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by

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On the Riemann theta-function of some trigonal curve and the solutions of the Boussinesque equation and the KP equation

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Abstract

Recently considerable progress has been obtaind in understanding the nature of the algebrogeometrical superposition principles for the solutions of the non-linear completely integrable evolution equations, mainly for the equations related with hyperelliptic Riemann surfaces. Here we find such a superposition formula for the particular real solutions of the KP and Boussinesque equation related with the non-hyperelliptic curve $\omega^4 = (\lambda - E_1)(\lambda - E_2)(\lambda - E_3)(\lambda - E_4)$ it is shown below that associated Riemann theta-function may be decomposed into a sum containing two terms, each term being the product of the three one-dimensional theta functions. The space and time variables of the KP and Boussinesque equation enters into the arguments of these one-dimensional theta functions in a linear way.

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1. Introduction

Up to the last time the simple examples of nonhyperelliptical solutions of the Boussinesque equation

$$3u_{yy} + (u_{xxx} + 6uu_x)_x = 0 \tag{1}$$

reducible to the one-dimensional theta-functions were unknown. The curve Γ: $\omega^4 = (\lambda - E_1)(\lambda - E_2)(\lambda - E_3)(\lambda - E_4)$, $\mathrm{Im}\, E_k = 0$, was discussed in the work /1/ as an example of the Krichever's reduction of the KP equation to the Boussinesque equation via a Weierstrass points. But at that time only the Weierstrass points coinciding with the branch points were explored. The possibility of the reduction of the associated three-dimensional theta-function to the one-dimensional theta-functions also was not remarked. Such a possibility arises from the existence of the conformal automorphism $\tau : (\omega, \lambda) \longrightarrow (i\omega, \lambda)$ interchanging the sheets of the associated Riemann surface - realized as a four-sheeted covering of the complex λ-plane -. The way to explore this automorphism for reduce 3-dimensional Riemann theta-function goes back to the methods of the recent works /2, 3/, - see also a review article /4/ - although the application of the matrix version of Appel's theorem used below simplifies the calculations. As a main result we obtain some family of the genus 3 solutions of the KP and Boussinesque equations expressed by means of the elliptic theta-functions.

2. The algebrogeometrical solutions of the KP and Boussinesque equation

The formula

$$u(x,y,t)=2\frac{\partial^2}{\partial x^2}\ln\Theta(x\vec U+y\vec V+t\vec W-\vec\ell\,|\,B)+C \qquad (2)$$

describes the solutions of the Kadomcev-Petviashvily equation

$$3u_{yy}+(u_{xxx}+6uu_x-4u_t)_x=0, \qquad (3)$$

generated by arbitrary compact Riemann surface Γ /5/. In (2) B means the matrix of b-periods of the surface Γ in some canonical basis of $H_1(\Gamma)$. Θ is g-dimensional theta-function defined by the formula

$$\Theta(\vec p|B)=\Theta\begin{bmatrix}0\\0\end{bmatrix}(\vec p|B), \qquad (4)$$

$$\Theta\begin{bmatrix}\vec\alpha\\ \vec\beta\end{bmatrix}(\vec p|B)=\sum_{\vec m\in Z^g}\exp\{\pi i\langle B(m+\alpha),m+\alpha\rangle+2\pi i\langle m+\alpha,p+\beta\rangle\}, \qquad (5)$$

g is the genus of the Riemann surface Γ. U, V, W are the vectors of b-periodes of some normalized abelian integrals of the second kind with the poles at the marked point $P_0\in\Gamma$. In the case of the hyperelliptic curve P_0 may coincide with one the branch points. In this case vector V turns to be equal to zero and formula (2) reduces to the solution of the KdV equation, given by Its-Matveev formula /1, 6, 7, 8/. If Γ is some trigonal curve, i.e. their exists a meromorphic function on with the unique pole at the point P_0 of the order 3 it turns out that W = 0. In this case the KP solution (2) is independent on t and satisfies the nonlinear Boussinesque equation (1). Trigonal curve 3 of the genus g ≍ 3 cannot be hyperelliptic. The curve Γ considered in this note is nonhyperelliptic of the genus 3.

It is of nondividing type. Its branch points are $P_j=(0,E_j)$. Let P_0 coincide with one of branch points. It is possible to construct a local parameter K(P), $P\in\Gamma$ in such a way that under the action of the antiholomorphic involution τ: $(w,\lambda)\longrightarrow(\bar w,\bar\lambda)$ it transforms to $\bar k$ or $-\bar k$. Let $\gamma_{\ell j}(m)$ is the path with the starting point at $(0,E_\ell)$ and the end at $(0,E_j)$, $\arg w(\lambda)=\frac{\pi m}{4}$ along the path. The canonical basis of the curve Γ now may be defined by the formula:

$$\begin{aligned}
a_1&=\gamma_{34}(1)+\gamma_{43}(-1),\quad a_2=\gamma_{34}(-3)+\gamma_{43}(3),\\
a_3&=\gamma_{23}(2)+\gamma_{32}(-2),\\
b_1&=\gamma_{34}(3)+\gamma_{43}(1)+\gamma_{23}(4)+\gamma_{32}(-2),\\
b_2&=\gamma_{34}(-1)+\gamma_{43}(-3)+\gamma_{23}(0)+\gamma_{23}(-2),\\
b_3&=\gamma_{21}(1)+\gamma_{21}(3).
\end{aligned} \qquad (6)$$

Let $\vec\ell$ satisfies the condition : $Re\,\ell_j=\frac{1}{2}Re\,B_{jj}$.

Under the assumptions of this point we have the following main theorem.

Theorem I The solution of the Boussinesque equation defined by the curve Γ is real-valued and may be reduced to the following form:

$$u=2\partial_x^2\ln\Big\{\Theta\begin{bmatrix}0\\ 1/2\end{bmatrix}(p_1|B_1)\,\Theta(p_2|B_2)\,\Theta(p_3|B_2)+\Theta\begin{bmatrix}1/2\\ 1/2\end{bmatrix}(p_1|B_1)\,\Theta\begin{bmatrix}1/2\\ 1/2\end{bmatrix}(p_2|B_2)\,\Theta\begin{bmatrix}1/2\\ 1/2\end{bmatrix}(p_3|B_2)\Big\}+C-2\pi i\frac{U_3^2}{B_{33}}, \qquad (7)$$

$$B_1 = 4B_{12} + B_{33} - B_{33}^{-1} + 2, \quad B_2 = -B_{33}^{-1},$$

$$p_1 = x(U_1+U_2) + y(V_1+V_2) + l_1 + l_2,$$

$$p_2 = x(U_1-U_2-U_3) + y(V_1-V_2-V_3) + l_1 - l_2 - l_3 - \frac{1}{2},$$

$$p_3 = B_{33}^{-1}(xU_3 + yV_3 + l_3). \tag{8}$$

3. Proof of the main theorem

The reality of the solution (7) of the Boussinesque equation follows directly from the recent results of B.A. Dubrovin /9/. The crucial step of the proof of the formula (7) is to demonstrate the associated decomposition formula for the Riemann theta-function of the curve Γ. For produce such a decomposition formula we start from specifying the structure of the Matrix B taking into account the symmetry properties of the curve Γ.

Lemma 1. Let τ_1, be a holomorphic automorphism of some compact Riemann surface Γ of the genus 3 which acts on basis of $H_1(\Gamma)$ in a following way:

$$\tau_1 \vec{a} = M\vec{a}, \quad \tau_1 \vec{b} = (M^T)^{-1}\vec{b} + L\vec{a} \tag{9}$$

Then B matrix satisfies the relation:

$$B = M^T B M - M^T L. \tag{10}$$

Proof of Lemma 1 follows directly from the general law - see /4/ - of transformation of the B matrix under the change of canonical basis of $H_1(\Gamma)$.

In the conditions of the Lemma 1 we can find a matrix T, $T_{ik} \in \mathbb{Z}$, in such a way that the condition

$$B' = GB'G, \quad G = T^{-1}MT, \tag{11}$$

$$B' = T^T(B-A)T, \tag{12}$$

- where A is some symmetric matrix with a rational matrix elements satisfying the same relation (10) as B-matrix -, leads to a block diagonal structure of the matrix B'.

Let the condition (11) is satisfied and the elements of the matrix A'-D defined by $A' = T^T A T$, $D = diag[A'_{ii}]$ are entries.

Now it is easy to prove the following

Theorem 2. Under the condition of this point g-dimensional theta-function generated by the B-matrix may be represented in the form:

$$\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(p|B) = \sum_{s\in \mathbb{Z}^g(T)} \exp\{-\pi i\langle (A'-D)\alpha(s), \alpha(s)\rangle\}\, \Theta\begin{bmatrix}\alpha(s)\\ \beta(s)\end{bmatrix}(T^T p|B'+D). \tag{13}$$

Summing over $\mathbb{Z}^g(T)$ means that $s \in \mathbb{Z}^g$ runs over all vectors constrained by the inequality $0 \leq (T^{-1}s)_j < 1$.

$$\alpha(s) = T^{-1}(s+\alpha), \quad \beta = (A'-D)\alpha(s) + T^T\beta.$$

Taking into account that B' is of the block structure and D is diagonal matrix it is evident that $\Theta\begin{bmatrix}\alpha(s)\\ \beta(s)\end{bmatrix}(T^{T}p\,|\,B'+D)$ may be represented in a form a product of the theta-functions of the lower dimensions.

Proof of the theorem 2 consists simply to go from summing over $m \in Z^{g}$ to summing over n and s related with m by the equality m = Tn + s. The number of the terms in the sum taken over $Z^{g}(T)$ is equal to $|\det T|$ because detT is the jacobian determinant of the transformation from m-lattice to n-lattice.

The theorem 2 is related with Theorem 1 in a following way. Let us introduce a new canonical basis in $H_1(\Gamma)$:

$$\tilde{a}_3 = -b_3, \quad \tilde{b}_3 = a_3, \quad \tilde{a}_k = a_k, \quad \tilde{b}_k = b_k, \quad k = 1,2.$$

Now the holomorphic automorphism appearing in Lemma I may be realized like so $\tau_1 : E_k \longrightarrow E_{5-k}, \; \omega \longrightarrow i\omega$. Then we find that the matrices A and T may be choosen in the form

$$T = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \tag{14}$$

Now the direct application of the Theorem 2 leads to the representation (7) for the solutions of the Boussinesque nonlinear equation.

Remark. If the point P_0 is not coincide with one of the Weierstrass points we get from decomposition (13) the solutions of KP equation represented in terms of the one dimensional theta-functions.

4. Weierstrass points of the curve Γ, which are not branch points

Weierstrass points of the curve Γ were studied in the work /10/ in which the complete description of the Weierstrass points of the genus 3 curves was presented. However, the variety of the Weierstrass points of the curve Γ pointed out in /10/ seems to be wrong because it is not invariant with respect to the action of some conformial automorphisms. That's why we recalculate the W-points of the curve Γ. This calculation may be performed out in a standard way looking at the behaviour of the wronskian of normalized abelian differentials. Here we give only the final answer, formulated for the curve

$$y^4 = x^4 - (m^2+1)x^2z^2 + m^2z^4 \tag{15}$$

This curve, given by the polynomial equation (15), were x,y,z are homogenous co-ordinates, is birationaly equivalent to the curve Γ. So it is sufficient to describe all Weierstrass points in this realization of Γ. There are two possibilities: $m^2 = -1$, $m^2 \neq -1$ which are essentially different in study of W-points. Let at first $m^2 = 1$. In this case number of Weierstrass points is 12. Their positions are $(0, i^k, \exp(i\pi/4)), (i^k, 0, 1), (i^k, 1, 0), \; k = 1,2,3,4$.

Meromorphic functions with the unique singulartiy-pole of the third order at these points are

$$x \cdot \{y e^{\frac{\pi i}{4}} - i^k z\}^{-1}, \; y(x - i^k z)^{-1}, \; z(x - i^k y)^{-1}, \tag{16}$$

respectively.

In the case $m^2 \neq -1$ we have 20 Weierstrass points. These W-points may be divided in two groups. First group contains all the branch points and the meromorphic functions with a third order pole as a unique singularity may be constructed as above. The second group contains 16 points with the positions $(x_k, y_{ik}, 1)$ defined from the system

$$y_{ik}^4 = x_k^4 - (m^2+1)x_k^2 + m^2$$

$$2(m^2+1)x_k^4 + (m^4-10m^2+1)x_k^2 + 2m^2(m^2+1) = 0. \qquad (17)$$

The related meromorphic functions are of the form $(x-\tilde{x}_k)(y-y_{ik}-y'_{ik}(x-x_k)^{-1})$, $'$ stays for the differentiation on x and

$$\tilde{x}_k = x_k + 4(x_k - y_{ik}(y'_{ik})^3[(y'_{ik})^4-1]^{-1}.$$

6. <u>The reduction of the basis abelian integrals of the first kind to the elliptic integrals</u>

$\int \frac{d\lambda}{\omega^3}$ and $\int \frac{\lambda d\lambda}{\omega^3}$ may be reduced to the linear combination of the integrals $I_{1,2}$:

$$I_1 = \int \frac{\mu d\mu}{[(\mu^2-1)(\mu^2-m^2)]^{3/4}}, \quad I_2 = \int \frac{d\mu}{[(\mu^2-1)(\mu^2-m^2)]^{3/4}}$$

by some fraction-linear transformation.

I_2 may be reduced to I_1 by the change of variables $\mu = m\nu^{-1}$

I_1 may be reduced by the transformation

$$y = \frac{2}{(1-m^2)}\sqrt{(\mu^2-1)(\mu^2-m^2)}$$

to the elliptic integral:

$$I_1 = \frac{(i)^k}{\sqrt{2(1-m^2)}} \int \frac{dy}{\sqrt{y(1+y^2)}}$$

where the choice of k depends on the path of integration. The third basis integral $\int \frac{d\lambda}{\omega^2}$ is just elliptic and needs no further reduction.

Conlcuding remarks.

This note is a part of a program /2, 3, 4/ of studies in theta-functions of Riemann surfaces with non-trivial automorphisms and their applications to soliton equations. Such a surfaces and their theta-functions are also of interest for the recent developpments of the quantum strings theories. Particularly such a surfaces appear in the description of the interactions on obrifolds involving the emission of twisted states /11/. For example the famous Klein curve with a simple group of birational automorphisms of the extremal order 168 - see /2/ for the study of its B-matrix and particular properties of its theta-function - is also encountered in the classification of obrifolds with SU(3) holonomy /12/.

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