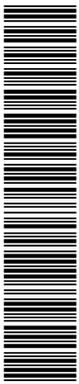


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March 1998**Inclusive Decay Rate for $B \rightarrow X_d + \gamma$ in Next-to-Leading Logarithmic Order and CP Asymmetry in the Standard Model**¹A. Ali^a, H. Asatrian^b and C. Greub^c^aDeutsches Elektronen-Synchrotron DESY, Hamburg, Germany^bYerevan Physics Institute, Alikhanyan Br., 375036-Yerevan, Armenia^cInst. f. Theor. Physik, Univ. Bern, Bern, Switzerland**Abstract**

We compute the decay rate for the Cabibbo-Kobayashi-Maskawa (CKM)-suppressed electromagnetic penguin decay $\bar{B} \rightarrow \bar{X}_d + \gamma$ (and its charge conjugate) in the next-to-leading order in QCD, including leading power corrections in $1/m_b^2$ and $1/m_c^2$ in the standard model. The average branching ratio $\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle$ of the decay $B \rightarrow X_d + \gamma$ and its charge conjugate $\bar{B} \rightarrow \bar{X}_d + \gamma$ is estimated to be in the range $6.0 \times 10^{-6} \leq \langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle \leq 2.6 \times 10^{-5}$, obtained by varying the CKM-Wolfenstein parameters ρ and η in the range $-0.1 \leq \rho \leq 0.4$ and $0.2 \leq \eta \leq 0.46$ and taking into account other parametric dependence. In the NLL approximation and in the stated range of the CKM parameters, we find the ratio $R(d\gamma/s\gamma) \equiv \langle \mathcal{B}(B \rightarrow X_d\gamma) \rangle / \langle \mathcal{B}(B \rightarrow X_s\gamma) \rangle$ to lie in the range $0.017 \leq R(d\gamma/s\gamma) \leq 0.074$. Theoretical uncertainties in this ratio are estimated and found to be small. Hence, this ratio is well suited to provide independent constraints on the CKM parameters. The CP-asymmetry in the decay rates, defined as $a_{CP}(B \rightarrow X_d\gamma) \equiv (\Gamma(B \rightarrow X_d\gamma) - \Gamma(\bar{B} \rightarrow \bar{X}_d\gamma)) / (\Gamma(B \rightarrow X_d\gamma) + \Gamma(\bar{B} \rightarrow \bar{X}_d\gamma))$, is found to be in the range (7 – 35)%. Both the decay rates and CP asymmetry are measurable in forthcoming experiments at B factories and possibly at HERA-B.

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1. Introduction

Electromagnetic penguins were first measured by the CLEO collaboration through the exclusive decay $B \rightarrow K^* + \gamma$ [1], followed by the measurement of the inclusive decay $B \rightarrow X_s + \gamma$ [2]. The present CLEO measurements can be summarized as [3]:

$$\begin{aligned}\langle \mathcal{B}(B \rightarrow X_s + \gamma) \rangle &= (2.32 \pm 0.57 \pm 0.35) \times 10^{-4}, \\ \langle \mathcal{B}(B \rightarrow K^* + \gamma) \rangle &= (4.2 \pm 0.8 \pm 0.6) \times 10^{-5}.\end{aligned}\tag{1}$$

Very recently, the inclusive radiative decay has also been reported by the ALEPH collaboration with a (preliminary) branching ratio [4]:

$$\langle \mathcal{B}(H_b \rightarrow X_s + \gamma) \rangle = (3.29 \pm 0.71 \pm 0.68) \times 10^{-4}.\tag{2}$$

The quantity $\langle \mathcal{B}(B \rightarrow X_s + \gamma) \rangle$ is the branching ratio averaged over the decays $B \rightarrow X_s + \gamma$ and its charge conjugate $\overline{B} \rightarrow \overline{X}_s + \gamma$. The branching ratio in (2) involves a different weighted average of the B -mesons and Λ_b baryons produced in Z^0 decays (hence the symbol H_b) than the corresponding one given in (1), which has been measured in the decay $\Upsilon(4S) \rightarrow B^+ B^-, B^0 \overline{B}^0$.

These measurements have stimulated an impressive theoretical activity, directed at improving the precision of the decay rates and distributions in the context of the standard model (SM) and beyond the standard model, in particular supersymmetry. In the SM-context, the complete next-to-leading-logarithmic (NLL) contributions have been painstakingly completed [5] - [14], and leading power corrections in $1/m_b^2$ [15, 16, 17] and $1/m_c^2$ [18, 19, 20] have also been calculated for the decay rate in $B \rightarrow X_s + \gamma$. This theoretical work allows to calculate the branching ratios in the SM with an accuracy of about $\pm 9\%$, yielding $\langle \mathcal{B}(B \rightarrow X_s + \gamma) \rangle = (3.50 \pm 0.32) \times 10^{-4}$ and $\langle \mathcal{B}(H_b \rightarrow X_s + \gamma) \rangle = (3.76 \pm 0.30) \times 10^{-4}$, in reasonable agreement with the CLEO and (preliminary) ALEPH measurements, respectively. The decay rates in eqs. (1) and (2) determine the ratio of the Cabibbo-Kobayashi-Maskawa (CKM) [22] matrix elements $|V_{ts}^* V_{tb}/V_{cb}|$. Since $|V_{cb}|$ and $|V_{tb}|$ have been directly measured, these measurements can be combined, yielding $|V_{ts}| = 0.033 \pm 0.007$ [21]. The central value of $|V_{ts}|$ is somewhat lower than the corresponding value of $|V_{cb}|$, $|V_{cb}| = 0.0393 \pm 0.0028$, but within errors the two matrix elements are found to be approximately equal, as expected from the CKM unitarity.

The interest in measuring the decay rate in $B \rightarrow X_d + \gamma$ (and its charge conjugate $\overline{B} \rightarrow \overline{X}_d + \gamma$) lies in that it will determine the CKM-Wolfenstein parameters ρ and η [23] in a theoretically reliable way. Likewise, this decay will enable us to search for new physics which may manifest itself through enhanced $bd\gamma$ and/or bdg effective vertices. These vertices are CKM-suppressed in the standard model, but new physics contributions may not follow the CKM pattern in flavor-changing-neutral-current transitions and hence new physics effects may become more easily discernible in $B \rightarrow X_d + \gamma$ (and its charge conjugate) than in the corresponding CKM-allowed vertices $bs\gamma$ and bsg . Closely related to this is the question of CP-violating asymmetry in the decay rates for $\overline{B} \rightarrow \overline{X}_d + \gamma$ and its charge conjugate $B \rightarrow X_d + \gamma$, which may provide us with the first measurements of the so-called direct CP violation in B physics. With the weak phase provided dominantly by the CKM matrix elements V_{td} and V_{ub} in the decay $B \rightarrow X_d + \gamma$, the perturbatively generated strong phases can be calculated by taking into account the charm and up quark loops in the electromagnetic penguins, which generate the necessary absorptive contributions. This calls for an improved theoretical estimate of $\mathcal{B}(B \rightarrow X_d + \gamma)$ and $\mathcal{B}(\overline{B} \rightarrow \overline{X}_d + \gamma)$ (hence a_{CP}) in the SM.

In what follows, we shall discuss for the sake of definiteness the decays of the b -quark $b \rightarrow s + \gamma(+g)$ and $b \rightarrow d + \gamma(+g)$, whose hadronic transcriptions are $\overline{B} \rightarrow \overline{X}_s + \gamma$ and $\overline{B} \rightarrow \overline{X}_d + \gamma$, respectively, but the discussion applies for the charge conjugate decays $B \rightarrow X_s + \gamma$ and $B \rightarrow X_d + \gamma$ as well with obvious changes. When it is essential to differentiate between the decay of a B meson and its charge conjugate \overline{B} , we shall do so. The branching ratio for the CKM-suppressed decay $\overline{B} \rightarrow \overline{X}_d + \gamma$ was calculated several years ago in partial next-to-leading order by two of us (A.A. and C.G.) [24]. Subsequent to that, the CP asymmetries in the decay rates in the leading logarithmic (LL) approximation were calculated in the SM [25, 26], and in some extensions of the SM [27, 28]. Much of the theoretical improvements carried out in the context of the decay $\overline{B} \rightarrow \overline{X}_s + \gamma$ mentioned above can be taken over for the decay $\overline{B} \rightarrow \overline{X}_d + \gamma$. Like the former decay, the NLL-improved and power-corrected decay rate for $\overline{B} \rightarrow \overline{X}_d + \gamma$ rests on firmer theoretical ground and consequently has much reduced theoretical uncertainty; in particular, the one arising from the scale-dependence which in the LL approximation is notoriously large, becomes drastically reduced in the complete NLL order, presented here. Hence, the improved theory for the decay rate would allow to extract more precisely the CKM parameters from the measured branching ratio $\overline{B} \rightarrow \overline{X}_d + \gamma$. Of particular theoretical interest is the ratio of the branching ratios, defined as

$$R(d\gamma/s\gamma) \equiv \frac{\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle}{\langle \mathcal{B}(B \rightarrow X_s + \gamma) \rangle}, \quad (3)$$

in which a good part of theoretical uncertainties (such as from m_t , μ_b , \mathcal{B}_{sl} etc.) cancel. Anticipating this, the ratio $R(d\gamma/s\gamma)$ was proposed in [24] as a good measure of $|V_{td}|$. We now calculate this ratio in the NLL accuracy, determine its CKM-parametric dependence precisely and estimate theoretical errors.

The CP-asymmetry in the decay rates, defined as

$$a_{CP}(B \rightarrow X_d + \gamma) \equiv \frac{\Gamma(B \rightarrow X_d + \gamma) - \Gamma(\overline{B} \rightarrow \overline{X}_d + \gamma)}{\Gamma(B \rightarrow X_d + \gamma) + \Gamma(\overline{B} \rightarrow \overline{X}_d + \gamma)} \quad (4)$$

has *not* so far been calculated in the NLL precision. We recall that, as opposed to the decay rates $\Gamma(\overline{B} \rightarrow \overline{X}_s + \gamma)$ and $\Gamma(\overline{B} \rightarrow \overline{X}_d + \gamma)$, which receive contributions starting from the lowest order, i.e., terms of the form $(\alpha_s^n(m_b) \ln^n(m_W/m_b))$, the CP-odd numerator in eq. (4) is suppressed by an extra factor α_s , i.e., it starts with terms of the form $\alpha_s(m_b) (\alpha_s^n \log^n(m_W/m_b))$. To simplify the language in the following, we refer to this statement by saying that the CP-odd numerator starts at order α_s . This results in a moderate scale dependence of a_{CP} , arising from the Wilson coefficients which contain a term proportional to $\alpha_s \ln(\mu_b/m_b)$ which is not compensated by the matrix elements in this order. We show the scale-dependence of a_{CP} numerically by varying the scale μ_b in the range $2.5 \text{ GeV} \leq \mu_b \leq 10 \text{ GeV}$. The compensation of this scale dependence requires the knowledge of the $O(\alpha_s^2)$ contributions in the matrix elements of the operators in the Wilson product expansion, which is not yet available. However, it is not unreasonable to anticipate that a judicious choice of the scale μ_b in B decays may reduce the NLL corrections. Since the results for the CP-even part, i.e., the denominator in eq. (4), are known in the LL approximation, and with the help of the present work now also in the NLL accuracy, this information can be used to guess the optimal scale. We make the choice $\mu_b = 2.5 \text{ GeV}$ for which we show that the NLL corrections in the decay rates become minimal (see Fig. 1). Of course, one can not insist that this feature must necessarily also hold for a_{CP} . Not having the

benefit of the complete $O(\alpha_s^2)$ calculation for a_{CP} , this particular choice of μ_b is an educated guess based on the inclusive decay rates presented here.

The branching ratio $\mathcal{B}(\overline{B} \rightarrow \overline{X}_d + \gamma)$ and a_{CP} depend on the parameters ρ and η and this dependence is the principal interest in measuring these quantities. To estimate them, we vary these parameters in the range $-0.1 \leq \rho \leq 0.4$ and $0.2 \leq \eta \leq 0.46$, which are the 95% C.L. ranges allowed by the present fits of the CKM matrix [29]. In addition to this, there are other well-known parametric dependences inherent to the theoretical framework being used here, such as α_s , m_t , m_b , m_c , α_{em} and \mathcal{B}_{sl} . We estimate the resulting uncertainty in the branching ratios for $B \rightarrow X_d + \gamma$ and $\overline{B} \rightarrow \overline{X}_d + \gamma$ in an analogous way as has been done for $\mathcal{B}(\overline{B} \rightarrow \overline{X}_s + \gamma)$ (and its charge conjugate). The resulting average branching ratio $\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle = (\mathcal{B}(B \rightarrow X_d + \gamma) + \mathcal{B}(\overline{B} \rightarrow \overline{X}_d + \gamma))/2$ and the CP rate asymmetry a_{CP} are found to be in the range $6.0 \times 10^{-6} \leq \langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle \leq 2.6 \times 10^{-5}$, and $a_{CP} = (7 - 35)\%$, with most of the dispersion arising due to the CKM parametric dependence of these quantities. For the central values of the fit-parameters [29] $\rho = 0.11, \eta = 0.33$ one obtains $\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle = (1.61 \pm 0.12) \times 10^{-5}$ and $a_{CP} = (11.5 - 16.5)\%$, where the errors reflect the uncertainties stemming from varying $\mu_b \in [2.5 \text{ GeV}, 10.0 \text{ GeV}]$ and the rest of the parameters. The ratio $R(d\gamma/s\gamma)$, defined in eq. (3), is largely free of parametric uncertainties; the residual theoretical error on this quantity is small but correlated with the values of ρ and η . Amusingly, the theoretical uncertainty on this ratio almost vanishes for the central values of the CKM-fit parameters! (see Fig. 5). However, as the presently allowed CKM-domain is large, one can take the largest uncertainty $\Delta R(d\gamma/s\gamma)/R(d\gamma/s\gamma) = \pm 7\%$, which is found for $\eta = 0.2$ and $\rho = 0.4$, as an upper limit on this uncertainty. The ratio $R(d\gamma/s\gamma)$ would then provide theoretically the most robust constraint on the CKM parameters. For the ratio itself, varying the CKM parameters in the 95% C.L. range, we find $0.017 \leq R(d\gamma/s\gamma) \leq 0.074$, with the central value being 0.046. Hence, even a first measurement of this ratio will provide a rather stringent constraint on the (ρ, η) domain. We show this for three assumed values, $R(d\gamma/s\gamma) = 0.017, 0.046, 0.074$ taking into account the theoretical errors (see Fig. 6).

Although the shape of the photon energy spectra in $B \rightarrow X_s + \gamma$ and $B \rightarrow X_d + \gamma$ is very similar, we think that a measurement of the much rarer decay $B \rightarrow X_d + \gamma$ should become feasible at future experiments like CLEO-III and B-factories, because these facilities will allow for a good K/π discrimination.

This paper is organized as follows: In section 2, we discuss the theoretical framework and present the salient features of the calculation for the decay rates in $\overline{B} \rightarrow \overline{X}_s + \gamma$ and its charge conjugate process and the CP asymmetry in the decay rates. In section 3, we work out the corresponding decay rates and CP asymmetry for $\overline{B} \rightarrow \overline{X}_d + \gamma$, and the ratio $R(d\gamma/s\gamma)$. Section 4 contains the numerical results and we conclude with a summary in section 5.

2. Decay rates and CP asymmetry in $B \rightarrow X_s + \gamma$ and $\overline{B} \rightarrow \overline{X}_s + \gamma$

The appropriate framework to incorporate QCD corrections is that of an effective theory obtained by integrating out the heavy degrees of freedom, which in the present context are the top quark and W^\pm bosons. The effective Hamiltonian depends on the underlying theory and for the SM one has (keeping operators up to dimension 6),

$$\mathcal{H}_{eff}(b \rightarrow s\gamma(+g)) = -\frac{4G_F}{\sqrt{2}}\lambda_t \sum_{i=1}^8 C_i(\mu)O_i(\mu), \quad (5)$$

where the operator basis and the corresponding Wilson coefficients $C_i(\mu)$ can be seen elsewhere [11]. The symbol $\lambda_t \equiv V_{tb}V_{ts}^*$ is the relevant CKM factor and G_F is the Fermi coupling constant.

The Wilson coefficients at the renormalization scale $\mu_b = O(m_b)$ are calculated with the help of the renormalization group equation whose solution requires the knowledge of the anomalous dimension matrix in a given order in α_s and the matching conditions, i.e., the Wilson coefficients $C_i(\mu = m_W)$, calculated in the complete theory to the commensurate order. The anomalous dimension matrix in the LL [30] and the NLL approximation [11] are known. The NLL matching conditions have also been worked out in the meanwhile by several groups. Of these, the first six corresponding to the four-quark operators have been derived in [31], and the remaining two, $C_7(\mu = m_W)$ and $C_8(\mu = m_W)$, were worked out in [8] and confirmed in [9], [13] and [14]. In addition, the NLL corrections to the matrix elements have also been calculated. Of these, the Bremsstrahlung corrections were obtained in [5, 24] in the truncated basis (involving the operators O_1 , O_2 , and O_7) and subsequently in the complete operator basis [6, 7]. The NLL virtual corrections were completed in [10]. This latter contribution plays a key role in reducing the scale-dependence of the LL inclusive decay width. All of these pieces have been combined to get the NLL decay width $\Gamma(\overline{B} \rightarrow \overline{X}_s + \gamma)$ and the details are given in the literature [11]-[13].

We recall that the operator basis in \mathcal{H}_{eff} is in fact larger than what is shown in eq. (5) in which operators multiplying the small CKM factor $\lambda_u \equiv V_{ub}V_{us}^*$ have been neglected. If the interest is in calculating the CP asymmetry, then they have to be put back. Doing this, and using the unitarity relation $\lambda_c = -\lambda_t - \lambda_u$, the effective Hamiltonian reads

$$\begin{aligned} \mathcal{H}_{eff}(b \rightarrow s\gamma(+g)) = & -\frac{4G_F}{\sqrt{2}} \{ \lambda_t [C_7(\mu)O_7(\mu) + C_8(\mu)O_8(\mu) + C_1(\mu)O_1(\mu) + C_2(\mu)O_2(\mu)] \\ & - \lambda_u [C_1(\mu)(O_{1u}(\mu) - O_1(\mu)) + C_2(\mu)(O_{2u}(\mu) - O_2(\mu))] + \dots \} . \end{aligned} \quad (6)$$

In this equation terms proportional to the small Wilson coefficients C_3, \dots, C_6 are dropped as indicated by the ellipses. The relevant operators are defined as:

$$\begin{aligned} O_1(\mu) &= (\bar{s}_L \gamma_\mu T^a c_L)(\bar{c}_L \gamma^\mu T^a b_L) \quad , \quad O_{1u}(\mu) = (\bar{s}_L \gamma_\mu T^a u_L)(\bar{u}_L \gamma^\mu T^a b_L) \quad , \\ O_2(\mu) &= (\bar{s}_L \gamma_\mu c_L)(\bar{c}_L \gamma^\mu b_L) \quad , \quad O_{2u}(\mu) = (\bar{s}_L \gamma_\mu u_L)(\bar{u}_L \gamma^\mu b_L) \quad , \\ O_7(\mu) &= \frac{e}{16\pi^2} m_b(\mu) (\bar{s}_L \sigma_{\mu\nu} b_R) F^{\mu\nu} \quad , \quad O_8(\mu) = \frac{g}{16\pi^2} m_b(\mu) (\bar{s}_L T^a \sigma_{\mu\nu} b_R) G^{a\mu\nu} . \end{aligned} \quad (7)$$

Note that the Wilson coefficients in eq. (6) are exactly the same as those in eq. (5). Moreover, the matrix elements $\langle s\gamma | O_{iu} | b \rangle$ and $\langle s\gamma g | O_{iu} | b \rangle$ of the additional operators O_{1u} and O_{2u} are obtained from those of O_1 and O_2 by obvious replacements.

For our intent and purpose, we write the amplitudes for the processes $b \rightarrow s\gamma$ and $b \rightarrow s\gamma g$ in a form where the dependence on the CKM matrix elements is manifest. The amplitude for the first process (including the virtual corrections) can be written as

$$\begin{aligned} A(b \rightarrow s\gamma) &= -\frac{4G_F}{\sqrt{2}} \langle s\gamma | O_7 | b \rangle_{tree} D(b \rightarrow s\gamma) \quad , \\ D(b \rightarrow s\gamma) &= \lambda_t (A_R^t + iA_I^t) + \lambda_u (A_R^u + iA_I^u) . \end{aligned} \quad (8)$$

It is straightforward to construct the real functions A_R^t, A_I^t, A_R^u and A_I^u from the expressions for the virtual correction in ref. [10] and the NLL Wilson coefficients in ref. [11]. The amplitude

of the charge conjugate decay $\bar{b} \rightarrow \bar{s}\gamma$ decay is then:

$$\begin{aligned} A(\bar{b} \rightarrow \bar{s}\gamma) &= -\frac{4G_F}{\sqrt{2}} \langle s\gamma | O_7 | b \rangle_{tree} D(\bar{b} \rightarrow \bar{s}\gamma), \\ D(\bar{b} \rightarrow \bar{s}\gamma) &= \lambda_t^*(A_R^t + iA_I^t) + \lambda_u^*(A_R^u + iA_I^u). \end{aligned} \quad (9)$$

The decay rate for the process $b \rightarrow s\gamma$ then reads ²

$$\begin{aligned} \Gamma(b \rightarrow s\gamma) &= \frac{m_b^5 G_F^2 \alpha_{em}}{32\pi^4} |D(b \rightarrow s\gamma)|^2, \\ |D(b \rightarrow s\gamma)|^2 &= |\lambda_t|^2 [(A_R^t)^2 + (A_I^t)^2] + |\lambda_u|^2 [(A_R^u)^2 + (A_I^u)^2] + \\ &\quad 2Re(\lambda_t^* \lambda_u) [A_R^t A_R^u + A_I^t A_I^u] - 2Im(\lambda_t^* \lambda_u) [A_R^t A_I^u - A_I^t A_R^u]. \end{aligned} \quad (10)$$

The order α_s^2 terms, which are generated when inserting the explicit expressions for the functions A_R^t , A_I^t , A_R^u , and A_I^u , are understood to be discarded. The corresponding expression for the $\bar{b} \rightarrow \bar{s}\gamma$ decay can be obtained from the preceding equation by changing the sign of the term proportional to $Im(\lambda_t^* \lambda_u)$.

An analogous expression for the decay width $\Gamma(b \rightarrow s\gamma g)$ of the Bremsstrahlung process, where the CKM dependence is explicit, is also easily obtained from the literature [5, 24, 6, 7, 10]. To get rid of the infrared singularity for $E_\gamma \rightarrow 0$, we included the virtual photonic correction to the process $b \rightarrow sg$, as discussed in these references. Another possibility, which was suggested in ref. [11], is to define the branching ratio in such a way that the photon energy E_γ has to be larger than some minimal value E_γ^{min} .

It is customary to express the branching ratio $\mathcal{B}(\bar{B} \rightarrow \bar{X}_s + \gamma)$ in terms of the measured semileptonic branching ratio $\mathcal{B}(B \rightarrow X\ell\nu_\ell)$,

$$\mathcal{B}(\bar{B} \rightarrow \bar{X}_s + \gamma) = \frac{\Gamma(\bar{B} \rightarrow \bar{X}_s + \gamma)}{\Gamma_{sl}} \mathcal{B}(B \rightarrow X\ell\nu_\ell). \quad (11)$$

The expression for Γ_{sl} (including radiative corrections) can be seen in refs. [33].

In addition to the perturbative QCD improvements discussed above, also the leading power corrections, which start in $1/m_b^2$, have been calculated to the decay widths appearing in the numerator and denominator of eq. (11) [15, 16, 17]. The power corrections in the numerator have been obtained assuming that the decay $\bar{B} \rightarrow \bar{X}_s + \gamma$ is dominated by the magnetic moment operator O_7 . Writing this correction in an obvious notation as

$$\frac{\Gamma(\bar{B} \rightarrow \bar{X}_s + \gamma)}{\Gamma^0(\bar{B} \rightarrow \bar{X}_s + \gamma)} = 1 + \frac{\delta_b}{m_b^2}, \quad (12)$$

one obtains $\delta_b = 1/2\lambda_1 - 9/2\lambda_2$, where λ_1 and λ_2 are, respectively, the kinetic energy and magnetic moment parameters of the theoretical framework based on heavy quark expansion. Using $\lambda_1 = -0.5 \text{ GeV}^2$ and $\lambda_2 = 0.12 \text{ GeV}^2$, one gets $\delta_b/m_b^2 \simeq -4\%$. However, the leading order ($1/m_b^2$) power corrections in the heavy quark expansion proportional to λ_1 are identical in the inclusive decay rates $\Gamma(\bar{B} \rightarrow \bar{X}_s + \gamma)$ and $\Gamma(B \rightarrow X\ell\nu_\ell)$. The corrections proportional to

²Note that we have absorbed the factor $F = 1 - (8\alpha_s)/(3\pi)$, present in ref. [10], into the term A_R^t .

λ_2 differ only marginally. Thus, including or neglecting the $1/m_b^2$ corrections makes a difference of only 1% in $\mathcal{B}(\overline{B} \rightarrow \overline{X}_s + \gamma)$.

The power corrections proportional to $1/m_c^2$, resulting from the interference of the operator O_2 (and O_1) with O_7 in $\overline{B} \rightarrow \overline{X}_s + \gamma$, have also been worked out [18, 19, 20]. Expressing this symbolically as

$$\frac{\Gamma(\overline{B} \rightarrow \overline{X}_s + \gamma)}{\Gamma^0(\overline{B} \rightarrow \overline{X}_s + \gamma)} = 1 + \frac{\delta_c}{m_c^2}, \quad (13)$$

one finds $\delta_c/m_c^2 \simeq +0.03$ [20].

It is convenient to express the branching ratio for $\overline{B} \rightarrow \overline{X}_s + \gamma$ in a form where the dependence on the CKM matrix factors is manifest:

$$\mathcal{B}(\overline{B} \rightarrow \overline{X}_s + \gamma) = \frac{|\lambda_t|^2}{|V_{cb}|^2} D_t + \frac{|\lambda_u|^2}{|V_{cb}|^2} D_u + \frac{Re(\lambda_t^* \lambda_u)}{|V_{cb}|^2} D_r + \frac{Im(\lambda_t^* \lambda_u)}{|V_{cb}|^2} D_i. \quad (14)$$

The quantities D_a ($a = t, u, r, i$), which depend on various input parameters such as m_t, m_b, m_c, μ_b and α_s , are calculated numerically and listed in Table 1. The averaged branching ratio $\langle \mathcal{B}(B \rightarrow X_s + \gamma) \rangle$ is obtained by discarding the last term on the right hand side of eq. (14). The CP-violating rate asymmetry has been defined earlier. In terms of the functions D_a it can be expressed as:

$$a_{CP}(B \rightarrow X_s + \gamma) = -\frac{Im(\lambda_t^* \lambda_u) D_i}{|\lambda_t|^2 D_t + |\lambda_u|^2 D_u + Re(\lambda_t^* \lambda_u) D_r}. \quad (15)$$

Since the function D_i in the numerator in eq. (15) only starts at order α_s , the complete NLL expression for a_{CP} requires D_i up to and including the $O(\alpha_s^2)$ term which is not known. Hence, in the LL approximation, a consistent definition of a_{CP} is the one in which only the LL result for the denominator is retained, i.e., in this approximation one should drop terms proportional to D_u and D_r and keep only the LL result for D_t , which is denoted as $D_t^{(0)}$ in the following. The expression for a_{CP} in this approximation then reduces to

$$a_{CP}(B \rightarrow X_s + \gamma) = -\frac{Im(\lambda_t^* \lambda_u) D_i}{|\lambda_t|^2 D_t^{(0)}}. \quad (16)$$

This is what we shall use in the numerical estimates of a_{CP} . Note that $D_t^{(0)}$, which is also shown in Table 1, is proportional to the square of the LL Wilson coefficient $C_7^{0, \text{eff}}(\mu_b)$. To be precise, this Wilson coefficient is obtained from the Wilson coefficients C_i ($i = 1, \dots, 8$) at the matching scale $\mu = m_W$ by using the 1-loop expression for $\alpha_s(\mu)$ in the renormalization group evolution. Moreover, notice that the power corrections, which are contained in the functions D_t and D_r in the NLL branching ratio (14), drop out in the LL expression for a_{CP} .

In the Wolfenstein parametrization [23], which will be used in the numerical analysis, the CKM matrix is determined in terms of the four parameters $A, \lambda = \sin \theta_C, \rho$ and η , and one can express the quantities λ_t, λ_u and $|V_{cb}|^2$ in the above equations as [32] (neglecting terms of $O(\lambda^6)$):

$$\lambda_u = A\lambda^4(\rho - i\eta), \quad \lambda_t = -A\lambda^2 \left(1 - \frac{\lambda^2}{2} + \lambda^2(\rho - i\eta) \right), \quad \lambda_c = -\lambda_u - \lambda_t, \quad |V_{cb}|^2 = A^2\lambda^4. \quad (17)$$

3. Decay rates and CP asymmetry in $B \rightarrow X_d + \gamma$ and $\bar{B} \rightarrow \bar{X}_d + \gamma$

In complete analogy with the $\bar{B} \rightarrow \bar{X}_s + \gamma$ case discussed earlier, the relevant set of dimension-6 operators for the processes $b \rightarrow d\gamma$ and $b \rightarrow d\gamma g$ can be written as

$$\begin{aligned} \mathcal{H}_{eff}(b \rightarrow d\gamma(+g)) = & -\frac{4G_F}{\sqrt{2}} \{ \xi_t [C_7(\mu)O_7(\mu) + C_8(\mu)O_8(\mu) + C_1(\mu)O_1(\mu) + C_2(\mu)O_2(\mu)] \\ & - \xi_u [C_1(\mu)(O_{1u}(\mu) - O_1(\mu)) + C_2(\mu)(O_{2u}(\mu) - O_2(\mu))] + \dots \} , \end{aligned} \quad (18)$$

where $\xi_j = V_{jb} V_{jd}^*$ with $j = u, c, t$. The operators are the same as in eq. (7) up to the obvious replacement of the s -quark field by the d -quark field. Moreover, the matching conditions $C_i(m_W)$ and the solutions of the RG equations, yielding $C_i(\mu_b)$, coincide with those needed for the process $b \rightarrow s\gamma(+g)$. The power corrections in $1/m_b^2$ and $1/m_c^2$ (besides the CKM factors) are also the same for $\Gamma(\bar{B} \rightarrow \bar{X}_d + \gamma)$ and $\Gamma(\bar{B} \rightarrow \bar{X}_s + \gamma)$. However, the so-called long-distance contributions from the intermediate u -quark in the penguin loops are *different* in the decays $\bar{B} \rightarrow \bar{X}_s + \gamma$ and $\bar{B} \rightarrow \bar{X}_d + \gamma$. These are suppressed in the decays $\bar{B} \rightarrow \bar{X}_s + \gamma$ due to the unfavorable CKM matrix elements. In $\bar{B} \rightarrow \bar{X}_d + \gamma$, however, there is no CKM-suppression and one has to include the long-distance intermediate u -quark contributions. It must be stressed that there is no spurious enhancement of the form $\ln(m_u/\mu_b)$ in the perturbative contribution to the matrix elements $\langle \bar{X}_d \gamma | O_{iu} | \bar{B} \rangle$ ($i = 1, 2$) as shown by the explicit calculation in [10] and also discussed more recently in [34]. In other words, the limit $m_u \rightarrow 0$ can be taken safely. The non-perturbative contribution generated by the u -quark loop can only be modeled at present. In this context, we recall that estimates based on the vector meson dominance indicate that these contributions are small [35]. Estimates of the long-distance contributions in exclusive decays $B \rightarrow \rho\gamma$ and $B \rightarrow \omega\gamma$ in the Light-Cone QCD sum rule approach put the corresponding corrections somewhere around $O(15\%)$ for the charged (B^\pm) decays and much smaller $O(5\%)$ for the neutral B decays [36, 37]. Model estimates based on final state interactions likewise give small long-distance contribution for the exclusive radiative B decays [38]. To take this uncertainty into account, we add an error proportional to D_t in LL approximation, viz. $\pm 0.1 D_t^{(0)}$, in the numerical estimate of the function D_r when calculating the branching ratio $\mathcal{B}(\bar{B} \rightarrow \bar{X}_d + \gamma)$ [35].

In analogy to eq. (14) the branching ratio $\mathcal{B}(\bar{B} \rightarrow \bar{X}_d + \gamma)$ in the SM can be written as

$$\mathcal{B}(\bar{B} \rightarrow \bar{X}_d + \gamma) = \frac{|\xi_t|^2}{|V_{cb}|^2} D_t + \frac{|\xi_u|^2}{|V_{cb}|^2} D_u + \frac{Re(\xi_t^* \xi_u)}{|V_{cb}|^2} D_r + \frac{Im(\xi_t^* \xi_u)}{|V_{cb}|^2} D_i , \quad (19)$$

where the functions D_a ($a = t, u, r, i$) are the same as in eq. (14). While these functions (or some combinations thereof) were obtained in partial NLL approximation some time ago [24], the complete NLL results are presented here for the first time. For numerical values of these functions we refer to Table 1. An expression for the averaged branching ratio $\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle$ is obtained by dropping the last term on the right hand side in eq. (19).

As the branching ratio $\langle \mathcal{B}(B \rightarrow X_s + \gamma) \rangle$ is very well approximated by $\langle \mathcal{B}(B \rightarrow X_s + \gamma) \rangle = |\lambda_t|^2 D_t / |V_{cb}|^2$, the ratio $R(d\gamma/s\gamma)$, defined in eq. (3), can be expressed as follows:

$$R(d\gamma/s\gamma) = \frac{|\xi_t|^2}{|\lambda_t|^2} + \frac{D_u}{D_t} \frac{|\xi_u|^2}{|\lambda_t|^2} + \frac{D_r}{D_t} \frac{Re(\xi_t^* \xi_u)}{|\lambda_t|^2} . \quad (20)$$

The leading term $|\xi_t|^2/|\lambda_t|^2$ in eq. (20) is obviously independent of any dynamical uncertainties; the subleading terms proportional to D_u/D_t and D_r/D_t are still uncertain by almost a factor 2, but numerically small compared to unity (see Table 1). Also, as we shall see in the next section, the allowed values of the CKM parameters provide a further suppression of these terms. Hence, the overall uncertainty in $R(d\gamma/s\gamma)$ is small.

Using again the LL expression for the denominator in eq. (4), the CP rate asymmetry can be written as

$$a_{CP}(B \rightarrow X_d + \gamma) = -\frac{\text{Im}(\xi_t^* \xi_u) D_i}{|\xi_t|^2 D_t^{(0)}}, \quad (21)$$

where $D_t^{(0)}$ stands for the LL expression of D_t . In the numerical analysis, we will use the following expressions for the quantities ξ_j in eqs. (19) and (21) (neglecting terms of $O(\lambda^7)$):

$$\xi_u = A \lambda^3 (\bar{\rho} - i\bar{\eta}), \quad \xi_t = A \lambda^3 (1 - \bar{\rho} + i\bar{\eta}), \quad \xi_c = -\xi_u - \xi_t, \quad (22)$$

with $\bar{\rho} = \rho(1 - \lambda^2/2)$ and $\bar{\eta} = \eta(1 - \lambda^2/2)$ [32]. Note that all three CKM-angle-dependent quantities ξ_j start at order λ^3 . Inserting these expressions into eq. (21), a simple form for the CP rate asymmetry is obtained:

$$a_{CP}(B \rightarrow X_d + \gamma) = \frac{D_i \bar{\eta}}{D_t^{(0)} [(1 - \bar{\rho})^2 + \bar{\eta}^2]}. \quad (23)$$

4. Numerical Estimates of branching ratios and CP asymmetries

We now proceed to the numerical analysis of our results. Based on present measurements and theoretical estimates, we take the following values for the input parameters: $\alpha_s(M_Z) = 0.118 \pm 0.003$, $m_b = 4.8 \pm 0.15$ GeV, $m_c/m_b = 0.29 \pm 0.02$, $m_t \equiv m_t(\text{pole}) = (175 \pm 6)$ GeV (corresponding to $\overline{m_t}(m_t) = (168 \pm 6)$ GeV), $\mathcal{B}_{sl} = (10.49 \pm 0.46)\%$, $\alpha_{em}^{-1} = (130.3 \pm 2.3)$. For the CKM matrix elements we note that the parameters A and λ are rather well determined. The parameters ρ and η are constrained from unitarity fits. The updated fits, taking into account also the lower bound on the mixing-induced mass difference ratio $\Delta M_s/\Delta M_d > 20.4$ yield (at $\pm 1\sigma$) [29],

$$\begin{aligned} A &= 0.81 \pm 0.057, & \lambda &= 0.22 \\ \eta &= 0.33 \pm 0.065, & \rho &= 0.11_{-0.11}^{+0.14}, \end{aligned} \quad (24)$$

where λ , being very accurately measured, was fixed to the value shown. Note that the allowed range of ρ is now asymmetric with respect to $\rho = 0$ due to the mentioned bound on $\Delta M_s/\Delta M_d$, which removes large negative- ρ values. We also note that the recent CKM fits reported in [39] yield an identical range for η but they find $\rho = 0.156 \pm 0.090$, which is more restrictive for the lower bound on ρ than the analysis in [29], that we use here.

In Table 1, we give the values of the functions D_t , D_u , D_r and D_i , evaluated for the central values of the parameters $\alpha_s(m_Z)$, m_t , m_b , α_{em} and the semileptonic branching ratio \mathcal{B}_{sl} . The other two parameters m_c/m_b and μ_b are varied as indicated. We note that the renormalization scale dependence of D_t is significantly reduced in the NLL compared to the LL result $D_t^{(0)}$. As D_u , D_r , and D_i start at order α_s only, their μ_b dependence is more significant, but their contribution to the branching ratio is rather small.

	$\mu_b = 2.5 \text{ GeV}$	$\mu_b = 5 \text{ GeV}$	$\mu_b = 10 \text{ GeV}$	m_c/m_b
$D_t^{(0)}/\lambda^4$	0.131	0.106	0.086	0.27
$D_t^{(0)}/\lambda^4$	0.142	0.114	0.093	0.29
$D_t^{(0)}/\lambda^4$	0.155	0.125	0.101	0.31
D_t/λ^4	0.150	0.147	0.140	0.27
D_t/λ^4	0.155	0.154	0.147	0.29
D_t/λ^4	0.161	0.163	0.157	0.31
D_u/λ^4	0.015	0.011	0.009	0.27
D_u/λ^4	0.016	0.012	0.009	0.29
D_u/λ^4	0.016	0.012	0.009	0.31
D_r/λ^4	-0.033	-0.021	-0.014	0.27
D_r/λ^4	-0.043	-0.028	-0.019	0.29
D_r/λ^4	-0.055	-0.036	-0.025	0.31
D_i/λ^4	0.056	0.039	0.028	0.27
D_i/λ^4	0.062	0.042	0.031	0.29
D_i/λ^4	0.068	0.047	0.034	0.31

Table 1: Values of the NLL functions D_t , D_u , D_r , D_i (divided by λ^4) for the indicated values of the scale parameter μ_b and the quark mass ratio m_c/m_b . Also tabulated are the values for the LL function $D_t^{(0)}/\lambda^4$.

For the values of the input parameters given above, the theoretical branching ratio for the decay $\overline{B} \rightarrow \overline{X}_s + \gamma$ in the SM is calculated by us as $\mathcal{B}(\overline{B} \rightarrow \overline{X}_s + \gamma) = (3.50 \pm 0.32) \times 10^{-4}$. This is to be compared with the recent result in ref. [40], where the central value 3.46×10^{-4} is quoted for the case in which the factor $1/\Gamma_{sl}$ is – like in the present work – expanded in α_s . Taking into account, that we use $|\lambda_t/V_{cb}|^2 = 0.96$ in the present CKM framework, whereas in ref. [40] a value 0.95 was used for the same quantity, the results here and in [40] are in agreement. To calculate a_{CP} in the decay rates for $B \rightarrow X_s + \gamma$ and its charge conjugate $\overline{B} \rightarrow \overline{X}_s + \gamma$, we shall use the LL approximation (16) for a_{CP} . For the central values of the parameters we obtain (neglecting corrections of $O(\lambda^2)$): $a_{CP}(B \rightarrow X_s + \gamma) = -(2.11)\eta\%$, which gives for $\eta = 0.33 \pm 0.13$ the following prediction for the decay rate asymmetry: $a_{CP}(B \rightarrow X_s + \gamma) = -(0.70 \pm 0.28)\%$. Note that these numbers correspond to the preferred scale $\mu_b = 2.5 \text{ GeV}$. For $\mu = 5 \text{ GeV}$, the asymmetry would be $a_{CP}(B \rightarrow X_s + \gamma) = -(0.59 \pm 0.25)\%$. Thus, the direct CP asymmetry in $\overline{B} \rightarrow \overline{X}_s + \gamma$ in the standard model turns out to be too small to be measurable.

We now discuss the decay $\overline{B} \rightarrow \overline{X}_d + \gamma$ and the CP conjugated process $B \rightarrow X_d + \gamma$. The averaged branching ratio $\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle$ strongly depends on the CKM parameters ρ , η . Taking the central values of the parameters $\alpha_s(M_Z)$, m_b , m_c/m_b , m_t , \mathcal{B}_{sl} , α_{em} and $\mu_b = 2.5 \text{ GeV}$, one obtains the following prediction:

$$\begin{aligned} \langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle &= 2.43 \left[(1 - \bar{\rho})^2 + \bar{\eta}^2 - 0.35(1 - \bar{\rho}) + 0.07 \right] \times 10^{-5}, \\ &\simeq 1.61 \times 10^{-5} \quad [\text{for } (\rho, \eta) = (0.11, 0.33), \text{ or } (\bar{\rho}, \bar{\eta}) = (0.107, 0.322)]. \end{aligned} \quad (25)$$

In comparison, the result in the LL approximation for $\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle$, for the same values of the parameters is: $\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle = 1.61 [(1 - \bar{\rho})^2 + \bar{\eta}^2] \times 10^{-5}$. This gives, $\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle = 1.45 \times 10^{-5}$ for $(\rho, \eta) = (0.11, 0.33)$. The difference between the LL and NLL results is $\sim 10\%$, increasing the branching ratio in the NLL case (see Fig. 1 for $\mu_b = 2.5$ GeV). The scale (μ_b)-dependence of $\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle$ in the LL and NLL accuracy is shown in Fig. 1, fixing all other parameters to their central values.

In Fig. 2 we give the ρ dependence of the branching ratio $\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle$ for $\eta = 0.20, 0.33$ and 0.46 , using $\mu_b = 2.5$ GeV and the central values for all other parameters. We note that the dependence on η is not very marked. The branching ratio is largest for the smallest allowed value of ρ (taken here as $\rho = -0.10$) and the largest allowed value of η (assumed here as $\eta = 0.46$), and may reach a value of 2.6×10^{-5} . The minimum value of $\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle$ in the SM is estimated as 6.0×10^{-6} .

The ratio $R(d\gamma/s\gamma)$ in eq. (20) can be expressed in terms of the CKM parameters $\bar{\rho}$ and $\bar{\eta}$ as follows (expanding $1/|\lambda_t|^2$ in powers of λ):

$$\begin{aligned} R(d\gamma/s\gamma) &= \lambda^2 [1 + \lambda^2(1 - 2\bar{\rho})] \left[(1 - \bar{\rho})^2 + \bar{\eta}^2 + \frac{D_u}{D_t}(\bar{\rho}^2 + \bar{\eta}^2) + \frac{D_r}{D_t}(\bar{\rho}(1 - \bar{\rho}) - \bar{\eta}^2) \right], \\ &\simeq 0.046 \quad [\text{for } (\rho, \eta) = (0.11, 0.33), \text{ or } (\bar{\rho}, \bar{\eta}) = (0.107, 0.322)]. \end{aligned} \quad (26)$$

Our prediction for the direct CP asymmetry $a_{CP}(B \rightarrow X_d + \gamma)$, based on the LL result (21) and for the central values of the input parameters, is:

$$\begin{aligned} a_{CP}(B \rightarrow X_d + \gamma)(\mu_b = 2.5 \text{ GeV}) &= \frac{0.44\bar{\eta}}{(1 - \bar{\rho})^2 + \bar{\eta}^2}, \\ &\simeq 0.16 \quad [\text{for } (\rho, \eta) = (0.11, 0.33), \text{ or } (\bar{\rho}, \bar{\eta}) = (0.107, 0.322)]. \end{aligned} \quad (27)$$

The scale dependence of this result is as follows: $a_{CP}(\mu_b = 5 \text{ GeV}) \simeq 0.13$ and $a_{CP}(\mu_b = 10 \text{ GeV}) \simeq 0.12$. As argued earlier, we prefer $\mu_b = 2.5$ GeV to estimate a_{CP} , as for this choice of the scale the NLL corrections in the decay rates are small.

In Fig. 3 we show the η dependence of the direct CP rate asymmetry for the $\bar{B} \rightarrow \bar{X}_d + \gamma$ decay for $\rho = -0.10, 0.11, 0.25$ and 0.40 , using again $\mu_b = 2.5$ GeV and the central values of all other parameters. The smallest value of $a_{CP}(B \rightarrow X_d + \gamma)$ is 7% (for $\rho = -0.10$ and $\eta = 0.20$) and may reach as high a value as 35% (for $\rho = 0.4$ and $\eta = 0.46$), as can be seen in Fig. 3. We want to stress that the ρ -dependence of the branching ratio $\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle$ and both the ρ - and η -dependence of $a_{CP}(B \rightarrow X_d + \gamma)$ are very marked. Hence, their measurements will help to determine these parameters more precisely.

To that end, it is important to estimate the theoretical uncertainties in the branching ratio, the ratio $R(d\gamma/s\gamma)$, and direct CP asymmetry in $B \rightarrow X_d + \gamma$ and $\bar{B} \rightarrow \bar{X}_d + \gamma$ for given values of η and ρ . We estimate these theoretical uncertainties by varying the scale $\mu_b \in [2.5 \text{ GeV}, 10.0 \text{ GeV}]$ and the input parameters in their respective $\pm 1\sigma$ -ranges given earlier. The procedure adopted is as follows: Individual errors $\Delta_i(\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle)$, $\Delta_i(R(d\gamma/s\gamma))$ and $\Delta_i(a_{CP})$ are estimated by varying each parameter at a time and the resulting errors are then added in quadrature, much the same way as it has been done for estimating the theoretical uncertainty in the branching ratio $\mathcal{B}(\bar{B} \rightarrow \bar{X}_s + \gamma)$. As mentioned earlier, we add an error of $\pm 0.1 D_i^{(0)}$ in the numerical estimate of the function D_r in order to take into account long-distance effects generated by intermediate u -quarks. The resulting theoretical uncertainty on

$\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle$ from all the sources is shown in Fig. 4 as $\pm 1\sigma$ bands for the central value $\eta = 0.33$ as a function of ρ . For a given value of ρ and η , the theoretical uncertainty is: $\Delta(\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle) / \langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle = \pm(6 - 10)\%$ on the branching ratio. This is much smaller than the factor 4 dispersion in $\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle$ due to the ρ and η dependence, shown in Fig. 2.

The uncertainty in the ratio $R(d\gamma/s\gamma)$ is even smaller, since the theoretical errors on $\langle \mathcal{B}(B \rightarrow X_s + \gamma) \rangle$ and $\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle$ tend to cancel. The residual theoretical uncertainty $\Delta R/R$ is correlated with the value of ρ and η , which is not difficult to see from the relation in eq. (26). For the central value of the CKM fits $\eta = 0.33$, this is shown in Fig. 5 where we plot $R(d\gamma/s\gamma)$ as a function of ρ . Interestingly, the theoretical uncertainties almost vanish for ρ in the proximity of the "best fit" value $\rho = 0.11$. The largest theoretical uncertainty $R(d\gamma/s\gamma)$ in the 95% C.L. allowed CKM-domain is for the point ($\rho = 0.4, \eta = 0.2$) where $\Delta R(d\gamma/s\gamma) / R(d\gamma/s\gamma) = \pm 7\%$, as the ratio $R(d\gamma/s\gamma)$ is smallest there. This study suggests that the impact of the measurement of $R(d\gamma/s\gamma)$ on the CKM parameters will be largely determined by experimental errors.

It is interesting to see with which theoretical accuracy the Wolfenstein parameters ρ and η get constrained assuming an ideal measurement of R . To illustrate this, we study the fixed- R contours in the (ρ, η) plane. Our procedure is as follows: We choose three hypothetical values $R = 0.017, 0.046, 0.074$, emerging from our NLL analysis. For each of these values, we solve eq. (26) for η . Fixing all input parameters, except ρ , leads to a curve in the (ρ, η) plane (fixed- R contour). Varying then the input parameters $\alpha_s(m_z), m_b, m_c/m_b, m_t$ and the scale μ_b (one at a time followed by adding the individual errors in quadrature), leads to a band in the (ρ, η) -plane for each value of R . In Fig. 6 these bands are shown for the values of R indicated above. The unitarity triangle corresponding to the "best fit" solution ($\rho = 0.11, \eta = 0.33$) is also drawn for orientation. One sees again that the theoretical uncertainties are minimal (practically vanishing) for the "best fit" solution.

In Fig. 7, we show the uncertainty on a_{CP} due to the scale variation and due to the input parameters as a function of η (with fixed $\rho = 0.11$). As mentioned, the power corrections drop out in the LL approximation. For given values of ρ and η , we find: $\Delta(a_{CP})/a_{CP} = \pm 17\%$. Since the asymmetry $a_{CP}(B \rightarrow X_d + \gamma)$ itself varies between 7% and 35% (see Fig. 3) in the presently allowed range of the parameters ρ and η , the residual theoretical uncertainty is not a serious hindrance in testing the CKM paradigm for CP violation in these decays. Of course, it will be nice to complete the calculation for a_{CP} in the NLL approximation, which hopefully will reduce the theoretical uncertainty on this quantity considerably.

5. Summary

To summarize, we have presented theoretical estimates of the branching ratio $\langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle$ and the ratio $R(d\gamma/s\gamma)$ in the NLL approximation, and $a_{CP}(B \rightarrow X_d + \gamma)$ in the LL approximation in SM, working out also theoretical errors. Varying the CKM-Wolfenstein parameters ρ and η in the range $-0.1 \leq \rho \leq 0.4$ and $0.2 \leq \eta \leq 0.46$ and taking into account other parametric dependences stated earlier, our numerical results can be summarized as follows:

$$\begin{aligned}
6.0 \times 10^{-6} &\leq \langle \mathcal{B}(B \rightarrow X_d + \gamma) \rangle \leq 2.6 \times 10^{-5} , \\
0.017 &\leq R(d\gamma/s\gamma) \leq 0.074 , \\
0.07 &\leq a_{CP}(B \rightarrow X_d + \gamma) \leq 0.35 .
\end{aligned} \tag{28}$$

The central values of these quantities corresponding to the "best fit" parameters ($\rho = 0.11$, $\eta = 0.33$) are: $\mathcal{B}(\overline{B} \rightarrow \overline{X}_d + \gamma) = (1.61 \pm 0.12) \times 10^{-5}$, $R(d\gamma/s\gamma) = 0.046$ and $a_{CP}(B \rightarrow X_d + \gamma) = (11.5 - 16.5)\%$, with practically no error on $R(d\gamma/s\gamma)$. This ratio is also otherwise found to be remarkably stable against variation in the input parameters, with the maximum uncertainty estimated as $\Delta R(d\gamma/s\gamma)/R(d\gamma/s\gamma) = \pm 7\%$ for ($\rho = 0.4$, $\eta = 0.2$). These quantities are expected to be measurable at the forthcoming high luminosity B facilities. The CP-violating asymmetry $a_{CP}(B \rightarrow X_s + \gamma)$ in the SM is found to be too small to measure. We emphasize the need to complete the NLL-improved calculation for $a_{CP}(B \rightarrow X_d + \gamma)$.

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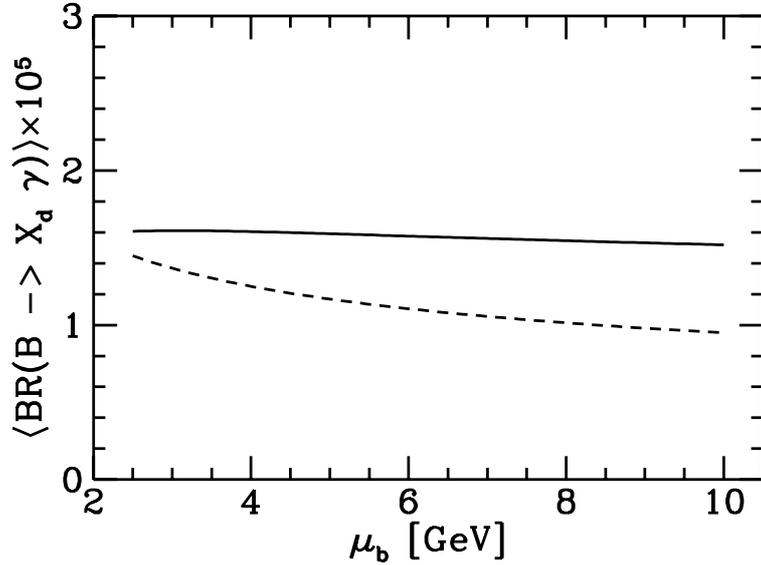


Figure 1: Average branching ratio of the processes $B \rightarrow X_d + \gamma$ and $\bar{B} \rightarrow \bar{X}_d + \gamma$, plotted as a function of the scale μ_b for the central values of the input parameters m_b , m_c/m_b , \mathcal{B}_{sl} , m_t , α_{em} and $\alpha_s(m_Z)$. The solid (dashed) curve shows the NLL (LL) result.

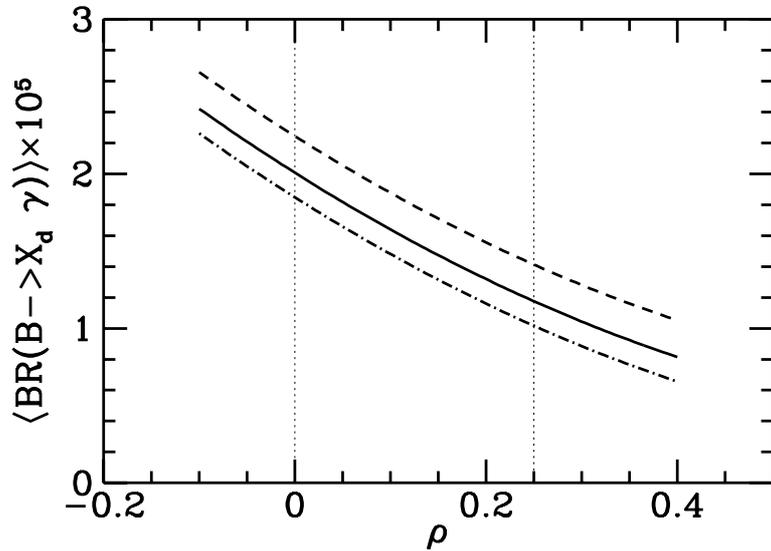


Figure 2: The ρ dependence of the average branching ratio for $B \rightarrow X_d + \gamma$ and $\bar{B} \rightarrow \bar{X}_d + \gamma$ is shown for different values of η : $\eta = 0.46$ (dashed curve); $\eta = 0.33$ (solid curve); $\eta = 0.20$ (dash-dotted curve). All three curves correspond to $\mu_b = 2.5$ GeV and to the central values of the input parameters m_b , m_c/m_b , \mathcal{B}_{sl} , m_t , α_{em} and $\alpha_s(m_Z)$. The vertical lines show the $\pm 1\sigma$ range for ρ from the CKM fits [29].

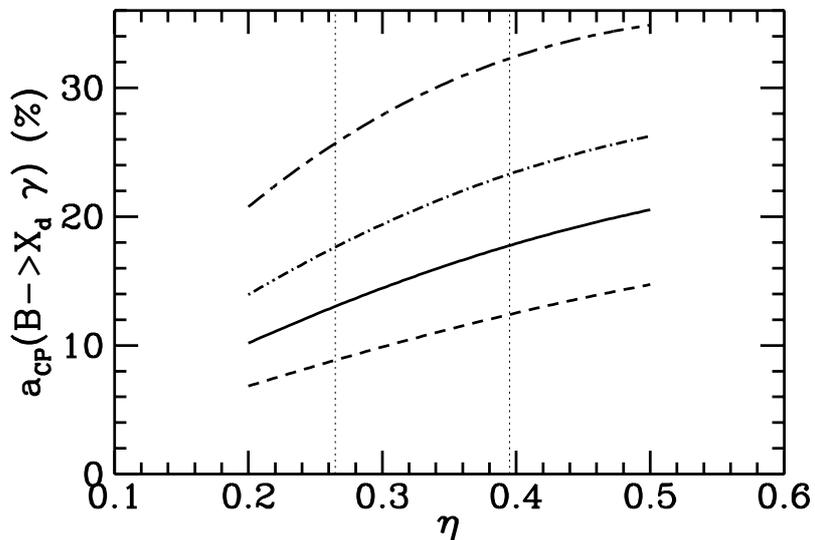


Figure 3: η dependence of the CP rate asymmetry $a_{CP}(B \rightarrow X_d + \gamma)$ for different values of ρ : $\rho = -0.1$ (dashed curve); $\rho = 0.11$ (solid curve); $\rho = 0.25$ (dash-dotted curve), $\rho = 0.4$ (long-short dashed curve). All four curves correspond to $\mu_b = 2.5$ GeV and to the central values of the input parameters m_b , m_c/m_b , \mathcal{B}_{sl} , m_t , α_{em} and $\alpha_s(m_Z)$. The vertical lines show the $\pm 1\sigma$ range for η from the CKM fits [29].

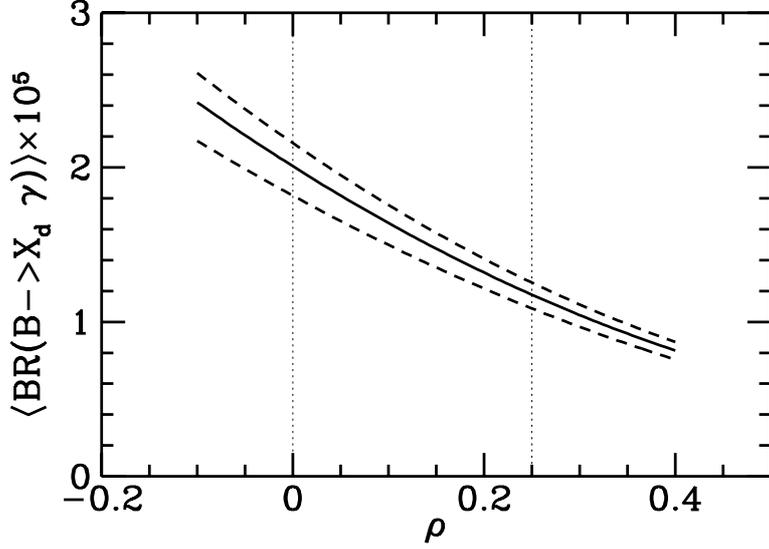


Figure 4: ρ dependence of the average branching ratio for $B \rightarrow X_d + \gamma$ and $\bar{B} \rightarrow \bar{X}_d + \gamma$ for fixed $\eta = 0.33$. The solid curve corresponds to $\mu_b = 2.5$ GeV and the central values of the input parameters. The upper and lower dashed curves show the theoretical dispersion due to the errors in the input parameters m_b , m_c/m_b , \mathcal{B}_{sl} , m_t , $\alpha_s(m_Z)$, α_{em} and due to the variation of the scale $\mu_b \in [2.5 \text{ GeV}, 10.0 \text{ GeV}]$. The long-distance contribution due to the u -quark loop is also included in estimating the errors (see text). The vertical lines show the $\pm 1\sigma$ range for ρ from the CKM fits [29].

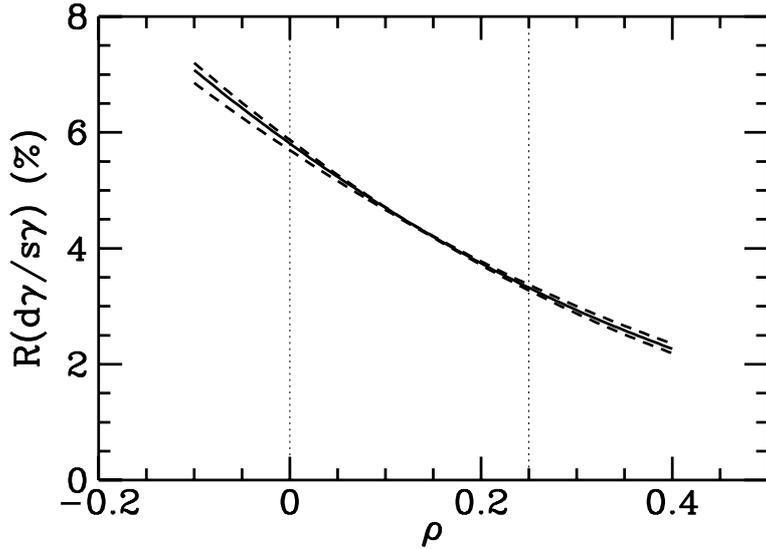


Figure 5: The ratio $R(d\gamma/s\gamma)$ (in %) as a function of the CKM parameter ρ for a fixed value of $\eta = 0.33$. The bands show the theoretical uncertainties following from the error estimates discussed in text. The vertical lines show the $\pm 1\sigma$ range for ρ from the CKM fits [29].

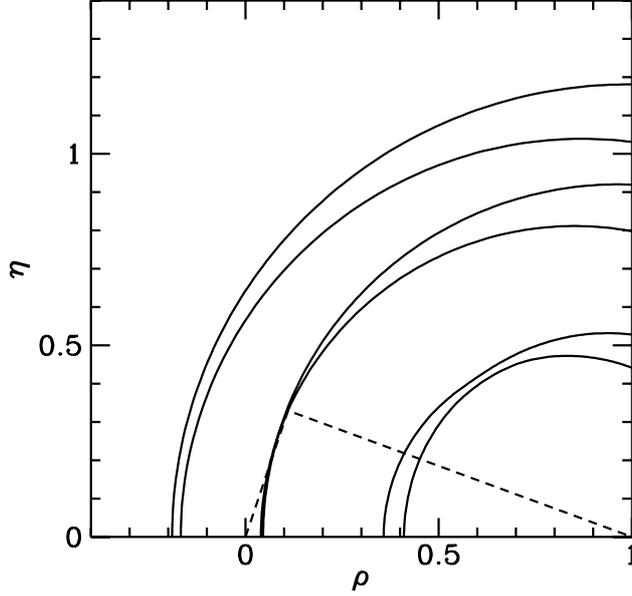


Figure 6: Fixed- R contours in the (ρ, η) plane, obtained by varying the input parameters and the scale $\mu_b \in [2.5 \text{ GeV}, 10.0 \text{ GeV}]$. The three bands shown in the figure correspond to $R = 0.017$ (bottom), $R = 0.046$ (middle) and $R = 0.074$ (top). The unitarity triangle corresponding to the "best fit" solution from the CKM fits [29] is also shown.

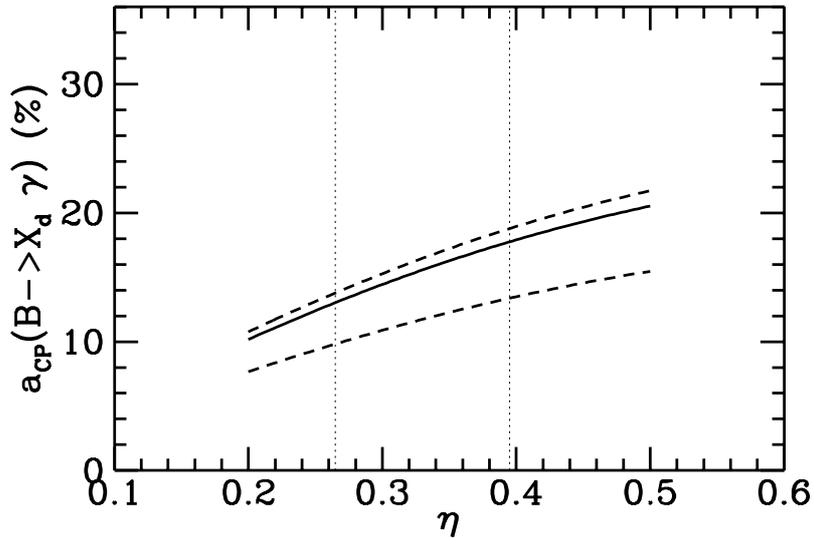


Figure 7: η dependence of the CP rate asymmetry $a_{CP}(B \rightarrow X_d + \gamma)$ for fixed $\rho = 0.11$. The solid curve corresponds to $\mu_b = 2.5 \text{ GeV}$ and the central values of the input parameters. The upper and lower dashed curves show the theoretical dispersion due to the errors in the parameters m_b , m_c/m_b , \mathcal{B}_{sl} , m_t , $\alpha_s(m_Z)$, α_{em} and due to the variation of the scale $\mu_b \in [2.5 \text{ GeV}, 10.0 \text{ GeV}]$. The vertical lines show the $\pm 1\sigma$ range for η from the CKM fits [29].