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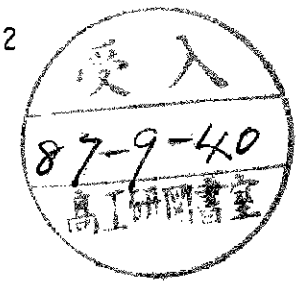

QUANTUM LIFETIME

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Quantum Lifetime

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Abstract

Quantum lifetimes in electron storage rings are evaluated within the frame work of the Kramers diffusion problem. The problem is rigourously formulated as the eigenvalue problem of the Fokker-Planck equation and/or the first-passage time problem. The domains of validity of each result are described. The main concern lies with the longitudinal stochastic dynamics and the lifetime of a particle, which differ in some respects from the transverse ones; the smooth potential barrier, diffusion across the characteristic boundary, and so on. In spite of these clear differences, the longitudinal quantum lifetime has been formulated in the same way as for the transverse case, using the energy as variable in the limit of extremely small friction. It is shown that another formulation called the transition state method which starts with the Kramers equation with the position and velocity as variables should give a more reliable estimate of the longitudinal lifetime than the formula normally used (by a factor of 2 - 5), and that the joint use of the two formulae is preferred.

1 Introduction

The particle distribution in an electron storage ring is determined by the balance between the quantum fluctuations and the radiation damping, both of which are caused by the quantum emission of photons. If the confining potential $\Phi(x)$ goes to infinity as the position x approaches $\pm$infinity, then a stationary distribution would be established after a long time. With this natural boundary condition, there is a finite number of particles for any large x, no matter what the form of $\Phi(x)$ is. In other words, the distribution has an infinitely long tail. In a real machine, however, there might be absorbing boundaries such as a vacuum chamber, or the potential well itself could be finite as in the case of the rf potential. Particles diffusing as a result of the stochastic photon emissions would be absorbed at the walls, or they could escape over a barrier from one potential well to another deeper well. The exit (escape) time required for the above processes is called the "quantum lifetime".

The exit problem for a Brownian particle in a potential well has been extensively studied since the work of Kramers[1], and the application of Kramers' method to the calculation of a quantum lifetime in a storage ring can be found in the text book of Bruck[2] or the article of Sands[3]. They solve the Fokker-Planck equation[4] with a certain boundary condition, and identify the lowest non-zero eigenvalue with the escape rate. In their treatments,

the particle distribution does not satisfy the boundary condition (actually, they use the stationary distribution function), and the lifetime is calculated simply as the the ratio of the probability current going through the boundary to the probability of finding particles near the bottom of the potential well. Nevertheless, the resulting formulae for lifetimes are good approximations when the thermal energy is much less than the height of the potential barrier (often referred to as activation energy). Chao[5] has reassessed this classical problem, and found the exact particle distribution which satisfies the boundary condition, and has given the corrected expression for the quantum lifetime in the one-dimensional case. The result converges to the familiar approximation in the limit where the ratio of the thermal energy to the potential height is small. Furthermore, he presented the approximate formula for the horizontal quantum lifetime where particles can reach the wall through diffusion not only in the horizontal-betatron phase space but also in the synchrotron phase space.

In all these treatments, however, lack of rigour makes it difficult to access in which regions of the parameters the method of solution and the final results are valid. Consequencely, for instance, the longitudinal quantum lifetime is treated in the exactly same way as the transverse lifetime, for which computations are done in energy or action space. There are, however, nontrivial differences between them. In the transverse exit problem, particles escape from an absorbing wall which has a sharp edge. They are then lost. But in the longitudinal case, particles cross a "smooth" potential barrier to another state, and they are not lost in a strict sense since they might have a small chance to come back to the initial state. In addition, their characteristic frequencies (the betatron and the synchrotron frequencies) differ by a factor of the order of $10^2 \sim 10^3$, while their damping times are of the same order.

The purpose of the paper is to give a rigourous treatment of the problem of quantum lifetime in electron storage rings and thus to contribute to a better understanding of stochastic dynamics in multi-particle systems. For this purpose, we start by deriving an appropriate Fokker-Planck equation[6] from the stochastic equation of motion with random photon emission (Langevin equation[7]). Most of the final results are not new, and could be considered to be no more than the rederivation of Chao's formulae. However, we want to find a physical interpretation of the diffusion process. The main concern of the paper lies with the longitudinal dynamics. One may ask:

- From which points do particles escape, and to where do they go ?

If they go to other rf buckets, then it follows that the longitudinal quantum lifetime becomes infinite in a machine where all the buckets are filled. It might be interesting to give answers to such fundamental questions to which little attention have been paid so far.

The paper is organized as follows. In Sec.2, we start by choosing which form for the Langevin equation we will solve. The Fokker-Planck equations are then set up by the usual formal procedure. We also introduce another approach to the exit problem which relates the lifetime to the first-passage time[8] from the domain of concern to the boundaries, and give a corresponding Fokker-Planck equation. We then solve these Fokker-Planck equations to calculate the vertical lifetime (one-variable case) in Sec.3. The result agrees with that of Chao, however the present method is applicable to a non-parabolic potential. The horizontal quantum lifetime (two-variable case) is treated in Sec.4. The resulting formula again agrees with that of Chao. Here we make a brief comment on the diffusion process in a multi-dimensional system. In Sec.5, we study the longitudinal quantum lifetime. There are three types of lifetimes. The first one is the familiar quantum lifetime derived from the one-variable

Fokker-Planck equation in terms of the energy, under the assumptiom that the friction is so small that the contours of constant probability can be approximated by the contours of constant energy. The second lifetime is derived using the so called transition state method[9], derived from the Kramers equation with no assumption on the value of the friction. We show that the second lifetime is relevant for machines in which normal parameters are used. The third lifetime is the lifetime of the oscillatory steady state[10] which can coexist with the stable equilibrium states (i.e., rf bucket). Particles which escape from an rf bucket eventually come to rest in this state, unless there are absorbing walls, and come back to the equilibrium state by fluctuation after staying there for some time. The paper is concluded in Sec.6.

2 Fokker-Planck equation

There are two languages for describing the stochastic process: the Langevin formalism and Fokker-Planck formalism. In the case of Gaussian δ-correlated white noise, they are stochastically equivalent. The following two methods have often been used in order to obtain a suitable Fokker-Planck equation:

1. One starts with the Kramers equation, a special form of the Fokker-Planck equation for the distribution function in position and velocity. Then, one introduces the polar-coordinate or the action-angle variables, and eliminates the angle dependence from the equation by the "homogenization" method to obtain an effective one-variable Fokker-Planck equation[5].

2. One makes the ansatz that the probability density is a function only of the action. One then derives a Fokker-Planck equation from the equation for the time evolution in the probability density[2].

In both of these methods, the Langevin equation corresponding to the final Fokker-Planck equation is not specified, and all the information about the angle variable is lost in the process of the homogenization. The Langevin equation has the merit that it allows an intuitive physical interpretation of the stochastic process and the roll of the fluctuation. Therefore we choose to begin with the Langevin description, and switch over to the corresponding Fokker-Planck equation later.

2.1 Langevin equation

In this subsection, we consider horizontal betatron motion. Other motions can be treated in a similar way. The simplified equation of motion of a single particle is given by Jowett[11] as

$$x'_\beta = p_\beta + \sqrt{Q}\frac{\eta}{c}\Gamma(s), \tag{2.1a}$$

$$p'_\beta = -(G^2(s) + K_1(s))x_\beta - 2\alpha_x\frac{p_\beta}{c} + \sqrt{Q}\frac{\eta'}{c}\Gamma(s), \tag{2.1b}$$

where $\Gamma(s)$ is the stochastic force with zero mean and with a δ-correlation:

$$< \Gamma(s) >= 0, \quad < \Gamma(s)\Gamma(s') >= 2\delta(s - s'). \tag{2.2}$$

This force has the frequency independent spectral density, and therefore is called a white noise force. The quantity $\Gamma(s)$ is not the energy change due to a single photon emission, but the fluctuating radiation "power" under the assumption that many photons are randomly emitted at position s. Then the Central Limit Theory assures us that "the special form of the photon energy spectrum will be washed out when the energies of all the individual photons are added, and the distribution of $\Gamma(s)$ is gaussian to a good approximation"[12]. This trick is the essential point of the statistical treatment of the synchrotron radiation as a tractable stochastic process. Here, η is the dispersion function, c is the velocity of light, G and K_1 are the normarized field strengths of the dipole and the quadrupole fields, respectively, α_x is the linear damping rate, and the prime denotes differentiation with respect to s. The constant Q is given by

$$Q = \frac{55}{48\sqrt{3}} r_e \hbar \frac{p_0^3}{(mc)^6} (p_0 c)^2 G^3, \tag{2.3}$$

where r_e is the classical electron radius, $\hbar$ is the Planck constant, p_0 is the longitudinal momentum of a reference particle, and m is the rest mass of the electron. As usual, we introduce normalized variables x and p by

$$x = \frac{x_\beta}{\sqrt{\beta}}, \tag{2.4a}$$

$$p = \sqrt{\beta}[p_\beta - x_\beta \cdot \frac{\beta'}{2\beta}], \tag{2.4b}$$

where β is the betatron function which satisfies the equation[13]

$$\frac{\beta\beta''}{2} - \frac{\beta'^2}{4} + (K_1 + G^2)\beta^2 = 1. \tag{2.5}$$

The equations of motion may be rewritten in the new variables as

$$x' = \frac{p}{\beta} + \sqrt{Q}\frac{\eta}{\sqrt{\beta}c}\Gamma(s), \tag{2.6a}$$

$$p' = -\frac{x}{\beta} - 2\frac{\alpha_x}{c}p + \frac{\sqrt{Q}}{\sqrt{\beta}c}[-\frac{\beta'\eta}{2} + \beta\eta']\Gamma(s), \tag{2.6b}$$

where we have omited the term $-\frac{\alpha_x}{c}\beta' x$ in the RHS of the second equation since it only slightly modifies the betatron frequency. If we now change the independent variable from s to a betatron phase advance $\psi(s)$ with $\psi = \int^s \frac{ds}{\beta}$, equations(2.6a,b) resemble those of a simple linear damped oscillator driven by white noise. This scheme, however, makes it difficult to extend the formalism to the multi-dimensional case including other motions. Therefore, we keep s as the independent variable.

In the absence of the stochastic force $\Gamma(s)$, Eqs(2.6a,b) can be written in the combined form as

$$x'' + [\frac{2\alpha_x}{c} + \frac{\beta'}{\beta}]x' + \frac{x}{\beta^2} = 0. \tag{2.7}$$

It is possible to generalize the problem by introducing a complex stochastic variable z which contains both amplitude and phase information[14]:

$$x = ze^{i\psi} + z^* e^{-i\psi}. \tag{2.8}$$

It should be noted that the fast variation in x due to the betatron oscillation is absorbed by the betatron phase advance ψ, and the slow drift and diffusion of particles in the azimuthal direction is represented by the phase of z. If we insert Eq.(2.8) into Eq.(2.7), and then neglect the small terms z' in x' and z'' in x'', we obtain the rotating wave approximation[15] to Eq.(2.7)

$$z' + \frac{\alpha_x}{c} z = 0, \tag{2.9}$$

or in the real notation $z = z_1 + iz_2$,

$$z_i' + \frac{\alpha_x}{c} z_i = 0. \quad i = 1, 2 \tag{2.10}$$

Adding the stochastic force $\Gamma(s)$, we have

$$z_i' + \frac{\alpha_x}{c} z_i = \sqrt{q}\Gamma_i, \quad i = 1, 2 \tag{2.11}$$

$$< \Gamma_i(s) >= 0, \ < \Gamma_i(s)\Gamma_j(s') >= 2\delta_{ij}\delta(s - s'), \tag{2.12}$$

where we have split $\Gamma(s)$ into the real and the imaginary parts: $\Gamma = \Gamma_1 + i\Gamma_2$. Since z_i varies on a time scale comparable to the damping time which is much larger than the revolution time, to use the rotating wave approximation, we can replace q by its average over the revolution time:

$$q = \langle \frac{Q}{2\beta c^2}((-\frac{\beta'\eta}{2} + \beta\eta')^2 + \eta^2) \rangle_{av}. \tag{2.13}$$

We introduce the action-angle variables I_s and φ_s defined by

$$2I_s = z_1^2 + z_2^2, \tag{2.14a}$$

$$\varphi_s = \tan^{-1} z_2/z_1. \tag{2.14b}$$

The Langevin equation(2.11) in these new variables reads

$$I_s' + 2\frac{\alpha_x}{c} I_s = \sqrt{2qI_s}\cos\varphi_s\Gamma_1 + \sqrt{2qI_s}\sin\varphi_s\Gamma_2, \tag{2.15a}$$

$$\varphi_s' = -\frac{\sin\varphi_s}{\sqrt{2I_s}}\sqrt{q}\Gamma_1 + \frac{\cos\varphi_s}{\sqrt{2I_s}}\sqrt{q}\Gamma_2. \tag{2.15b}$$

2.2 Fokker-Planck equation

The transition probability $P(I, \varphi, I_0, \varphi_0, s)$ of the stochastic variables of Eqs.(2.15a,b) defined as

$$P(I, \varphi, I_0, \varphi_0, s) = \Pr(I_s(s) \in (I, I + dI), \varphi_s(s) \in (\varphi, \varphi + d\varphi) | I_s(0) = I_0, \varphi_s(0) = \varphi_0) \tag{2.16}$$

is the solution of the Fokker-Planck equation

$$\frac{\partial P}{\partial s} = [-\frac{\partial}{\partial I} D_I - \frac{\partial}{\partial \varphi} D_\varphi + \frac{\partial^2}{\partial I^2} D_{II} + \frac{\partial^2}{\partial I \partial \varphi} D_{I\varphi} + \frac{\partial^2}{\partial \varphi^2} D_{\varphi\varphi}]P \tag{2.17}$$

with the initial condition

$$P(I, \varphi, I_0, \varphi_0, 0) = \delta(I - I_0)\delta(\varphi - \varphi_0). \tag{2.18}$$

Given an initial particle distribution $f(I_0, \varphi_0)$, the function $W(I, \varphi, s)$

$$W(I, \varphi, s) = \int P(I, \varphi, I_0, \varphi_0, s) f(I_0, \varphi_0) dI_0 d\varphi_0 \tag{2.19}$$

is the probability density for I_s, φ_s at time $t = s/c$ which also satisfies the Fokker-Planck eq.(2.17). Since the Langevin forces Γ_i have Gaussian distributions with δ-correlation[12], the Fokker-Planck equation is the exact equation for the probability density $W(I, \varphi, s)$, i.e., all the terms higher than the second derivatives vanish (Pawula theorem[16]).

The drift and the diffusion coefficients are read from Eqs.(2.15a,b) (see Appendix A)

$$D_I = -\frac{2\alpha_x}{c} I + 2q, \tag{2.20a}$$

$$D_\varphi = 0, \tag{2.20b}$$

$$D_{II} = 2qI, \tag{2.20c}$$

$$D_{\varphi^2} = \frac{q}{2I}, \tag{2.20d}$$

$$D_{I\varphi} = 0. \tag{2.20e}$$

Note that there is no drift in the φ-direction. The explicit expression of the Fokker-Planck equation is then,

$$\frac{\partial W}{\partial s} = [-\frac{\partial}{\partial I}(-\frac{2\alpha_x}{c} I + 2q) + \frac{\partial^2}{\partial I^2} 2qI + \frac{q}{2I} \frac{\partial^2}{\partial \varphi^2}] W \tag{2.21}$$

which agrees with that in Bruck's text book except for the last term in the RHS, which expresses the diffusion in the φ-direction. In the present formalism, the Fokker-Planck equation is still two dimensional. The probability density is normalized at $s = 0$ so that

$$\int_0^{I_{max}} \int_0^{2\pi} W(I, \varphi, 0) dI d\varphi = 1. \tag{2.22}$$

The inverse of the lowest eigenvalue of Eq.(2.21) is the average lifetime for absorption by the walls. Obviously, the corresponding eigenfunction should have no phase dependence[17], therefore Eq.(2.21) can be reduced to a one-variable Fokker-Planck equation

$$\frac{\partial W}{\partial s} = [-\frac{\partial}{\partial I}(-\frac{2\alpha_x}{c} I + 2q) + \frac{\partial^2}{\partial I^2} 2qI] W. \tag{2.23}$$

The form (2.23) in which the diffusion coefficient depends on I is inconvenient to handle. By the change of the variable from I to y

$$y = \sqrt{2I}, \tag{2.24}$$

the Fokker-Planck equation is transformed to

$$\frac{\partial W'}{\partial s} = [-\frac{\partial}{\partial y}(-\frac{\alpha_x}{c} y + \frac{q}{y}) + q \frac{\partial^2}{\partial y^2}] W', \tag{2.25}$$

where the diffusion coefficient is now constant. The new probability density is related to the old one by

$$W'(y, s) = y W(I, s). \tag{2.26}$$

The equation(2.25) is the Smoluchowski equation[18] with the potential $f(y)$, with

$$f'(y) = \frac{\alpha_x}{c} y - \frac{q}{y}. \tag{2.27}$$

Equation(2.25) may be also written in the form of a continuity equation (to simplify the notation we omit the prime unless confusion can occur)

$$\frac{\partial W}{\partial s} + \frac{\partial S}{\partial y} = 0, \tag{2.28}$$

where

$$S(y,s) = [-f'(s) - q\frac{\partial}{\partial y}]W \tag{2.29}$$

is called the probability current.

Finally, we have to specify appropriate boundary conditions. In the problem of transverse quantum lifetime, the vacuum chamber acts as a completely absorbing wall where the probability density should vanish:

$$W(y_{max}) = 0. \tag{2.30}$$

At the lower boundary $y = 0$, there is a reflecting wall into which particles cannot penetrate, and the probability current should vanish there:

$$S(y_{min}) = 0. \tag{2.31}$$

The boundary conditions for the longitudinal case is discussed in Sec.5.

This much is all we have to do to formulate the problem. For the remainder, we must solve Eq.(2.25) with the boundary conditions(2.30) and (2.31).

Before concluding this subsection, we wish to mention some points which we carelessly ignored in the above procedure. The first question is whether the action-angle variables are just as suitable for describing the system as the simple position and velocity variables (note that Eqs.(2.15a,b) are only the approximations to Eqs.(2.6a,b), neglecting all the higher order terms). The hypothesis behind the use of action (or energy) is that the friction α_x is so small that a particle's motion can be approximated by the fluctuation about the trajectory of the constant action which is invariant in the limit of no damping and no fluctuation. In this approximation, particles no longer damp (or are attracted) to the origin $I = 0$. The condition for this is, according to Bruck,

$$\alpha_x \ll \omega_\beta, \tag{2.32}$$

where ω_β is the betatron angular frequency. In fact, the above condition is very weak. The correct condition is:

$$\alpha_x \ll \frac{\varepsilon_x}{y_{max}^2} \frac{\omega_\beta}{2\pi}, \tag{2.33}$$

where $\varepsilon_x = \frac{qc}{\alpha_x}$ is the beam emittance. The condition(2.33) and its derivation are discussed in more detail in Sec.5. In most machines, the condition(2.33) is satisfied, therefore the use of action-angle variables is allowed. If inequality(2.33) is not satisfied, we should use the original variables x and p, and then solve the two-dimensional Kramers equation[19]-[21]. In this case, we need the Wang-Uhlenbeck boundary condition[22] at $x = x_{max}$:

$$W(x_{max}, p, s) = 0 \quad \text{for } p > 0, \tag{2.34}$$

where the probability density vanishes only for the particles leaving the domain with positive velocity. For the lower boundary at $x = x_{min}$, we have similarly

$$W(x_{min}, p, s) = 0 \quad \text{for } p < 0. \tag{2.35}$$

The second point is that we have implicitly assumed that the aperture limit y_{max} is constant over the ring. In reality, it may vary along the ring as the betatron function and the size of the vacuum chamber vary. It seems that the variation of the aperture limit gives particles a chance to avoid being absorbed by the walls even if their amplitudes reach the minimum of y_{max}, with the result that the lifetime could be longer. However, since we already know from experiments that the assumption of constant aperture limit leads to a reliable formula, the correction due to the variation of the aperture limit seems likely to be negligible.

2.3 First-passage time problem

Calculation of the lowest eigenvalue of the Fokker-Planck equation is not only the way to obtain the quantum lifetime. In fact, another approach which relates the lifetime to the first-passage time[8] is more popular in other fields of physics, and sometimes more powerful. The first-passage time T is the time at which the stochastic variable leaves a given domain Ω and reaches a boundary $\partial\Omega$ for the first time. The time T depends on the initial position x' of a particle, and is also a random variable, so that it has a distribution. The first moment T_1 of the first-passage time distribution is called the mean first-passage time, and agrees approximately with the inverse of the lowest eigenvalue.

The mean first-passage time $T_1(x')$ is the solution of the Dirichlet boundary value problem (see Appendix B)[23]

$$L_{BK}(x')T_1(x') = -1/c \quad \text{in } \Omega, \tag{2.36a}$$

$$T_1(x') = 0 \quad \text{on } \partial\Omega, \tag{2.36b}$$

where L_{BK} is the backward Kolmogorov operator[6] whose explicit form for the present problem reads

$$L_{BK} = (-\frac{\alpha_x}{c}x' + \frac{q}{x'})\frac{\partial}{\partial x'} + q\frac{\partial^2}{\partial x'^2}, \tag{2.37}$$

which is actually the adjoint form of the Fokker-Planck operator of Eq.(2.25). This method extends to higher dimension more easily than the eigenvalue approach. In fact, most of the recent progress[24]-[28] in the problem of diffusion over a potential barrier has been based on this method. We show an example of a solution in the next section.

3 Vertical lifetime

The Fokker-Planck equation for the vertical lifetime is also given by Eq.(2.25), while the diffusion coefficient q is a parameter which is in most cases calculated "backwards" from the beam size. The Fokker-Planck operator of Eq.(2.25) can be written in the form

$$L_{FP} = q\frac{\partial}{\partial y}e^{-\Phi(y)}\frac{\partial}{\partial y}e^{\Phi(y)}, \tag{3.1}$$

where

$$\Phi = \frac{f(y)}{q} = \frac{y^2}{2\varepsilon_x} - \log y + \log 0_+ \tag{3.2}$$

with an arbitrary infinitsimal constant 0_+. The probability current S is written accordingly as

$$S(y) = -qe^{-\Phi(y)}\frac{\partial}{\partial y}[e^{\Phi(y)}W]. \tag{3.3}$$

To avoid the divergence problem of the potential at $y = 0$, we slightly modify the boundary condition so that the lower reflecting boundary is now at $y = 0_+$, not $y = 0$: $S(0_+) = 0$.

A separation ansatz for $W(y, s)$:

$$W(y, s) = v(y)e^{-\lambda s/c} \tag{3.4}$$

leads to

$$q\frac{\partial}{\partial y}e^{-\Phi}\frac{\partial}{\partial y}e^{\Phi}v = -\lambda v/c. \tag{3.5}$$

We solve this eigenvalue problem by Miller's asymptotic method[6,29]. Integration of Eq.(3.5) yields

$$qe^{-\Phi(y)}\frac{\partial}{\partial y}e^{\Phi(y)}v(y) - q[e^{-\Phi(y)}\frac{\partial}{\partial y}e^{\Phi(y)}v(y)]_{y=0_+} = -\lambda/c\int_{0_+}^{y} v(z)dz. \tag{3.6}$$

The second term in the LHS vanishes due to the boundary condition $S(0_+) = 0$. Integrating the above equation again, we have the integral equation

$$v(y) = e^{-\Phi(y)}[e^{\Phi(0_+)}v(0_+) - \frac{\lambda}{cq}\int_{0_+}^{y} dxe^{\Phi(x)}\int_{0_+}^{x} dzv(z)]. \tag{3.7}$$

We solve Eq.(3.7) by the following iteration method, assuming that the second term in the RHS is quite small over a wide range of y. As the zeroth approximation,

$$v_0(y) = e^{-\Phi(y)+\Phi(0_+)}v_0(0_+), \quad \lambda_0 = 0. \tag{3.8}$$

Inserting the zeroth approximation into the RHS of Eq.(3.7), we obtain the first approximation

$$v_1(y) = v_0(y)[1 - \frac{\lambda}{qc}\int_{0_+}^{y} dxe^{\Phi(x)}\int_{0_+}^{x} dze^{-\Phi(z)}]. \tag{3.9}$$

The boundary condition $v_1(y_{max}) = 0$ leads to the first order approximation of λ :

$$\lambda_1 = qc/\int_{0_+}^{y_{max}} dxe^{\Phi(x)}\int_{0_+}^{x} dze^{-\Phi(z)}. \tag{3.10}$$

If we carry out the double integral with the potential(3.2), we obtain the lifetime τ

$$\tau \cong \frac{1}{\lambda_1} = \frac{1}{2\alpha_x}[\mathrm{E}_i(\frac{y_{max}^2}{2\varepsilon_x}) - \gamma - \log\frac{y_{max}^2}{2\varepsilon_x}], \tag{3.11}$$

where $\mathrm{E}_i(x)$ is is the exponential integral[30], and γ is the Euler constant. The lifetime τ can be expressed in terms of the series expansion of $\mathrm{E}_i(x)$ as

$$\tau = \frac{1}{2\alpha_x}h(\frac{y_{max}^2}{2\varepsilon_x}), \tag{3.12}$$

where

$$h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!} \tag{3.13}$$

is the function which Chao defines in Ref.5.

The eigenfunction in the first order is written in terms of the function $h(x)$ as

$$v_1(y) = v_0(y)[1 - \frac{h(y^2/2\varepsilon_x)}{h(y_{max}^2/2\varepsilon_x)}]. \tag{3.14}$$

All of these results agree with those of Chao, while our method is applicable to a non-parabolic potential. It should be noted that as seen from Eq.(3.14), the first approximation(3.10) is valid only when $h(y_{max}^2/2\varepsilon_x) \gg 1$, or $y_{max}^2/2\varepsilon_x \gg 1$. As $y_{max}^2/2\varepsilon_x$ approaches 1, we need higher order correction terms.

If $y_{max}^2/2\varepsilon_x \gg 1$, we can derive a simple formula from Eq.(3.10)[31]. The main contribution to the double integral comes from the sharp maximum of $\Phi(y)$ at $y = y_{max}$, while the first integral $\int_{0_+}^{x} dz e^{-\Phi(z)}$ is extremely close to 1 in the vicinity of $y = y_{max}$. Therefore, we can write approximately

$$\lambda_1 \simeq qc / \int_{0_+}^{y_{max}} dx e^{\Phi(x)} \tag{3.15}$$

which yields the familiar Kramers' formula

$$\tau \cong \frac{\varepsilon_x}{\alpha_x y_{max}^2} e^{\frac{y_{max}^2}{2\varepsilon_x}}. \tag{3.16}$$

It should be emphasized that it is the behaviour of $\Phi(y)$ around $y = y_{max}$ that is most crucial in determining the lifetime. If the uniformity of the quadrupole gradient fails at large amplitude, we can no longer use the formula(3.16), but must carry out the integration(3.10) with the correct potential $\Phi(y)$. For any potential, however, it is proved[32] that we can write the lifetime in the form

$$\tau = \omega e^{\frac{\Delta U}{kT}}, \tag{3.17}$$

where ΔU is the potential height, and kT is the thermal energy. The problem is then to find a suitable frequency ω.

Let us solve the same problem as a first-passage time problem. The backward Kolmogolov operator(2.37) can be written in the form

$$L_{BK} = q e^{\Phi(y')} \frac{\partial}{\partial y'} e^{-\Phi(y')} \frac{\partial}{\partial y'}. \tag{3.18}$$

Integrating Eq.(2.36a) after dividing both sides by $qe^{\Phi(y')}$, and integrating again after dividing both sides by $e^{-\Phi(y')}$, we obtain

$$T_1(y') = \frac{1}{qc} \int_{y'}^{y_{max}} dx e^{\Phi(x)} \int_{0_+}^{x} dz e^{-\Phi(z)} \tag{3.19}$$

which is equal to the inverse of λ_1 for $y' = 0_+$ (see Eq.(3.10)). Therefore, the formula(3.12) is exact as the mean first-passage time and is valid for any $y_{max}^2/2\varepsilon_x$, while it is only the first-order approximation of the lowest eigenvalue valid for $y_{max}^2/2\varepsilon_x \gg 1$, although the deviation from the first-passage time might be quite large if $y_{max}^2/2\varepsilon_x \approx 1$.

The quantity y_{max} in the physical cooordinate system is given by

$$y_{max}^2 = \left(\frac{A^2}{\beta}\right)_{min}, \tag{3.20}$$

where A is the vertical aperture of the vacuum chamber.

4 Horizontal lifetime

In the case of horizontal motion, the amplitude receives contribution from synchrotron motion at the places where the dispersion is non-zero. Thus synchrotron motion can also lead to loss of particles in the horizontal space. The Fokker-Planck equation is now 4-dimensional. However, we reduce the number of variables to two under the following assumptions:

1. The horizontal damping rate satisfies the condition(2.33).
2. The absorbing boundary is in a region of the longitudinal space where the rf potential can be linearized.
3. The longitudinal damping rate satisfies the condition(2.33).

The longitudinal equation of motion in position, z, and the momentum deviation, ε, is given by Jowett[11] as

$$z' = -\alpha\varepsilon, \tag{4.1a}$$

$$\varepsilon' = \frac{\omega_s^2}{\alpha c^2} z - \frac{2\alpha_\varepsilon}{c}\varepsilon - \frac{\sqrt{Q}}{c}\Gamma(s), \tag{4.1b}$$

where α is the momentum compaction factor, ω_s is the synchrotron angular frequency at zero amplitude, and α_ε is the linear longitudinal damping rate. The position z is measured from the center of bunch, z_s. Through the same procedure as in Sec.3, we arrive at the rotating wave approximation to Eqs.(4.1a,b)

$$\xi_i' = -\frac{\alpha_\varepsilon}{c}\xi_i + \sqrt{q_\varepsilon}\Lambda_i, \quad i = 1,2 \tag{4.2}$$

$$< \Lambda_i(s) >= 0, \quad < \Lambda_i(s)\Lambda_j(s') >= 2\delta_{ij}\delta(s-s'), \tag{4.3}$$

where

$$\varepsilon = \xi e^{-i\omega_s s/c} + \xi^* e^{i\omega_s s/c}, \tag{4.4}$$

$$\xi = \xi_1 + i\xi_2, \quad \Gamma(s) = \Lambda_1 + i\Lambda_2, \tag{4.5}$$

and

$$q_\varepsilon = \langle \frac{Q}{2c^2} \rangle_{av}. \tag{4.6}$$

Since in this formalism, motions in the betatron and synchrotron planes are uncorrelated, we can assume that the stochastic forces $\Gamma_1, \Gamma_2, \Lambda_1$, and Λ_2 are independent. (Note that there must be a coupling between the betatron and the synchrotron motions, since all the stochastic forces come from the emission of the same photon.) Then the Fokker-Planck operator for the synchrotron motion is

$$L_{FPL} = [-\frac{\partial}{\partial J}(-\frac{2\alpha_\varepsilon}{c}J + 2q_\varepsilon) + \frac{\partial^2}{\partial J^2}2q_\varepsilon J + \frac{q_\varepsilon}{2J}\frac{\partial^2}{\partial\phi^2}], \tag{4.7}$$

where

$$2J = \xi_1^2 + \xi_2^2, \tag{4.8a}$$

$$\phi = \tan^{-1}\xi_2/\xi_1. \tag{4.8b}$$

Again, since it is expected that the eigenfunction of the lowest eigenvalue has no phase dependence, Eq.(4.7) reduces to a one-variable Fokker-Planck operator. If we change the variables from I and J to $y_1 = \sqrt{2I}$ and $y_2 = \sqrt{2J}$, and write the resulting Fokker-Planck equation in the symmetrical form, we obtain

$$\partial_s W' = [-\partial_i b^i + \partial_i \partial_j a^{ij}] W', \tag{4.9}$$

where we have used the simpler notations $\partial_i = \frac{\partial}{\partial y_i}$ and Einstein's summation convention that repeated indices are summed from 1 to 2. The drift and the diffusion coefficients are

$$b^1 = -\frac{\alpha_x}{c} y_1 + \frac{q}{y_1}, b^2 = -\frac{\alpha_\epsilon}{c} y_2 + \frac{q_\epsilon}{y_2}, \tag{4.10}$$

$$a^{11} = q, \quad a^{22} = q_\epsilon, \quad a^{12} = a^{21} = 0, \tag{4.11}$$

and the new and old probability densities are related by

$$W' = y_1 y_2 W. \tag{4.12}$$

Before solving the eigenvalue problem(4.9), let us summarize the essential points of Miller's method in the one-variable case. The approximation(3.16) is directly obtained from Eq.(4.9) in the following way[29] ($i = j = 1$, and the indices are omitted for simpler notation). If we integrate Eq.(4.9) from 0_+ to y_{max} after substituting $W' = ve^{-\lambda s/c}$, the result is

$$\lambda/c = \frac{[bv - \partial_y(av)]_{0_+}^{y_{max}}}{\int_{0_+}^{y_{max}} v dy}. \tag{4.13}$$

From the boundary condition $bv = 0$ at $y = y_{max}$ and $S = bv - \partial_y(av) = 0$ at $y = 0_+$, Eq.(4.13) becomes

$$\lambda/c = \frac{-\partial_y(av)|_{y=y_{max}}}{\int_{0_+}^{y_{max}} v dy}. \tag{4.14}$$

Then, we substitute the stationary distribution $v = e^{-\Phi(y)}$ into Eq.(4.14), and obtain

$$\lambda/c = \frac{a \cdot e^{-\Phi(y)} \partial_y \Phi|_{y=y_{max}}}{\int_{0_+}^{y_{max}} e^{-\Phi(y)} dy} \tag{4.15}$$

which leads to the formula(3.16).

The above procedure can be carried out in the same way in the two-dimensional case. Let v be the solution of the eigenvalue problem:

$$[-\partial_i b^i + \partial_i \partial_j a^{ij}] v = -\lambda v/c. \tag{4.16}$$

The geometry of the domain Ω and its boundary conditions are shown in Fig.1. At the boundaries $y_i = 0_+$ ($\partial\Omega_i$), the probability currents $S_i = b^i - \partial_j a^{ij}$ should vanish, while at the boundary $y_1\sqrt{\beta} + y_2\eta = A$, the probability density v vanishes. If we integrate Eq.(4.16) over the domain Ω with the boundary $\partial\Omega_i$, we obtain

$$\lambda/c = \frac{-\int_{\partial\Omega_3} \partial_j(a^{ij} v) dy_i}{\int_\Omega v dy_1 dy_2}. \tag{4.17}$$

It should be noted that only the diffusion term remains in the numerator, since the drift term $-b^i v$ is directed towards the centre of Ω, and represents a force which brings particles back to the domain. As was suggested by the one dimensional case, we can use the stationary distribution as an approximation to v, which, however, does not satisfy the boundary condition $v = 0$ at $\partial\Omega_3$:

$$\lambda/c = \frac{\int_{\partial\Omega_3} a^{ij} e^{-\Phi(y_1,y_2)} \partial_j \Phi dy_i}{\int_\Omega e^{-\Phi} dy_1 dy_2}, \tag{4.18}$$

where

$$\Phi = \frac{y_1^2}{2\varepsilon_x} - \log y_1 + \frac{y_2^2}{2\sigma_\epsilon^2} - \log y_2 + \log 0_+ \tag{4.19}$$

with the standard deviation of momentum error $\sigma_\epsilon = \sqrt{\frac{q_\epsilon c}{\alpha_\epsilon}}$.

The integration of the numerator can be carried out by Laplace's method unless $1 \ll r \equiv \frac{\sigma_\epsilon \eta}{\sqrt{\varepsilon_x \beta}}$ or $r \ll 1$. The result is

$$\frac{1}{\tau} = \lambda \cong \sqrt{2\pi} \left(\frac{A}{\sigma_h}\right)^3 \frac{r}{(1+r^2)^2} (\alpha_x + \alpha_\epsilon r^2) e^{-\frac{A^2}{2\sigma_h^2}}, \tag{4.20}$$

where we have defined the total horizotal beam size $\sigma_h = \sqrt{\varepsilon_x \beta + \sigma_\epsilon^2 \eta^2}$. The formula(4.20) was first derived by Chao[5] and Kohaupt[33], independently. Equation(4.20) does not converge to the one-dimensional result of Eq.(3.16) as the ratio r approaches infinity or zero. In these cases, one should use the one dimensional formula(3.16). The expression is, however, as pointed out by Chao, an excellent approximation to Eq.(4.18) in quite a wide range

$$\frac{\sigma_h}{A} < r < \frac{A}{\sigma_h}. \tag{4.21}$$

The proof of the validity of the approximate formula(4.18) was given by Matkowsky and Schuss in terms of the asymptotic singular perturbation method[25].

Let us consider how particles escape from the domain. It was pointed out by Wentzel and Freidlin[34] that if there is a point on the boundary where the minimum of the potential is attained, particles escape from that point with almost unit probability. In the present case, this point is the closest point of contact of $\partial\Omega$ with the largest ellipse $y_1^2/2\varepsilon_x + y_2^2/2\sigma_\epsilon^2 =$ constant, centered at the origin inscribed in Ω. The flow of probability across this exit point is illustrated in Fig.2. Laplace's method is applied to this singular point from which the main contribution to the numerator of Eq.(4.18) comes.

5 Longitudinal lifetime

The problem of the transverse lifetime is that of an absorbing boundary where the potential suddenly jumps to an infinitely large negative value. The boundary is unrelated to the shape of the potential. The problem of the longitudinal lifetime differs from that of the transverse one in the following ways:

1. Particles move in the periodic potential with a constant external force.

2. The potential barrier is smooth.

3. Particles diffuse across the characteristic boundary, i.e., the boundary curve consists of a trajectory of the deterministic motion of particles without noise (separatrix).

4. The ratio of the synchrotron frequency to the longitudinal damping rate is much smaller than that for the transverse case.

In spite of these clear differences, the longitudinal quantum lifetime has been formulated in the same way as for the transverse case, using the energy as variable as shown later. In fact, this can be justified in the limit of extremely small friction. There is another formulation which starts with the Kramers equation with the position and velocity as variables, and leads to an expression for the lifetime without assumptions for the value of the friction. This later formulation turns out to be relevant in most storage rings.

In addition to the equilibrium steady state, there is the non-equilibrium oscillatory steady state in which particles escaping from the equilibrium states will be trapped[10]. Particles stay there for some time, and come back to the equilibrium state by fluctuation. Therefore, the non-equilibrium state also has a lifetime. In order to give a clear picture to the reader, we begin this section by reviewing the deterministic dynamics of a particle without noise.

5.1 Deterministic dynamics of a particle

The equation of motion for the coordinate z of a particle without noise is given by

$$z'' + \frac{2\alpha_\varepsilon}{c} z' + \frac{e\hat{V}\alpha}{2\pi R p_0 c}[\sin\frac{h}{R}(z+z_s) - \sin\frac{h}{R}z_s] = 0, \tag{5.1}$$

where $\hat{V}$ is the rf voltage, h is the harmonic number,and $\phi_s = \frac{h}{R}z_s$ is the stable phase angle. Introducing a dimensionless variable $\phi = \frac{h}{R}z$, we transform the period of the potential to 2π:

$$\phi'' + G\phi' + U'(\phi) = 0, \tag{5.2}$$

where we have used the abbreviation $G = \frac{2\alpha_\varepsilon}{c}$, and the potential $U(\phi)$ is given by

$$U(\phi) = -d(\cos(\phi+\phi_s) - \cos\phi_s) - F\phi \tag{5.3}$$

with $d = \frac{e\hat{V}h\alpha}{2\pi R^2 p_0 c}$, and $F = d\sin\phi_s$. The sketch of the potential is shown in Fig.3. Equation(5.2) is equivalent to that of the underdamped forced pendulum with the friction G and external torque F. If the torque F is small, the pendulum will oscillate around its downside position, while if the torque is large and the value of G is in the range $0 < G < G_M = \frac{\pi F}{4\sqrt{d}}$, the pendulum will whirl continuously[10]. The former and the latter motions correspond to the equilibrium state and the non-equilibrium oscillatory state, respectively. When noise is included, transitions between these states occur[35]. The typical phase space plot is shown in Fig.4. The points E and S are the solutions of Eq.(5.2), each corresponding to the stable equilibrium point (attractor) and the unstable equilibrium point (saddle point). The curve N is the phase space trajectory of the non-equilibrium steady state. Two states are separated from each other by the separatrix going through the saddle point. All the trajectories either spiral into one of the stable equilibrium points E (shadowed region), or merge to the trajectory of the non-equilibrium state (white region), depending on the initial position. In the presence of random noise, any trajectory inside of the separatrix can cross to the domain

of the non-equilibrium state Ω_N in finite time. They will spend a long time fluctuating in the vicinity of E, but eventually reach the separatrix, and cross to Ω_N through the saddle point with almost unit probability. This is because the particles spend longer near the saddle point, and have more opportunity to be pushed out by fluctuation than other points on the separatrix. It should be emphasized that the resulting particle distribution outside of the separatrix is one-sided in the positive velocity direction[36].

5.2 Energy-angle variable formalism

The Langevin equation is written as a system of two first-order equations:

$$z' = -\alpha\varepsilon, \tag{5.4a}$$

$$\varepsilon' = -\alpha K'(z) - \frac{2\alpha_\varepsilon}{c}\varepsilon, + \frac{\sqrt{Q}}{c}\Gamma(s), \tag{5.4b}$$

where we have defined a dimensionless potential

$$K(z) = \frac{e\hat{V}}{2\pi p_0 ch\alpha}[\cos\frac{h}{R}(z+z_s) - \cos\frac{h}{R}z_s + \frac{h}{R}z\sin\frac{h}{R}z_s]. \tag{5.5}$$

The distribution function $W(z,\varepsilon,s)$ in position and velocity space obeys the Kramers equation:

$$\frac{\partial W}{\partial s} = L_K W, \tag{5.6}$$

$$L_K = \frac{\partial}{\partial z}\alpha\varepsilon + \frac{\partial}{\partial\varepsilon}[\frac{2\alpha_\varepsilon}{c}\varepsilon + \alpha K'(z)] + 2q_\varepsilon\frac{\partial^2}{\partial\varepsilon^2}]. \tag{5.7}$$

In the limit of no friction and no dissipation, the energy

$$E = \frac{\varepsilon^2}{2} + K(z) \tag{5.8}$$

is the invariant of motion. For very small friction and dissipation, the energy will vary slowly during the time of oscillation, and then "the effect of Brownian motion consists of the gradual change of the distribution function over the different energy values"[1]. In this case, we can solve Eq.(5.6) in terms of the energy variable, closely following the method of Kramers[1]. Firstly, we define the action I by $I = \oint\varepsilon dz$ which denotes the area inside a curve of constant energy. The quantity WdI then denotes the fraction of the particle density lying inside of the ring-shaped area dI. The equation for $W(E,s)$ is given by averaging Eq.(5.6) over a trajectory of constant E. Using $\langle y^2\rangle_{av} = I(E)\omega(E)/2\pi$, where the frequency $\frac{\omega}{2\pi} = \frac{dE}{dI}$, we have

$$\frac{\partial W}{\partial s} = \frac{\alpha_\varepsilon}{\pi c}\frac{\partial}{\partial E}(I\omega + \frac{I\omega}{\sigma_\varepsilon^2}\frac{\partial}{\partial E})W. \tag{5.9}$$

It should be noticed that Eq.(5.9) is slightly different from Eq.(26.12) in Bruck's text book where $I\omega/2\pi$ is replaced by E. This is true only if the potential is parabolic. The potential field which we consider now has the shape as shown in Fig.3 with a smooth barrier. The top of the potential barrier is the saddle point. The biggest problem with the smooth potential barrier is that one cannot specify the appropriate boundary conditions of W. The particle distribution does not have to vanish on the separatrix, in contrast to a potential with an edge

shaped barrier. We then have to make an assumption that there is an absorbing wall in the vicinity of the saddle point. Particles which reach this point will never come back.

The probability current is properly defined in terms of the gradient with respect to I as

$$\frac{\partial W}{\partial s} + \frac{\partial S}{\partial I} = 0, \tag{5.10}$$

where

$$\begin{aligned} S &= -\frac{2\alpha_\epsilon}{c}(I + \frac{I}{\sigma_\epsilon^2}\frac{\partial}{\partial E})W, \\ &= -\frac{2\alpha_\epsilon}{c} I\sigma_\epsilon^2 e^{-\frac{E}{\sigma_\epsilon^2}}\frac{\partial}{\partial E}(e^{\frac{E}{\sigma_\epsilon^2}}W). \end{aligned} \tag{5.11}$$

If $E_{max}/\sigma_\epsilon^2 \gg 1$, we can expect that the quasi-stationary state has been established near the equilibrium point $E = E_{min}$, and the probability current must be approximately constant. Integrating Eq.(5.11) between $E_{min} = 0_+$ and E_{max} along the E coordinate, we obtain

$$S \cong \frac{2\alpha_\epsilon}{c}\sigma_\epsilon^2 W(0_+)/[\int_{0_+}^{E_{max}} \frac{1}{I} e^{E/\sigma_\epsilon^2} dE]. \tag{5.12}$$

Since the main contribution to the integral in the denominator stems from the region around E_{max}, we get

$$\int_{0_+}^{E_{max}} \frac{1}{I} e^{E/\sigma_\epsilon^2} dE \cong \frac{\sigma_\epsilon^2}{I_{max}} e^{\frac{E_{max}}{\sigma_\epsilon^2}}, \tag{5.13}$$

where I_{max} is the action along the separatrix. The probability p to find particles near E_{min} is

$$p \cong \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(0_+) e^{-\frac{\epsilon^2 + K(z)}{2\sigma_\epsilon^2}} d\epsilon dz = \frac{2\pi\sigma_\epsilon^2}{\omega_s} W(0_+). \tag{5.14}$$

The lifetime τ is given by the ratio of p to S:

$$\tau = \frac{p}{Sc} = \frac{\pi}{\alpha_\epsilon}\frac{\sigma_\epsilon^2}{I_{max}\omega_s} e^{\frac{E_{max}}{\sigma_\epsilon^2}}. \tag{5.15}$$

In the case of a parabolic potential, $I_{max}\omega_s/2\pi = E_{max}$, then the formula(5.15) reduces to the familiar one(3.16). For a large amplitude, such as at the separatrix, the cosine potential can no longer be approximated by the parabolic potential. Even without such approximation, the expression(5.15) can be easily calculated by remembering that I_{max} is the area of the rf bucket. However, in any case, no high accuracy can be expected from the formula(5.15), since we make the arbitrary assumption that there is an absorbing wall in the vicinity of the saddle point, and consequently, that the smoothness of the potential barrier is not taken into account. The result might easily be wrong by a factor of a few units. Therefore, we may use E_{max} as a rough measure of $I_{max}\omega_s/2\pi$ in Eq.(5.15):

$$\tau \simeq \frac{1}{2\alpha_\epsilon}\frac{\sigma_\epsilon^2}{E_{max}} e^{\frac{E_{max}}{\sigma_\epsilon^2}}. \tag{5.16}$$

The explicit expression of E_{max} follows from Eq.(5.8)

$$E_{max} = \frac{e\hat{V}}{\pi p_0 ch\alpha}[-\cos\phi_s + (\frac{\pi}{2} - \phi_s)\sin\phi_s]. \tag{5.17}$$

If we take account of the possibility to escape from the potential barrier on the left hand side, the total lifetime reads

$$1/\tau_{tot} = 2\alpha_\varepsilon \frac{E_{max}}{\sigma_\varepsilon^2} e^{-\frac{E_{max}}{\sigma_\varepsilon^2}} + 2\alpha_\varepsilon \frac{E_{max}+\Delta E}{\sigma_\varepsilon^2} e^{-\frac{E_{max}+\Delta E}{\sigma_\varepsilon^2}}, \tag{5.18}$$

where

$$\Delta E = \frac{e\hat{V}\sin\phi_s}{p_0 ch\alpha}. \tag{5.19}$$

5.3 Transition state method formalism

The use of the energy variable is valid if α_ε is much smaller than ω_s. Actually, without this assumption, we can directly solve the Kramers equation(5.6) for any value of α_ε. The trouble is that the resulting formula does not reduce to the formula(5.16) for small α_ε. Rather, it yields the result

$$\tau_{tr} = \frac{2\pi}{\omega_s} e^{\frac{E_{max}}{\sigma_\varepsilon^2}} \tag{5.20}$$

which is called the lifetime in the transition state method[9]. The formula(5.20) can be also derived by the following simple consideration[1]. We assume that perfect thermal equilibrium is attained over the phase space so that we can use the stationary distribution W_{st} for W. The number of particles S which pass the transition point S from left to right in unit time is

$$\begin{aligned} S &= \int_0^\infty W_{st}(z,\varepsilon)\varepsilon d\varepsilon = e^{-\frac{E_{max}}{\sigma_\varepsilon^2}} \int_0^\infty e^{-\frac{\varepsilon^2}{2\sigma_\varepsilon^2}} \varepsilon d\varepsilon \cdot W(0_+) \\ &= \sigma_\varepsilon^2 W(0_+) e^{-\frac{E_{max}}{\sigma_\varepsilon^2}}. \end{aligned} \tag{5.21}$$

With the number of particles near E_{min} calculated in Eq.(5.14), we obtain the result(5.20)

$$\tau_{tr} = \frac{p}{S} = \frac{2\pi}{\omega_s} e^{\frac{E_{max}}{\sigma_\varepsilon^2}}. \tag{5.22}$$

A more precise formula is derived by Kramers[1,32] on the basis of the Kramers equation, and is given by

$$\tau = \frac{2\pi}{\omega_s}\{[1+(\frac{\alpha_\varepsilon}{\omega_s})^2]^{1/2} - \frac{\alpha_\varepsilon}{\omega_s}\}^{-1} e^{\frac{E_{max}}{\sigma_\varepsilon^2}}. \tag{5.23}$$

For normal machine parameters, the formula(5.20) may be sufficiently accurate.

The three lifetimes are illustrated in Fig.5 as a function of the damping constant α_ε. For a very small $\alpha_\varepsilon < \frac{\sigma_\varepsilon^2}{2E_{max}}\frac{\omega_s}{2\pi}$, the formula(5.16) gives a longer lifetime, whereas for α_ε larger than $\frac{\sigma_\varepsilon^2}{2E_{max}}\frac{\omega_s}{2\pi}$, the lifetime is almost constant. In most large storage rings, α_ε lies in the constant lifetime region. For instance, the longitudinal quantum lifetime in LEP at 55 GeV has been estimated to be 24 hours from the formula(5.16)[37]. The correct value is about 50 hours.

5.4 Oscillatory steady state

As was mentioned in subsection 5.1, particles which escape from rf buckets come to rest in the oscillatory non-equilibrium steady state. Then the thermal noise causes transition from the oscillatory steady state to the equilibrium states and visa-versa. This unfamiliar state

has been studied by Ben-Jacob et al.[10] for the hysteretic Josephson junction. According to them, the effective potential in action space can be represented as in Fig.6; it is linear in the rf bucket, and harmonic in the region of the oscillatory state. The lifetime of the oscillatory state is given by

$$\tau_o \cong \sqrt{\pi}\frac{1}{2\alpha_\varepsilon}[\frac{T}{\Delta W}]^{1/2}\exp[\frac{\Delta W}{T}], \tag{5.24}$$

where $T = (\frac{h\sigma_l}{R})^2$,

$$\Delta W \cong \frac{1}{2}[\frac{\omega_s \sin\phi_s}{2\alpha_\varepsilon\sqrt{\cos\phi_s}} - \frac{4}{\pi}]^2, \tag{5.25}$$

and σ_l is the bunch length. If we insert some practical values for the parameters, we find that τ_o becoms very large. Therefore, we can conclude that we do not have to take into account the return of escaped particles at all, in comparison with the loss of particles with very large energy deviation which hit the vacuum chamber.

6 Discussions

The quantum lifetime problem is a good exercise in understanding the stochastic dynamics in a multi-particle system. The method of analysis is classified according to the shape of potential and the value of the damping constant. The transverse and the longitudinal quantum lifetime calculations relate to different domains, and therefore require different treatments. The reexamination confirms the familiar formulae(3.16) and (4.20) for the transverse lifetimes, while for the longitudinal lifetime, we show that the formula(5.20) for the transition state method is relevant for normal machine parameters in large storage rings, or that the joint use with the formula(5.16) is preferable.

Experimental verification of this result for the longitudinal lifetime would be no easy problem, for the values of lifetime from the formulae(5.20) and (5.16) differ from each other only by a factor of 2 - 5. If one changes the rf parameters in order to bring down the lifetime to the order of 10 minutes, the difference might become smaller, or formula(5.16) might turn out to be relevant. The contributions of other loss mechanisms such as Touschek effect to the lifetime have to be rather precisely substracted from a measured value in order to see which formula gives the best description. If the present analysis were to be verified as correct, we would have a longer lifetime than that from the conventional formula with the same parameters. However, it should be mentioned that the influence of the exponential factor in the formulae is so strong that for a given lifetime, the new formula predicts little improvement in the required overvoltage ratio compared to that predicted with the conventional formula.

We analysed the longitudinal quantum lifetime as a purely longitudinal problem, and we did not consider the existence of the absorbing vacuum chamber wall and the horizontal dispersion. This is justified for the following reason: as mentioned in Sec.4, the most likely exit point in the betatron and the synchrotron amplitude space is normally located near the periphery of the space. Particles which hit the vacuum chamber must have relatively large betatron and synchrotron amplitudes. On the contrary, most of the particles which escape from the rf separatrix have large synchrotron amplitudes, but small betatron amplitudes, since the particle distribution in the betatron phase space is concentrated near the origin (Gaussian distribution). Therefore, the vacuum chamber acts as a selectively absorbing wall

which absorbs all particles with sufficiently large betatron amplitude, but is transparent to other particles.

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Appendix A: The drift and the diffusion coefficients

The general Langevin equation for N variables $\xi = \xi_1, \xi_2, ..., \xi_N$ reads

$$\xi_i' = h_i(\xi, s) + g_{ij}(\xi, s)\Gamma_j(s), \tag{6.1}$$

where

$$< \Gamma_i(s) >= 0, \quad < \Gamma_i(s)\Gamma_j(s') >= 2\delta_{ij}\delta(s - s'), \tag{6.2}$$

and we have used Einstein's summation convention. If the Langevin force $\Gamma_i(s)$ has a δ-correlated Gaussian distribution, the Langevin equation is equivalent to the Fokker-Planck equation

$$\frac{\partial W(\mathbf{x}, s)}{\partial s} = [-\frac{\partial}{\partial x_i}D_i(\mathbf{x}, s) + \frac{\partial^2}{\partial x_i \partial x_j}D_{ij}(\mathbf{x}, s)]W(\mathbf{x}, s) \tag{6.3}$$

for the probability density $W(\mathbf{x}, s)$ to find particles in the interval $(\mathbf{x}, \mathbf{x} + d\mathbf{x})$. The drift and the diffusion coefficients are given by

$$D_i(\mathbf{x}, s) = h_i(\mathbf{x}, s) + g_{kj}(\mathbf{x}, s)\frac{\partial}{\partial x_k}g_{ij}(\mathbf{x}, s), \tag{6.4a}$$

$$D_{ij}(\mathbf{x}, s) = g_{ik}(\mathbf{x}, s)g_{jk}(\mathbf{x}, s), \tag{6.4b}$$

respectively. The reader should refer Ref.6 for more details.

Appendix B: Differential equation for the first-passage time

We start with the Fokker-Planck equation for the transition probability $P(\mathbf{x}, s|\mathbf{x_0}, 0)$

$$\frac{\partial}{\partial s}P(\mathbf{x}, s|\mathbf{x_0}, 0) = L(\mathbf{x})P(\mathbf{x}, s|\mathbf{x_0}, 0), \tag{6.5}$$

where L is given by the inside of the square bracket in Eq.(6.3). The formal solution of the above equation is

$$P(\mathbf{x}, s|\mathbf{x_0}, 0) = \int_\Omega d\mathbf{x}'\delta(\mathbf{x} - \mathbf{x}')\exp[sL(\mathbf{x}')]\delta(\mathbf{x}' - \mathbf{x_0}), \tag{6.6}$$

where Ω is the bounded domain of concern. Equation(6.6) can be rewritten as

$$P(\mathbf{x}, s|\mathbf{x_0}, 0) = \int_\Omega d\mathbf{x}'\delta(\mathbf{x}' - \mathbf{x_0})\exp[sL^+(\mathbf{x}')]\delta(\mathbf{x} - \mathbf{x}'), \tag{6.7}$$

where L^+ is the adjoint operator of L:

$$L^+ = D_i(\mathbf{x}', s)\frac{\partial}{\partial x_i'} + D_{ij}(\mathbf{x}', s)\frac{\partial^2}{\partial x_i' \partial x_j'}. \tag{6.8}$$

Differentiation of this equation with respect to s yields the backward Kolmogorov equation

$$\frac{\partial}{\partial s}P(\mathbf{x}, s|\mathbf{x_0}, 0) = L^+(\mathbf{x_0})P(\mathbf{x}, s|\mathbf{x_0}, 0). \tag{6.9}$$

The boundary condition for P is, obviously,

$$P(\mathbf{x}, s|\mathbf{x_0}, 0) = 0 \quad \text{if } \mathbf{x_0} \in \partial\Omega. \tag{6.10}$$

We define the probability $\Sigma(\mathbf{x_0}, t)$ that a particle initially at $\mathbf{x_0}$ has not yet reached the absorbing boundary up to time $t = s/c$ as;

$$\Sigma(\mathbf{x_0}, t) = \int_\Omega d\mathbf{x}P(\mathbf{x}, s|\mathbf{x_0}, 0). \tag{6.11}$$

The mean time required for absorbtion is

$$T_1(\mathbf{x_0}) = \int_0^\infty dt\Sigma(\mathbf{x_0}, t). \tag{6.12}$$

The differential equation for $\Sigma(\mathbf{x_0}, t)$ is derived by integrating Eq.(6.9) over $\mathbf{x}$, with the result,

$$\frac{\partial}{\partial s}\Sigma(\mathbf{x_0}, t) = L^+(\mathbf{x_0})\Sigma(\mathbf{x_0}, t). \tag{6.13}$$

If we integrate Eq.(6.13) over all time, and recall that

$$\begin{aligned}\int_0^\infty dt\frac{\partial}{\partial s}\Sigma(\mathbf{x_0}, t) &= \frac{1}{c}[\Sigma(\mathbf{x_0}, \infty) - \Sigma(\mathbf{x_0}, 0)] \\ &= -\frac{1}{c},\end{aligned} \tag{6.14}$$

we obtain the differential equation(2.36a)

$$L^+(\mathbf{x_0})T_1(\mathbf{x_0}) = -1/c \quad \text{in } \Omega \tag{6.15}$$

with the boundary condition

$$T_1(\mathbf{x_0}) = 0 \qquad \text{on } \partial\Omega. \tag{6.16}$$

References

[1] H. A. Kramers, Physica, **7**, 284 (1940).

[2] H. Bruck, Accelerateurs Circulaires de Particles (Presses Universitaires de France, Paris, 1966).

[3] M. Sands, in *Physics with Intersecting Storage Rings* (B. Touschek, Ed., Academic Press, New York, 1971), pp.257-411.

[4] S. Chandrasekhar, Rev. Mod. Phys., **15**, 1 (1943).

[5] A. W. Chao, IEEE Trans. Nucl. Sci., **NS-24**, 1885 (1977), and see also PEP-136 (1975), and PEP-214 (1976) by the same author.

[6] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984).

[7] P. Langevin, Comptes rendus, **146**, 530 (1908).

[8] R. L. Stratonovich, *Topics in the Theory of Random Noise*, Vol.1 (Gordon and Breach, New York, 1963), p.79.

[9] H. Pelzer, and E. Wigner, Z. Phys. Chem., **B15**, 445 (1932).

[10] E. Ben-Jacob et al., Phys. Rev. A, **26**, 2805 (1982).

[11] J. M. Jowett, *Proc. the Joint US-CERN School on Particle Accelerators* (Sardinia 1985), p.343.

[12] J. M. Jowett, SLAC-PUB-4033 (1986).

[13] E. D. Courant, and H. S. Snyder, Ann. Phys., **3**, 1 (1958).

[14] A. Schenzle, and H. Brand, Phys. Rev. A, **20**, 1628 (1979).

[15] H. Haken, *Laser Theory, Encyclopedia of Physics*, Vol 25/2c (Springer-Verlag, Berlin, Heidelberg, New York, 1970).

[16] R. F. Pawula, Phys. Rev., **162**, 186 (1967).

[17] A. A. Kolomensky, and A. N. Levedev, *Theory of Cyclic Accelerators.*

[18] M. v. Smoluchowski, Ann. Physik, **48**, 1103 (1915).

[19] M. A. Burschka, and U. M. Titulaer, J. Stat. Phys., **25**, 569 (1981).

[20] M. A. Burschka, and U. M. Titulaer, J. Stat. Phys., **26**, 59 (1981).

[21] M. A. Burschka, and U. M. Titulaer, Physica, **112A**, 315 (1982).

[22] M. C. Wang, and G. E. Uhlenbeck, Rev. Mod. Phys, **17**, 323 (1945). For the recent progress, see R. Beals and V. Protopopescu, J. Stat. Phys., **32**, 565 (1983).

[23] A. Szabo et al., J. Chem. Phys., **72**, 4350 (1980).

[24] B. J. Matkowski, and Z. Schuss, SIAM J. Appl. Math., **33**, 588 (1977).

[25] Z. Schuss, and B. J. Matkowski, SIAM J. Appl. Math., **35**, 604 (1979).

[26] B. J. Matkowski, and Z. Schuss, SIAM J. Appl. Math., **40**, 242 (1981).

[27] B. J. Matkowski, and Z. Schuss, SIAM J. Appl. Math., **42**, 822 (1981).

[28] Z. Schuss, SIAM Rev., **22**, 119 (1980).

[29] D. Ludwig, SIAM Rev., **17**, 605 (1975).

[30] M. Abramowitz, and I. Stegun, *Handbook of Mathematical Functions* (Dover, 1965).

[31] C. W. Gardiner, J. Stat. Phys., **30**, 157 (1983).

[32] B. J. Matkowski, Z. Schuss, and E. Ben-Jacob, SIAM J. Appl. Math., **42**, 835 (1982).

[33] R. D. Kohaupt,DESY Technische Note H2-76/10 (1976).

[34] A. D. Wentzel, and M. I. Freidlin, Russian Math. Surveys, **25**, 1 (1970).

[35] H. D. Vollmer, and H. Risken, Z. Physik, **B37**, 343 (1980).

[36] H. D. Vollmer, and H. Risken, Z. Physik, **B34**, 313 (1979).

[37] LEP Design Report, CERN-LEP/84-01 (1984).

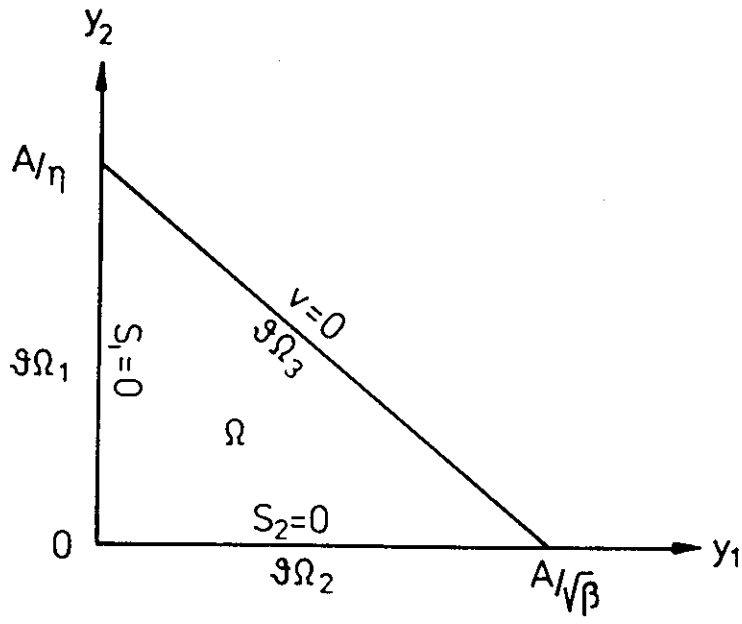

Figure 1: The geometry of the domain Ω and its boundary conditions.

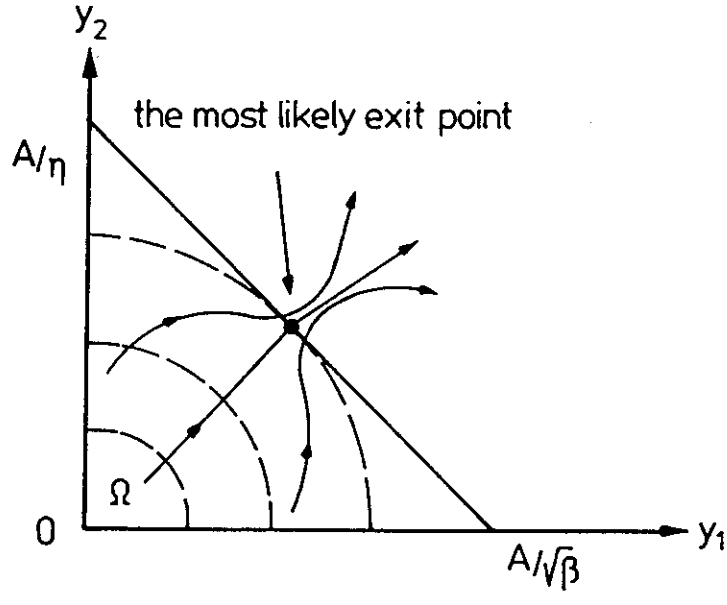

Figure 2: Schematic illustration of the flow of probability across the most likely exit point (the solid line). The broken line denotes the equipotential contours.

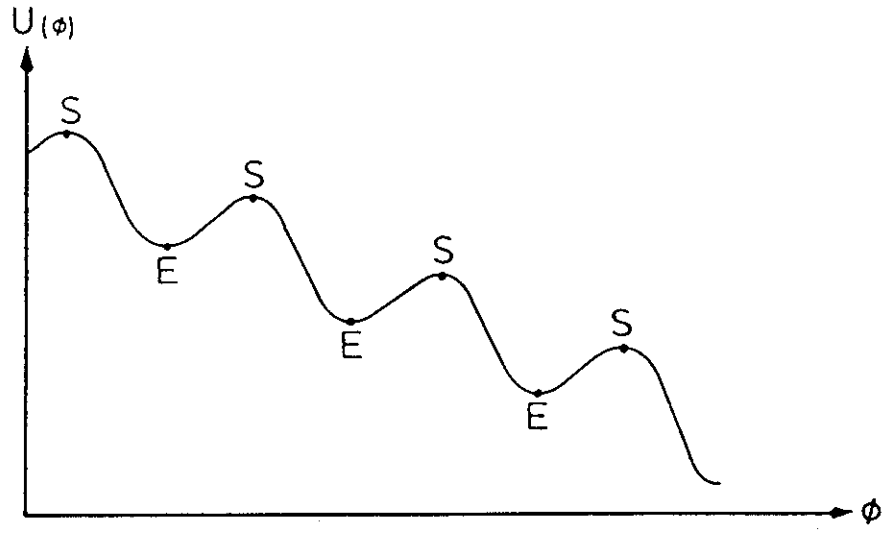

Figure 3: Sketch of the rf potential.

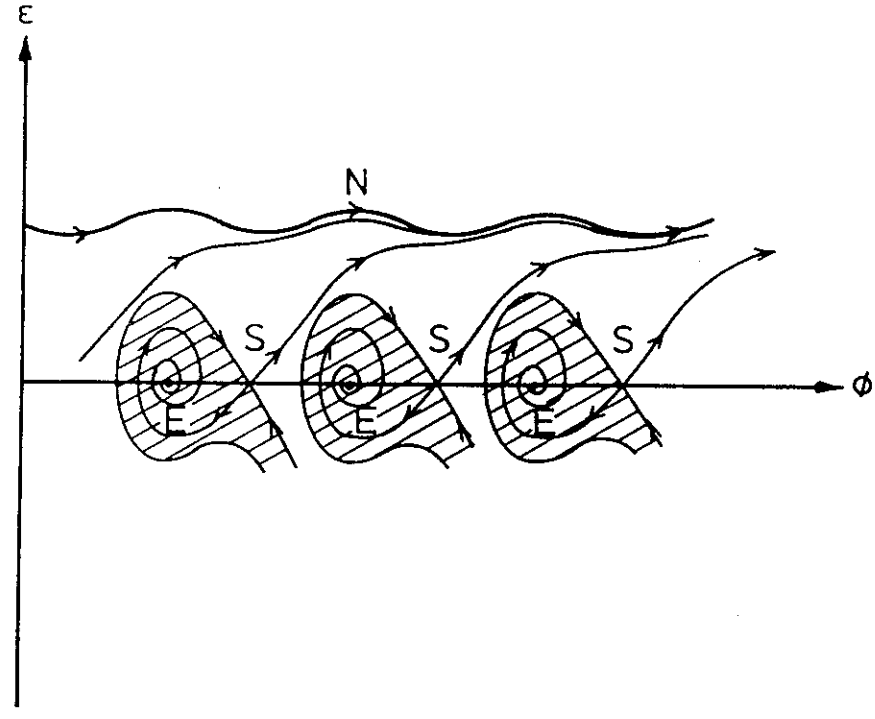

Figure 4: Typical phase space plot for an underdamped forced pendulum. The curve N is the trajectory of the non-equilibrium oscillatory steady state.

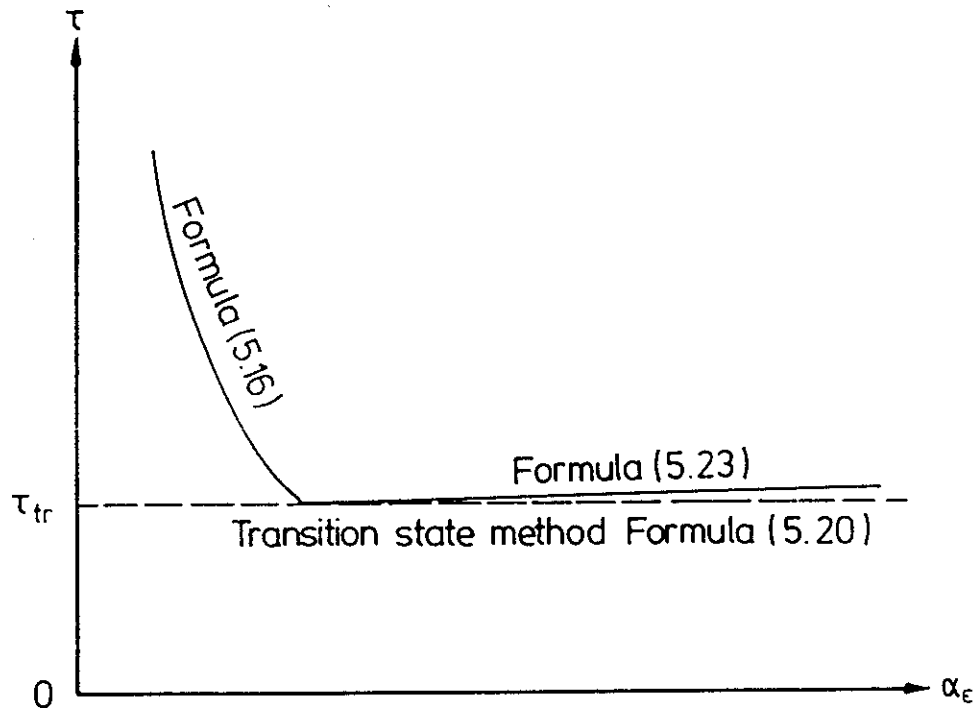

Figure 5: The longitudinal quantum lifetime τ as a function of the damping rate α_ϵ. The broken line shows τ for the transition state method.

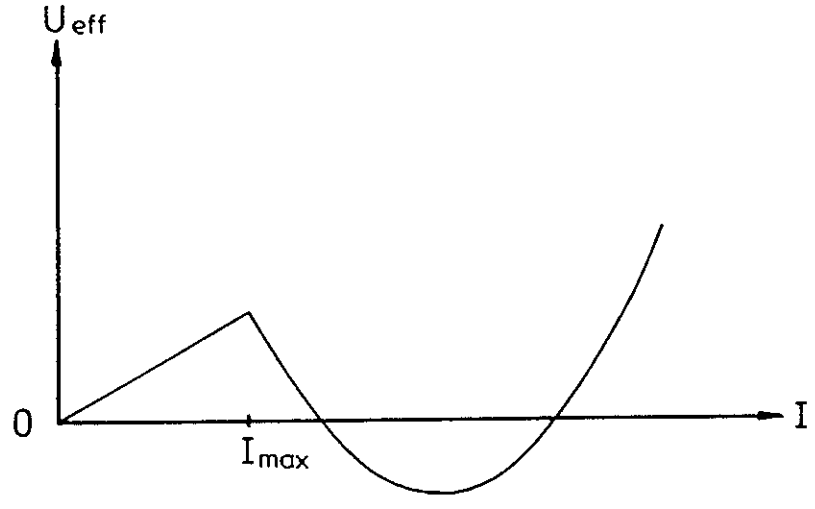

Figure 6: Sketch of the effective potential of the forced pendulum in action space.