\beta H}$$

$$= \ldots$$

$$= V \int \frac{d^d p}{(2\pi)^d}\, e^{-\beta \frac{\vec{p}^{\,2}}{2m}}\, \mathrm{tr}\left(e^{\alpha\beta\sigma_3 B}\right)$$

$$= V\left(\frac{mT}{2\pi}\right)^{\frac{d}{2}} 2\cosh\left(\alpha\frac{B}{T}\right). \tag{2.7}$$

Here d is the number of space-dimensions,

$$H = \frac{\vec{p}^{\,2}}{2m} - \alpha\sigma_3 B \tag{2.8}$$

is the one particle Hamiltonian and Tr denotes the trace in the one particle state space. For the particle number we obtain

$$\langle N \rangle = e^{\beta\mu}\, \mathrm{Tr}\, e^{-\beta H} = e^{\beta\mu} V \left(\frac{mT}{2\pi}\right)^{\frac{d}{2}} 2\cosh\left(\alpha\frac{B}{T}\right)$$

$$= V\left(\frac{mT}{2\pi}\right)^{\frac{d}{2}}\left[1 + \mathcal{O}\left(\frac{1}{T}\right)\right] \tag{2.9}$$

and the total magnetization per unit volume is

$$M(T,\mu,B) = \frac{1}{V\alpha\beta}\frac{\partial}{\partial B}\ln\Xi = \frac{1}{V} e^{\beta\mu}\, \mathrm{Tr}\left(\sigma_3 e^{-\beta H}\right)$$

$$= \frac{\langle N \rangle}{V}\tanh\left(\alpha\frac{B}{T}\right) = e^{\beta\mu}\left(\frac{mT}{2\pi}\right)^{\frac{d}{2}} 2\sinh\left(\alpha\frac{B}{T}\right)$$

$$= \left(\frac{m}{2\pi}\right)^{\frac{d}{2}} 2\alpha B\, T^{\frac{d-2}{2}}\left[1 + \mathcal{O}\left(\frac{1}{T}\right)\right]. \tag{2.10}$$

The temperature dependence is now modified significantly due to the factor $\langle N \rangle$. Note that the result crucially depends on the number of spatial dimensions. If we evaluate (2.10) for a planar gas in d=2 dimensions we obtain for $T \to \infty$ a finite constant magnetization

$$\lim_{T\to\infty} M(T,\mu,B) = \frac{\alpha}{\pi} m B \tag{2.11}$$

and susceptibility

$$\lim_{T\to\infty} \chi(T,\mu,B) = \frac{\alpha}{\pi} m \quad . \tag{2.12}$$

Boltzmann statistics is of course inappropriate for the treatment of electrons and the discussion above only serves to exhibit the essential points in the most simple way. The results, however, also hold for the case of Fermi-Dirac statistics as will be shown next. Equations (2.6), (2.7) are to replaced by /6/

$$\ln \Xi(T,\mu,V) = \mathrm{Tr}\ \ln\left(1 + e^{\beta\mu} e^{-\beta H}\right)$$

$$= V \left(\frac{mT}{2\pi}\right)^{\frac{d}{2}} \mathrm{tr}\ f_{\frac{d+2}{2}}\left(e^{\beta\mu} e^{\alpha\beta\sigma_3 B}\right) \tag{2.13}$$

with the Fermi integrals

$$f_{\frac{d+2}{2}}(z) = \frac{2}{\Gamma(\frac{d}{2})} \int_0^\infty dp\ p^{d-1} \ln\left(1 + z e^{-p^2}\right)$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} z^n n^{-\frac{d+2}{2}} \quad . \tag{2.14}$$

The particle number and magnetization are

$$\langle N \rangle = \mathrm{Tr}\left(1 + e^{-\beta\mu} e^{\beta H}\right)^{-1}$$

$$= V \left(\frac{mT}{2\pi}\right)^{\frac{d}{2}} \mathrm{tr}\ f_{\frac{d}{2}}\left(e^{\beta\mu} e^{\alpha\beta\sigma_3 B}\right) \tag{2.15}$$

$$= V \left(\frac{mT}{2\pi}\right)^{\frac{d}{2}} 2 f_{\frac{d}{2}}(1) \left[1 + O\left(\frac{1}{T}\right)\right]$$

$$M(T,\mu,B) = \frac{1}{V} \mathrm{Tr}\left[\sigma_3 \left(1 + e^{-\beta\mu} e^{\beta H}\right)^{-1}\right] \tag{2.16}$$

$$= \left(\frac{mT}{2\pi}\right)^{\frac{d}{2}} \mathrm{tr}\left[\sigma_3 f_{\frac{d}{2}}\left(e^{\beta\mu} e^{\alpha\beta\sigma_3 B}\right)\right]$$

$$= \left(\frac{mT}{2\pi}\right)^{\frac{d}{2}} 2 f_{\frac{d-2}{2}}(1)\ \alpha \frac{B}{T} \left[1 + O\left(\frac{1}{T}\right)\right]$$

$$= \frac{\alpha}{2\pi} m B \left[1 + O\left(\frac{1}{T}\right)\right] \quad \text{for} \quad d=2 \quad . \tag{2.17}$$

Although M in this case is not just given by the product of the one-particle magnetization and the particle number $\langle N \rangle$ as in Boltzmann statistics (2.10), the high temperature behaviour again leads to a finite limit for M and χ as $T \to \infty$ if the number of dimensions is d=2.

A finite result is obtained because the effect of the thermal fluctuations which tend to decrease the magnetization for increasing temperature is compensated for by the increasing particle number density. In the next section we will argue that this "fine tuning" is the basic mechanicsm leading to anomalies.

At this point a remark about the Hamiltonian is in order. The full one-particle Hamiltonian for our model should properly be

$$H = \frac{1}{2m}\left(\vec{p} + e\vec{A}\right)^2 - \alpha\sigma_3 B \tag{2.18}$$

instead of (2.8). The coupling to the vector potential $\vec{A}$ is important for the Landau diamagnetism. However, for the average spin value which we define by (2.16) our previous results on the high temperature behaviour are unchanged. The spectrum of H is well known /7/. In d=2 the eigenvalues are

$$E_{k,s} = \frac{e}{m} B\left(k + \tfrac{1}{2}\right) - \alpha B s \quad ; \quad k = 0,1,2,\ldots \quad , \quad s = \pm 1 \tag{2.19}$$

and the degeneracy is such that the trace in the one particle state space is

$$\mathrm{Tr} = V \frac{eB}{2\pi} \sum_{k,s} \quad . \tag{2.20}$$

With the help of the Euler summation formula in the form

$$\sum_{k=0}^{\infty} g(\beta k) = \frac{1}{\beta} \int_0^{\infty} dy\, g(y) + \frac{1}{2} g(0) + \mathcal{O}(\beta) \tag{2.21}$$

it is easy to see that in the high temperature limit the formulae (2.15), (2.17) are reproduced.

3. Chiral anomalies in 2n dimensions

What do our two-dimensional considerations have to do with anomalies? The connection comes through the fact that the expression for the magnetization in the grand canonical ensemble (2.16) is the same as for the regularized two-dimensional anomaly $\mathcal{A}_2(x;T)$, see (1.3), if we choose the cutoff function to be

$$f(s) = 2\,(1 + e^{s})^{-1} \tag{3.1}$$

and let $\mu = 0$ (in the limit $T \to \infty$ the quantities under consideration are independent of μ). In the following we shall make this connection explicit and extend the discussion to systems in all even dimensions. Working in 2n dimensions we consider an electron which can exist in 2^n spin states which are energetically degenerate in field free space. (Recall that a 2n-dimensional Dirac spinor has 2^n components.) An external magnetic field is described by the 2n-dimensional space-part $F_{\mu\nu}$, $\mu,\nu = 1, \ldots, 2n$, of the field strength tensor, which we assume to be constant. The generalization of the interaction term of the spin to the external field in higher dimensions is well known to be of the form $\sigma_{\mu\nu} F^{\mu\nu}$, where $\sigma_{\mu\nu}$ is related to the Dirac matrices for 2n dimensions by

$$\sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu] \quad . \tag{3.2}$$

Thus the single particle Hamiltonian reads

$$H = \frac{p_\mu^2}{2m} - \alpha\, \sigma_{\mu\nu} F^{\mu\nu} \quad . \tag{3.3}$$

In the two-dimensional example magnetization was defined through the average value of σ_3, i.e. the spin component belonging to the fictitious time direction. In the general case this average will be formed using the analog of γ_5:

$$\gamma_{2n+1} = -i^{n+1} \prod_{\mu=1}^{2n} \gamma^\mu \quad . \tag{3.4}$$

The magnetization due to a single particle is given by

$$\langle \gamma_{2n+1} \rangle = \frac{\mathrm{tr}\left[\gamma_{2n+1}\, e^{\beta\alpha\sigma_{\mu\nu}F^{\mu\nu}}\right]}{\mathrm{tr}\left[e^{\beta\alpha\sigma_{\mu\nu}F^{\mu\nu}}\right]} \quad . \tag{3.5}$$

Using the trace identity

$$\mathrm{tr}\left[\gamma_{2n+1}\gamma^{\mu_1}\cdots\gamma^{\mu_j}\right] = \begin{cases} 0, & \text{for } 1 \le j < 2n \\ (-2i)^n \varepsilon^{\mu_1\cdots\mu_{2n}}, & \text{for } j = 2n \end{cases} \tag{3.6}$$

one obtains for its high temperatur limit

$$\langle \gamma_{2n+1} \rangle = \frac{1}{n!}\left(\frac{\alpha}{2}\right)^n \frac{1}{T^n}\, \mathrm{tr}\left[\gamma_{2n+1} (\sigma_{\mu\nu}F^{\mu\nu})^n\right] \left(1 + \mathcal{O}\left(\frac{1}{T}\right)\right). \tag{3.7}$$

Considering now a grand canonical ensemble with Fermi-Dirac statistics, the particle number density becomes

$$\begin{aligned} n(T,\mu) &\equiv \frac{1}{V}\langle N \rangle = \frac{1}{V}\,\mathrm{Tr}\left(1 + e^{-\beta\mu} e^{\beta H}\right)^{-1} \\ &= \left(\frac{mT}{2\pi}\right)^n \mathrm{tr}\, f_n\left(e^{\beta\mu} e^{\alpha\beta\sigma_{\mu\nu}F^{\mu\nu}}\right) \end{aligned} \tag{3.8}$$

with the Fermi integrals f_n as in (2.14). Its high temperature limit is independently of the chemical potential μ and the field strength given by

$$n(T) = \left(\frac{mT}{\pi}\right)^n f_n(1)\left[1 + \mathcal{O}\left(\frac{1}{T}\right)\right] \quad . \tag{3.9}$$

This T^n-behaviour is just what is needed to cancel the $1/T^n$ fall-off of the magnetization due to a single particle. In fact for the magnetization per unit volume we get a finite, but non-zero high temperature limit

$$\begin{aligned} M(T,\mu) &= \frac{1}{V}\,\mathrm{Tr}\left[\gamma_{2n+1}\left(1 + e^{-\beta\mu}\, e^{\beta H}\right)^{-1}\right] \\ &= \left(\frac{mT}{2\pi}\right)^n \mathrm{tr}\left[\gamma_{2n+1}\, f_n\left(e^{\beta\mu}\, e^{\alpha\beta\,\sigma_{\mu\nu}F^{\mu\nu}}\right)\right] \\ &= \left(\frac{m\alpha}{2\pi}\right)^n \frac{1}{n!}\,\frac{1}{2}\,\mathrm{tr}\left[\gamma_{2n+1}\left(\sigma_{\mu\nu}F^{\mu\nu}\right)^n\right]\left(1+\mathcal{O}\left(\frac{1}{T}\right)\right) . \end{aligned} \tag{3.10}$$

Next we show that the above model correctly describes the essential features of the vacuum fluctuations $\varphi_i(x)$ contributing to the anomaly (1.3) when the regulator is removed. Recall that the expression (1.3), which is essentially the trace of the cutoff factor, can be evaluated by going to a plane wave basis $\langle x|p\rangle$ = exp(ipx) and then imitating Dyson's expansion up to a certain order, which depends on n /3/. In this way one finds

$$\mathcal{A}_{2n} = 2 \lim_{M\to\infty} \int \frac{d^{2n}p}{(2\pi)^{2n}}\, \mathrm{tr}\left(\gamma_{2n+1}\, f\left(\not{D}^2/M^2\right)\right) . \tag{3.11}$$

With Fujikawa's choice for f (1.5) the integral can be performed easily and one arrives at

$$\mathcal{A}_{2n} = 2\, \frac{(-1)^n}{(2\pi)^n\, n!} \left(\frac{1}{4}\right)^n \mathrm{tr}\left[\gamma_{2n+1}\left(\sigma_{\mu\nu}F^{\mu\nu}\right)^n\right] \tag{3.12}$$

or

$$\mathcal{A}_{2n} = \frac{(-1)^n}{2^{2n-1}\,\pi^n\, n!}\, \varepsilon_{\mu_1\nu_1\ldots\mu_n\nu_n}\, F^{\mu_1\nu_1}\ldots F^{\mu_n\nu_n} . \tag{3.13}$$

The result, however, does not depend on this particular choice of f. Comparing (3.12) to (3.10) we see that the anomaly is precisely four times(*) the high temperature limit of the magnetization in the above model if one chooses $\alpha = -1/4m$:

$$\mathcal{A}_{2n} = 4 \lim_{T\to\infty} M(T) \quad . \tag{3.14}$$

This is not an accident since if the cutoff factor $f(\not{D}^2/M^2)$ is considered a Fermi-Dirac weight factor (3.1) the corresponding Hamiltonian is given by (**)

$$H = \frac{1}{2m}\not{D}^2 = \frac{1}{2m}\left(p_\mu - A_\mu\right)^2 + \frac{1}{4m}\sigma_{\mu\nu}F^{\mu\nu} . \tag{3.15}$$

The mass scale m was introduced to ensure that H has the correct dimension. Apart from the A_μ-term appearing in the covariant derivative, the Hamiltonian (3.15) coincides with (3.3) for $\alpha = -1/4m$. Straightforwardly applying Fujikawa's procedure to (1.2) one finds that all terms coming from the covariant derivative cancel and that it is the $\sigma_{\mu\nu}F^{\mu\nu}$-term alone which gives rise to the anomaly. In the context of our model this was verified explicitly for the case d=2 in section 2. The coincidence of the high temperature limit of (3.10) with (3.13) suggests that in a basis were the fermionic field modes $\varphi_i(x^\mu)$ are represented by plane waves exp(i $k\cdot x$) there is a one-to-one correspondence between these field modes and the Pauli electrons with momentum k^μ populating a world with 2n spatial dimensions. In a (2n+1)-dimensional sense these particles are in thermal and diffusive contact with a reservoir, so that their energy and particle number density is a function of temeperature. (This is analogous to the black body radiation in the case of the photon.) In this thermodynamic picture the existence of the anomaly is equivalent to a magnetization which even in the infinite temperature limit is not

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(*) One factor of 2 in (3.14) is due to the fact that Ψ and $\overline{\Psi}$ independently contribute to the anomalous Jacobian for a chiral rotation. The other factor of 2 is due to the normalization (3.1).

(**) The gauge coupling is set equal to unity.

completely destroyed by the thermal agitation. For this interpretation to be possible it is crucial to use non-relativistic particles with a kinetic term quadratic in p^μ. Using a relativistic kinetic term linear in p^μ would yield a wrong answer.

4. The trace anomaly

Also the trace (or conformal) anomaly of Dirac fermions can be interpreted in the above framework. In Fujikawa's treatment /8/ the trace anomaly appears as an expression which is very similar to (1.2) with (1.3), viz.,

$$\langle 0 | T^\mu_\mu (x) | 0 \rangle \equiv \mathcal{A}_{tr}(x) = \lim_{M\to\infty} \mathcal{A}_{tr}(x;M) \tag{4.1}$$

where

$$\mathcal{A}_{tr}(x;M) = \sum_i \varphi_i^+(x)\, f\left(\frac{\not{D}^2}{M^2}\right) \varphi_i(x) \ . \tag{4.2}$$

The only difference between (4.2) and (1.3) is the absence of the matrix γ_{2n+1}. Restricting the discussion to 4 dimensions (n=2), Fujikawa obtains for (4.2)

$$\mathcal{A}_{tr}(x;M) = \frac{M^4}{(2\pi)^4}\int_0^\infty ds\, s f(s) + \frac{1}{24\pi^2} F_{\mu\nu}F^{\mu\nu} + \mathcal{O}\left(\frac{1}{M}\right) . \tag{4.3}$$

Renormalizing $\langle T^\mu_\mu \rangle$ at vanishing background field, i.e., subtracting the divergent M^4 piece from (4.3), the renormalized anomaly is given by $1/(24\pi^2) F_{\mu\nu}F^{\mu\nu}$. It is independent of the choice of f(s). To make contact with statistical mechanics we use the form (3.1) corresponding to Fermi-Dirac statistics. Employing (3.8) with (3.15) we then find

$$\mathcal{A}_{tr}(x;M) = 2\,\frac{1}{V}\,\mathrm{Tr}\left(1 + e^{\not{D}^2/M^2}\right)^{-1} = 2n\left(\frac{M^2}{2m}, 0\right) . \tag{4.4}$$

Obviously the unrenormalized trace anomaly (4.3) is equal to the particle number density for $T = M^2/2m$. Its leading contribution (3.9) which corresponds to the M^4-term in (4.3) is independent of $F_{\mu\nu}$ and hence is irrelevant for the renormalized anomaly. Thus the renormalized trace anomaly can be obtained as the temperature independent part of the particle number density.

5. Conclusion

We have shown that for a large, but finite, value of the cutoff M the field modes $\varphi_i(x^\mu)$ behave like a grand canonical ensemble of Pauli electrons. The regularized chiral anomaly $\mathcal{A}_{2n}(x;M)$ can be obtained within this quantum statistical mechanics system as the average of the generalized magnetization defined above. A finite, but non-zero, infinite temperature limit is obtained because for high temperatures (i.e., for a large cutoff) the increasing particle number density exactly compensates for the decreasing magnetization of a single particle. Similarly, the renormalized trace anomaly is the temperature independent, but field dependent, part of the particle number density.

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