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Abstract

We investigate the motion of protons of arbitrary energy (below and above transition energy) in a storage ring. The motion is described both in terms of the fully six-dimensional formalism with the canonical variables x, $\hat{p}_x$, z, $\hat{p}_z$, $\sigma = s - v_0 \cdot t$, $\eta = \Delta E/E_0 \equiv p_\sigma$ and in terms of a dispersion formalism with new variables $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma$. Since the dispersion function is introduced into the equations of motion via a canonical transformation the symplectic structure of these equations is completely preserved. In this formulation it is then possible to define three uncoupled linear (unperturbed) oscillation modes which are described by phase ellipses. Perturbations manifest themselves as deviations from these ellipses. The equations of motion are solved within the framework of the fully six-dimensional formalism.

The equations, so derived, could be used to develop a non-linear, six-dimensional (symplectic) tracking program for protons of arbitrary velocity.

Contents

1. Introduction

In DESY Report 85-084 we considered the non-linear equations of coupled synchro-betatron oscillations of ultrarelativistic protons and their solution within the framework of a six-dimensional formalism[1]. These equations, which are already canonical, can form the basis of a non-linear six-dimensional (symplectic) tracking program[2,3,4].

The aim of the present work is to extend this formalism to the case where:

1) The protons have arbitrary constant average velocity. Such a generalization is needed, for example, for tracking studies in DESY III[2,3].

2) The equations of motion are written using dispersion. This formalism then allows one to split the problem into an "unperturbed" part which describes linear uncoupled oscillations and a "perturbation" part representing the remaining terms. We are then in the position to investigate the influence of these "perturbation" terms by studying the deviations of the phase diagrams from the ellipses describing the linear uncoupled oscillations.

We assume that the ring contains no skew quadrupoles and solenoids so that in linear approximation the betatron oscillations are uncoupled.

2. Equations of motion

The (canonical) equations of motion for a particle in a storage ring are:

$$x' = \frac{\partial K}{\partial p_x} \quad ; \quad p_x' = -\frac{\partial K}{\partial x} \quad ;$$

$$z' = \frac{\partial K}{\partial p_z} \quad ; \quad p_z' = -\frac{\partial K}{\partial z} \quad ; \tag{2.1a}$$

$$t' = -\frac{\partial K}{\partial E} \quad ; \quad E' = \frac{\partial K}{\partial t} \quad .$$

where the Hamiltonian is the same as that introduced in Ref. 1 and is given by

$$K = -(1 + K_x \cdot x + K_z \cdot z) \cdot \left\{ \frac{E^2}{c^2} - m_0^2 c^2 - (p_x - \frac{e}{c} A_x)^2 - (p_z - \frac{e}{c} A_z)^2 \right\}^{1/2}$$

$$- (1 + K_x \cdot x + K_z \cdot z) \cdot \frac{e}{c} A_s \quad . \tag{2.1b}$$

$$\vec{A} = (A_x, A_z, A_s)$$

is the vector potential and the electric and magnetic fields are obtained from:

$$\vec{\varepsilon} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} \quad ; \tag{2.2a}$$

$$\vec{B} = \mathrm{rot}\, \vec{A} \quad . \tag{2.2b}$$

By replacing the energy, E, and time, t, by the variables

$$\eta = \frac{E - E_0}{E_0} \qquad \text{and} \qquad v_0 \cdot t$$

(E_0 = design energy ; $\beta_0 = \sqrt{1 - \left(\frac{m_0 c^2}{E_0} \right)^2}$; v_0 = design speed = $c\beta_0$)

equations (2.1) can be written in the form

$$x' = \frac{\partial \hat{K}}{\partial \hat{p}_x} \quad ; \quad \hat{p}_x' = -\frac{\partial \hat{K}}{\partial x} \quad ;$$

$$z' = \frac{\partial \hat{K}}{\partial \hat{p}_z} \quad ; \quad \hat{p}_z' = -\frac{\partial \hat{K}}{\partial z} \quad ;$$

$$(-v_o \cdot t)' = \frac{\partial \hat{K}}{\partial \eta} \quad ; \quad \eta' = -\frac{\partial \hat{K}}{\partial(-v_o \cdot t)}$$

where

$$\hat{p}_x = \frac{v_o}{E_o} \cdot p_x \quad ; \quad \hat{p}_z = \frac{v_o}{E_o} \cdot p_z \; ; \tag{2.5}$$

and
$$\hat{K} = \frac{v_o}{E_o} \cdot K = -(1 + K_x \cdot x + K_z \cdot z) \cdot \Big\{\beta_o^2 \cdot [(1+\eta)^2 - \Big(\frac{m_o c^2}{E_o}\Big)^2] -$$

$$- (\hat{p}_x - \beta_o \frac{e}{E_o} A_x)^2 - (\hat{p}_z - \beta_o \frac{e}{E_o} A_z)^2\Big\}^{1/2}$$

$$- (1 + K_x \cdot x + K_z \cdot z) \cdot \beta_o \cdot \frac{e}{E_o} A_s \; . \tag{2.6}$$

We now replace the monotonically increasing quantity $v_o \cdot t$ by the (small) oscillating quantity, σ, which describes the distance of the particle from the center of the bunch:

$$\sigma = s - v_o \cdot t \; .$$

This can be achieved by means of the canonical transformation

$$(-v_o \cdot t), \eta \longrightarrow \sigma, \bar{\eta}$$

for which we use the generating function

$$F_3(p, \bar{q}, s) \equiv F_3(\eta, \sigma, s) = -\sigma \cdot \eta + s \cdot \eta \; .$$

The transformation equations resulting:

$$(-v_0 \cdot t) = -\frac{\partial F_3}{\partial \eta} \quad \Rightarrow -v_0 \cdot t = \sigma - s \;;\; \sigma = s - v_0 \cdot t$$

$$\bar{\eta} = -\frac{\partial F_3}{\partial \sigma} \quad \Rightarrow \bar{\eta} = \eta$$

lead finally to canonical equations of the form

$$x' = \frac{\partial \mathcal{H}}{\partial \hat{p}_x} \;;\; \hat{p}_x' = -\frac{\partial \mathcal{H}}{\partial x} \;;$$

$$z' = \frac{\partial \mathcal{H}}{\partial \hat{p}_z} \;;\; \hat{p}_z' = -\frac{\partial \mathcal{H}}{\partial z} \;; \qquad (2.7a)$$

$$\sigma' = \frac{\partial \mathcal{H}}{\partial \eta} \;;\; \eta' = -\frac{\partial \mathcal{H}}{\partial \sigma} \;.$$

with the Hamiltonian

$$\mathcal{H} = \hat{K} + \frac{\partial F_3}{\partial s}$$

$$= \eta - (1 + K_x \cdot x + K_z \cdot z) \cdot \beta_0 \cdot \sqrt{(1+\eta)^2 - \left(\frac{m_0 c^2}{E_0}\right)^2} \;\times$$

$$\times \left\{1 - \frac{(\hat{p}_x - \beta_0 \frac{e}{E_0} A_x)^2 + (\hat{p}_z - \beta_0 \frac{e}{E_0} A_z)^2}{\beta_0^2 \cdot [(1+\eta)^2 - \left(\frac{m_0 c^2}{E_0}\right)^2]}\right\}^{1/2}$$

$$- (1 + K_x \cdot x + K_z \cdot z) \cdot \beta_0 \cdot \frac{e}{E_0} A_s \;. \qquad (2.7b)$$

In order to make calculations tractible the square root in (2.7b):

$$W = \left\{1 - \frac{(\hat{p}_x - \beta_0 \frac{e}{E_0} A_x)^2 + (\hat{p}_z - \beta_0 \frac{e}{E_0} A_z)^2}{\beta_0^2 \cdot [(1+\eta)^2 - \left(\frac{m_0 c^2}{E_0}\right)^2]}\right\}^{1/2}$$

must be developed in a power series:

$$W = 1 - \frac{1}{2}\, \frac{(\hat{p}_x - \beta_o \frac{e}{E_o} A_x)^2 + (\hat{p}_z - \beta_o \frac{e}{E_o} A_z)^2}{\beta_o^2 \cdot [(1+\eta)^2 - \left(\frac{m_o c^2}{E_o}\right)^2]} + \cdots . \qquad (2.8)$$

The power at which the series is truncated defines the the order of the approximation to the particle motion.

In the following, only terms up to quadratic in $\hat{p}_x$ and $\hat{p}_z$ will be kept, whence:

$$\mathcal{H} = \eta - (1 + K_x \cdot x + K_z \cdot z) \cdot \beta_o^2 \cdot (1 + \hat{\eta}) +$$

$$+ \frac{1}{2\beta_o^2} \cdot \frac{(\hat{p}_x - \beta_o \frac{e}{c} A_x)^2}{(1+\hat{\eta})} + \frac{1}{2\beta_o^2} \cdot \frac{(\hat{p}_z - \beta_o \frac{e}{c} A_z)^2}{(1+\hat{\eta})} -$$

$$- (1 + K_x \cdot x + K_z \cdot z) \cdot \beta_o \cdot \frac{e}{E_o} A_s + \cdots . \qquad (2.9)$$

with $\hat{\eta}$ defined by

$$(1 + \hat{\eta}) = \frac{1}{\beta_o} \sqrt{(1+\eta)^2 - \left(\frac{m_o c^2}{E_o}\right)^2} = \frac{1}{\beta_o} \cdot \frac{p \cdot c}{E_o} = \frac{p}{p_o} \qquad (2.10)$$

p = momentum corresponding to energy $E = E_o(1 + \eta)$;
p_o = " " " " E_o. (2.11)

In terms of the variables x, z, s, σ the eqns. (2.2) for the fields can be written as

$$\vec{\mathcal{E}} = \beta_o \cdot \frac{\partial}{\partial \sigma} \vec{A} \quad ; \qquad (2.11a)$$

$$\left\{\begin{aligned} B_x &= \frac{1}{h}\cdot\left\{\frac{\partial}{\partial z}(h\cdot A_s) - \frac{\partial}{\partial s}A_z\right\}; \\ B_z &= \frac{1}{h}\cdot\left\{\frac{\partial}{\partial s}A_x - \frac{\partial}{\partial x}(h\cdot A_s)\right\}; \\ B_s &= \frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_x \end{aligned}\right. \tag{2.11b}$$

where

$$h = 1 + K_x \cdot x + K_z \cdot z .$$

Since vector potentials will not appear in the final equations of motion, we may use any form for the vector potential that leads via (2.11) to the correct fields for the various magnet types.

We will assume here that the ring only contains quadrupoles, dipoles, sextupoles and cavities. The vector potentials suitable for describing these elements are:

a) Cavity

$$\varepsilon(s,\sigma) = V(s)\cdot\sin\left[k\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right] \quad ;$$

$$A_s = -\frac{1}{\beta_o}\cdot\frac{L}{k\cdot 2\pi}\cdot V(s)\cdot\cos\left[k\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right] \;;\; A_x = A_z = 0 \quad .$$

b) Quadrupole

$$K_x = K_z = 0 \;;$$

$$B_x = z\cdot\left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \;;$$

$$B_z = x\cdot\left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \;;$$

$$\frac{e}{E_o}A_s = \frac{1}{2}\cdot\frac{c\cdot p_o}{E_o}\cdot g_o\cdot(z^2-x^2) = \frac{1}{2}\beta_o\cdot g_o\cdot(z^2-x^2)$$

$$\text{with}\quad g_o = \frac{e}{p_o\cdot c}\cdot\left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \;;\quad A_x = A_z = 0 \;.$$

c) Sextupole

$$K_x = K_z = 0 \; ;$$

$$B_x = x \cdot z \cdot \left(\frac{\partial^2 B_z}{\partial x^2} \right)_{x=z=0} \; ;$$

$$B_z = \frac{1}{2} (x^2 - z^2) \cdot \left(\frac{\partial^2 B_z}{\partial x^2} \right)_{x=z=0} \; ;$$

$$\frac{e}{E_o} A_s = - \lambda_o \cdot \frac{1}{6} (x^3 - 3xz^2) \cdot \frac{cp_o}{E_o} = - \beta_o \cdot \lambda_o \cdot \frac{1}{6} (x^3 - 3xz^2)$$

with $\lambda_o = \frac{e}{p_o \cdot c} \cdot \left(\frac{\partial^2 B_z}{\partial x^2} \right)_{x=z=0}$.

d) Bending magnet

$$(K_x, K_z) \neq (0,0) \; ; \quad K_x \cdot K_z = 0 \; ;$$

$$\frac{e}{p_o \cdot c} \cdot B_x = - K_z \; ;$$

$$\frac{e}{p_o \cdot c} \cdot B_z = + K_x \; ;$$

$$\frac{e}{p_o \cdot c} A_s = - \frac{1}{2} (1 + K_x \cdot x + K_z \cdot z) \; .$$

The Hamiltonian (2.9) can then be written in the form

$$\begin{aligned} \mathcal{H} = p_\sigma &- (1 + K_x \cdot x + K_z \cdot z) \cdot \beta_o^2 \cdot f(p_\sigma) + \frac{1}{2\beta_o^2} \cdot \frac{\hat{p}_x^2 + \hat{p}_z^2}{1 + f(p_\sigma)} \\ &+ \frac{1}{2} \beta_o^2 \cdot [(K_x^2 + g_o) \cdot x^2 + (K_z^2 - g_o) \cdot z^2] \\ &+ \beta_o^2 \cdot \lambda_o \cdot \frac{1}{6} (x^3 - 3xz^2) \\ &+ \frac{L}{k \cdot 2\pi} \cdot \frac{eV(s)}{E_o} \cdot \cos[k \cdot \frac{2\pi}{L} \cdot \sigma + \varphi] \end{aligned} \tag{2.12}$$

with

$$p_\sigma = \eta$$

$$f(p_\sigma) = \hat{\eta}(s) \equiv \frac{\Delta p}{p_o} \quad \text{(see eqn. 2.10)} \tag{2.13}$$

(a constant term, $(-\frac{1}{2}\beta_o^2)$, in the Hamiltonian, which has no influence on the motion has been neglected)

For the equations of motion we now have

$$x' = \frac{\partial \mathcal{H}}{\partial \hat{p}_x} = \frac{1}{\beta_o^2} \cdot \frac{\hat{p}_x}{1 + f(p_\sigma)} ; \tag{2.14a}$$

$$\hat{p}_x' = -\frac{\partial \mathcal{H}}{\partial x} = -\beta_o^2 \cdot (K_x^2 + g_o) \cdot x$$
$$- \frac{1}{2}\beta_o^2 \cdot \lambda_o \cdot (x^2 - z^2) + K_x \cdot \beta_o^2 \cdot f(p_\sigma) ; \tag{2.14b}$$

$$z' = \frac{\partial \mathcal{H}}{\partial \hat{p}_z} = \frac{1}{\beta_o^2} \cdot \frac{\hat{p}_z}{1 + f(p_\sigma)} ; \tag{2.14c}$$

$$\hat{p}_z' = -\frac{\partial \mathcal{H}}{\partial z} = -\beta_o^2 \cdot (K_z^2 - g_o) \cdot z + \beta_o^2 \cdot \lambda_o \cdot xz + K_z \cdot \beta_o^2 \cdot f(p_\sigma) ; \tag{2.14d}$$

$$\sigma' = \frac{\partial \mathcal{H}}{\partial \hat{p}_\sigma} = 1 - (1 + K_x \cdot x + K_z \cdot z) \cdot \beta_o^2 \cdot f'(p_\sigma)$$
$$- \frac{1}{2\beta_o^2} \cdot \frac{\hat{p}_x^2 + \hat{p}_z^2}{[1 + f(p_\sigma)]^2} \cdot f'(p_\sigma) ; \tag{2.14e}$$

$$p_\sigma' = -\frac{\partial \mathcal{H}}{\partial \sigma} = \frac{eV(s)}{E_o} \cdot \sin[k \cdot \frac{2\pi}{L} \cdot \sigma + \varphi] . \tag{2.14f}$$

Remark

In equ. (2.14) the first four equations describe betatron oscillations and the last two synchrotron motion. Equation (2.14f) relates to energy conservation. Using equ. (2.14a,c), (2.14e) can also be written in the form

$$\sigma' = 1 - [1 + K_x \cdot x + K_z \cdot z + \frac{1}{2}(x'^2 + z'^2)] \cdot \beta_o^2 \cdot f'(p_\sigma). \quad (2.15)$$

This result can also be obtained directly from the defining equation for σ:

$$\sigma = s - v_o \cdot t ;$$
$$\frac{d\sigma}{ds} = 1 - v_o \cdot \frac{dt}{ds} \quad (2.16)$$

with[1]

$$dt = \frac{d\ell}{v} \quad (2.17)$$

$$d\ell = [1 + K_x \cdot x + K_z \cdot z + \frac{1}{2}(x'^2 + z'^2)] \cdot ds + \cdots . \quad (2.18)$$

From eqns. (2.16, 2.17, 2.18) one then gets

$$\sigma' = 1 - \frac{v_o}{v} \cdot [1 + K_x \cdot x + K_z \cdot z + \frac{1}{2}(x'^2 + z'^2)] . \quad (2.19)$$

This agrees with eqn. (2.15) since

$$\frac{df(p_\sigma)}{dp_\sigma} = \frac{1}{\beta_o} \cdot \frac{(1 + p_\sigma)}{\sqrt{(1 + p_\sigma)^2 - \left(\frac{m_o c^2}{Eo}\right)^2}} = \frac{1}{\beta_o} \cdot \frac{E}{p \cdot c} = \frac{1}{\beta_o \cdot \beta}$$

3. Introduction of dispersion

Equations (2.14) describe coupled synchro-betatron motion. The longitudinal and transverse coupling described by the terms

$$- (K_x \cdot x + K_z \cdot z) \cdot \beta_o^2 \cdot f(p_\sigma) \tag{3.1a}$$

and

$$\frac{1}{2\beta_o^2} \cdot \frac{\hat{p}_x^2 + \hat{p}_z^2}{1 + f(p_\sigma)} \tag{3.1b}$$

in the Hamiltonian (eqn. (2.12)) depends on the curvature of the orbit in the bending magnets and on the energy deviation of the particles.

We now introduce dispersion (see later: eqn. (3.5))

$$\vec{D}^T = (D_1, D_2, D_3, D_4)$$

and replace the quantitites x, $\hat{p}_x$, z, $\hat{p}_z$, σ, p_σ by new variables $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma$ which according to the definition of dispersion satisfy

$$\begin{aligned} \tilde{x} &= x - f(p_\sigma) \cdot D_1 \; ; \\ \tilde{p}_x &= p_x - f(p_\sigma) \cdot D_2 \; ; \\ \tilde{z} &= z - f(p_\sigma) \cdot D_3 \; ; \\ \tilde{p}_z &= p_z - f(p_\sigma) \cdot D_4 \; ; \end{aligned} \tag{3.2}$$

This replacement

$$(x, \hat{p}_x, z, \hat{p}_z, \sigma, p_\sigma) \longrightarrow (\tilde{x}, \tilde{p}_x, \tilde{z}, \tilde{p}_z, \tilde{\sigma}, \tilde{p}_\sigma)$$

can be achieved using the generating function

$$\begin{aligned} F_2(x, z, \sigma, \tilde{p}_x, \tilde{p}_z, \tilde{p}_\sigma) = \; & \tilde{p}_x \cdot [x - f(\tilde{p}_\sigma) \cdot D_1] + f(\tilde{p}_\sigma) \cdot D_2 \cdot x \\ & + \tilde{p}_z \cdot [z - f(\tilde{p}_\sigma) \cdot D_3] + f(\tilde{p}_\sigma) \cdot D_4 \cdot z \\ & - \frac{1}{2} [D_1 \cdot D_2 + D_3 \cdot D_4] \cdot f^2(\tilde{p}_\sigma) + \tilde{p}_\sigma \cdot \sigma \, . \end{aligned}$$

with the result that:

$$\tilde{x} = \frac{\partial F_2}{\partial \tilde{p}_x} = x - f(\tilde{p}_\sigma) \cdot D_1 \quad ; \tag{3.3a}$$

$$\hat{p}_x = \frac{\partial F_2}{\partial x} = \tilde{p}_x + f(\tilde{p}_\sigma) \cdot D_2 \; ; \tag{3.3b}$$

$$\tilde{z} = \frac{\partial F_2}{\partial \tilde{p}_z} = z - f(\tilde{p}_\sigma) \cdot D_3 \quad ; \tag{3.3c}$$

$$\hat{p}_z = \frac{\partial F_2}{\partial z} = \tilde{p}_z + f(\tilde{p}_\sigma) \cdot D_4 \; ; \tag{3.3d}$$

$$\tilde{\sigma} = \frac{\partial F_2}{\partial \tilde{p}_\sigma} = \sigma + \frac{df(\tilde{p}_\sigma)}{d\tilde{p}_\sigma} \cdot \{- \tilde{p}_x \cdot D_1 + \tilde{x} \cdot D_2 - \tilde{p}_z \cdot D_3 + \tilde{z} \cdot D_4\} \; ; \tag{3.3e}$$

$$p_\sigma = \frac{\partial F_2}{\partial \sigma} = \tilde{p}_\sigma \; ; \tag{3.3f}$$

$$\tilde{\mathcal{H}} = \mathcal{H} + \frac{\partial F_2}{\partial s} \; . \tag{3.4}$$

These in turn lead to eqn. (3.2).

On taking into account the defining equations for the dispersion in the general case of arbitrary velocity β_0 (see eqns. (2.14)):

$$\left.\begin{aligned} D_1' &= \frac{D_2}{\beta_0^2} \; ; \\ D_2' &= - \beta_0^2 \cdot (K_x^2 + g_0) \cdot D_1 + \beta_0^2 \cdot K_x \; ; \end{aligned}\right\} \Longrightarrow D_1'' = -(K_x^2 + g_0) \cdot D_1 + K_x \; ;$$

$$\left.\begin{aligned} D_3' &= \frac{D_4}{\beta_0^2} \; ; \\ D_4' &= - \beta_0^2 \cdot (K_z^2 - g_0) \cdot D_3 + \beta_0^2 \cdot K_z \; ; \end{aligned}\right\} \Longrightarrow D_3'' = -(K_z^2 - g_0) \cdot D_3 + K_z \tag{3.5}$$

we have the new Hamiltonian (3.4):

$$\tilde{\mathcal{H}} = \frac{1}{2\beta_0^2} \cdot [\tilde{p}_x^2 + \tilde{p}_z^2] + \frac{1}{2}\beta_0^2 \cdot \{(K_x^2 + g_0) \cdot \tilde{x}^2 + (K_z^2 - g_0) \cdot \tilde{z}^2\}$$

$$+ \frac{eV(s)}{E_0} \cdot \frac{L}{k \cdot 2\pi} \cdot \cos\{k \cdot \frac{2\pi}{L} \cdot [\tilde{\sigma} - \frac{df(\tilde{p}_\sigma)}{d\tilde{p}_\sigma}$$

$$\times (-\tilde{p}_x \cdot D_1 + \tilde{x} \cdot D_2 - \tilde{p}_z \cdot D_3 + \tilde{z} \cdot D_4)] + \varphi\}$$

$$- \frac{1}{2}\beta_0^2 \cdot f^2(\tilde{p}_\sigma) \cdot [K_x \cdot D_1 + K_z \cdot D_3]$$

$$+ \tilde{p}_\sigma - \beta_0^2 \cdot f(\tilde{p}_\sigma) - \frac{1}{2\beta_0^2} \cdot \{[\tilde{p}_x + f(\tilde{p}_\sigma) \cdot D_2]^2$$

$$+ [\tilde{p}_z + f(\tilde{p}_\sigma) \cdot D_4]^2\} \cdot \frac{f(\tilde{p}_\sigma)}{1 + f(\tilde{p}_\sigma)}$$

$$+ \beta_0^2 \cdot \frac{\lambda_0}{6} \cdot \{[\tilde{x} + f(\tilde{p}_\sigma) \cdot D_1]^3$$

$$- 3 \cdot [\tilde{x} + f(\tilde{p}_\sigma) \cdot D_1][\tilde{z} + f(\tilde{p}_\sigma) \cdot D_3]^2\} . \qquad (3.6)$$

In this Hamiltonian, the coupling term (3.1a) which arose from the orbit curvature no longer appears. Instead, there appears a term for the cavities

$$A = \frac{eV(s)}{E_0} \cdot \frac{L}{k \cdot 2\pi} \cdot \cos\{k \cdot \frac{2\pi}{L} \cdot [\tilde{\sigma} - \frac{df(\tilde{p}_\sigma)}{d\tilde{p}_\sigma} \times$$

$$\times (-\tilde{p}_x \cdot D_1 + \tilde{x} \cdot D_2 - \tilde{p}_z \cdot D_3 + \tilde{z} \cdot D_4) + \varphi]\}$$

representing a coupling between the longitudinal and transverse motion which disappears if

$$V(s) \cdot D_1(s) = V(s) \cdot D_3(s) = 0 ;$$
$$V(s) \cdot D_2(s) = V(s) \cdot D_4(s) = 0 .$$

(- no dispersion in the cavities).

If the term, A, and the function $f(\tilde{p}_\sigma)$ in equ. (3.6) are developed as a power series in σ and $\tilde{p}_\sigma$ respectively (see (2.10), (2.13)), then we obtain

$$A = \frac{eV(s)}{E_0} \cdot \{\frac{L}{k \cdot 2\pi} \cdot \cos\varphi - \sigma \cdot \sin\varphi - \frac{1}{2}\sigma^2 \cdot \frac{k \cdot 2\pi}{L} \cdot \cos\varphi \pm \dots\} \qquad (3.7a)$$

with $\sigma = \tilde{\sigma} - \frac{df(\tilde{p}_\sigma)}{d\tilde{p}_\sigma} \cdot \{- \tilde{p}_x \cdot D_1 + \tilde{x} \cdot D_2 - \tilde{p}_z \cdot D_3 + \tilde{z} \cdot D_4]\}$;

$$f(\tilde{p}_\sigma) = \frac{1}{\beta_o^2} \cdot \tilde{p}_\sigma - \frac{1}{\beta_o^4 \cdot \gamma_o^2} \cdot \frac{1}{2} \tilde{p}_\sigma^2 \pm \ldots \qquad (3.7b)$$

In this expansion the first (constant) term in A has no influence on the motion and the second term vanishes since $\sin\varphi = 0$ (no energy uptake in the cavities for protons). Thus, the Hamiltonian $\tilde{\mathcal{H}}$ can be separated into three uncoupled oscillation modes

$$\tilde{\mathcal{H}}_{ox} = \frac{1}{2\beta_o^2} \cdot \tilde{p}_x^2 + \frac{1}{2} \beta_o^2 \cdot (K_x^2 + g_o) \cdot \tilde{x}^2 \; ; \qquad (3.8a)$$

$$\tilde{\mathcal{H}}_{oz} = \frac{1}{2\beta_o^2} \cdot \tilde{p}_z^2 + \frac{1}{2} \beta_o^2 \cdot (K_x^2 - g_o) \cdot \tilde{z}^2 \; ; \qquad (3.8b)$$

$$\tilde{\mathcal{H}}_{o\sigma} = - \frac{1}{2\beta_o^2} \cdot \tilde{p}_\sigma^2 \cdot [(K_x \cdot D_1 + K_z \cdot D_3) - \frac{1}{\gamma_o^2}] - $$
$$- \frac{1}{2} \frac{eV(s)}{E_o} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\varphi \cdot \tilde{\sigma}^2 \; . \qquad (3.8c)$$

and a (small) perturbation term $\tilde{\mathcal{H}}_1$ which contains the remaining terms:

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_{ox} + \tilde{\mathcal{H}}_{oz} + \tilde{\mathcal{H}}_{o\sigma} + \tilde{\mathcal{H}}_1 \; . \qquad (3.8)$$

$\tilde{\mathcal{H}}_{ox}$, $\tilde{\mathcal{H}}_{oz}$, $\tilde{\mathcal{H}}_{o\sigma}$ describe linear oscillations in the x, z, σ directions respectively and the term $\tilde{\mathcal{H}}_1$ describes a (small) perturbation.

The components $\tilde{\mathcal{H}}_{ox}$, $\tilde{\mathcal{H}}_{oz}$, $\tilde{\mathcal{H}}_{o\sigma}$ have the general form

$$\tilde{\mathcal{H}}_{oy} = \frac{1}{2} F(s) \cdot p_y^2 + \frac{1}{2} G(s) \cdot y^2 \; ; \qquad (3.9)$$

$$(y \equiv x, z, \sigma) \; .$$

and from the transfer matrix $\underline{M}(s+L,s)$ resulting from this unperturbed Hamiltonian:

$$\underline{M}(s+L,s) = \begin{bmatrix} \cos 2\pi Q_y + \alpha(s) \cdot \sin 2\pi Q_y & \beta(s) \cdot \sin 2\pi Q_y \\ - \gamma(s) \cdot \sin 2\pi Q_y & \cos 2\pi Q_y - \alpha(s) \cdot \sin 2\pi Q_y \end{bmatrix}$$

we can extract the Twiss parameters α, β, γ. The motion can then be represented in the form[7]

$$y(s) = \sqrt{2 \cdot \beta(s) \cdot I} \cdot \cos[\Phi(s) + \Phi_0] \quad ;$$
$$p_y(s) = \sqrt{\frac{2 \cdot I}{\beta(s)}} \cdot \left\{ \sin[\Phi(s) + \Phi_0] + \alpha(s) \cdot \cos[\Phi(s) + \Phi_0] \right\} \qquad (3.10)$$

with $$I = \frac{1}{2} \cdot [\gamma \cdot y^2 + \beta \cdot p_y^2 + 2\alpha \cdot p_y y] = \text{const.} \qquad (3.11)$$

Eqn. (3.11) predicts that for all three modes y and p_y lie on an ellipse and that the equation for the ellipse is s dependent and periodic. As pointed out in the introduction these properties provide a way of using the motion in the phase plane for recognizing linear motion. Perturbations to the linear motion can then be characterized by their influence in causing distortion of the original ellipses.

The linear oscillation modes can only be discerned when the theory is written in terms of the variables $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma$. Nevertheless, the equations of motion are easier to solve when written in terms of the original set x, $\hat{p}_x$, z, $\hat{p}_z$, σ, p_σ. This is the subject of the next section.

Remarks:

1) If the quantities $(K_x \cdot D_1 + K_z \cdot D_3)$ in (3.8c) and $\frac{eV(s)}{E_0} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\varphi$ in (3.8c) are replaced by their averages[7]

$$\kappa = \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot [K_x(\tilde{s}) \cdot D_1(\tilde{s}) + K_z(\tilde{s}) \cdot D_3(\tilde{s})] \qquad (3.12a)$$

(momentum compaction factor)

and

$$\frac{\Omega^2 \cdot \beta_0^2}{(\kappa - \frac{1}{\gamma_0^2})} = k \cdot \frac{2\pi}{L} \cdot \cos\varphi \cdot \frac{1}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot \frac{eV(\tilde{s})}{E_0} \qquad (3.12b)$$

$\tilde{\mathcal{H}}_{o\sigma}$ takes the form

$$\tilde{\mathcal{H}}_{o\sigma} = -\frac{1}{2\beta_0^2} \cdot \tilde{p}_\sigma^2 \cdot [\kappa - \frac{1}{\gamma_0^2}] - \frac{1}{2}\tilde{\sigma}^2 \cdot \frac{\Omega^2 \cdot \beta_0^2}{(\kappa - \frac{1}{\gamma_0^2})} \quad . \qquad (3.13)$$

The canonical equations are then

$$\frac{d\tilde{\sigma}}{ds} = + \frac{\partial\tilde{\mathcal{H}}_{o\sigma}}{\partial\tilde{p}_\sigma} = - \frac{\tilde{p}_\sigma}{\beta_o^2} \cdot (\kappa - \frac{1}{\gamma_o^2}) \quad ; \tag{3.14a}$$

$$\frac{d\tilde{p}_\sigma}{ds} = - \frac{\partial\tilde{\mathcal{H}}_{o\sigma}}{\partial\tilde{\sigma}} = \frac{eV(s)}{E_o} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\varphi \cdot \tilde{\sigma} \tag{3.14b}$$

or

$$\frac{d^2\tilde{\sigma}}{ds^2} = - \Omega^2 \cdot \tilde{\sigma} \quad ; \tag{3.15a}$$

$$\frac{d^2\tilde{p}_\sigma}{ds^2} = - \Omega^2 \cdot \tilde{p}_\sigma \ . \tag{3.15b}$$

where $$\Omega^2 = \frac{1}{L} \int_{s_o}^{s_o+L} d\tilde{s} \cdot \frac{eV(s)}{E_o} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\varphi \cdot \frac{1}{\beta_o^2} \cdot (\kappa - \frac{1}{\gamma_o^2}) \ . \tag{3.16}$$

From eqn. (3.15) it follows that the synchrotron oscillations are stable only if

$$\Omega^2 > 0 \ . \tag{3.17}$$

For protons ($eV(s) > 0$) with $\sin\varphi = 0$ (no energy uptake in the cavities) this corresponds to the usual conditions

$$\begin{aligned} \varphi &= 0 \quad \text{for} \quad \kappa > \frac{1}{\gamma_o^2} \quad \text{(above "transition")}; \\ \varphi &= \pi \quad \text{for} \quad \kappa < \frac{1}{\gamma_o^2} \quad \text{(below "transition")}. \end{aligned} \tag{3.18}$$

2) According to eqn. (3.10)

$$y^2 + [\alpha(s) \cdot y + \beta(s) \cdot p_y]^2 = 2I \cdot \beta(s) \ . \tag{3.19}$$

If instead of p_y one uses the variable $\hat{p}_y$:

$$\hat{p}_y = \alpha(s) \cdot y + \beta(s) \cdot p_y \ , \tag{3.20}$$

the trajectory in the $(y, \hat{p}_y)$-plane is a circle:

$$y^2 + \hat{p}_y^2 = 2I \cdot \beta(s) \ . \tag{3.21}$$

3) In defining the linear oscillator Hamiltonians $\tilde{\mathcal{H}}_{oy}$, $\tilde{\mathcal{H}}_{oz}$, $\tilde{\mathcal{H}}_{o\sigma}$ only the factor $\tilde{\sigma}^2$ was extracted from the term σ^2 appearing in eqn. (3.7a).
   It is of course also possible to take into account further quadratic terms in σ^2, namely

$$\frac{1}{\beta_o^4} \cdot (\tilde{p}_x \cdot D_1 - \tilde{x} \cdot D_2)^2 \quad \text{for } \tilde{\mathcal{H}}_{ox}$$

$$\frac{1}{\beta_o^4} \cdot (\tilde{p}_z \cdot D_3 - \tilde{z} \cdot D_4)^2 \quad \text{for } \tilde{\mathcal{H}}_{oz}$$

so that $\tilde{\mathcal{H}}_{oy}$ takes the more general form

$$\tilde{\mathcal{H}}_{oy} = \frac{1}{2} F(s) \cdot p_y^2 + R(s) \cdot y \cdot p_y + \frac{1}{2} G(s) \cdot y^2 .$$

As was shown in Ref. 7, this also leads to equations of the form (3.10) and (3.11) so that the motion in the (y, p_y)-plane can be still described by phase ellipses.

4) For $\beta_o \longrightarrow 1$ (ultrarelativistic particles) $f(\tilde{p}_\sigma)$ becomes (see eqn. (3.7b)

$$f(\tilde{p}_\sigma) = \tilde{p}_\sigma .$$

In this case the transformation formulas (3.3) take the same form as used in Ref. 6.

4. Solution of the equations of motion

As pointed out above, the equations of motion are most easily solved in terms of the variables x, $\hat{p}_x$, z, $\hat{p}_z$, σ, p_σ.
The variables $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma$ can then be calculated using eqns. (3.3) and the dispersion. In particular eqn. (3.3e) can be rewritten as:

$$\tilde{\sigma} = \sigma + \frac{df(p_\sigma)}{dp_\sigma} \cdot \left\{- \hat{p}_x \cdot D_1 + x \cdot D_2 - \hat{p}_z \cdot D_3 + z \cdot D_4\right\} \quad ; \qquad (4.1)$$

In the following we give the solution of the equations of motion (2.14) and of the dispersion equation (3.5) for various types of magnets and for cavities.

4.1 Quadrupole

The equations of motion are

$$x' = \frac{1}{\beta_o^2} \cdot \frac{\hat{p}_x}{1 + f(p_\sigma)} \quad ; \qquad (4.2a)$$

$$\hat{p}_x' = - \beta_o^2 \cdot g_o \cdot x \quad ; \qquad (4.2b)$$

$$z' = \frac{1}{\beta_o^2} \cdot \frac{\hat{p}_z}{1 + f(p_\sigma)} \qquad (4.2c)$$

$$\hat{p}_z' = \beta_o^2 \cdot g_o \cdot z \quad ; \qquad (4.2d)$$

$$\sigma' = 1 - \beta_o^2 \cdot \frac{df(p_\sigma)}{dp_\sigma} - \frac{\beta_o^2}{2} \cdot \frac{df(p_\sigma)}{dp_\sigma} \cdot [(x')^2 + (z')^2] \quad ; \qquad (4.2e)$$

$$p_\sigma' = 0 \ . \qquad (4.2f)$$

From eqn. (4.2f) we have

$$p_\sigma(s) = p_\sigma(0) \qquad (4.3)$$

If we write the solution as

$$\vec{y}(s) = \underline{M}(s,0)\ \vec{y}(0) \qquad (4.4)$$

with

$$\vec{y}^T = (x,\ x',\ z,\ z')\ ; \tag{4.5}$$

$$\hat{p}_x = \beta_o^2 \cdot [1 + f(p_\sigma)] \cdot x'\ ;$$
$$\hat{p}_z = \beta_o^2 \cdot [1 + f(p_\sigma)] \cdot z' \tag{4.6}$$

and introduce

$$g = \frac{g_o}{1 + f(p_\sigma)} \tag{4.7}$$

then eqns. (4.2a,b,c,d) give:

a) $g > 0$:

$$M_{11}(s,0) = \cos(\sqrt{g} \cdot s)\ ;$$
$$M_{12}(s,0) = \frac{1}{\sqrt{g}} \cdot \sin(\sqrt{g} \cdot s)\ ;$$
$$M_{21}(s,0) = -\sqrt{g} \cdot \sin(\sqrt{g} \cdot s)\ ;$$
$$M_{22}(s,0) = \cos(\sqrt{g} \cdot s)\ ;$$
$$M_{31}(s,0) = \cosh(\sqrt{g} \cdot s)\ ;$$
$$M_{32}(s,0) = \frac{1}{\sqrt{g}} \cdot \sinh(\sqrt{g} \cdot s)\ ;$$
$$M_{41}(s,0) = \sqrt{g} \cdot \sinh(\sqrt{g} \cdot s)\ ;$$
$$M_{42}(s,0) = \cosh(\sqrt{g} \cdot s)\ ;$$

b) $g < 0$

$$M_{11}(s,0) = \cosh(\sqrt{|g|} \cdot s)\ ;$$
$$M_{12}(s,0) = \frac{1}{\sqrt{|g|}} \cdot \sinh(\sqrt{|g|} \cdot s)\ ;$$
$$M_{21}(s,0) = \sqrt{|g|} \cdot \sinh(\sqrt{|g|} \cdot s)\ ;$$
$$M_{22}(s,0) = \cosh(\sqrt{|g|} \cdot s)\ ;$$
$$M_{31}(s,0) = \cos(\sqrt{|g|} \cdot s)\ ;$$
$$M_{32}(s,0) = \frac{1}{\sqrt{|g|}} \sin(\sqrt{|g|} \cdot s)\ ;$$
$$M_{41}(s,0) = -\sqrt{|g|} \sin(\sqrt{|g|} \cdot s)\ ;$$
$$M_{42}(s,0) = \cos(\sqrt{|g|} \cdot s)\ . \tag{4.8b}$$

Finally from eqn. (4.2c) (for g $\gtrless$ 0) we have:

$$\sigma(s) = \sigma(0) + s \cdot [1 - \beta_0^2 \cdot \frac{df(p_\sigma)}{dp_\sigma}]$$
$$- \frac{9}{4} \cdot \beta_0^2 \cdot \frac{df(p_\sigma)}{dp_\sigma} \cdot \{x^2(0) \cdot [s - M_{11}(s,0) \cdot M_{12}(s,0)]$$
$$- z^2(0) \cdot [s - M_{33}(S,0) \cdot M_{34}(s,0)]$$
$$- \frac{1}{4} \cdot \beta_0^2 \cdot \frac{df(p_\sigma)}{dp_\sigma} \cdot \{x'^2(0) \cdot [s + M_{11}(s,0) \cdot M_{12}(s,0)]$$
$$+ z'^2(0) \cdot [s + M_{33}(s,0) \cdot M_{43}(s,0)]\}$$
$$- \frac{1}{2} \cdot \beta_0^2 \cdot \frac{df(p_\sigma)}{dp_\sigma} \cdot \{x(0) \cdot x'(0) \cdot M_{12}(s,0) \cdot M_{21}(s,0)$$
$$+ z(0) \cdot z'(0) \cdot M_{34}(s,0) \cdot M_{43}(s,0)\}. \quad (4.9)$$

The dispersion obeys the equations (see eqn. (3.5)

$$D_1' = \frac{D_2}{\beta_0^2} \quad ;$$
$$D_2' = - \beta_0^2 \cdot g_0 \cdot D_1 \quad ;$$
$$D_3' = \frac{D_4}{\beta_0^2} \quad ;$$
$$D_4' = + \beta_0^2 \cdot g_0 \cdot D_3$$

and has the solution:

a) $g_0 > 0$;

$$D_1(s) = D_1(0) \cdot \cos(\sqrt{g_0} \cdot s) + \frac{D_2(0)}{\beta_0^2 \cdot \sqrt{g_0}} \cdot \sin(\sqrt{g_0} \cdot s) \quad ;$$

\# $$D_2(s) = - D_1(0) \cdot \beta_0^2 \cdot \sqrt{g_0} \cdot \sin(\sqrt{g_0} \cdot s) + D_2(0) \cdot \cos(\sqrt{g_0} \cdot s) \quad ;$$

$$D_3(s) = D_3(0) \cdot \cosh(\sqrt{g_0} \cdot s) + \frac{D_4(0)}{\beta_0^2 \cdot \sqrt{g_0}} \cdot \sinh(\sqrt{g_0} \cdot s) \quad ;$$

$$D_4(s) = D_3(0) \cdot \beta_0^2 \cdot \sqrt{g_0} \cdot \sinh(\sqrt{g_0} \cdot s) + D_4(0) \cdot \cosh(\sqrt{g_0} \cdot s) \quad ; \quad (4.10a)$$

b) $g_0 < 0$

$$D_1(s) = D_1(0) \cdot \cosh(\sqrt{|g_0|} \cdot s) + \frac{D_2(0)}{\beta_0^2 \cdot \sqrt{|g_0|}} \cdot \sin(\sqrt{|g_0|} \cdot s) \;;$$

$$D_2(s) = D_1(0) \cdot \beta_0^2 \cdot \sqrt{|g_0|} \cdot \sinh(\sqrt{|g_0|} \cdot s) + D_2(0) \cdot \cosh(\sqrt{|g_0|} \cdot s) \;;$$

$$D_3(s) = D_3(0) \cdot \cos(\sqrt{|g_0|} \cdot s) + \frac{D_4(0)}{\beta_0^2 \cdot \sqrt{|g_0|}} \cdot \sin(\sqrt{|g_0|} \cdot s) \;;$$

$$D_4(s) = - D_3(0) \cdot \beta_0^2 \cdot \sqrt{|g_0|} \cdot \sin(\sqrt{|g_0|} \cdot s) + D_4(0) \cdot \cos(\sqrt{|g_0|} \cdot s) \;;$$

(4.10b)

4.2 Bending magnet

From equ. (2.14) we have

$$x' = \frac{1}{\beta_0^2} \cdot \frac{\hat{p}_x}{1 + f(p_\sigma)} \;; \qquad (4.11a)$$

$$\hat{p}_x' = - \beta_0^2 \cdot K_x^2 \cdot x + K_x \cdot \beta_0^2 \cdot f(p_\sigma) \;; \qquad (4.11b)$$

$$z' = \frac{1}{\beta_0^2} \cdot \frac{\hat{p}_z}{1 + f(p_\sigma)} \;; \qquad (4.11c)$$

$$\hat{p}_z = - \beta_0^2 \cdot K_z^2 \cdot z + K_z \cdot \beta_0^2 \cdot f(p_\sigma) \;; \qquad (4.11d)$$

$$\sigma' = 1 - \beta_0^2 \cdot \frac{df(p_\sigma)}{dp_\sigma} \cdot \left\{(1 + K_x \cdot x + K_z \cdot z) + \frac{1}{2}\left[(x')^2 + (z')^2\right]\right\}; \qquad (4.11e)$$

$$p_\sigma' = 0 \;. \qquad (4.11f)$$

From (4.11f) we obtain

$$p_\sigma(s) = p_\sigma(0) \qquad (4.12)$$

and to solve eqns. (4.11a,b,c,d) we write

$$\vec{y}(s) = M(s,0)\,\vec{y}(0) + \vec{q} \;; \qquad (4.13)$$

$$\vec{y}^{T} = (x,\ x',\ z,\ z')$$

and obtain

$$M_{11}(s,0) = \cos(\sqrt{G_X} \cdot s) ;$$

$$M_{12}(s,0) = \frac{1}{\sqrt{G_X}} \cdot \sin(\sqrt{G_X} \cdot s) ;$$

$$M_{21}(s,0) = - \sqrt{G_X} \cdot \sin(\sqrt{G_X} \cdot s) ;$$

$$M_{22}(s,0) = M_{11}(s,0) ;$$

$$M_{33}(s,0) = \cos(\sqrt{G_Z} \cdot s) ;$$

$$M_{34}(s,0) = \frac{1}{\sqrt{G_Z}} \cdot \sin(\sqrt{G_Z} \cdot s) ;$$

$$M_{43}(s,0) = - \sqrt{G_Z} \cdot \sin(\sqrt{G_Z} \cdot s) ;$$

$$M_{44}(s,0) = M_{33}(S,0) ;$$

$$q_1(s,0) = \frac{1}{K_X} \cdot f(p_\sigma) \cdot [1 - \cos(\sqrt{G_X} \cdot s)] ;$$

$$q_2(s,0) = \frac{\sqrt{G_X}}{K_X} \cdot f(p_\sigma) \cdot \sin(\sqrt{G_X} \cdot s) ;$$

$$q_3(s,0) = \frac{1}{K_Z} \cdot f(p_\sigma) \cdot [1 - \cos(\sqrt{G_Z} \cdot s)] ;$$

$$q_4(s,0) = \frac{\sqrt{G_Z}}{K_Z} \cdot f(p_\sigma) \cdot \sin(\sqrt{G_Z} \cdot s) . \qquad (4.14)$$

with

$$G_X = \frac{K_X^2}{1 + f(p_\sigma)} ;$$

$$G_Z = \frac{K_Z^2}{1 + f(p_\sigma)} ; \qquad (4.15)$$

Finally from (4.11e), σ(s) can be written as:

$$\sigma(s) = \sigma(0) + s \cdot [1 - \beta_o^2 \cdot \frac{df(p_\sigma)}{dp_\sigma}]$$

$$- \beta_o^2 \cdot \frac{df(p_\sigma)}{dp_\sigma} \cdot \Big\{K_x \cdot x(0) \cdot M_{12}(s,0) + K_x \cdot \frac{x'(0)}{G_x} \cdot [1 - M_{11}(s,0)]$$

$$+ f(p_\sigma) \cdot [s - M_{12}(s,0)]$$

$$+ K_z \cdot z(0) \cdot M_{34}(s,0) + K_z \cdot \frac{z'(0)}{G_z} \cdot [1 - M_{33}(s,0)]$$

$$+ f(p_\sigma) \cdot [s - M_{34}(s,0)]\Big\}$$

$$- \frac{\beta_o^2}{2} \cdot \frac{df(p_\sigma)}{dp_\sigma} \cdot \Big\{\frac{1}{2} \cdot [s - M_{11}(s,0) \cdot M_{12}(s,0)] \times$$

$$\times [x^2(0) \cdot G_x + \frac{G_x}{K_z^2} \cdot f^2(p_\sigma) - 2x(0) \cdot \frac{G_x}{K_x} \cdot f(p_\sigma)]$$

$$+ \frac{1}{2} \cdot [s + M_{11}(s,0) \cdot M_{12}(s,0)] \cdot x'^2(0)$$

$$- M_{12}(s,0) \cdot M_{21}(s,0) \times$$

$$\times [-x(0) \cdot x'(0) + x'(0) \cdot \frac{1}{K_x} \cdot f(p_\sigma)]$$

$$+ \frac{1}{2} \cdot [s - M_{33}(S,0) \cdot M_{34}(s,0)] \times$$

$$\times [z^2(0) \cdot G_z + \frac{G_z}{K_z^2} \cdot f^2(p_\sigma) - 2z(0) \cdot \frac{G_z}{K_z} \cdot f(p_\sigma)]$$

$$+ \frac{1}{2} \cdot [s + M_{33}(s,0) \cdot M_{34}(s,0)] \cdot z'^2(0)$$

$$- M_{34}(s,0) \cdot M_{43}(s,0) \times$$

$$\times [-z(0)\, z'(0) + z'(0) \cdot \frac{1}{K_z} \cdot f(p_\sigma)]\Big\} \, . \qquad (4.16)$$

By eqn. (3.5) the dispersion obeys the equations

$$D_1' = \frac{D_2}{\beta_0^2} ;$$

$$D_2' = - \beta_0^2 \cdot K_x^2 \cdot D_1 + \beta_0^2 \cdot K_x ;$$

$$D_3' = \frac{D_4}{\beta_0^2} ;$$

$$D_4' = - \beta_0^2 \cdot K_z^2 \cdot D_3 + \beta_0^2 \cdot K_z$$

and has the solution

$$D_1(s) = D_1(0) \cdot \cos(|K_x| \cdot s) + \frac{D_2(0)}{\beta_0^2 \cdot |K_x|} \cdot \sin(|K_x| \cdot s) + \frac{1}{K_x} \cdot [1 - \cos(|K_x| \cdot s)] ;$$

$$D_2(s) = - D_1(0) \cdot \beta_0^2 \cdot |K_x| \cdot \sin(|K_x| \cdot s) + D_2(0) \cdot \cos(|K_x| \cdot s) + \frac{|K_x|}{K_x} \cdot \sin(|K_x| \cdot s) ;$$

$$D_3(s) = D_3(0) \cdot \cos(|K_z| \cdot s) + \frac{D_4(0)}{\beta_0^2 \cdot |K_z|} \cdot \sin(|K_z| \cdot s) + \frac{1}{K_z} \cdot [1 - \cos(|K_z| \cdot s)] ;$$

$$D_4(s) = - D_3(0) \cdot \beta_0^2 \cdot |K_z| \cdot \sin(|K_z| \cdot s) + D_4(0) \cdot \cos(|K_z| \cdot s) + \frac{|K_z|}{K_z} \cdot \sin(|K_z| \cdot s) ; \tag{4.17}$$

4.3 Sextupole

The equations of motion in a thin lens sextupole are

$$x' = \frac{1}{\beta_o^2} \cdot \frac{\hat{p}_x}{1 + f(p_\sigma)} \; ; \qquad (4.18a)$$

$$\hat{p}_x' = -\frac{1}{2}\,\beta_o^2 \cdot \lambda_o \cdot (x^2 - z^2) \; ; \qquad (4.18b)$$

$$z' = \frac{1}{\beta_o^2} \cdot \frac{\hat{p}_z}{1 + f(p_\sigma)} \; ; \qquad (4.18c)$$

$$\hat{p}_z' = +\,\beta_o^2 \cdot \lambda_o \cdot xz \; ; \qquad (4.18d)$$

$$\sigma' = 1 - \beta_o^2 \cdot \frac{df(p_\sigma)}{dp_\sigma} - \frac{\beta_o^2}{2} \cdot \frac{df(p_\sigma)}{dp_\sigma} \cdot [(x')^2 + (z')^2] \; ; \qquad (4.18e)$$

$$p_\sigma' = 0 \, . \qquad (4.18f)$$

with

$$\lambda(s) = \frac{\lambda_o(s)}{1 + f(p_\sigma)} = \hat{\lambda} \cdot \delta(s - s_o) \qquad (4.19)$$

and have the solutions

$$\sigma(s_o + 0) = \sigma(s_o - 0) \; ;$$

$$p_\sigma(s_o + 0) = p_\sigma(s_o - 0) \; ;$$

$$x(s_o + 0) = x(s_o - 0) \; ;$$

$$x'(s_o + 0) = x'(s_o - 0) - \frac{\hat{\lambda}}{2} \cdot [x^2(s_o - 0) - z^2(s_o - 0)] \; ;$$

$$z(s_o + 0) = z(s_o - 0) \; ;$$

$$z'(s_o + 0) = z'(s_o - 0) + \hat{\lambda} \cdot x(s_o - 0) \cdot z(s_o - 0) \, . \qquad (4.20)$$

For the dispersion we obtain (eqn. (3.5)):

$$\vec{D}(s_o + 0) = \vec{D}(s_o - 0) \, . \qquad (4.21)$$

4.4 Cavity

For a pointlike cavity with

$$V(s) = \hat{V} \cdot \delta(s - s_o) \tag{4.22}$$

the equations of motion are

$$x' = \frac{1}{\beta_o^2} \cdot \frac{\hat{p}_x}{1 + f(p_\sigma)} ; \tag{4.22a}$$

$$\hat{p}_x' = 0 ; \tag{4.22b}$$

$$z' = \frac{1}{\beta_o^2} \cdot \frac{\hat{p}_z}{1 + f(p_\sigma)} ; \tag{4.22c}$$

$$\hat{p}_z' = 0 ; \tag{4.22d}$$

$$\sigma' = 1 - \beta_o^2 \cdot \frac{df(p_\sigma)}{dp_\sigma} - \frac{\beta_o^2}{2} \cdot \frac{df(p_\sigma)}{dp_\sigma} \cdot [(x')^2 + (z')^2] ; \tag{4.22e}$$

$$p_\sigma' = \frac{e\hat{V}}{E_o} \cdot \sin[k \cdot \frac{2\pi}{L} \cdot \sigma + \varphi]] \delta(s - s_o) . \tag{4.22f}$$

From equ. (4.22e) and (4.22f) we obtain

$$\sigma(s_o + 0) = \sigma(s_o - 0) ; \tag{4.23a}$$

$$p_\sigma(s_o + 0) = p_\sigma(s_o - 0) + \frac{e\hat{V}}{E_o} \cdot \sin[k \cdot \frac{2\pi}{L} \cdot \sigma(s_o - 0) + \varphi] \tag{4.23b}$$

and from equ. (4.22a) and (4.22b) we find

$$\frac{d}{ds} \left\{[1 + f(p_\sigma)] \cdot x'\right\} = 0 \quad \Longrightarrow \quad [1 + f(p_\sigma)] \cdot x'(s) = \text{const.}$$

Then by recalling the relation

$$\hat{\eta}(s) = f(p_\sigma) \tag{4.24}$$

we get:

$$[1 + \hat{\eta}(s_o + 0)] \cdot x'(s_o + 0) = [1 + \hat{\eta}(s_o - 0)] \cdot x'(s_o - 0)$$

$$\Longrightarrow \quad x'(s_o + 0) = \frac{1 + \hat{\eta}(s_o - 0)}{1 + \hat{\eta}(s_o + 0)} \cdot x'(s_o - 0) \; . \qquad (4.25a)$$

Correspondingly, from eqn. (4.18c) and (4.18d) we find

$$z'(s_o + 0) = \frac{1 + \hat{\eta}(s_o - 0)}{1 + \hat{\eta}(s_o + 0)} \cdot z'(s_o - 0) \; . \qquad (4.25b)$$

Furthermore, from (4.22a) and (4.22b) we also have

$$x(s_o + 0) = x(s_o - 0) \; ; \qquad (4.26a)$$

$$z(s_o + 0) = z(s_o - 0) \; . \qquad (4.26b)$$

Finally, for the dispersion (eqn. (3.5)):

$$\vec{D}(s_o + 0) = \vec{D}(s_o - 0) \; . \qquad (4.27)$$

5. Summary

We have investigated the motion of protons in a storage ring of arbitrary energy E_0 (below and above transition energy). Two different ways to describe the motion of the particles have been presented:

a) The fully six-dimensional description of the motion with the canonical variables x, $\hat{p}_x$, z, $\hat{p}_z$, $\sigma = s - v_0 \cdot t$, $\eta = \Delta E/E_0 \equiv p_\sigma$.

b) The dispersion formalism with the variables $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma$ defined by eqn. (3.3).

After using the fully six-dimensional description to derive the Hamiltonian and the canonical equations of motion, the dispersion function is introduced via a canonical transformation so that the symplectic structure of the equations of motion is completely preserved. The coupling in the synchro-betatron oscillations now appears in the cavities and vanishes if the dispersion in the cavities is equal to zero. In this dispersion formalism it is possible to define three linear uncoupled oscillation modes.

The equations of motion are solved by using the fully six-dimensional description.

For studying the influence of perturbations it was shown that it is useful to describe the motion by the variable $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma$ (dispersion formalism). In this case, the linear oscillation modes are described by phase ellipses and perturbations are characterized by deviations from these phase ellipses.

To avoid linear perturbations we have assumed that the ring only contains quadrupoles, bending magnets, cavities and sextupoles.

The Hamiltonian $\tilde{\mathcal{H}}$ in section 3 can be used as the starting point for a nonlinear theory of coupled synchro-betatron oscillations for protons with arbitrary velocity[7].

The solutions given in section 4 can be used as the basis for a nonlinear tracking program for protons with arbitrary velocity (see e.g. Refs. 2,3,4).

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References

1) G. Ripken: "Non-Linear Canonical Equations of Coupled Synchro-Betatron Motion and their Solutions within the Framework of a Non-Linear Six-Dimensional (Symplectic) Tracking Program for Ultrarelativistic Protons", DESY 85-084

2) F. Schmidt: "Simulationsrechnungen für DESY III", DESY HERA 86-10

3) F. Schmidt: Phd. Theses (to be published)

4) F. Schmidt: "Erweiterung des Racetrack Codes auf einen 6-dimensionalen symplektischen Formalismus" (to be published)

5) C.J.A. Corsten and H.L. Hagedorn, Nuclear Instruments and Methods 212 (1983), 37-46

6) H. Mais, G. Ripken: "Spin-Orbit Motion in a Storage Ring in the Presence of Synchrotron Radiation using a Dispersion Formalism", DESY 86-029

7) D.P. Barber, H. Mais, G. Ripken, F. Willeke: "Nonlinear Theory of Coupled Synchro-Betatron Motion", DESY 86-147