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TOPOLOGY AND DYNAMICS OF THE CONFINEMENT MECHANISM

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Abstract: We carry out the Abelian projection of SU(N) gauge theories, both in the continuum and on the lattice. Then the degrees of freedom are Abelian gauge fields, gluons, quarks, and magnetic monopoles. No approximation is involved, just gauge fixing. We discuss the topology of the monopoles in detail and investigate their role in the confinement mechanism using numerical simulations.

I. Introduction and motivation

Many years ago several authors, starting with 't Hooft [1] and Mandelstam [2], conjectured that quark confinement could be understood by analogy to superconductivity. In such a scenario magnetic charges condense, and the dual Meissner effect constricts color electric fields into flux tubes, ensuring quark confinement. Although these authors shared a scenario, there was little consensus in the definition of the monopoles. In addition, there was, as a rule, little discussion of the dynamics in a nonperturbative context [3].

The above view of the confinement mechanism is encouraged by the analysis of the phase transition in the gauge coupling β of compact U(1) lattice gauge theory [4-6]. At high values of β free electric charges can exist, whereas at low values of β magnetic monopoles condense and electric charges are confined. Once the analytic framework was established [4, 5], numerical simulations confirmed the picture [6].

If quantum chromodynamics describes the strong interactions, it should confine color, at least at low temperatures. An economic description of the phenomenon would confine quarks and gluons by the same mechanism. 't Hooft's Abelian projection [7] of a nonabelian gauge theory provides a promising framework. By choosing a particular gauge, 't Hooft formulates the theory in terms of Abelian gauge fields. With respect to these "photons" both quarks and gluons are (color) electrically charged. Moreover, there are topological excitations that can be identified as (color) magnetic monopoles. If these condense, then the above scenario implies that electric charges, i.e. quarks and gluons, are confined.

Another feature of the Abelian projection is that some of the basic excitations (the monopoles) are intrinsically nonperturbative. Since confinement appears because of nonperturbative effects, it is useful to incorporate them from the outset.

In order to understand nonperturbative physics one must formulate the theory on a lattice. The central new result of this paper is the extension of the Abelian projection to lattice gauge theory. This equips us with a formulation amenable to numerical simulations. Qualitative and quantitative aspects are accessible. Our initial results demonstrate the presence of Abelian projected monopoles on the lattice, and they lend support to 't Hooft's and Mandelstam's conjectures. In fact there are indications that the loss of confinement is accompanied by a transition from a magnetic monopole-antimonopole plasma to an electric quark-gluon plasma.

In sec. II we present a thorough discussion of the Abelian projection in the continuum, making the electric nature of quarks and gluons clear. In sec. III we elucidate the physical origin of the magnetic monopoles, and we show that topology requires them to obey a generalized Dirac quantization condition [8]. Section IV extends the Abelian projection to nonabelian lattice gauge theory, and we present the results of our numerical simulations in sec. V. Finally, we offer some concluding remarks in sec. VI.

II. Abelian projection in the continuum

We consider SU(N) gauge fields on a compact manifold M with field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{2.1}$$

where

$$A_\mu = A_\mu{}^a T^a \tag{2.2}$$

is the Lie algebra valued gauge potential. The generators T^a of SU(N) are antihermitian. Under local SU(N) gauge transformations g the potential transforms as

$$A'_\mu = g\,[A_\mu + \partial_\mu]\,g^{-1}, \tag{2.3}$$

and the field strength as

$$F'_{\mu\nu} = g\,F_{\mu\nu}\,g^{-1}. \tag{2.4}$$

We also consider matter fields, either in the fundamental representation

$$\psi' = g\,\psi, \tag{2.5}$$

or in the adjoint representation

$$\Phi' = g\,\Phi\,g^{-1}. \tag{2.6}$$

The art of choosing a gauge is to isolate the relevant degrees of freedom, so that calculations are simpler and the physics is clearer. To understand the physics of confinement, 't Hooft [7] suggested using the device of a (possibly composite) field X that transforms as

$$X' = g\,X\,g^{-1} \tag{2.7}$$

under local SU(N) gauge transformations. For example, X might be an adjoint matter field in a grand unified theory, the Polyakov loop in finite temperature gauge theory, any other Wilson loop, or the symmetric product $F_{\mu\nu}F_{\mu\nu}$. Indeed, X need not be a Lorentz scalar: F_{12} and the antisymmetric product $\epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ are also possibilities. Only the transformation law (2.7) is essential.

The nonabelian part of the gauge is completely fixed if one chooses a gauge transformation V so that X becomes diagonal everywhere,

$$X = V X V^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N), \tag{2.8}$$

with some ordering prescription for the eigenvalues λ_i. If X lives in the Lie algebra of SU(N), then the natural choice is $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$; if X lives in the gauge group SU(N), then one can write

$$\lambda_i = \exp(i\,\phi_i), \quad \sum_i \phi_i = 0, \quad |\phi_i - \phi_j| \leq 2\pi \quad \forall\ i, j, \tag{2.9}$$

and the natural choice is $\phi_1 \geq \phi_2 \geq \cdots \geq \phi_N$. However, V is only determined up to left multiplication by factors

$$d = \mathrm{diag}[\exp(i\alpha_1), \exp(i\alpha_2), \ldots, \exp(i\alpha_N)], \quad \sum_i \alpha_i = 0, \tag{2.10}$$

because if X is diagonal and eigenvalue-ordered, so is dXd^{-1}. {d} is the Cartan, or largest Abelian, subgroup: $\{d\} = U(1)^{N-1} \subset SU(N)$, and forms a residual local gauge group. Its appearance yields the terminology Abelian projection.

In the following discussion we shall refer to fields in the gauge where X is diagonalized and its eigenvalues are ordered. In other words, the gauge potential has been transformed

$$\tilde{A}_\mu = V\,[A_\mu + \partial_\mu]\,V^{-1}, \tag{2.11}$$

and the quark field in the fundamental representation has been transformed

$$\tilde{\psi} = V\psi. \tag{2.12}$$

It is instructive to discuss how matter and gauge fields, $\tilde{\psi}$ and $\tilde{A}_\mu$, transform under the residual $U(1)^{N-1}$ local gauge symmetry. The diagonal components of the vector potential

$$a_\mu{}^i = -i\,[\tilde{A}_\mu]_{ii} \tag{2.13}$$

transform as

$$a'_\mu{}^i = a_\mu{}^i - \partial_\mu \alpha_i, \tag{2.14}$$

i. e. as N Abelian potentials, subject to the constraint $\sum_i a_\mu{}^i = 0$. (Note that $\tilde{A}_\mu$ is traceless.) For convenience we shall follow 't Hooft's slight abuse of language and refer to the $a_\mu{}^i$ as "photons" [7]. A little rearrangement shows that their field strengths,

$$f_{\mu\nu}{}^i = \partial_\mu a_\nu{}^i - \partial_\nu a_\mu{}^i, \tag{2.15}$$

are given by

$$f_{\mu\nu}{}^i = -i\,[\,V F_{\mu\nu} V^{-1} - [V(A_\mu + \partial_\mu)V^{-1}, V(A_\nu + \partial_\nu)V^{-1}]\,]_{ii} \tag{2.16}$$

in terms of the original fields. If X were the Higgs field of a spontaneously broken grand unified theory with breaking $SU(N) \longrightarrow U(1)^{N-1}$, (2.16) would be the correct extension for the field strength of the famous 't Hooft-Polyakov monopole [9]. As in that case, the magnetic currents

$$K_\mu{}^i = \frac{1}{8\pi}\epsilon_{\mu\nu\rho\sigma}\partial_\nu f_{\rho\sigma}{}^i \tag{2.17}$$

do not vanish, owing to stringularities in the $[V\partial_\mu V^{-1}, V\partial_\nu V^{-1}]$ piece of (2.16). The magnetic currents are (topologically) conserved [10],

$$\partial_\mu K_\mu{}^i = 0 \tag{2.18}$$

because two derivatives are contracted with $\epsilon_{\mu\nu\rho\sigma}$. Integration of the current density over a three dimensional region Ω yields

$$m^i(\Omega) = \int_\Omega d^3\sigma_\mu\, K_\mu{}^i = \frac{1}{8\pi}\int_{\partial\Omega} d^2\sigma_{\mu\nu}\, \epsilon_{\mu\nu\rho\sigma} f_{\mu\nu}{}^i. \tag{2.19}$$

Taking Ω at constant time ($\mu = 4$) we see that

$$m^i(\Omega) = \frac{1}{4\pi}\int_{\partial\Omega} d^2\sigma_j\, b_j{}^i, \quad b_j{}^i = \frac{1}{2}\epsilon_{jkl} f_{kl}{}^i \tag{2.20}$$

is the magnetic flux through $\partial\Omega$, and hence counts the magnetic charge inside Ω. In the following section we shall show that the magnetic charges obey the Dirac quantization condition [8], i.e. $m^i(\Omega)$ is always $0, \pm\frac{1}{2}, \pm 1, \ldots$

The remaining particles are electrically charged with respect to the photons. The off-diagonal elements

$$c_\mu{}^{ij} = -\, i\, [\tilde{A}_\mu]_{ij} \quad (i \neq j) \tag{2.21}$$

transform as

$$c'_\mu{}^{ij} = \exp[\, i(\alpha_i - \alpha_j)]\, c_\mu{}^{ij}, \tag{2.22}$$

i. e. as $N(N-1)$ charged vector fields. (Note that $[\tilde{A}_\mu]_{ji} = -\,[\tilde{A}_\mu]_{ij}{}^*$.) The electric charge of $c_\mu{}^{ij}$ is $+1$ (-1) with respect to the Abelian potential $a_\mu{}^i$ $(a_\mu{}^j)$. We shall call the $c_\mu{}^{ij}$ gluons. Quark fields transform as

$$\tilde{\Psi}'_i = \exp(\, i\alpha_i)\, \tilde{\Psi}_i \tag{2.23}$$

with respect to $U(1)^{N-1}$; each of the N components of the quark field has electric charge $+1$ with respect to the appropriate photon. However, since $\sum_i a_\mu{}^i = 0$, the composition of N different components of the quark field is neutral.

It is physically attractive that in this framework both the quarks and gluons have electric charges. That means that any mechanism that confines electric charges will confine both quarks and gluons. If we had fixed the gauge further, leaving, for example, the Z_N center symmetry, the gluons would have been uncharged, and the physics of their confinement would be left obscure.

III. Topology of the gauge condition

From eqs. (2.15) and (2.17) the magnetic currents $K_\mu{}^i$ vanish if $a_\mu{}^i$ is twice differentiable. Since A_μ is (at least) twice differentiable, the only term in (2.16) contributing to $K_\mu{}^i$ is $[V\partial_\rho V^{-1}, V\partial_\sigma V^{-1}]$. Thus the flux integral is

$$m^i(\Omega) = \frac{i}{4\pi}\int_\Omega d^3\sigma_\mu \epsilon_{\mu\nu\rho\sigma}\partial_\nu\, [V\partial_\rho V^{-1}\; V\partial_\sigma V^{-1}]_{ii}. \tag{3.1}$$

When X has two degenerate eigenvalues, the corresponding eigenvectors composing V are not well defined. Then V has a line of directional singularities, which one can interpret as the world line of a magnetic monopole (in Euclidean space). In the generic case the world lines intersect the three dimensional region Ω in a discrete set of points. Because the eigenvalues of X are ordered, only adjacent pairs can become degenerate. If $\lambda_i = \lambda_{i+1}$, we shall label such a point $x^{(i)}$. Should X be an element of the group SU(N), one must keep in mind that (2.9) also admits $\lambda_1 = \lambda_N$ with $\phi_1 = \phi_N + 2\pi$. We shall label such points $x^{(0)}$ or $x^{(N)}$. Away from all $x^{(i)}$ the currents vanish because V is then differentiable often enough. It is therefore sufficient to restrict attention to infinitesimal balls $B_\epsilon(x^{(i)})$ around each $x^{(i)}$:

$$m^i(\Omega) = \sum_{x^{(i)}\in\Omega} m^i(B_\epsilon(x^{(i)})) + \sum_{x^{(i-1)}\in\Omega} m^i(B_\epsilon(x^{(i-1)})). \tag{3.2}$$

As an example, focus on the case $\lambda_1 = \lambda_2$, which contributes to m^1 and m^2. Using Gauß' theorem,

$$m^i(B_\epsilon(x^{(1)})) = \frac{i}{4\pi}\int_{S^2_\epsilon(x^{(1)})} d^2\sigma_{\mu\nu}\, \epsilon_{\mu\nu\rho\sigma}[V\partial_\rho V^{-1}\, V\partial_\sigma V^{-1}]_{ii}, \tag{3.3}$$

where $S^2_\epsilon(x^{(1)}) = \partial B_\epsilon(x^{(1)})$ is the infinitesimal sphere surrounding $x^{(1)}$, and now $i = 1, 2$. The integrand is again a total divergence:

$$\epsilon_{\mu\nu\rho\sigma}[V\partial_\rho V^{-1}\, V\partial_\sigma V^{-1}]_{ii} = -\, \epsilon_{\mu\nu\rho\sigma}\,\partial_\rho[V\partial_\sigma V^{-1}]_{ii}, \tag{3.4}$$

but one should not naively apply Gauß' theorem, because $V\partial_\sigma V^{-1}$ has singularities.

To exhibit explicitly the structure of these singularities we write

$$V = W\begin{pmatrix} V^{(2)} & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.5}$$

where $W \in SU(N)$ is constant in the neighborhood of $x^{(1)}$, and $V^{(2)}$ is an SU(2) matrix. A convenient parameterization* of $V^{(2)}$ is

$$V^{(2)} = \cos\tfrac{1}{2}\theta + i\vec{\sigma}\cdot\vec{e}_\phi \sin\tfrac{1}{2}\theta \tag{3.6}$$

where θ and ϕ are the polar and azimuthal angles in the SU(2) subspace. Combining (3.3)-(3.6) yields

$$m^i(B_\epsilon(x^{(1)})) = \frac{1}{8\pi}\int_{S^2_\epsilon(x^{(1)})} d^2\sigma_{\mu\nu}\, \epsilon_{\mu\nu\rho\sigma}\,\partial_\rho(1 - \cos\theta)\, \partial_\sigma\phi\, [\sigma_3]_{ii}. \tag{3.7}$$

The integrand is the Jacobian of the transformation from coordinates on $S^2_\epsilon(x^{(1)})$ to (θ, ϕ). Since

$$\Pi_2(SU(2)/U(1)) = Z \tag{3.8}$$

the magnetic charge of the (anti-)monopole is

$$m^i(B_\epsilon(x^{(1)})) = \pm\, \tfrac{1}{2}, \tag{3.9}$$

in accordance with the Dirac quantization condition.

* Other parameterizations of $V^{(2)}$ are $U(1)^{N-1}$ gauge transforms of (3.5).

Alternatively, one can integrate (3.7) further, excluding the point where $\theta = \pi$, at which a Dirac string crosses S^2_ϵ, leaving a line integral

$$m^i(B_\epsilon(x^{(1)})) = -\frac{i}{4\pi}\int_{S^1} d\sigma_{\mu\nu\rho}\,\epsilon_{\mu\nu\rho\sigma}[V\partial_\sigma V^{-1}]_{ii} \quad (3.10)$$
$$= \frac{1}{4\pi}\int_{S^1} d\sigma_{\mu\nu\rho}\,\epsilon_{\mu\nu\rho\sigma}\,\partial_\sigma\phi\,[\sigma_3]_{ii}$$

around the string. Equation (3.10) also yields (3.9), because

$$\Pi_1(U(1)) = Z. \quad (3.11)$$

It is also possible to avoid Dirac strings by introducing two different gauges for V or $V^{(2)}$, as in Wu and Yang's treatment [11] of the Dirac monopole.

Finally, at points $x^{(1)}$ eqs. (3.7) and (3.10) both show that

$$m^2(B_\epsilon(x^{(1)})) = -m^1(B_\epsilon(x^{(1)})). \quad (3.12)$$

Generalizing to all other monopole locations and using (3.2) one notices that

$$\sum_i m^i(\Omega) = 0 \quad (3.13)$$

for all Ω, which is a consequence of the representation of the $U(1)^{N-1}$ gauge group by N parameters with one constraint. Indeed, the topology of a single SU(2) subgroup can also be generalized. Considering (3.3) as denoting a diagonal matrix, the appropriate homotopy group is $\Pi_2(SU(N)/U(1)^{N-1})$, from the definition of V. We have shown that

$$\Pi_2(SU(N)/U(1)^{N-1}) = Z^{N-1} \quad (3.14)$$

by reducing $SU(N)/U(1)^{N-1}$ to $SU(2)/U(1)$. Actually (3.14) is a consequence of group homomorphisms, because

$$\Pi_1(SU(N)) = \{0\} \quad (3.15)$$

implies that

$$\Pi_2(SU(N)/U(1)^{N-1}) = \Pi_1(U(1)^{N-1}) = Z^{N-1}. \quad (3.16)$$

The singularities at the points $x^{(i)}$ have all the properties of magnetic monopoles: they produce the appropriate electric and magnetic fields of the gauge group $U(1)^{N-1}$, and their charge is quantized in the way consistent with quantum mechanics. Hence we shall call them monopoles. The generalization of the Dirac quantization condition to $U(1)^{N-1}$ is

$$\sum_i q^{(i)} m^{(i)} = 0, \pm\tfrac{1}{2}, \pm 1, \ldots, \quad (3.17)$$

where $q^{(i)}$ and $m^{(i)}$ are electric and magnetic charges with respect to $a_\mu{}^i$. The electromagnetic duality expressed by eq. (3.17) is neatly summarized by introducing (N-1)-dimensional electric and magnetic charge lattices [12], shown in fig. 1 for SU(2) and SU(3). The lattices incorporate the $\sum_i a_\mu{}^i = 0$ constraint geometrically and exhibit clearly that an electromagnetic duality transformation exchanges gluons and monopoles.

In contrast to most well known monopoles the Abelian projected monopoles arise from quantum fluctuations. X need not have a vacuum expectation value, nor need it be a solution of the classical equations of motion. Indeed, for many interesting choices of X, especially in pure gauge theories or QCD, the vacuum or classical value of X is trivial to diagonalize; then the monopoles arise solely from the fluctuations. Whether the monopoles describe the important fluctuations depends on the dynamics, and also on the choice of X.

IV. Abelian projection on the lattice

To attain nonperturbative control of the dynamics one must formulate the Abelian projection on the lattice. For simplicity we restrict our attention to a hypercubic lattice with periodic boundary conditions, i.e. the manifold $\mathbf{M}$ is the 4-torus T^4. The lattice is defined by

$$\Lambda = \{ s \in \mathbf{M} \mid s_\mu \in \mathbf{Z},\ \mu = 1, 2, 3, 4 \}, \tag{4.1}$$

and it induces a natural covering of $\mathbf{M}$ by hypercubical cells

$$\mathbf{M} = \bigcup_{s \in \Lambda} h(s), \quad h(s) = \{ y \in \mathbf{M} \mid s_\mu \le y_\mu \le s_\mu + 1,\ \forall\ \mu \}. \tag{4.2}$$

The lattice gauge field consists of parallel transporters $U(s, \hat{\mu})$ defined on the link from s to $s + \hat{\mu}$. Under local $SU(N)$ gauge transformations these transform as

$$U'(s, \hat{\mu}) = g(s)\, U(s, \hat{\mu})\, g^{-1}(s + \hat{\mu}), \tag{4.3}$$

Lattice matter fields are defined on the sites s and transform as in eqs. (2.5) and (2.6).

As in the continuum Abelian projection we wish to consider a (possibly composite) field X that transforms as

$$X'(s) = g(s)\, X(s)\, g^{-1}(s), \tag{4.4}$$

and we choose the gauge where X is diagonal and eigenvalue-ordered:

$$X(s) = V(s)\, X(s)\, V^{-1}(s) = \mathrm{diag}(\ \lambda_1(s),\ \lambda_2(s), \cdots,\ \lambda_N(s)). \tag{4.5}$$

Once again V is only determined up to left multiplication by factors

$$d(s) = \mathrm{diag}\{\exp[i\alpha_1(s)], \exp[i\alpha_2(s)], \ldots, \exp[i\alpha_N(s)]\}, \quad \sum_i \alpha_i(s) = 0, \tag{4.6}$$

which also forms a $U(1)^{N-1}$ gauge group on Λ.

In this gauge the link variables are

$$\tilde{U}(s, \hat{\mu}) = V(s)\, U(s, \hat{\mu})\, V^{-1}(s + \hat{\mu}), \tag{4.7}$$

and the quark field is as in (2.12). We want to extract Abelian parallel transporters $u(s, \hat{\mu}) \in U(1)^{N-1}$ and matter fields $c(s, \hat{\mu})$ from $U(s, \hat{\mu})$:

$$U(s, \hat{\mu}) = c(s, \hat{\mu})\, u(s, \hat{\mu}), \tag{4.8}$$

with the appropriate gauge transformation properties under $U(1)^{N-1}$:

$$u'(s, \hat{\mu}) = d(s)\, u(s, \hat{\mu})\, d^{-1}(s + \hat{\mu}) \tag{4.9u}$$

$$c'(s, \hat{\mu}) = d(s)\, c(s, \hat{\mu})\, d^{-1}(s) \tag{4.9c}$$

But this is not straightforward, essentially because the fundamental gauge variables $U(s, \hat{\mu})$ transform bilocally. The Abelian projection (4.8) is therefore not unique, although sensible choices should not be ambiguous in the continuum limit. This is a familiar story in lattice theories; for instance, the choice of the action or the second Chern number is not unique at finite lattice spacing.

Equation (4.8) is a coset decomposition of SU(N) with respect to its Cartan subgroup $U(1)^{N-1}$. A particularly nice parameterization is

$$u(s, \hat{\mu}) = \exp(\omega^d(s, \hat{\mu})\, T^d), \qquad (4.10)$$
$$c(s, \hat{\mu}) = \exp(\omega^n(s, \hat{\mu})\, T^n),$$

where the T^d (T^n) are the diagonal (nondiagonal) generators of SU(N). Insisting that $u(s, \hat{\mu}) \longrightarrow 1$ if $U(s, \hat{\mu}) \longrightarrow 1$ fixes $u(s, \hat{\mu})$ and $c(s, \hat{\mu})$ uniquely. In numerical work the decomposition (4.10) is inconvenient for $N \geq 3$, so we have adopted it only for SU(2). A convenient, but less symmetric, alternative to (4.10) is

$$u(s, \hat{\mu}) = \mathrm{diag}[u_i(s, \hat{\mu})]$$
$$u_i(s, \hat{\mu}) = \exp\{\, i \arg[U_{ii}(s, \hat{\mu})]\}, \; i = 1, N-1 \qquad (4.11)$$
$$u_N(s, \hat{\mu}) = \prod_{i=1}^{N-1} u_i(s, \hat{\mu})^*.$$

This is equally valid for all SU(N) and coincides with (4.10) in the special case SU(2). In the naive continuum limit both choices agree, and in fact

$$u_i(s, \hat{\mu}) \longrightarrow \exp(\, i \int_s^{s+\hat{\mu}} dx_\mu\, a_\mu{}^i) \qquad (4.12)$$

where $a_\mu{}^i$ is the continuum Abelian potential, and where the line integral extends along the link.

From (4.9) we see that $u_i(s, \hat{\mu})$ and the elements of $c(s, \hat{\mu})$ really are the

lattice analogs of $a_\mu{}^i$ and $c_\mu{}^{ij}$. Under $U(1)^{N-1}$ gauge transformations

$$u'_i(s, \hat{\mu}) = \exp[i\alpha_i(s)]\, u_i(s, \hat{\mu})\, \exp[i\alpha_i(s + \hat{\mu})], \qquad (4.13)$$
$$c'_{ij}(s, \hat{\mu}) = \exp[i\alpha_i(s) - i\alpha_j(s)]\, c_{ij}(s, \hat{\mu}). \qquad (4.14)$$

Hence, in this gauge the nonabelian lattice theory is equivalent to Abelian gauge fields (photons), charged vector matter fields (gluons), and, as we shall see, magnetic monopoles.

To study the magnetic currents on the lattice, it is most reasonable to integrate the current density over elementary faces

$$f(s, \mu) = h(s - \hat{\mu}) \cap h(s) \qquad (4.15)$$

in Λ. The flux integral (2.19) leads us to consider the flux through plaquettes. The Abelian parallel transport around a plaquette,

$$u_i(s, \hat{\mu}, \hat{\nu}) = u_i(s, \hat{\mu})\, u_i(s + \hat{\mu}, \hat{\nu})\, u_i{}^*(s + \hat{\nu}, \hat{\mu})\, u_i{}^*(s, \hat{\nu}), \qquad (4.16)$$

is related to the field strength $f_{\mu\nu}{}^i$ such that on the lattice (2.19) reads [4]

$$m^i(f(s, \mu)) = \frac{1}{4\pi} \sum_{p \in \partial f(s,\mu)} \arg u_i(p), \qquad (4.17)$$

where the phases $\arg u_i(p)$ are chosen so that

$$\sum_i \arg u_i(p) = 0, \quad |\arg u_i(p) - \arg u_j(p)| \leq 2\pi \quad \forall\, i, j \qquad (4.18)$$

and p inherits its orientation from $\partial f(s, \mu)$. Notice that $m^i(f(s, \mu))$ takes values

$0, \pm\frac{1}{2}, \pm 1, \ldots$, just as the topology of the continuum demands. This arises from the compact representation of $U(1)^{N-1}$ descendent from SU(N), and from the definition of the phases.

The physics is most transparent if we introduce the dual lattice $^*\Lambda$. The dual sites $^*s \in {}^*\Lambda$ are the centers of the hypercubes $h(s)$ in Λ. We can then define

$$m^i(^*s, \hat{\mu}) = m^i(f(s + \hat{\mu}, \mu)) \qquad (4.19)$$

and view $m^i(^*s, \hat{\mu})$ as half-integer valued magnetic currents on the links of the dual lattice. They are conserved on the dual sites $^*s$:

$$\sum_\mu \left[m^i(^*s, \hat{\mu}) - m^i(^*s - \hat{\mu}, \hat{\mu}) \right] = 0, \qquad (4.20)$$

in analogy with local continuum current conservation (2.18). One obtains the total magnetic charge by summing over all oriented faces in a 3-torus (a time slice), but this vanishes due to $\partial T^3 = \emptyset$ and the magnetic Gauß' law.

The lattice magnetic currents enjoy several other nice properties. They are gauge invariant. By construction $\sum_i m^i(^*s, \hat{\mu}) = 0$, i.e. (3.13) holds. They have a genuine topological significance: small deformations of the lattice fields do not change the currents. Finally, the lattice expression (4.17) has the correct continuum limit (2.19). We are therefore justified in interpreting activity in the magnetic currents as evidence for the monopoles introduced in sec. II.

V. Numerical investigation of monopole dynamics

The previous section provides the computational tools to study 't Hooft's ideas on confinement [7]. He conjectured that the monopole degrees of freedom condense, causing a dual Meissner effect. This means that the Abelian magnetic fields are screened, and that the Abelian electric fields are constricted to flux tubes whose ends are capped by electric charges. Recall that the gluons c_μ^{ij} as well as the quarks carry electric charge. The energy of the electric flux tube rises linearly with its length, which leads to confinement. This picture hinges not only on the Abelian projection, but also on the choice of the adjoint field X. The best X should allow the confinement picture to emerge without obscuring the rest of the physics. Barring theoretical input suggesting a candidate X, we advocate testing candidates in numerical simulations, which one needs for a detailed understanding of the dynamics anyway.

't Hooft suggested [7] considering $X = F_{12}$ for SU(2) or $X = F_{\mu\nu}F_{\mu\nu}$ for general SU(N), $N \geq 3$. On the lattice we therefore consider

$$X(s) = U(s, \hat{1}, \hat{2}) \qquad (5.1)$$

for SU(2), where the nonabelian plaquette $U(s, \hat{\mu}, \hat{\nu})$ is defined by analogy to (4.16). For $N \geq 3$ we wish to preserve the parity of $F_{\mu\nu}F_{\mu\nu}$ on the lattice. From a plaquette we thus define

$$F(s, \hat{\mu}, \hat{\nu}) = -i \log U(s, \hat{\mu}, \hat{\nu}) \qquad (5.2)$$

using the Cayley-Hamilton procedure to obtain the logarithm [13]. Then

$$X(s) = \sum_{\mu,\nu} \left[F^2(s, \hat{\mu}, \hat{\nu}) + F^2(s, -\hat{\mu}, \hat{\nu}) + F^2(s, \hat{\mu}, -\hat{\nu}) + F^2(s, -\hat{\mu}, -\hat{\nu}) \right]. \qquad (5.3)$$

The Polyakov loop is an order parameter of the pure glue deconfinement phase transition; hence it is also an obvious candidate at finite temperature:

$$X(s) = \prod_{t=0}^{L_4-1} U(s + t\hat{4}, \hat{4}) \tag{5.4}$$

where L_4 is the extent of the lattice in the 4-direction. Incidentally, the order parameter can be reconstructed from the Abelian parallel transporters:

$$X(s) = \prod_{t=0}^{L_4-1} \mathrm{diag}[u_1(s + t\hat{4}, \hat{4}), \ldots, u_N(s + t\hat{4}, \hat{4})] \tag{5.5}$$

when X is the Polyakov loop. In the continuum, this choice corresponds to treating A_4 as a "Higgs field," which is familiar from the study of nonabelian magnetic monopoles at finite temperature [14].

We have turned to the pure compact U(1) lattice gauge theory for hints, because it is known to have a phase transition driven by magnetic monopoles. At strong coupling monopole loops condense, and the vacuum is a plasma of quasifree monopoles and antimonopoles. Electric charges are confined. At weak coupling the monopole loops become smaller and more dilute. This can be made very explicit by transforming the partition function of the compact U(1) theory into a noncompact theory with magnetic monopoles [4, 5]. However, the monopoles decouple in the continuum limit. In the Abelian projection of nonabelian gauge theory there are gluon degrees of freedom in addition to the photons and monopoles, and one hopes that gluon interactions are sufficiently intense so that monopoles survive the continuum limit. This could then explain why the T = 0 nonabelian theory has no phase transition.

In the framework of the pure compact U(1) lattice gauge theory Cardy [5] suggested the polarizability of the monopole gas,

$$\chi^i_m = \sum_{*s,\mu} (^*s)^2 \langle m^i(^*s, \hat{\mu})\, m^i(0, \hat{\mu}) \rangle, \tag{5.6}$$

as a probe of the phase structure. In the Coulomb phase χ^i_m is finite and renormalizes the coupling by a finite amount. In the confining phase it diverges, and the infinite renormalization of the coupling results in the breakdown of the Coulomb force law. The polarizability should also be a significant quantity in the Abelian projection of SU(N). Unfortunately, we are not yet in a position to determine χ^i_m in a Monte Carlo calculation, owing to dramatic cancellations.

Instead of the polarizability we have looked at three quantities of more heuristic interest. First, the density of the total length of loops of monopole current [6]:

$$\ell^i = \frac{1}{4V} \sum_{*s,\mu} |m^i(^*s, \hat{\mu})|, \tag{5.7}$$

where $V = L_1 L_2 L_3 L_4$ is the lattice volume. Second, the number density of monopoles and antimonopoles:

$$\rho^i_m = \frac{1}{V} \sum_{*s} |m^i(^*s, \hat{4})|. \tag{5.8}$$

Third, the number density $\rho^i_{m\bar{m}}$ of monopole-antimonopole neighbors, i.e. the number density of instances where a monopole and antimonopole, both of type i, are in adjacent spacelike cubes.

Using the Wilson action, we have performed numerical simulations on a 5^4 lattice for the simplest case, SU(2), and the most interesting case, SU(3). We have not included dynamical fermions. We are especially interested in the deconfinement phase transition, which is also known to appear on symmetric lattices [15]. In figs. 2 and 3 we plot the length of loops vs. gauge coupling β and the neighbor density vs. number density. Figure 2 is for SU(2), and fig. 3 for SU(3). For SU(2) figs. 2a,b are for X as in (5.1), and figs. 2c,d are for the Polyakov loop (5.4). Here the label i is superfluous, because of the unitary constraint, so it is omitted. For SU(3) figs. 3a,b are for X as in (5.3), and figs 3c,d are again for the Polyakov loop. The label i is now needed, but we have averaged the numerical data over i.

Let us first discuss the length of loops vs. β (figs. 2a,c and 3a,c). Large ℓ means that there are large and/or many monopole loops. If monopoles drive confinement one expects that their loops condense, i.e. ℓ is large, in the confined phase. On the other hand, monopoles should be dilute in the deconfined phase, i.e. ℓ should be smaller. Indeed, for all choices of X and for both gauge groups we see the anticipated behavior: ℓ is larger in the confined phase (to the left of the dashed line) than in the deconfined phase (to the right). The effect is especially noticeable when we choose X to be the Polyakov loop. Moreover, notice that in that case the falloff in ℓ is steeper for SU(3) than for SU(2). For these gauge theories the phase transition is known to be first and second order, respectively. Of course, we cannot use our results on these 5^4 lattices to determine the order of the transition, but it is reassuring that the qualitative behavior is compatible.

In figs. 2b,d and 3b,d we plot $\rho_{m\bar{m}}$ vs. ρ_m. If the monopoles condense or form a monopole-antimonopole plasma one expects the distribution of monopoles and antimonopoles to be more or less independent, so $\rho_{m\bar{m}} \sim \rho_m^2$ at low values of β. On the other hand, one expects tightly bound monopole-antimonopole pairs in the deconfined phase, so $\rho_{m\bar{m}} \sim \rho_m$ at high values of β. Again, our results support the expectations in the confined phase, yet only when X is the Polyakov loop do they support the expectations in the deconfined phase.

It is not hard to explain the special paucity of monopoles in the deconfined phase when X is chosen to be the Polyakov loop. It exhibits long range order by freezing to an element of the center, $Z_N \subset SU(N)$, in the deconfined phase. The fluctuations are small, and since the monopoles arise from fluctuations, they are very dilute. Our other choices of X are not order parameters, so they fluctuate more, yielding more monopoles in the Abelian projection.

Recently there has been other work in this direction [16]. There a number density of magnetic monopoles was computed, after diagonalizing the Polyakov loop. Owing to a lack of details describing the algorithm used in ref. [16], we cannot be sure that it has defined a genuine Abelian projection in the sense of 't Hooft. Our calculations do not support the conclusions drawn by those authors. In particular, we find no evidence for a cusp in ρ_m.

In the continuum theory Abelian projected monopoles based on the Polyakov loop are static. In the lattice simulations this property is lost at small values of β due to "small scale fluctuations." These are familiar from studies of the topological (instanton) charge [17, 18], and in general arise whenever topological quantities are defined in terms of lattice fields that are not sufficiently smooth. Indeed, at higher values of β we see milder small scale fluctuations, because the fields are smoother. Before making quantitative conclusions, one must carefully analyze the potential contribution of small scale fluctuations. However, we have not analyzed ℓ, ρ_m, or $\rho_{m\bar{m}}$ in this light, because of the essentially qualitative nature of this initial study.

VI. Discussion and conclusions

Although the Abelian projection of nonabelian gauge theory offers an appealing framework for studying problems like the confinement mechanism, until now it has not led to compelling results because of a lack of computational tools. After a detailed reexamination of the projection of the continuum theory, we have succeeded in implementing it in lattice gauge theory. In the Abelian projected version the degrees of freedom have been rearranged into photons, gluons, and, as manifestation of the topology, magnetic monopoles. In particular, we have a genuine $U(1)^{N-1}$ gauge model on the lattice: the photons are defined on the links, the gluons are defined on the sites, and the monopole currents are defined on the dual links.

The advantage of formulating the Abelian projection on the lattice is that it allows one to study its implications in a nonperturbative setting. In particular, widely accepted ideas on (color) electromagnetic duality can be tested quantitatively using numerical simulations. Our simulations show clearly that Abelian projected monopoles are present in nonabelian gauge theories, and that their properties are significantly different in the two phases. The phase with electric confinement is a plasma of magnetic charges, whereas the other phase is a plasma of electric charges (quarks and gluons) with tightly bound magnetic dipoles. Perhaps the monopoles are even confined. Thus, our simulations constitute the first numerical evidence for electromagnetic duality in nonabelian gauge theories.

With this formalism a number of projects come to mind, of which we mention only a few. A calculation of the polarizability suggested by Cardy (eq. (5.6)) [5] would place the present results on a more solid footing, so it is worth overcoming the challenge of the concomitant cancellations. In grand unified theories with adjoint Higgs fields one could take $X = \Phi$ and use the Abelian projected version to investigate the phase structure and symmetry breaking patterns in terms of the models' monopoles. Finally, one could use our explicit expressions for the degrees of freedom of Abelian projected nonabelian gauge theories to make analytical progress, for example along the lines of ref. [4].

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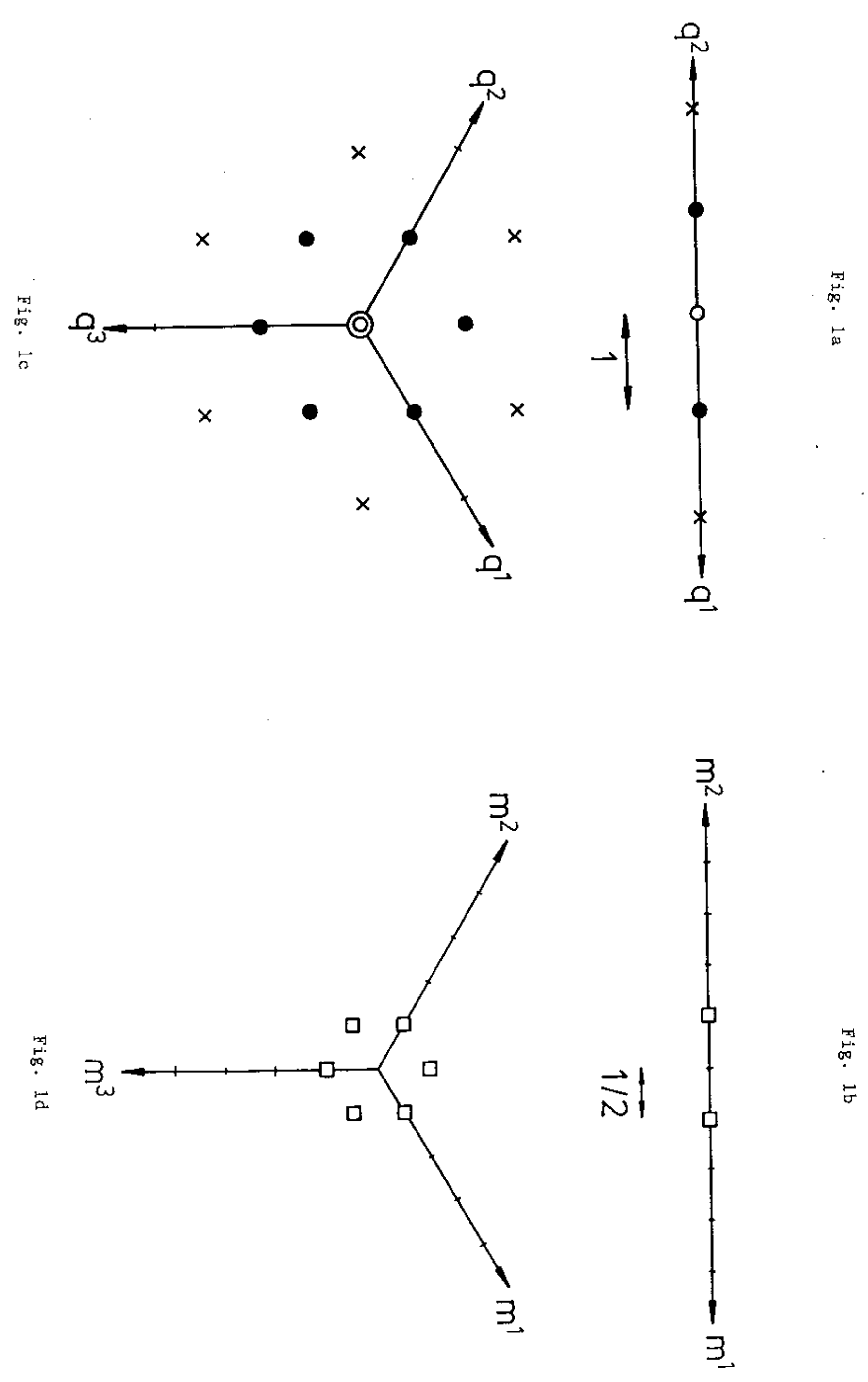
Figure Captions:

Fig. 1. Electric (a) and magnetic (b) charge lattices for SU(2). (c) and (d) are for SU(3). Open circles o represent photons, crosses x gluons, dots • quarks and antiquarks, and squares □ magnetic monopoles and antimonopoles.

Fig. 2. Results of numerical simulations in SU(2) at β = 0.1, 1.0, 1.9, 2.0, 2.1, 2.2, 2.3, 2.4, 2.5, and 3.0. X is taken from (5.1) for (a,b) and from (5.4: Polyakov loop) for (c,d). (a) and (c) show the density of the length of monopole loops vs. β; the dashed line indicates the location of the deconfinement phase transition; in (c) the absolute value |L| of the Polyakov loop is also shown. (b) and (d) show the neighbor density vs. the number density, which both decrease with increasing β; in (d) the straight line is a fit through the origin and the first two (high β) points, and the parabola $\rho_{mm} \sim \rho_m^2$ is forced through the last (β = 0.1) point.

Fig. 3. Same as fig. 2 for SU(3) at β = 0.1, 4.0, 5.3, 5.5, 5.65, 5.8, 6.0, 7.0, 10, and 20, <u>except</u> X is taken from (5.3) for (a) and (b). In (b) and (d) the densities ρ_{mm} and ρ_m at β = 20 turn out to be larger than the corresponding β = 10 values; otherwise the qualitative dependence of the densities on β is as in SU(2).



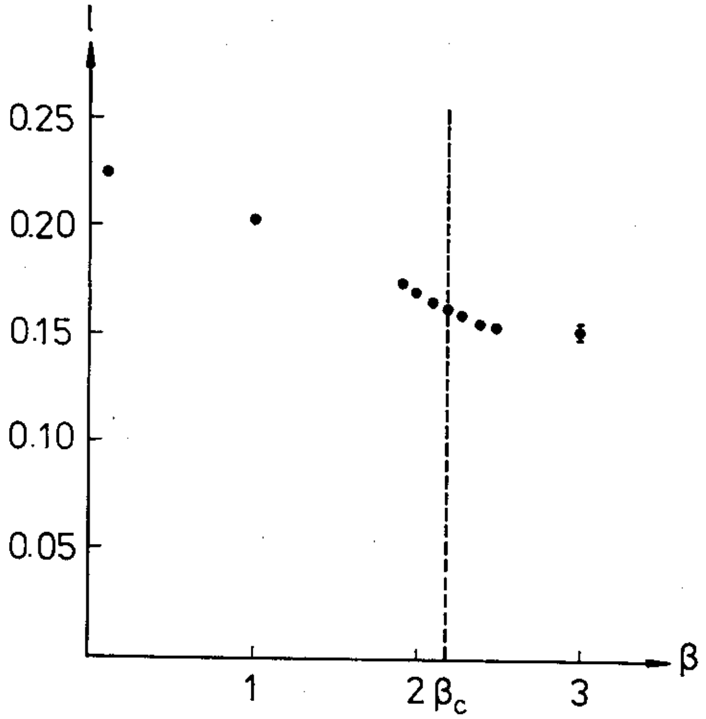


Fig. 2a

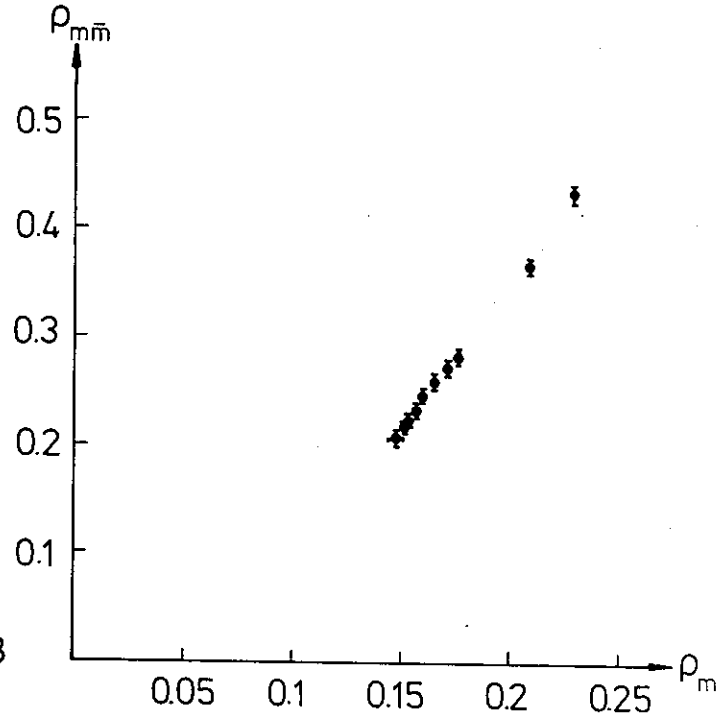


Fig. 2b

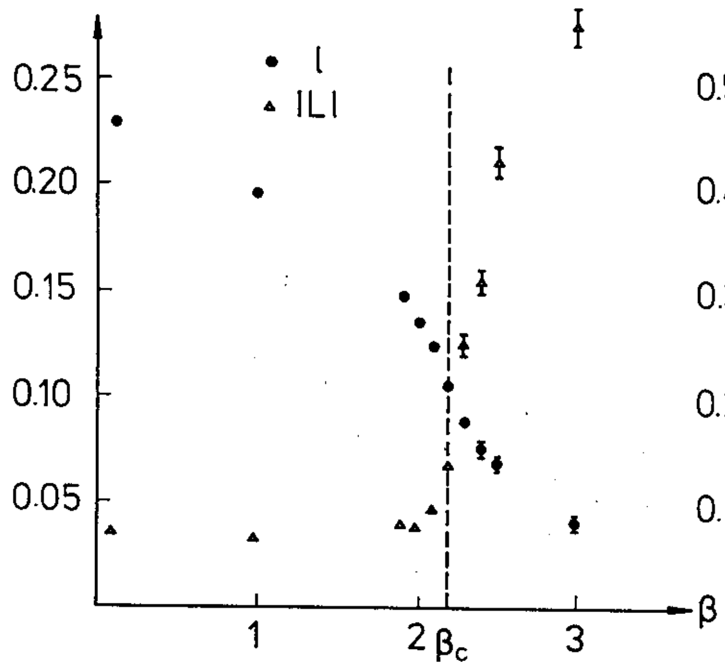


Fig. 2c

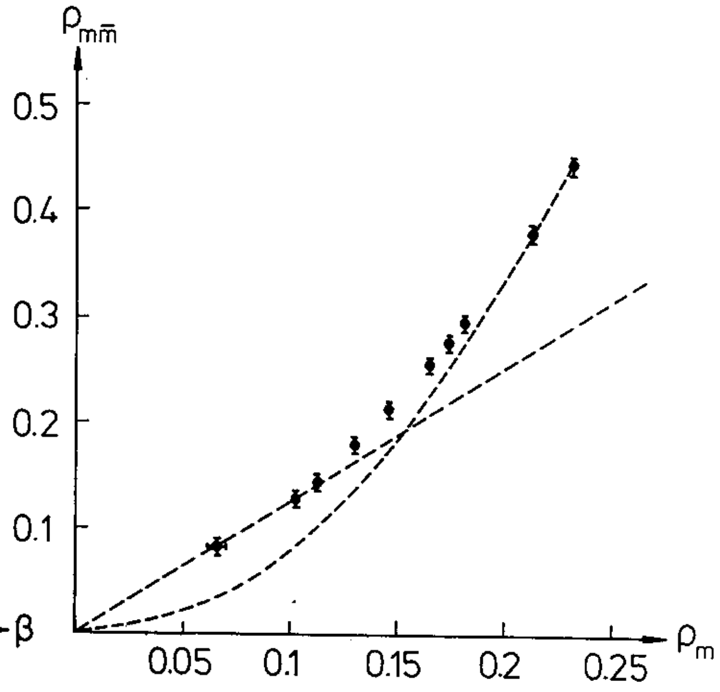


Fig. 2d

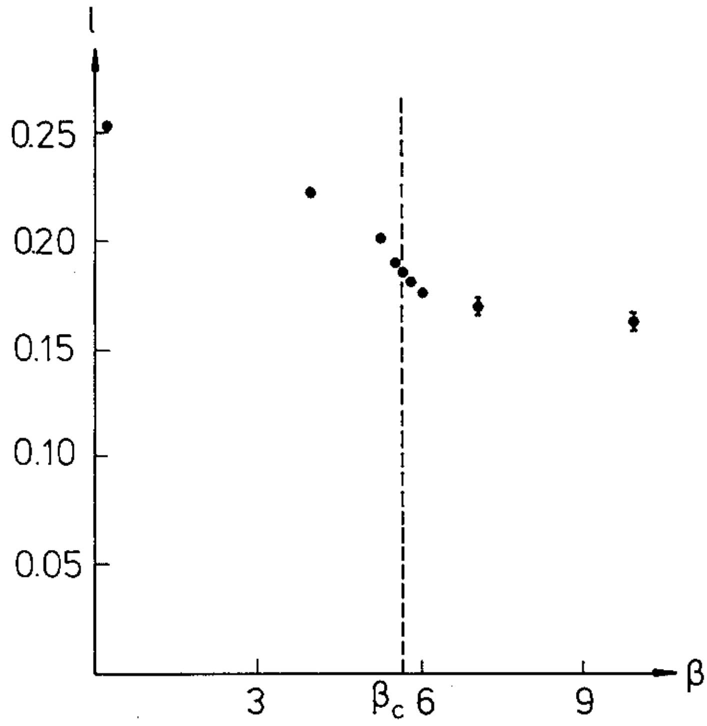


Fig. 3a

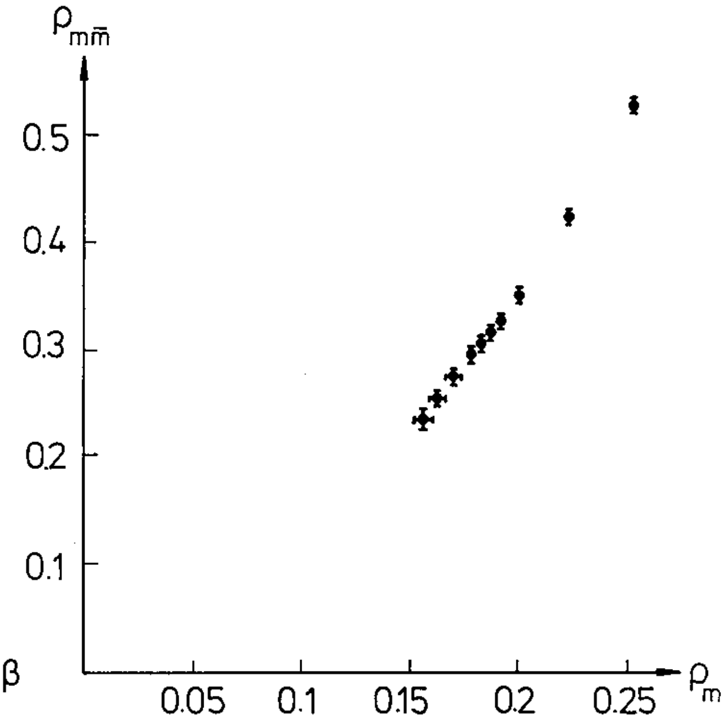


Fig. 3b

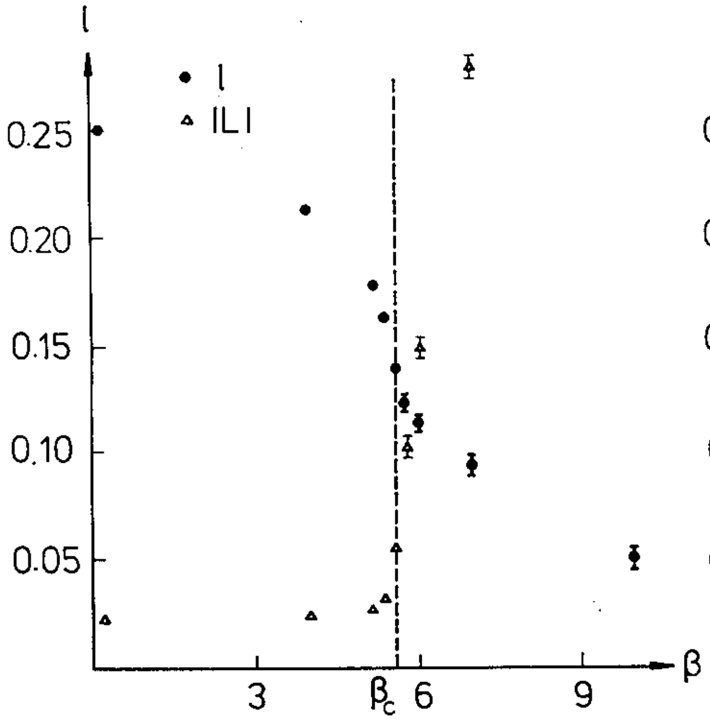


Fig. 3c

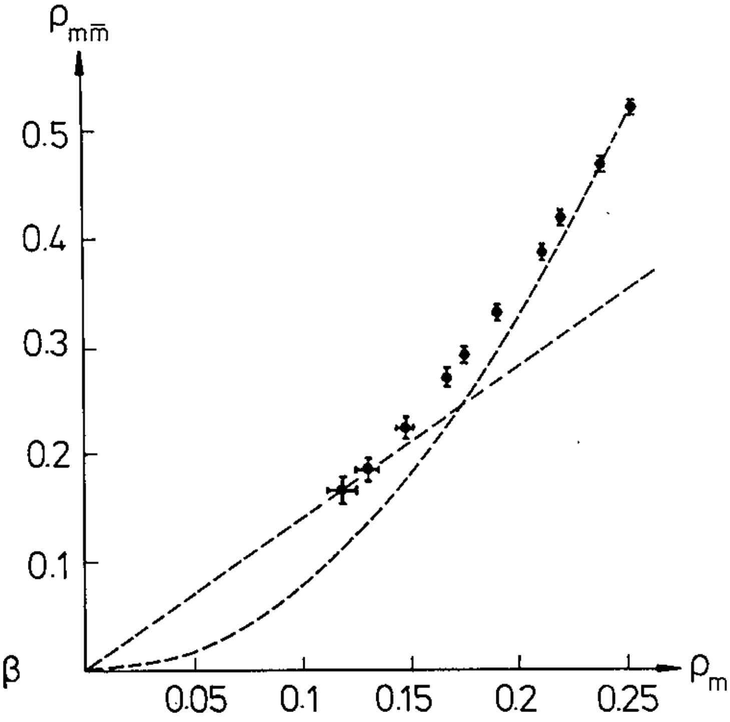


Fig. 3d