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Weak gauge coupling expansion near the critical line in the standard SU(2) Higgs model

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Abstract

The behaviour of the lattice-regularized SU(2) Higgs model with a scalar doublet field is investigated near the critical line at vanishing gauge coupling. By the use of the expansion in the gauge coupling the determination of the renormalization group trajectories is reduced to a similar problem in the pure ϕ^4 model. The shape of the critical surface separating the confining- and the Higgs-phase can also be obtained by the weak gauge coupling expansion from the shape of the ϕ^4 critical line.

1 Introduction

The large cut-off behaviour of a gauge theory with scalar matter fields is in general quite different from a theory with spin-$\frac{1}{2}$ fermion matter fields. The scalar fields can have renormalizable self-couplings, in contrary to the fermion fields, but these couplings are typically not asymptotically free, therefore the question of the large cut-off behaviour is a non-perturbative problem. In the simplest case of the $O(n)$-symmetric n-component scalar ϕ^4 model (without gauge fields) information about the large cut-off behaviour can be obtained by combining the hopping parameter expansion ("high temperature expansion" in the terminology of statistical physics) and Callan-Symanzik renormalization group equations [1]. This procedure is particularly succesful in the symmetric phase, where the high order hopping parameter expansion has, for intermediate cut-off's, a similar (or even better) precision than a good Monte Carlo calculation [2]. The renormalized scalar ϕ^4 coupling vanishes logarithmically with the cut-off near the critical line. As a consequence, for very high cut-off's the possible values of the physical parameters are severely constrained. In particular, in the limit of an infinite cut-off ("continuum limit") the only possibility is a free (trivial, non-interacting) theory. In the case of the single component ϕ^4 model the triviality of the continuum limit is also supported by several important exact results (for an incomplete list of references see [3]). Of course, the triviality of the continuum limit does not necessarily mean that such theories are physically uninteresting, since probably every quantum field theory is physically valid only up to some high but finite cut-off.

It is an interesting question, how does the inclusion of an asymptotically free non-abelian gauge coupling change the large cut-off behaviour. In the present paper this question is investigated in the simple and important case of the SU(2) Higgs model with a scalar doublet (in terms of real field components this is equivalent to the gauging of an $O(3)$ subgroup of the $O(4)$ symmetric 4-component ϕ^4 model). The technical tool which will be used is the weak gauge coupling expansion (WGCE) [4] in the vicinity of the critical line at vanishing gauge coupling.

In the next Section first the lattice action will be defined with the lattice version of the covariant gauge fixing and then the WGCE master formula for connected Green's functions will be derived. In Section 3 the general procedure of deriving differential equations for the curves of constant physics will be discussed and applied to the standard Higgs model at arbitrary scalar self-coupling and small gauge coupling. In Section 4 the phase transition surface separating the confining phase and the Higgs phase will be investigated. Section 5 contains the discussion.

2 Weak gauge coupling expansion with gauge fixing

In Ref. [4] WGCE was derived in a gauge invariant formalism for gauge invariant Green's functions of composite fields. For the discussion of renormalization it is, however, more convenient to consider the gauge dependent Green's functions in the renormalizable 't Hooft gauges [5]. In this Section first the gauge fixed lattice action will be considered and then the master formula of WGCE for connected Green's functions will be derived.

2.1 Lattice action with covariant gauge fixing

Let us first consider the gauge action. The SU(2) gauge variable $U(x,\mu)$ on the link $(x, x+\hat{\mu})$ from the point x to the neighbouring point $x+\hat{\mu}$ will be described by 3 real components $a_{rx\mu}$ $(r = 1,2,3;\ \mu = 1,2,3,4)$ in the same way as in [4] :

$$U(x,\mu) = 1 - a_{x\mu} + i\tau_r a_{rx\mu} \qquad (1)$$

Here τ_r $(r = 1,2,3)$ are isospin Pauli matrices (over repeated isospin indices $r, s, \ldots$ an automatic summation is understood). The real variable $a_{x\mu}$ is given in general by

$$a_{x\mu} \equiv 1 - a_{0x\mu} = 1 - z_{x\mu}\sqrt{1 - a_{rx\mu}a_{rx\mu}} \qquad (2)$$

where $z_{x\mu} = \pm 1$ is an Ising variable. In perturbation theory it is assumed that $U(x,\mu) \simeq 1$ dominates, therefore one can put $z_{x\mu} \equiv 1$. In this case we have

$$a_{x\mu} = \frac{1}{2}a_{rx\mu}a_{rx\mu} + \sum_{n=2}^{\infty} \frac{(2n-3)!!}{2^n n!} \left(a_{rx\mu}a_{rx\mu}\right)^n \equiv \frac{1}{2}a_{rx\mu}a_{rx\mu} + \bar{a}_{x\mu} \qquad (3)$$

For the gauge part of the lattice action S_g we take the Wilson action:

$$S_g = \beta \sum_P \left(1 - \frac{1}{2} Tr\, U_P\right) \qquad (4)$$

where $\beta \equiv 4/g^2$ gives the bare gauge coupling, $\sum_P$ stands for a summation over positively oriented plaquettes and U_P is the product of the link variables around the plaquette.

The gauge field propagator and vertices can be obtained by substituting Eq. (1) into Eq. (4) and collecting terms according to the powers of $a_{rx\mu}$:

$$S_g = S_g^{(2)} + S_g^{(3)} + S_g^{(4)} + \cdots \tag{5}$$

The second order term $S_g^{(2)}$ will be discussed later together with the propagators. The third and fourth order vertices can be simply expressed in terms of the Fourier-transformed variables $\tilde{a}_{rk\mu}$:

$$\tilde{a}_{rk\mu} \equiv \sum_x e^{-i(k,x)-\frac{i}{2}k_\mu} a_{rx\mu} \qquad a_{rx\mu} = \frac{1}{N}\sum_k e^{i(k,x)+\frac{i}{2}k_\mu}\tilde{a}_{rk\mu} \tag{6}$$

Here $N = N_1N_2N_3N_4$ is the number of lattice points and $\sum_k$ means a summation over the Brillouin zone (we always assume periodic boundary conditions). The scalar product of momentum and position is defined as

$$(x,k) \equiv 2\pi\left(\frac{\nu_1 x_1}{N_1} + \cdots \frac{\nu_4 x_4}{N_4}\right) \tag{7}$$

where $\nu_1, \ldots, \nu_4$ are the integers characterizing lattice momenta. In terms of the variables in Eq. (6) the three-point gauge vertex is given by

$$S_g^{(3)} = 4ig^{-2}\epsilon_{rst} \sum_{k_1k_2k_3}^{0} \sum_{\mu,\nu} \tilde{a}_{rk_1\mu}\tilde{a}_{sk_2\nu}\tilde{a}_{tk_3\nu} \cos\left(\frac{k_{1\nu}}{2}\right) 2\sin\left(\frac{k_{3\mu}-k_{2\mu}}{2}\right) \tag{8}$$

The momentum sums are always taken over momenta which sum up to zero. Similarly, the four point gauge vertex is:

$$S_g^{(4)} = g^{-2} \sum_{k_1k_2k_3k_4}^{0} \sum_{\mu,\nu} \left\{ \frac{1}{2}\tilde{a}_{rk_1\mu}\tilde{a}_{rk_2\mu}\tilde{a}_{sk_3\mu}\tilde{a}_{sk_4\mu}\, 4\sin^2\left(\frac{k_{1\nu}+k_{2\nu}}{2}\right) \right.$$

$$+4\left(\tilde{a}_{rk_1\mu}\tilde{a}_{rk_2\mu}\tilde{a}_{sk_3\nu}\tilde{a}_{sk_4\nu} - \tilde{a}_{rk_1\mu}\tilde{a}_{sk_2\mu}\tilde{a}_{rk_3\nu}\tilde{a}_{sk_4\nu}\right)\cos\left(\frac{k_{1\nu}-k_{2\nu}}{2}\right)\cos\left(\frac{k_{3\mu}-k_{4\mu}}{2}\right)$$

$$-2\tilde{a}_{rk_1\mu}\tilde{a}_{rk_2\mu}\tilde{a}_{sk_3\nu}\tilde{a}_{sk_4\nu}\cdot$$

$$\cdot\left[\cos\left(\frac{k_{1\nu}+k_{2\nu}}{2}\right) - \cos\left(\frac{k_{1\nu}-k_{2\nu}}{2}\right)\right]\left[\cos\left(\frac{k_{3\mu}+k_{4\mu}}{2}\right) - \cos\left(\frac{k_{3\mu}-k_{4\mu}}{2}\right)\right]$$

$$\left. -2\tilde{a}_{rk_1\mu}\tilde{a}_{rk_2\mu}\tilde{a}_{sk_3\nu}\tilde{a}_{sk_4\nu}\, 2\sin\left(\frac{k_{4\mu}}{2}\right) 2\sin\left(k_{3\nu}+\frac{k_{4\nu}}{2}\right)\right\} \tag{9}$$

This expression is somewhat simpler than the general formula for SU(n) with the usual exponential parametrization of $U(x,\mu)$ [6].

The integration measure for the gauge variables is originally the invariant SU(2) Haar measure $d^3U(x,\mu)$. In terms of the real variables $a_{rx\mu}$ this can be written as

$$d^3U(x,\mu) = \frac{\theta(1-a_{rx\mu}a_{rx\mu})}{2\pi^2\sqrt{1-a_{rx\mu}a_{rx\mu}}} d^3a_{rx\mu}da_{0x\mu} \sum_{z_{x\mu}} \delta\left(a_{0x\mu} - z_{x\mu}\sqrt{1-a_{rx\mu}a_{rx\mu}}\right)$$

$$\Rightarrow d^3a_{rx\mu}\exp\left\{\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}(a_{rx\mu}a_{rx\mu})^n\right\} \tag{10}$$

The second line is obtained from the first one by omitting an inessential constant factor and taking only $z_{x\mu} = +1$. The exponent will be included in the action by the "measure" term

$$S_m = -\sum_{(x\mu)}\sum_{n=1}^{\infty}\frac{1}{2n}(a_{rx\mu}a_{rx\mu})^n \tag{11}$$

Here $\sum_{(x\mu)}$ denotes a sum over positively oriented links.

The Higgs field variables can be chosen in several different ways. Here we shall use the real components σ_{0x} and π_{rx} $(r = 1,2,3)$ which are connected to the 2×2 matrices φ_x and $\alpha_x \in SU(2)$ used, for instance, in [4] by

$$\varphi_x \equiv \rho_x\alpha_x \equiv \sigma_{0x} + i\tau_r\pi_{rx} \tag{12}$$

The connection to the SU(2) doublet field ϕ_x^α $(\alpha = 1,2)$ is:

$$\sigma_{0x} = \frac{1}{2}(\phi_x^2 + \phi_x^{2*})$$

$$\pi_{1x} = \frac{i}{2}(-\phi_x^1 + \phi_x^{1*}) \quad \pi_{2x} = \frac{1}{2}(\phi_x^1 + \phi_x^{1*}) \quad \pi_{3x} = \frac{i}{2}(\phi_x^2 - \phi_x^{2*}) \tag{13}$$

The σ-field σ_{0x} can have a non-zero vacuum expectation value, which will be denoted by v. The fluctuation part σ_x is defined by

$$\sigma_x \equiv \sigma_{0x} - v \tag{14}$$

The pure Higgs scalar action S_h can be obtained in terms of σ_x and π_{rx} from the usual O(4) symmetric form with four components σ_{0x} and π_{rx} by shifting the σ-field according to Eq. (14):

$$S_h = 2v(1-2\lambda-8\kappa+2\lambda v^2)\sum_x \sigma_x - 2\kappa\sum_{(x\mu)}(\sigma_x\sigma_{x+\hat{\mu}} + \pi_{rx}\pi_{rx+\hat{\mu}})$$

$$+\sum_x\left\{\sigma_x^2(1-2\lambda+6\lambda v^2) + \pi_{rx}\pi_{rx}(1-2\lambda+2\lambda v^2) + 4\lambda v\sigma_x(\sigma_x^2+\pi_{rx}\pi_{rx}) + \lambda(\sigma_x^2+\pi_{rx}\pi_{rx})^2\right\} \tag{15}$$

This is the lattice action of the linear σ-model [7], which is the limit of the standard SU(2) Higgs model for vanishing gauge coupling $g^2 = 0$. The bare parameters λ and κ stand, respectively, for the scalar self-coupling and hopping parameter.

Besides the gauge term S_g, the measure term S_m and the σ-model term S_h, the whole action S contains also the gauge fixing term S_{gf}, the Faddeev-Popov term S_{fp} and the interaction term S_i describing the interaction between the gauge field and the scalar matter fields:

$$S \equiv S^{(\alpha)}_{\lambda,\beta,\kappa} \equiv S[\sigma,\pi,a] = S_g + S_m + S_h + S_{gf} + S_{fp} + S_i \tag{16}$$

The gauge fixing function f_{rx} is chosen in such a way that the mixed second order term in $a_{rx\mu}$ and π_{rx} be absent in S. This can be achieved by taking [5,6]:

$$f_{rx} \equiv \sum_{\mu>0}(a_{rx\mu} - a_{rx-\hat{\mu}\mu}) - \frac{\alpha\kappa}{2}g^2 v\pi_{rx} \tag{17}$$

Here α is the usual arbitrary gauge parameter. The resulting gauge fixing term in the action is:

$$S_{gf} = \frac{2}{\alpha g^2}\sum_x\sum_{\mu,\nu}(a_{rx\mu} - a_{rx-\hat{\mu}\mu})(a_{rx\nu} - a_{rx-\hat{\nu}\nu}) + \frac{\alpha\kappa^2}{2}g^2v^2\pi_{rx}\pi_{rx} \tag{18}$$

The corresponding Faddeev-Popov term, involving the Grassmann variables c_{rx} and $\bar{c}_{rx}$ is:

$$S_{fp} \equiv \sum_{xy} \bar{c}_{rx} M[\sigma,\pi,a]_{rx,sy} c_{sy}$$

$$= \sum_{(x\mu)} (\bar{c}_{rx} - \bar{c}_{rx+\hat{\mu}})(c_{rx} - c_{rx+\hat{\mu}}) + \frac{\alpha\kappa}{2} g^2 v \sum_{x} \{v\bar{c}_{rx}c_{rx} + \bar{c}_{rx}(\delta_{rs}\sigma_x + \epsilon_{rst}\pi_{tx})c_{sx}\}$$

$$- \sum_{(x\mu)} \left\{ (\frac{1}{2}a_{rx\mu}a_{rx\mu} + \bar{a}_{x\mu})(\bar{c}_{rx} - \bar{c}_{rx+\hat{\mu}})(c_{rx} - c_{rx+\hat{\mu}}) + \epsilon_{rst}a_{rx\mu}(\bar{c}_{sx} + \bar{c}_{sx+\hat{\mu}})(c_{tx} - c_{tx+\hat{\mu}}) \right\} \quad (19)$$

The interaction piece S_i in the action is given by the hopping term $Tr(\varphi^{\dagger}_{x+\hat{\mu}} U(x,\mu)\varphi_x)$. In terms of the variables $a_{rx\mu}$ and σ_x, π_{rx} it is:

$$S_i = 2\kappa \sum_{(x\mu)} \left\{ v^2 \bar{a}_{x\mu} + (\frac{1}{2}a_{rx\mu}a_{rx\mu} + \bar{a}_{x\mu}) \left[v(\sigma_x + \sigma_{x+\hat{\mu}}) + \sigma_x\sigma_{x+\hat{\mu}} + \pi_{rx}\pi_{rx+\hat{\mu}} \right] \right\}$$

$$- 2\kappa \sum_{(x\mu)} \{\sigma_x a_{rx\mu}\pi_{rx+\hat{\mu}} + \pi_{rx}a_{rx\mu}\sigma_{x+\hat{\mu}} + \epsilon_{rst}\pi_{rx}a_{sx\mu}\pi_{tx+\hat{\mu}}\} + \kappa v^2 \sum_{(x\mu)} a_{rx\mu}a_{rx\mu} \quad (20)$$

The corresponding vertices in momentum space can be easily obtained from the above forms of $S_{m,\cdots,i}$. Besides the variables in Eq. (6), the Fourier transformation for the scalar field σ is defined as

$$\tilde{\sigma}_k \equiv \sum_x e^{-i(k,x)}\sigma_x \qquad \sigma_x = \frac{1}{N}\sum_k e^{i(k,x)}\tilde{\sigma}_k \quad (21)$$

and similarly for π_{rx}, c_{rx}, $\bar{c}_{rx}$.

2.2 Propagators

The propagator matrix is the inverse of the quadratic part of the action. For the scalar fields σ and π we have the well known lattice propagators

$$\Delta^{\sigma}_{xy} = \frac{1}{N}\sum_k e^{-i(k,x-y)} \left[2\kappa(\mu^2_{\sigma} + \hat{k}^2)\right]^{-1}$$

$$\Delta^{\pi}_{xy} = \frac{1}{N}\sum_k e^{-i(k,x-y)} \left[2\kappa(\mu^2_{\pi} + \alpha\mu^2_W + \hat{k}^2)\right]^{-1} \quad (22)$$

where

$$\hat{k}^2 \equiv 4\sum_{\mu>0} \sin^2\frac{k_\mu}{2} \qquad \hat{k}_\mu \equiv 2\sin\frac{k_\mu}{2} \quad (23)$$

and the squared masses $\mu^2_{\sigma,\pi}$ are given by

$$\mu^2_{\sigma} = \frac{1 - 2\lambda + 6\lambda v^2}{\kappa} - 8 \qquad \mu^2_{\pi} = \frac{1 - 2\lambda + 2\lambda v^2}{\kappa} - 8 \quad (24)$$

The W-mass squared μ^2_W will be given below.

The quadratic part of the action in the gauge field $a_{rx\mu}$ is

$$S^{(q)}_g \equiv S^{(2)}_g + \kappa v^2 \sum_{(x\mu)} a_{rx\mu}a_{rx\mu} + \frac{2}{\alpha g^2}\sum_x\sum_{\mu,\nu}(a_{rx\mu} - a_{rx-\hat{\mu}\mu})(a_{rx\nu} - a_{rx-\hat{\nu}\nu}) \quad (25)$$

Here the $n = 1$ piece coming from the measure term S_m is not taken into account. It is left as a two-point interaction vertex, in the same way as in Ref. [6]. Because of the factors g^{-2}, in the $g^2 \to 0$ limit the appropriate gauge variable is

$$A_{rx\mu} \equiv \frac{2}{g} a_{rx\mu} \tag{26}$$

In terms of this let us write the quadratic gauge part as

$$S_g^{(q)} \equiv \frac{1}{2} \sum_{(x\mu)(y\nu)} A_{rx\mu} K^{(\alpha)}_{rx\mu,sy\nu} A_{sy\nu} \tag{27}$$

Going to the momentum space by

$$K^{(\alpha)}_{rx\mu,sy\nu} \equiv \frac{1}{N} \sum_k e^{-i(k,x-y)-\frac{i}{2}(k_\mu - k_\nu)} \tilde{K}^{(\alpha)}(k)_{r\mu,s\nu} \tag{28}$$

we have

$$\tilde{K}^{(\alpha)}(k)_{r\mu,s\nu} = \delta_{rs} \left\{ \delta_{\mu\nu}[\mu_W^2 + \hat{k}^2] - (1 - \frac{1}{\alpha})\hat{k}_\mu \hat{k}_\nu \right\} \tag{29}$$

According to Eq. (25) the W-mass squared is

$$\mu_W^2 = \frac{\kappa}{2} g^2 v^2 \tag{30}$$

The inverse of Eq. (29) gives for the gauge propagator

$$\Delta^{(\alpha)}_{rx\mu,sy\nu} = \frac{\delta_{rs}}{N} \sum_k \frac{e^{-i(k,x-y)}}{\mu_W^2 + \hat{k}^2} \left[\delta_{\mu\nu} - (1-\alpha) \frac{\hat{k}_\mu \hat{k}_\nu}{\alpha \mu_W^2 + \hat{k}^2} \right] \tag{31}$$

Finally, the propagator of the Faddeev-Popov ghost field is

$$\Delta^{FP}_{rx,sy} = \frac{\delta_{rs}}{N} \sum_k e^{-i(k,x-y)} \left[\alpha \mu_W^2 + \hat{k}^2 \right]^{-1} \tag{32}$$

The squared masses μ_σ^2, μ_π^2 and μ_W^2 appearing in the propagators can be arbitrarily shifted according to $\mu^2 \to \mu^2 + \delta\mu^2$. This corresponds to the freedom of splitting up the action differently into a free part and an interaction part. In order to compensate for the shift $\delta\mu^2$ of the propagator mass, in perturbation theory one has to take into account also two-point vertices ("insertions") proportional to $-\delta\mu^2$. The convergence properties of the bare perturbation theory do, of course, in general depend on the choice of $\delta\mu^2$. The freedom of choosing the propagator mass was, in fact, already exploited in Eq. (30), where a negative piece coming from the measure term S_m was not included. Correspondingly, a non-zero gluon propagator mass can also be introduced in the confining phase where $v = 0$ or in pure gauge theory (at $\kappa = 0$). It is possible that low order bare perturbation theory gives always a better approximation with some effective non-zero propagator mass.

2.3 WGCE for the generating function

The aim of the weak gauge coupling expansion is to express the expectation values at an arbitrary point (λ, β, κ) of the bare parameter space in terms of a series of expectation values at some point $(\lambda, \beta = \infty, \kappa_0)$ with vanishing gauge coupling. This is achieved by performing the integration over the gauge field variables in perturbation theory, thereby explicitly displaying the dependence on the gauge field propagators and vertices. As an example, one can consider the generating function of connected Green's functions defined as

$$W[h,i]^{\alpha}_{\lambda\beta\kappa} = \log \left\{ \frac{I[h,i]^{\alpha}_{\lambda\beta\kappa}}{I[0,0]^{\alpha}_{\lambda\beta\kappa}} \right\} \tag{33}$$

where

$$I[h,i]^{\alpha}_{\lambda\beta\kappa} \equiv \int \left[d\sigma_{0x} d^3\pi_{rx} d^3U(x,\mu) d^3\bar{c}_{rx} d^3c_{rx} \right]$$
$$\cdot \exp \left\{ -S^{(\alpha)}_{\lambda,\beta,\kappa} + \sum_x (h_{0x}\sigma_{0x} + h_{rx}\pi_{rx}) + \sum_{(x\mu)} i_{rx\mu} a_{rx\mu} \right\} \tag{34}$$

Of course, more complicated generating functions can also be considered, for instance containing also external currents coupled to gauge invariant composite fields. Such a generating function was considered in [4] in the framework of a gauge invariant formalism. The general procedure is, however, always the same, therefore it is enough to consider here the above simple case.

Since the procedure to derive WGCE for $W[h,i]$ is the same as the one applied in Ref. [4], it is enough to indicate the main steps of the derivation and to give the result. The relation between the action at the point (λ, β, κ) and $(\lambda, \beta = \infty, \kappa)$ is

$$S^{(\alpha)}_{\lambda,\beta,\kappa} = S_{\lambda,\beta=\infty,\kappa_0} + S^{(\alpha)}_g - (\kappa - \kappa_0) \sum_{(x\mu)} s_{x\mu} - \kappa \sum_{(x\mu)} [-s_{x\mu} a_{x\mu} + i u_{rx\mu} a_{rx\mu}] \tag{35}$$

Here $S_{\lambda,\beta=\infty,\kappa_0}$ is the ϕ^4 action at $\beta = \infty$ and the complete gauge action $S^{(\alpha)}_g$ is defined from the pieces in Eq. (16) as

$$S^{(\alpha)}_g = S_g + S_m + S_{gf} + S_{fp} \tag{36}$$

The composite fields $s_{x\mu}$ and $u_{rx\mu}$ are the same as in Ref. [4]. In terms of the real field variables we have

$$s_{x\mu} = 2(\sigma_{0x}\sigma_{0x+\hat{\mu}} + \pi_{rx}\pi_{rx+\hat{\mu}})$$
$$u_{rx\mu} = 2i(\pi_{rx}\sigma_{0x+\hat{\mu}} - \sigma_{0x}\pi_{rx+\hat{\mu}} + \epsilon_{rst}\pi_{sx}\pi_{tx+\hat{\mu}}) \tag{37}$$

Note that, for simplicity, the vacuum expectation value of the σ-field is not displayed here. In most cases we shall treat the symmetric case with $v = 0$ and only discuss the changes (mainly of technical nature) which occur in the derivation if the broken phase is explicitly considered.

Substituting the decomposition in Eq. (35) into Eq. (34), the gauge integral one has to perform turns out to be the following:

$$I[j]^{\alpha}_g \equiv \int \left[d^3U(x,\mu) d^3\bar{c}_{rx} d^3c_{rx} \right] \exp \left\{ -S^{(\alpha)}_g + \sum_{(x\mu)} (j_{x\mu} a_{x\mu} + j_{rx\mu} a_{rx\mu}) \right\} \tag{38}$$

The composite fields ("currents") $j_{x\mu}$ and $j_{rx\mu}$ are defined as

$$j_{x\mu} \equiv -\kappa s_{x\mu} \qquad j_{rx\mu} \equiv i_{rx\mu} + i\kappa u_{rx\mu} \tag{39}$$

Using the integration variables $A_{rx\mu}$ in Eq. (26) and taking into account Eq. (3), this can be written like

$$I[j]_g^\alpha = const. \int \left[d^3 A_{rx\mu} d^3\bar{c}_{rx} d^3 c_{rx}\right] \cdot$$
$$\cdot \exp\left\{-S_g^{(\alpha)} + \sum_{(x\mu)}\left[\frac{g}{2} j_{rx\mu} A_{rx\mu} + \frac{g^2}{8}(1 + j_{x\mu}) A_{rx\mu} A_{rx\mu} + \right.\right.$$
$$\left.\left. + \sum_{n=2}^{\infty} (A_{rx\mu} A_{rx\mu})^n \left(\frac{g}{2}\right)^{2n} \left(\frac{1}{2n} + j_{x\mu} \frac{(2n-3)!!}{n! 2^n}\right)\right]\right\} \tag{40}$$

The last sum in the exponent is included here only for completeness. It contains higher dimensional vertices which are negligible in the large cut-off limit.

The result of the gauge integration can be exponentiated by expressing it in terms of the connected expectation values of the gauge fields:

$$I[j]_g^\alpha = const. \; \exp\left\{\sum_{mn} \sum_{[rx\mu]_m [y\nu]_n} \frac{g^{m+2n}}{2^{m+3n} m! n!} (j.)_{rx\mu}^m (1 + j.)_{y\nu}^n \left\langle (A.)_{rx\mu}^m (A_s. A_s.)_{y\nu}^n \right\rangle_{\alpha g}^c + \cdots\right\} \tag{41}$$

Here the higher dimensional terms are already neglected and a shorthand notation for index repetitions, similar to the one used in [4], is introduced:

$$(f.)_\nu^n \equiv f_{\nu_1} f_{\nu_2} \cdots f_{\nu_n}$$
$$\sum_{[\nu]_n} (f.)_\nu^n \equiv \sum_{\nu_1 \cdots \nu_n} f_{\nu_1} f_{\nu_2} \cdots f_{\nu_n} \tag{42}$$

In the definition of the connected gauge field expectation values $< \ldots >_{\alpha g}^c$ the products in parentheses like $(A_{sy\nu} A_{sy\nu})$ have to be considered as a single entity. Replacing in Eq. (35) $S_g^{(\alpha)}$ by the exponent in Eq. (41) one obtains the "effective ϕ^4 action" as a power series in g^2. In order to obtain the terms of this series, the connected gauge field expectation values have to be inserted as obtained from pure gauge field perturbation theory with the action $S_g^{(\alpha)}$.

It is now straightforward to obtain the master formula for $W[h,i]$ in terms of the connected expectation values in the ϕ^4 model containing, in addition to the original fields σ_{0x}, π_{rx}, also the composite fields $s_{x\mu}$ and $u_{rx\mu}$ defined in Eq. (37). Let us first define the notations

$$C[i]_{[rx\mu]_m [y\nu]_n}^{mn} \equiv \frac{1}{2^{m+3n} m! n!} (i. + i\kappa u.)_{rx\mu}^m (1 - \kappa s.)_{y\nu}^n$$
$$A_{[rx\mu]_m [y\nu]_n}^{(\alpha)mn} \equiv \left\langle (A.)_{rx\mu}^m (A_s. A_s.)_{y\nu}^n \right\rangle_{\alpha g}^c \tag{43}$$

The result for the generating function of connected Green's functions W is in this notation (applying the trick (42) twice):

$$W[h,i]_{\lambda\beta\kappa}^\alpha = \sum_{LMN} \sum_{[X]_L [RY]_M [Z\lambda]_N} \sum_K \sum_{[m]_K [n]_K} \sum_{[[rx\mu]_m [y\nu]_n]_K} \frac{(h_0.)_X^L (h.)_{RY}^M (\kappa - \kappa_0)^N}{K! L! M! N!} \cdot$$

$$\cdot \left(g^{m+2n} A^{(\alpha)mn}_{..}\right)^K_{[rx\mu]_m[y\nu]_n} \left\langle (\sigma_{0.})^L_X (\pi_.)^M_{RY} (s_.)^N_{Z\lambda} \left(C[i]^{mn}_{..}\right)^K_{[rx\mu]_m[y\nu]_n} \right\rangle^c_{\lambda\kappa_0} \tag{44}$$

Here in the connected ϕ^4 expectation value $< \cdots >^c_{\lambda\kappa_0}$ the contents of the parentheses have to be considered as single entities for connectedness. By taking derivatives of $W[h,i]$ with respect to h and i one can obtain the WGCE for individual Green's functions of the fields σ_{0x}, π_{rx} and $a_{rx\mu}$.

2.4 Graphical rules

The content of the formula (44) can also be summarized by formulating graphical rules for the calculation of individual Green's functions. This has to be so, because in the limit $\lambda \to 0$ ordinary Feynman perturbation theory has to come out if the ϕ^4 Green's functions are expanded in powers of λ. In short, in the WGCE formula for the connected Green's functions the gauge field dependent parts of the Feynman graphs (like gauge and ghost propagators and vertices) are displayed explicitly, whereas the remaining parts are lumped together into the ϕ^4 expectation values $< \cdots >^c_{\lambda\kappa_0}$. These latter can be denoted graphically by blobs connecting external scalar lines to the composite fields $s_{x\mu}$ and $u_{rx\mu}$.

Let us formulate the graphical rules in the simpler case with $\kappa = \kappa_0$ (i. e. the expansion done only in g^2, for constant λ and κ):

- draw all Feynman graphs up to the given order of g^2 for the connected Green's function in question;
- identify connected subgraphs consisting only of scalar lines and vertices and replace them by "scalar blobs" (Feynman graphs with identical gauge field parts and identical scalar blob structure belong to a single WGCE graph);
- the scalar blobs are connected to each other and to the gauge parts by gauge field lines ending on the blobs either as a single gauge line or as a pair of gauge lines in the same point; the former belong to a factor $u_{rx\mu}$ in the blob, the latter to a factor $s_{x\mu}$ (see Fig. 1);
- write down the factors to the gauge field and ghost parts in the same way as in ordinary perturbation theory; the scalar blobs represent factors like $< \sigma_0 \ldots \pi \ldots u \ldots s \ldots >_{\lambda\kappa_0}$;
- in order to have all the constant factors (including the combinatorial ones) correctly, compare to the corresponding term in the master formula Eq. (44).

Sometimes the more general expansion with $\kappa_0 \neq \kappa$ is also useful, in particular if $(\kappa - \kappa_0)$ is of the order g^2. As it can be seen from Eq. (44), the additional graphs are proportional to $(\kappa - \kappa_0)^N$ and contain N external composite fields $s_{z_1\lambda_1} \ldots s_{z_N\lambda_N}$ entering some scalar blobs.

With the help of WGCE the behaviour of the Higgs model at (λ, g^2, κ) is given, for weak gauge coupling (g^2 small), in terms of the properties of the ϕ^4 model in the point (λ, κ_0). The convergence of the g^2-expansion will, in general, depend on the choice of κ_0. A good convergence of WGCE cannot be expected if the mass scales at (λ, g^2, κ) and (λ, κ_0) are very different, or if one of the points is in the symmetric phase and the other in the broken phase. Of course, κ_0 has to be chosen in such a way, that the necessary information about the ϕ^4 expectation values be available. For the study of the critical behaviour one has to choose the expansion point (λ, κ_0) near to the critical line of the ϕ^4 model. In the present paper we shall use the knowledge about the critical behaviour of the O(4)-symmetric ϕ^4 model in order to obtain a qualitative description of some aspects of the critical behaviour in the Higgs model.

3 Curves of constant physics

Both in numerical Monte Carlo studies and in analytical calculations an important step is to find the "curves of constant physics" (or "renormalization group trajectories"). By definition, along these curves the physics described by the lattice-regularized quantum field theory is constant, only the value of the cut-off (or lattice spacing) is changing. In this Section we discuss the differential equations determining the curves of constant physics (CCP's) in the standard SU(2) Higgs model for arbitrary scalar self-coupling and small gauge coupling. We shall assume that the behaviour of the O(4)-symmetric ϕ^4 model near the critical line at vanishing gauge coupling is known, in particular that the CCP's of ϕ^4 are known in both phases.

3.1 General procedure and a simple example

Before going to the Higgs model, let us first formulate the differential equations for the curves of constant physics in the general case. Let us consider a lattice quantum field theory with n bare couplings $g_1, g_2, \ldots, g_n$. In order to define the CCP's one has to keep $(n-1)$ independent physical quantities $F_2, F_3, \ldots, F_n$ constant (we are assuming here that the number of relevant couplings is n):

$$F_j(g_1, \ldots, g_n) = F_{j0} = const. \qquad (j = 2, \ldots, n) \tag{45}$$

The CCP's are characterized by the constant values F_{j0}. The points of a singled out CCP can be parametrized, for instance, by the first bare coupling g_1: $g_j = g_j(g_1)$ $(j = 2, \ldots, n)$. In this case we have

$$\frac{dg_j(g_1)}{dg_1} = \frac{\det^{[1,j]}_{n-1}\left(\frac{\partial F}{\partial g}\right)}{\det^{[1,1]}_{n-1}\left(\frac{\partial F}{\partial g}\right)} \tag{46}$$

Here $\det^{[i,k]}_{n-1}(\frac{\partial F}{\partial g})$ denotes the $(n-1) \times (n-1)$ subdeterminant of the $n \times n$ derivative matrix $\frac{\partial F}{\partial g}$ belonging to the matrix element $\frac{\partial F_i}{\partial g_k}$.

Another possibility is to parametrize the points of a CCP by the value of some reference physical quantity F_1. (In practical cases F_1 is usually some physical mass in lattice units.) In this case the differential equations for $g_i(F_1)$ $(i = 1, \ldots, n)$ are:

$$\frac{dg_i(F_1)}{dF_1} = \frac{\det^{[1,i]}_{n-1}\left(\frac{\partial F}{\partial g}\right)}{\det_n\left(\frac{\partial F}{\partial g}\right)} \tag{47}$$

where $\det_n(\cdots)$ is the $n \times n$ determinant of the derivative matrix.

Sometimes it is also useful to consider curves in subspaces of the bare parameter space which belong to constant values of an appropriately smaller number of physical quantities. These "curves of partially constant physics" (CPCP's) are defined by fixing $(n-k)$ physical quantities $F_2, \ldots, F_{n-k+1}$ and $(k-1)$ bare parameters $g_{n-k+2}, \ldots, g_n$. The differential equations for CPCP's have the same form as Eqs. (46-47). For simplicity, let us consider here only the case with $n = 3$ bare parameters (as we have in the standard Higgs model) and look at the plane with constant bare coupling g_3. Keeping the value of some physical quantity $F_2(g_1, g_2, g_3) = F_{20}$ fixed and parametrizing the points of the curve by the reference quantity F_1, the differential equation for the function $g_2(F_1)$ is:

$$\frac{dg_2(F_1)}{dF_1} = \left(\frac{-\frac{\partial F_2}{\partial g_1}}{\frac{\partial F_1}{\partial g_1}\frac{\partial F_2}{\partial g_2} - \frac{\partial F_1}{\partial g_2}\frac{\partial F_2}{\partial g_1}} \right)_{g_2 = g_2(F_1, g_2, g_3)} \tag{48}$$

As an example in the standard Higgs model, one can take $g_1 = \kappa$, $g_2 = \lambda$, $g_3 = g^2$ and $F_1 = \mu_W$ (the W-mass), $F_2 = R_{HW} \equiv \mu_H/\mu_W$ (the ratio of Higgs- to W-mass). In this case Eq. (48) gives the curves with constant Higgs- to W-mass ratio in the $g^2 = const.$ planes.

In order to illustrate how these equations work, let us consider, as an exercise, the well known case of the $O(N)$-symmetric ϕ^4 model in ordinary perturbation theory (i. e. for small self-coupling λ). In this case we have $n = 2$, and the CCP's can be defined by keeping the renormalized ϕ^4 coupling λ_r fixed: $F_2 = \lambda_r$. The reference quantity can be the renormalized mass squared: $F_1 = \mu_r^2$. (We consider the symmetric phase, where λ_r and μ_r^2 can be defined in the usual way at vanishing four-momenta: see e. g. Ref. [8].) Up to 1-loop order we have

$$\mu_r^2 = \mu_0^2 + 4(N+2)\lambda_0 \mathcal{I}_1(\mu_0^2) + o(\lambda_0^2)$$

$$\lambda_r = \lambda_0 - 4(N+8)\lambda_0^2 \mathcal{I}_2(\mu_0^2) + o(\lambda_0^3) \tag{49}$$

Here we used the bare parameters (λ_0, μ_0^2) which are connected to (λ, κ) in Eq. (15) (with $v = 0$) by

$$\mu_0^2 = \frac{1-2\lambda}{\kappa} - 8 \qquad \lambda_0 = \frac{\lambda}{4\kappa^2} \tag{50}$$

On a finite lattice the function $\mathcal{I}_k$ is a finite sum, on an infinite lattice an integral:

$$\mathcal{I}_n(\mu^2) \equiv \frac{1}{N}\sum_k (\mu^2 + \hat{k}^2)^{-n} \Longrightarrow \frac{1}{(2\pi)^4}\int_{-\pi}^{\pi} d^4k(\mu^2 + \hat{k}^2)^{-n} \tag{51}$$

The differential equation corresponding to Eq. (47) for the function $\lambda_0(\mu_r^2)$ is, in leading order:

$$\frac{d\lambda_0}{d\mu_r^2} = -\left(\frac{\partial \lambda_r}{\partial \mu_0^2}\right)_{\mu_0^2 \to \mu_r^2} + o(\lambda_0^3) = -8(N+8)\lambda_0^2 \mathcal{I}_3(\mu_r^2) + o(\lambda_0^3) \tag{52}$$

Using the logarithmic scale variable

$$\tau \equiv \log \mu_r^{-1} \tag{53}$$

we have for large cut-off (small lattice spacing, i.e. $\mu_r^2 \simeq 0$):

$$\frac{d\lambda_0(\tau)}{d\tau} = \frac{(N+8)}{2\pi^2}\lambda_0^2 + o(\lambda_0^3, \mu_r^2) \tag{54}$$

On the right hand side the well known universal 1-loop Callan-Symanzik β-function appears. Since for $\tau \to \infty$ $\lambda_0(\tau)$ is growing, at some point the higher order terms on the right hand side become important, therefore Eq. (54) is not suitable for the infinite cut-off limit.

The other equation in (47) is:

$$\frac{d\mu_0^2}{d\mu_r^2} = \left(\frac{\partial \mu_r^2}{\partial \mu_0^2}\right)^{-1} + o(\lambda_0^2) = 1 + 4(N+2)\lambda_0 \mathcal{I}_2(\mu_0^2) + o(\lambda_0^2) \tag{55}$$

Since $\mathcal{I}_2(\mu^2)$ is logarithmically divergent for $\mu^2 \to 0$, this equation is not as useful as Eq. (52). If besides $\lambda_0(\tau)$ one is also interested in $\mu_0^2(\tau)$, the better way to obtain it is to solve the first of Eq. (49) for μ_0^2 in terms of μ_r^2 and the solution $\lambda_0(\mu_r^2)$ of Eq. (52). (In fact, for this one has first to shift the propagator mass μ_0^2 to $(\mu_0^2 - \mu_{cr}^2)$. Here $\mu_{cr}^2(\lambda_0)$ is the critical line obtained from the condition $\mu_r^2 = 0$.) In WGCE we shall only consider in this paper the equations analogous to Eq. (54) and leave the behaviour of μ_0^2 (or κ) implicit. Equations like Eq. (46) will not be considered either. This will, however, be enough to determine the qualitative behaviour for large cut-off's, if the lowest order terms of WGCE give a reliable approximation.

3.2 Renormalization at non-zero constant σ-field

In order to determine the CCP's in perturbation theory or in WGCE one has to find suitable physical quantities to be kept constant. For instance, the renormalized ϕ^4 coupling at zero four-momentum is, in principle, a possibility also in the Higgs model, but one has to be careful in the definition to avoid infrared singularities. It will be necessary to define renormalized quantities in both the symmetric and spontaneously broken phase of the O(4) ϕ^4 model and in both the confining and Higgs-phase of the Higgs model. Infrared singularities at zero four-momenta certainly occur in the spontaneously broken phase of the ϕ^4 model because of the Goldstone bosons. In the confining phase infrared singularities are produced by the zero mass gluons. Perturbative infrared singularities can appear also in the Higgs phase,for instance in the Landau gauge ($\alpha = 0$), where the propagator mass in Δ_{xy}^{π} in Eq. (22) is zero, because μ_π vanishes at the tree level. In principle, one could use in the different phases different renormalization schemes, but it is simpler to choose a unique scheme which is appropriate everywhere. Such a universal possibility is to define the renormalized quantities at zero four momenta but at some non-zero constant value μ_σ of the σ-field (see, for instance, [9]). A non-zero constant σ-field acts as an infrared regulator in both the scalar and gauge field propagators. The scale introduced by μ_σ plays in the renormalization scheme a similar role as the momentum scale would play if the renormalized quantities would be defined at non-zero momentum.

In the Higgs model we need two dimensionless renormalized quantities. These can be chosen as the renormalized ϕ^4 coupling (λ_R) and the renormalized gauge coupling squared (g_R^2). For the dimensionful (reference) quantity one can take either the renormalized W-mass (μ_{WR}) or Higgs-mass (μ_{HR}). In the μ_σ-scheme all of these quantities are defined by appropriate derivatives of the effective action $\Gamma[\sigma,\pi,a]$ at $\sigma_x = \mu_\sigma$, $\pi_{rx} = a_{rx\mu} = 0$. In order to see in more detail how things work out, let us briefly consider the well known case of $\lambda_R(\mu_\sigma)$ in ordinary perturbation theory. The 1-loop effective action is given by

$$\Gamma[\sigma,\pi,a] = S[\sigma,\pi,a] + \frac{1}{2} Tr \log(D[\sigma,\pi,a]\Delta) - Tr \log(M[\sigma,\pi,a]\Delta^{FP}) \tag{56}$$

Here D is the second derivative matrix of the action with respect to the bosonic (scalar and gauge field) variables and Δ is the bosonic propagator matrix. The last term is the contribution of the Faddeev-Popov ghost loop. The 1-loop effective σ-field potential $V_{eff}(\sigma)$ in the Landau gauge ($\alpha = 0$) is easy to obtain with the Feynman rules in Section 2:

$$V_{eff}(\sigma) \equiv \frac{1}{N}\Gamma[\sigma_x = \sigma, \pi = 0, a = 0] = (1 - 2\lambda - 8\kappa)\sigma(\sigma + 2v) + \lambda[(\sigma + v)^4 - v^4]$$

$$+\frac{1}{N}\sum_{k}\left\{\frac{1}{2}\log\left[1+\frac{6\lambda\sigma(\sigma+2v)}{\kappa(\mu_\sigma^2+\hat{k}^2)}\right]+\frac{3}{2}\log\left[1+\frac{2\lambda\sigma(\sigma+2v)}{\kappa(\mu_\pi^2+\hat{k}^2)}\right]+\frac{9}{2}\log\left[1+\frac{\kappa g^2\sigma(\sigma+2v)}{2(\mu_W^2+\hat{k}^2)}\right]\right\} \quad (57)$$

The vacuum expectation value v is determined by the requirement that $dV_{eff}(\sigma)/d\sigma|_{\sigma=0}=0$. Using, instead of v, the more natural variable

$$w_0 \equiv v\sqrt{\frac{2\lambda}{\kappa}} \quad (58)$$

with Eqs. (50-51) the above condition gives

$$0 = w_0\left\{w_0^2+\mu_0^2+12\lambda_0\mathcal{I}_1(\mu_0^2+3w_0^2)+12\lambda_0\mathcal{I}_1(\mu_0^2+w_0^2)+\frac{9}{4}g^2\mathcal{I}_1(w_0^2\frac{g^2}{16\lambda_0})\right\} \quad (59)$$

The solution is either $w_0 = 0$ or otherwise $w_0^2 \neq 0$ is such that the content of the curly brackets vanishes. The first solution is relevant to the confining phase, the second one to the Higgs-phase. The phase transition occurs on the surface, where the two solutions belong to equally deep minima of the effective potential.

The explicit form of $\lambda_R(\mu_\sigma)$ is the same in both phases if one takes the constant value of σ_x such that $\sigma_{0x}=\sigma_x - v = \mu_\sigma$. Let us consider here explicitly only the confining phase with $v=0$, and let us introduce a suitable normalization factor in front of the constant σ-field by defining μ_σ to be the value of $\sqrt{2\kappa}\sigma_x$. The definition of the renormalized ϕ-mass squared and renormalized ϕ^4-coupling is in this case:

$$\frac{\mu_R^2(\mu_\sigma)}{Z_R(\mu_\sigma)} = \frac{1}{2\kappa}\frac{d^2V_{eff}}{d\sigma^2}\bigg|_{\sigma\sqrt{2\kappa}=\mu_\sigma}$$

$$\frac{\lambda_R(\mu_\sigma)}{Z_R(\mu_\sigma)^2} = \frac{1}{4\kappa^2}\frac{d^4V_{eff}}{d\sigma^4}\bigg|_{\sigma\sqrt{2\kappa}=\mu_\sigma} \quad (60)$$

It is straightforward to obtain the one-loop expressions from Eq. (57), but we shall not consider them explicitly here. The wave function renormalization factor $Z_R(\mu_\sigma)$ can be obtained from the small momentum behaviour of the function

$$Z(\sigma,k) \equiv \frac{1}{N}\sum_{xy} e^{i(k,x-y)}\frac{\partial^2\Gamma}{\partial\sigma_x\partial\sigma_y}\bigg|_{\sigma_x=\sigma,\pi=a=0} \quad (61)$$

The definition is

$$Z_R(\mu_\sigma) \equiv \left\{\frac{1}{16\kappa}\sum_\mu \frac{\partial^2}{\partial k_\mu \partial k_\mu}\bigg|_{k=0} Z(\sigma,k)\right\}_{\sigma\sqrt{2\kappa}=\mu_\sigma} \quad (62)$$

The derivation of the differential equations for the CCP's in the μ_σ renormalization scheme is, in principle, quite similar to the case considered before. One has only to take into account the additional variable μ_σ introduced by the renormalization. In the specific case of the standard Higgs model the choice of the bare and physical parameters can be as follows:

$$g_1=\mu_\sigma^2, \quad g_2=\mu_0^2, \quad g_3=\lambda_0, \quad g_4=g^2;$$

$$F_1=\mu_R^2, \quad F_2=\frac{\mu_R^2}{\mu_\sigma^2}, \quad F_3=\lambda_R, \quad F_4=g_R^2. \quad (63)$$

This means that on the CCP's, besides λ_R and g_R^2, one has to keep constant also the ratio of the renormalized mass μ_R to μ_σ. In this case we have, for instance,

$$\mu_R \frac{d\lambda_0(\mu_R)}{d\mu_R} = -\left(\mu_0 \frac{\partial}{\partial\mu_0} + \mu_\sigma \frac{\partial}{\partial\mu_\sigma}\right) \lambda_R(\mu_\sigma) + o(\lambda_0^3, \lambda_0^2 g^2, \lambda_0 g^4, g^6) \qquad (64)$$

It is straightforward to work out the one-loop expression for $\lambda_R(\mu_\sigma)$ from the above definitions and Eqs. (56-57). We do not include here the somewhat lengthy result, but the industrious reader can verify that in the large cut-off limit, with $\tau = \log \mu_R^{-1}$, we have

$$\frac{d\lambda_0(\tau)}{d\tau} = \frac{6}{\pi^2}\lambda_0^2 + \frac{9}{512\pi^2}g^4 - \frac{9}{16\pi^2}\lambda_0 g^2 + \cdots$$

$$\frac{dg^2(\tau)}{d\tau} = -\frac{43}{48\pi^2}g^4 + \cdots \qquad (65)$$

The second line is the analogous result for the bare gauge coupling. The right hand sides are again the one-loop universal Callan-Symanzik β-functions.

3.3 The curves of constant physics in WGCE

In order to obtain differential equations for the CCP's in the weak gauge coupling expansion, one has to proceed very similarly to ordinary perturbation theory. As discussed in Section 2, the expectation values at a point (λ, g^2, κ) with weak bare gauge coupling and arbitrary bare scalar self-coupling are given in WGCE by explicit gauge propagators and vertices and by scalar blobs representing expectation values in the pure ϕ^4 model at $g^2 = 0$ and (λ, κ_0). Let us now assume that the continuum limit in the pure ϕ^4 model is trivial [3]. Then, if the point (λ, κ_0) is close to the critical line, the renormalized ϕ^4 coupling λ_r in the pure ϕ^4 model is small and the renormalized ϕ^4 Green's functions can be well approximated by a low order perturbative expansion in λ_r. In WGCE we need unrenormalized $g^2 = 0$ expectation values, which are schematically connected to the renormalized ones by

$$< \sigma_0 \ldots \pi \ldots u \ldots s \ldots >_{\lambda\kappa_0} = Z_r^{\cdots} Z_u^{\cdots} Z_s^{\cdots} < \sigma_0 \ldots \pi \ldots u \ldots s \ldots >_{\lambda\kappa_0}^{renormalized} \qquad (66)$$

Here Z_r denotes the (identical) wave function renormalization factor for the scalar fields σ_0 and π, Z_u and Z_s are the multiplicative renormalization factors belonging to the composite fields $u_{rx\mu}$ and $s_{x\mu}$, respectively. In a leading order calculation the renormalized truncated one-particle irreducible vertex functions of the σ_0, π, u and s fields can be obtained from the lowest order ϕ^4 Feynman graphs belonging to the vertex function in question. In this way the low order terms in WGCE are easily obtained in terms of g^2, λ_r and the wave function renormalization factors $Z_{r,u,s}$.

Since the $g^2 = 0$ expectation values in WGCE are given in terms of the pure ϕ^4 renormalized coupling λ_r, it is natural to parametrize the points of the bare parameter space, instead of (λ, g^2, κ), by (λ_r, g^2, κ). (In this case the ϕ^4 Z-factors have to be considered also as functions of λ_r and κ: $Z_{r,s,u} = Z_{r,s,u}(\lambda_r, \kappa)$.) In this way the problem of determining the CCP's for small bare gauge coupling is reduced to the problem of finding the CCP's, with $\lambda_r = const.$, in the ϕ^4 model at $g^2 = 0$. The renormalization is also decomposed in two steps: after going to the renormalized variables at $g^2 = 0$, λ_r is considered as one of the bare parameters for WGCE in the Higgs model. The renormalized quantities of the Higgs model are introduced in WGCE in the same way as in ordinary perturbation theory. The CCP's in the Higgs model can be defined by the requirement that the renormalized ϕ^4 coupling λ_R and renormalized gauge coupling squared g_R^2 be constant. (Note that capital R denotes renormalized quantities in the Higgs model, whereas small r is reserved for the renormalized quantities at $g^2 = 0$.) As a parameter along the CCP's, one can take the renormalized ϕ-mass squared μ_R^2 (or $\tau \equiv \log \mu_R^{-1}$). Our aim is to find the differential equations for $\lambda_r(\tau)$ and $g^2(\tau)$.

The relevant leading order WGCE graphs are depicted in Fig. 2. The graphs for g_R^2 without scalar blobs are the same as in pure gauge theory, therefore their contribution need not be recalculated. The differential equations corresponding to Eq. (47) are, in the leading order approximation, in the large cut-off (small μ_R^2) limit:

$$\frac{d\lambda_r(\tau)}{d\tau} = -\frac{9}{16\pi^2}\lambda_r g^2 + Z_s^{-2}\frac{9}{512\pi^2}g^4 + \cdots$$

$$\frac{dg^2(\tau)}{d\tau} = -\frac{43}{48\pi^2}g^4 + \cdots \tag{67}$$

The absence of the λ_r^2 term, as compared to Eq. (65), is due to the fact that, instead of λ_0, λ_r is considered to be the bare parameter. The multiplicative renormalization factor Z_u for the conserved current $u_{rx\mu}$ was put equal in Eq. (67) to $Z_u = 1$. This follows from the Ward-Takahashi identities for the vertex functions containing $u_{rx\mu}$. It is clear from Eq. (67) that under rather mild assumptions about the behaviour of Z_s^{-2} near the $g^2 = 0$ line, both $\lambda_r(\tau)$ and $g^2(\tau)$ tend to zero for $\tau \to 0$. It is, for instance, enough to assume that Z_s^{-2} is bounded. In fact, the assumption of triviality and scaling in the O(4) symmetric ϕ^4 model implies that $Z_s^{-2} \to C_{s\lambda}\lambda_r$ for λ fixed and $\kappa \to \kappa_{cr}(\lambda)$ (with some λ-dependent constant $C_{s\lambda}$). This can be easily deduced from a Callan-Symanzik equation following Refs. [1,8]. Therefore, the term proportional to Z_s^{-2} in Eq. (67) is negligible for $\tau \to \infty$, and the asymptotic solution is:

$$g^2(\tau) = \left[g_0^{-2} + \frac{43}{48\pi^2}(\tau - \tau_0)\right]^{-1}$$

$$\lambda_r(\tau) = \lambda_{r0}\left[1 + \frac{43 g_0^2}{48\pi^2}(\tau - \tau_0)\right]^{-\frac{27}{43}} \tag{68}$$

Here g_0^2 and λ_{r0} are the initial values at $\tau = \tau_0$. A better approximation, which takes into account non-leading terms, can be obtained numerically once $C_{s\lambda}$ is known. The λ-dependence of $C_{s\lambda}$ can be determined, for instance, by the strong self-coupling expansion [10].

According to Eq. (68), for $\tau \to \infty$ both $g^2(\tau)$ and $\lambda_r(\tau)$ tend to zero. Therefore, if the leading order approximation is qualitatively correct, the WGCE is an *asymptotically free* expansion. The question is, whether the higher order graphs can indeed be neglected? Unfortunately, the answer to this question is not easy, because the internal momenta in the WGCE graphs like in Fig. 2a (or in more complicated graphs) are integrated over all momenta. The renormalized ϕ^4 perturbation theory is expected to be reliable for low momenta about the mass scale, but non-reliable for momenta near the cut-off if the cut-off is much larger than the mass. This could perhaps imply that the high momentum behaviour of the correct ϕ^4 expectation values is such, that their contribution to the higher order WGCE graphs is not negligible even at small g^2 and λ_r. In the rest of this paper it will be assumed that this does not actually happen. Nevertheless, one has to keep in mind that the conclusions rely on this assumption. The question of the higher orders is obviously interesting and has to be investigated in the future.

4 Critical surface in WGCE

In the standard Higgs model there is a critical surface separating the Higgs phase and the confining phase. The $g^2 = 0$ edge of this surface is the second order critical line of the O(4) ϕ^4 model. For finite g^2 recent Monte Carlo calculations suggest a first order phase transition [11,12]. Irrespective of the order, the shape of the critical surface can be questioned in perturbation theory. Before going to WGCE, which gives the perturbation of the ϕ^4 critical line by a small gauge coupling, let us recall how the position of the critical surface can be obtained for small couplings in ordinary perturbation theory.

Starting from the Higgs phase with non-zero σ-field vacuum expectation value, the critical surface can be localized in the large cut-off region up to one-loop by requiring that the non-trivial solution ($w_0^2 > 0$) of Eq. (59) should tend to zero. In the Higgs phase we have at tree level $\mu_0^2 = -w_0^2 < 0$, therefore in the vicinity of $w_0^2 = 0$ we have to shift the scalar propagator mass squared according to $\mu_0^2 \to \mu_0^2 - \mu_{cr}^2$. Here $\mu_{cr}^2 \equiv \mu^2(\lambda_0, g^2)_{cr}$ is the critical value of μ_0^2 we are going to determine now. In this case the one-loop equation for w_0^2 is:

$$w_0^2 = -\mu_0^2 - 12\lambda_0\mathcal{I}_1(\mu_0^2 - \mu_{cr}^2 + 3w_0^2) - 12\lambda_0\mathcal{I}_1(\mu_0^2 - \mu_{cr}^2 + w_0^2) - \frac{9}{4}g^2\mathcal{I}_1(w_0^2\frac{g^2}{16\lambda_0}) \qquad (69)$$

The solution for $w_0^2 \to 0$ is:

$$\mu_0^2 \equiv \mu^2(\lambda_0, g^2)_{cr} = -(24\lambda_0 + \frac{9}{4}g^2)\mathcal{I}_1(0) \qquad (70)$$

Going over to the variables (λ, κ) by Eq. (50) and taking the infinite lattice value $\mathcal{I}_1(0) = 0.1549\ldots$, the result for the critical hopping parameter κ_{cr} is [13]:

$$\kappa(\lambda, g^2)_{cr} = \frac{1}{8} + \lambda 0.6796\ldots + g^2 0.005447\ldots + \cdots \qquad (71)$$

An identical result can also be obtained by starting from the confining phase. For instance, the second derivative of the effective potential at $\sigma = 0$ with $v = 0$ is the renormalized σ-mass squared μ_R^2 in the confining phase at zero four-momentum. Requiring $\mu_R^2 = 0$ leads again to Eq. (70).

The coincidence of the perturbative result for the critical surface as found by starting from the two phases may seem at the first sight surprising in view of the known fact that the one-loop effective potential implies a first order phase transition [9,14]. The expectation for a first order phase transition is illustrated by Fig. 3. According to this κ_{cr} should be slightly larger if defined from the confining phase (from below the critical surface) rather than from the Higgs-phase (from above the critical surface). As we have seen, this difference turns out to be zero in perturbation theory. In other words, the difference is an exponentially small non-perturbative effect.

After this preparation let us consider the critical surface in leading order WGCE for small bare gauge coupling and any scalar self-coupling. As usual in WGCE, we assume that the critical line $\kappa_{cr}(\lambda, g^2 = 0)$ at $g^2 = 0$ is known. In this case we need the option of expanding at different hopping parameter: $\kappa_0 \neq \kappa$. The shift $(\kappa - \kappa_0)$ in the master formula (44) is assumed to be of the order g^2. The condition for the critical surface is that the renormalized mass squared μ_R^2 should be zero. This requires that the sum of the two WGCE graphs in Fig.

4 has to vanish at zero external four-momentum. From this condition a simple calculation gives

$$\kappa_{cr}(\lambda, g^2) = \kappa_{cr}(\lambda, 0) + \kappa_{cr}(\lambda, 0)\frac{9g^2}{32}\mathcal{I}_1(0) + \cdots = \kappa_{cr}(\lambda, 0)\{1 + g^2 0.04358\ldots + \cdots\} \quad (72)$$

The question of the higher order contributions remains to be sattled also here (see the discussion at the end of the last Section).

5 Discussion

In the previous Sections the leading order WGCE predictions for the curves of constant physics and for the shape of the phase transition surface were derived. Let us now proceed under the assumption that the leading order approximation is qualitatively correct in the high cut-off region. The behaviour of the curves of constant physics for large cut-off's near the $g^2 = 0$ critical line is determined by the asymptotic behaviour (68) of g^2 and λ_r as a function of the logarithmic scale parameter $\tau = \log \mu_R^{-1}$. (See Fig. 5.) Since both $g^2(\tau)$ and $\lambda_r(\tau)$ tend to zero for $\tau \to \infty$, WGCE is an *asymptotically free* expansion. This does not, however, mean that for $\tau \to \infty$ a non-trivial continuum limit exists. The reason is that on the (τ, λ_r) plane not every point is possible. The triviality of the continuum limit of ϕ^4 implies that the CCP's in ϕ^4 with $\lambda_r = const.$ are ending at $\lambda = \infty$ near the critical point $\kappa_{cr}(\lambda = \infty)$ for some finite cut-off. This means that on the (τ, λ_r) plane there is a limiting curve and the allowed points are below this (at smaller values of τ and λ_r). To obtain the exact shape of the limiting curve is a non-perturbative problem in the four-component $O(4)$-symmetric ϕ^4 model. In the 1-component ϕ^4 model the limiting curve was determined from high order hopping parameter expansion and the Callan-Symanzik equation [1]. This method can also be extended to the O(4) model, but at present the only non-perturbative information about the limiting curve in the four-component model comes from an approximate block-spin transformation scheme [15]. In order to have a rough qualitative estimate, one can take the limiting curve for large τ from the position of the "Landau-pole" in one-loop perturbation theory:

$$\lambda_r(\tau)_{max} \simeq \frac{\pi^2}{6\tau} \quad (73)$$

The intersection of the curve $\lambda_r(\tau)$ in Eq. (68) with $\lambda_r(\tau)_{max}$ determines the maximal cut-off τ_{max} which belongs to the Higgs model CCP given by $(g^2(\tau), \lambda_r(\tau))$. If the maximal cut-off is required to be the Planck mass ($\tau \simeq 20$), this crude estimate for a CCP with gauge coupling $g_R^2 = 0.5$, roughly equal to the physical value in the standard electroweak model, gives λ_R about a factor of 2 larger than a one-loop perturbative calculation [16] would give.

According to Eq. (68) $\lambda_r(\tau)$ goes to zero slower than τ^{-1}, therefore every CCP with a finite λ_R and g_R^2 has a finite maximal cut-off $\tau_{max} < \infty$. The only possibility to reach an infinite cut-off is to put $\lambda_R = g_R^2 = 0$. In other words, the continuum limit in the standard Higgs model at the $g^2 = 0$ critical line is trivial in both the confining- and the Higgs-phase. (As discussed in Section 3, in the μ_σ renormalization scheme at non-zero constant σ-field the derivation of Eq. (67) is the same in both phases.) The triviality of the $g^2 = 0$ continuum limit in the Higgs phase was recently concluded also in Ref. [13] on the basis of perturbation theory near the Gaussian fixed point $(\lambda = 0, g^2 = 0, \kappa = 1/8)$. Perturbation theory is, however, not applicable for large bare self-coupling λ, therefore Ref. [13] did not exclude the possibility of a non-trivial fixed point in the combined gauge-scalar system at $g^2 = 0$ and large λ.

In order to determine the two parameter family of CCP's quantitatively, the dependence on the third bare parameter (κ or μ_0^2) has to be considered, too. (See the discussion in Section 3.) For this, and for the transformation of variables $(\lambda_r, g^2, \kappa) \rightarrow (\lambda, g^2, \kappa)$, a detailed knowledge of the CCP's in the $O(4)$-symmetric ϕ^4 model (in particular, their κ-dependence) is needed. This non-perturbative problem in ϕ^4 can be solved by the methods developped in Ref. [1]. Since the κ-dependence in Eqs. (67-68) is implicit, the problem of the order of the confinement-Higgs phase transition is avoided. The question of the order can be translated into a question about the allowed set of initial values (g_0^2, λ_{r0}). Alternatively, one can ask, what is the allowed set of the physical parameters (λ_R, g_R^2)? If the phase transition is first order, some (λ_R, g_R^2) values belong to metastable situations. (Note in this respect that, because of the scale breaking lattice artifacts, the phase transition surface itself does not coincide exactly with a one-parameter subset of CCP's.) In principle, a more detailed κ-dependent treatment with WGCE can also give non-trivial information concerning the small g^2 behaviour of the phase transition, too.

According to Fig. 5 the behaviour of CCP's near $\beta = \infty$ is qualitatively different from the conjectured picture in Ref. [17]. The WGCE implies that the non-trivial λ-independent continuum limit suggested there is not possible. The only open possibility for a search of a non-trivial continuum limit in the standard Higgs model is to go inside the bare parameter space, to points where also the gauge coupling is non-perturbative. However, even if such a fixed point would exist, it would not necessarily be adequate for the description of the standard electroweak physics. The absence of a λ-independent continuum limit at the $g^2 = 0$ critical line also implies that a really strongly interacting standard Higgs sector is impossible. The reason is that once the cut-off is required to be reasonably high (say, $> 10 m_W$), the upper limit for the renormalized ϕ^4 self-coupling (or for the Higgs boson mass to W-boson mass ratio) becomes relatively low. For a numerical study of the upper limit in the standard Higgs model see [19].

The framework of WGCE is obviously more general than the specific case of the standard $SU(2)$ Higgs model. It would certainly be interesting to consider in the future more general Higgs models, too. In particular, as one can see from Eq. (68), there is an interesting class of models, where the Callan-Symanzik β-function coefficients are such that the power of the squared brackets in $\lambda_r(\tau)$ is, instead of $\frac{-27}{43}$, equal to -1. In this case the leading asymptotic behaviour of $\lambda_r(\tau)$ coincides with the asymptotics of the limiting curve in Eq. (73). The question of a possible non-trivial continuum limit at $\lambda = \infty$ is then decided on the next-to-leading order level. In any case, even if the strict continuum limit would turn out to be trivial, such models are interesting, because they can easily allow for very large cut-off's in a wide range of physical situations. A simple example of a model with a τ^{-1} leading λ_r-behaviour is an $SU(2)$ Higgs model with 1 scalar doublet and 4 vector-like spin-$\frac{1}{2}$ fermion doublets. Namely, in this case the coefficient of the g^4 term in Eq. (67) is equal to $-27/(48\pi^2)$. Because of the vector-like fermions, Yukawa-couplings are forbidden. In cases with Yukawa-couplings and chiral fermions (as in the standard model) the appropriate lattice formulation has to be constructed first, and similar questions can be asked only afterwards.

The Monte Carlo calculations are complementary to the perturbative information obtained from WGCE. The shape of the regions where Monte Carlo calculations are interesting and where WGCE can give a good approximation is schematically shown in Fig. 6, on a $\lambda = const.$

plane. The figure is optimistic in the sense that the MC and WGCE regions touch. In reality there might be some no-man's-land inbetween, where the correlation lengths are too large for a numerical investigation but not large enough to make the couplings small enough for a low order WGCE. In this sense the situation could be similar to the relationship between asymptotically free perturbation theory and the Monte Carlo calculations in QCD.

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Figure captions

Fig. 1. The gauge field lines end on the scalar blobs either as a single line (Fig. 1a) or as a pair of lines in the same point (Fig. 1b).

Fig. 2. The WGCE graphs for 1PI vertex functions which have to be calculated for the leading order equations of CCP's. Wavy lines denote gauge fields, full lines the scalar field.

Fig. 3. The critical structure corresponding to a first order phase transition. On the left the qualitative behaviour of the effective potential is shown. Coming from either phase 1 or from phase 2, metastable states occur in the regions where the arrows are hatched. At the phase transition the two minima of the effective potential are equal. At the limits of the metastability the second derivative at one of the local minima becomes zero.

Fig. 4. The lowest order WGCE graphs for the determination of the critical surface. The second graph has an external composite field $s_{x\mu}$ and is proportional to $(\kappa - \kappa_0)$. (It corresponds to $N = 1$ in the master formula (44).)

Fig. 5. The qualitative picture of CCP's in the standard Higgs model projected on the (g^2, λ) plane. The small g^2 behaviour is the result of WGCE. The extension to larger g^2 is a guess supported by some approximate numerical Monte Carlo calculations at $\lambda = \infty$, $\beta = 2 - 3$ [17,18]. Note that in reality there is a two-parameter family of CCP's, but here only a one-parameter subset is shown for simplicity.

Fig. 6. The schematic lay-out of the regions where interesting Monte Carlo calculations can be done (MC) and where WGCE can be expected to give a good approximation (WGCE). The uninteresting region of dominant lattice artifacts is denoted by LA. The confining-Higgs phase transition is at the dashed line. The whole picture is for $\lambda = const.$.

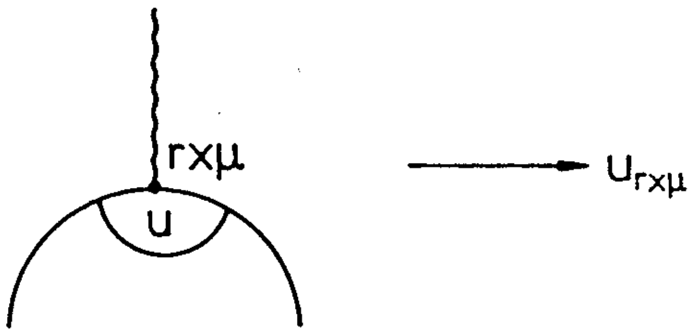

Fig. 1a

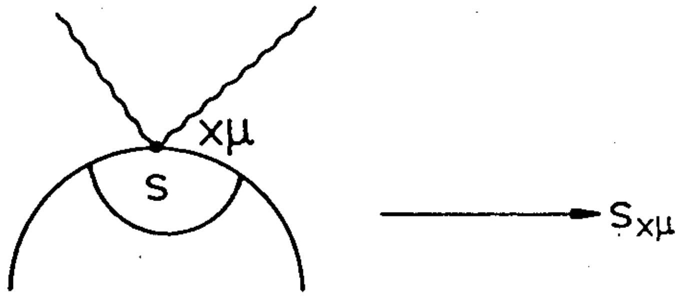

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Fig. 1b

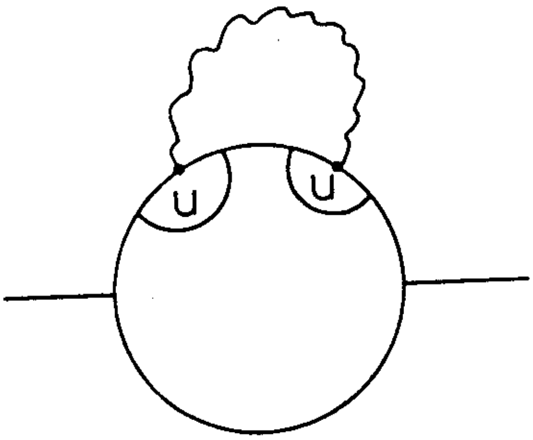
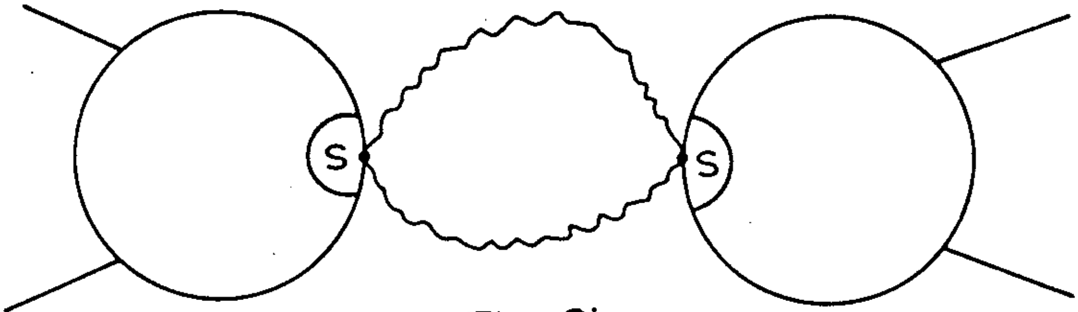


Fig. 2a



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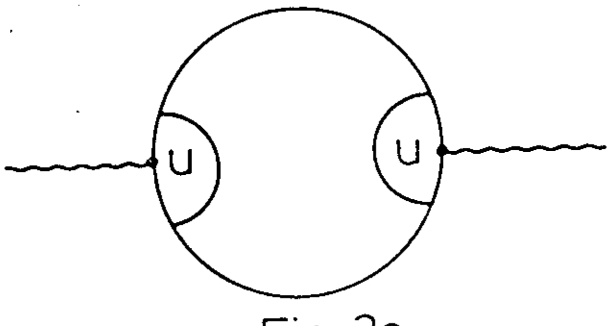

Fig. 2c

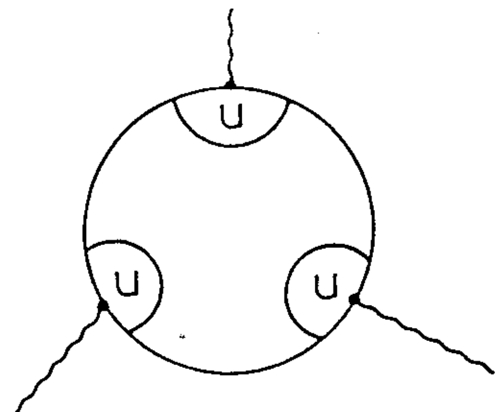

Fig. 2d

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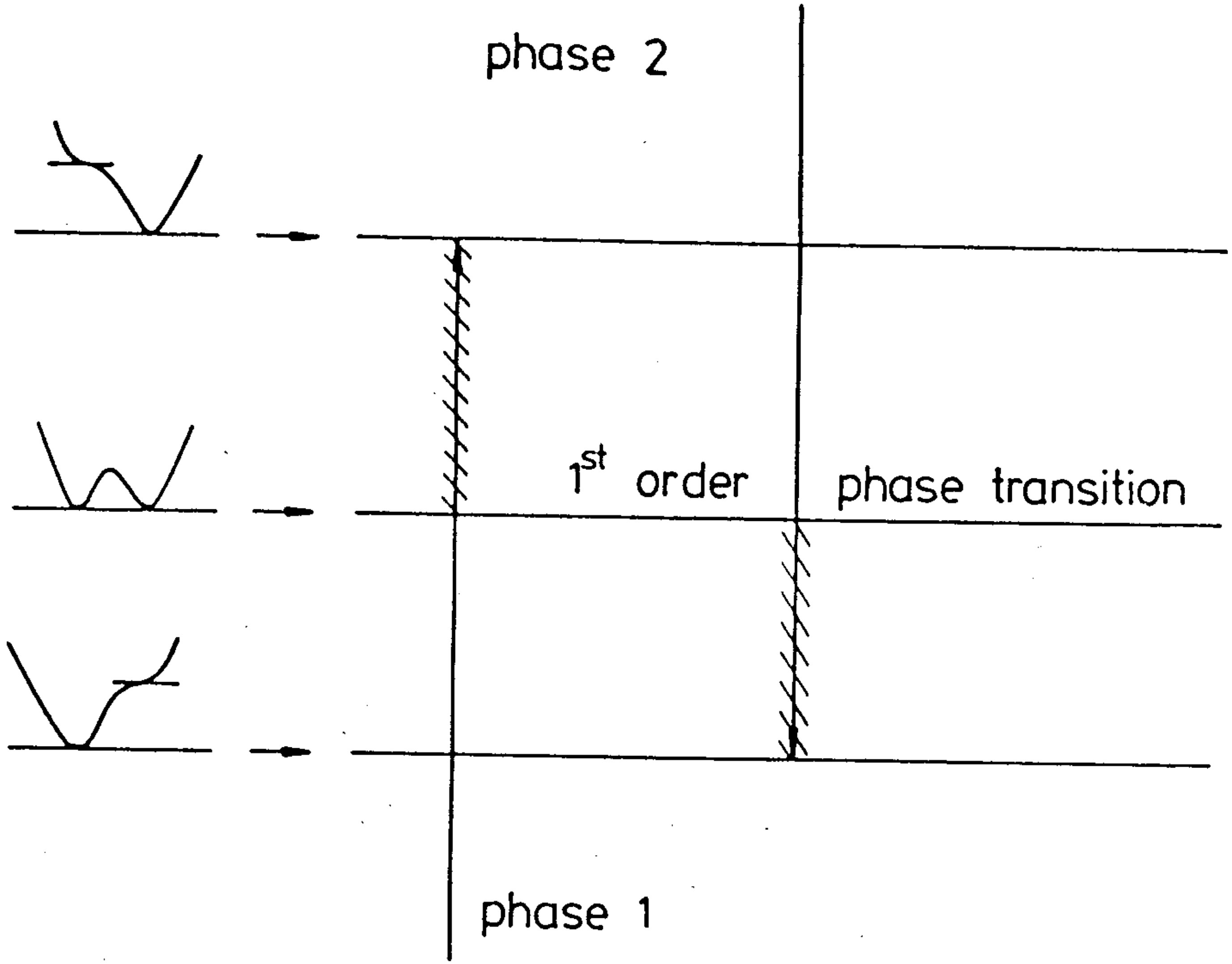

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Fig. 3

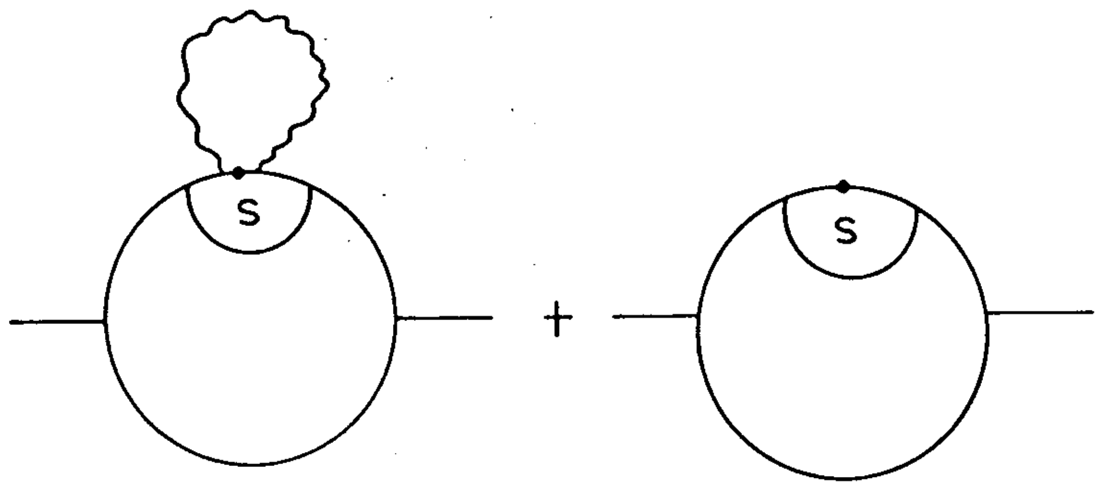

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Fig. 4

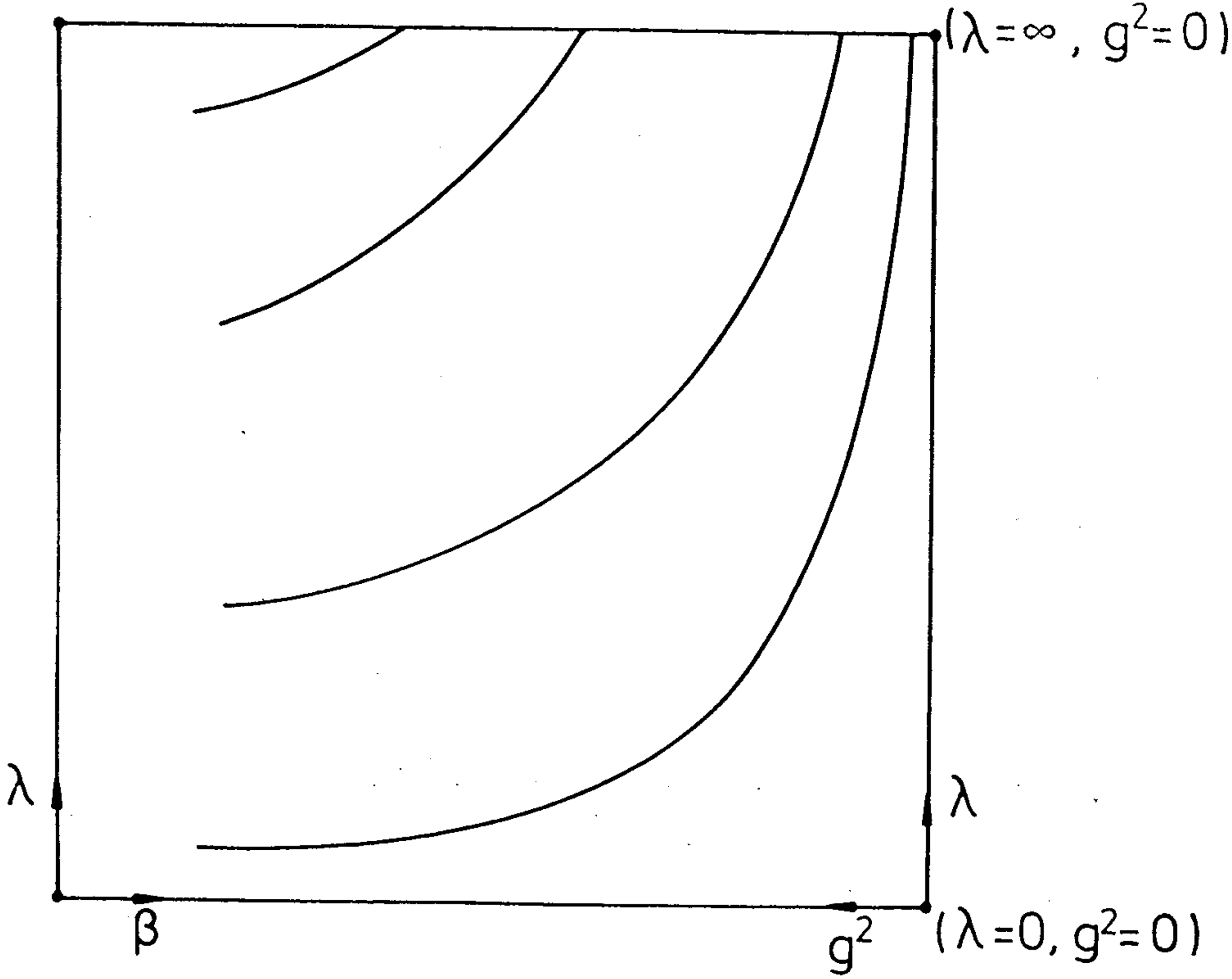

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Fig. 5

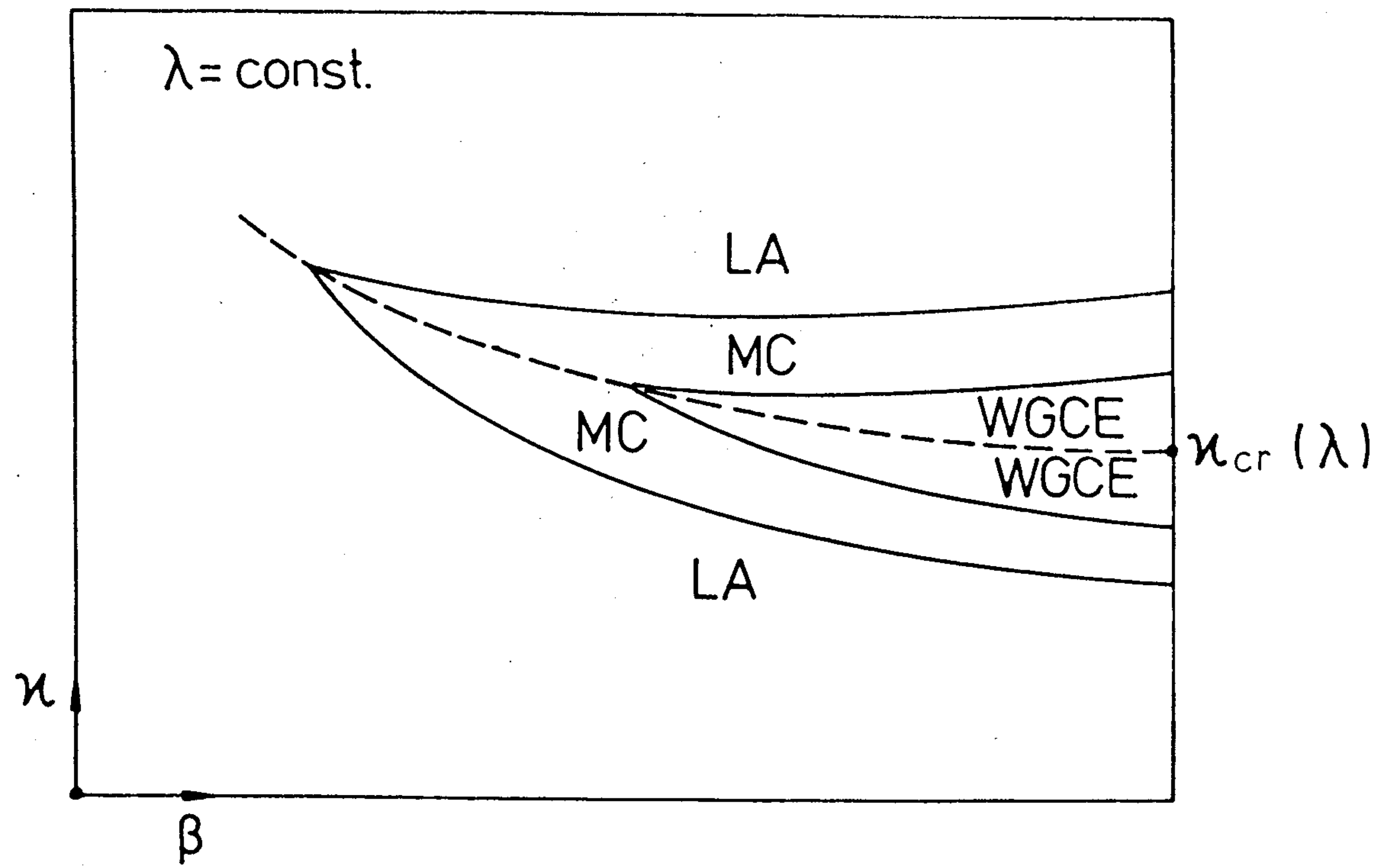

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Fig. 6