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Chiral Anomaly from the Fokker-Planck Formalism

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Abstract

We show how chiral anomalies arise in the Fokker-Planck formulation of stochastic quantization. Starting from a noise correlation function which is non-local in real space-time, a gauge invariantly regularized Fokker-Planck Hamiltonian is derived and used to compute the anomaly.

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I. Introduction

Recently various authors /1/ treated the problem of anomalous symmetry breaking in the framework of stochastic quantization /2,3,4,5/. All these derivations of the chiral anomaly, for instance, have been done using the Langevin formulation. In this letter we look at the same problem from the Fokker-Planck point of view. At first glance, it might seem that this can be done trivially because the relationship between the Fokker-Planck and the Langevin formulation is well-established. However, actually this is not the case. The reason is that, to regulate the quantum field theory, one uses the Breit-Gupta-Zaks /3/ regularized noise which basically makes the process non-Markov. Thus, it is not evident if there is a Fokker-Planck formulation at all because the equivalence between the two formulations is defined by a single-"time" equation /4/. To circumvent this problem we propose to use the following different regularization scheme: instead of smearing out the noise correlation in the τ -direction, we replace the space-time δ -function by a smooth, Lorentz-invariant regulator function. This leads to a Fokker-Planck Hamiltonian which is a non-local, second order functional operator. Calculating the anomaly in this scheme turns out to be basically equivalent to the well-known point-splitting method /6,7/.

II. The Gauge-Invariant, Regularized Fokker-Planck Hamiltonian

In this section we will derive the gauge-invariant, regularized Fokker-Planck Hamiltonian using the canonical procedure.

For illustrative purposes consider the Euclideanized action of massless QED_4

$$S = -\int d^4x \; \bar{\psi}\,(i\not{D})\,\psi \; , \tag{1}$$

where $\not{D} = \not{\partial} + i\not{A}$ and we follow the convention $g_{\mu\nu} = -\delta_{\mu\nu}$, $\gamma_\mu^+ = -\gamma_\mu$.
The Langevin equations are then

$$\frac{\partial \psi}{\partial \tau} = -\not{D}^2 \psi + i\not{D}\,\eta \; , \tag{2a}$$

$$\frac{\partial \bar{\psi}}{\partial \tau} = -\bar{\psi}\,\overleftarrow{\not{D}}^2 + \bar{\eta} \; , \tag{2b}$$

where $\overleftarrow{\not{D}} = \overleftarrow{\not{\partial}} - i\not{A}$. The Langevin equations (2a,b) are gauge invariant provided the noise terms η and $\bar{\eta}$ transform like ψ and $\bar{\psi}$, respectively.

At this point, we specify how the noise is regularized. It is a common practice to regularize the noise in the τ direction. As pointed out already in the introduction, this makes the stochastic process non-Markov. As an alternative, we propose to regulate the noise in the x_μ direction. Because of the gauge transformation properties of η and $\bar{\eta}$, we propose the following correlation

$$\langle \eta_\alpha(x,\tau)\,\bar{\eta}_\beta(x',\tau') \rangle = 2\,\delta_{\alpha\beta}\,\delta(\tau-\tau')\,\beta_\Lambda(x-x')\,\phi(x,x') \; , \tag{3}$$

where $\beta_\Lambda(x-x')$ is a Lorentz-invariant smearing function which approaches the Dirac delta as Λ (which has the dimensionality of a mass) approaches ∞. We also assume that $\int \beta_\Lambda(x-x')\, d^4x' = 1$. The phase factor

$$\phi(x,x') = e^{\,i\int_x^{x'} A\cdot dy} \; , \tag{4}$$

is necessary because of the gauge transformation properties of the noises. The correlation (3) implies that the noise have the distribution

$$\exp\left\{-\frac{1}{2}\int d\tau\, d^4x\, d^4x'\; \bar{\eta}(x,\tau)\,\beta_\Lambda(x-x')\,\phi(x,x')\,\eta(x',\tau)\right\} .$$

To derive the Fokker-Planck Hamiltonian, we consider the Wiener integral corresponding to the Langevin process given by (2a,b).

$$\langle f,\tau=\tau_f | i,\tau=0\rangle = \int_{\substack{\psi_\eta(x,0)=\psi_i(x)\\ \psi_\eta(x,\tau)=\psi_f(x)}} [d\bar{\eta}(x,\tau)][d\eta(x,\tau)]\; e^{-\frac{1}{2}\int d\tau\, d^4x\, d^4x'\, \bar{\eta}(x,\tau)\,\beta_\Lambda(x-x')\,\phi(x,x')\,\eta(x',\tau)} \; . \tag{5}$$

Using the Langevin equations, we transform this into integrals over 4 and $\bar{4}$.

$$\langle f,\tau=\tau_f | i,\tau=0\rangle = \int_{\text{end points}} [d\bar{\psi}(x,\tau)][d\psi(x,\tau)]\; J\!\left(\frac{\eta,\bar{\eta}}{\psi,\bar{\psi}}\right) \exp\left\{-\frac{1}{2}\int d\tau\, d^4x\, d^4x' \right.$$
$$\left. \bar{\psi}\left[\frac{\overleftarrow{\partial}}{\partial\tau} + \overleftarrow{\not{D}}^2\right]\beta_\Lambda(x-x')\,\phi(x,x')\left[\frac{1}{i\not{D}}\left(\frac{\partial}{\partial\tau} + \not{D}^2\right)\psi\right]_{x',\tau}\right\} , \tag{6}$$

and the Jacobian of transformation is

$$J = \det\begin{bmatrix} \delta(x-x')\left(\frac{\partial}{\partial\tau} + \not{D}^2(x)\right)\delta(\tau-\tau') & 0 \\ 0 & \delta(x-x')\left(\frac{\partial}{\partial\tau} + \overleftarrow{\not{D}}^2(x')\right)\delta(\tau-\tau') \end{bmatrix} . \tag{7}$$

In (7), g(x-x') is the Greens function of $i\not{\mathcal{D}}$. Factoring out

$$J_0 = \det \begin{bmatrix} g(x-x')\frac{\partial}{\partial\tau}\delta(\tau-\tau') & 0 \\ 0 & \frac{\partial}{\partial\tau}\delta(x-x')\delta(\tau-\tau') \end{bmatrix},$$

and using the midpoint rule $\Theta(0) = 1/2$, we find

$$J \sim \exp\left\{\frac{1}{2}\delta^4(0)\int d\tau\, d^4x \left[\mathcal{D}_x^2 + \overleftarrow{\mathcal{D}}_x^2\right]\right\}. \tag{8}$$

From (8) and the exponential term in (6) we read-off the Fokker-Planck Lagrangian /5/

$$L_{FP} = \frac{1}{2}\int d^4x\, d^4x' \left[\frac{\partial\bar\psi}{\partial\tau} + \bar\psi\overleftarrow{\mathcal{D}}^2\right]_x \beta_\Lambda(x-x')\phi(x,x')\left[\frac{1}{i\not{\mathcal{D}}}\left(\frac{\partial\psi}{\partial\tau} + \mathcal{D}^2\psi\right)\right]_{x'}$$
$$- \frac{1}{2}\delta^4(0)\int d^4x \left[\mathcal{D}_x^2 + \overleftarrow{\mathcal{D}}_x^2\right]. \tag{9}$$

We now use the canonical procedure. The conjugate momenta are

$$\Pi_\psi = \frac{\delta L_{FP}}{\delta(\partial_\tau\psi)} = \frac{1}{2}\int d^4x_1\, d^4x_2 \left[\frac{\partial\bar\psi}{\partial\tau} + \bar\psi\overleftarrow{\mathcal{D}}^2\right]_{x_1}\beta_\Lambda(x_1-x_2)\phi(x_1,x_2)\, g(x_2-x), \tag{10a}$$

$$\Pi_{\bar\psi} = \frac{\delta L_{FP}}{\delta(\partial_\tau\bar\psi)} = \frac{1}{2}\int d^4x_1\, d^4x_2\, \beta_\Lambda(x-x_1)\phi(x,x_1)\, g(x_1-x_2)\left(\frac{\partial\psi}{\partial\tau} + \mathcal{D}^2\psi\right)_{x_2}. \tag{10b}$$

To solve for the velocities, we use

$$\int d^4x_1\, \beta_\Lambda(x-x_1)\phi(x,x_1)\beta_\Lambda^{-1}(x_1-x_2)\phi(x_1,x_2) = \delta^4(x-x_2).$$

The "Hamiltonian" is given by the Legendre transformation

$$H_{FP} = \int d^4x \left[\Pi_{\psi_\alpha}\frac{\partial\psi_\alpha}{\partial\tau} + \Pi_{\bar\psi_\alpha}\frac{\partial\bar\psi_\alpha}{\partial\tau}\right] - L_{FP}, \tag{11}$$

and after operator ordering gives

$$\hat H_{FP} = \int d^4x\, d^4x' \left\{\frac{\delta}{\delta\psi(x)}\, i\not{\mathcal{D}}_x\left(\beta_\Lambda^{-1}(x-x')\phi(x,x')\right)\frac{\delta}{\delta\bar\psi(x')} - \frac{\delta}{\delta\bar\psi(x)}\left(\phi^{-1}(x,x')\beta_\Lambda^{-1}(x-x')\right) i\overleftarrow{\not{\mathcal{D}}}_x\frac{\delta}{\delta\psi(x')}\right\}$$
$$+ \int d^4x \left\{\frac{\delta}{\delta\psi(x)}\mathcal{D}_x^2\psi(x) + \frac{\delta}{\delta\bar\psi(x)}\left(\bar\psi(x)\overleftarrow{\mathcal{D}}_x^2\right)\right\}. \tag{12}$$

Equation (12) is the gauge-invariant, regularized Fokker-Planck Hamiltonian. By doing a similarity transformation

$$\hat H'_{FP} = e^{\hat A}\,\hat H_{FP}\, e^{-\hat A}, \tag{13a}$$

$$\hat{A} = \int d^4x\, d^4x' \frac{\delta}{\delta\psi(x)}\, g(x-x')\, \beta_\Lambda^{-1}(x-x')\, \phi(x,x') \frac{\delta}{\delta\bar{\psi}(x')} \quad , \tag{13b}$$

we find

$$\hat{H}'_{FP} = \int d^4x \left\{ \frac{\delta}{\delta\psi(x)} \not{D}_x^{2} \psi(x) + \frac{\delta}{\delta\bar{\psi}(x)} \left(\bar{\psi}(x) \overleftarrow{\not{D}}_x^{2}\right) \right\} . \tag{14}$$

Note that $\hat{H}'_{FP}$ does not depend on the regulator and thus its spectrum is independent of Λ. Sakita had shown that $\hat{H}'_{FP}$ is a positive-definite operator and thus it is also true that the point-splitted, gauge-invariant Fokker-Planck Hamiltonian (12) is also positive definite. Another prerequisite is the existence of a mass gap between the zero mode and the non-zero modes. Making the usual assumption that $\hat{H}'_{FP}$ has a mass gap then $\hat{H}_{FP}$ also has a mass gap.

III. Derivation of the anomaly

We determine the anomaly as the Jacobian $J[\alpha]$ associated with an infinitesimal chiral transformation

$$\begin{aligned} \psi'(x) &= e^{i\alpha(x)\gamma_5}\, \psi(x) \; , \\ \bar{\psi}'(x) &= \bar{\psi}(x)\, e^{i\alpha(x)\gamma_5} \; . \end{aligned} \tag{15}$$

For this Jacobian we make the ansatz

$$J[\alpha] = \exp\left(-i \int d^4x\, \alpha(x)\, \mathcal{A}(x)\right) . \tag{16}$$

Now one considers the normalization condition

$$\int [d\psi][d\bar{\psi}]\; P[\psi,\bar{\psi};\tau] = 1 \quad , \tag{17}$$

and changes the integration variables from ψ and $\bar{\psi}$ to ψ' and $\bar{\psi}'$ according to (15) for infinitesimal $\alpha(x)$. This implies

$$i \int d^4x\, \alpha(x)\, \mathcal{A}(x) = \int [d\psi][d\bar{\psi}]\; \delta P[\psi,\bar{\psi};\tau] \; , \tag{18}$$

where $\delta P[\psi,\bar{\psi};\tau] = P[\psi',\bar{\psi}';\tau] - P[\psi,\bar{\psi};\tau]$. To make this equation more transparent, we introduce a complete set $\{F_n[\psi,\bar{\psi}]\}$ of eigenfunctionals of the Fokker-Planck Hamiltonian: $\hat{H}_{FP} F_n = E_n F_n$. In terms of the F_n's the distribution function may be expanded as

$$P[\psi,\bar{\psi};\tau] = \sum_n{}' c_n\, e^{-E_n \tau}\, F_n[\psi,\bar{\psi}] \; . \tag{19}$$

The coefficients $\{c_n\}$ are to be determined from the initial conditions. Thus we have

$$i \int d^4x\, \alpha(x)\, \mathcal{A}(x) = \sum_n{}' c_n\, e^{-E_n \tau} \int [d\psi][d\bar{\psi}]\; \delta F_n[\psi,\bar{\psi}] \; . \tag{20}$$

This equation nicely exhibits the τ-evolution of the anomaly. Since in general the eigenfunctionals F_n for $E_n > 0$ are not known explicitly, we can evaluate the RHS of (20) only in the limit $\tau \to \infty$. In this case only the zero-mode $F_0 \equiv Z^{-1} \exp(-S_\varepsilon)$ with the partition function $Z \equiv \int [d\psi][d\bar{\psi}] \exp(-S_\varepsilon)$ contributes. Hence one obtains

$$i\int d^4x\, \alpha(x)\mathcal{A}(x) = -Z^{-1}\int [d\psi][d\bar\psi]\, \delta S_\varepsilon \exp(-S_\varepsilon) = -\langle \delta S_\varepsilon \rangle. \qquad (21)$$

Contrary to Fujikawa's /8/ approach, where $\mathcal{A}$ is evaluated directly from the definition of the Jacobian $\delta(\psi',\bar\psi')/\delta(\psi,\bar\psi)$, we here determine $\mathcal{A}$ by calculating $\langle \delta S_\varepsilon \rangle$.

From the Hamiltonian (12) it is easy to see that S_ε is given by

$$S_\varepsilon = -\frac{1}{2}\int d^4x\, d^4\varepsilon\, \beta_\Lambda(\varepsilon)\left\{\bar\psi(x+\varepsilon)\phi(x+\varepsilon,x)\, i\not\partial_x \psi(x) + \bar\psi(x)\, i\not\partial_x \phi(x,x+\varepsilon)\psi(x+\varepsilon)\right\}. \qquad (22)$$

Recalling that for $\Lambda \to \infty$, the Lorentz invariant function β_Λ approaches a δ-function, we may replace $\lim_{\Lambda\to\infty}\int d^4\varepsilon\, \beta_\Lambda(\varepsilon)(\ldots)$ by $\lim_{\varepsilon\to 0}(\ldots)$ where an average over the directions of ε^μ according to $\frac{\varepsilon^\mu\varepsilon^\nu}{\varepsilon^2} \to \frac{g^{\mu\nu}}{4}$, etc. is understood. Expanding the bracket of (22) to first order in ε, which is equivalent to the first order in $1/\Lambda$, one obtains

$$S_\varepsilon = -\lim_{\varepsilon\to 0}\int d^4x\, \bar\psi(x+\tfrac{\varepsilon}{2})\left\{i\not\partial - \not A + \varepsilon^\mu A_\mu \not\partial + i\not A\varepsilon^\mu A_\mu + \tfrac{1}{2}(\not\partial\varepsilon^\mu A_\mu)\right\}\psi(x-\tfrac{\varepsilon}{2}). \qquad (23)$$

The change of (23) under an infinitesimal chiral rotation reads

$$\delta S_\varepsilon = -\int d^4x\, \bar\psi(x+\tfrac{\varepsilon}{2})\left\{-(\not\partial\alpha) + \varepsilon^\mu(\partial_\mu\alpha)(\not\partial + i\not A) + \tfrac{1}{2}\not\partial(\varepsilon^\mu\partial_\mu\alpha) + i\varepsilon^\mu A_\mu(\not\partial\alpha)\right\}\gamma_5\psi(x-\tfrac{\varepsilon}{2}). \qquad (24)$$

Next, one has to compute the vacuum expectation value of (24) in the limit $\varepsilon\to 0$. Using the well known matrix element /7/

$$\langle\bar\psi(x+\tfrac{\varepsilon}{2})\gamma_\mu\gamma_5\psi(x-\tfrac{\varepsilon}{2})\rangle = \frac{1}{4\pi^2}\frac{\varepsilon^\nu}{\varepsilon^2}\epsilon_{\nu\mu\alpha\beta}F^{\alpha\beta} + O(\varepsilon^0), \qquad (25)$$

and the Heisenberg equations for ψ it is straightforward to arrive at

$$\langle\delta S_\varepsilon\rangle = -i\lim_{\varepsilon\to 0}\int d^4x\, F_{\mu\nu}\,\varepsilon^\mu\langle\bar\psi(x+\tfrac{\varepsilon}{2})\gamma^\nu\gamma_5\psi(x-\tfrac{\varepsilon}{2})\rangle = -i\int d^4x\, \alpha(x)\mathcal{A}(x), \qquad (26)$$

with

$$\mathcal{A}(x) = \frac{1}{8\pi^2}F^*_{\mu\nu}F^{\mu\nu}. \qquad (27)$$

Together with (16) this yields the standard result /8/ for the Jacobian of the chiral transformations (15):

$$J[\alpha] = \exp\left(-\frac{i}{8\pi^2}\int d^4x\, F^*_{\mu\nu}F^{\mu\nu}\right). \qquad (28)$$

To obtain the divergence of the axial vector current one notes that using the equations of motion for ψ the expectation value of (24) can be written as

$$\langle\delta S_\varepsilon\rangle = -\int d^4x\, \alpha(x)\lim_{\varepsilon\to 0}\partial_\mu\langle\bar\psi(x+\tfrac{\varepsilon}{2})\gamma^\mu\gamma_5\phi(x+\tfrac{\varepsilon}{2},x-\tfrac{\varepsilon}{2})\psi(x-\tfrac{\varepsilon}{2})\rangle. \qquad (29)$$

Inserting this together with (27) into (21), one ends up with the desired relation /7/

$$\lim_{\varepsilon\to 0}\partial_\mu\langle\bar\psi(x+\tfrac{\varepsilon}{2})\gamma^\mu\gamma_5\phi(x+\tfrac{\varepsilon}{2},x-\tfrac{\varepsilon}{2})\psi(x-\tfrac{\varepsilon}{2})\rangle = \frac{i}{8\pi^2}F^*_{\mu\nu}(x)F^{\mu\nu}(x). \qquad (30)$$

This completes the proof that the Fokker-Planck dynamics described by our $\hat{H}_{FP}$ leads to the correct anomaly for $\tau \to \infty$.

IV. Conclusions

In this letter we proposed a new way of regulating the noise in stochastic quantization. We find that this scheme gives rise to a (gauge-invariant) Fokker-Planck Hamiltonian whereas in the Breit-Gupta-Zaks scheme it is not evident if there is a simple Fokker-Planck formulation. As a first application, we derive the chiral anomaly of QED_4. The generalization to more complicated theories is in principle straightforward.

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