ANALYTIC TOOLS TO BRANE TECHNOLOGY IN $N = 2$ GAUGE THEORIES WITH MATTER [1]

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Abstract

Exact solutions to the low-energy effective action (LEEA) of the four-dimensional $N = 2$ supersymmetric gauge theories are known to be obtained either by quantum field theory methods from S-duality in the Seiberg-Witten approach, or by the Type-IIA superstring/M-Theory methods of brane technology. After a brief review of the standard field-theoretical results for the $N = 2$ *gauge* (Seiberg-Witten) LEEA, we consider a field-theoretical derivation of the exact *hypermultiplet* LEEA by using the $N = 2$ harmonic superspace methods. We illustrate our techniques on a number of explicit examples. Our main purpose, however, is to discuss the existing *analytical* (calculational) support for the alternative methods of brane technology. We summarize known exact solutions to the eleven-dimensional and ten-dimensional type-IIA supergravities, which describe classical configurations of intersecting BPS branes with eight supercharges relevant to the non-perturbative $N = 2$ gauge theory with fundamental hypermultiplet matter. The crucial role of the M-Theory in providing a classical resolution of singularities in the ten-dimensional (Type-IIA superstring) brane picture, as well as the $N = 2$ extended supersymmetry in four dimensions, are

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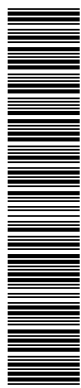
made *manifest.* The two approaches to a derivation of the exact $N = 2$ gauge theory LEEA are thus seen to be complementary to each other and mutually dependent.

CONTENT

1   Introduction

Quantum Field Theory (QFT) is the current theoretical foundation of the elementary particles physics, including the Standard Model (SM) and its (minimal) supersymmetric extension (MSSM). An experimental success of the SM and a theoretical attractiveness of the MSSM imply some general lessons to the field theory practicioners. Among them are: (i) not just an arbitrary QFT is of particular relevance for physical applications, but a renormalizable, unitary and asymptotically-free gauge theory, (ii) gauge symmetry and internal symmetry play the crucial role in maintaining consistency of a 'good' QFT at the quantum level, (iii) a 'good' QFT should be supersymmetric at sufficiently high energies, while the supersymmetry must be broken at lower energies.

The standard textbook description of quantum gauge theories is often limited to *perturbative* considerations, whereas many physical phenomena (e.g., confinement) are essentially *non*-perturbative. It is usually straightforward (although, it may be quite non-trivial !) to develop a quantum perturbation theory with all the fundamental symmetries to be *manifestly* (i.e. linearly) realized. Unfortunately, a perturbative expansion usually does not make sense when the field coupling becomes strong. A path integral representing the quantum generating functional of QFT has to be defined in practical terms which, generally speaking, can be (and, in fact, has been) done in *many* ways beyond the perturbation theory (e.g., lattice regularization, instantons, duality). Because of this reasoning, it was, until recently, common to believe among many field theorists that a non-perturbative gauge QFT is not well-defined enough, in order to make definitive predictions and non-perturbative calculations from the first principles (*cf.* the current status of QCD).

Nowadays, since the discovery of *exact* non-perturbative QFT solutions to the low-energy effective action (LEEA) in certain $N = 2$ supersymmetric four-dimensional ($D = 4$) gauge field theories, pioneered by Seiberg and Witten in ref. [1], and subsequent advances in non-perturbative M-Theory 'formerly known as the theory of superstrings', initiated in another Witten's paper [2], [3] the conventional wisdom briefly outlined above may have to be revised. Though non-trivial exact solutions were only found in a certain class of $N = 2$ *supersymmetric* gauge QFTs having no immediate phenomenological applications, they are, nevertheless, of great theoretical value. Indeed, unlike the SM or MSSM, the $N = 2$ extended supersymmetric gauge field theories in four dimensions cannot directly serve for phenomenological applications

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[3]See e.g., refs. [3, 4] for an introductory review.

at (low) energies of order 100 GeV, essentially because an $N = 2$ supersymmetric matter can only be defined in *real* representations of the gauge group and, hence, parity is conserved. The $N = 2$ gauge theories may, nevertheless, appear as a kind of effective QFT description at some intermediate energies provided that the ultimate (yet unknown at the microscopic level) unified theory of Nature at Planckian energies (e.g. M-Theory !) lives in higher spacetime dimensions and has even more supersymmetries. A solvable (in the LEEA sense) gauge QFT may be a good starting point for further symmetry breaking towards getting phenomenologically applicable QFT models at lower energies, which are supposed to resemble SM or MSSM, while maintaining the integrability properties of the initial supersymmetric field theory and thus keeping its non-perturbative LEEA under control.

Exact solutions to the LEEA of the four-dimensional $N = 2$ supersymmetric gauge field theories can be obtained either by the conventional QFT methods after taking into account S-duality in the Seiberg-Witten approach [1], or by the alternative Type-IIA superstring/M-Theory methods of *brane technology* [5, 6, 7]. [4] Unlike ref. [8], where the brane technology in various dimensions and with various amount of supersymmetry was considered, we restrict ourselves to the case of four (uncompactified) spacetime dimensions with the $N = 2$ extended supersymmetry there. Our interest to this situation is partially motivated by some general lessons out of the recent results about the non-perturbative behaviour of the four-dimensional gauge QFTs, namely,

- in order to be solvable in the low-energy approximation, a $D = 4$ gauge QFT has to be a 'good' one, i.e. it should have $N = 2$ extended supersymmetry or, equivalenntly, eight conserved supercharges,

- exact internal symmetries within the $N = 2$ extended supersymmetry severely restrict the form of allowed non-perturbative solutions to the LEEA in $N = 2$ supersymmetric gauge theories, with the Seiberg-Witten solution [1] being an example,

- the non-abelian gauge symmetry is always broken in the full (non-perturbative) $N = 2$ supersymmetric QFT to an abelian subgroup, while the $N = 2$ supersymmetry is unbroken.

The paper is organized as follows. After a brief review of the standard field-theoretical results for the $N = 2$ *gauge* (Seiberg-Witten) LEEA in sect. 2, we consider

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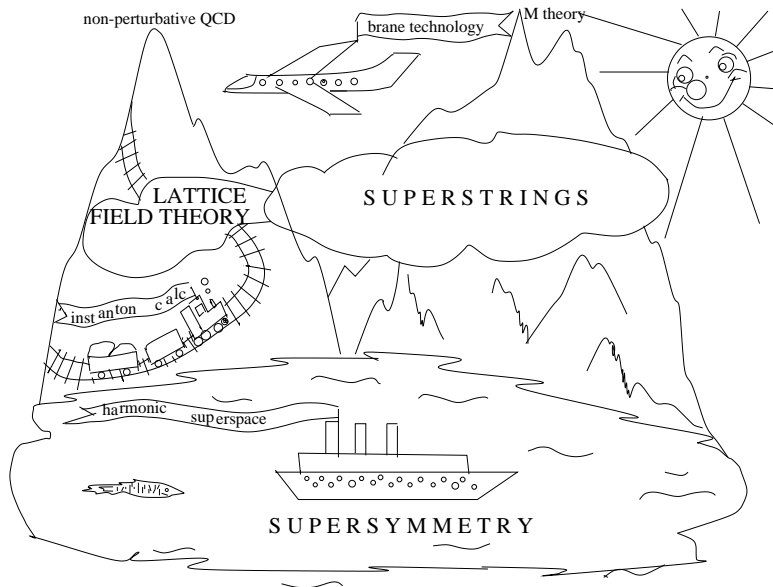
[4]See ref. [8] for a recent review.

Fig. 1. Thanks to my children, Denise(7) and Michael(9). No comments.

a field-theoretical derivation of the exact *hypermultiplet* LEEA in sect. 3 by using the $N = 2$ harmonic superspace methods. We illustrate our techniques on a number of explicit examples. The brane technology with eight conserved supercharges is discussed in Sect. 4. The calculational support to the brane technology is provided by exact BPS-brane solutions to the maximally extended supergravities in eleven and ten dimensions, and the harmonic superspace. M-Theory plays the crucial role in the brane technology, by providing a classical resolution of singularities in the ten-dimensional (Type-IIA superstring) brane picture. The extended supersymmetry with eight supercharges is made manifest, which results in many simplifications, transparency and an analytic control (Fig. 1).

1.1   Basic facts about $N = 2$ supersymmetry in $D = 4$

Four-dimensional ($D = 4$), $N = 2$ supersymmetric gauge field theories are not integrable, either classically or quantum-mechanically. [5] The full quantum effective action Γ in these theories is highly non-local and intractable. Nevertheless, it can be decomposed into a sum of local terms in powers of space-time derivatives or momenta divided by some dynamically generated scale Λ (in components). The leading kinetic terms of this expansion are called the *low-energy effective action* (LEEA). Determining the exact LEEA is a great achievement because it provides the information about

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[5]It is the *self-dual* sector of their Euclidean versions that is integrable in the classical sense [9, 10].

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a non-perturbative spectrum and exact static couplings in the full quantum theory at energies which are well below Λ. Since we are only interested in the $D = 4$, $N = 2$ gauge theories with spontaneously broken gauge symmetry via the Higgs mechanism, the effective low-energy field theory may include only *abelian* massless vector particles. All the massive fields (like the charged W-bosons) are supposed to be integrated out. This very general concept of LEEA is sometimes called the *Wilsonian* LEEA since it is familiar from statistical mechanics. There is a difference between the quantum effective action Γ, usually defined as the quantum generating functional of the one-particle-irreducible (1PI) Green's functions, and the Wilsonian effective action, as far as the gauge theories with massless particles are concerned. [6]

The $N = 2$ supersymmetry severely restricts the form of the LEEA. The very presence of $N = 2$ supersymmetry in the full non-perturbatively defined quantum $N = 2$ gauge theory follows from the fact that its Witten index [12] does not vanish, $\Delta_W = \text{tr}(-1)^F \neq 0$. It just means that $N = 2$ supersymmetry cannot be dynamically broken. Alternatively, one may try to consistently formulate the whole theory in a manifestly $N = 2$ supersymmetric way, e.g., in $N = 2$ superspace.

There are only two basic supermultiplets (modulo classical duality transformations) in the rigid $N = 2$ supersymmetry: an $N = 2$ *vector* multiplet and a *hypermultiplet*. The $N = 2$ vector multiplet components (in a WZ-gauge) are

$$\{ \, a \, , \quad \lambda_\alpha^i \, , \quad V_\mu \, , \quad D^{(ij)} \, \} \, , \tag{1.1}$$

where a is a complex Higgs scalar, λ^i is a chiral spinor ('gaugino') $SU(2)_A$ doublet, V_μ is a real vector gauge field, and D^{ij} is an auxiliary scalar $SU(2)_A$ triplet. [7] Similarly, the on-shell physical components of the *Fayet-Sohnius* (FS) [13] version of a hypermultiplet are

$$\text{FS}: \qquad \{ \, q^i \, , \quad \psi_\alpha \, , \quad \bar{\psi}_{\dot{\alpha}} \, \} \, , \tag{1.2}$$

where q^i is a complex scalar $SU(2)_A$ doublet, and ψ is a Dirac spinor. There exists another (dual) *Howe-Stelle-Townsend* (HST) version [14] of a hypermultiplet, whose on-shell physical components are

$$\text{HST}: \qquad \{ \, \omega \, , \quad \omega^{(ij)} \, , \quad \chi_\alpha^i \, \} \, , \tag{1.3}$$

where ω is a real scalar, $\omega^{(ij)}$ is a scalar $SU(2)_A$ triplet, and χ^i is a chiral spinor $SU(2)_A$ doublet. The hypermultiplet spinors are sometimes referred to as 'quarks',

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[6]See e.g., ref. [11] and references therein.

[7]The internal symmetry $SU(2)_A$ is the automorphism symmetry of the $N = 2$ supersymmetry algebra, which rotates its two spinor supercharges.

even though $N = 2$ supersymmetry implies (apparently absent in experiment) extra 'mirror' particle for each 'true' quark in the $N = 2$ super-QCD.

The manifestly supersymmetric formulation of supersymmetric field theories is provided by *superspace* [15]. Since superfields are generically reducible representations of supersymmetry, they have to be restricted by imposing certain superspace constraints. The standard constraints defining the $N = 2$ *super-Yang-Mills* (SYM) theory in the ordinary $N = 2$ superspace [16] essentially amount to the existence of a restricted chiral $N = 2$ superfield strength W, whose leading component is the Higgs field, $W| = a$. The $N = 2$ superfield W also contains the usual Yang-Mills field strength $F_{\mu\nu}(V)$ among its bosonic components, as well as the $SU(2)_A$ auxiliary triplet $D^{(ij)}$. Since the latter has to be real in the sence $\overline{D^{ij}} = \varepsilon_{ik}\varepsilon_{jl}D^{kl}$, this leads to (non-chiral) $N = 2$ superspace constraints on W, which are not easy to solve in terms of unconstrained $N = 2$ superfields in the non-abelian case. The situation is even more dramatic in the case of the FS hypermultiplet whose universal off-shell formulation does not even exist in the ordinary $N = 2$ superspace (i.e. with a finite number of auxiliary fields). Though a particular (dual) form of the HST hypermultiplet can be defined off-shell in the ordinary $N = 2$ superspace where it is known as an $N = 2$ *tensor* (or linear) multiplet (see subsect. 3.3), its self-couplings are very restricted and not universal there. For instance, in order to be coupled to the $N = 2$ gauge superfields, the $N = 2$ tensor multiplet has to be generalized to a reducible (relaxed) version (known as the *relaxed* HST multiplet) which is highly complicated and cumbersome. The most general (universal) off-shell formulation of a hypermultiplet is, however, necessary in order to write down its most general couplings, which may appear e.g. in the LEEA, in a model-independent way.

The universal off-shell solution to all $N = 2$ supersymmetric field theories in $D = 4$ was proposed in 1984 by Galperin, Ivanov, Kalitzin, Ogievetsky and Sokatchev [17]. They introduced the so-called $N = 2$ *harmonic superspace* (HSS) by adding extra bosonic variables (=harmonics), parameterizing a sphere $S^2 = SU(2)/U(1)$, to the ordinary $N = 2$ superspace coordinates. It amounts to the introduction of infinitely many auxiliary fields in terms of the ordinary $N = 2$ superfields. By the use of harmonics, one can rewrite the standard $N = 2$ superspace constraints to another form (that may be called a 'zero-curvature representation') in which the hidden *analytical* structure of the constraints becomes manifest. In the HSS approach, the harmonics play the role of twistors or spectral parameters which are well-known in the theory of integrable models. As a result [17], all the $N = 2$ supersymmetric field theories can be naturally formulated in terms of *unconstrained* and *analytic* $N = 2$ super-

fields, i.e. fully off-shell. [8] In particular, an off-shell FS hypermultiplet is naturally described by an analytic superfield q^+ of the $U(1)$ charge $(+1)$, whereas the analytic (sub)superspace measure has the $U(1)$ charge (-4). A generic hypermultiplet Lagrangian in the analytic HSS has to be an analytic function of q^+ and ω, and it may also depend upon harmonics $u_i^\pm$.

In the next subsect. 1.2. we are going to discuss the most general form of the LEEA, which is dictated by $N = 2$ supersymmetry alone. The rest of the paper will be devoted to the question how to fix the $N = 2$ supersymmetric Ansatz for the vector *and* hypermultiplet LEEA completely, by using all available methods of calculation (Fig. 1).

1.2   Basic facts about $N = 2$ supersymmetric LEEA

We are now already in a position to formulate the general *Ansatz* for the $N = 2$ supersymmetric LEEA. As regards the $N = 2$ vector multiplet terms, they can only be in the form of $N = 2$ superspace integrals,

$$\Gamma_V[W, \bar{W}] = \int_{\text{chiral}} \mathcal{F}(W) + \text{h.c.} + \int_{\text{full}} \mathcal{H}(W, \bar{W}) + \dots \, , \qquad (1.4)$$

where we have used the fact that the abelian $N = 2$ superfield strength W is an $N = 2$ chiral and gauge-invariant superfield. The leading term in eq. (1.4) is given by the chiral $N = 2$ superspace integral of a *holomorphic* function $\mathcal{F}$ of the gauge superfield strength W which is supposed to be valued in the Cartan subalgebra of the gauge group. The *next-to-leading-order* term is given by the full $N = 2$ superspace integral over the real function $\mathcal{H}$ of W *and* $\bar{W}$. The dots in eq. (1.4) stand for higher-order terms containing the derivatives of W and $\bar{W}$.

Similarly, the leading term in the hypermultiplet LEEA is of the form

$$\Gamma_H[q^+, \overset{*}{\bar{q}}{}^+; \omega] = \int_{\text{analytic}} \mathcal{K}^{(+4)}(q^+, \overset{*}{\bar{q}}{}^+; \omega; u_i^\pm) + \dots \, , \qquad (1.5)$$

where $\mathcal{K}^{(+4)}$ is an analytic function of the FS-type superfields q^+, their conjugates $\overset{*}{\bar{q}}{}^+$, the HST-type superfields ω and, perhaps, harmonics $u_i^\pm$. The action (1.5) is supposed to be added to the standard kinetic hypermultiplet action whose analytic Lagrangian is quadratic in q^+ and ω, and of $U(1)$-charge $(+4)$ (see sect. 3.1 for details). The function $\mathcal{K}$ is called a *hyper-Kähler potential.* An arbitrary choice of this function in eq. (1.5) automatically leads to the $N = 2$ supersymmetric *non-linear sigma-model*

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[8]See subsect. 3.1. for an introduction to $N = 2$ HSS.

(NLSM) with a *hyper-Kähler* metric, because of the $N = 2$ extended supersymmetry by construction (see Appendices A and B for some explicit examples).

When being expanded in components, the first term in eq. (1.4) also leads, in particular, to the certain Kähler NLSM in the scalar Higgs sector $(a, \bar{a})$. The corresponding NLSM Kähler potential $K_{\mathcal{F}}(a, \bar{a})$ is dictated by the holomorphic function $\mathcal{F}$ as $K_{\mathcal{F}} = \mathrm{Im}[\bar{a}\mathcal{F}'(a)]$, so that the function $\mathcal{F}$ plays the role of a potential for this *special* NLSM Kähler (but not a hyper-Kähler) geometry $K_{\mathcal{F}}(a, \bar{a})$. As regards the hypermultiplet NLSM of eq. (1.5), the relation between a hyper-Kähler potential $\mathcal{K}$ and the corresponding Kähler potential $K_{\mathcal{K}}$ of the same NLSM is much more involved. Indeed, it is easy to see that the hyper-Kähler condition on a Kähler potential amounts to a non-linear (Monge-Ampéré) partial differential equation which is not easy to solve. It is remarkable that the HSS approach allows one to formally get a general 'solution' to the hyper-Kähler constraints, in terms of an analytic scalar potential $\mathcal{K}$. Of course, the real problem is now being translated into the precise relation between $\mathcal{K}$ and the corresponding Kähler potential (or metric) in components, whose determination amounts to solving infinitely many linear differential equations altogether, just in order to eliminate the infinite number of the auxiliary fields involved (see Appendices A and B for examples). Nevertheless, the notion of hyper-Kähler potential in HSS turns out to be very useful in dealing with the hypermultiplet LEEA (sect. 3).

The LEEA gauge-invariant functions $\mathcal{F}(W)$ and $\mathcal{H}(W, \bar{W})$ generically receive both perturbative and non-perturbative contributions,

$$\mathcal{F} = \mathcal{F}_{\mathrm{per.}} + \mathcal{F}_{\mathrm{inst.}} \ , \qquad \mathcal{H} = \mathcal{H}_{\mathrm{per.}} + \mathcal{H}_{\mathrm{non-per.}} \ , \tag{1.6}$$

while the non-perturbative corrections to the holomorphic function $\mathcal{F}$ are entirely due to instantons. The last observation is in a sharp contrast with the situation in the (bosonic) non-perturbative QCD whose LEEA is dominated by instanton-antiinstanton contributions.

Unlike the gauge LEEA, the exact (charged) hypermultiplet LEEA is essentially perturbative (see sects. 3 and 4), i.e. it does not receive any instanton corrections,

$$\mathcal{K}[q^+] = \mathcal{K}_{\mathrm{per.}}[q^+] \ . \tag{1.7}$$

It is quite remarkable that the perturbative contributions to the leading and (in some cases) even subleading terms in the $N = 2$ supersymmetric LEEA entirely come from the *one* loop only. For example, as regards the leading holomorphic contribution to the gauge LEEA, there exists a simple argument: supersymmetry puts the trace of the energy-momentum tensor $T_\mu{}^\mu$ and the anomaly $\partial_\mu j_R^\mu$ of the abelian R-symmetry

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into a single $N = 2$ supermultiplet. The trace $T_\mu{}^\mu$ is essentially determined by the perturbative renormalization group β-function [9], $T_\mu{}^\mu \sim \beta(g)FF$, whereas the one-loop contribution to the R-anomaly, $\partial \cdot j_R \sim C_{1-\mathrm{loop}}F^*F$, is well-known to saturate the exact solution to the Wess-Zumino consistency condition for the same anomaly (e.g. the one to be obtained from the index theorem). Hence, $\beta_{\mathrm{per.}}(g) = \beta_{1-\mathrm{loop}}(g)$ by $N = 2$ supersymmetry also. Finally, since $\beta_{\mathrm{per.}}(g)$ is effectively determined by the second derivative of $\mathcal{F}_{\mathrm{per.}}$, one concludes that $\mathcal{F}_{\mathrm{per.}} = \mathcal{F}_{1-\mathrm{loop}}$ too.

The naive component argument can be extended to a proof [18] in a manifestly $N = 2$ supersymmetric way by using the $N = 2$ HSS approach, where the whole chiral perturbative contribution $\int_{\mathrm{chiral}} \mathcal{F}_{\mathrm{per.}}(W)$ arises as an anomaly. The non-vanishing central charges of the $N = 2$ supersymmetry algebra in the $N = 2$ gauge field theory under consideration turn out to be primarily responsible for the non-vanishing leading holomorphic contribution to the LEEA. A perturbative part of the LEEA thus takes the form $\mathcal{F}_{\mathrm{per.}}(W) \sim W^2 \log(W^2/M^2)$, where M is the renormalization scale, with the coefficient being fixed by the one-loop β-function of the renormalization group (see sect. 5).

The usual strategy in determining the exact LEEA exploits exact symmetries of a given $N = 2$ quantum gauge theory together with a certain physical input. As the particularly important example of an $N = 2$ supersymmetric gauge theory, one can use the $N = 2$ supersymmetric QCD with the gauge group $G_c = SU(N_c)$ and $N = 2$ matter described by some number (N_f) of hypermultiplets in the fundamental representation $\underline{N_c} + \underline{N_c}^*$ of the gauge group $SU(N_c)$. A quantum consistency of the non-abelian gauge theory requires asymptotic freedom which, in its turn, implies $N_f < 2N_c$ in this case.

All possible $N = 2$ supersymmetric vacua can be classified as follows:

- *Coulomb branch*: $\langle q \rangle = \langle \omega \rangle = 0$, whereas $\langle a \rangle \neq 0$; the gauge group G_c is broken to its abelian subgroup $U(1)^{\mathrm{rank}\ G_c}$; non-vanishing 'quark' masses are allowed;

- *Higgs branch*: $\langle q \rangle \neq 0$ or $\langle \omega \rangle \neq 0$ for some hypermultiplets, whereas $\langle a \rangle = 0$ and all the 'quark' masses vanish; the gauge group G_c is completely broken;

- *mixed (Coulomb-Higgs) branch*: some $\langle q \rangle \neq 0$ *and* $\langle a \rangle \neq 0$; it requires $N_c > 2$, in particular.

In the Coulomb branch, one has to specify the both equations (1.6) and (1.7), whereas in the Higgs branch only eq. (1.7) really needs to be fixed. In addition,

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[9]Here and in what follows g denotes the gauge coupling constant.

there may be less symmetric vacua when e.g., a non-vanishing *Fayet-Iliopoulos* (FI) term is present, i.e. $\langle D^{ij} \rangle = \xi^{ij} \neq 0$. Though a FI-term is usually associated with a spontaneous or soft breaking of supersymmetry, it does not imply the supersymmetry breaking automatically. We will only consider a FI term for the fictitious (i.e. non-dynamical) $N = 2$ vector multiplet to be introduced as a Lagrange multiplier $N = 2$ superfield (sect. 6).

2   Gauge LEEA in Coulomb branch

Seiberg and Witten [1] gave a full solution to the holomorphic function $\mathcal{F}(W)$ by using certain physical assumptions about the global structure of the quantum moduli space $\mathcal{M}_{\text{qu}}$ of vacua *and* electric-magnetic duality, i.e. not from the first principles. Their main assumption was the precise value of the Witten index [10], $\Delta_W = 2$, which implies just two physical singularities in $\mathcal{M}_{\text{qu}}$. The electric-magnetic duality (also known as *S-duality*) was used in ref. [1] to connect the weak and strong coupling regions of $\mathcal{M}_{\text{qu}}$.

The Seiberg-Witten solution [1] in the simplest case of the $SU(2)$ gauge group (no fundamental $N = 2$ matter) reads

$$a_D(u) = \frac{\sqrt{2}}{\pi} \int_1^u \frac{dx \sqrt{x - u}}{\sqrt{x^2 - 1}} \ , \quad a(u) = \frac{\sqrt{2}}{\pi} \int_{-1}^1 \frac{dx \sqrt{x - u}}{\sqrt{x^2 - 1}} \ , \tag{2.1}$$

where the renormalization-group independent (Seiberg-Witten) scale $\Lambda^2 = 1$ and, by definition,

$$a_D = \frac{\partial \mathcal{F}(a)}{da} \ . \tag{2.2}$$

The solution (2.1) is thus written down in the parametric form. Its holomorphic parameter u can be identified with the second Casimir eivenvalue, $u = \langle \text{tr}\, a^2 \rangle$, that parameterizes $\mathcal{M}_{\text{qu}}$. We find convenient to use the same lower-case letter (a) to denote the leading component (scalar field) of the $N = 2$ vector multiplet and its expectation value (complex constant) simultaneously. The holomorphic function $\mathcal{F}$ is thus defined over the quantum moduli space of vacua, while the S-duality can be identified with the action of the modular group $SL(2, \mathbf{Z})$ there [1]. The monodromies of the multi-valued function $\mathcal{F}$ around the singularities are supplied by the perturbative β-functions, whereas the whole function $\mathcal{F}$ is a (unique) solution to the corresponding Riemann-Hilbert problem. [11]

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[10]A formal derivation of Witten's index Δ_W from the path integral is plagued with ambiguities.

[11]See refs. [19, 11] for a review.

In order to make contact with our general discussion in sect. 1, let's consider an expansion of the SW solution in the semiclassical region, i.e. when $|W| \gg \Lambda$,

$$\mathcal{F}(W) = \frac{i}{2\pi} W^2 \log \frac{W^2}{\Lambda^2} + \frac{1}{4\pi i} W^2 \sum_{m=1}^{\infty} c_m \left(\frac{\Lambda^2}{W^2} \right)^{2m} , \qquad (2.3)$$

where we have restored the Λ-dependence and written down the interacting terms only. It is now obvious that the first term in eq. (2.3) represents the perturbative (one-loop) contribution whereas the rest is just the sum over the non-perturbative instanton contributions (see subsect. 2.1). It is straightforward to calculate the numerical coefficients $\{c_m\}$ from the explicit solution (2.1) and (2.2) [20]:

$$\begin{array}{c|cccccc} m & 1 & 2 & 3 & 4 & 5 & \dots \\ \hline c_m & 1/2^5 & 5/2^{14} & 3/2^{18} & 1469/2^{31} & 4471/5 \cdot 2^{34} & \dots \end{array} \qquad (2.4)$$

From the technical point of view, the SW solution is nothing but eq. (2.4). It is a challenge for field theorists to reproduce this solution from the first principles.

2.1   On instanton calculations

The SW solution predicts that the non-perturbative holomorphic contributions to the $N = 2$ vector gauge LEEA are entirely due to instantons. It is therefore quite natural to try to reproduce them 'from the first principles', e.g. from the path integral approach. The $N = 2$ supersymmetric instantons are solutions of the classical self-duality equations

$$F = {}^*F , \qquad i\gamma^{\mu} D_{\mu} \lambda = 0 , \qquad D^{\mu} D_{\mu} a = [\bar{\lambda}, \lambda] , \qquad (2.5)$$

whose Higgs scalar a approaches a non-vanishing constant at the spacial infinity so that the whole configuration has a non-vanishing topological charge $m \in \mathbf{Z}$.

From the path-integral point of view, the sum over instantons should be of the form

$$\mathcal{F}_{\text{inst.}} = \sum_{m=1}^{\infty} \mathcal{F}_m , \quad \text{where} \quad \mathcal{F}_m = \int d\mu_{\text{inst.}}^{(m)} \exp \left[-S_{(m)-\text{inst.}} \right] . \qquad (2.6)$$

Each term $\mathcal{F}_m$ in this sum can be interpreted as the partition function in the multi(m)-instanton background. The non-trivial measure $d\mu_{\text{inst.}}^{(m)}$ in eq. (2.6) appears as the result of changing variables from the original fields to the collective instanton coordinates in the path integral, whereas the action $S_{(m)-\text{inst.}}$ is just the Euclidean action of an $N = 2$ superinstanton configuration of charge (m). More details about the instanton calculus can be found e.g., in ref. [21]. One usually assumes that the scalar surface

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term ($\sim \operatorname{tr} \int dS^\mu a^\dagger D_\mu a$) is the only relevant term in the action $S_{(m)-\text{inst.}}$ that contributes. In particular, the bosonic and fermionic determinants, which always appear in the saddle-point expansion and describe small fluctuations of the fields, actually cancel in a supersymmetric self-dual gauge background [22]. Supersymmetry is thus also in charge for the absence of infra-red divergences present in the determinants.

The functional dependence $\mathcal{F}_m(a)$ easily follows from the integrated renormalization group (RG) equation for the one-loop β-function,

$$\exp\left(-\frac{8\pi^2 m}{g^2}\right) = \left(\frac{\Lambda}{a}\right)^{4m} , \tag{2.7}$$

and dimensional reasons as follows:

$$\mathcal{F}_m(a) = \frac{a^2}{4\pi i}\left(\frac{\Lambda}{a}\right)^{4m} c_m , \tag{2.8}$$

as it should have been expected, up to a numerical coefficient c_m. It is therefore the exact values of the coefficients $\{c_m\}$ that is the issue here, as was already noticed above. Their evaluation can thus be reduced to the problem of calculating the finite-dimensional multi-instanton measure $\{d\mu_{\text{inst.}}^{(m)}\}$.

A straightforward computation of the measure naively amounts to an explicit solution of the $N = 2$ supersymmetric self-duality equations in terms of the collective $N = 2$ instanton coordinates for any positive integer instanton charge. As is well known, the Yang-Mills self-duality differential equations of motion (as well as their supersymmetric counterparts) can be reduced to the purely *algebraic* (though highly non-trivial) set of equations when using the standard ADHM construction [23]. Unfortunately, an explicit solution to the algebraic ADHM equations is known for only $m = 1$ [24] and $m = 2$ [25], but it is unknown for $m > 2$. Nevertheless, as was recently demonstrated by Dorey, Khoze and Mattis [26], the correct multi-instanton measure for any instanton number can be fixed indirectly, by imposing $N = 2$ supersymmetry and the cluster decomposition requirements together, and without using the electric-magnetic duality ! This is closed enough to a derivation 'from the first principles'. In particular, in the Seiberg-Witten model with the $SU(2)$ gauge group considered above, there exists an instanton solution for $\{c_m\}$ in quadratures [26]. It was demonstrated in refs. [27] that the leading instanton corrections ($m = 1, 2$) agree with the exact Seiberg-Witten solution of eq. (2.4).

2.2   Seiberg-Witten curve

From the mathematical point of view, the Seiberg-Witten exact solution (2.1) in the case of two colors ($N_c = 2$) or the $SU(2)$ gauge group is a solution to the standard

Riemann-Hilbert problem of fixing a holomorphic multi-valued function $\mathcal{F}$ by its given monodromy and singularities. The number (and nature) of the singularities is the physical input: they are identified with the appearance of massless non-perturbative BPS-like physical states (dyons) like the famous t'Hooft-Polyakov magnetic monopole. The monodromies are supplied by perturbative beta-functions and S-duality (see ref. [11] for an introduction).

The $SU(2)$ solution (2.1) can be encoded in terms of the auxiliary (Seiberg-Witten) *elliptic curve* (or torus) Σ_{SW} defined by the algebraic equation [1]:

$$\Sigma_{\text{SW}} : \qquad y^2 = (v^2 - u)^2 - \Lambda^4 \ . \tag{2.9}$$

The multi-valued functions $a_D(u)$ and $a(u)$ then appear by integration of a certain abelian (Seiberg-Witten) differential λ_{SW} (of the 3rd kind) over the torus periods A and B of Σ_{SW}:

$$a(u) = \oint_A \lambda_{\text{SW}} \ , \qquad a_D(u) = \oint_B \lambda_{\text{SW}} \ , \tag{2.10}$$

while the SW differential λ_{SW} itself is simply related to a unique holomorphic 1-form ω on the torus Σ_{SW},

$$\frac{\partial \lambda_{\text{SW}}}{\partial u} = \omega \ , \qquad \omega \equiv \frac{dv}{y(v,u)} \ . \tag{2.11}$$

In the case of eq. (2.9) one easily finds that $\lambda_{\text{SW}} = v^2 dv / y(v, u)$ up to a total derivative.

The fundamental relation to the theory of Riemann surfaces can be generalized further to more general simply-laced gauge groups and $N = 2$ super-QCD as well [28, 29]. For instance, a solution to the LEEA of the purely $N = 2$ gauge theory with the gauge group $SU(N_c)$ is encoded in terms of a *hyperelliptic curve* of genus $(N_c - 1)$, whose algebraic equation reads [28]

$$\Sigma_{\text{SW}} : \qquad y^2 = W^2_{A_{N_c-1}}(v, \vec{u}) - \Lambda^{2N_c} \ . \tag{2.12}$$

The polynomial $W_{A_{N_c-1}}(v, \vec{u})$ is known in mathematics [30] as the *simple singularity* associated with $A_{N_c-1} \sim SU(N_c)$. In two-dimensional conformal field theory, the same polynomial is known as the $N = 2$ supersymmetric *Landau-Ginzburg* potential [31]. Its explicit form is given by

$$W_{A_{N_c-1}}(v, \vec{u}) = \sum_{l=1}^{N_c} \left(v - \vec{\lambda}_l \cdot \vec{a} \right) = v^{N_c} - \sum_{l=0}^{N_c-2} u_{l+2}(\vec{a}) v^{N_c-2-l} \ , \tag{2.13}$$

where $\vec{\lambda}_l$ are the weights of $SU(N_c)$ in the fundamental representation, and $\vec{u}$ are the Casimir eigenvalues, i.e. the Weyl group-invariant polynomials in $\vec{a}$ to be constructed by a classical Miura transformation [11]. The simple singularity seems to be the only remnant of the fundamental non-abelian gauge symmetry in the Coulomb branch.

Adding fundamental $N = 2$ matter does not pose a problem in calculating the corresponding Seiberg-Witten curve. The result reads [29]

$$\Sigma_{\mathrm{SW}} \; : \qquad y^2 = W^2_{A_{N_c-1}}(v, \vec{u}) - \Lambda^{2N_c-N_f} \prod_{j=1}^{N_f} (v - m_j) \; , \qquad (2.14)$$

where $\{m_j\}$ are the bare hypermultiplet masses of N_f hypermultiplets ($N_f < N_c$), in the fundamental representation of the gauge group $SU(N_c)$.

In the canonical first homology basis (A_α, B^β), $\alpha, \beta = 1, \ldots, g$, of the general genus-$g$ Riemann surface Σ_{SW}, the multi-valued sections $a(u)$ and $a_{\mathrm{D}}(u)$ are determined by the equations

$$\frac{\partial a_\alpha}{\partial u_\beta} = \oint_{A_\alpha} \omega^\beta \; , \qquad \frac{\partial a_{\mathrm{D}}^\alpha}{\partial u_\beta} = \oint_{B^\alpha} \omega^\beta \; , \qquad (2.15)$$

in terms of g independent holomorphic 1-forms ω^α on Σ_{SW}. Eq. (2.15) is quite similar to eq. (2.10) after taking into account that the Seiberg-Witten differential λ_{SW} is defined by a relation very similar to that of eq. (2.11), namely,

$$\frac{\partial \lambda_{\mathrm{SW}}}{\partial u_\alpha} = \omega^\alpha \; . \qquad (2.16)$$

The minimal data $(\Sigma_{\mathrm{SW}}, \lambda_{\mathrm{SW}})$ needed to reproduce the Seiberg-Witten exact solution to the *four-dimensional* LEEA can be associated with a certain *two-dimensional* integrable system [32, 33]. In particular, the SW potential $\mathcal{F}$ appears to be a solution to the Dijkgraaf-Verlinde-Verlinde-Witten-type [34] non-linear differential equations known in the two-dimensional (conformal) topological field theory [35]:

$$\mathcal{F}_i \mathcal{F}_k^{-1} \mathcal{F}_j = \mathcal{F}_j \mathcal{F}_k^{-1} \mathcal{F}_i \; , \quad \text{where} \quad (\mathcal{F}_i)_{jk} \equiv \frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k} \; . \qquad (2.17)$$

There also exists another non-trivial equation for $\mathcal{F}$ which is a consequence of the anomalous (chiral) $N = 2$ superconformal Ward identities in $D = 4$ [36].

Though the mathematical relevance of the Seiberg-Witten curve is quite clear from what was already written above, its geometrical origin and physical interpretation are still obscure at this point. This issue can be most naturally understood in the context of M-Theory-based brane technology (sect. 4).

3   Hypermultiplet LEEA in the Coulomb branch

The previous sect. 2 was entirely devoted to the *holomorphic* function $\mathcal{F}$ appearing in the gauge LEEA (1.4) in the Coulomb branch. In this sect. 3 we are going to discuss

another *analytic* function $\mathcal{K}$ dictating the hypermultiplet LEEA (1.5). The function $\mathcal{K}$ is known as a *hyper-Kähler potential,* [12] and its role in the hypermultiplet LEEA is quite similar to that of $\mathcal{F}$ in the vector gauge LEEA. Since the very notion of the hyper-Kähler potential requires an introduction of the *harmonic superspace* (HSS), we begin with a brief introduction into the $N = 2$ HSS in the next subsect. 3.1 (see refs. [17, 38] for more).

Being intimately related to the hyper-Kähler geometry, the use of a hyper-Kähler potential is by no means limited to the $N = 2$ extended supersymmetry in four spacetime dimensions. It fact, it is necessary to make the hyper-Kähler structure manifest in field theory as well as in brane technology. A hyper-Kähler structure naturally appears in M-Theory (subsect. 4.5), and it is also known to be the important ingredient of an integrable dynamical system [32, 33]. For instance, the use of a hyper-Kähler potential allows one to make manifest the symmetry enhancement in the case of two 6-branes 'on top of each other' (sect. 4.7).

3.1 $N = 2$ harmonic superspace

The supersymmetric field theories can be formulated in superspace [15], usually in terms of constrained superfields. Unfortunately, the constraints defining a (non-abelian) $N = 2$ vector multiplet or a hypermultiplet in the ordinary $N = 2$ superspace in $D = 4$ do not have a manifestly holomorphic (or analytic) structure. Accordingly, they do not have a simple solution in terms of unconstrained $N = 2$ superfields which are needed for supersymmetric quantization. The situation is even more dramatic for the hypermultiplets whose known off-shell formulations in the ordinary $N = 2$ superspace are not universal so that their practical meaning is limited.

In the HSS formalism, the standard $N = 2$ superspace [13] $Z^M = (x^m, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha} i})$, $\alpha = 1, 2$, and $i = 1, 2$, is extended by adding the bosonic variables (or 'zweibeins') $u^{\pm i}$ parameterizing the sphere $S^2 \sim SU(2)/U(1)$. By using these extra variables one can make manifest the hidden analyticity structure of all the standard $N = 2$ superspace constraints as well as find their solutions in terms of unconstrained (analytic) superfields. The harmonic variables have the following fundamental properties:

$$\begin{pmatrix} u^{+i} \\ u^{-i} \end{pmatrix} \in SU(2) , \quad \text{so that} \quad u^{+i}u_i^- = 1 , \quad \text{and} \quad u^{+i}u_i^+ = u^{-i}u_i^- = 0 . \quad (3.1)$$

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[12]Any $D = 4$, globally $N = 2$ supersymmetric NLSM with the highest physical spin 1/2 necessarily has a hyper-Kähler metric in its kinetic terms [37].

[13]Since our $D = 4$ spacetime is supposed to be flat in this section we identify the flat ($m = 0, 1, 2, 3$) and curved ($\mu = 0, 1, 2, 3$) spacetime vector indices.

Instead of using an explicit parameterization of the sphere S^2, it is convenient to deal with functions of zweibeins, that carry a definite $U(1)$ charge q to be defined by $q(u_i^{\pm}) = \pm 1$, and use the following integration rules [17]:

$$\int du = 1 , \qquad \int du\, u^{+(i_1} \cdots u^{+i_m} u^{-j_1} \cdots u^{-j_n)} = 0 , \quad \text{when} \quad m + n > 0 . \quad (3.2)$$

It is obvious that any integral over a $U(1)$-charged quantity vanishes.

The usual complex conjugation does not preserve analyticity (see below). However, when being combined with another (star) conjugation that only acts on the $U(1)$ indices as $(u_i^+)^* = u_i^-$ and $(u_i^-)^* = -u_i^+$, it does preserve analyticity. One easily finds [17]

$$\overset{*}{\overline{u^{\pm i}}} = -u_i^{\pm} , \qquad \overset{*}{\overline{u_i^{\pm}}} = u^{\pm i} . \quad (3.3)$$

The covariant derivatives with respect to the zweibeins, which preserve the defining conditions (3.1), are given by

$$D^{++} = u^{+i} \frac{\partial}{\partial u^{-i}} , \quad D^{--} = u^{-i} \frac{\partial}{\partial u^{+i}} , \quad D^0 = u^{+i} \frac{\partial}{\partial u^{+i}} - u^{-i} \frac{\partial}{\partial u^{-i}} . \quad (3.4)$$

It is easy to check that they satisfy the $SU(2)$ algebra,

$$[D^{++}, D^{--}] = D^0 , \quad [D^0, D^{\pm\pm}] = \pm 2 D^{\pm\pm} . \quad (3.5)$$

The key feature of the $N = 2$ HSS is the existence of the so-called *analytic* subspace parameterized by the coordinates

$$(\zeta, u) = \left\{ x_A^m = x^m - 2i\theta^{(i} \sigma^m \bar{\theta}^{j)} u_i^+ u_j^- , \ \theta_\alpha^+ = \theta_\alpha^i u_i^+ , \ \bar{\theta}_{\dot{\alpha}}^+ = \bar{\theta}_{\dot{\alpha}}^i u_i^+ , \ u_i^{\pm} \right\} , \quad (3.6)$$

which is invariant under $N = 2$ supersymmetry and closed under the combined conjugation of eq. (3.3) [17]. This allows one to define *analytic* superfields of any non-negative and integer $U(1)$ charge q, by the analyticity conditions

$$D_\alpha^+ \phi^{(q)} = \bar{D}_{\dot{\alpha}}^+ \phi^{(q)} = 0 , \quad \text{where} \quad D^+{}_\alpha = D_\alpha^i u_i^+ \quad \text{and} \quad \bar{D}_{\dot{\alpha}}^+ = \bar{D}_{\dot{\alpha}}^i u_i^+ . \quad (3.7)$$

An analytic measure is given by $d\zeta^{(-4)} du \equiv d^4 x_A d^2\theta^+ d^2\bar{\theta}^+ du$. It carries the $U(1)$ charge (-4), whereas the full neutral measure in the $N = 2$ HSS takes the form

$$d^4 x d^4\theta d^4\bar{\theta} du = d\zeta^{(-4)} du (D^+)^4 , \quad (3.8)$$

where

$$(D^+)^4 = \frac{1}{16}(D^+)^2 (\bar{D}^+)^2 = \frac{1}{16}(D^{+\alpha} D_\alpha^+)(\bar{D}_{\dot{\alpha}}^+ \bar{D}^{+\dot{\alpha}}) . \quad (3.9)$$

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In the analytic subspace, the harmonic derivative D^{++} reads

$$D^{++}_{analytic} = D^{++} - 2i\theta^+ \sigma^m \bar{\theta}^+ \partial_m \ , \tag{3.10}$$

it preserves analyticity, and it allows one to integrate by parts. Both the original (central) basis and the analytic one can be used on equal footing in the HSS. In what follows we omit the subscript *analytic* at the covariant derivatives in the analytic basis, in order to simplify the notation.

It is the advantage of the analytic $N = 2$ HSS compared to the ordinary $N = 2$ superspace that both an off-shell $N = 2$ vector multiplet and an off-shell hypermultiplet can be introduced there on equal footing. There exist two off-shell hypermultiplet versions in HSS, which are dual to each other. The so-called *Fayet-Sohnius-type* (FS) hypermultiplet is defined as an unconstrained complex analytic superfield q^+ of $U(1)$-charge $(+1)$, whereas its dual, called the *Howe-Stelle-Townsend-type* (HST) hypermultiplet, is a real unconstrained analytic superfield ω with the vanishing $U(1)$-charge. [14] The on-shell physical components of the FS hypermultiplet comprise an $SU(2)_A$ doublet of complex scalars and a Dirac spinor which is a singlet w.r.t. the $SU(2)_A$. The on-shell physical components of the HST hypermultiplet comprise real singlet and triplet of scalars, and a doublet of chiral spinors. The FS hypermultiplet is natural for describing a charged $N = 2$ matter (e.g. in the Coulomb branch), whereas the HST hypermultiplet is natural for describing a neutral $N = 2$ matter in the Higgs branch. Similarly, an $N = 2$ vector multiplet is described by an unconstrained analytic superfield V^{++} of the $U(1)$-charge $(+2)$. The V^{++} is real in the sense $\overline{V^{++}}^* = V^{++}$, and it can be naturally introduced as a connection to the harmonic derivative D^{++}.

A free FS hypermultiplet HSS action is given by

$$S[q] = -\int d\zeta^{(-4)} du \ \overset{*}{\bar{q}}{}^+ D^{++} q^+ \ , \tag{3.11}$$

whereas its minimal coupling to an $N = 2$ gauge superfield reads

$$S[q, V] = -\mathrm{tr} \int d\zeta^{(-4)} du \ \overset{*}{\bar{q}}{}^+ (D^{++} + iV^{++}) q^+ \ , \tag{3.12}$$

where the both superfields, q^+ and V^{++}, are now Lie algebra-valued.

It is not difficult to check that the free FS hypermultiplet equations of motion, $D^{++}q^+ = 0$, imply $q^+ = q^i(Z)u_i^+$ as well as the usual (on-shell) Fayet-Sohnius con-

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[14]It is worth mentioning here that the FS and HST multiplets were originally introduced in the ordinary $N = 2$ superspace [13, 14], whereas we use the same names to denote *different* $N = 2$ *harmonic* superfields, just in order to keep track of their on-shell connection.

straints [13] in the ordinary $N = 2$ superspace,

$$D_\alpha^{(i} q^{j)}(Z) = D_{\dot\alpha}^{(i} q^{j)}(Z) = 0 \ . \tag{3.13}$$

Similarly, a free HSS action of the HST hypermultiplet is given by

$$S[\omega] = -\tfrac{1}{2} \int d\zeta^{(-4)} du \, (D^{++}\omega)^2 \ , \tag{3.14}$$

and it is equivalent (dual) to the standard $N = 2$ tensor (linear) multiplet action (see subsect. 3.3).

The standard *Grimm-Sohnius-Wess* (GSW) constraints [16] defining the $N = 2$ super-Yang-Mills theory in the ordinary $N = 2$ superspace imply the existence of a (covariantly) chiral [15] and gauge-covariant $N = 2$ SYM field strength W satisfying, in addition, the reality condition (or the Bianchi 'identity')

$$\mathcal{D}^\alpha_{(i} \mathcal{D}_{j)\alpha} W = \bar{\mathcal{D}}_{\dot\alpha(i} \bar{\mathcal{D}}^{\dot\alpha}_{j)} \bar{W} \ . \tag{3.15}$$

Unlike the $N = 1$ SYM theory, an $N = 2$ supersymmetric solution to the non-abelian $N = 2$ SYM constraints in the ordinary $N = 2$ superspace is not known in an analytic form. It is the $N = 2$ HSS reformulation of the $N = 2$ SYM theory that makes it possible [17]. An exact non-abelian relation between the constrained, harmonic-independent superfield strength W and the unconstrained analytic (harmonic-dependent) superfield V^{++} is given in refs. [17, 38], and it is highly non-linear. It is merely its abelian version that is needed for calculating the perturbative LEEA in the Coulomb branch. The abelian relation is given by

$$W = \frac{1}{4}\{\bar{\mathcal{D}}^+_{\dot\alpha}, \bar{\mathcal{D}}^{-\dot\alpha}\} = -\frac{1}{4}(\bar{D}^+)^2 A^{--} \ , \tag{3.16}$$

where the non-analytic harmonic superfield connection $A^{--}(Z, u)$ to the derivative D^{--} has been introduced, $\mathcal{D}^{--} = D^{--} + iA^{--}$. As a consequence of the $N = 2$ HSS abelian constraint $[\mathcal{D}^{++}, \mathcal{D}^{--}] = \mathcal{D}^0 = D^0$, the connection A^{--} satisfies the relation

$$D^{++}A^{--} = D^{--}V^{++} \ , \tag{3.17}$$

whereas eq. (3.15) can be rewritten to the form

$$(D^+)^2 W = (\bar{D}^+)^2 \bar{W} \ . \tag{3.18}$$

A solution to the A^{--} in terms of the analytic unconstrained superfield V^{++} easily follows from eq. (3.17) when using the identity [38]

$$D_1^{++}(u_1^+ u_2^+)^{-2} = D_1^{--}\delta^{(2,-2)}(u_1, u_2) \ , \tag{3.19}$$

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[15]A covariantly-chiral superfield can be transformed into a chiral superfield by field redefinition.

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where we have introduced the harmonic delta-function $\delta^{(2,-2)}(u_1, u_2)$ and the harmonic distribution $(u_1^+ u_2^+)^{-2}$ according to their definitions in refs. [17, 38], hopefully, in the self-explaining notation. One finds [39]

$$A^{--}(z, u) = \int dv \, \frac{V^{++}(z, v)}{(u^+ v^+)^2} \ , \tag{3.20}$$

and

$$W(z) = -\frac{1}{4} \int du (\bar{D}^-)^2 V^{++}(z, u) \ , \quad \bar{W}(z) = -\frac{1}{4} \int du (D^-)^2 V^{++}(z, u) \ , \tag{3.21}$$

by using the identity

$$u_i^+ = v_i^+ (v^- u^+) - v_i^- (u^+ v^+) \ , \tag{3.22}$$

which is the obvious consequence of the definitions (3.1).

The free equations of motion of an $N = 2$ vector multiplet are given by the vanishing analytic superfield

$$(D^+)^4 A^{--}(Z, u) = 0 \ , \tag{3.23}$$

while the corresponding action reads [39]

$$\begin{aligned} S[V] = &\frac{1}{4} \int d^4x \, d^4\theta \, W^2 + \text{h.c.} = \frac{1}{2} \int d^4x \, d^4\theta \, d^4\bar{\theta} \, du \, V^{++}(Z, u) A^{--}(Z, u) \\ = &\frac{1}{2} \int d^4x \, d^4\theta \, d^4\bar{\theta} \, du_1 du_2 \, \frac{V^{++}(Z, u_1) V^{++}(Z, u_2)}{(u_1^+ u_2^+)^2} \ . \end{aligned} \tag{3.24}$$

In a WZ-like gauge, the abelian analytic HSS prepotential V^{++} amounts to the following explicit expression [17]:

$$\begin{aligned} V^{++}(x_A, \theta^+, \bar{\theta}^+, u) = &\bar{\theta}^+ \bar{\theta}^+ a(x_A) + \bar{a}(x_A) \theta^+ \theta^+ - 2i\theta^+ \sigma^m \bar{\theta}^+ V_m(x_A) \\ &+ \bar{\theta}^+ \bar{\theta}^+ \theta^{\alpha+} \psi_\alpha^i (x_A) u_i^- + \theta^+ \theta^+ \bar{\theta}_{\dot{\alpha}}^+ \bar{\psi}^{\dot{\alpha} i}(x_A) u_i^- \\ &+ \theta^+ \theta^+ \bar{\theta}^+ \bar{\theta}^+ D^{(ij)}(x_A) u_i^- u_j^- \ , \end{aligned} \tag{3.25}$$

where $(a, \psi_\alpha^i, V_m, D^{ij})$ are the usual $N = 2$ vector multiplet components [16].

The (BPS) mass of a hypermultiplet can only come from the central charges of the $N = 2$ SUSY algebra since, otherwise, the number of the massive hypermultiplet components has to be increased. The most natural way to introduce central charges $(Z, \bar{Z})$ is to identify them with spontaneously broken $U(1)$ generators of dimensional reduction from six dimensions via the Scherk-Schwarz mechanism [40]. Indeed, after being written down in six dimensions, eq. (3.10) implies an additional 'connection' term in the associated four-dimensional harmonic derivative,

$$\mathcal{D}^{++} = D^{++} + v^{++} \ , \quad \text{where} \quad v^{++} = i(\theta^+ \theta^+)\bar{Z} + i(\bar{\theta}^+ \bar{\theta}^+)Z \ . \tag{3.26}$$

Comparing eq. (3.26) with eqs. (3.12) and (3.21) clearly shows that the $N = 2$ central charges can be equivalently treated as a non-trivial $N = 2$ gauge background with the covariantly constant chiral superfield strength

$$\langle W \rangle = \langle a \rangle = Z \ , \tag{3.27}$$

where eq. (3.25) has been used too. See refs. [41, 42, 43, 44] for more details.

3.2   Taub-NUT metric or KK-monopole

Since the HSS formulation of hypermultiplets has the manifest off-shell $N = 2$ supersymmetry, it is perfectly suitable for discussing possible hypermultiplet self-interactions which are highly restricted by $N = 2$ supersymmetry. Moreover, the manifestly $N = 2$ supersymmetric Feynman rules can be derived in HSS. The latter can be used to actually calculate the perturbative hypermultiplet LEEA (see below).

To illustrate the power of HSS, let's consider a single FS hypermultiplet for simplicity. Its free action in HSS can be rewritten in the pseudo-real notation, $q_a^+ = (q^+, \overset{*}{\bar{q}}{}^+)$, $q^a = \varepsilon^{ab} q_b$, $a = 1, 2$, as follows:

$$S[q] = -\frac{1}{2} \int_{\text{analytic}} q^{a+} \mathcal{D}^{++} q_a^+ \ , \tag{3.28}$$

where the derivative $\mathcal{D}^{++}$ (in the analytic basis) includes central charges in accordance with eq. (3.26). It is obvious from eq. (3.28) that the action $S[q]$ has the extended internal symmetry given by

$$SU(2)_A \otimes SU(2)_{\text{PG}} \ , \tag{3.29}$$

where the $SU(2)_A$ is the automorphism symmetry of the $N = 2$ supersymmetry algebra, [16] whereas the additional *Pauli-Gürsey* $SU(2)_{\text{PG}}$ symmetry acts on the extra indices (a, b) only. Adding a minimal interaction with an abelian $N = 2$ vector superfield V^{++} in eq. (3.28) obviously breaks the internal symmetry (3.29) down to a subgroup

$$SU(2)_A \otimes U(1)_{\text{PG}} \ . \tag{3.30}$$

It is now easy to see that the *only* FS hypermultiplet self-interaction consistent with the internal symmetry (3.30) is given by the hyper-Kähler potential

$$\mathcal{K}^{(+4)} = \frac{\lambda}{2} \left(\overset{*}{\bar{q}}{}^+ q^+ \right)^2 \ , \tag{3.31}$$

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[16]It is easy to keep track of the $SU(2)_A$ symmetry in the $N = 2$ HSS where this symmetry amounts to the absence of an explicit dependence of a HSS lagrangian upon the harmonic variables $u_i^{\pm}$.
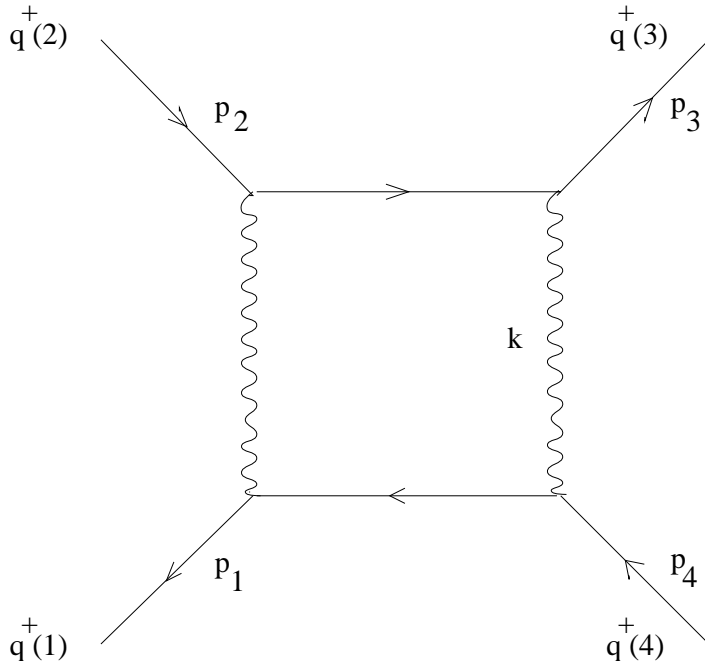
Fig. 2 . The one-loop harmonic supergraph contributing

to the induced hypermultiplet self-interaction.

since it is the only admissible term of the $U(1)$-charge $(+4)$ which can be added to the FS hypermultiplet action (3.28). We thus get the answer for the LEEA of a single matter hypermultiplet in the Coulomb branch almost for free, up to the induced NLSM coupling constant λ.

Similarly, the unique FS hypermultiplet self-interaction in the $N = 2$ super-QCD with $N_c = 3$ colors and N_f flavors, and vanishing bare hypermultiplet masses, which is consistent with the $SU(N_f) \otimes SU(2)_A \otimes U(1)^2$ symmetry, is given by

$$\mathcal{K}_{\text{QCD}}^{(+4)} = \frac{\lambda}{2} \sum_{i,j=1}^{N_f} \left(\overset{*}{\tilde{q}}{}^{i+} \cdot q_j^+ \right) \left(\overset{*}{\tilde{q}}{}^{j+} \cdot q_i^+ \right) \;, \qquad (3.32)$$

where the dots stand for contractions of color indices.

The induced coupling constant λ in eq. (3.31) is entirely determined by the one-loop HSS graph shown in Fig. 2. Since the result vanishes $(\lambda = 0)$ in the absence of central charges, [17] let's assume that $Z = \langle a \rangle \neq 0$, i.e. we are in the Coulomb branch. The free HSS action of an $N = 2$ vector multiplet and that of a hypermultiplet given above are enough to compute the corresponding $N = 2$ superpropagators. The

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[17]The same conclusion also follows from the $N = 1$ superspace calculations [45].

$N = 2$ vector multiplet action takes the particularly simple form in the $N = 2$ super-Feynman gauge (there are no central charges for the $N = 2$ vector multiplet),

$$S[V]_{\text{Feynman}} = \frac{1}{2} \int_{\text{analytic}} V^{++} \square V^{++} \; , \tag{3.33}$$

so that the corresponding analytic HSS propagator (the wave lines in Fig. 2) reads

$$i \left\langle V^{++}(1) V^{++}(2) \right\rangle = \frac{1}{\square_1} (D_1^+)^4 \delta^{12}(Z_1 - Z_2) \delta^{(-2,2)}(u_1, u_2) \; , \tag{3.34}$$

where the harmonic delta-function $\delta^{(-2,2)}(u_1, u_2)$ has been introduced [38]. The FS hypermultiplet HSS propagator (solid lines in Fig. 2) with non-vanishing central charges is more complicated [39, 44]:

$$i \left\langle q^+(1) q^+(2) \right\rangle = \frac{-1}{\square_1 + a\bar{a}} \frac{(D_1^+)^4 (D_2^+)^4}{(u_1^+ u_2^+)^3} e^{\tau_3 [v(2) - v(1)]} \delta^{12}(Z_1 - Z_2) \; , \tag{3.35}$$

where v is the so-called 'bridge' satisfying the equation $\mathcal{D}^{++} e^v = 0$. One easily finds that

$$iv = -a(\bar{\theta}^+ \bar{\theta}^-) - \bar{a}(\theta^+ \theta^-) \; . \tag{3.36}$$

The rest of the $N = 2$ HSS Feynman rules is very similar to that of the ordinary $(N = 0)$ Quantum Electrodynamics (QED).

A calculation of the HSS graph in Fig. 2 is now straightforward, while the calculational details are given in ref. [44]. One finds the predicted form of the induced hyper-Kähler potential as in eq. (3.31) indeed, with the induced NLSM coupling constant given by

$$\lambda = \frac{g^4}{\pi^2} \left[\frac{1}{m^2} \ln \left(1 + \frac{m^2}{\Lambda^2} \right) - \frac{1}{\Lambda^2 + m^2} \right] \; , \tag{3.37}$$

where g is the gauge coupling constant, $m^2 = |a|^2$ is the hypermultiplet BPS mass, and Λ is the IR-cutoff parameter. Note that $\lambda \to 0$ when the central charge $a \to 0$.

The only technical problem left is how to decode the HSS result (3.31) in the conventional component form. In other words, we have to deduce an explicit hyper-Kähler metric which corresponds to the hyper-Kähler potential (3.31). A general procedure of getting the component form of the bosonic NLSM from a hypermultiplet self-interaction in HSS consists of the following steps:

- expand the equations of motion in the Grassmann (anticommuting) coordinates, and ignore all the fermionic field components,

- solve the kinematical linear differential equations for all the auxiliary fields, thus eliminating the infinite tower of them in the harmonic expansion of the hypermultiplet HSS analytic superfields,

- substitute the solution back into the HSS hypermultiplet action, and integrate over all the anitcommuting and harmonic HSS coordinates.

Of course, it is not always possible to actually perform this procedure. For instance, just the second step above would amount to solving infinitely many linear differential equations altogether. However, just in the case of eq. (3.31), the explicit solution exists [46, 44]. When using the parametrization

$$
q^+\big|_{\theta=0} = f^i(x)u_i^+ \exp\left[\lambda f^{(j)}(x)\bar{f}^{k)}(x)u_j^+ u_k^-\right] \ , \tag{3.38}
$$

one finds the following bosonic $D = 4$ NLSM action (see Appendix A for details of calculation):

$$
S_{\text{NLSM}} = \int d^4x \left\{ g_{ij}\partial_m f^i \partial^m f^j + \bar{g}^{ij}\partial_m \bar{f}_i \partial^m \bar{f}_j + h^i{}_j \partial_m f^j \partial^m \bar{f}_i - V(f) \right\} \ , \tag{3.39}
$$

whose metric is given by [46]

$$
g_{ij} = \frac{\lambda(2 + \lambda f\bar{f}}{4(1 + \lambda f\bar{f})}\bar{f}_i \bar{f}_j \ , \quad \bar{g}^{ij} = \frac{\lambda(2 + \lambda f\bar{f}}{4(1 + \lambda f\bar{f})}f^i f^j \ ,
$$

$$
h^i{}_j = \delta^i{}_j(1 + \lambda f\bar{f}) - \frac{\lambda(2 + \lambda f\bar{f})}{2(1 + \lambda f\bar{f}}f^i \bar{f}_j \ , \quad f\bar{f} \equiv f^i \bar{f}_i \ , \tag{3.40}
$$

and the induced scalar potential reads [44]

$$
V(f) = |Z|^2 \frac{f\bar{f}}{1 + \lambda f\bar{f}} \quad . \tag{3.41}
$$

In the form (3.40) the induced metric is apparently free from singularities.

It is usually non-trivial to compare a given metric with any standard hyper-Kähler metric since the metrics themselves are defined modulo field redefinitions, i.e. modulo four-dimensional diffeomorphisms in the case under considerarion. Fortunately, it is known how to transform the metric (3.40) to the standard *Euclidean Taub-NUT* (ETN) form :

$$
ds^2 = \frac{r + M}{2(r - M)}dr^2 + \frac{1}{2}(r^2 - M^2)(d\vartheta^2 + \sin^2\vartheta d\varphi^2) + 2M^2\left(\frac{r - M}{r + M}\right)(d\psi + \cos\vartheta d\varphi)^2 \ , \tag{3.42}
$$

by using the following change of variables [46]:

$$
f^1 = \sqrt{2M(r - M)}\cos\frac{\vartheta}{2}\exp\frac{i}{2}(\psi + \varphi) \ ,
$$

$$
f^2 = \sqrt{2M(r - M)}\sin\frac{\vartheta}{2}\exp\frac{i}{2}(\psi - \varphi) \ , \tag{3.43}
$$

$$ f\bar{f} = 2M(r - M) \ , \qquad r \geq M = \frac{1}{2\sqrt{\lambda}} \ , $$

where $M = \frac{1}{2}\lambda^{-1/2} \sim g^{-2}$ is the instanton mass. The ETN instanton is also known in the literature as the KK-instanton [47, 48, 49] (see sect. 4 for more).

We thus showed in this subsection that the induced NLSM metric of the hypermultiplet LEEA in the Coulomb branch is generated in the one-loop order of quantum perturbation theory, and it is given by the Taub-NUT or its higher-dimensional generalizations. The non-trivial scalar potential is also generated by one-loop quantum corrections, with the hypermultiplet (BPS) mass being unrenormalized as it should.

3.3 Duality transformation and $N = 2$ tensor multiplet

There exists an interesting connection between the FS hypermultiplet Taub-NUT self-interaction in the $N = 2$ *harmonic* superspace and the $N = 2$ tensor (or linear) multiplet self-interaction in the *ordinary* $N = 2$ superspace. Namely, the $N = 2$ supersymmetric Taub-NUT NLSM is equivalent to a sum of the naive (quadratic in the fields and non-conformal) and *improved* (non-polynomial in the fields and $N = 2$ superconformally invariant) actions for the $N = 2$ tensor multiplet in the ordinary $N = 2$ superspace !

The $N = 2$ *tensor* multiplet in the ordinary $N = 2$ superspace is defined by the constraints

$$ D_\alpha{}^{(i}L^{ik)}(Z) = \bar{D}_{\dot{\alpha}}{}^{(i}L^{jk)}(Z) = 0 \ , \tag{3.44} $$

and the reality condition

$$ \overline{L^{ij}} = \varepsilon_{ik}\varepsilon_{jl}L^{kl} \ . \tag{3.45} $$

Unlike the FS hypermultiplet in the ordinary $N = 2$ superspace, the constraints (3.44) are off-shell, i.e. they do not imply the equations of motion for the components of the $N = 2$ tensor multiplet. The $N = 2$ tensor multiplet itself can be identified as a restricted HST hypermultiplet (i.e. as an analytic ω superfield subject to extra off-shell constraints), while its $N = 2$ supersymmetric self-interactions are a subclass of those for ω [50]. The $N = 2$ tensor multiplet has $8_B \oplus 8_F$ off-shell components:

$$ \vec{L}, \quad \zeta_\alpha^i \ , \quad B \ , \quad E'_m = \frac{1}{2}\varepsilon_{mnpq}\partial_n E_{pq} \ , \tag{3.46} $$

where $\vec{L}$ is the scalar $SU(2)_A$ triplet, $\vec{L} = \mathrm{tr}(\vec{\tau}L)$ and $\vec{\tau}$ are Pauli matrices, ζ_α^i is a chiral spinor doublet, B is a complex auxiliary scalar, and E_{mn} is a gauge antisymmetric tensor whose field strength is E'_m.

Let's start with our induced hypermultiplet LEEA

$$S[q^+]_{\text{Taub-NUT}} = \int_{\text{analytic}} \left[\overset{*}{\overline{q}}{}^+ D^{++} q^+ + \frac{\lambda}{2} (q^+)^2 (\overset{*}{\overline{q}}{}^+)^2 \right] , \qquad (3.47)$$

and make the following substitution of the HSS superfield variables [50]:

$$\sqrt{\lambda} q^+ = -i \left(2u_1^+ + ig^{++} u_1^- \right) e^{-i\tilde{\omega}/2} , \quad \sqrt{\lambda} \overset{*}{\overline{q}}{}^+ = i \left(2u_2^+ - ig^{++} u_2^- \right) e^{i\tilde{\omega}/2} , \quad (3.48)$$

where

$$g^{++}(l, u) \equiv \frac{2(l^{++} - 2iu_1^+ u_2^+)}{1 + \sqrt{1 - 4u_1^+ u_2^+ u_1^- u_2^- - 2il^{++} u_1^- u_2^-}} , \qquad (3.49)$$

and (l^{++}, ω) are the new dimensionless analytic superfieds. It is not difficult to check that eqs. (3.48) and (3.49) imply, in particular, that

$$\lambda \overset{*}{\overline{q}}{}^+ q^+ = 2il^{++} , \qquad (3.50)$$

whereas the action (3.47) takes the form (after the rescaling $l^{++} \equiv \sqrt{\lambda} L^{++}$ and $\tilde{\omega} = \sqrt{\lambda} \omega$):

$$S[L^{++}; \omega]_{\text{Taub-NUT}} = S_{\text{free}}[L^{++}; \omega] + S_{\text{impr.}}[L^{++}] , \qquad (3.51)$$

where

$$S_{\text{free}}[L^{++}; \omega] = \frac{1}{2} \int_{\text{analytic}} \left[(L^{++})^2 + \omega D^{++} L^{++} \right] , \qquad (3.52)$$

and

$$S_{\text{impr.}}[L^{++}] = \frac{1}{2\lambda} \int_{\text{analytic}} \left[g^{++}(L; u) \right]^2 . \qquad (3.53)$$

The action (3.51) or (3.52) has the superfield ω as a Lagrange multiplier. Hence, on the one hand side, varying the action (3.51) with respect to ω yields the constraint

$$D^{++} L^{++} = 0 , \qquad (3.54)$$

which, in its turn, implies $L^{++} = u_i^+ u_j^+ L^{ij}(Z)$ *and* eq. (3.44). Therefore, the actions (3.51) and (3.52) describe an $N = 2$ tensor multiplet in the $N = 2$ HSS. On the other hand side, one can vary e.g. the action (3.52) with respect to L^{++} first. Then one finds that

$$L^{++} = D^{++} \omega . \qquad (3.55)$$

Hence, L^{++} can be removed in favor of ω. This example is a manifestation of the classical *duality* between the FS hypermultiplet q^+ and the HST hypermultiplet ω in the $N = 2$ HSS.

The action (3.53) describes the so-called *improved* $N = 2$ tensor multiplet [51]. It can be shown that it is fully invariant under the rigid $N = 2$ superconformal

symmetry, while the associated hyper-Kähler metric is equivalent to the flat metric up to a $D = 4$ diffeomorphism [51]. However, the sum of the actions (3.52) and (3.53) describes an interacting theory, and it is just the NLSM with the Taub-NUT instanton (or KK-monopole) metric.

Because of this connection between certain $N = 2$ supermultiplets and their self-interactions in the HSS, it should not be very surprising that the Taub-NUT self-interaction can also be reformulated in the *ordinary* $N = 2$ superspace in terms of the $N = 2$ tensor multiplet alone, just as the sum of its naive and improved actions. The most elegant formulation of the latter exists in the *projective* $N = 2$ superspace [52, 53] in which the harmonic variables are replaced by a single complex projective variable $\xi \in CP(1)$. Unlike the $N = 2$ HSS, the projective $N = 2$ superspace does not have to introduce extra auxiliary fields beyond those already present in the off-shell $N = 2$ tensor multiplet. The starting point are the defining constraints (3.44) for the $N = 2$ tensor multiplet in the ordinary $N = 2$ superspace. It is not difficult to check that they imply (see ref. [53] for more details and generalizations)

$$\nabla_\alpha G \equiv (D^1_\alpha + \xi D^2_\alpha)G = 0 , \qquad \Delta_{\dot\alpha} G \equiv (\bar D^1_{\dot\alpha} + \xi \bar D^2_{\dot\alpha})G = 0 , \qquad (3.56)$$

for *any* function $G(Q(\xi), \xi)$ which is a function of $Q(\xi) \equiv \xi_i \xi_j L^{ij}(Z)$ and $\xi_i \equiv (1, \xi)$ only.

It follows that we can build an $N = 2$ superinvariant just by integrating G over the rest of the $N = 2$ superspace coordinates in the directions which are 'orthogonal' to those in eq. (3.56), namely,

$$S_{\text{inv.}}[L] = \int d^4x \frac{1}{2\pi i} \oint_C d\xi \, \tilde\nabla^2 \tilde\Delta^2 G(Q, \xi) , \qquad (3.57)$$

where we have introduced the new derivatives

$$\tilde\nabla_\alpha = \xi D^1_\alpha - D^2_\alpha , \quad \tilde\Delta_{\dot\alpha} = \xi \bar D^1_{\dot\alpha} - \bar D^2_{\dot\alpha} . \qquad (3.58)$$

The choice of the function $G(Q, \xi)$ and the contour C in the complex ξ-plane, which yields the Taub-NUT self-interaction in eq. (3.57), is given by [52, 53]

$$S_{\text{Taub-NUT}}[L] = \int d^4x \, \tilde\nabla^2 \tilde\Delta^2 \frac{1}{2\pi i} \left\{ \oint_{C_1} d\xi \frac{Q^2}{2\xi} + \frac{1}{\sqrt\lambda} \oint_{C_2} d\xi \, Q \ln(\sqrt\lambda Q) \right\} , \qquad (3.59)$$

where the contour C_1 goes around the origin, whereas the contour C_2 encircles the roots of the quadratic equation $Q(\xi) = 0$ in the complex ξ-plane.

The $N = 2$ superfield Feynman rules can also be developed in the $N = 2$ projective superspace [54], where they are likely to be deducible from the more general HSS rules. Partial (abelian) results are also available in the ordinary $N = 2$ superspace [53].

Finally, one may wonder, in which sense an $N = 2$ tensor multiplet action describes a $D = 4$, $N = 2$ supersymmetric NLSM with the highest physical spin $1/2$, because of the apparent presence of the gauge antisymmetric tensor E_{mn} among the $N = 2$ tensor multiplet components — see eq. (3.46). A detailed investigation of the component action, which follows from the superspace action (3.59), shows that the tensor E_{mn} and its field strength E'_m enter the action only in the combination

$$\left(1 + \frac{1}{\lambda \vec{L}^2}\right)(E'_m)^2 + \frac{1}{2}\varepsilon_{mnpq}E_{pq}F_{mn}(L) \; , \tag{3.60}$$

where the tensor

$$F_{mn}(L) \equiv \left(\partial_m \vec{L} \times \partial_n \vec{L}\right) \cdot \frac{\vec{L}}{\left|\vec{L}\right|^3} \tag{3.61}$$

is formally identical to the electromagnetic field strength of a magnetic monopole. Therefore, there exists a vector potential A_m such that $F_{mn}(L) = \partial_m A_n - \partial_n A_m$. An explicit magnetic monopole solution for the locally defined potential $A_m(\vec{L})$ cannot be 'rotationally' invariant with respect to the $SO(3) \sim SU(2)_A/Z_2$ symmetry, though it can be written down as a function of the $SO(2)$-irreducible L^{ij}-components defined by $L^{ij} = \delta^{ij}S + P^{(ij)}_{\text{traceless}}$. After integrating by parts and introducing a Lagrange multiplier V as

$$* EF = {}^*EdA \rightarrow -d^*EA = -E'_m A_m \rightarrow -E_m A_m - E_m \partial_m V \; , \tag{3.62}$$

we can integrate out the full vector E_m. It results in the bosonic NLSM action in terms of four real scalars $(S, P^{(ij)}_{\text{traceless}}, V)$, as it should.

3.4 Induced multicentre Taub-NUT metrics

The *Euclidean Taub-NUT* (ETN) metric is a unique non-trivial hyper-Kähler four-dimensional metric having the $U(2)$ isometry whose transformation laws are linear and holomorphic [55]. Without the holomorphicity requirement, there exists another hyper-Kähler solution known as the *Eguchi-Hanson* (EH) four-dimensional instanton [56]. The EH metric also possess the $U(2)$ isometry, and its possible appearance in the hypermultiplet LEEA will be discussed in sect. 6 (see also Appendix B for a derivation of the EH metric from harmonic superspace).

From the viewpoint of $N = 2$ supersymmetry in four spacetime dimensions, the origin of the $U(2) = SU(2) \times U(1)$ internal symmetry in the ETN- and EH-type hypermultiplet LEEA is, however, quite different. In the ETN-case, the $SU(2)$ isometry factor is identified with the $SU(2)_A$ automorphisms of $N = 2$ supersymmetry,

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whereas the $U(1)$ isometry factor is identified with the unbroken subgroup $U(1)_{\text{PG}}$ of the Pauli-Gürsey (PG) symmetry $SU(2)_{\text{PG}}$ of the free FS-type hypermultiplet action (3.28). In the EH-case, the $SU(2)$ isometry factor has to be identified with the $SU(2)_{\text{PG}}$ symmetry, whereas the $U(1)$ isometry factor is just the unbroken abelian part of the $SU(2)_A$ automorphisms. All these identifications are quite obvious from the viewpoint of HSS: the ETN-type hypermultiplet self-interaction in HSS maintains $SU(2)_A$ but breaks $SU(2)_{\text{PG}}$ down to its abelian subgroup $U(1)_{\text{PG}}$ (see subsect. 3.2 and Appendix A), whereas the EH-type hypermultiplet self-interaction (see sect. 6 and Appendix B) breaks $SU(2)_A$ down to its abelian subgroup $U(1)_A$ but maintains $SU(2)_{\text{PG}}$.

A natural generalization is provided by the FS-type hypermultiplet LEEA whose induced hyper-Kähler metric merely has the abelian $U(1) \times U(1)$ isometry, where the first $U(1)$ isometry factor is supposed to be identified with $U(1)_A$ whereas the second isometry $U(1)$ factor is $U(1)_{\text{PG}}$. Since the $SU(2)_A$ internal symmetry is not anomalous in $N = 2$ supersymmetric quantum perturbation theory, its breaking can only be caused either (i) by $SU(2)_A$ non-invariant terms in the fundamental lagrangian, with the Fayet-Iliopoulos (FI) term (see sect. 6) being an example, or (ii) non-perturbatively, with the BPS branes intersecting at angles (see sect. 6) being an example.

The most general $U(1)_A \times U(1)_{\text{PG}}$-invariant hyper-Kähler potential $\mathcal{K}$ is given by a general, of overall charge $(+4)$, analytic function of the product $(\overset{*}{\bar{q}}\,q)^{(+2)}$, i.e.

$$S[q] = \int_{\text{analytic}} \left\{ \overset{*}{\bar{q}}{}^{+} D^{++} q^{+} + \mathcal{K}^{(+4)}(\overset{*}{\bar{q}}\,q) \right\} \ , \tag{3.63}$$

with

$$\mathcal{K}^{(+4)}(\overset{*}{\bar{q}}\,q) = \sum_{l=0}^{\infty} \xi^{(-2l)} \frac{(\overset{*}{\bar{q}}{}^{+} q^{+})^{l+2}}{l+2} \ , \tag{3.64}$$

where the harmonic-dependent 'coefficients' $\xi^{(-2l)}(u)$ are defined by

$$\xi^{(-2l)} = \xi^{(i_1 \cdots i_{2l})} u_{i_1}^{-} \cdots u_{i_{2l}}^{-} \ , \quad l = 1, 2, \ldots \ , \tag{3.65}$$

and satisfy the reality condition

$$\overset{*}{\bar{\xi}}{}^{(-2l)} = (-1)^l \xi^{(-2l)} \ . \tag{3.66}$$

As was shown in refs. [57, 58], in fact, *any* four-dimensional hyper-Kähler metric having the $U(1)_{\text{PG}}$ isometry belongs to the class of the multicentre *Gibbons-Hawking* (GH) metrics [59], which includes the ETN and EH metrics as the particular cases. Since the multicentre metrics are of special importance to the brane technology (see

the next sect. 4), where they are usually described in terms of a harmonic function H
(see eqs. (3.70)–(3.72) below), it is useful to establish a correspondence between the
two equivalent descriptions, the one in terms of an *analytic* hyper-Kähler potential $\mathcal{K}$
and another one in terms of a *harmonic* (singular) function H satisfying the Laplace
equation

$$\Delta H = 0 \tag{3.67}$$

outside the origin ($r = 0$) in three Euclidean dimensions. [18] This correspondence was
established in ref. [58]. Given a general solution to eq. (3.67) in spherical coordinates
(r, ϑ, φ), which reads

$$H = \frac{1}{2r} + \frac{U(\vec{r})}{2} \equiv \frac{1}{2r} + \frac{1}{2} \sum_{l=0}^{+\infty} \sum_{m=-l}^{m=+l} c_{lm} r^l Y_l^m(\vartheta, \varphi) , \tag{3.68}$$

in terms of the standard momentum eigenfunctions $Y_l^m(\vartheta, \varphi)$, the one-to-one corre-
spondence between the integration constants c_{lm} in eq. (3.68) and the hyper-Kähler
potential coefficients (3.65) is given by [58]

$$\xi^{i_1=1,\dots,i_{l-m}=1,i_{l-m+1}=2,\dots,i_{2l}=2} = \frac{c_{lm}}{C} \frac{(2l+1)}{(l+1)} , \tag{3.69}$$

which some normalization-dependent constant C.

A multicentre GH metric is usually written down in the form

$$ds^2 = H d\vec{r} \cdot d\vec{r} + H^{-1}(d\varrho + \vec{C} \cdot d\vec{r})^2 , \tag{3.70}$$

where the vector $\vec{C}(\vec{r})$ satisfies the equation

$$\vec{\nabla} H = \vec{\nabla} \wedge \vec{C} , \tag{3.71}$$

while the harmonic function H has the 'milticentre' form

$$H(\vec{r}) = \frac{\lambda}{2} + \sum_{i=1}^{n} \frac{|k_i|}{2 |\vec{r} - \vec{r_i}|} . \tag{3.72}$$

The harmonic function (3.72) can always be put into the form (3.68), when choos-
ing the coordinate system whose origin coincides with one of $\vec{r_i}$, extracting the leading
singular term and then expanding the regular rest in the spherical harmonics.

A multicentre metric can therefore be naturally interpreted as the metric describ-
ing a static multi-monopole configuration whose monopoles have magnetic charges

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[18]Because of the $U(1)_{\text{PG}}$ global isometry, the function H can be chosen to be independent upon
one coordinate.

k_i and sit at the space points $\vec{r}_i$. The $4n$-parameters $\{\vec{r}_i, k_i\}$ in eq. (3.72), i.e. the *moduli* of this hyper-Kähler configuration, thus have the very clear physical meaning in terms of the harmonic function description, while the harmonic function H itself has singularities at the positions of the monopoles. In the alternative HSS description of the same multi-monopole configuration, though the HSS moduli $\xi^{(i_1 \cdots i_{2l})}$ have no direct physical interpretation, the description itself in terms of the analytic hyper-Kähler potential is apparently *non-singular*. The latter will be useful for the brane technology, when describing the internal symmetry enhancement of coinciding branes in non-singular terms (subsect. 4.2).

4 Brane technology

It is quite remarkable that one can also get exact LEEA solutions to the $D = 4$, $N = 2$ *quantum* gauge field theories from the *classical* M-theory brane dynamics [7] (see refs. [60, 61] also). Among a few things one knows about M-Theory, only two general statements are going to be used in what follows, namely, that (i) M-theory is the strong coupling limit of the type-IIA superstring theory (which is rather non-constructive), and (ii) the low-energy limit of M-theory is $D = 11$ supergravity. Because of (ii), stable and unique BPS states of the $D = 11$ supergravity can be exploited to extract non-perturbative information about M-theory. This information can then be applied to study the effective supersymmetric gauge field theories in the BPS brane world-volumes. These 'in-M-brane' gauge theories are not quite the same as the ones we studied in the previous sections (see the end of subsect. 4.6 and sect. 5), though they can share the same LEEA under certain circumstances to be discussed below. The supersymmetric BPS branes, which are relevant to this *brane technology*, are introduced in the next subsections, along the lines of the existing reviews [62, 63, 64] (see refs. [62, 63, 64] and references therein for more details).

4.1 $D = 11$ supergravity and its BPS solutions

$D = 11$ is believed to be the maximum dimension of spacetime (with Lorentzian signature), where a consistent interacting supersymmetric field theory exists [65]. This is essentially the consequence of the fact that the massless physical particles mediating long-range forces in our $D = 4$ spacetime can have spin 2 at most, while there is only one type of particles of spin 2 (i.e. gravitons). A supersymmetry charge (a component of $D = 4$ spinor) changes helicity λ of a massless particle by a half, so

that the maximal non-vanishing product of N supersymmetry charges changes λ by $N/2$. Hence, in a massless representation of the N-extended $D = 4$ supersymmetry, the helicity varies from λ to $\lambda + \frac{1}{2}N$. It immediately implies $N \leq 8$ provided that $|\lambda| \leq 2$. The maximal $N = 8$ supersymmetry in $D = 4$ has $8 \times 4 = 32$ real component charges, while the maximal spacetime dimension (with Lorentzian signature), where a minimal spinor representation also has real 32 components, is just $D = 11$. The simplest supersymmetry algebra in $D = 11$ takes the form

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^M)_{\alpha\beta}P_M \ , \tag{4.1}$$

where Q_α is a Majorana spinor supersymmetry charge, $\alpha = 1, 2, \ldots, 32$, P_M is a $D = 11$ spacetime momentum operator, Γ^M are $D = 11$ real gamma matrices, and C is the charge conjugation matrix in $D = 11$, while $M = 0, 1, 2, \ldots, 10$, and Minkowski flat metric is $\eta^{MN} = \mathrm{diag}(-, +, \ldots, +)$.

The $D = 11$ supergravity [66] is described in terms of three fields: a metric g_{MN}, a gravitino $\psi_{M\alpha}$ and a 3-form gauge potential A_{MNP} with a gauge transformation $\delta A_{(3)} = d\Lambda_{(2)}$ and a field strength $F_{(4)} = dA_{(3)}$. The theory has $128_{\mathrm{B}} + 128_{\mathrm{F}}$ on-shell physical components. The bosonic part of the $D = 11$ supergravity action reads

$$S_{\mathrm{bosonic}} = \frac{1}{2} \int d^{11}x \left\{ \sqrt{-g}(R - \frac{1}{48}F^2) + \frac{1}{6}F_{(4)} \wedge F_{(4)} \wedge A_{(3)} \right\} \ , \tag{4.2}$$

where we have taken the gravitational coupling constant to be equal to one. The last (Chern-Simons) term in eq. (4.2) is required by $D = 11$ supersymmetry of the total action including the gravitino-dependent terms.

A quantized $D = 11$ supergravity is not expected to be a consistent theory e.g., because of its apparent non-renormalizability. Instead, it should rather be interpreted as the *effective* low-energy approximation (LEEA) to the presumably consistent M-Theory. This fact alone leads to many far-reaching consequences. Though we are unable to describe underlying (microscopic) dynamics of the M-Theory from its LEEA alone, knowing the latter allows us to determine the spectrum of BPS states in the M-theory, just by constructing the classical solutions to the $D = 11$ supergravity equations of motion, which preserve some part (ν) of the $D = 11$ supersymmetry and, hence, some part of the translational invariance as well. Since the supersymmetry variations of the bosonic fields of $D = 11$ supergravity are all proportional to the gravitino field, the latter should vanish in a supersymmetric solution, $\psi_{M\alpha} = 0$. The vanishing supersymmetry variation of the gravitino field itself,

$$\delta\psi_M|_{\psi=0} = \widetilde{D}_M\varepsilon \equiv \left(D_M - \frac{1}{288}[\Gamma_M{}^{NPQS} - 8\delta_M^N\Gamma^{PQS}]F_{NPQS} \right) \varepsilon = 0 \ , \tag{4.3}$$

where $D_M\varepsilon = (\partial_M + \frac{1}{4}\omega_M{}^{BC}\Gamma_{BC})\varepsilon$, then implies the existence of a Majorana *Killing spinor* field ε which satisfies the first-order differential equation (4.3). In eq. (4.3) we denoted by $\Gamma^{M_1\cdots M_p}$ the antisymmetric products of the gamma matrices with unit weight.

Assuming the existence of asymptotic states with a supersymmetric vacuum, and requiring the $D = 11$ metric to be asymptotically Minkowskian, it is easy to see that the only BPS states with respect to the supersymmetry algebra (4.1) are just massless particles, since [19]

$$0 = \det \langle \{Q_\alpha, Q_\beta\}\rangle = \langle \det(\Gamma \cdot P)\rangle = \left\langle (P^2)^{16} \right\rangle \ , \tag{4.4}$$

while the matrix $\{Q_\alpha, Q_\beta\}$ has 16 independent zero eigenvalues ($\nu = \frac{1}{2}$). This simply means that a massless representation of $D = 11$ supersymmetry is $\frac{1}{2}$-shorter than the massive one, as is well-known in supersymmetry *without* central charges. The corresponding asymptotically flat classical BPS solution of $D = 11$ supergravity with $P^2 = 0$ (called M-wave) was found e.g. in ref. [67].

It is one of the lessons of $D = 4$ gauge field theory that the massless particles appearing in a perturbative spectrum may not be the only BPS states. Non-perturbative (massive) BPS states in extended $D = 4$ supersymmetry carry electric and magnetic charges saturating the Bogomolnyi bound, whereas these charges appear as the central charges on the right-hand-side of the supersymmetry algebra. The symmetric matrix on the left-hand-side of eq. (4.1) belongs to the adjoint representation **528** of the Lie algebra of $Sp(32)$, which is decomposed with respect to its (Lorentz) subgroup $SO(1,10)$ as

$$\mathbf{528} \rightarrow \mathbf{11} \oplus \mathbf{55} \oplus \mathbf{462} \ . \tag{4.5}$$

The **11** is apparently associated with P_M in eq. (4.1), whereas the rest has to be associated with some additional 'central' charges commuting with supersymmetry charges and monenta, but not commuting with Lorentz rotations. The $D = 11$ Lorentz representations **55** and **462** are associated with a 2-form $Z_{(2)}$ and a 5-form $Y_{(5)}$, respectively, so that the maximal $D = 11$ supersymmetry algebra reads [68]

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^M)_{\alpha\beta}P_M + \tfrac{1}{2}(\Gamma^{MN}C)_{\alpha\beta}Z_{MN} + \tfrac{1}{5!}(\Gamma^{MNPQS}C)_{\alpha\beta}Y_{MNPQS} \ , \tag{4.6}$$

where Z_{MN} represent the 'electric' charges and Y_{MNPQS} are the 'magnetic' ones. The BPS object carrying non-vanishing electric charges is known as a *supermembrane* or an electric *M-2-brane* [69]. Associated with the $D = 11$ spacetime symmetries broken

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[19]It is the asymptotical form of the local $D = 11$ supersymmetry algebra that is given by the rigid superalgebra (4.1). The Killing spinor $\varepsilon(x)$ should also be constant at infinity.

by the supermembrane are the *Nambu-Goldstone* (NG) modes. The three-dimensional LEEA action describing the dynamics of small fluctuations of the NG fields about the supermembrane in a $D = 11$ supergravity background was discovered by Bergshoeff, Sezgin and Townsend [70]. [20]

The BPS object, which is magnetically *dual* to the M-2-brane in eleven dimensions, is a magnetically charged 5-brane called *M-5-brane*. Indeed, according to Gauss's law, the electric charge of a particle (i.e. 0-brane) in some number (D) of spacetime dimensions is measured by the dual gauge field strength according to the integral $Q_{electric} = \int_{S^{D-2}} {}^*F$ over the sphere S^{D-2} surrounding the particle, where $F_{(2)} = dA_{(1)}$ is the abelian field strength of a $U(1)$ gauge field $A_{(1)}$ and ${}^*F_{(D-2)}$ is the Hodge dual to $F_{(2)}$ in D dimensions. In the case of an 'electric' p-brane charged with respect to a gauge $(p + 1)$-form $A_{(p+1)}$ in D dimensions, the field strength is $F_{(p+2)} = dA_{(p+1)}$ and its dual is ${}^*F_{(D-p-2)}$. For *magnetically* charged objects the roles of F and ${}^*F$ are supposed to be interchanged. For example, the object carrying a magnetic charge in $D = 4$ is again a 0-brane (i.e. particle or monopole) since the dual potential $\tilde{A}$ defined by ${}^*F_{(2)} = d\tilde{A}$ is a 1-form, whereas the charge of the D=4 monopole is measured by $F_{(2)}$ as $Q_{magnetic} = \int_{S^2} F$. Similarly, since the potential $\tilde{A}$ of the dual field strength $*F_{(D-p-2)}$ is a $(D - p - 3)$-form, ${}^*F_{(D-p-2)} = d\tilde{A}_{(D-p-3)}$, it is a $(D - p - 4)$-brane that can support magnetic charges. The well-known 'golden rule' for an electrically charged p-brane and its dual, magnetically charged q-brane thus reads

$$p + q = D - 4 . \tag{4.7}$$

Given $D = 11$ and $p = 2$, one has $q = 5$. The magnetic charge of an M-5-brane is proportional to the integral $\int_{S^4} F_{(4)}$ over the sphere S^4 surrounding the brane at spacial infinity in five directions transverse to its six-dimensional worldvolume. The integral is obviously topological (i.e. homotopy invariant) due to the Bianchi identity $dF_{(4)} = 0$.

The explicit form of electric (M-2-brane) and magnetic (M-5-brane) BPS solutions to the $D = 11$ supergravity is known (see, e.g., the reviews [62, 63, 64] and references therein). For our purposes we only need a solitonic 5-brane solution found by Güven [71], which reads

$$ds^2 = H^{-1/3}(y)dx^\mu dx^\nu \eta_{\mu\nu} + H^{2/3}(y)dy^m dy^n \delta_{mn} , \qquad F_{(4)} = {}^*_5 dH , \tag{4.8}$$

where the $D = 11$ spacetime coordinates have been split into the 'worldvolume' coordinates labeled by $\mu, \nu = 0, 1, 2, 3, 4, 5$ and the 'transverse to the worldvolume'

---

[20]An action of NG fields is entirely determined by the broken symmetries and, hence, it is unique.

coordinates labeled by $m, n = 6, 7, 8, 9, 10$, according to the spacetime decomposition $R^{1,10} = R^{1,5} \times R^5$. In eq. (4.8), the Hodge dual $(\overset{*}{_5})$ in the five transverse dimensions has been introduced, whereas $H(y)$ is supposed to be a *harmonic* function in R^5, i.e. $\vec{\nabla}^2 H(y) \equiv \Delta H(y) = 0$. All the other components of $F_{(4)}$ are zero. For a single M-5-brane of magnetic charge k, the harmonic function $H(y)$ is given by

$$H(y) = 1 + \frac{|k|}{r^3} \ , \quad \text{where} \quad r^2 = y^m y_m \ . \tag{4.9}$$

This M-5-brane solution is completely regular (i.e. truly solitonic) and it, in fact, interpolates between the two maximally supersymmetric $D = 11$ 'vacua', the one being asymptotically flat in the limit $r \to \infty$ while another approaching $(AdS)_7 \times S^4$ in the limit $r \to 0$ [72, 73].

When one chooses instead the harmonic function

$$H(y) = 1 + \sum_{s=1}^{n} \frac{|k_s|}{|\vec{y} - \vec{y}_s|^3} \ , \tag{4.10}$$

one arrives at the classical configuration of n *parallel* and similarly oriented M-5-branes of magnetic charges k_s, located at $\vec{y}_s$ in the R^5-space. This *multicentre* BPS solution also admits 16 Killing spinor fields by construction, so that it preserves $\nu = \frac{1}{2}$ of supersymmetry in $D = 11$. The existence of the multicentre brane solutions can be physically interpreted as a result of cancellation of gravitational and *anti-gravitational* (due to the antisymmetric tensor) forces, which is quite similar to the well-known 'no force condition' (zero binding energy) in D=4 physics of monopoles.

4.2   NS and D branes in $D = 10$ dimensions

In eleven dimensions there are only M-waves, M-2- and M-5-branes as the 'elementary' BPS states preserving exactly a half of the $D = 11$ supersymmetry. In order to make contact with the type-IIA superstring theory in $D = 10$, let's now assume that one of the *transverse* (to the brane worldvolume) dimensions is compactified on a circle S^1 of radius $R_{[11]}$. An M-5-brane can now be either (i) Kaluza-Klein-like 'reduced' to ten dimensions, which results in a solitonic *NS-5-brane*, or (ii) it can be 'wrapped' around the circle S^1, which results in a *D-4-brane*.

By construction, the NS-5-brane is *magnetically* charged with respect to a gauge NS-NS 2-form (Kalb-Ramond field) $B_{(2)}^{[10]}$ descending from the gauge 3-form $A_{(3)}^{[11]}$ in eleven dimensions. In accordance with eq. (4.7), the NS-5-brane is magnetically dual to the 'fundamental' $D = 10$ superstring. Since the NS-5-brane still depends upon the

compactified (periodic) coordinate ϱ of S^1, it thus contains all the associated *Kaluza-Klein* (KK) physical modes. In order to become a BPS solution to the Type-IIA $D = 10$ supergravity, the NS-5-brane solution should therefore be 'averaged' over the compactified coordinate ϱ, which just amounts to dropping all the *massive* KK modes. Though the latter is fully legitimate for a small compactification radius $R_{[11]}$ of S^1 (i.e for weakly coupled superstrings — see eq. (4.15) below), it becomes illegitimate for large $R_{[11]}$ (i.e. for strongly coupled superstrings) when some massive KK modes become light. From the viewpoint of the ten-dimensional type-IIA superstring theory, all the KK-modes appear as non-perturbative states.

The wrapped M-5-brane(=D-4-brane) is (RR) charged with respect to the gauge 3-form $A_{(3)}^{[10]}$ of the type-IIA supergravity, which is also descending from $A_{(3)}^{[11]}$, so that it is a *Dirichlet*-4-brane indeed. [21]

The KK Ansatz for the bosonic fields of $D = 11$ supergravity, which leads to the $D = 10$ type-IIA action (in the so-called string frame), reads [62, 63, 64]

$$
\begin{aligned}
ds_{[11]}^2 &= e^{-\frac{2}{3}\phi} ds_{[10]}^2 + e^{\frac{4}{3}\phi} (d\varrho + C_M dx^M)^2 \ , \\
A_{(3)}^{[11]} &= A_{(3)}^{[10]} + B_{(2)}^{[10]} \wedge d\varrho \ , \quad M = 0, 1, 2, \ldots, 9 \ ,
\end{aligned}
\tag{4.11}
$$

where the S^1 coordinate ϱ is supposed to be periodic (with period 2π), and the $D = 10$ *dilaton* ϕ and KK vector C_M have been introduced.

The $D = 10$ bosonic action descending from eq. (4.2) reads

$$
\begin{aligned}
S_{\text{IIA}} =& \tfrac{1}{2} \int d^{10}x \sqrt{-g} \left\{ e^{-2\phi} \left[R + 4\nabla_M \phi \nabla^M \phi - \tfrac{1}{12} F_{MNP} F^{MNP} \right] \right. \\
& \left. - \tfrac{1}{48} F_{MNPQ} F^{MNPQ} - \tfrac{1}{4} F_{MN} F^{MN} \right\} + (\text{Chern} - \text{Simons terms}) \ ,
\end{aligned}
\tag{4.12}
$$

where $F_{(2)} = dC_{(1)}$, $F_{(3)} = dB_{(2)}$ and $F_{(4)} = dA_{(3)}$. The kinetic terms of the type-IIA superstring NS-NS fields (g_{MN}, B_{MN}, ϕ) in the first line of eq. (4.12) are thus uniformly coupled to the dilaton factor $e^{-2\phi}$ (*cf.* the familiar factor g^{-2} in front of the Yang-Mills action), whereas the field strengths of the RR-fields (C_M, A_{MNP}) in the second line of eq. (4.12) do *not* couple to the dilaton at all. Therefore, the superstring coupling constant g_{string} is given by the asymptotical value of e^ϕ,

$$
g_{\text{string}} = \left\langle e^\phi \right\rangle \ ,
\tag{4.13}
$$

---

[21]As is well-known [74], the D-branes have a simple interpretation in the perturbative superstring theory as the spacetime topological defects on which the open type-I superstrings with Dirichlet boundary conditions can end. The charges carried by the D-branes are known to be the *Ramond-Ramond* (RR) charges in superstring theory.

while the RR-field couplings to the D-branes in the type-IIA supergravity should contain non-perturbative information about the type-IIA superstring/M-Theory.

It follows from the KK-Ansatz (4.11) that the compactification radius $R_{[11]}$ is also related to the dilaton as

$$R_{[11]} = \left\langle e^{\frac{2}{3}\phi} \right\rangle \ . \tag{4.14}$$

Combining eqs. (4.13) and (4.14) results in the remarkable relation [2]

$$R_{[11]} = g_{\text{string}}{}^{2/3} \ . \tag{4.15}$$

The strong coupling limit of the type-IIA superstring theory is, therefore, *eleven-dimensional* [2] ! Since there is only one $D = 11$ supersymmetric field theory (of the second-order in spacetime derivatives), namely, the $D = 11$ supergravity, it should thus be interpreted as the LEEA of some eleven-dimensional M-Theory which is supposed to be $D = 11$ supersymmetric by consistency.

The solitonic (i.e. regular) NS-5-brane BPS solution to the type-IIA, $D = 10$ supergravity, which is obtained by plain *dimensional reduction* of the M-5-brane solution (4.8) down to $R^{1,5} \times R^4$, is given by [62, 63]

$$ds^2_{[10]} = dx^\mu dx^\nu \eta_{\mu\nu} + H(y) dy^m dy^n \delta_{mn} \ , \qquad F_{(3)} = {}^*_4 dH \ , \qquad e^{2\phi} = H \ , \tag{4.16}$$

where x^μ ($\mu = 0,1,2,3,4,5$) parameterize $R^{1,5}$ and y^m ($m = 6,7,8,9$) parameterize R^4, $\binom{*}{4}$ is the Hodge dual in R^4, and $H(y)$ is a harmonic function in R^4. A BPS configuration of n parallel and similarly oriented NS-5-branes is obtained after choosing the harmonic function $H(y)$ as in eq. (4.10).

Similarly, the D-4-brane solution to the type-IIA supergravity reads [62, 63]

$$ds^2_{[10]} = H^{-\frac{1}{2}} dx^\mu dx^\nu \eta_{\mu\nu} + H^{\frac{1}{2}} dy^m dy^n \delta_{mn} \ , \qquad F_{(3)} = 0 \ , \qquad e^{-4\phi} = H \ , \tag{4.17}$$

where x^μ ($\mu = 0,1,2,3,4$) parameterize $R^{1,4}$, whereas y^m ($m = 5,6,7,8,9$) parameterize R^5, and $H(y)$ is a harmonic function in R^5. Given the choice of eq. (4.10) for the harmonic H-function in eq. (4.17), one arrives at a BPS configuration of n parallel and similarly oriented D-4-branes in static equilibrium. The 'no force condition' in this case can again be physically interpreted as the result of mutual cancellation of the gravitational (NS-NS) and anti-gravitational (R-R) forces between the D-4-branes. However, unlike the similar NS-5-brane solution (4.16), the solution (4.17) has isolated singularities at the positions $\{\vec{y}_s\}$ of the D-4-branes in R^5.

In the case of n parallel and similarly oriented D-4-branes, there are n abelian gauge fields in their common worldvolume, which originate as the zero modes of the

open superstrings stretched between the D-4-branes [74]. In the coincidence limit, where all n D-4-branes collapse, i.e. when they are 'on top of each other', the gauge symmetry $U(1)^n$ enhances to $U(n)$ [75]. This gauge symmetry enhancement can be understood from the viewpoint of the perturbative open (T-dual, or subject to Dirichlet boundary conditions) superstring theory [74] due to the appearance of extra massless vector bosons in the coincidence limit.

However, it is not yet the end of the $D = 10$ BPS brane story. The KK massive particles associated with the compactification circle S^1 can be naturally interpreted in $D = 10$ as the D-0-branes charged with respect to the RR (and KK) gauge field $C_{(1)}$ [75]. Indeed, the eleventh component of the spacetime momentum in the eleven-dimensional supersymmetry algebra (4.1) plays the role of an abelian central charge in the compactified theory, whereas this central charge in $D = 10$ originates from the RR charges of D-0-branes. From the viewpoint of the type-IIA superstring, all these BPS states are truly non-perturbative. [22]

According to the golden rule (4.7), magnetically dual to the D-0-branes in $D = 10$ are D-*6-branes*, which are, therefore, also of KK origin in the type-IIA supergravity. The corresponding classical BPS solution of the type-IIA supergravity equations of motion in $D = 10$ reads [76]

$$ds^2_{[10]} = H^{-\frac{1}{2}} dx^\mu dx^\nu \eta_{\mu\nu} + H^{\frac{1}{2}} dy^m dy^n \delta_{mn} \ , \qquad F_{(2)} = {}^*_3 dH \ , \qquad e^{-4\phi} = H^3 \ , \quad (4.18)$$

where x^μ ($\mu = 0, 1, 2, 3, 4, 5, 6$) parameterize $R^{1,6}$ and y^m ($m = 7, 8, 9$) parameterize R^3, $\binom{*}{3}$ is the Hodge dual in R^3, and $H(y)$ is a harmonic function in R^3. A BPS configuration of n parallel D-6-branes is described by the harmonic function $H(y)$ similar to that of eq. (4.10) but with the different power (-1) instead of (-3) there. Like a D-4-brane, a single D-6-brane is singular in $D = 10$ at the position of the D-6-brane in R^3. An M-Theory resolution of the D-4-brane singularity will be discussed in subsect. 4.5. As regards the D-6-brane singularity in $D = 10$, its M-Theory resolution in eleven dimensions is provided by the following *non-singular* $D = 11$ supergravity solution [49]:

$$ds^2_{[11]} = dx^\mu dx^\nu \eta_{\mu\nu} + H d\vec{y} \cdot d\vec{y} + H^{-1}(d\varrho + \vec{C} \cdot d\vec{y})^2 \ , \qquad (4.19)$$

where $H = 1 + \frac{1}{2} r^{-1}$, $r = |\vec{y}|$, and $\vec{\nabla} \times \vec{C} = \vec{\nabla} H$ (*cf.* eq. (3.70) !). The $D = 11$ spacetime (4.19) is given by the product of the flat space $R^{1,6}$ with the Euclidean Taub-NUT (ETN) space (sect. 3). The ETN space can be thought of as a non-trivial bundle (Hopf fibration) with the base R^3 and the fiber S^1. After dimensional

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[22]There are no RR charged states in the perturbative superstring spectrum.

reduction to $D = 10$ dimensions, eq. (4.19) results in a single D-6-brane located at the origin of R^3. Though the Taub-NUT metric seems to be singular at $r = 0$, as is well-known [47, 48], it is just a coordinate singularity provided that ϱ is periodic with the period 2π. [23]

Therefore, the M-theory interpretation of a D-6-brane is given by the ETN (or KK [47, 48]) monopole which interpolates between the two maximally supersymmetric M-Theory 'vacua': the flat $D = 11$ spacetime near $r = 0$ and the KK spacetime $R^{1,9} \times S^1$ near $r \to \infty$ [49]. It is straightforward to generalize this result to a system of n parallel and similarly oriented D-6-branes in $D = 10$, whose M-theory interpretation is given by a Euclidean *multi-Taub-NUT* monopole (subsect. 3.4) described by the multi-centered harmonic function (*cf.* eq. (3.72))

$$H_{\text{multi}-\text{ETN}}(\vec{y}) = 1 + \sum_{i=1}^{n} \frac{|k_i|}{2\,|\vec{y} - \vec{y_i}|} \ . \tag{4.20}$$

This solution is non-singular in $D = 11$ provided that no two centers coincide, i.e. $\vec{y_i} \neq \vec{y_j}$ for all $i \neq j$. In the coincidence limit of parallel and similarly oriented D-6-branes with equal RR charges (=1), non-isolated singularities appear.

To investigate the coincidence limit in some more details, one may have to generalize the harmonic function H of eq. (4.20) to the form (3.72). For instance, in the case of just two centers with equal charges, a good choice is given by [62]

$$H(\vec{y}) = \frac{\lambda}{2} + \frac{1}{2\,|\vec{y} - \epsilon\vec{y_0}|} + \frac{1}{2\,|\vec{y} + \epsilon\vec{y_0}|} \ , \tag{4.21}$$

where $\{\lambda, \epsilon\}$ are some positive constants. The double ETN metric is then obtained by choosing $\lambda/2 = 1$ in eq. (4.21), whereas the limit $\lambda \to 0$ results in the Eguchi-Hanson (EH) metric. In the coincidence limit $|\epsilon| \to 0$, near the singularity of H, the value of λ is obviously irrelevant, whereas the gauge symmetry in the common worldvolume of the D-6-branes is *enhanced* from $U(1) \times U(1)$ to $U(2)$ [2, 75]. From the M-theory perspective, the M-2-branes can wrap around the compactification circle S^1, being simultaneously stretched between the D-6-branes. It is the massless modes of these M-2-branes that play the role of the additional (non-abelian) massless vector particles in the coincidence limit [62]. In order to make this symmetry enhancement manifest, it is useful to employ the non-singular HSS description of the mixed EH-ETN metric in terms of the harmonic potential (4.21) — see subsect. 4.7.

The $D = 10$ singularity of the single D-6-brane solution can be physically interpreted as the result of an illegitimate neglect of the KK particles which become *massless* at the D-6-brane core and whose inclusion resolves the singularity in D=11 [49].

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[23]The asymptotic ETN metric near the singularity of H is diffeomorphism-equivalent to the flat R^4 metric.

This phenomenon is known as the *M-Theory resolution* of short-distance singularities in the $D = 10$ type-IIA supergravity by relating them (via a strong-weak coupling duality) to the long-distance effects of the massless modes of the M-2-brane wrapped around S^1 [62]. The M-theory resolution is one of the cornerstones of the brane technology in non-perturbative $D = 4$, $N = 2$ supersymmetric gauge field theories [7] (see subsect. 4.5).

4.3   Intersecting branes

Each of the BPS (single or parallel) 'elementary' brane solutions to $D = 11$ or $D = 10$ (type-IIA) supergravity considered in the previous subsections breaks exactly 1/2 of the maximal supersymmetry having 32 supercharges, and it is governed by a single harmonic function of transverse spacial coordinates. More BPS brane solutions preserving some part $\nu \leq 1/2$ of the maximal supersymmetry can be obtained by a superposition of the (intersecting) 'elementary' branes. A procedure of constructing the corresponding classical supergravity solutions depending upon several harmonic functions is outlined in ref. [77], and it is known as the 'harmonic function rule'. [24]

For our purposes of describing the brane technology towards the gauge field theories with $N = 2$ supersymmetry in $D = 4$ (i.e. having $2 \times 4 = 8$ supercharges), we only need 'marginal' (i.e. with vanishing binding energy) BPS brane configurations which preserve exactly $8/32 = 1/4$ of the maximal supersymmetry and have an (uncompactified) flat spacetime $R^{1,3}$ as their intersection. This limits our discussion (e.g. in $D = 10$ dimensions) to the *orthogonally* intersecting NS-branes, D-4-branes and D-6-branes, having $R^{1,3}$ as their common worldvolume, where the effective four-dimensional $N = 2$ supersymmetric gauge field theory in question lives. From the M-theory perspective (i.e. $D = 11$ dimensions), we want the NS- and D-4-branes to be represented by a *single* M-5-brane described by a single harmonic function and, possibly, in the background of a (multi-ETN) monopole described by another harmonic function or the corresponding hyper-Kähler potential. The relevant 1/4-supersymmetric BPS brane solution is then parameterized by two functions, as in the $N = 2$ gauge field theory with a hypermultiplet matter. Unlike the $D = 10$ configuration of the intersecting BPS branes (which is singular), the corresponding M-theory brane configuration in $D = 11$ is non-singular, and it carries some non-perturbavive information about the $N = 2$, $D = 4$ gauge theory in question. Our immediate tasks are, therefore, (i) to establish a correspondence (i.e. a dictionary) between the

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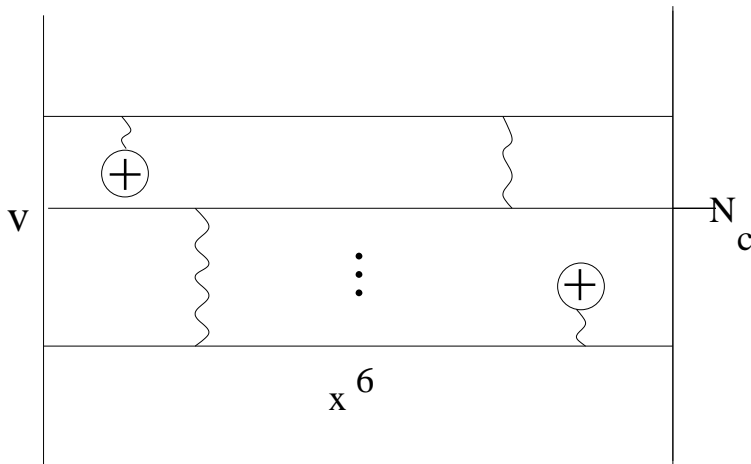[24]See e.g., ref. [64] and references therein for details.

Fig. 3. The configuration of NS-5-branes (two vertical lines),

N_c D-4-branes (horizontal lines and dots), and N_f

D-6-branes (encircled crosses) in type-IIA picture.

(classical) brane- and (quantum) field theory quantities, and (ii) to fix the form of the M-5-brane. Both problems were solved by Witten in ref. [7] (see also refs. [5, 6] for some earlier work on brane technology).

4.4 Effective field theory in worldvolume of type-IIA branes

Exact solutions to the LEEA in $D = 4$, $N = 2$ supersymmetric gauge field theories (for definiteness, the $N = 2$ super-QCD with N_c colors and N_f flavors) can be interpreted (and, in fact, derived) in a nice geometrical way, when considering the effective field theory in the *common* worldvolume (to be identified with our $D = 4$ spacetime $R^{1,3}$) of the magnetically (or RR) charged BPS branes of the type IIA superstring and then resolving classical singularities of these $D = 10$ branes in $D = 11$ M-theory [7].

The relevant 1/4-supersymmetric configuration of the orthogonally intersecting branes in ten dimensions $R^{1,9}$ parameterized by $(x^0, x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9)$ is schematically depictured in Fig. 3. It consists of two parallel (magnetically charged) NS-5-branes, N_c parallel Dirichlet-4-branes orthogonally stretching between the NS-5-branes, and N_f Dirichlet-6-branes which are orthogonal to both NS and D-branes.

The two parallel 5-branes are located at $\vec{w} = (x^7, x^8, x^9) = 0$ and have classically fixed x^6 values. Being parallel to each other but orthogonal to the 5-branes, the 4-branes have their worldvolumes to be parameterized by $(x^0, x^1, x^2, x^3) \in R^{1,3}$ and
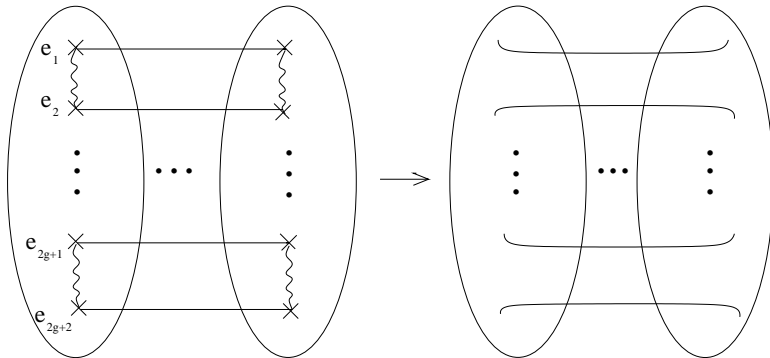
Fig. 4. A hyperelliptic curve of genus g .

x^6. Being orthogonal to both 5-branes and 4-branes, the 6-branes are located at fixed values of (x^4, x^5, x^6), while their worldvolumes are parameterized by $(x^0, x^1, x^2, x^3) \in R^{1,3}$ and $\vec{w} \in R^3$.

After 'blowing-up' the intersecting NS-5- and D-4-brane configuration depictured in Fig. 3, its two-dimensional projection (in the directions orthogonal to the $D = 4$ effective spacetime) looks like as a *hyperelliptic* curve Σ of genus $g = N_c - 1$, depictured in Fig. 4. Indeed, as is well-known in the theory of Riemann surfaces [78], a hyperelliptic (compact) Riemann surface of genus g can be defined by taking two Riemann spheres, cutting each of them between $2g + 2$ ramification (Weierstrass) points e_i, and then identifying the cuts as shown in Fig. 4. The corresponding algebraic (complex) equation reads $y^2 = \prod_{i=1}^{2g+2}(z - e_i)$, with $e_i \neq e_j$ for $i \neq j$. In other words, a two-sheeted cover of the sphere branched over $2g + 2$ points is just the (essentially unique) realization of a hyperelliptic surface of genus $g \geq 1$. Though the surface obtained by the projection of the M-5-brane worldvolume is actually non-compact (it goes through infinity), nevertheless, we are going to formally apply the theory of compact Riemann surfaces to our case (a justification of this procedure is beyond the scope of this paper).

Back to ten dimensions, the 5-brane worldvolumes are, therefore, given by the *local* product $R^{1,3} \times \Sigma_0$, where $R^{1,3}$ is our $D = 4$ spacetime parameterized by the coordinates (x^0, x^1, x^2, x^3), whereas Σ_0 is the singular, of genus $g = N_c - 1$, Riemann surface parameterized by the real coordinates (x^4, x^5) or, equivalently, the complex variable $v \equiv x^4 + ix^5$.

The type-IIA brane interaction in ten dimensions can be visualized as an exchange of open superstrings, even though the ultimate force between some static branes may

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vanish. Associated with the zero modes of such open superstrings are BPS multiplets of a supersymmetric field theory. In particular, a gauge $N = 2$ vector multiplet in the effective $D = 4$ spacetime $R^{1,3}$ may be identified with massless modes of an open $(4 - 4)$ superstring carrying Chan-Paton factors at its ends and stretching between two D-4-branes (Fig. 3), whereas the spacetime matter hypermultiplets are just zero modes of open $(6 - 4)$ superstrings connecting the D-6-branes to the D-4-branes. The BPS mass of a hypermultiplet is determined by the distance (in $x^{4,5}$ directions) between the corresponding D-6-brane and D-4-brane (see also the next subsect. 5). [25]

On physical reasons, the effective gauge coupling constant g_{gauge} of the N=2 supersymmetric gauge field theory in our effective $D = 4$ spacetime should be proportional to a distance between two NS-5-branes [7],

$$\frac{1}{g_{\text{gauge}}^2} = \frac{x_1^{[6]} - x_2^{[6]}}{g_{\text{string}}} \ , \tag{4.22}$$

where g_{string} is the type-IIA superstring coupling constant. Indeed, as was explained by Witten [7], the 5-brane x^6-coordinate should be thought of as a function of v by minimizing the total (BPS !) 5-brane worldvolume. The BPS condition for large v is given by a two-dimensional Laplace equation on x^6, whose solution has a logarithmic dependence upon v for large values of v. Having interpreted $|v|$ as a mass scale in our theory, eq. (4.22) is apparently consistent with the well-known logarithmic behaviour of the four-dimensional effective gauge coupling at high energies in an asymptotically free gauge QFT. The presence of the superstring coupling g_{string} in eq. (4.22) can be justified by the way it appears in the the D-brane effective action induced by open superstrings (ending on a D-brane) in the brane worldvolume. This effective action (or, at least, its bosonic part) is calculable, e.g. by the use of the standard sigma-model approach to open string theory [79]. As a leading contribution, one finds the *Born-Infeld*-type effective action [80]

$$S_{\text{BI}} = T \int_{\text{worldvolume}} e^{-\phi} \sqrt{\det(g_{\mu\nu} + B_{\mu\nu} + 2\pi F_{\mu\nu})} \ , \tag{4.23}$$

where T is a constant brane tension, ϕ is the dilaton field, $g_{\mu\nu}$ is the induced metric in the worldvolume, $B_{\mu\nu}$ and $F_{\mu\nu}$ are the pull-backs of the 2-form B and the abelian field strength F, respectively. The factor $\left\langle e^{-\phi} \right\rangle = 1/g_{\text{string}}$ contributing to the effective brane tension in eq. (4.23) is dictated by the disc topology of the relevant open superstring tree diagram. Extracting from eq. (4.23) the term quadratic in F leads to the denominator on the right-hand-side of eq. (4.22).

---

[25]The $(6 - 6)$ open superstrings decouple from the NS-5- and D-4-branes in the LEEA limit.

To this end, let's summarize the fundamental properties of the brane configuration depictured in Fig. 3, from the 10-dimensional type-IIA point of view:

- its common worldvolume is $(1 + 3)$-dimensional, being infinite in all directions there,

- it is the BPS (stable and unique) supersymmetric solution to the type-IIA supergravity equations of motion,

- it is invariant under $1/2 \times 1/2 = 1/4$ of the maximal supersymmetry, with the first $1/2$ factor being due to the parallel NS-5-branes and the second $1/2$ factor due to the parallel D-4-branes orthogonal to the NS-5-branes; this results in $32/4 = 8$ conserved supercharges of the $N = 2$ extended supersymmetry in the effective $D = 4$ spacetime,

- the D-6-branes, which are orthogonal to both NS-5- and D-4-branes, do not break the $N = 2$ supersymmetry any more; they correspond to the matter hypermultiplets in $D = 4$,

- the ten-dimensional Lorentz group $SO(1,9)$ is broken down to

$$SO(1,3) \otimes SU(2)_A \otimes U(1)_{\text{c.c.}} , \qquad (4.24)$$

in accordance to the local decomposition $R^{1,9} = R^{1,3} \times R^2_v \times R^1_{x^6} \times R^3$, respectively; the $SO(1,3)$ factor of eq. (4.24) can be identified with the Lorentz group of our $D = 4$ spacetime $R^{1,3}$, the rotational symmetry $SO(3)$ of R^3 implies the $N = 2$ supersymmetry automorphisms $SU(2)_A = SO(3)/Z_2$ in $D = 4$, whereas the $U(1) = SO(2)$ factor of eq. (4.24) can be identified with the central charge (v) transformations [44].

These are all the fundamental properties that, in fact, uniquely determine the general BPS brane configuration of Fig. 3. One gets the effective gauge field theory in the common brane worldvolume as the effective spacetime, having the $N = 2$ extended supersymmetry, whose LEEA is under control. In order to accommodate non-perturbative quantum gauge field theory dynamics, the classical brane configuration of Fig. 3 should be 'blown-up'. In the type-IIA picture considered above, as we already know from the previous subsections, the corresponding BPS solution to the type-IIA supergravity suffers from singularities. These singularities cannot be described semi-classically in $D = 10$, but they can be resolved in D=11 after reinterpreting the brane configuration of Fig. 3 in M-Theory (see the next subsection).

4.5 M-Theory resolution

It follows from eq. (4.22) that one can keep the effective $D = 4$ gauge coupling constant g_{gauge} fixed while *increasing* the distance $L = x_1^6 - x_2^6$ between the two NS-5-branes and, simultaneously, the type-IIA superstring coupling constant g_{string} accordingly. As we already know, at strong coupling the type-IIA superstring should be replaced by M-theory. This means that the additional compact dimension (x^{10}) represented by the circle S^1 of radius $R \sim g_{\text{string}}^{2/3}$, can no longer be ignored. Associated with the S^1-rotations is a non-perturbative $U(1)_M$ gauge symmetry.

The low-energy classical description of M-theory and its PBS branes turns out to be sufficient (see the next subsection) for a purely geometrical derivation of exact solutions to the four-dimensional LEEA of the effective $N = 2$ supersymmetric gauge field theories in the M-theory 5-brane worldvolume, just because *all* the relevant distances in the non-perturbative eleven-dimensional BPS brane configuration become *large* while no singularity appears, unlike that in the type-IIA picture considered in the previous subsection. In particular, the D-4-branes and NS-5-branes in the type-IIA picture are replaced in M-Theory by a *single* and *smooth* M-5-brane whose worldvolume is given by the *local* product $R^{1,3} \times \Sigma$, where Σ is the genus $g = N_c - 1$ hyperelliptic Riemann surface *holomorphically* embedded into a four-dimensional hyper-Kähler manifold Q given by the local product of R^3 and S^1. [26] The manifold Q is thus topologically a bundle $Q \sim R^3 \times S^1$ parameterized by the coordinates (x^4, x^5, x^6) and ϱ, whose base R^3 can be interpreted as a part of the D-6-brane worldvolume in the type-IIA picture and whose fiber S^1 is the eleventh dimension of M-Theory [49].

The hyperelliptic curve Σ lies at a single point in R^3. After unifying the real coordinates x^6 and ϱ ($R = 1$) into a single complex coordinate $s = x^6 + i\varrho$ as in ref. [7], the analytic equation defining the Riemann surface Σ should be of the form

$$F(s, v) = 0 \qquad (4.25)$$

with a holomorphic function F. Given a finite number of branes, the function F has to be a polynomial in v and $t = e^{-s}$ [7]. This polynomial can be fixed in terms of the $N = 2$ gauge field theory data $(SU(N_c), N_f, m_i)$ by using standard techniques of the singularity theory (see, e.g. ref. [30]). Eq. (4.25) then takes the form of the Seiberg-Witten curve Σ_{SW} described in subsect. 2.2 [7].

---

[26]The holomorphicity of the embedding as well as the hyper-Kähler nature of the manifold Q are both required by $N = 2$ supersymmetry of the effective field theory in the $R^{1,3}$ spacetime.

Without hypermultiplet matter, the hyper-Kähler manifold Q is flat. In the presence of (magnetically charged) hypermultiplets, the manifold Q is given by a multi-ETN monopole. The BPS bound for any Riemann surface Σ embedded into a hyper-Kähler manifold Q is given by [60]

$$Area_\Sigma \geq \left| \int_\Sigma \Omega \right| \ , \tag{4.26}$$

where the Kähler form Ω of Q [81] has been introduced. The bound (4.26) becomes saturated if and only if Σ is holomorphically embedded into Q, i.e. if the holomorphic description (4.25) of Σ is valid.

The origin of the abelian $N = 2$ vector multiplets in the Coulomb branch of the effective $D = 4$ gauge field theory also becomes more transparent from the M-theory point of view [5]. The effective field theory in the six-dimensional worldvolume of an M-5-brane should have chiral six-dimensional $N = 2$ supersymmetry [82], while the only admissible $N = 2$ chiral supermultiplet in six dimensions is given by the *tensor* supermultiplet having a two-form $B_{(2)}$ with the *self-dual* field strength $T_{(3)}$ (see, e.g. ref. [83]). Indeed, being invariant under 16 linearly realized supersymmetries and having $11 - 6 = 5$ scalar fields describing transverse fluctuations to the 5-brane, the right supermultiplet must have $\frac{1}{2} \cdot 16 - 5 = 3$ additional bosonic on-shell degrees of freedom, which can only be delivered by a bosonic gauge 2-form with self-dual field strength, belonging to an $N = 2$ chiral tensor multiplet in six dimensions. Since, in our case, the M-5-brane is wrapped around the Riemann surface, we can decompose the self-dual 3-form $T_{(3)}$ as

$$T_{(3)} = F_{(2)} \wedge \omega_{(1)} + \overset{*}{_4} F_{(2)} \wedge \overset{*}{_2} \omega_{(1)} \ , \tag{4.27}$$

where $F_{(2)}$ is the two-form in $R^{1,3}$, whereas $\omega_{(1)}$ is the one-form on the Riemann surface Σ_{N_c-1} of genus $N_c - 1$. The equations of motion $dT = 0$ then imply

$$dF = d^* F = 0 \ , \tag{4.28}$$

and

$$d\omega = d^* \omega = 0 \ . \tag{4.29}$$

Eq. (4.29) means that the one-form ω is harmonic on Σ_{N_c-1}. Since the number of the independent harmonic one-forms on a Riemann surface exactly equals to its genus [78], one also has $(N_c - 1)$ two-forms F, while each of them satisfies eq. (4.28). Since eq. (4.28) is nothing but the Maxwell equations for an electro-magnetic field strength F, this explains the origin of the abelian gauge group $U(1)^{N_c-1}$ in the Coulomb branch of the effective $D = 4$ gauge field theory.

To summarize the above-mentioned in this subsection, we conclude that the geo-metrical M-Theory interpretation of the $N = 2$ *gauge* LEEA in the Coulomb branch is given by the identification [5]

$$\Sigma_{N_c-1} = \Sigma_{SW} \ . \tag{4.30}$$

To understand the *hypermultiplet* LEEA (sect. 3) in the similar way, one first notices that the D-6-branes are *magnetically* charged with respect to the non-perturbative $U(1)_M$ gauge symmetry. Hence, the fiber S^1 of Q has to be non-trivial (i.e. of non-vanishing magnetic charge $m \neq 0$). After taking into account the $U(1)_{\rm PG}$ isometry of the internal hypermultiplet NLSM target space $Q_{(m)}$ in the Coulomb branch, one concludes that $Q_{(m)}$ has to be the Euclidean multicentre Taub-NUT space or a multi-KK monopole whose metric was described in subsects. (3.4) and (4.2). In particular, when the magnetic charge $m = 1$, the isometry of $Q_{(1)}$ is enhanced to $U(2)$ and one arrives at the Taub-NUT space whose metric was described in subsect. 3.2 (see Appendix A also).

One may wonder about the M-Theory reinterpretation of the BPS multiplets of the effective four-dimensional $N = 2$ gauge field theory from the $D = 11$ viewpoint, since there are no strings in eleven dimensions, whereas the open superstring zero modes (in $D = 10$) were identified with the $N = 2$ vector and hyper-multiplets in the previous subsection. In $D = 11$ dimensions, there are, however, M-2-branes which are also BPS and can end on an M-5-brane. When considering these M-2-branes as the BPS (of minimal area) deformations of the M-5-brane, one can identify the BPS states in the effective $N = 2$ supersymmetric four-dimensional field theory with zero modes of these M-2-branes. The type of an $N = 2$ supermultiplet is determined by a static M-2-brane topology: a cylinder corresponds to an $N = 2$ vector multiplet, whereas a disc corresponds to a hypermultiplet [60].

4.6   SW solution from classical M-5-brane dynamics

We are now in a position to ask an *educated* question towards a derivation of the Seiberg-Witten exact solution from the brane technology. [27]   We should consider a single M-5-brane in eleven dimensions, whose worldvolume is given by the local product of a flat spacetime $R^{1,3}$ with the hyperelliptic curve Σ of genus $g = N_c - 1$ (see subsect. 4.4), where Σ is supposed to vary in the effective $D = 4$ spacetime $R^{1,3}$. Moreover, in order to be consistent with the rigid $N = 2$ supersymmertry in the

---

[27]Essentially the same reasoning was first suggested in ref. [61]. See also refs. [60, 84].

$D = 4$ spacetime, the Riemann surface Σ should be *holomorphically* embedded into a *hyper-Kähler* four-dimensional manifold Q (say, a milti-ETN), which is a part of the whole eleven-dimensional spacetime given by the local-product $R^{1,6} \times Q$. Since the Nambu-Goldstone-type effective action of the M-5-brane is *uniquely* determined by its symmetries, it is, in principle, straightforward to calculate it explicitly (see refs. [85, 86] for a fully supersymmetric and covariant form of the action). The KK reduction of the six-dimensional M-5-brane action on the complex curve Σ then gives rise to the effective $N = 2$ supersymmetric gauge field theory action which is nothing but the Seiberg-Witten effective action ! This approach to a derivation of the SW result can be considered as the particular application of the theory of integrable systems [32, 33]. To the end of this subsection, we discuss its simplest technical realization (without hypermultiplets, i.e. with a flat hyper-Kähler manifold Q).

The fully covariant and supersymmetric M-5-brane action of refs. [85, 86] is not really needed for this purpose. Even its bosonic part in a flat $D = 11$ supergravity background is too complicated, because of the need to accommodate off-shell the self-duality condition for the field strength $T_{(3)} \equiv dB_{(2)}$ of the 2-form $B_{(2)}$ present in the six-dimensional chiral $N = 2$ tensor multiplet. Rather superficially, the bosonic part of the M-5-brane action has the structure

$$S_{M-5,\ \text{bosonic}} = S_{\text{N--G}} + S_{\text{self--dual}} + S_{\text{WZ}} \ , \qquad (4.31)$$

where $S_{\text{N--G}}$ is the standard 5-brane *Nambu-Goto* (N-G) action, $S_{\text{self--dual}}$ stands for the naive worldvolume integral over T^2 subject to an (implicit) self-duality constraint, whereas S_{WZ} is a higher-derivative *Wess-Zumino* (WZ) term which can be ignored anyway if we are only interested in the leading contribution to the M-5-brane LEEA with two spacetime derivatives at most. Since we are going to restrict ourselves to a calculation of the *scalar* sector of the effective $N = 2$ supersymmetric LEEA in $D = 4$ (the rest of the field theory LEEA depending upon the vector and fermionic components of $N = 2$ vector supermultiplets is entirely determined by the special geometry of the scalar field components due to the $N = 2$ extended supersymmetry), we are allowed to ignore even the second term in the bosonic M-5-brane effective action (4.31). [28]

The N-G action in eq. (4.31) reads

$$S_{\text{N--G}} = T \int \text{Vol}(g) = T \int d^6\xi \sqrt{-\det(g_{\mu\nu})} \ , \qquad (4.32)$$

---

[28]If Σ has a component Σ_f of *finite* volume, the 2-form B may be proportional to the Σ_f volume form, thus leading to an extra scalar. This scalar is, however, compact [84], so that is does not afftect the spacetime part of the effective action that we are interested in here.

in terms of the 5-brane tension T and the induced metric

$$g_{\mu\nu} = \eta_{MN}\partial_\mu x^M \partial_\nu x^N \tag{4.33}$$

in the M-5-brane worldvolume parameterized by the coordinates ξ^μ, where the functions $x^M(\xi)$ describe the embedding of the six-dimensional M-5-brane worldvolume into a flat eleven-dimensional spacetime, $M = 0, 1, \ldots, 10$, and $\mu = 0, 1, \ldots, 5$.

To simplify the form of the N-G action (4.32) before its KK reduction down to four dimensions, we make use of (i) the reparametrizational invariance of this action in six worldvolume dimensions, and (ii) the geometrical information about the M-5-brane configuration collected in subsects. 4.4 and 4.5. The local symmetry (i) allows us to choose a static gauge

$$x^{\underline{\mu}} = \xi^{\underline{\mu}} \ , \tag{4.34}$$

where we have introduced the notation

$$\mu = \{\underline{\mu}, 4, 5\} \ , \quad \text{with} \quad \underline{\mu} = 0, 1, 2, 3 \ . \tag{4.35}$$

We remind the reader that our M-5-brane has $\vec{w} = (x^7, x^8, x^9) = \vec{0}$, the Riemann surface Σ is parameterized by the two coordinates $(v, \bar{v})$, whereas the flat (hyper-Kähler) manifold Q is parameterized by four coordinates $(s, v, \bar{s}, \bar{v})$.

Since we are interested in the M-Theory limit, where the supergravity decouples and the central charge $v = x^4 + ix^5$ of the $D = 4$, $N = 2$ supersymmetry algebra is constant, we are going to assume in what follows that v is $D = 4$ spacetime independent. Moreover, since Σ is supposed to be holomorphically embedded into Q, while Σ is also holomorphically dependent upon its complex moduli u_α, $\alpha = 1, \ldots, g$, an actual dependence of the single remaining non-trivial function s among the M-5-brane embedding functions $x^M(\xi)$ should be *holomorphic*, i.e. of the form

$$s = s(v, u_\alpha(x^{\underline{\mu}})) \ , \tag{4.36}$$

where we have taken into account that $u_\alpha = u_\alpha(x^{\underline{\mu}})$. From the viewpoint of the effective $D = 4$ gauge theory, the complex moduli $u_\alpha(x)$ of the Riemann surface $\Sigma(x)$ are just the scalar field components of $N = 2$ vector multiplets in $D = 4$ spacetime.

The induced metric (4.33) now takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \tfrac{1}{2}\left(\partial_\mu s \partial_\nu \bar{s} + \partial_\nu s \partial_\mu \bar{s}\right) \ , \tag{4.37}$$

as in ref. [61], so that its direct consequences for the N-G action of eq. (4.32) can be simply 'borrowed' from ref. [61]. When keeping only the terms of the second order in the derivatives $\partial_{\underline{\mu}}$ after a substitution of eq. (4.37) into eq. (4.32), one gets the action

$$S[s] = \frac{T}{2} \int d^6\xi \, \eta^{\mu\nu} \partial_\mu s \partial_\nu \bar{s} \ \rightarrow \ S[u] = \frac{T}{2} \int d^6\xi \, \partial_{\underline{\mu}} s \partial^{\underline{\mu}} \bar{s} \ . \tag{4.38}$$

The KK-reduction of the action (4.38) on Σ gives rise to the following four-dimensional NLSM action:

$$S[u] = \frac{T}{4i} \int d^4x \, \partial_{\underline{\mu}} u_\alpha \partial^{\underline{\mu}} \overline{u}_\beta \int_\Sigma \omega^\alpha \wedge \overline{\omega}^\beta \, , \tag{4.39}$$

where the holomorphic 1-forms ω_α on Σ have been introduced as

$$\omega^\alpha = \frac{\partial s}{\partial u_\alpha} dv \, . \tag{4.40}$$

The NLSM metric in eq. (4.39) can be put into another equivalent form, by using the Riemann bilinear identity [78]

$$\int_\Sigma \omega^\alpha \wedge \overline{\omega}^\beta = \sum_{\gamma=1}^g \left(\int_{A_\gamma} \omega^\alpha \int_{B^\gamma} \overline{\omega}^\beta - \int_{A_\gamma} \overline{\omega}^\beta \int_{B^\gamma} \omega^\alpha \right) \, , \tag{4.41}$$

where a canonical (symplectic) basis (A_α, B^β) of the first homology class has been introduced on the Riemann surface Σ_g of genus g, $\alpha, \beta = 1, \ldots, g$, and $g = N_c - 1$. Substituting eq. (4.41) into eq. (4.39) and using the definitions (2.15) of the multi-valued functions $a_\alpha(u)$ and their duals $a_{\mathrm{D}\alpha}(u)$ yields [60, 61, 84]

$$
\begin{aligned}
S[u] &= \frac{T}{4i} \int d^4x \sum_{\alpha=1}^{N_c-1} \left(\partial_{\underline{\mu}} a_\alpha \partial^{\underline{\mu}} \overline{a}_\mathrm{D}^\alpha - \partial_{\underline{\mu}} \overline{a}_\alpha \partial^{\underline{\mu}} a_\mathrm{D}^\alpha \right) = \\
&= -\frac{T}{2} \operatorname{Im} \left[\int d^4x \, \partial_\mu \overline{a}_\alpha \partial^{\underline{\mu}} a_\beta \, \tau^{\alpha\beta} \right] \, ,
\end{aligned}
\tag{4.42}
$$

where the period matrix $\hat{\tau}$ of Σ has been introduced,

$$\tau^{\alpha\beta} = \frac{\partial a_\mathrm{D}^\alpha}{\partial a_\beta} = \frac{\partial^2 \mathcal{F}}{\partial a_\alpha \partial a_\beta} \, . \tag{4.43}$$

Eq. (4.42) precisely gives the scalar part of the full SW effective action in the $N = 2$ super-QCD, as derived in ref. [1]. Because of the NLSM special geometry described by eq. (4.43), a unique $N = 2$ supersymmetric extension of eq. (4.42), including all fermionic- and vector-dependent terms, is just given by the superfield function $\mathcal{F}(W)$ integrated over the $N = 2$ chiral superspace (see sect. 1).

It is straightforward to generalize this derivation of the exact, purely gauge, Seiberg-Witten LEEA to a more general case with hypermultiplet matter by replacing the flat background space Q above with a non-flat hyper-Kähler manifold Q described by a multicentre ETN metric [84].

Being applied to a derivation of the *hypermultiplet* LEEA in $D = 4$, the brane technology suggests to *dimensionally reduce* the effective action of a D-6-brane (see

subsects. 4.2 and 4.4) down to four spacetime dimensions. We already know that in M-Theory the D-6-brane is just a KK monopole described by the non-singular eleven-dimensional metric (4.19). Hence, the induced metric in the 6-brane worldvolume (in a static gauge) is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + G_{ij}(y)\partial_\mu y^i \partial_\nu y^j \ , \tag{4.44}$$

where G_{ij} is the four-dimensional ETN-metric, $\mu, \nu = 0, 1, \ldots, 6$ and $i, j = 1, 2, 3, 4$. Substituting eq. (4.44) into the LEEA (Nambu-Goto) part of the 6-brane effective action,

$$S[y] = \int d^7\xi \sqrt{-\det(g_{\mu\nu})} \ , \tag{4.45}$$

expanding it up to the second-order in derivatives,

$$\sqrt{-\det(g_{\mu\nu})} = \text{const.} - \tfrac{1}{2}\eta^{\mu\nu}g_{\mu\nu} + \ldots \ , \tag{4.46}$$

where the dots stand for higher-derivative terms, and performing plain dimensional reduction from seven to four dimensions (in fact, our fields do not depend upon three irrelevant coordinates already) results in the NLSM action with the Taub-NUT metric,

$$S[y] = -\tfrac{1}{2} \int d^4x \, G_{ij}(y)\partial_{\underline{\mu}} y^i \partial^{\underline{\mu}} y^j \ , \tag{4.47}$$

in full agreement with the QFT results obtained in the $D = 4$, $N = 2$ harmonic superspace (subsect. 3.2).

Therefore, as regards the *leading* (i.e. the second-order in spacetime derivatives, in components) contributions to the $N = 2$ supersymmetric LEEA, the brane technology offers very simple classical tools for their derivation, when compared to the conventional and considerably more complicated QFT methods. Being either *holomorphic* in the case of $N = 2$ vector multiplets or *analytic* in the case of hypermultiplets, the leading contributions are always given by integrals over a half of the $N = 2$ superspace anticommuting coordinates on dimensional reasons, while they are *protected* due to their 'anomalous' nature against higher loop quantum corrections and mixed instanton/anti-instanton contributions. All the higher-order terms (in the spacetime derivatives) in the $N = 2$ supersymmetric LEEA are given by integrals over the whole $N = 2$ superspace, and they are, therefore, 'unprotected' which casts some doubts on the applicability of the brane technology to their derivation. In sect. 5 we discuss the next-to-leading-order correction to the gauge (SW) $N = 2$ LEEA as an example.
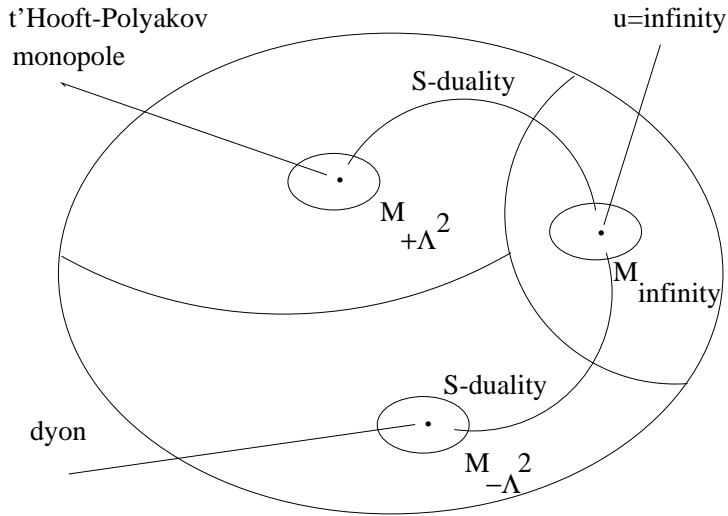
Fig. 5. The quantum moduli space in the Seiberg-Witten model.

4.7 Relation to the HSS results and S duality

The relation between the HSS results in subsect. 3.2 and that of brane technology in subsect. 4.6 towards the hypermultiplet LEEA (in fact, their equivalence) is provided by the *S-duality* in field theory (Fig. 5).

Consider, for simplicity, the famous Seiberg-Witten model [1] whose fundamental action describes the purely gauge $N = 2$ super-Yang-Mills theory with the $SU(2)$ gauge group spontaneously broken to its $U(1)_e$ subgroup.

On the one hand, in the strong coupling region of the Coulomb branch, e.g. near a singularity in the quantum moduli space where a BPS-like (t'Hooft-Polyakov) monopole becomes massless, the Seiberg-Witten (SW) theory is just described by the S-dual $N = 2$ supersymmetric QED. In particular, the t'Hooft-Polyakov (HP) monopole belongs to a *magnetically* charged hypermultiplet q^+_{HP} which represents the non-perturbative degrees of freedom in the strongly coupled (Coulomb) branch of the SW theory under consideration (Fig. 5). The HSS results of subsect. 3.2 imply that the HP hypermultiplet self-interaction in the vicinity of the HP-monopole singularity is regular in terms of the magnetically dual variables,

$$\mathcal{K}^{(+4)}_{\text{Taub-NUT}}(q^+_{HP}) = \frac{\lambda_{\text{dual}}}{2} \left(\overset{*}{q^+}_{HP} q^+_{HP} \right)^2 , \tag{4.48}$$

i.e. it is given by the NLSM with the Taub-NUT metric.

On the other hand, from the type-IIA superstring (or M-theory) point of view, the HP-hypermultiplet is just the zero mode of the open superstring stretching between

a magnetically charged D-6-brane and a D-4-brane. Therefore, the magnetically charged (HP) hypermultiplet is the only $N = 2$ matter that survives in the effective four-dimensional $N = 2$ gauge theory (given by the $N = 2$ super-QED) near the HP singularity after taking the proper LEEA limit of the brane configuration where the supergravity decouples. According to the preceeding subsections, the NLSM target space geometry governing the induced HP hypermultiplet self-interaction has to be that of Taub-NUT (or KK-monopole) again !

To conclude this section, we would like to return to the well-known and important phenomenon of the *symmetry enhancement,* by using the particular case of two coinciding D-6-branes 'on top of each other' as an example. Their M-Theory resolution was described in subsect. 4.2 in terms of the hyper-Kähler metric governed by the harmonic potential (4.21) and describing two nearly coinciding ETN monopoles with equal charges put on a line going through the origin. The associated *double* (i.e. two-centered) ETN metric is, in fact, equivalent to the *mixed* EH-ETN metric defined by the following NLSM action [87]:

$$
S_{\text{mixed}}[q_A, V_{\text{L}}] = \int_{\text{analytic}} \left\{ q_A^{a+} D^{++} q_{aA}^+ + V_{\text{L}}^{++} \left(\tfrac{1}{2} \varepsilon^{AB} q_A^{a+} q_{aB}^+ + \xi^{++} \right) \right.
$$
$$
\left. + \tfrac{1}{8} \lambda \left(q_A^{a+} q_{aA}^+ \right)^2 \right\} ,
\tag{4.49}
$$

written down in terms of a gauged $O(2)$ analytic superfield q_A^+, $A = 1, 2$, and the associated $O(2)$ vector gauge analytic superfield (Lagrange multiplier) V_{L}^{++} having no kinetic term. The hyper-Kähler metric in components, corresponding to the HSS action (4.49) leads to the harmonic potential (4.21) of the double ETN monopole, as was shown by an explicit calculation in ref. [58]. Moreover, the action (4.49) obviously respects the $U(1)_A \times U(1)_{\text{PG}}$ symmetry. In the limit $\lambda = 0$, eq. (4.49) leads to the EH metric (see sect. 6), whereas in another limit $\xi^{++} \equiv \xi^{ij} u_i^+ u_j^+ = 0$ it results in the ETN metric up to a superfield redefinition [58]. Stated differently, the HSS action (4.49) interpolates between the ETN action (see Appendix A for details) and the EH action (see Appendix B for details), as well as between the corresponding hyper-Kähler metrics.

Unlike the description in terms of the harmonic potential (4.21) which is not manifestly $N = 2$ supersymmetric and looks singular, the equivalent description in terms of the action (4.49) is fully non-singular and manifestly $N = 2$ supersymmetric. In particular, the coincidence limit $\epsilon \to 0$ in eq. (4.21) corresponds to the limit $\xi \to 0$ in eq. (4.49) where it obviously gives rise to the enhanced $U(2)$ isometry since any explicit dependence of the HSS lagrangian of eq. (4.49) upon harmonics drops in this limit.

Fig. 6. The one-loop series for the N=2 vector gauge

LEEA . The internal loop is made out of the

hypermultiplet or N=2 superghost propagators .

5 Next-to-leading-order correction to the Seiberg-Witten LEEA

The next-to-leading-order correction to the $N = 2$ gauge (Seiberg-Witten) LEEA in the Coulomb branch is determined by a real function $\mathcal{H}$ of W *and* $\overline{W}$, which is supposed to be integrated over the whole $N = 2$ superspace. Since the full $N = 2$ superspace measure $d^4 x d^8 \theta$ is dimensionless, the function $\mathcal{H}(W, \overline{W})$ should also be dimensionless, i.e. without any $N = 2$ superspace derivatives [88]. Moreover, the exact function

$$\mathcal{H}(W, \overline{W}) = \mathcal{H}_{\mathrm{per.}}(W, \overline{W}) + \mathcal{H}_{\mathrm{non-per.}}(W, \overline{W}) \tag{5.1}$$

has to be S-duality invariant [89].

It is also worth mentioning that the non-holomorphic function $\mathcal{H}$ is only defined modulo Kähler gauge transformations

$$\mathcal{H}(W, \overline{W}) \to \mathcal{H}(W, \overline{W}) + f(W) + \bar{f}(\overline{W}) \ , \tag{5.2}$$

with an arbitrary holomorphic function $f(W)$ as a gauge parameter.

Within the manifestly $N = 2$ supersymmetric background-field approach in $N = 2$ HSS, the one-loop contribution to $\mathcal{H}_{\mathrm{per.}}$ is given by a sum of the $N = 2$ HSS graphs schematically pictured in Fig. 6. The sum goes over the external V^{++}-legs, whereas the loop consists of the $N = 2$ matter (and $N = 2$ ghost) superpropagators (sect. 3).

$N = 2$ ghosts contribute in very much the same way as $N = 2$ matter does, since the $N = 2$ ghosts are also described in terms of the FS- and HST-type hypermultiplets (with the opposite statistics of components) in the $N = 2$ HSS [43]. Because of the (abelian) gauge invariance, the result can only depend upon the abelian $N = 2$ superfield strength W and its conjugate $\overline{W}$ via eq. (3.21). In fact, Fig. 6 also determines the one-loop perturbative contribution to the leading holomorphic LEEA, which appears as the *anomaly* associated with the non-vanishing central charges [18]. The self-energy $N = 2$ HSS supergraph with only two external legs in Fig. 6 is the only one which is UV-divergent there. The IR-divergences of all the HSS graphs in Fig. 6 can be regularized by introducing the IR-cutoff Λ, which is proportional to the Seiberg-Witten scale Λ_{SW} defined in sect. 2, with the relative coefficient being dependent upon the actual renormalization scheme used (see e.g., ref. [90] for more details). In the case of a single q^+-type (FS) matter hypermultiplet in an abelian $N = 2$ gauge theory, an evaluation of the infinite HSS series depictured in Fig. 6 yields [42, 43, 90, 91] [29]

$$\mathcal{F}_q(W) = -\frac{1}{32\pi^2} W^2 \ln \frac{W^2}{M^2} , \qquad (5.3)$$

where the renormalization scale M has been fixed by the condition $\mathcal{F}_q(M) = 0$, and

$$\mathcal{H}_q(W, \overline{W}) = \frac{1}{(16\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \left(\frac{W\overline{W}}{\Lambda^2} \right)^k = \frac{1}{(16\pi)^2} \int_0^{W\overline{W}/\Lambda^2} \frac{d\xi}{\xi} \ln(1 + \xi) , \quad (5.4)$$

where we have used the standard integral representation for the dilogarithm function. It is not difficult to verify that the asymptotical perturbation series (5.4) can be rewritten as

$$\mathcal{H}_q(W, \overline{W}) = \frac{1}{(16\pi)^2} \ln \left(\frac{W}{\Lambda} \right) \ln \left(\frac{\overline{W}}{\Lambda} \right) , \qquad (5.5)$$

or, equivalently, as

$$\mathcal{H}_q(W, \overline{W}) = \frac{1}{2(16\pi)^2} \ln^2 \left(\frac{W\overline{W}}{\Lambda^2} \right) , \qquad (5.6)$$

modulo irrelevant Kähler gauge terms, see eq. (5.2). The non-holomorphic contribution of eq. (5.5) or (5.6) does not really depend upon the scale Λ, again due to the Kähler invariance (5.2) [92].

The HSS result (5.3) for the perturbative part of the SW $N = 2$ gauge LEEA agrees with the well-known Seiberg argument [93] based on the perturbative $U(1)_R$ symmetry and an integration of the associated chiral anomaly. As is obvious from eq. (5.5), the next-to-leading-order non-holomorphic contribution to the SW gauge

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[29]Our overall normalization differs from that in refs. [42, 43, 91] by some factors of 2.

LEEA satisfies a simple differential equation [90]

$$W\overline{W}\partial_W\partial_{\overline{W}}\mathcal{H}_q(W,\overline{W}) = const. \tag{5.7}$$

which can be considered as the direct consequence of scale and $U(1)_R$ invariances (*cf.* ref. [92]).

In a more general case of N_f FS-type hypermultiplets in the fundamental representation of the gauge group $SU(N_c)$, i.e. the $N = 2$ super-QCD, the extra coefficient in front of the holomorphic contribution $\mathcal{F}$ is proportional to the one-loop RG beta-function $(N_f - 2N_c)$, whereas the extra coefficient in front of the non-holomorphic contribution $\mathcal{H}$ is proportional to $(2N_f - N_c)$, in the $N = 2$ super-Feynman gauge [90]. In another interesting case of the $N = 4$ super-Yang-Mills theory, whose $N = 2$ matter content is given by a single HST-type hypermultiplet in the *adjoint* representation of the gauge group, the numerical coefficient in front of the holomorphic function $\mathcal{F}$ vanishes together with the RG beta-function, whereas the numerical coeffient in front of the non-holomorphic contribution $\mathcal{H}$ always appears to be positive, in agreement with the earlier calculations in terms of $N = 1$ superfields (see page 390 in the second reference of [15]) and some recent $N = 2$ supersymmetric calculations by different methods [91, 94]. In the case of finite and $N = 2$ *superconformally* invariant gauge field theories $(N_f = 2N_c)$, the leading non-holomorphic contribution to the LEEA is given by eq. (5.5) multiplied by $3N_c$, so that it never vanishes too.

It is also straightforward to check in the HSS approach that there are no *two-loop* contributions to $\mathcal{F}_{\text{per.}}$ *and* $\mathcal{H}_{\text{per.}}$, since all the relevant HSS graphs shown in Fig. 7 do not actually contribute in the local limit. This conclusion is in agreement with calculations in terms of the $N = 1$ superfields [95], and it is also consistent with the general perturbative structure of the $N = 2$ supersymmetric gauge field theories within the manifestly $N = 2$ supersymmetric framework of the background-field method in $N = 2$ HSS [18]. It seems, therefore, to be conceivable that *all* higher-loop (perturbative) contributions to $\mathcal{H}_{\text{per.}}(W, \overline{W})$ may be absent too. It should definitely be the case in the $N = 2$ superconformally invariant gauge field theories and the $N = 4$ SYM theory as well, since any higher loop perturbative contributions would break the scale and $U(1)_R$ invariances of these theories. As regards the non-perturbative (instanton) contributions to $\mathcal{H}$, they are also expected to vanish in the four-dimensional gauge field theories with extended *superconformal* invariance by the same reasoning. Possible non-perturbative instanton corrections in the finite $N = 2$ supersymmetric gauge field theories were studied in refs. [96, 97], where their absence was argued within the standard instanton approach. Stated differently, the one-loop
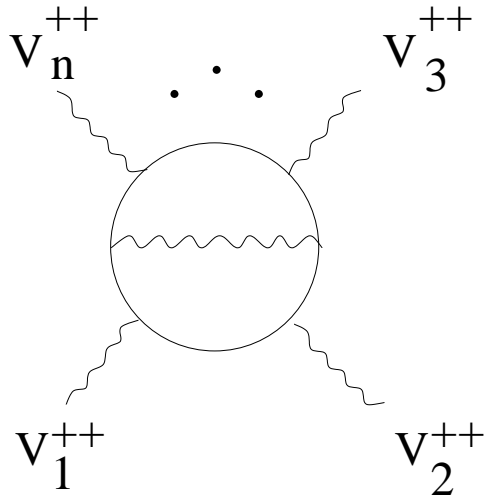
Fig. 7. A two-loop N=2 HSS graph for the N=2

vector gauge LEEA in the N=2 HSS .

non-holomorphic result above represents the exact leading contribution to the LEEA in the $N = 2$ superconformally invariant gauge field theories in $D = 4$.

An exact result for the real function $\mathcal{H}(W, \overline{W})$ in a non-superconformally invariant $N = 2$ gauge field theory is still unknown. There is, however, an interesting proposal due to Matone [98] that the exact function $\mathcal{H}(W, \overline{W})$ may satisfy a non-linear differential equation

$$\partial_{\overline{W}} \partial_W \ln \left[\mathcal{H} \partial_W \partial_{\overline{W}} \ln \mathcal{H} \right] = 0 \ , \tag{5.8}$$

which can be interpreted as the fully non-perturbative non-chiral superconformal 'Ward identity'. [30] The leading one-instanton contribution in the pure $N = 2$ gauge (SW) theory was already calculated in ref. [99], and it does not vanish. The full non-perturbative contribution is unlikely to be given by the sum over instanton contributions alone, since it can also include (multi)anti-instanton and mixed (instanton-anti-instanton) contributions, which are allowed in a non-conformal theory.

The brane technology of sect. 4 might offer the alternative way of calculating the exact next-to-leading-order non-holomorphic correction, e.g. by using the covariant M-5-brane action describing the classical M-theory 5-brane dynamics, simply by expanding it further in powers of the spacetime derivatives, up to the fourth-order terms in components. Such an investigation was done by de Boer, Hori, Ooguri and

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[30]An explicit solution to eq. (5.8) was also proposed in ref. [98].

Oz in ref. [84], who showed that the actual results obtained from the brane technology considerably differ from those of the $N = 2$ quantum gauge field theory, as regards the non-holomorphic terms in the LEEA. There is only one dynamically generated dimensionful scale Λ in the four-dimensional LEEA of the QFT under consideration, whereas there are at least two dimensionful parameters in M-Theory, namely, the radius R of the compactified 11-th dimension and the typical scale L of the brane configuration. As was explained in ref. [84], the non-holomorphic contributions derived from the M-5-brane effective action are dependent upon the radius R in a highly non-trivial way, while this dependence does not decouple in any simple limit. From the physical viewpoint, this implies the absence of a natural field theory limit where the supergravity and the associated KK modes decouple from the bulk.

Yet another possible approach to an explicit calculation of non-perturbative non-holomorphic terms in the $N = 2$ field theory LEEA may be based on the use of $N = 2$ HSS in instanton-type calculations, which is, however, yet to be developed.

6   Hypermultiplet LEEA in baryonic Higgs branch

The Higgs branch can be naturally divided into two phases called 'baryonic' and 'non-baryonic' according to ref. [100]. In the baryonic phase, Fayet-Iliopoulos (FI) terms are present so that the color symmetry is completely broken. In the non-baryonic phase, all the FI terms vanish so that a non-trivial gauge subgroup of the color symmetry survives. In this section we are going to discuss the baryonic phase only. It intersects with the Coulomb branch at a *single* point in the quantum moduli space of the Coulomb branch [100].

As was already mentioned in sect. 3, the most natural and manifestly $N = 2$ supersymmetric description of hypermultiplets in the Higgs branch is provided by HSS in terms of the HST-type analytic superfield ω of vanishing $U(1)$ charge. The $N = 2$ HSS is also the quite natural framework to address possible symmetry breakings.

The free action of a single ω superfield reads

$$S[\omega] = -\frac{1}{2} \int_{\text{analytic}} \left(D_A^{++}\omega \right)^2 \ . \tag{6.1}$$

Similarly to the free action (3.11) for a q^+-type analytic superfield, the action (6.1) also possesses the extended internal symmetry

$$SU(2)_A \otimes SU(2)_{\text{PG}} \ , \tag{6.2}$$

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where $SU(2)_A$ is well-known automorphism symmetry of the $N = 2$ supersymmetry algebra (sometimes also called the $SU(2)_R$ symmetry). The additional $SU(2)_{\text{PG}}$ symmetry of eq. (6.1) is less obvious [44] :

$$\delta\omega = c^{--} D_A^{++}\omega - c^{+-}\omega \ , \tag{6.3}$$

where $c^{--} = c^{(ij)}u_i^- u_j^-$ and $c^{+-} = c^{(ij)}u_i^+ u_j^-$, and $c^{(ij)}$ are the infinitesimal parameters of $SU(2)_{\text{PG}}$.

It is quite clear now that it is not possible to construct any non-trivial self-interaction in terms of the $U(1)$-chargeless superfield ω alone, without breaking the $SU(2)_A$ symmetry, simply because a hyper-Kähler potential has to have the non-vanishing $U(1)$ charge $(+4)$. Hence, when $N = 2$ supersymmetry and the $SU(2)_A$ internal symmetry are not broken, one gets the well-known result [1]:

$$\mathcal{K}^{(+4)}{}_{\text{Higgs}}(\omega) = 0 \ , \tag{6.4}$$

i.e. the induced hyper-Kähler metric in the fully $N = 2$ supersymmetric Higgs branch is flat and, in particular, it does not receive quantum corrections.

It is, however, possible to break the internal symmetry (6.2) down to

$$U(1)_A \otimes SU(2)_{\text{PG}} \ , \tag{6.5}$$

by introducing the FI term

$$\left\langle D^{ij} \right\rangle = \xi^{ij} = const. \neq 0 \ , \tag{6.6}$$

which may softly break the $N = 2$ supersymmetry too [101]. This way of symmetry breaking still allows one to maintain control over the $N = 2$ supersymmetric hypermultipet LEEA due to the presence of the non-abelian internal symmetry (6.5). The only non-trivial hyper-Kähler potential, which is invariant with respect to the symmetry (6.5), is given by [44]

$$\mathcal{K}^{(+4)}{}_{\text{EH}}(\omega) = -\frac{(\xi^{++})^2}{\omega^2} \ , \tag{6.7}$$

where $\xi^{++} = \xi^{ij}u_i^+ u_j^+$. It is straightforward to deduce the corresponding hyper-Kähler metric from eq. (6.7) by using the procedure already described in subsect. 3.2. One finds that the metric is equivalent to the standard *Eguchi-Hanson* (EH) instanton metric in four dimensions [87, 102] (see Appendix B also).

In order to understand better the origin of the EH-metric from HSS, it is quite useful to employ the *gauging* procedure of generating hyper-Kähler metrics (see ref. [103]

for a review). In the $N = 2$ HSS, the additional resources for generating new hyper-Kähler potentials are given by (i) gauging isometries and (ii) adding (electric) FI terms. For instance, given two FS-type hypermultiplets $q_A^+ \in \mathbf{2}$ of $SU(2)_f$, one can gauge a $U(1)$ subgroup of $SU(2)_f$ and simultaneously add a FI term as follows [87]:

$$S_{\text{EH}} = \int_{\text{analytic}} \left\{ q_A^{a+} D^{++} q_{aA}^+ + V_{\text{L}}^{++} \left(\tfrac{1}{2} \varepsilon^{AB} q_A^{a+} q_{aB}^+ + \xi^{++} \right) \right\} , \qquad (6.8)$$

where V_{L}^{++} is the corresponding $N = 2$ vector gauge potential without a kinetic term (i.e. the Lagrange multiplier), $\xi^{++} = \xi^{(ij)} u_i^+ u_j^+$, and $A, B = 1, 2$. The action (6.8) has the manifest PG symmetry $SU(2)_{\text{PG}}$, which rotates the lower-case Latin indices only. The $SU(2)_A$ symmetry is explicitly broken down to its $U(1)_A$ subgroup by a non-vanishing FI-term ξ^{++}. After some algebra, the Lagrange multiplier $N = 2$ superfield V_{L}^{++} and one of the hypermultiplet superfields can be eliminated, while the resulting action equivalent to eq. (6.8) takes the form [87]

$$S_{\text{EH}} = \int_{\text{analytic}} \left\{ q^{a+} D^{++} q_a^+ - \frac{(\xi^{++})^2}{(q^{a+} u_a^-)^2} \right\} . \qquad (6.9)$$

Changing the variables $q_a^+ = u_a^+ \omega + u_a^- f^{++}$, in terms of the dual ω-type hypermultiplet and yet another Lagrange multiplier f^{++}, gives us the HSS action with the hyper-Kähler potential (6.7) after eliminating f^{++} according to its algebraic HSS superfield equations of motion. It should be noticed that the hyper-Kähler potential (6.7) already implies that $\langle \omega \rangle \neq 0$, so that we are in the Higgs branch indeed.

Let's now try to understand how a FI-term could be non-perturbatively generated. First, we can slightly generalize this problem by allowing non-vanishing vacuum expectation values for all bosonic components of the abelian $N = 2$ superfield strength W,

$$\langle W \rangle = \left\{ \ \langle a \rangle = Z \ , \quad \langle F_{\mu\nu} \rangle = n_{\mu\nu} \ , \quad \langle \vec{D} \rangle = \vec{\xi} \ \right\} , \qquad (6.10)$$

where all the parameters $(Z, n_{\mu\nu}, \vec{\xi})$ are constants. Generally speaking, it also implies *soft* $N = 2$ supersymmetry breaking [101]. We already know about the physical meaning of Z — it is just the complex central charge in the $N = 2$ supersymmetry algebra. The related gauge-invariant quantity $u \sim \langle \text{tr} \, a^2 \rangle$ parameterizes the quantum moduli space of vacua in the Coulomb branch. The central charge Z can be naturally generated via the standard Scherk-Schwarz mechanism of dimensional reduction from six dimensions [44]. Similarly, $n_{\mu\nu} \neq 0$ can be interpreted as a *toron* background after replacing the effective spacetime R^{1+3} by a hypertorus T^{1+3} and imposing t'Hooft's twisted boundary conditions [104]. The $\vec{\xi} \neq 0$ is just a FI term.

The brane technology can help us to address the question of *dynamical* generation of both $n_{\mu\nu}$ and $\vec{\xi}$ in a very geometrical way [44]. Namely, let's deform the brane

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configuration of Fig. 3 by allowing some of the NS-5- or D-4-branes to *intersect at angles* instead of being parallel ! Indeed, the vector $\vec{w} = (x^7, x^8, x^9)$ is the same in Fig. 3 for both solitonic 5-branes. Its non-vanishing value

$$\vec{\xi} = \vec{w}_1 - \vec{w}_2 \neq 0 \tag{6.11}$$

effectively generates the FI term. Similarly, when allowing the D-4-branes to intersect at angles, some non-trivial values of $\langle F_{\mu\nu} \rangle = n_{\mu\nu} \neq 0$ are generated [105].

Since the spacetime LEEA of BPS branes is governed by a gauge field theory, it does not seem to be very surprising that torons can also be understood as the BPS bound states of certain D-branes in the field theory limit $M_{\text{Planck}} \to \infty$ [105]. Moreover, torons are known to generate a gluino condensate [106]

$$\left\langle \lambda^i \lambda^j \right\rangle = \Lambda^3 (\xi^2)^{ij} , \quad \xi^{ij} \sim \delta^{ij} \exp\left(-\frac{2\pi^2}{g^2} \right) , \tag{6.12}$$

where $\vec{\xi} \sim \{\xi^{ij}\}$ have to be constant [107], so that they can be identified with the FI term by $N = 2$ supersymmetry.

Finally, it can be useful to understand the origin of the hypermultiplet EH-type self-interaction in the Higgs branch from the viewpoint of brane technology. It is worth mentioning here that the D-4-branes can also end on the D-6-branes (in the type-IIA picture), while these D-4-branes actually support hypermultiplets, not $N = 2$ vector multiplets [7]. It results in the hyper-Kähler manifold Q with has the different (versus ETN) topology $\sim S^3/Z_2$ in its spacial infinity. It now suffices to mention that the EH-instanton is the only hyper-Kähler manifold having this topology among the four-dimensional ALF spaces !

7   Conclusion

Though being very different, the three approaches depictured in Fig. 1, namely,

  (i) instanton calculus,

 (ii) Seiberg-Witten approach and M theory (=brane technology),

(iii) harmonic superspace,

lead to the *consistent* results, as regards the *leading* terms in the LEEA of the four-dimensional $N = 2$ supersymmetric gauge field theories. It is the second (ii) and third

(iii) approaches that were extensively discussed in this paper, whereas the instanton calculus (i) was merely mentioned. Apparently, there is no single universal method to handle the whole range of problems associated with non-perturbative $D = 4$ supersymmetric gauge field theories in a natural and easy way. Instead, each approach has its own advantages and disadvantages. For example, in the Seiberg-Witten approach, the physical information is encoded in terms of functions defined over the quantum moduli space whose modular group is identified with the S-duality group. The SW approach is based on knowing exact perturbative limits of the non-abelian $N = 2$ gauge field theory under consideration, whereas the HSS approach is the most efficient one in quantum perturbative calculations. At the same time, the HSS approach up to now cannot be directly applied to address truly non-perturbative phenomena yet. In this paper, the HSS description of some non-perturbative features related to hypermultiplets was only used in combination with the strong-weak couling duality (=S-duality). In its turn, the instanton calculus is very much dependent upon applicability of its own basic assumptions, while it is not manifestly supersymmetric at any rate if it is supersymmetric at all. Moreover, the known instanton methods sometimes need an additional input. Being geometrically very transparent, the M-Theory brane technology still has a limited analytic support, whereas its successful applications were limited so far to only those terms in the LEEA which are protected by anomalous symmetries, i.e. which are either holomorphic or analytic. In any case, some care should be exercised to play safely with the brane technology. I believe, it is merely a clever combination of all the methods available that has the strongest potential for further progress, while it can simultaneously teach us how to proceed with each particular approach.

I would like to conclude with a few comments about $N = 2$ supersymmetry breaking and confinement, in order to indicate on a possible importance of the exact hypermultiplet low-energy effective action towards a solution to these problems. Indeed, it seems to be quite natural to take advantage of the existence of exact solutions to the low-energy effective action in $N = 2$ supersymmetric gauge field theories, and apply them to the old problem of color confinement in QCD. In fact, it was one of the main motivations in the original work of Seiberg and Witten [1]. The most attractive mechanism for color confinement is known to be the dual Meissner effect or the dual (Type II) superconductivity [108]. It takes three major steps to connect an ordinary BCS superconductor to the simplest Seiberg-Witten model in quantum field theory: first, one defines a *relativistic* version of the superconductor, known as the (abelian) Higgs model in field theory, second, one introduces a *non-abelian* version of the Higgs model, known as the Georgi-Glashow model, and, third, one *N=2 supersymmetrizes*

the Georgi-Glashow model in order to get the Seiberg-Witten model [1]. Since the t'Hooft-Polyakov monopole of the Georgi-Glashow model belongs to a (HP) hypermultiplet in its $N = 2$ supersymmetric (Seiberg-Witten) generalization, it is quite natural to explain confinement as the result of a monopole condensation (= the dual Meissner effect as the consequence of the dual Higgs effect), due to a non-vanishing vacuum expectation value for the magnetically charged (dual Higgs) scalar belonging to the HP hypermultiplet. Of course, it is only possible after a judicious $N = 2$ supersymmetry breaking.

It is worth emphasizing that our whole discussion in this paper was essentially based on having eight conserved supercharges or, equivalently, the $N = 2$ extended supersymmetry in the $D = 4$ spacetime. Accordingly, most of the results are not applicable to a more interesting case of $D = 4$ gauge field theories with only $N = 1$ supersymmetry (see e.g., ref. [109] for a review of the known field theory results, and ref. [8] for a review of brane technology with less than eight supercharges). An exact solution to the low-energy effective action of a $D = 4$ quantum gauge field theory is generically available only with $N = 2$ supersymmetry which is directly connected to integrable systems, and neither with merely $N = 1$ supersymmetry nor in the bosonic QCD. Therefore, on the one hand side, it is the $N = 2$ supersymmetry that crucially simplifies an evaluation of the $D = 4$ QFT low-energy effective action. However, on the other hand side, it is the same $N = 2$ supersymmetry in $D = 4$ that is obviously incompatible with phenomenology e.g., because of equal masses of bosons and fermions inside $N = 2$ supermultiplets (this is applicable, in fact, to any $N \geq 1$ supersymmetry), and the non-chiral nature of the $N = 2$ supersymmetry (e.g. 'quarks' can only appear in *real* representations of the gauge group). Therefore, if one believes in the fundamental role of $N = 2$ supersymmetry in high energy physics, one has to find a way of judicious $N = 2$ supersymmetry breaking. The associated with supersymmetry breaking, dual Higgs mechanism may then be responsible for the chiral symmetry breaking and the appearance of the pion effective Lagrangian too, provided that the dual Higgs field has flavor charges [1]. In fact, Seiberg and Witten [1] used a mass term for the $N = 1$ chiral multiplet which is a part of the $N = 2$ vector multiplet, in order to *softly* break $N = 2$ supersymmetry to $N = 1$ supersymmetry 'by hand'. As a result, they found a non-trivial vacuum solution with a monopole condensation and, hence, a confinement. The weak point of their approach is an *ad hoc* assumption about the existence of the mass gap, i.e. the mass term itself.

The $N = 2$ supersymmetry can be broken either softly or spontaneously, if one wants to preserve the benefits of its presence (e.g. maintaining the full control over the

low-energy effective action) at high energies. A detailed investigation of *soft* $N = 2$ supersymmetry breaking in the $N = 2$ supersymmetric QCD was done by Alvarez-Gaumé, Mariño and Zamora [101]. [31] The soft $N = 2$ supersymmetry breaking is most naturally done by the use of FI-terms [101]. Though being pragmatic, the soft $N = 2$ supersymmetry breaking has a limited predictive power because of many parameters, whose number, however, is significatly less than that in the $N = 1$ case. It may be more reasonable to search for *spontaneous* $N = 2$ supersymmetry breaking, where the non-vanishing FI-terms would appear as stationary solutions to the dynamically generated scalar potential. This would imply the existence of a non-supersymmetric vacuum solution for the exact $N = 2$ supersymmetric scalar potential which is quite difficult to get. Since the $N = 2$ supersymmetry remains unbroken for any exact Seiberg-Witten solution in the gauge sector, we may have to consider the induced scalar potentials in the hypermultiplet sector of an $N = 2$ gauge theory in $D = 4$. Indeed, given non-trivial kinetic terms in the hypermultiplet low-energy effective action to be represented by a hyper-Kähler NLSM, in the presence of non-vanishing central charges they also imply a non-trivial hypermultiplet *scalar* potential whose form is entirely determined by the hyper-Kähler metric of the kinetic terms and $N = 2$ supersymmetry (see Appendices A and B for two explicit examples). Though it may not be easy to search for the most general solution with spontaneously broken $N = 2$ supersymetry, partly because of complications associated with general hyper-Kähler geometries having no isometries, our toy examples presented in the Appendices are enough to demonstrate a richness of opportunities there.

Acknowledgements

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[31]See e.g., ref. [110] for a similar analysis in $N = 1$ supersymmetric gauge field theories.

Appendix A: Taub-NUT hypermultipet self-interaction in components from harmonic superspace

In this Appendix we explicitly calculate the induced (effective) metric and the induced (effective) scalar potential in components from the $N = 2$ harmonic superspace results (subsect. 3.2) for a single charged FS-type hypermultiplet in the Coulomb branch. The corresponding hypermultiplet LEEA is given by the NLSM, whose action in the analytic subspace $\zeta^M = (x_A^m, \theta^+{}_\alpha, \bar{\theta}^+{}_{\dot{\alpha}})$ of the $N = 2$ HSS reads [32]

$$S_{\text{ETN}}[q] = -\int d\zeta^{(-4)} du \left\{ \overset{*}{q}{}^+ D^{++} q^+ + \frac{\lambda}{2}(q^+)^2(\overset{*}{q}{}^+)^2 \right\} , \tag{1}$$

where the HSS covariant derivative D^{++}, in the analytic basis and with non-vanishing central charges Z and $\overline{Z}$, has been introduced,

$$D^{++} = \partial^{++} - 2i\theta^+\sigma^m\bar{\theta}^+\partial_m + i\theta^+\bar{\theta}^+\overline{Z} + i\bar{\theta}^+\bar{\theta}^+ Z , \tag{2}$$

whereas λ is the induced (Taub-NUT) NLSM coupling constant. Eq. (2) can be most naturally obtained by (Scherk-Schwarz) dimensional reduction from six dimensions [44]. For simplicity, we ignore here possible couplings to an abelian $N = 2$ vector superfield. An explicit expression for λ in terms of the fundamental gauge coupling and the hypermultiplet BPS mass was calculated in ref. [44]. Our q^+ superfields are of dimension minus one (in units of length), while the coupling constant λ is of dimension two.

Our aim here is to find the component form of the action (1). Without central charges it was done in ref. [46]. The non-vanishing central charges were incorporated in ref. [44]. Here we are going to concentrate on a derivation of the component metric *and* the scalar potential originating from the $N = 2$ HSS action (1). The corresponding equations of motion are

$$D^{++}q^+ + \lambda(q^+ \overset{*}{q}{}^+)q^+ = 0 \quad \text{and} \quad D^{++} \overset{*}{q}{}^+ - \lambda(q^+ \overset{*}{q}{}^+) \overset{*}{q}{}^+ = 0 . \tag{3}$$

Since we are only interested in the purely bosonic part of the action (1), we drop all fermionic fields in the θ^+, $\bar{\theta}^+$ expansion of q^+,

$$
\begin{aligned}
q^+(\zeta, u) =\ & F^+(x_A, u) + i\theta^+\sigma^m\bar{\theta}^+ A_m^-(x_A, u) \\
& + \theta^+\theta^+ M^-(x_A, u) + \bar{\theta}^+\bar{\theta}^+ N^-(x_A, u) \\
& + \theta^+\theta^+\bar{\theta}^+\bar{\theta}^+ P^{(-3)}(x_A, u) .
\end{aligned}
\tag{4}
$$

---

[32] We use a separate numeration of equations in each Appendix.

After being substituted into eq. (3), eq. (4) yields

$$\partial^{++}F^+ + \lambda(F^+ \overset{*}{\overline{F}}{}^+)F^+ \;=\; 0\ , \qquad (5)$$

$$\partial^{++}A_m^- - 2\partial_m F^+ + 2\lambda A_m^- \overset{*}{\overline{F}}{}^+ F^+ + \lambda(F^+)^2 \overset{*}{\overline{A}}{}_m{}^- \;=\; 0\ , \qquad (6)$$

$$\partial^{++}M^- + 2\lambda M^- \overset{*}{\overline{F}}{}^+ F^+ + \lambda \overset{*}{\overline{N}}{}^- (F^+)^2 + i\overline{Z}F^+ \;=\; 0\ , \qquad (7)$$

$$\partial^{++}N^- + 2\lambda N^- \overset{*}{\overline{F}}{}^+ F^+ + \lambda \overset{*}{\overline{M}}{}^- (F^+)^2 + iZF^+ \;=\; 0\ , \qquad (8)$$

$$\partial^{++}P^{(-3)} + \partial^m A_m^- + 2\lambda F^+ \overset{*}{\overline{F}}{}^+ P^{(-3)} + \lambda(F^+)^2 \overset{*}{\overline{P}}{}^{(-3)}$$

$$-\frac{\lambda}{2}A^{-m}A_m^- \overset{*}{\overline{F}}{}^+ - \lambda A^{-m} \overset{*}{\overline{A}}{}_m{}^- F^+ + 2\lambda F^+(M^- \overset{*}{\overline{M}}{}^- + N^- \overset{*}{\overline{N}}{}^-)$$

$$+2\lambda \overset{*}{\overline{F}}{}^+ M^- N^- + i\overline{Z}N^- + iZM^- \;=\; 0\ , \qquad (9)$$

as well as their conjugates. Integrating over $\theta^+, \bar{\theta}^+$ in eq. (1) results in the bosonic action

$$S_T \;=\; -\int d\zeta^{(-4)}du \left\{ \overset{*}{\overline{q}}{}^+ D^{++}q^+ + \frac{\lambda}{2}(q^+)^2(\overset{*}{\overline{q}}{}^+)^2 \right\}$$

$$\rightarrow \;\; S = -\tfrac{1}{2}\int d^4x\,du \left\{ (\overset{*}{\overline{A}}{}_m{}^- \partial^m F^+ - A_m^- \partial^m \overset{*}{\overline{F}}{}^+) \right.$$

$$\left. +iF^+(\overline{Z}\,\overset{*}{\overline{M}}{}^- + Z\,\overset{*}{\overline{N}}{}^-) + i\,\overset{*}{\overline{F}}{}^+(ZM^- + \overline{Z}N^-) \right\}\ . \qquad (10)$$

Since the action (1) has the global $U(1)$ invariance

$$q^{+\,'} = e^{i\alpha}q^+, \qquad \overset{*}{\overline{q}}{}^{+\prime} = e^{-i\alpha}\,\overset{*}{\overline{q}}{}^+\ , \qquad (11)$$

there exists the conserved Noether current j^{++} ,

$$D^{++}j^{++} = 0, \qquad j^{++} = iq^+ \overset{*}{\overline{q}}{}^+\ . \qquad (12)$$

It implies, in particular, that $\partial^{++}(F^+ \overset{*}{\overline{F}}{}^+) = 0$ and, hence,

$$F^+(x,u)\,\overset{*}{\overline{F}}{}^+(x,u) = C^{(ij)}u_i^+ u_j^+\ , \qquad (13)$$

$$(F^+ \overset{*}{\overline{F}}{}^+)^{\overset{*}{=}} = -F^+ \overset{*}{\overline{F}}{}^+ \rightarrow \overline{C^{(ij)}} = -\epsilon_{il}\epsilon_{jn}C^{(ln)}\ , \qquad (14)$$

where the new function $C^{(ij)}(x)$ has been introduced. Changing the variables as

$$F^+(x,u) = f^+(x,u)e^{\lambda\varphi}, \qquad \varphi(x,u) = -C^{(ij)}(x)u_i^+ u_j^- = -\overset{*}{\overline{\varphi}}(x,u)\ , \qquad (15)$$

reduces eq. (5) to the linear equation

$$\partial^{++}f^+(x,u) = 0 \rightarrow f^+(x,u) = f^i(x)u_i^+\ . \qquad (16)$$

After taking into account that

$$F^+ \overset{*}{\overline{F}}{}^+ = f^+ \overset{*}{\overline{f}}{}^+ \rightarrow C^{(ij)}(x) = -f^{(i}(x)\bar{f}^{j)}(x) ,\qquad(17)$$

where $\bar{f}^i = \epsilon^{ij}\bar{f}_j$ and $\bar{f}_j \equiv \overline{(f^j)}$, we obtain a general solution in the form

$$\begin{aligned}
F^+(x,u) &= f^i u^+_i e^{\lambda\varphi} \\
&= f^i(x)u^+_i \exp\{\lambda f^{(j}\bar{f}^{k)}u^+_j u^-_k\} .
\end{aligned}\qquad(18)$$

The same conclusion appears when using the *Ansatz*

$$F^+ = e^C[f^i u^+_i + B^{ijk}u^+_i u^+_j u^-_k]\qquad(19)$$

in terms of functions C and B^{ijk} at our disposal. After substituting eq. (19) into the equation of motion (5), we find

$$B^{ijk} = 0 \qquad \text{and} \qquad C = \lambda f^{(i}\bar{f}^{j)}u^+_i u^-_j ,\qquad(20)$$

so that

$$F^+ = f^i u^+_i \exp\{\lambda f^{(j}\bar{f}^{k)}u^+_j u^-_k\}\qquad(21)$$

again. To get a similar equation for A^-_m, we use eq. (6) and the *Ansatz*

$$\begin{aligned}
A^-_m &= e^{\lambda\varphi}\Big\{a\lambda f^i u^+_i \partial_m(f^{(k}\bar{f}^{j)}u^-_k u^-_j) \\
&\quad + b\partial_m f^i u^-_i + c\frac{\lambda f^i u^-_i}{1+\lambda f\bar{f}}(f^j\partial_m\bar{f}_j - \bar{f}_j\partial_m f^j)\Big\}
\end{aligned}\qquad(22)$$

with some coefficients (a,b,c) to be determined. After substituting eq. (22) into eq. (6), we find the relations

$$b = 2 , \quad 2a = 2+b , \quad \text{and} \quad c = b-a+1 ,\qquad(23)$$

so that $a = b = 2$ and $c = 1$. Therefore, we have

$$\begin{aligned}
A^-_m &= e^{\lambda\varphi}\Big\{2\lambda f^i u^+_i \partial_m(f^{(k}\bar{f}^{j)}u^-_k u^-_j) \\
&\quad + 2\partial_m f^i u^-_i + \frac{\lambda f^i u^-_i}{1+\lambda f\bar{f}}(f^j\partial_m\bar{f}_j - \bar{f}_j\partial_m f^j)\Big\} .
\end{aligned}\qquad(24)$$

To solve the remaining equations of motion (7) and (8) for the auxiliary fields M^- and N^- (the rest of equations of motion in eq. (9) is irrelevant for our purposes), we introduce the *Ansatz*

$$\begin{aligned}
M^- &= e^{\lambda\varphi}R^- \equiv Re^{\lambda\varphi}f^i u^-_i , &(25) \\
N^- &= e^{\lambda\varphi}S^- \equiv Se^{\lambda\varphi}f^i u^-_i , &(26)
\end{aligned}$$

with some coefficient functions R and S to be determined. After substituting eq. (25) into eq. (7) we get

$$\partial^{++}R^- - \lambda f^{(j}\bar{f}^{k)}u^+_j u^+_k R^- + \lambda f^i f^j u^+_i u^+_j \overset{*}{S}{}^- + i\overline{Z}f^i u^+_i \;=\; 0 \;,$$

$$Rf^i u^+_i - R\lambda f^{(m}\bar{f}^{n)}f^i u^+_m u^+_n u^-_i + i\overline{Z}f^i u^+_i - \lambda\bar{S}f^m f^n \bar{f}^i u^+_m u^+_n u^-_i \;=\; 0 \;,$$

$$Rf^i u^+_i + R\lambda f^m \bar{f}_n f^i(u^{-\,n}u^+_i + \delta^n_i)u^+_m + i\overline{Z}f^i u^+_i - \lambda\bar{S}f^m f^n \bar{f}^i u^+_m u^+_n u^-_i \;=\; 0 \;,$$

$$f^i u^+_i[R(1 + \lambda f^j \bar{f}_j) + i\overline{Z}] - \lambda(R + \bar{S})f^m \bar{f}^n f^i u^+_m u^+_i u^-_n \;=\; 0 \;.$$

It follows

$$R = -\frac{i\overline{Z}}{1 + \lambda f\bar{f}} \quad \text{and, hence,} \quad M^- = -\frac{i\overline{Z}}{1 + \lambda f\bar{f}}\, e^{\lambda\varphi}f^i u^-_i \;. \tag{27}$$

Similarly, we find from eqs. (8) and (26) that

$$N^- = -\frac{iZ}{1 + \lambda f\bar{f}}\, e^{\lambda\varphi}f^i u^-_i \;. \tag{28}$$

Substituting now the obtained solutions for the auxiliary fields F^+, A^-_a, M^- and N^- into the action (10) yields the bosonic NLSM action

$$S = \tfrac{1}{2}\int d^4x \left\{ g_{ij}\partial_m f^i \partial^m f^j + \bar{g}^{ij}\partial_m \bar{f}_i \partial^m \bar{f}_j + 2h^i{}_j\partial_m f^j \partial^m \bar{f}_i) - V(f) \right\} \;, \tag{29}$$

whose metric takes the form [46]

$$g_{ij} \;=\; \frac{\lambda(2 + \lambda f\bar{f})}{2(1 + \lambda f\bar{f})}\bar{f}_i \bar{f}_j \;, \quad \bar{g}^{ij} = \frac{\lambda(2 + \lambda f\bar{f})}{2(1 + \lambda f\bar{f})}f^i f^j \;, \tag{30}$$

$$h^i_j \;=\; \delta^i_j(1 + \lambda f\bar{f}) - \frac{\lambda(2 + \lambda f\bar{f})}{2(1 + \lambda f\bar{f})}f^i \bar{f}_j \;.$$

This metric is known to be equivalent to the standard Taub-NUT metric up to a field redefinition [46]. The scalar potential in eq. (31) takes the form [44]

$$V(f) = \frac{Z\overline{Z}}{1 + \lambda f\bar{f}}\, f\bar{f} \;. \tag{31}$$

By construction, the effective scalar potential (31) for a single charged hypermultiplet is generated in the one-loop perturbation theory (subsect. 3.2), and it is exact in the Coulomb branch. The vacuum expectation values for the scalar hypermultiplet components, which are to be calculated from this effective potential, all vanish. Notably, the BPS mass $m^2_{\mathrm{BPS}} = |Z|^2$ is not renormalized, as it should. However, the exact effective scalar potential (31) is not merely the quadratic (BPS mass) contribution.

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Appendix B: Eguchi-Hanson hypermultipet self-interaction in components from harmonic superspace

As was argued in sect. 6, a non-trivial hypermultiplet self-interaction can be non-perturbatively generated in the Higgs branch, in the presence of the non-vanishing constant FI-term $\xi^{(ij)} = \frac{1}{2}(\vec{\tau} \cdot \vec{\xi})^{ij}$, where $\vec{\tau}$ are Pauli matrices. The FI-term is given by the vacuum expectation value of the $N = 2$ vector multiplet auxiliary components (in a WZ-like gauge), and it has a nice geometrical interpretation in the underlying ten-dimensional type-IIA superstring brane picture (sect. 6).

The simplest non-trivial LEEA for a single dimensionless ω-hypermultiplet in the baryonic Higgs branch reads

$$S_{\mathrm{EH}}[\omega] = -\frac{1}{2\kappa^2} \int d\zeta^{(-4)} du \left\{ \left(D^{++}\omega \right)^2 - \frac{(\xi^{++})^2}{\omega^2} \right\} \ , \tag{1}$$

where $\xi^{++} = u_i^+ u_j^+ \xi^{(ij)}$ is the FI-term, and κ is the coupling constant of dimension one (in units of length). After changing the variables to $q_a^+ = u_a^+ \omega + u_a^- f^{++}$ and eliminating the Lagrange multiplier f^{++} via its algebraic equation of motion, one can rewrite eq. (1) to the equivalent form, in terms of the dual q^+-hypermultiplet, as [87]

$$S_{EH}[q] = -\frac{1}{2\kappa^2} \int d\zeta^{(-4)} du \left\{ q^{a+} D^{++} q_a^+ - \frac{(\xi^{++})^2}{(q^{a+}u_a^-)^2} \right\} \ , \tag{2}$$

where we have used the notation $q_a^+ = (\overset{*}{q}{}^+, q^+)$ and $q^{a+} = \varepsilon^{ab} q_b^+$. In its turn, eq. (2) is classically equivalent to the following gauge-invariant action in terms of *two* FS hypermultiplets q_{aA}^+ ($A = 1, 2$) and the auxiliary (acting as Lagrange multiplier) real analytic $N = 2$ vector gauge superfield V^{++} [87]:

$$S_{EH}[q, V] = -\frac{1}{2\kappa^2} \int d\zeta^{(-4)} du \left\{ q_A^{a+} D^{++} q_{aA}^+ + V^{++} \left(\frac{1}{2}\varepsilon^{AB} q_A^{a+} q_{Ba}^+ + \xi^{++} \right) \right\} \ . \tag{3}$$

We now calculate the component form of this hypermultiplet self-interaction by using eq. (3) as our starting point. In a bit more explicit form, it reads

$$\begin{aligned} S \ = \ & -\frac{1}{2\kappa^2} \int d\zeta^{(-4)} du \left\{ \overset{*}{q}_1{}^+ D^{++} q_1^+ + \overset{*}{q}_2{}^+ D^{++} q_2^+ \right. \\ & \left. V^{++} (\overset{*}{q}_1{}^+ q_2^+ - \overset{*}{q}_2{}^+ q_1^+ + \xi^{++}) \right\} \ . \end{aligned} \tag{4}$$

The equations of motion are given by

$$D^{++} q_1^+ + V^{++} q_2^+ \ = \ 0 \ , \tag{5}$$

$$D^{++} q_2^+ - V^{++} q_1^+ \ = \ 0 \ , \tag{6}$$

$$\overset{*}{q}_1{}^+ q_1^+ - \overset{*}{q}_2{}^+ q_1^+ + \xi^{++} \ = \ 0 \ , \tag{7}$$

while the last equation is clearly the algebraic constraint on the two FS hypermultiplets. In what follows, we ignore fermionic contributions and use a WZ-gauge for the $N = 2$ vector superfield V^{++}, so that D^{++} and q^+ are still given by eqs. (2) and (4), whereas

$$V^{++} = -2i\theta^+\sigma^m\bar{\theta}^+ V_m(x_A) + \theta^+\theta^+\bar{a}(x_A) + \bar{\theta}^+\bar{\theta}^+ a(x_A) \tag{8}$$

$$+\theta^+\theta^+\bar{\theta}^+\bar{\theta}^+ D^{(ij)}(x_A)u^-_i u^-_j \ . \tag{9}$$

The equation of motion (5) in components reads

$$\partial^{++} F^+_1 = 0 \ , \tag{10}$$

$$-2\partial_m F^+_1 + \partial^{++} A^-_{1m} - 2V_m F^+_2 = 0 \ , \tag{11}$$

$$i\bar{Z} F^+_1 + \partial^{++} M^-_1 + \bar{a} F^+_2 = 0 \ , \tag{12}$$

$$i Z F^+_1 + \partial^{++} N^-_1 + a F^+_2 = 0 \ , \tag{13}$$

$$\partial^{++} P^{(-3)}_1 + \partial^m A^-_{1m} + i\bar{Z} N^-_1 + i Z M^-_1$$
$$+V^m A^-_{2m} + \bar{a} N^-_2 + a M^-_2 + D^{(ij)}u^-_i u^-_j F^+_2 = 0 \ , \tag{14}$$

whereas eq. (6) gives

$$\partial^{++} F^+_2 = 0 \ , \tag{15}$$

$$-2\partial_m F^+_2 + \partial^{++} A^-_{2m} + 2V_m F^+_1 = 0 \ , \tag{16}$$

$$i\bar{Z} F^+_2 + \partial^{++} M^-_2 - \bar{a} F^+_1 = 0 \ , \tag{17}$$

$$i Z F^+_2 + \partial^{++} N^-_2 - a F^+_1 = 0 \ , \tag{18}$$

$$\partial^{++} P^{(-3)}_2 + \partial^m A^-_{2m} + i\bar{Z} N^-_2 + i Z M^-_2$$
$$-V^m A^-_{1m} - \bar{a} N^-_1 - a M^-_1 - D^{(ij)}u^-_i u^-_j F^+_1 = 0 \ . \tag{19}$$

The constraint (7) in components is given by

$$\overset{*}{\overline{F}}_1{}^+ F^+_2 - \overset{*}{\overline{F}}_2{}^+ F^+_1 + \xi^{++} = 0 \ , \tag{20}$$

$$\overset{*}{\overline{A}}_{1a}{}^- F^+_2 + \overset{*}{\overline{F}}_1{}^+ A^-_{2a} - \overset{*}{\overline{A}}_{2a}{}^- F^+_1 - \overset{*}{\overline{F}}_2{}^+ A^-_{1a} = 0 \ , \tag{21}$$

$$\overset{*}{\overline{F}}_1{}^+ M^-_2 - \overset{*}{\overline{F}}_2{}^+ M^-_1 + \overset{*}{\overline{N}}_1{}^- F^+_2 - \overset{*}{\overline{N}}_2{}^- F^+_1 = 0 \ , \tag{22}$$

$$\overset{*}{\overline{F}}_1{}^+ N^-_2 - \overset{*}{\overline{F}}_2{}^+ N^-_1 + \overset{*}{\overline{M}}_1{}^- F^+_2 - \overset{*}{\overline{M}}_2{}^- F^+_1 = 0 \ , \tag{23}$$

$$-\tfrac{1}{2}\overset{*}{\overline{A}}_1{}^{m-} A^-_{2m} + \overset{*}{\overline{M}}_1{}^- M^-_2 + \overset{*}{\overline{N}}_1{}^- N^-_2 + \overset{*}{\overline{P}}_1{}^{(-3)} F^+_2 + \overset{*}{\overline{F}}_1{}^+ P^{(-3)}_2$$
$$+\tfrac{1}{2}\overset{*}{\overline{A}}_2{}^{m-} A^-_{1m} - \overset{*}{\overline{M}}_2{}^- M^-_1 - \overset{*}{\overline{N}}_2{}^- N^-_1 - \overset{*}{\overline{P}}_2{}^{(-3)} F^+_1 - \overset{*}{\overline{F}}_2{}^+ P^{(-3)}_1 = 0 \ . \tag{24}$$

Substituting the component expressions for q_A^+ and V^{++} into the action (4) results in the following bosonic action:

$$
\begin{aligned}
S \;=\; & -\frac{1}{2\kappa^2}\int d^4x\,du\,\Big\{ \overset{*}{\overline{F}}_1{}^+\partial^m A_{1m}^- + \overset{*}{\overline{F}}_2{}^+\partial^m A_{2m}^- + V^m(\overset{*}{\overline{F}}_1{}^+ A_{2m}^- - \overset{*}{\overline{F}}_2{}^+ A_{1m}^-) \\
& + \overset{*}{\bar a}\,(\overset{*}{\overline{F}}_1{}^+ N_2^- - \overset{*}{\overline{F}}_2{}^+ N_1^-) + a(\overset{*}{\overline{F}}_1{}^+ M_2^- - \overset{*}{\overline{F}}_2{}^+ M_1^-) \\
& + iD^{(ij)}u^-_i u^-_j(\xi^{++} + \overset{*}{\overline{F}}_1{}^+ F_2^+ - \overset{*}{\overline{F}}_2{}^+ F_1^+) \\
& + \overset{*}{\overline{F}}_1{}^+(i\overline{Z}N_1^- + iZM_1^-) + \overset{*}{\overline{F}}_2{}^+(i\overline{Z}N_2^- + iZM_2^-) \Big\}\;.
\end{aligned}
\tag{25}
$$

The next step in our calculation is to fix the harmonic dependence of the fields F_i^+, A_{ia}^-, M_i^- and N_i^-. Eqs. (10) and (15) imply

$$
F_1^+ = f_1^i u_i^+ \quad\text{and}\quad F_2^+ = f_2^i u_i^+\;,
\tag{26}
$$

whereas eq. (11) yields

$$
-2\partial_m F_1^+ + \partial^{++}A_{1m}^- - 2V_m F_2^+ = 0\;.
\tag{27}
$$

After introducing the *Ansatz*

$$
A_{1m}^- = A_{1m}^i u_i^- + B_{1m}^{ijk}u_i^+ u_j^- u_k^-
\tag{28}
$$

we find that

$$
A_{1m}^- = (2\partial_m f_1^i + 2V_m f_2^i)u_i^-\;.
\tag{29}
$$

Similarly, it follows from eq. (16) that

$$
A_{2m}^- = (2\partial_m f_2^i - 2V_m f_1^i)u_i^-\;.
\tag{30}
$$

Eqs. (12), (13), (17) and (18) now imply

$$
\begin{aligned}
M_1^- &= -(\overset{*}{\bar a}\,f_2^i + i\overline{Z}f_1^i)u_i^-\;, &\tag{31}\\
N_1^- &= -(a f_2^i + iZ f_1^i)u_i^-\;, &\tag{32}\\
M_2^- &= (\overset{*}{\bar a}\,f_1^i - i\overline{Z}f_2^i)u_i^-\;, &\tag{33}\\
N_2^- &= (a f_1^i - iZ f_2^i)u_i^-\;. &\tag{34}
\end{aligned}
$$

After substituting all the component solutions into the action (25), we find the (abelian) gauged NLSM action

$$
\begin{aligned}
S \;=\; & \frac{1}{2\kappa^2}\int d^4x\,\Big\{ (\partial_m f_1{}^i + V_m f_2{}^i)(\partial^m \bar f_{1i} + V^m \bar f_{2i}) \\
& + (\partial_m f_2{}^i - V_m f_1{}^i)(\partial^m \bar f_{2i} - V^m \bar f_{1i}) \\
& - \frac{Z\overline{Z}}{(f_1\bar f_1 + f_2\bar f_2)}\left[(f_1^i \bar f_{2i} - f_2^i \bar f_{1i})^2 + (f_1^i \bar f_{1i} + f_2^i \bar f_{2i})^2\right] \Big\}\;,
\end{aligned}
\tag{35}
$$

where the scalar hypermultiplet components $f_{1,2}^i$ are still subject to the constraint

$$\xi^{(ij)} \quad = \quad \bar{f}_1^{(i} f_2^{j)} - f_1^{(i} \bar{f}_2^{j)} \ .$$ (36)

In calculating the action (35) we have also used the equation of motion for the $N = 2$ vector multiplet auxiliary field a, whose solution reads

$$a \quad = \quad -iZ \frac{f_1^i \bar{f}_{2i} - f_2^i \bar{f}_{1i}}{f_1^i \bar{f}_{1i} + f_2^i \bar{f}_{2i}} \ .$$ (37)

A solution to the equation of motion for the vector gauge field V_m is given by

$$2V_m \quad = \quad \frac{\partial_m \bar{f}_{1j} f_2^j - \bar{f}_{1j} \partial_m f_2^j - \partial_m \bar{f}_{2j} f_1^j + \bar{f}_{2j} \partial_m f_1^j}{\bar{f}_{1j} f_1^j + \bar{f}_{2j} f_2^j} \ .$$ (38)

In terms of the two complex scalar $SU(2)$ doublets $f_{1,2}^i$ subject to the three real constraints (36) and one abelian gauge invariance, we have $2 \times 2 \times 2 - 3 - 1 = 4$ independent degrees of freedom, as it should. After eliminating the auxiliary vector potential V_m via eq. (38) and solving the constraint (36), one finds the NLSM with a non-trival hyper-Kähler metric (by construction, as the consequence of $N = 2$ supersymmetry) *and* a non-trivial scalar potential as well,

$$V = \frac{Z\overline{Z}}{(f_1 \bar{f}_1 + f_2 \bar{f}_2)} \left[(f_1^i \bar{f}_{2i} - f_2^i \bar{f}_{1i})^2 + (f_1^i \bar{f}_{1i} + f_2^i \bar{f}_{2i})^2 \right] \ .$$ (39)

The kinetic terms in the NLSM (35) are known to represent the *Eguchi-Hanson* instanton metric up to a field redefinition [87], so that to this end we concentrate on the scalar potential (39) only. Let's introduce the notation

$$\bar{f}_{(1,2)}^1 = \overset{*}{f}_{(1,2)}{}^2 \ , \qquad \bar{f}_{(1,2)}^2 = - \overset{*}{f}_{(1,2)}{}^1 \ ,$$ (40)

and keep the positions of indices as above. The operator $*$ denotes the usual complex conjugation. The constraints (36) now take the form

$$\xi^{11} \quad = \quad \bar{f}_1^1 f_2^1 - f_1^1 \bar{f}_2^1 = \overset{*}{f}_1{}^2 f_2^1 - f_1^1 \overset{*}{f}_2{}^2 \ ,$$
$$\xi^{12} \quad = \quad \tfrac{1}{2}(\bar{f}_1^1 f_2^2 + \bar{f}_1^2 f_2^1) - \tfrac{1}{2}(f_1^1 \bar{f}_2^2 + f_1^2 \bar{f}_2^1) \ ,$$
$$\xi^{22} \quad = \quad \bar{f}_1^2 f_2^2 - f_1^2 \bar{f}_2^2 \ .$$

After mulitplying these constraints with Pauli matrices $(\tau_1, 1, \tau_3)_{ij}$, we get

$$\xi^1 \quad = \quad \bar{f}_1^1 f_2^2 + \bar{f}_1^2 f_2^1 - (f_1^1 \bar{f}_2^2 + f_1^2 \bar{f}_2^1) \ ,$$ (41)
$$\xi^2 \quad = \quad \bar{f}_1^1 f_2^1 - f_1^1 \bar{f}_2^1 + \bar{f}_1^2 f_2^2 - f_1^2 \bar{f}_2^2 \ ,$$ (42)
$$\xi^3 \quad = \quad \bar{f}_1^1 f_2^1 - f_1^1 \bar{f}_2^1 - \bar{f}_1^2 f_2^2 + f_1^2 \bar{f}_2^2 \ ,$$ (43)

while we have $\vec{\xi}^2 \equiv (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 \neq 0$. We now choose the direction $\xi^2 = \xi^3 = 0$ and $\xi^1 = 2i$, so that our constraints now take the form

$$\bar{f}_1^{\,1} f_2^{\,2} + \bar{f}_1^{\,2} f_2^{\,1} - (f_1^{\,1} \bar{f}_2^{\,2} + f_1^{\,2} \bar{f}_2^{\,1}) \;=\; 2i \;, \tag{44}$$

$$(-(f_1^{\,1})^* f_2^{\,1} + f_1^{\,1} (f_2^{\,1})^*) + ((f_1^{\,2})^* f_2^{\,2} - f_1^{\,2}(f_2^{\,2})^*) \;=\; 2i \;, \tag{45}$$

and

$$f_2^{\,1}(f_1^{\,2})^* = f_1^{\,1}(f_2^{\,2})^* \;\;, \quad f_2^{\,2}(f_1^{\,1})^* = f_1^{\,2}(f_2^{\,1})^* \;. \tag{46}$$

We thus end up with only two+one real constraints and one gauge invariance

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}' = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \;. \tag{47}$$

In the parameterization

$$f_i^{\,j} = m_i^{\,j} exp(i\varphi_i^{\,j}) \tag{48}$$

the constraints (45) and (46) read

$$m_1^1 m_2^2 = m_1^2 m_2^1 e^{-i\varphi_2^1 - i\varphi_2^2 + i\varphi_1^1 + i\varphi_1^2} \;, \tag{49}$$

$$m_1^1 m_2^1 \sin(\varphi_1^1 - \varphi_2^1) + m_2^2 m_1^2 \sin(\varphi_2^2 - \varphi_1^2) = 1 \;. \tag{50}$$

We now want to fix the local $U(1)$ symmetry by imposing the gauge condition

$$\varphi_2^1 + \varphi_2^2 \;=\; \varphi_1^1 + \varphi_1^2 \;. \tag{51}$$

When using

$$\left| f_2^1 \right| \equiv m, \qquad \left| f_2^2 \right| \equiv n, \qquad \varphi_1^1 \equiv \theta \;, \qquad \varphi_2^2 \equiv \phi \;, \tag{52}$$

as the independent fields, our constraints above can be easily solved as

$$-\varphi_2^1 = \varphi_2^2 = \phi, \qquad \varphi_1^1 = -\varphi_1^2 = \theta \;, \qquad m_2^1 = m, \quad m_2^2 = n \;, \tag{53}$$

and

$$m_1^1 = \frac{m}{(m^2 + n^2)\sin(\theta + \phi)} \;, \quad m_1^2 = \frac{n}{(m^2 + n^2)\sin(\theta + \phi)} \;. \tag{54}$$

It is straightforward to deduce the other fields F_i^+, A_{im}^-, M_i^- and N_i^- in terms of the independent components (52). The scalar potential (39) in terms of these independent field variables takes the form (no indices and constraints any more !)

$$V = \frac{|Z|^2 \sin^2(\theta + \phi)}{m^2 + n^2} \left[\frac{4(m^2 - n^2)^2}{1 + (m^2 + n^2)^2 \sin^2(\theta + \phi)} + \frac{1 + (m^2 + n^2)^2 \sin^2(\theta + \phi)}{\sin^4(\theta + \phi)} \right] \;. \tag{55}$$

It is clear from this equation that the potential V is positively definite, and it is only non-vanishing due to the non-vanishing central charge $|Z|$. This signals spontaneous breaking of $N = 2$ supersymmetry in our model.

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