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QED Corrections for Polarized Elastic μe Scattering[a,b]

Dima Bardin [1] and Lida Kalinovskaya [1,2]

[1] Laboratory for Nuclear Problems, JINR, ul. Joliot-Curie 6, RU–141980 Dubna, Russia

[2] DESY-Zeuthen, Platanenallee 6, D-15735, Zeuthen, Germany

ABSTRACT

We present a new study of polarized elastic muon-electron scattering. The Born cross-section is calculated for arbitrary polarization of muon and electron. The complete photonic $\mathcal{O}(\alpha)$ radiative corrections are determined for the case of longitudinally polarized muons and electrons. All calculations are done by two methods: semianalytic, which allows an implementation of the experimental cuts used for the analysis of μe scattering data from the beam polarimeter of the SMC experiment at CERN and completely analytic, which is used for cross checks. The `FORTRAN` code μela realizes formulae of both approaches. We prove that certain experimental cuts lead to negligible radiative corrections in the muon beam polarization experiment.

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Emails: `bardin@nusun.jinr.ru`
`kalinov@nusun.jinr.ru`

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1 Introduction

Polarized elastic μe scattering is being measured by the SMC collaboration at CERN as a monitor of muon beam polarization [1]. Since the measurement pretends to be very precise, the photonic corrections have to be taken into account.

The differential cross-section for this process in lowest order may be cast into the simple form [2]

$$\frac{d\sigma^{\rm BORN}}{dy} = \frac{2\pi\alpha^2}{m_e E_\mu}\left[\frac{(Y-y)}{y^2 Y}(1-yP_eP_\mu)+\frac{1}{2}(1-P_eP_\mu)\right], \tag{1.1}$$

where the following notation is used:

m_μ, m_e – muon and electron masses,

P_μ, P_e – longitudinal polarizations of muon beam and electron target,

$y = y_\mu = 1 - \frac{E'_\mu}{E_\mu}$ – the measured energy loss of the muon,

$Y = \left(1 + \frac{m_\mu}{2E_\mu}\right)^{-1}$ – its kinematical maximum,

E_μ, E'_μ, E'_e – muon (initial, final), electron final energies in the laboratory frame.

The polarization dependence of $d\sigma$ is used to calculate the measured electron spin-flip asymmetry $A^{\rm exp}_{\mu e}$

$$A^{\rm exp}_{\mu e} = \frac{\dfrac{d\sigma(\uparrow\downarrow)}{dy} - \dfrac{d\sigma(\uparrow\uparrow)}{dy}}{\dfrac{d\sigma(\uparrow\downarrow)}{dy} + \dfrac{d\sigma(\uparrow\uparrow)}{dy}} . \tag{1.2}$$

The asymmetry is measured as a function of the variable y_μ.

Previous calculations [3]-[6], presented results in terms of the variable $y_e = E'_e/E_\mu$, and, only Ref. [6] took into account the polarizations.

If elastic μe scattering is treated in the Born approximation then

$$y_\mu = y_e = y, \tag{1.3}$$

and eq. (1.1) may be written in terms of either y_μ or y_e.

The situation changes drastically if one calculates QED corrections. Due to the emission of non-observed photons the identity (1.3) does not hold anymore, and one has to specify the variable to be used for the calculation of radiative corrections. Their numerical values may be very different in y_μ and y_e.

Since the measurement and the analysis were performed in terms of y_μ [1], the calculation of QED corrections must be done, of course, in terms of the same variable. This is why a new calculation was neccessary.

Our new calculation is the theoretical basis for the Fortran program μe*la*, [7]. It is a complete, order $\mathcal{O}(\alpha^3)$, calculation. It takes into account longitudinal polarizations of both μ and e, finite muon mass effects (the electron mass is neglected wherever possible). In the semianalytic approach it is possible to apply all experimental cuts which were used in the analysis of the experimental data:

– a recoil electron energy cut, $E'_e \geq E^{RC}$ ($E^{RC} = 35$ GeV);

– an energy balance cut, $\left|E - E'_\mu - E'_e\right| \geq E^{BC}$ ($E^{BC} = 40$ GeV);

– angular cuts on both μ and e, θ_μ and θ_e in the laboratory system: $\left|\theta_e^{\mathrm{meas}} - \theta_e^{\mathrm{BORN}}\right| \leq \theta_{\mathrm{min}}$, $\left|\theta_\mu^{\mathrm{meas}} - \theta_\mu^{\mathrm{BORN}}\right| \leq \theta_{\mathrm{min}}$ ($\theta_{\mathrm{min}} = 1$ mrad). In the above, θ_e^{meas} and $\theta_\mu^{\mathrm{meas}}$ are the measured angles, while $\theta_\mu^{\mathrm{BORN}}$ and θ_e^{BORN} are angular values calculated using BORN kinematics.

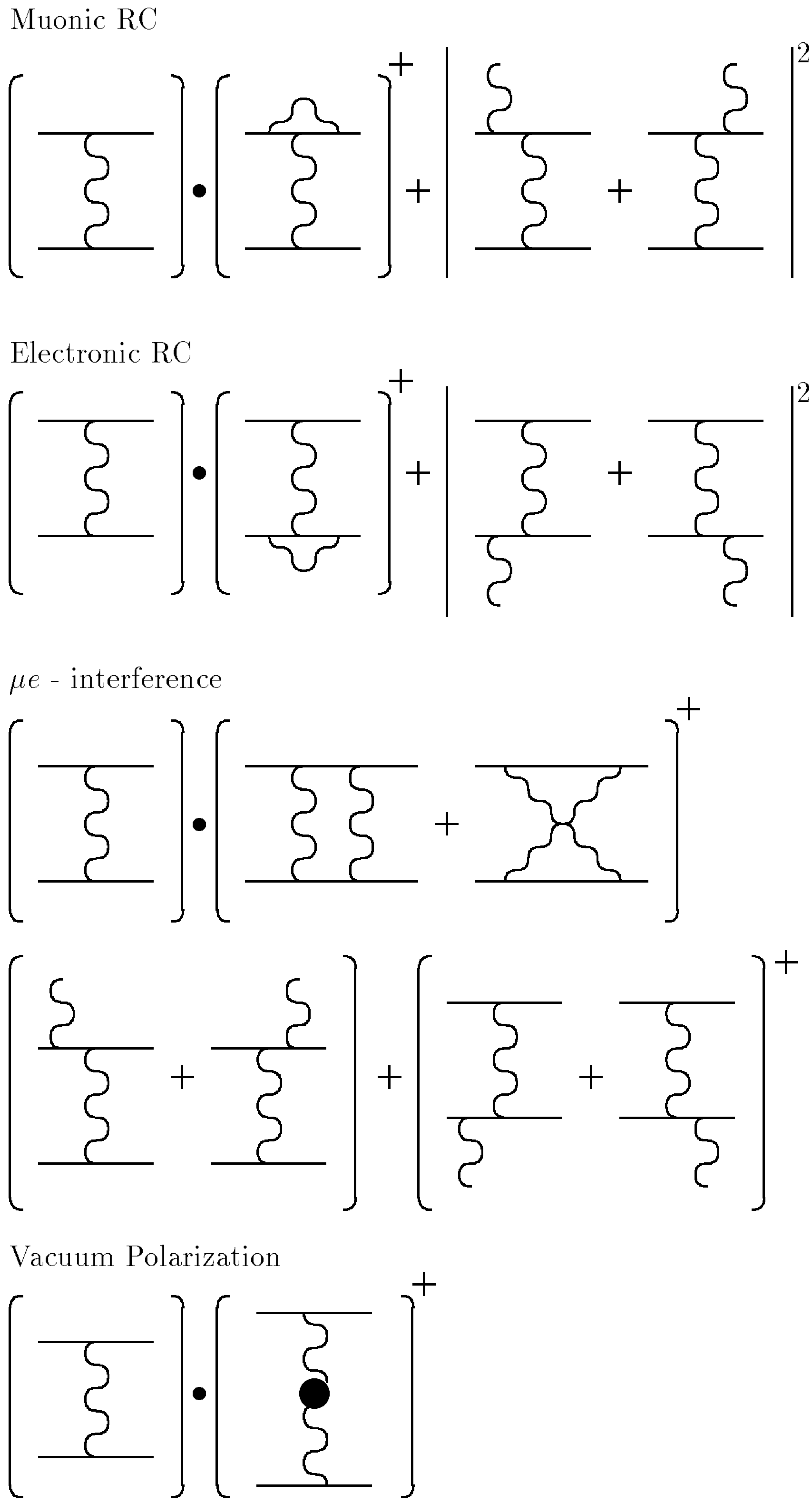

Figure 1: Feynman diagrams for the elastic μe scattering in order $\mathcal{O}(\alpha^3)$.

It order $\mathcal{O}(\alpha^3)$, the 14 Feynman graphs, shown in Fig. 1, contribute to the cross-section. The latter may be subdivided into **12=2×6** separately gauge invariant contributions:

$$\frac{d\sigma^{\rm QED}}{dy_\mu} = \sum_{l=1}^{2}\sum_{k=1}^{6}\frac{d\sigma_k^l}{dy_\mu}, \tag{1.4}$$

where the indices k and l have the following meaning

$l=$	1	–	unpolarized contribution, $l=$ unpol;
	2	–	polarized contribution (the terms proportional to P_eP_μ in (1.1)) , $l=$ pol.
$k=$	1	–	Born cross-section, $k=b$;
	2	–	Radiative corrections (RC) for the muonic current: vertex + bremsstrahlung, $k=\mu\mu$;
	3	–	contribution of the anomalous magnetic moment of the muon, $k=$ amm;
	4	–	RC for the electronic current: vertex + bremsstrahlung, $k=ee$;
	5	–	μe interference: two-photon exchange + muon-electron bremsstrahlung interference, $k=\mu e$;
	6	–	Vacuum polarization correction, running α, $k=$ vp.

The resulting QED corrected cross-section is given by the sum

$$\frac{d\sigma^{\rm QED}}{dy_\mu} = \sum_k\left(\frac{d\sigma_k^{\rm unpol}}{dy_\mu} + P_eP_\mu\frac{d\sigma_k^{\rm pol}}{dy_\mu}\right). \tag{1.5}$$

The cross-sections with $k=\mu\mu, ee, \mu e$ have similar generic structure

$$\frac{d\sigma_k}{dy_\mu} = \frac{\alpha}{\pi}\delta_k^{\rm VR}\frac{d\sigma^{\rm BORN}}{dy_\mu} + \frac{d\sigma_k^{\rm BREM}}{dy_\mu}, \tag{1.6}$$

with a factorized part $\delta_k^{\rm VR}$ originating from infrared divergent virtual (V) and real soft photon (R) contributions[1].

The main conclusion of this study is illustrated in Figs. 2 and 3, which present radiative corrections to the asymmetry as a function of the variable y_μ for two cases: without any cuts and with experimental cuts described above.

The asymmetry $A^a_{\mu e}$, and the radiative correction to it, $\delta^A_{y_\mu}$ are defined as follows:

$$A^a_{\mu e} = \frac{\dfrac{d\sigma^a(\uparrow\downarrow)}{dy_\mu} - \dfrac{d\sigma^a(\uparrow\uparrow)}{dy_\mu}}{\dfrac{d\sigma^a(\uparrow\downarrow)}{dy_\mu} + \dfrac{d\sigma^a(\uparrow\uparrow)}{dy_\mu}}, \qquad a = \text{BORN, QED}, \qquad \delta^A_{y_\mu} = \frac{A^{\rm QED}_{\mu e}}{A^{\rm BORN}_{\mu e}} - 1. \tag{1.7}$$

As is seen from the figures, the corrections without cuts are very large and reach up to -20%. When the four above mentioned cuts are taken into account they reduce δ to values below 1%. Actually, for a wide range of y_μ they are even well below 1%.

The main conclusion of our investigation is that one may safely neglect radiative corrections in the determination of the muon beam polarization with the SMC set-up.

[1] In the following we will always present the formulae in the form (1.5), i.e. summed over l.

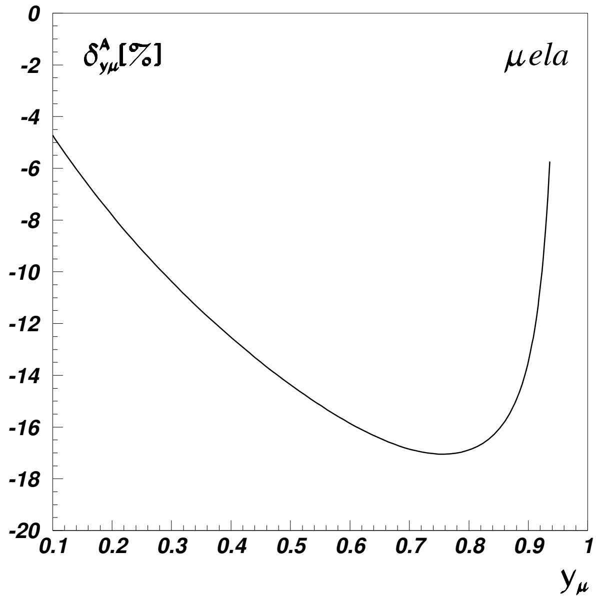

Figure 2: *QED corrections to the polarization asymmetry without experimental cuts.*

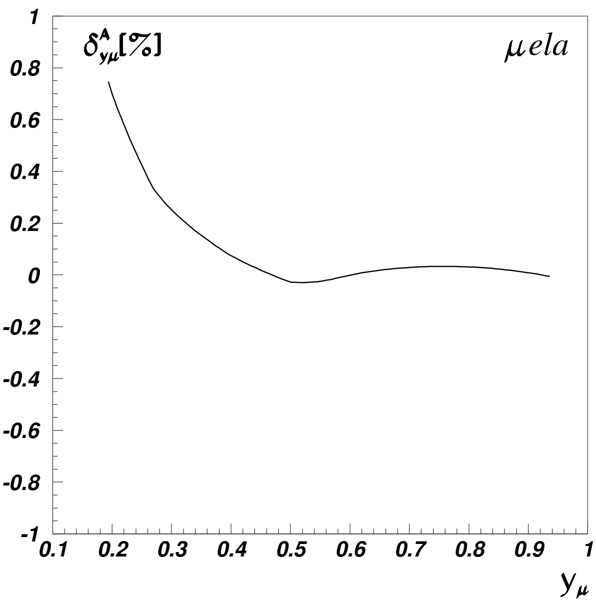

Figure 3: *QED corrections to the polarization asymmetry with experimental cuts:* $E^{RC} = 35\ GeV,\ \ E^{BC} = 40,\ \ \theta_{e,\min} = \theta_{\mu.\min} = 1\ mrad.$

2 The Born cross-section

2.1 Kinematics and phase space

We consider the elastic scattering process

$$\mu(k_1) + e(p_1) \to \mu(k_2) + e(p_2), \tag{2.8}$$

in a fixed target experiment, i.e. with the initial state electron at rest, $k_1 = (\vec{0}, im_e)$.

Since the typical incident muon energy, E_μ, in present-day fixed target experiments is $\mathcal{O}(10^2 - 10^3\ \mathrm{GeV})$, the maximal c.m.s. energy,

$$s \;=\; -(k_1+p_1)^2 = m_\mu^2 + m_e^2 + 2m_e E_\mu, \tag{2.9}$$

is very small, $\sqrt{s} \le 1$ GeV. Therefore, we may completely neglect Z-boson exchange.

In fact, for the energy used by the SMC collaboration, $E_\mu = 190$ GeV, the invariant s is only 20 times bigger than the muon mass squared. Therefore, we *can not* neglect effects of the finite muon mass. Of course, the electron mass may be completely neglected.

While calculating the Born cross-section, we will perform all derivations exactly even in m_e, since the resulting expressions are very compact even if m_e is kept, but at the end of calculations we will neglect the electron mass[2]. The Born process is characterized by one kinematical variable, besides s. We will use the dimensionless variable y:

$$y = \frac{p_1(k_1-k_2)}{p_1 k_1} = \frac{k_1^0 - k_2^0}{k_1^0} \equiv 1 - \frac{E'_\mu}{E_\mu}. \tag{2.10}$$

We will introduce also the transferred momentum squared

$$Q^2 = (k_1-k_2)^2 = -t. \tag{2.11}$$

It is easy to derive the identity:

$$Q^2 = Sy, \tag{2.12}$$

where

$$S = s - m_e^2 - m_\mu^2. \tag{2.13}$$

For the Born cross-section, we have

$$d\sigma^{\mathrm{BORN}} = \frac{1}{2\sqrt{\lambda_S}} |M^{\mathrm{BORN}}|^2 d\Gamma_2, \tag{2.14}$$

with

$$\lambda_S = S^2 - 4m_e^2 m_\mu^2, \tag{2.15}$$

$$d\Gamma_2 = (2\pi)^4 \frac{d^3\vec{k}_2}{(2\pi)^3 2k_2^0} \frac{d^3\vec{p}_2}{(2\pi)^3 2p_2^0} \delta(k_1+p_1-k_2-p_2). \tag{2.16}$$

In terms of y the differential phase space reads

$$d\Gamma_2 = \frac{1}{16\pi^2}\frac{dQ^2}{\sqrt{\lambda_S}} d\varphi = \frac{S}{16\pi^2\sqrt{\lambda_S}} dy\, d\varphi. \tag{2.17}$$

[2]We refer to this approximation as to the "Ultra-Relativistic Approximation (URA) in m_e".

2.2 Spin degrees of freedom

Since we are going to deal with the scattering of polarized particles, there will be additional essential variables, besides s and y, which are supposed to describe the *spin degrees of freedom* of the problem. Their description uses the language of spin density matrix (for details we refer e.g. to Appendix C of [8]).

For non-polarized particles, we use projection operators in trace calculations, i.e. summing and averaging over spin indices looks as

$$\overline{\sum_s} u^s(p)\bar{u}^s(p) = \frac{1}{2}\Lambda(p), \tag{2.18}$$

with

$$\Lambda(p) = -i\hat{p} + m. \tag{2.19}$$

For polarized particles, we use the spin density matrix instead

$$\sum_s u^s(p)\bar{u}^s(p) = \frac{1}{2}(1 + i\gamma_5\hat{\xi})\Lambda(p), \tag{2.20}$$

where ξ is the *polarization* four-vector [3].

In the particle rest frame, $\vec{p} = 0$, it is:

$$\xi = (P\vec{n}, 0), \tag{2.22}$$

where $\vec{n}$ is a unit vector in the direction of spin quantization, and P is the *polarization*, defining the degree of spin orientation along the direction $\vec{n}$. For instance, $P = 1$ means that the probability of a particle to have its spin projection *along* the direction $\vec{n}$ is equal to 1 (right handed longitudinal polarization, if vector $\vec{n}$ is chosen along particle momentum $\vec{p}$). From (2.22) in the particle rest frame, we have

$$\begin{aligned} \xi p &= 0, \\ \xi^2 &= P^2. \end{aligned} \tag{2.23}$$

Due to Lorenz invariance, the properties (2.23) are fulfilled in *any* Lorenz frame.

The initial electron with the four-momentum p_1 is at rest in the laboratory frame. Using then the direction of incoming muon as the direction of spin quantization, i.e. $\vec{n} = \vec{k_1}/|\vec{k_1}|$, we get the four-vector of the electron polarization from (2.22)

$$\xi_e = P_e \left(\frac{\vec{k_1}}{|\vec{k_1}|}, 0 \right). \tag{2.24}$$

The four-vector ξ_μ may be obtained from the expression similar to (2.24) in the muon rest frame

$$\xi_\mu = P_\mu \left(\frac{\vec{k_1}}{|\vec{k_1}|}, 0 \right) \tag{2.25}$$

[3] A naive use of longitudinal polarizations from the early beginning of calculations, i.e. use of the spin density matrices in the form

$$\sum_s u^s(p)\bar{u}^s(p) = \frac{1}{2}(1 + \lambda\gamma_5)\Lambda(p), \tag{2.21}$$

does not properly reproduce the finite muon mass terms in the $P_e P_\mu$ part of the cross-section.

by Lorenz boost to the electron rest frame along the beam axis:

$$\xi_\mu = P_\mu \frac{k_1^0}{m_\mu} \left(\frac{\vec{k}_1}{|\vec{k}_1|}, \frac{|\vec{k}_1|}{k_1^0} \right) . \qquad (2.26)$$

We will consider the Born cross-section with arbitrary orientations of electron and muon spins. We choose the laboratory frame with z-axis oriented along the incoming muon 3-momentum $\vec{k}_1$ and with the plane (x,z) coinciding with the reaction plane. Another plane is spanned by the vectors $\vec{k_1}$ and the projection $(\vec{\xi_e})_{xy}$ of vector $\vec{\xi_e}$ to a plane perpendicular to z-axis. In this frame, the relevant 4-vectors are written as follows:

$$\begin{aligned} k_1 &= \left(0,\ 0,\ |\vec{k}_1|, k_1^0\right) , \\ k_2 &= \left(|\vec{k}_2|\sin\theta_\mu,\ 0,\ |\vec{k}_2|\cos\theta_\mu, k_2^0\right) , \\ p_1 &= (0,\ 0,\ 0,\ m_e) , \\ p_2 &= \left(|\vec{p}_2|\sin\theta_e,\ 0,\ |\vec{p}_2|\cos\theta_e, p_2^0\right) . \end{aligned} \qquad (2.27)$$

For the spin vector ξ_e arbitrarily oriented in 3D-space, we have in the choosen laboratory frame instead of (2.24) the following generalization

$$\xi_e = P_e \left(\sin\vartheta_e \cos\varphi_e,\ \ \sin\vartheta_e \sin\varphi_e,\ \ \cos\vartheta_e,\ 0\right) . \qquad (2.28)$$

We may identify the angle φ_e in (2.27) with φ of the phase space parametrization in (2.17) and therefore φ becomes an essential degree of freedom in presence of a transverse polarization.

For arbitrarily oriented ξ_μ , we have instead of (2.25) in the corresponding rest frame

$$\xi_\mu = P_\mu \left(\sin\vartheta_\mu \cos\varphi_\mu,\ \ \sin\vartheta_\mu \sin\varphi_\mu,\ \ \cos\vartheta_\mu,\ 0\right) . \qquad (2.29)$$

Now we boost ξ_μ from the muon rest frame to the laboratory frame

$$\xi_\mu \;=\; P_\mu \left(\sin\vartheta_\mu \cos\varphi_\mu,\ \ \sin\vartheta_\mu \sin\varphi_\mu,\ \frac{k_1^0}{m_\mu}\cos\vartheta_\mu,\ \frac{|\vec{k}_1|}{m_\mu}\cos\vartheta_\mu \right) . \qquad (2.30)$$

Using the explicit representations (2.27), (2.28) and (2.30), we can easily derive all scalar products involving polarization vectors.

The doubly differential (in y and φ [4]) cross-section exact in both masses reads:

$$\begin{aligned} \frac{d\sigma^{\text{BORN}}}{dy d\varphi} \;=\; & \frac{2\alpha^2 S}{\lambda_S} \Biggl\{ \frac{1}{y^2} - \frac{s}{yS} + \frac{1}{2} + P_e P_\mu \Biggl[\left(-\frac{1}{y} + \frac{1}{Y} - \frac{1}{2} \right) \cos\vartheta_e \cos\vartheta_\mu \\ & + \frac{m_\mu |\vec{p}_2|}{y\sqrt{\lambda_S}} \left(1 + \frac{2m_e^2}{S} \right) \sin\theta_e \cos\vartheta_e \sin\vartheta_\mu \cos\varphi_\mu \\ & - \frac{m_e |\vec{k}_2|}{y\sqrt{\lambda_S}} \left(1 + \frac{2m_\mu^2}{S} \right) \sin\theta_\mu \cos\vartheta_\mu \sin\vartheta_e \cos\varphi_e \\ & - 2\frac{m_e m_\mu}{y^2 S} \sin\vartheta_e \sin\vartheta_\mu \Biggl(\frac{|\vec{k}_2||\vec{p}_2|}{S} \sin\theta_e \sin\theta_\mu \cos\varphi_e \cos\varphi_\mu \\ & \qquad + y\cos\delta_{(\varphi_e - \varphi_\mu)} \Biggr) \Biggr] \Biggr\} . \end{aligned} \qquad (2.31)$$

[4] For φ two choices are possible: 1) $\varphi = \varphi_e$, then $\varphi_\mu = \varphi_e - \delta_{(\varphi_e - \varphi_\mu)}$ or 2) $\varphi = \varphi_\mu$, then $\varphi_e = \varphi_\mu + \delta_{(\varphi_e - \varphi_\mu)}$.

The expressions for $\sin\theta_e$ and $\sin\theta_\mu$ exact in m_e are

$$\sin\theta_e = \frac{2m_e\sqrt{S\hat{y}}}{\sqrt{\lambda_e^0}}, \tag{2.32}$$

$$\sin\theta_\mu = \frac{2m_e\sqrt{S\hat{y}}}{\sqrt{\lambda_l}}, \tag{2.33}$$

where

$$\begin{aligned} \lambda_l &= S^2(1-y)^2 - 4m_e^2m_\mu^2, \\ \lambda_e^0 &= S^2y^2 + 4m_e^2Sy, \qquad (2.34) \\ \hat{y} &= y\left(1-\frac{y}{Y}\right) \qquad (2.35) \end{aligned}$$

and

$$Y = \frac{\lambda_S}{sS} \approx \left(1+\frac{m_\mu^2}{S}\right)^{-1} \tag{2.36}$$

is the kinematical maximum of y-variation.

The substitution of these variables into (2.31) exhibits an interesting property of the general Born cross-section which becomes

$$\begin{aligned} \frac{d\sigma^{\rm BORN}}{dyd\varphi} &= \frac{2\alpha^2S}{\lambda_S}\Biggl\{\frac{1}{y^2}-\frac{s}{yS}+\frac{1}{2}+P_eP_\mu\Biggl[\left(-\frac{1}{y}+\frac{1}{Y}-\frac{1}{2}\right)\cos\vartheta_e\cos\vartheta_\mu \\ &+\frac{m_\mu\sqrt{S\hat{y}}}{y\sqrt{\lambda_S}}\left(1+\frac{2m_e^2}{S}\right)\cos\vartheta_e\sin\vartheta_\mu\cos\varphi_\mu \\ &-\frac{m_e\sqrt{S\hat{y}}}{y\sqrt{\lambda_S}}\left(1+\frac{2m_\mu^2}{S}\right)\cos\vartheta_\mu\sin\vartheta_e\cos\varphi_e \\ &-2\frac{m_em_\mu}{yS}\sin\vartheta_e\sin\vartheta_\mu\left(\left(1-\frac{y}{Y}\right)\cos\varphi_e\cos\varphi_\mu+\cos\delta_{(\varphi_e-\varphi_\mu)}\right)\Biggr]\Biggr\}. \end{aligned} \tag{2.37}$$

From the last presentation it is clearly seen that while terms related to the transverse electron polarization are small since they are suppressed by the electron mass (third and fourth lines), the term induced by the transverse muon beam polarization (second line) is **not small**, since it appears to be proportional to the **muon mass**.

By trivial algebra one may show that the expression (2.31) reduces to a very short form in two particular cases. In case of tranverse electron and longitudinal muon polarizations in the URA in the electron mass we obtain

$$\frac{d\sigma^{\rm BORN}}{dyd\varphi} = \frac{2\alpha^2}{S}\left[\frac{1}{y^2}-\frac{1}{yY}+\frac{1}{2}+P_eP_\mu\cos\varphi\sin\theta_\mu\frac{1-y}{y}\left(\frac{1}{2}-\frac{1}{Y}\right)\right]. \tag{2.38}$$

The corresponding expression for the case of longitudinal polarization of both particles, but exact in both masses reads:

$$\frac{d\sigma^{\rm BORN}}{dy} = \frac{4\pi\alpha^2S}{\lambda_S}\left[\frac{1}{y^2}-\frac{s}{yS}+\frac{1}{2}+P_eP_\mu\left(-\frac{1}{y}+\frac{1}{Y}-\frac{1}{2}\right)\right]. \tag{2.39}$$

Having in mind that $r_e = \alpha/m_e$ and $S = 2m_e E_\mu$, we immediately identify (2.39) with the corresponding expressions from Section 2.2 of ref. [2] if one neglects terms of $\mathcal{O}(m_e^2)$ here. In the URA in m_e, equation (2.39) may be rewritten in a form, which is explicitly positive definite:

$$\frac{d\sigma^{\rm BORN}}{dy} = \frac{4\pi\alpha^2}{S}\left[\frac{(Y-y)}{y^2 Y}(1-yP_eP_\mu)+\frac{1}{2}(1-P_eP_\mu)\right]. \tag{2.40}$$

3 Complete $\mathcal{O}(\alpha)$ Radiative Corrections

3.1 Kinematics of $\mu e \to \mu e\gamma$

The reaction

$$\mu(k_1)+e(p_1)\to\mu(k_2)+e(p_2) \tag{3.1}$$

is accompanied by the bremsstrahlung of non-observed photon(s)

$$\mu(k_1)+e(p_1)\to\mu(k_2)+e(p_2)+(n)\gamma(p). \tag{3.2}$$

First of all we will study the kinematics of one-photon bremsstrahlung. We want y_μ to be the last integration variable out of a set of four variables (besides S). There is some freedom in doing this.

We will use the definitions:

$$Q_\mu^2=(k_1-k_2)^2, \qquad y_\mu=\frac{p_1(k_1-k_2)}{p_1k_1}, \tag{3.3}$$

and

$$Q_e^2=(p_2-p_1)^2, \qquad y_e=\frac{p_1(p_2-p_1)}{p_1k_1}. \tag{3.4}$$

The other invariants are:

$$\begin{aligned} z_1 &= -2pk_1, & z_2 &= -2pk_2,\\ V_1 &= -2pp_1, & V_2 &= -2pp_2. \end{aligned} \tag{3.5}$$

Of course, only four of the invariants are independent.

Using 4-momentum conservation, it is easy to derive the following relations among them:

$$\begin{aligned} z_1+V_1 &= z_2+V_2,\\ V_1 &= Sy_\mu-Q_e^2,\\ V_2 &= Sy_\mu-Q_\mu^2,\\ Q_e^2 &= Sy_e. \end{aligned} \tag{3.6}$$

We will use the following set of independent variables

$$S, \quad Q_\mu^2, \quad y_\mu, \quad Q_e^2, \quad z_{2(1)}. \tag{3.7}$$

The last of eqs. (3.6) deserves a comment. Our choice of the 4-momentum p_1 in definitions (3.3) and (3.4) introduces the asymmetry between y_μ and y_e. This is reason why Q_μ^2 and y_μ may be chosen as independent variables, while there exists a relation between Q_e^2 and y_e.

For the moduli of particle momenta and the energies, in the electron rest system $\vec{p}_1 = 0$, we obtain

$$
\begin{aligned}
|\vec{p}| &= \frac{\sqrt{\lambda_p}}{2m_e}, & p^0 &= \frac{S(y_\mu - y_e)}{2m_e}, \\
|\vec{k}_1| &= \frac{\sqrt{\lambda_S}}{2m_e}, & k_1^0 &= \frac{S}{2m_e}, \\
|\vec{k}_2| &= \frac{\sqrt{\lambda_l}}{2m_e}, & k_2^0 &= \frac{S_1}{2m_e}, \\
|\vec{Q}_l| &= \frac{\sqrt{\lambda_\mu}}{2m_e}, & Q_\mu^0 &= \frac{S y_\mu}{2m_e}, \\
|\vec{p}_2| &= |\vec{Q}_e| = \frac{\sqrt{\lambda_e}}{2m_e}, & p_2^0 &= m_e + \frac{S y_e}{2m_e}. \\
\vec{k}_1 \cdot \vec{k}_2 &= \frac{S S_1}{4m_e^2} - \frac{Q_\mu^2}{2} - m_\mu^2, & \vec{k}_1 \cdot \vec{p}_2 &= \frac{S(Q_e^2 + 2m_e^2)}{4m_e^2} - \frac{S - Q_e^2 - z_2}{2},
\end{aligned}
\tag{3.8}
$$

with

$$S_1 = S(1 - y_\mu). \tag{3.9}$$

The corresponding equations for the Born kinematics ($|\vec{p}| = 0$) can be easily derived from (3.8) setting

$$V_1 = V_2 = z_1 = z_2 = 0. \tag{3.10}$$

In this limit

$$Q_e^2 = Q_\mu^2 = Q^2 \qquad \text{and} \qquad y_e - y_\mu = y. \tag{3.11}$$

Many relations and useful notations can now be taken from [9]. As usual, we introduce the relevant kinematical λ-functions:

$$
\begin{aligned}
\lambda_p &\equiv \lambda[-(p_1 + p)^2, -p_1^2, -p^2] &&= S^2 (y_\mu - y_e)^2, \\
\lambda_S &\equiv \lambda[-(p_1 + k_1)^2, -p_1^2, -k_1^2] &&= S^2 - 4m_\mu^2 m_e^2, \\
\lambda_l &\equiv \lambda[-(p_1 + k_2)^2, -p_1^2, -k_2^2] &&= S_1^2 - 4m_\mu^2 m_e^2, \\
\lambda_\mu &\equiv \lambda[-(p_1 + Q_\mu)^2, -p_1^2, -Q_\mu^2] &&= S^2 {y_\mu}^2 + 4m_e^2 Q_\mu^2, \\
\lambda_e &\equiv \lambda[-(p_1 + p_2)^2, -p_1^2, -p_2^2] &&= S^2 {y_e}^2 + 4m_e^2 Q_e^2,
\end{aligned}
\tag{3.12}
$$

where

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \tag{3.13}$$

3.2 Kinematic boundaries

The boundary conditions may be taken from [9]. The first one, (B.3) of [9], remains unchanged

$$m_e^2 Q_\mu^4 + S^2 y_\mu Q_\mu^2 + m_\mu^2 S^2 {y_\mu}^2 - \lambda_S Q_\mu^2 = 0, \tag{3.14}$$

while the second condition, (B.4) of [9], changes, since, contrary to [9], y_e is not an independent variable here. This is due to the fact that we are dealing now with elastic scattering rather than with the deep inelastic scattering in [9]. Using the last of eqs. (3.6), eq. (B.4) of [9] takes the form:

$$S^2 {y_\mu}^2 Q_e^2 + Q_e^4 Q_\mu^2 - m_e^2 (Q_\mu^2 - Q_e^2)^2 - S y_\mu Q_e^2 (Q_\mu^2 + Q_e^2) = 0. \tag{3.15}$$

The physical region $\mathcal{E}_\mu = (Q_\mu^2, y_\mu)$ is given by two inequalities (see [9], subsection B.2.1), which are derived from (3.14)

$$0 \leq Q_\mu^2 \leq \frac{\lambda_S}{S + m_\mu^2 + m_e^2} \equiv \bar{Q}_\mu^2. \tag{3.16}$$

where

$$y_\mu^{\min}(Q_\mu^2) = \frac{Q_\mu^2}{S} \leq y_\mu \leq y_\mu^{\max}(Q_\mu^2), \tag{3.17}$$

$$y_\mu^{\max}(Q_\mu^2) = \frac{1}{2m_\mu^2} \left[\frac{1}{S} \sqrt{\lambda_S \lambda_m} - Q_\mu^2 \right], \tag{3.18}$$

and

$$\lambda_m = Q_\mu^2 (Q_\mu^2 + 4m_\mu^2). \tag{3.19}$$

The solution of eq.(3.15) is

$$(Q_e^2)^{\max,\min} = \frac{S y_\mu (S y_\mu - Q_\mu^2) + 2 m_e^2 Q_\mu^2 \pm (S y_\mu - Q_\mu^2)\sqrt{\lambda_\mu}}{2(S y_\mu - Q_\mu^2 + m_e^2)}. \tag{3.20}$$

3.3 Another set of independent variables

Besides of (3.7) we use

$$S, \quad y_\mu, \quad V_2, \quad V_1, \quad z_{2(1)}, \tag{3.21}$$

To write down the limits in these invariants, we reorder first the physical region $\mathcal{E}_\mu = (Q_\mu^2, y_\mu) \to (y_\mu, Q_\mu^2)$. Trivial manipulations with (3.16)–(3.18) lead to

$$0 \leq y_\mu \leq y_\mu^{\max}, \tag{3.22}$$

$$(Q_\mu^2)^{\min} \leq Q_\mu^2 \leq \min\{(Q_\mu^2)^{\max}, S y_\mu\}. \tag{3.23}$$

Here

$$(Q_\mu^2)^{\max,\min} = \frac{\lambda_S - S^2 y_\mu \pm \sqrt{\lambda_S \lambda_l}}{2m_e^2}. \tag{3.24}$$

Solving the equation

$$(Q_\mu^2)^{\max} = S y_\mu, \tag{3.25}$$

we find a maximal value $y_\mu^{\max}$

$$y_\mu^{\max} \equiv \frac{\lambda_S}{S(S + m_\mu^2 + m_e^2)}, \tag{3.26}$$

where the two upper limit branches of (3.23) meet each other.

From (3.23) and the definitions of V_i, we easily derive the limits of V_2 as function of y_μ

$$0 \le V_2 \le \frac{Sy_\mu(S+2m_e^2) - \lambda_S + \sqrt{\lambda_S \lambda_l}}{2m_e^2}. \tag{3.27}$$

The second solution

$$V_2^{\min} = \frac{Sy_\mu(S+2m_e^2) - \lambda_S - \sqrt{\lambda_S \lambda_l}}{2m_e^2}, \tag{3.28}$$

is unphysical. It is negative in the physical region (3.22) of y_μ. Finally, from (3.20) we derive the limits of V_1 as functions of y_μ and V_2

$$V_1^{\min} \le V_1 \le V_1^{\max}, \tag{3.29}$$

where

$$V_1^{\max,\min} = V_2 \frac{Sy_\mu + 2m_e^2 \pm \sqrt{\lambda_\mu}}{2(V_2+m_e^2)} \, . \tag{3.30}$$

Examining (3.27), one may see that the invariant V_2 is positive only in the interval

$$0 \le y_\mu \le y_\mu^{\max}. \tag{3.31}$$

To complete the study of kinematics of the reaction (3.2), we have to give the limits of variation of the variable $z_{1(2)}$. We may take all the relevant formulae from [9] and simply list them for completenes:

$$z_{1(2)}^{\max,\min}(y_\mu, Q_\mu^2, y_e) = \frac{B_{1(2)} \pm \sqrt{D_z}}{A_{1(2)}} = \frac{C_{1(2)}}{B_{1(2)} \mp \sqrt{D_z}}, \tag{3.32}$$

and the Gram determinant

$$\begin{aligned}
R_z &= -A_1 z_1^2 + 2B_1 z_1 - C_1 \equiv -A_2 z_2^2 + 2B_2 z_2 - C_2, && (3.33)\\
A_2 &= \lambda_\mu \equiv A_1, && (3.34)\\
B_2 &= \Big\{2m_e^2 Q_\mu^2 (Q_\mu^2 - Q_e^2) + S^2(1-y_\mu)(y_\mu Q_e^2 - y_e Q_\mu^2) + S^2(1) Q_\mu^2 (y_\mu - y_e)\Big\} \\
&\equiv \; - \; B_1 \Big\{(1) \leftrightarrow -(1-y_\mu)\Big\}, && (3.35)\\
C_2 &= \Big\{S(1-y_\mu)Q_e^2 - SQ_\mu^2\,[(1)-y_e]\Big]^2 + 4m_\mu^2 \Big[S^2(y_\mu - y_e)(y_\mu Q_e^2 - y_e Q_\mu^2) - m_e^2(Q_e^2 - Q_\mu^2)^2\Big\} \\
&\equiv C_1 \Big\{(1) \leftrightarrow -(1-y_\mu)\Big\}, && (3.36)\\
D_z &= B_{1(2)}^2 - A_{1(2)} C_{1(2)}. && (3.37)
\end{aligned}$$

Here we now understand that $Q_e^2 \equiv Sy_e$. This makes no simplifications, therefore we did not substitute it in order not to destroy a nice symmetry of these equations.

The inequalities (3.31), (3.27), (3.30) and (3.32) are the kinematical limits in the sequence: $y_\mu, \ V_2, \ V_1, \ z_2$.

3.4 Phase spaces

In order to have a convincing proof that all the boundaries are correctly derived, we always write a supporting `FORTRAN` program to numerically check that the phase space volume in any sequence of variables is exactly the same. We calculated as many phase space integrals analytically as is possible.

For example, we realized the cross check of our sequence (3.21). Three integrations may be easily performed analytically, yielding:

$$\Gamma = \frac{\pi^2 S}{4\sqrt{\lambda_S}} \int_0^{y_\mu^{\max}} dy_\mu \left(V_2^{\max} - m_e^2 \ln \frac{V_2^{\max} + m_e^2}{m_e^2} \right). \tag{3.38}$$

The numerical check returned for this integral exactly the same value as for the basic sequence, (C.1) of [9].

We will always integrate over z_2 first. Then, only three variables V_1, V_2, y_μ remain. The physical regions $\mathcal{E} = (y_\mu, V_2)$ and $\mathcal{I} = (V_2, V_1)$, as derived from (3.27) and (3.30), are shown in Figs. 4 and 5.

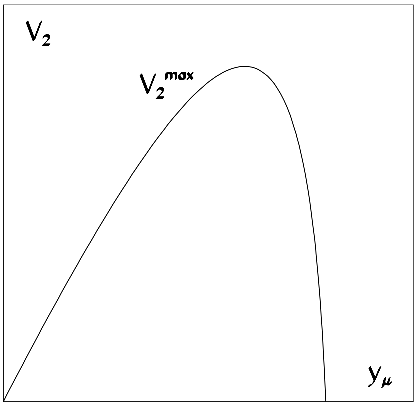

Figure 4: (V_2, y_μ)-*plot*

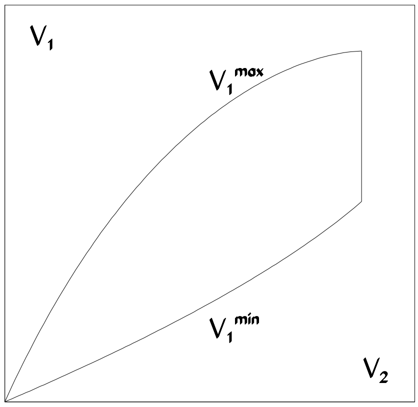

Figure 5: (V_1, V_2)-*plot*

3.5 Bremsstrahlung cross section

For the normalized bremsstrahlung cross-section, we have

$$d\sigma^{\text{BREM}} = \frac{1}{2\sqrt{\lambda_S}} \left| M^{\text{BREM}} \right|^2 d\Gamma_3, \tag{3.39}$$

with the $2 \to 3$ phase space

$$d\Gamma_3 = (2\pi)^4 \frac{d^3\vec{k_2}}{(2\pi)^3 2k_2^0} \frac{d^3\vec{p_2}}{(2\pi)^3 2p_2^0} \frac{d^3\vec{p}}{(2\pi)^3 2p^0} \delta(k_1 + p_1 - k_2 - p_2 - p). \tag{3.40}$$

In terms of the variables (3.21) the differential phase space looks as follows

$$d\Gamma_3 = \frac{S}{2^7\pi^4\sqrt{\lambda_S}} dy_\mu dV_2\ dV_1 \frac{dz_1}{\sqrt{R_z}}. \tag{3.41}$$

We developped two branches to present the final result:

- calculation within the *Numerical Approach*, with the possibility to impose arbitrary experimental cuts;
- calculation within an *Analytic Approach* with limited possibility to apply cuts.

We note that the integrals over z_1 were calculated analytically with arbitrary limits $\hat{z}_1^{\min,\max}$ (see, Appendix (A)), while integrals over $V_{2(1)}$ were computed purely numerically. This is the essence of our *numerical approach*, the `muenum` branch of μe*la*.

We also performed the complete $\mathcal{O}(\alpha)$ RC calculations within an *Analytic Approach*, the `mueana` branch of μe*la*. In this description, we will present only a sketch of the calculations.

Within the analytic approach we have integrated the bremsstrahlung contribution over the variables z_1 and V_1 over the full photonic phase space without imposing experimental cuts. The aim of this is twofold: first, we performed an independent calculation of the bremsstrahlung cross section and its integration which was used to cross check the formulae of the numerical approach and their coding in the `FORTRAN` program μe*la*. Second, these analytic formulae are rather elegant, although lengthy, in the SIngle Ultra Relativistic Approximation (SIURA) in the electron mass. Due to analytic integration, they run at the computer incredibly fast, several orders of magnitude faster than the numerical integration within the numerical approach. We note that this relative simplicity may be reached only if we integrate over the full phase space of photons. From formulae in the SIURA it is easy to derive the formulae in the DOubly Ultra Relativistic Approximation, DOURA. It is important to stress, that in the latter case satisfactory precision of the calculations is not ensured.

The derivation of analytic results is quite straightforward. We consequently integrate in the sequence $dV_2dV_1dz_1$. In reality, it is convenient to calculate and substitute the twofold integrals: dV_1dz_1. These are presented in the Table in Appendix (D.1). After substituting the dV_1dz_1 integrals, we arrive at another Table of integrals over V_2 which are listed in Appendix (D.2). They are much more complicated than the first, two fold integrals. The latter integrals were calculated in limits $[0, \hat{V}_2^{\max}]$ with an arbitrary upper limit. This allows us to impose V_2-dependent experimental cuts even within the analytic approach. An example is the muon angular cut.

A comment on DOURA is in due here. Since here we neglect m_μ^2, a problem might arise from the fact that $Q_\mu^{2\min}$ is proportional to the muon mass. Apparently, m_μ^2 can't be neglected in $Q_\mu^{2\min}$, since it enters in the denominator of the photon propagator. However, we assume that a cut is imposed on V_2, which prevents $Q_\mu^{2\min}$ from reaching its kinematical limit. In other words we assume that

$$Q_\mu^2 \geq (Q_\mu^2)^{cut} \equiv (Q_\mu^2)^c \gg m_\mu^2. \tag{3.42}$$

3.5.1 Final expression for the normalized bremsstrahlung cross section

Defining the *z-integrated* contributions,

$$\left[\left|M^{\rm BREM}\right|^2\right]_z = \int_{\hat{z}_1^{\min}}^{\hat{z}_1^{\max}} \frac{dz_1}{\pi\sqrt{R_z}} \left|M^{\rm BREM}\right|^2 \tag{3.43}$$

and neglecting m_e wherever possible, we finally obtain

$$\frac{d\sigma^{\rm BREM}}{dy_\mu} = \frac{\alpha^3}{S}\int_0^{\hat{V}_2^{\max}} dV_2 \int_{V_1^{\min}}^{\hat{V}_1^{\max}} dV_1 \left[\left|M^{\rm BREM}\right|\right]^2_z. \tag{3.44}$$

In (3.44), the limits $\hat{V}_{2(1)}^{\max}$, $\hat{z}_1^{\min,\max}$ are functions of experimental cuts. In this way, the cuts are implemented within our numerical approach.

There are three gauge-invariant contributions to the bremsstrahlung differential cross section: *muonic*, *electronic* and *μe interference*, each of them is represented as the sum of an InfraRed divergent contribution 'IR' and a finite (Regular) part 'R':

$$\left[\left|M^{\rm BREM}\right|^2_{\mu\mu}\right]_z = \mathsf{Q}_\mu^2\left(\mathcal{B}F^{\rm IR}_{\mu\mu} + S^{\rm R}_{\mu\mu}\right), \tag{3.45}$$

$$\left[\left|M^{\rm BREM}\right|^2_{ee}\right]_z = \mathsf{Q}_e^2\left(\mathcal{B}F^{\rm IR}_{ee} + S^{\rm R}_{ee}\right), \tag{3.46}$$

$$\left[\left|M^{\rm BREM}\right|^2_{\mu e}\right]_z = \mathsf{Q}_\mu\mathsf{Q}_e\left(\mathcal{B}F^{\rm IR}_{\mu e} + S^{\rm R}_{\mu e}\right), \tag{3.47}$$

where $\mathcal{B}$ is the Born factor

$$\mathcal{B} = 4\frac{Y-y}{y^2Y}\left(1-yP_eP_\mu\right) + 2\left(1-P_eP_\mu\right) \equiv \frac{1}{16\pi^2\alpha^2}\left|M^{\rm BORN}\right|^2. \tag{3.48}$$

The $F_i^{\rm IR}$ are infrared factors:

$$F^{\rm IR}_{\mu\mu} = \left(Sy_\mu + 2m_\mu^2\right)\left[\frac{1}{z_1z_2}\right]_z - m_\mu^2\left[\frac{1}{z_1^2}\right]_z - m_\mu^2\left[\frac{1}{z_2^2}\right]_z, \tag{3.49}$$

$$F^{\rm IR}_{ee} = \left(\frac{Sy_\mu}{V_1V_2} - \frac{m_e^2}{V_1^2} - \frac{m_e^2}{V_2^2}\right)\left[1\right]_z, \tag{3.50}$$

$$F^{\rm IR}_{\mu e} = -\frac{S}{V_1}\left[\frac{1}{z_1}\right]_z - \frac{S}{V_2}\left[\frac{1}{z_2}\right]_z + \frac{S_1}{V_1}\left[\frac{1}{z_2}\right]_z + \frac{S_1}{V_2}\left[\frac{1}{z_1}\right]_z. \tag{3.51}$$

In (3.49)-(3.51) and in the cumbersome functions $S_i^{\rm R}$ given in Appendix (A), the z-integration (3.43) with arbitrary limits is assumed to be done.

Tables of z-integrals with cuts and equations for limits $\hat{V}_{2(1)}^{\max}$, $\hat{z}_1^{\min,\max}$ in terms of experimental cuts are presented below in Appendix B: (B.93), (B.85), (B.107)-(B.108).

3.5.2 Treatment of the infrared divergent part

The three terms with $F^{\rm IR}$ in (3.45)-(3.47) cannot be simply integrated in (3.44) because of the infrared divergency at $V_2 = 0$. It is treated by dimensional regularization.

Substituting all terms with $F^{\rm IR}$ into (3.41), we define the **IR Part** of the bremsstrahlung cross-section.

$$\begin{aligned} d\sigma^{\rm IR} &\equiv \frac{2^7\pi^3\alpha^3}{\sqrt{\lambda_S}}\mathcal{B}\left(\mathsf{Q}_\mu^2F^{\rm IR}_{\mu\mu} + \mathsf{Q}_\mu\mathsf{Q}_eF^{\rm IR}_{\mu e} + \mathsf{Q}_e^2F^{\rm IR}_{ee}\right)d\Gamma_3 \\ &= \frac{2^7\pi^3\alpha^3}{\sqrt{\lambda_S}}\mathcal{B}\left(\mathsf{Q}_\mu^2F^{\rm IR}_{\mu\mu} + \mathsf{Q}_\mu\mathsf{Q}_eF^{\rm IR}_{\mu e} + \mathsf{Q}_e^2F^{\rm IR}_{ee}\right)\left[\theta(\varepsilon-p^0)+\theta(p^0-\varepsilon)\right]d\Gamma_3 \\ &\equiv d\sigma^{\rm IR,soft} + d\sigma^{\rm IR,hard}. \end{aligned} \tag{3.52}$$

We will treat $d\sigma^{\rm IR}$ in the R-frame which is defined in Appendix C by the condition

$$\vec{p}_2 + \vec{p} = 0. \tag{3.53}$$

In this frame we find

$$V_2 = -2p.p_2 = -(p_2+p)^2 - m_e^2 = (p_2^0 + p^0)^2 - m_e^2. \tag{3.54}$$

For sufficiently small ε we have at the point of separation of *soft* and *hard* photons, $p^0 = \varepsilon$,

$$V_2 = \bar{V}_2 = 2m_e\varepsilon, \tag{3.55}$$

which can be choosen to be much smaller then any typical invariant of the process.

From a study of the R-system it may be understood that there is an unique correspondence between angular integrations in the R-system and invariant integrations, i.e. we may use invariant limits of z in (3.43) and of V_1 in (3.44) instead of angular limits in the R-frame in order to compute the *hard* part of $d\sigma^{\rm IR}$

$$d\sigma^{\rm IR,hard} = \frac{2^7\pi^3\alpha^3}{\sqrt{\lambda_S}}\mathcal{B}\left(\mathsf{Q}_\mu^2 F_{\mu\mu}^{\rm IR} + \mathsf{Q}_\mu\mathsf{Q}_e F_{\mu e}^{\rm IR} + \mathsf{Q}_e^2 F_{ee}^{\rm IR}\right) d\Gamma_3\theta(p^0-\varepsilon). \tag{3.56}$$

This is very convenient, since *hard* photons are in general affected by experimental cuts and the cuts are implemented in our approach by means of *limits* of a numerical integration over z_1 and V_1. Finally, we get from (3.52)

$$\begin{aligned}\frac{d\sigma^{\rm IR,hard}}{dy_\mu} &= \frac{\alpha^3}{S}\mathcal{B}\int_{\bar{V}_2}^{\hat{V}_2^{\max}} dV_2 \int_{V_1^{\min}}^{\hat{V}_1^{\max}} dV_1 \left(\mathsf{Q}_\mu^2\left[F_{\mu\mu}^{\rm IR}\right]_z + \mathsf{Q}_\mu\mathsf{Q}_e\left[F_{\mu e}^{\rm IR}\right]_z + \mathsf{Q}_e^2\left[F_{ee}^{\rm IR}\right]_z\right)\\ &= \frac{d\sigma^{\rm BORN}}{dy_\mu}\frac{\alpha}{\pi}\delta^{\rm IR,hard}\end{aligned} \tag{3.57}$$

with

$$\delta^{\rm IR,hard} = \int_{\bar{V}_2}^{\hat{V}_2^{\max}} dV_2 \int_{V_1^{\min}}^{\hat{V}_1^{\max}} dV_1 \left(\mathsf{Q}_\mu^2\left[F_{\mu\mu}^{\rm IR}\right]_z + \mathsf{Q}_\mu\mathsf{Q}_e\left[F_{\mu e}^{\rm IR}\right]_z + \mathsf{Q}_e^2\left[F_{ee}^{\rm IR}\right]_z\right). \tag{3.58}$$

Again, in (3.57)-(3.58) the z-integration (3.43) with experimental cuts is assumed to have been done.

3.5.3 A short form of the completely differential bremsstrahlung cross-section

The IR finite z-integrated contributions (A.71-A.73) are rather cumbersome indeed. For the unpolarized squared matrix elements of (3.45-3.47) with not yet separated infrared parts (3.49-3.51) a compact representation is known in the literature [10]. We present it here, in notations of our paper, even in a more compact form:

$$\begin{aligned}\left[\left|M_{\rm form}^{\rm BREM}\right|^2_{\mu\mu}\right]^{\rm unpol} &= \mathsf{Q}_\mu^2\left[\frac{\mathcal{F}^R}{Q_e^2 z_1 z_2} + \frac{2}{Q_e^4}\left(m_\mu^2\Delta_{\mu\mu}^\mu + m_e^2\Delta_{\mu\mu}^e\right)\right],\\ \left[\left|M_{\rm form}^{\rm BREM}\right|^2_{ee}\right]^{\rm unpol} &= \mathsf{Q}_e^2\left[\frac{\mathcal{F}^R}{Q_\mu^2 V_1 V_2} + \frac{2}{Q_\mu^4}\left(m_e^2\Delta_{ee}^e + m_\mu^2\Delta_{ee}^\mu\right)\right],\\ \left[\left|M_{\rm form}^{\rm BREM}\right|^2_{\mu e}\right]^{\rm unpol} &= \mathsf{Q}_\mu\mathsf{Q}_e\left[-\frac{\mathcal{F}^R}{Q_e^2 Q_\mu^2}\left(\frac{S}{V_1 z_1} + \frac{S'}{V_2 z_2} + \frac{U}{V_1 z_2} + \frac{U'}{z_1 V_2}\right) - \frac{2m_\mu^2}{Q_e^2 Q_\mu^2}\Delta_{\mu e}\right],\end{aligned} \tag{3.59}$$

where

$$
\begin{aligned}
\mathcal{F}^{R} &= S^2 + S'^2 + U^2 + U'^2 \\
\Delta^{\mu}_{\mu\mu} &= -\frac{S'^2+U^2}{z_1^2} - \frac{S^2+U'^2}{z_2^2} - 2\frac{SU+S'U'}{z_1 z_2} \\
&\quad -2Q_e^2\left(\frac{1}{z_1}-\frac{1}{z_2}\right) + 2m_\mu^2 Q_e^2\left(\frac{1}{z_1}-\frac{1}{z_2}\right)^2, \\
\Delta^{e}_{\mu\mu} &= -\frac{S+U'}{z_1} + \frac{S'+U}{z_2} + 4\frac{m_\mu^2 Q_e^2}{z_1 z_2}, \\
\Delta^{e}_{ee} &= -\frac{S'^2+U'^2}{V_1^2} - \frac{S^2+U^2}{V_2^2} + 2m_\mu^2 Q_\mu^2\left(\frac{1}{V_1^2}+\frac{1}{V_2^2}\right), \\
\Delta^{\mu}_{ee} &= -\frac{(S+U)^2+(S'+U')^2}{V_1 V_2}, \\
\Delta_{\mu e} &= -\left(Q_e^2+Q_\mu^2\right)\left(\frac{S}{V_1 z_1}+\frac{S'}{V_2 z_2}+\frac{U}{V_1 z_2}+\frac{U'}{z_1 V_2}\right) \\
&\quad +2\frac{S'-U}{z_1} + 2\frac{U'-S}{z_2},
\end{aligned}
\tag{3.60}
$$

and

$$
\begin{aligned}
S' &= S - z_1 - V_1, \\
U &= -S + Q_e^2 + V_1, \\
U' &= -S + Q_e^2 + z_2.
\end{aligned}
\tag{3.61}
$$

The representation (3.60)-(3.61) is really very elegant, and it may be used in a Monte Carlo code. It can't be used, however, within the numerical approach of this paper for the following reasons:

1. It does not take into account polarizations (main reason).
2. It includes the IRD part which has to be separated out.
3. Our bremsstrahlung cross-section, as we emphazed already, is integrated once over the invariant $z_{1,2}$ within arbitrary limits (by making use of a table of indefinite integrals). To do so, we need the canonical representation

   $$S_i^R = \sum_{l,j} f_l(z_j) K_{il}(S, y_\mu, V_2, V_1). \quad i = ee, e\mu, \mu\mu, \quad j = 1,2, \quad l = 1-24, \tag{3.62}$$

   see Appendix (D). We gain at least one order of magnitude in CPU-time, since instead of a three-fold numerical integration over V_2, V_1 and $z_{1,2}$, we need only two-fold: (V_2, V_1). This gain of CPU-time makes our code very fast and user friendly. In order to arrive at this level, however, one has to substitute S', U, U', and the compact expressions, actually, become much more cumbersome (see Appendix A).
4. Finally, the canonical representation (3.62) seems to be the only road towards an analytically integrated differential cross-section, see the discussion at the beginning of this section.

3.6 The muon anomalous magnetic moment contribution

The contribution of the anomalous magnetic moment of the muon to the cross-section can be expressed as (here we follow decomposition of ref. [11]):

$$\frac{d\sigma^{\rm amm}}{dy} = \frac{\alpha^3}{S}\,F_\mu^{\rm amm}\left(-\frac{4m_\mu^2}{Sy^2}\right)\left(\frac{1}{y}-\frac{1}{Y}\right)(2-yP_eP_\mu), \tag{3.63}$$

where

$$F_\mu^{\rm amm} = -\frac{1}{\beta}\log\frac{\beta+1}{\beta-1}, \tag{3.64}$$

and

$$\beta = \sqrt{1+\frac{4m_\mu^2}{Q^2}}. \tag{3.65}$$

3.7 The running electromagnetic coupling

The correction due to the running QED coupling (vacuum polarization) can be implemented as

$$\frac{d\sigma_{\rm vp}}{dy} = \left[\left(\frac{\alpha(Q^2)}{\alpha}\right)^2 - 1\right]\frac{d\sigma^{\rm BORN}}{dy} = \delta_{\rm vp}\frac{d\sigma^{\rm BORN}}{dy}, \tag{3.66}$$

where

$$\alpha(Q^2) = \frac{\alpha}{1-\Delta\alpha(Q^2)}. \tag{3.67}$$

The correction $\Delta\alpha$ contains two parts:

$$\Delta\alpha(Q^2) = \Delta\alpha_l + \Delta\alpha_{udcsb}, \tag{3.68}$$

the first contribution is due to the charged leptons, the second one is due to the light quarks (u,d,c,s,b). All details about the calculation of $\delta\alpha_l$ are described in [12]. For the calculation of the $\Delta\alpha_{udcsb}$ we use the parametrization of ref. [13].

3.8 The net $\mathcal{O}(\alpha^3)$ cross-section

Now we have collected all the ingredients to construct the net QED cross-section up to order $\mathcal{O}(\alpha^3)$. It has the form

$$\begin{aligned}\frac{d\sigma^{\rm QED}}{dy_\mu} &= \frac{d\sigma^{\rm BORN}}{dy_\mu}(1+\delta_{\rm vp}) + \frac{d\sigma^{\rm amm}}{dy_\mu} \\ &\quad + \sum_{k=\mu\mu,ee,\mu e}\left(\frac{\alpha}{\pi}\delta_k^{\rm VR}\frac{d\sigma^{\rm BORN}}{dy_\mu} + \frac{d\sigma_k^{\rm BREM}}{dy_\mu}\right),\end{aligned} \tag{3.69}$$

where the corrections $\delta_k^{\rm VR}$ are given by (E.186), (E.196) and(E.215) and the last sum in (3.69) reads

$$\sum_k \frac{d\sigma_k^{\rm BREM}}{dy_\mu} = \frac{\alpha^3}{S}\int_{\bar V_2}^{\hat V_2^{max}} dV_2 \int_{V_1^{min}}^{\hat V_1^{max}} dV_1 \left(\mathtt{Q}_\mu^2 S_{\mu\mu}^{\rm R} + \mathtt{Q}_e^2 S_{ee}^{\rm R} + \mathtt{Q}_\mu\mathtt{Q}_e S_{\mu e}^{\rm R}\right), \tag{3.70}$$

with $S_k^{\rm R}$ given by (A.71), (A.72) and (A.73). The limits of integration in (3.70) are discussed in the Appendix B.

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A Finite contributions to the bremsstrahlung cross-section

Here we list three finite contributions, S_i^{R}, to the hard part of $d\sigma$ which enter the formulae (3.45-3.47):

$$
\begin{aligned}
S^{\mathrm{R}}_{\mu\mu} = P_eP_\mu \Bigg\{ & \left[\frac{1}{z_1^2}\right]_z 2R_\mu \Bigg[\frac{-2m_e^2 S}{Q_e^4}\left(V_1 y_\mu - 2V_2 y_\mu + \frac{V_2^2}{S}\right) \\
& \qquad + \frac{S}{Q_e^2}\left(V_1\left(2y_\mu - 3 + \frac{2}{y_\mu}\right) + V_2\right) - V_1 \Bigg] \\
& + \left[\frac{1}{z_2^2}\right]_z \frac{4R_\mu S V_1}{Q_e^2 y_\mu} \\
& + \left[\frac{1}{z_1 z_2}\right]_z \Bigg[\frac{2m_e^2}{Q_e^4}\Big(-SV_1y_\mu^2(1+2R_\mu)^2 + SV_2y_\mu^2(3+8R_\mu+4R_\mu^2) \\
& \qquad - V_2^2 y_\mu(3+4R_\mu) + \frac{V_2^3}{S}\Big) \\
& + \frac{S^2 y_{\mu_1}}{Q_e^4}\left[V_1\left(y_\mu + 2(2+y_\mu)R_\mu\right) - 2V_2\left(y_\mu + (2+y_\mu)R_\mu\right) + \frac{V_2^2}{S}\right] \\
& - \frac{S}{Q_e^2}\left[V_1\left(1 - 2y_\mu - 2\left(2y_\mu + 3 - \frac{4}{y_\mu}\right)R_\mu\right) - V_2(-3y_\mu + 2 - 2R_\mu)\right] \\
& - V_1(1+2R_\mu) + V_2(1-4R_\mu) \Bigg] \\
& + \left[\frac{1}{z_1}\right]_z \Bigg[\frac{4m_e^2}{Q_e^4}\left(V_1 y_\mu\left(1 + R_\mu + 2R_\mu^2\right) - V_2 y_\mu(2+R_\mu) + \frac{V_2^2}{S}\right) \\
& - \frac{S y_{\mu_1}}{Q_e^4}\left[V_1\left(1 + \frac{2(2+y_\mu)}{y_\mu}R_\mu\right) - V_2\right] \\
& - \frac{1}{Q_e^2}\left[\frac{2V_1}{y_\mu}\left(1 - \left(1 - \frac{2}{y_\mu}\right)R_\mu\right) + V_2\right] \\
& + 2\left(1 - \frac{1}{y_\mu} + \frac{1}{y_\mu}\left(1 - \frac{2}{y_\mu}\right)R_\mu\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{1}{z_2}\right]_z \left[\frac{2m_e^2}{Q_e^4}\left(-V_1 y_\mu (1+2R_\mu)^2 + 2V_2 y_\mu (1+2R_\mu) - \frac{V_2^2}{S}\right)\right. \\
&+\frac{S y_{\mu_1}}{Q_e^4}\left(V_1\left(1+\frac{2(2+y_\mu)}{y_\mu}R_\mu\right) - V_2\right) + \frac{1}{Q_e^2}\left(2S\left(1-\left(1-\frac{2}{y_\mu}\right)R_\mu\right) - V_2\right)\Bigg] \\
&+\left[1\right]_z \frac{4m_e^2}{S Q_e^4}(V_1 - V_2)\Bigg\} \\
&-\left[\frac{1}{z_1^2}\right]_z \frac{4R_\mu S V_1}{Q_e^2 y_\mu}\left(\frac{S y_{\mu_1}^2}{Q_e^2} + \frac{y_{\mu_1}}{y_\mu} - R_\mu\right) \\
&-\left[\frac{1}{z_2^2}\right]_z \frac{4R_\mu S V_1}{Q_e^2 y_\mu}\left(\frac{S}{Q_e^2} + \frac{y_{\mu_1}}{y_\mu} - R_\mu\right) \\
&+\left[\frac{1}{z_1 z_2}\right]_z \left[-\frac{4m_e^2}{Q_e^4}\left(S V_1 (y_\mu + 2R_\mu) - 2 S V_2 (y_\mu + R_\mu) + V_2^2\right)\right. \\
&+\frac{S^2}{Q_e^4}\left[V_1\left(4y_{\mu_1} + y_\mu^2 + 2\left(y_\mu + \frac{4y_{\mu_1}}{y_\mu}\right)R_\mu\right) - V_2\left(4y_{\mu_1} + 2y_\mu R_\mu\right) - \frac{V_2^2 y_\mu}{S}\right] \\
&+\frac{S}{Q_e^2}\left[4V_1\left(\frac{y_{\mu_1}}{y_\mu} - R_\mu\right)\left(1+\frac{2R_\mu}{y_\mu}\right) + V_2(y_\mu + 8R_\mu)\right] - 2V_2\Bigg] \\
&+\left[\frac{1}{z_1}\right]_z \left[\frac{2m_e^2}{Q_e^4}\left(V_1\left(1+\frac{4}{y_\mu}\right) - V_2\right) - \frac{S}{Q_e^4}\left(V_1\left(\frac{4y_{\mu_1}}{y_\mu} + y_\mu + 2R_\mu\right) + V_2 y_\mu\right)\right. \\
&-\frac{1}{Q_e^2}\left(\frac{2V_1}{y_\mu}\left(\frac{y_{\mu_1}(2+y_\mu)}{y_\mu} - R_\mu\right) + V_2\right) + 1 + \frac{2}{y_\mu} - \frac{4}{y_\mu^2} + \frac{2R_\mu}{y_\mu}\Bigg] \\
&+\left[\frac{1}{z_2}\right]_z \left[-\frac{2m_e^2}{Q_e^4}\left(V_1\left(1+\frac{4R_\mu}{y_\mu}\right) - V_2\right) + \frac{S}{Q_e^4}\left(V_1\left(\frac{4y_{\mu_1}}{y_\mu} + y_\mu + 2R_\mu\right) + V_2 y_\mu\right)\right. \\
&-\frac{1}{Q_e^2}\left(\frac{2V_1}{y_\mu}\left(3 - \frac{2}{y_\mu} + R_\mu\right) - V_2\right) + 1 - \frac{6}{y_\mu} + \frac{4}{y_\mu^2} - \frac{2R_\mu}{y_\mu}\Bigg] \\
&+\left[1\right]_z 2\left(-\frac{2m_e^2 V_1}{Q_e^4 S y_\mu} + \frac{1}{Q_e^2}\right), \qquad\qquad (A.71)
\end{aligned}
$$

$$
\begin{aligned}
S_{ee}^{\mathrm{R}} \;=\; & P_e P_\mu \Bigg\{\left[z_1^2\right]_z \frac{4m_e^2}{S V_1^2 Q_\mu^2} \\
&+\left[z_1\right]_z \left[\frac{2m_e^2}{S V_1^2}\left(1 - \frac{2S}{Q_\mu^2}(1 - y_\mu R_\mu)\right) - \frac{2}{V_2 Q_\mu^2} + \frac{2}{V_1 V_2}\right] \\
&+\left[1\right]_z \left[-\frac{2m_e^2}{V_1^2 V_2 Q_\mu^2}\left[1 - \frac{2}{y_\mu} + 2\left(\frac{S y_\mu}{Q_\mu^2} + 1\right)R_\mu\right] - \frac{V_1}{Q_\mu^2 V_2}(1 - 2R_\mu) + \frac{2}{Q_\mu^2 y_\mu}\right. \\
&+2\left(\frac{S y_\mu V_2}{V_1} - V_1\right)\frac{R_\mu}{Q_\mu^4} + \frac{V_2}{V_1 Q_\mu^2} - 2\left(\frac{S y_\mu}{V_1 Q_\mu^2} - \frac{1}{V_2}\right)\left(\frac{1}{y_\mu} - 2R_\mu\right)\Bigg]\Bigg\} \\
&+\left[z_1^2\right]_z 2\left(-\frac{2m_e^2}{V_1^2 Q_\mu^2} + \frac{1}{V_1 V_2}\right)\frac{1}{Q_\mu^2} \qquad\qquad (A.72) \\
&+\left[z_1\right]_z 2\left[\left(\frac{2m_e^2}{V_1^2 Q_\mu^2} - \frac{1}{V_1 V_2}\right)\left(\frac{2S}{Q_\mu^2} - 1\right) + \frac{1}{V_2 Q_\mu^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\Big[1\Big]_z\Bigg[\frac{4m_e^2V_2}{V_1^2Q_\mu^2y_\mu}\left(1-\frac{1}{y_\mu}-\frac{S}{Q_\mu^2}+R_\mu\right)\\
&+\frac{V_1}{V_2Q_\mu^2}\left(1-\frac{2R_\mu}{y_\mu}\right)-\frac{2}{y_\mu Q_\mu^2}(1-2R_\mu)-2\left(\frac{SV_2}{V_1}+\frac{V_1}{y_\mu}\right)\frac{R_\mu}{Q_\mu^4}\\
&+\frac{V_2}{V_1Q_\mu^2}\left(1-\frac{2}{y_\mu}+\frac{4}{y_\mu^2}-\frac{4R_\mu}{y_\mu}\right)-\frac{2}{V_1y_\mu}\left(1-\frac{2}{y_\mu}+2R_\mu\right)-\frac{2}{V_2y_\mu}(1-2R_\mu)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
S_{\mu e}^{\rm R} \;=\;& P_eP_\mu\Bigg\{\Big[1\Big]_z 2\left[\frac{1}{Sy_\mu}\left(2+\frac{V_1}{Q_e^2}+\frac{V_2}{Q_\mu^2}\right)-\frac{1}{V_1}-\frac{1}{V_2}\right]\\
&+\left[\frac{1}{z_1}\right]_z\Bigg[S\left(\frac{V_1y_{\mu_1}}{Q_e^2V_2}-\frac{V_2}{V_1Q_\mu^2}\right)\left(1-\frac{2}{y_\mu}+2R_\mu\right)\\
&+\frac{1}{Q_e^2}\frac{y_{\mu_1}}{y_\mu}\left[2V_1\left(\frac{y_{\mu_1}}{y_\mu}+R_\mu\right)+V_2\right]\\
&+\frac{V_1}{Q_\mu^2}\left[\frac{1}{y_\mu}-2\left(2+\frac{1}{y_\mu}\right)R_\mu\right]-\frac{2V_2}{Q_\mu^2y_\mu}\left(\frac{1}{y_\mu}-2y_{\mu_1}R_\mu\right)\\
&+\frac{V_2}{V_1y_\mu}+\frac{V_1y_{\mu_1}}{V_2y_\mu}(1-2R_\mu)+2\left(1-\frac{2}{y_\mu}-\left(1-\frac{3}{y_\mu}\right)R_\mu\right)\Bigg]\\
&+\left[\frac{1}{z_2}\right]_z\Bigg[-\frac{V_1}{Q_e^2}\left[\frac{S}{V_2}\left(1-\frac{2}{y_\mu}+2R_\mu\right)+\frac{2}{y_\mu}\left(\frac{1}{y_\mu}+R_\mu\right)\right]\\
&+\frac{V_2}{Q_e^2y_\mu}+\frac{V_1}{y_\mu}\left(\frac{y_{\mu_1}}{Q_\mu^2}+\frac{1}{V_2}\right)(1+2R_\mu)+\frac{V_2y_{\mu_1}}{V_1}\left[\frac{S}{Q_\mu^2}\left(1-\frac{2}{y_\mu}+2R_\mu\right)+\frac{1}{y_\mu}\right]\\
&+\frac{2V_2}{Q_\mu^2y_\mu}\left(\frac{y_{\mu_1}^2}{y_\mu}-2R_\mu\right)+2\left(1-\frac{2}{y_\mu}+\left(2-\frac{3}{y_\mu}\right)R_\mu\right)\Bigg]\Bigg\}\\
&+\Big[z_1\Big]_z\left[-2\left(\frac{1}{Q_e^2V_2}+\frac{1}{V_1Q_\mu^2}\right)\right]\\
&+\Big[1\Big]_z\Bigg[-\frac{2}{Sy_\mu^2}\left(\frac{V_1y_{\mu_1}}{Q_e^2}+\frac{V_2}{Q_\mu^2}\right)\\
&+2\left(\frac{1}{Sy_\mu}-\frac{1}{V_1}-\frac{1}{V_2}\right)\left(1-\frac{2}{y_\mu}\right)-\frac{2V_1}{Q_e^2V_2}\left(2-\frac{3}{y_\mu}\right)-\frac{2V_2}{V_1Q_\mu^2}\left(1-\frac{3}{y_\mu}\right)\Bigg]\\
&+\left[\frac{1}{z_1}\right]_z\Bigg[-\frac{8SV_2y_{\mu_1}R_\mu}{Q_e^2Q_\mu^2y_\mu}+\frac{SV_1}{Q_e^2V_2}\left(-3y_\mu+9-\frac{10}{y_\mu}+\frac{4}{y_\mu^2}-\frac{2y_{\mu_1}R_\mu}{y_\mu}\right)\\
&+\frac{2V_1y_{\mu_1}}{Q_e^2y_\mu^2}(y_{\mu_1}-3R_\mu)+\frac{V_2y_{\mu_1}}{Q_e^2y_\mu}-\frac{V_1}{y_\mu}\left(\frac{1}{Q_\mu^2}+\frac{y_{\mu_1}}{V_2}\right)\\
&+\frac{V_2}{V_1}\left[\frac{S}{Q_\mu^2}\left(-1+\frac{2}{y_\mu}-\frac{4}{y_\mu^2}+\frac{2R_\mu}{y_\mu}\right)+\frac{1}{y_\mu}\right]\\
&+\frac{2V_2}{Q_\mu^2y_\mu^2}[1-(1+2y_\mu)R_\mu]+2\left[1-\frac{2}{y_\mu}+\frac{2}{y_\mu^2}+\frac{1}{y_\mu}\left(1-\frac{4}{y_\mu}\right)R_\mu\right]\Bigg]\\
&+\left[\frac{1}{z_2}\right]_z\Bigg[\frac{2V_1}{Q_e^2y_\mu^2}(1+3R_\mu)+\frac{SV_1}{Q_e^2V_2}\left(-1+\frac{2}{y_\mu}-\frac{4}{y_\mu^2}+\frac{2R_\mu}{y_\mu}\right)
\end{aligned}
$$

$$+\frac{V_2}{Q_e^2 y_\mu}\left(-1+\frac{8SR_\mu}{Q_\mu^2}\right)+\frac{V_2}{V_1}\left[\frac{S}{Q_\mu^2}\left(-3y_\mu+9-\frac{10}{y_\mu}+\frac{4}{y_\mu^2}-\frac{2y_{\mu_1}R_\mu}{y_\mu}\right)-\frac{y_{\mu_1}}{y_\mu}\right]$$
$$+\frac{V_1}{y_\mu}\left(\frac{y_{\mu_1}}{Q_\mu^2}+\frac{1}{V_2}\right)+\frac{2V_2}{Q_\mu^2 y_\mu^2}\left(y_{\mu_1}^2+(1-3y_\mu)R_\mu\right)$$
$$+2\left(1-\frac{2}{y_\mu}+\frac{2}{y_\mu^2}-\frac{1}{y_\mu}\left(3-\frac{4}{y_\mu}\right)R_\mu\right)\Bigg]. \tag{A.73}$$

The indefinite integrals, in terms of which the quantities S_k^{R}, are presented here, is the first series of integrals over z_1 or z_2, which is given in this paper. It correspontds to the first, innermost and the only semianalytical integration, within our *numerical approach*. In the R-frame, it corresponds to the integration over the angle φ_R of the photon, see eq. (C.127).

The three first integrals of this series are presented in Appendix D.1 of [9], eqns.(D.6)-(D.8). Here we recall the definition of integration, and present two additional integrals.

$$\Big[\mathcal{A}\Big]_z = \frac{1}{\pi}\int_{\hat{z}_{1(2)}^{\min}}^{\hat{z}_{1(2)}^{\max}} \frac{dz_{1(2)}}{\sqrt{R_z}}\mathcal{A},$$

4) $$\Big[z_{1(2)}\Big]_z = -\frac{1}{\pi\lambda_q}\sqrt{R_z(z_{1(2)})}\Big|_z + \frac{B_{1(2)}}{\lambda_q}\Big[1\Big]_z,$$

5) $$\Big[z_{1(2)}^2\Big]_z = -\frac{1}{2\pi\lambda_q}\left(z_{1(2)}+3\frac{B_{1(2)}}{\lambda_q}\right)\sqrt{R_z(z_{1(2)})}\Big|_z + \frac{1}{2\lambda_q^2}\left(3B_{1(2)}^2-\lambda_q C_{1(2)}\right)\Big[1\Big]_z.$$

The functions A, B, C are defined in (3.34)-(3.36).

In the table, we introduced the abbreviation

$$\Big|_z \equiv \Big|_{\hat{z}_{1(2)}^{\min}}^{\hat{z}_{1(2)}^{\max}}. \tag{A.74}$$

The limits of the integration $\hat{z}_{1(2)}^{\min,\max}$ may be arbitrary. The particular realization of these limits, depending on realistic experimental cuts, which we implement within our numerical approach, has been derived in Appendix B.3.

B Numerical approach. Implementation of experimental cuts

Within our numerical approach, the cuts are applied via integration limits in (3.70). There are cuts in variables V_2, V_1 and z_1.

B.1 V_2-dependent cut

- Angular cut of muon

  Here we consider the cut on the angle of the scattered muon. This angle may be easily calculated from the Born and bremsstrahlung kinematics. In both cases

$$-2k_1k_2 = 2\left(k_1^0k_2^0 - |\vec{k}_1||\vec{k}_2|\cos\theta_\mu\right). \tag{B.75}$$

For the case of Born kinematics one can easily derive

$$\cos\theta_\mu^{\text{BORN}} = \frac{S^2(1-y_\mu) - 2m_e^2(Sy_\mu + 2m_\mu^2)}{\sqrt{\lambda_S}\sqrt{\lambda_l}}. \tag{B.76}$$

From eq. (B.76), keeping only terms $\mathcal{O}(m_e^2)$, we get

$$\sin^2\theta_\mu^{\text{BORN}} = 1 - \cos^2\theta_\mu^{\text{BORN}} = \frac{4m_e^2 V_2^{\max}}{S^2(1-y_\mu)}, \tag{B.77}$$

so, one may derive the θ_μ^{BORN}. Here $V_2^{\max}$

$$V_2^{\max} = Sy_\mu - \frac{m_\mu^2 {y_\mu}^2}{1-y_\mu} = \frac{Sy_\mu}{1-y_\mu}\left(1 - \frac{y_\mu}{Y}\right), \tag{B.78}$$

is the upper limit of V_2 neglecting m_e. The exact expression was given in (3.27). Using (3.8), one may receive from (B.75) the following expression for $\cos\theta_\mu^{\text{BREM}}$:

$$\cos\theta_\mu^{\text{BREM}} = \frac{S^2(1-y_\mu) - 2m_e^2\left(Sy_\mu - V_2 + 2m_\mu^2\right)}{\sqrt{\lambda_S}\sqrt{\lambda_l}}. \tag{B.79}$$

Keeping again only terms linear in m_e^2, we may derive the value of sine of the muon scattering angle in the bremsstrahlung process:

$$\sin^2\theta_\mu^{\text{BREM}} = \frac{4m_e^2}{S^2(1-y_\mu)}\left(V_2^{\max} - V_2\right). \tag{B.80}$$

It is obvious that:

$$\theta_\mu^{\text{BREM}} \le \theta_\mu^{\text{BORN}}. \tag{B.81}$$

Therefore, we may introduce the upper limit of the difference

$$\theta_\mu^{\text{BORN}} - \theta_\mu^{\text{BREM}} \le \bar\theta, \tag{B.82}$$

or

$$(\theta_\mu^{\text{BREM}})^2 \ge (\theta_\mu^{\text{BORN}})^2 - 2\theta_\mu^{\text{BORN}}\bar\theta + \bar\theta^2. \tag{B.83}$$

By substitution of expressions (B.77) and (B.80), we receive

$$V_2 \le V_2^c = V_2^{\max}\frac{\bar\theta(2\theta_\mu^{\text{BORN}} - \bar\theta)}{(\theta_\mu^{\text{BORN}})^2}. \tag{B.84}$$

- Kinematical limitation

  The kinematical limit on $V_2^{\max}$ was written in (3.27).

So, the absolute max of V_2 should be a minimum of the two possible maximal values of V_2

$$\hat V_2^{\max} = \min\left[V_2^{\max}, V_2^c\right]. \tag{B.85}$$

B.2 V_1-dependent cuts

- Energy recoil cut E^{RC}.

  The electron energy in the final state is limited from below by: $E'_{el} = p_2^0 \geq E^{RC}$, where E^{RC} is some cut on recoil electron energy. In numerical calculation we use the SMC value $E^{RC} = 35$ GeV. So,

$$2m_e p_2^0 \geq 2m_e E^{RC}. \tag{B.86}$$

  From the other hand:

$$-2p_1p_2 = 2m_e p_2^0 = Q_e^2 + 2m_e^2. \tag{B.87}$$

  This corresponds to

$$Sy_\mu - V_1 \geq 2m_e(E^{RC} - m_e), \tag{B.88}$$

  and we receive one of the possible cuts on V_1:

$$V_1 \leq V_1^{ERC} = Sy_\mu - 2m_e(E^{RC} - m_e). \tag{B.89}$$

- Energy balance cut E^{BC}.

  Photon energy p^0 also could be implicitely limited by an experimental condition, emerging from checking of overall energy balance:

$$E_\mu + m_e - E'_\mu - E'_e = p_0 \leq E^{BC}. \tag{B.90}$$

  The SMC value is $E^{BC} = 40$ GeV. It is another energy cut related to the invariant V_1:

$$2m_e p^0 \leq 2m_e E^{BC}, \tag{B.91}$$

$$V_1 \equiv -2p_1 p = 2m_e p^0 \leq 2m_e E^{BC} \equiv V_1^{EBC}. \tag{B.92}$$

- Kinematical limitation

  Still we must take into account the kinematical limitation on $V_1^{\max}$. The boundaries of the allowed phase-space region are defined by (3.30).

These three cuts allow us to make the right choice for an absolute maximum of V_1

$$\hat{V}_1^{\max} = \min\left[V_1^{\max}, V_1^{ERC}, V_1^{EBC}\right]. \tag{B.93}$$

B.3 z_1-dependent cut

- Cut on the electron angle The electron angle can be determined from

$$-2k_1p_2 = 2k_1^0p_2^0 - 2|\vec{k}_1||\vec{p}_2|\cos\theta_e \tag{B.94}$$

  It differs for the Born and bremsstrahlung kinematics. We consider first the radiative case, where it may be rewritten as:

$$\begin{aligned}\cos\theta_e^{\text{BREM}} &= \frac{S\left[(Sy_\mu - V_1) + 2m_e^2\right] - \left[S(1-y_\mu) + V_2 - z_1\right]2m_e^2}{\sqrt{\lambda_S}\sqrt{\lambda_e}} \\ &= \frac{(S+2m_e^2)(Sy_\mu - V_1) + 2m_e^2 z_2}{\sqrt{\lambda_S}\sqrt{(Sy_\mu - V_1)^2 + 4m_e^2(Sy_\mu - V_1)}}.\end{aligned} \tag{B.95}$$

and

$$\sin^2\theta_e^{\rm BREM} \;=\; \frac{4m_e^2}{\lambda_S\lambda_e}\Big\{Q_e^2\Big[\lambda_S - Q_e^2(S+m_e^2+m_\mu^2) - z_2(S+2m_e^2)\Big] - m_e^2 z_2\Big\}. \tag{B.96}$$

In the URA in m_e^2, it becomes

$$\sin^2\theta_e^{\rm BREM} \approx \frac{4m_e^2}{S^2Q_e^2}\Big[S^2 - Q_e^2 m_\mu^2 - S(Sy_\mu - V_1 + z_2)\Big]. \tag{B.97}$$

We may extract the correct boundaries for the invariant z_1, from three quantities: $\left(\theta_e^{\rm BORN}\right)^2$, $\left(\theta_e^{\rm BREM}\right)^2$ and the difference $\bar\theta$.
From (B.97) we get:

$$\left(\theta_e^{\rm BREM}\right)^2 = \frac{4m_e^2}{S}\left[\frac{S(1-y_\mu)+V_2-z_1}{Sy_\mu - V_1} - \frac{m_\mu^2}{S}\right]. \tag{B.98}$$

In the Born approximation $V_1 = V_2 = z_1 = 0$ it reduces to

$$\left(\theta_e^{\rm BORN}\right)^2 = \frac{4m_e^2}{S}\left[\frac{1-y_\mu}{y_\mu} - \frac{m_\mu^2}{S}\right]. \tag{B.99}$$

Due to the fact that the differences between $\left|\theta_e^{\rm BORN} - \theta_e^{\rm BREM}\right|$ has to be smaller than some value, which is determined by experiment, we have:

$$\left|\theta_e^{\rm BORN} - \theta_e^{\rm BREM}\right| \leq \bar\theta. \tag{B.100}$$

One gets two cases

$$\theta_e^{\rm BORN} \geq \theta_e^{\rm BREM}. \tag{B.101}$$

and

$$\theta_e^{\rm BREM} \geq \theta_e^{\rm BORN}. \tag{B.102}$$

With (B.98), (B.99) and (B.101) we arrive at

$$(z_1^c)^{\max} \;=\; V_2 + V_1\frac{(1-y_\mu)}{y_\mu} + \frac{SQ_e^2}{4m_e^2}\,\bar\theta\,(2\theta_e^{\rm BORN} - \bar\theta), \tag{B.103}$$

while from (B.102) we receive

$$(z_1^c)^{\min} \;=\; V_2 + V_1\frac{(1-y_\mu)}{y_\mu} - \frac{SQ_e^2}{4m_e^2}\,\bar\theta\,(\bar\theta + 2\theta_e^{\rm BORN}). \tag{B.104}$$

- Kinematical limitation

  Also the kinematic boundaries ${z_1}^{\max}$ and ${z_1}^{\min}$ defined in (3.32), should be taken into account.

The θ_e cut conditions is active, if at least one of

$$(z_1^c)^{\min}, (z_1^c)^{\max} \in [z_1^{\min}, z_1^{\max}], \tag{B.105}$$

and

$$(z_1^c)^{\min} \le (z_1^c)^{\max}. \tag{B.106}$$

Then the absolute maximum of z_1 should be a minimun of the two possible maxima of z_1

$$\hat{z}_1^{\max} = \min\left[z_1^{\max}, (z_1^c)^{\max}\right], \tag{B.107}$$

and the absolute minimum of z_1 should be a maximum of the two possible minima of z_1

$$\hat{z}_1^{\min} = \max\left[z_1^{\min}, (z_1^c)^{\min}\right]. \tag{B.108}$$

C R-frame kinematics

In two applications in this paper, we will need another frame than the laboratory frame. When calculating the soft photon contribution and a table of two-fold integrals within the completely analytic approach, we will also make use of the so called R-frame, which is defined by

$$\vec{p_2} + \vec{p} = 0, \tag{C.109}$$

or

$$\vec{Q} = \vec{p_1} + \vec{k_1} - \vec{k_2} = 0. \tag{C.110}$$

First, we introduce coordinates of 4-vectors p_1, k_1, k_2, p, shown in the figure,

$$\begin{aligned}
p_1 &= \left(0, 0, |\vec{p_1}|, p_1^0\right) \\
k_i &= \left(0, |\vec{k_i}|\sin\theta_i, |\vec{k_i}|\cos\theta_i, k_i^0\right) \\
p &= p^0\left(\sin\theta_R\sin\varphi_R, \sin\theta_R\cos\varphi_R, \cos\theta_R, 1\right), \\
p_2 &= \left(-p^0\sin\theta_R\sin\varphi_R, -p^0\sin\theta_R\cos\varphi_R, -p^0\cos\theta_R, p_2^0\right).
\end{aligned} \tag{C.111}$$

As may be seen from figure 6, we have choosen the z-axis of the R-frame along the vector $\vec{p_1}$ and matched the R-frame (z, y)-plane with the plane spanned by vector $\vec{p_1}$ and one of the vectors $\vec{k_1}$ or $\vec{k_2}$. This trick is possible since we will use the R-frame only in analytical calculations of tables of bremsstrahlung hard and soft integrals, in which we will always 'decouple' z_1 and z_2 using an invariant partial fraction decomposition while calculating hard integrals and Feynman parametrization for the soft. Finally, the photonic vector $\vec{p}$ is directed arbitrarily and its angles are unlimited

$$\begin{aligned}
0 \le \varphi_R \le 2\pi, \\
0 \le \theta_R \le \pi.
\end{aligned} \tag{C.112}$$

Actually we are using two R-frames, different for z_1 and z_2, containing integrands with different pairs of θ_R, φ_R, both varying within the full solid angle 4π.

The vector coordinates (C.111) contain, apart from angular variables, the energies $p_1^0, k_{1,2}^0$ and vector moduli $|\vec{p_1}|, |\vec{k}_{1,2}|$. They can be derived using an invariant language. First we consider

$$\tau = -(p_2 + p)^2 = (p_2^0 + p^0)^2. \tag{C.113}$$

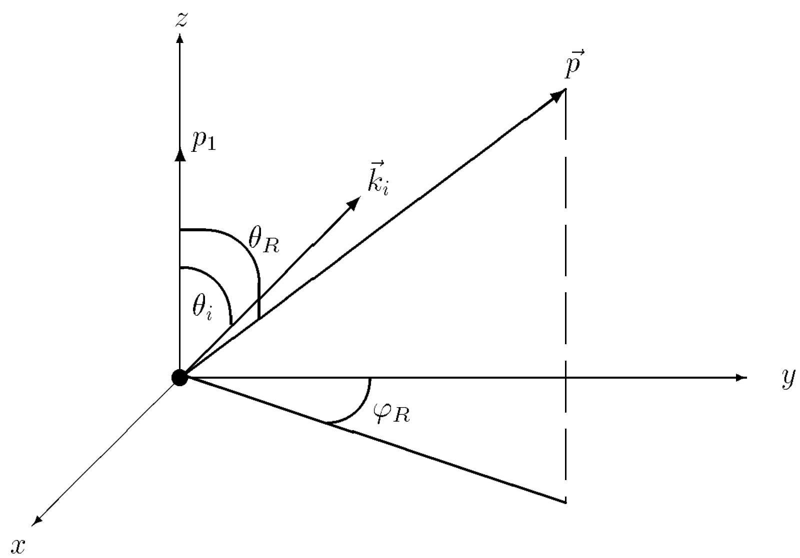


Figure 6: *The R-frame.*

Therefore,

$$p_2^0 + p^0 = \sqrt{\tau}. \tag{C.114}$$

To derive an energetic coordinate of a 4-momentum, we consider the 4-scalar product of this 4-momentum with the four vector $-2(p_2 + p)$ and write it twice: through invariants, and in the R-frame, using (C.114). We give the simplest example:

$$\begin{aligned} -2(p_2+p).p &= V_2 \quad - \quad \text{through invariants,} \\ &= 2(p_2^0+p^0)p^0 = 2\sqrt{\tau}p^0 \quad - \quad \text{in the R-frame.} \end{aligned} \tag{C.115}$$

Let us introduce the following λ-functions:

$$\begin{aligned} \lambda_{k_1} &\equiv \lambda\left[-(p_2+p-k_1)^2, \tau, -k_1^2\right] = \left[S(1-y_\mu)+V_2\right]^2 - 4m_\mu^2\tau, \\ \lambda_{k_2} &\equiv \lambda\left[-(p_2+p-k_2)^2, \tau, -k_2^2\right] = (S-V_2)^2 - 4m_\mu^2\tau, \\ \lambda_{p1} &\equiv \lambda\left[-(p_2+p-p_1)^2, \tau, -p_1^2\right] = (Sy_\mu+2m_e^2)^2 - 4m_e^2\tau = \lambda_\mu. \end{aligned} \tag{C.116}$$

In this way, the following table had been derived:

$$\begin{aligned} |\vec{k_1}| &= \frac{\sqrt{\lambda_{k_1}}}{2\sqrt{\tau}}, & k_1^0 &= \frac{S(1-y_\mu)+V_2}{2\sqrt{\tau}} = \frac{S_{k1}}{2\sqrt{\tau}}, \\ |\vec{k_2}| &= \frac{\sqrt{\lambda_{k_2}}}{2\sqrt{\tau}}, & k_2^0 &= \frac{S-V_2}{2\sqrt{\tau}} = \frac{S_{k2}}{2\sqrt{\tau}}, \\ |\vec{p_1}| &= \frac{\sqrt{\lambda_\mu}}{2\sqrt{\tau}}, & p_1^0 &= \frac{Sy_\mu+2m_e^2}{2\sqrt{\tau}} = \frac{S_{p1}}{2\sqrt{\tau}}, \\ |\vec{p_2}| &= \frac{V_2}{2\sqrt{\tau}}, & p_2^0 &= \frac{V_2+2m_e^2}{2\sqrt{\tau}}, \\ |\vec{p}| &= |\vec{p_2}|, & p^0 &= \frac{V_2}{2\sqrt{\tau}}. \end{aligned} \tag{C.117}$$

Now we write invariant S in terms of R-frame variables

$$S = -2p_1k_1 = 2(p_1^0k_1^0 - |\vec{p_1}||\vec{k_1}|\cos\theta_1). \tag{C.118}$$

By using (C.117), we get

$$S = \frac{S_{p1}S_{k1} - \sqrt{\lambda_\mu\lambda_{k_1}}\cos\theta_1}{2\tau}, \tag{C.119}$$

from which we derive an expression for $\cos\theta_1$ in the R-frame in terms of invariants:

$$\cos\theta_1 = \frac{S_{p1}S_{k1} - 2S\tau}{\sqrt{\lambda_\mu}\sqrt{\lambda_{k_1}}}. \tag{C.120}$$

Then we write the invariant S_1

$$S_1 = -2p_1k_2 = 2(p_1^0k_2^0 - |\vec{p_1}||\vec{k_2}|\cos\theta_2). \tag{C.121}$$

By using (C.117), we get

$$S_1 = \frac{S_{p1}S_{k2} - \sqrt{\lambda_\mu\lambda_{k_2}}\cos\theta_2}{2\tau}, \tag{C.122}$$

from where we derive an expression for $\cos\theta_2$ in R-frame:

$$\cos\theta_2 = \frac{S_{p1}S_{k2} - 2S_1\tau}{\sqrt{\lambda_\mu}\sqrt{\lambda_{k_2}}}. \tag{C.123}$$

Now we can write down invariants z_1 and z_2 in their two own R-frames correspondingly

$$z_1 = -2k_1p = 2p^0\left(k_1^0 - |\vec{k}_1|\cos\theta_1\cos\theta_R - |\vec{k}_1|\sin\theta_1\sin\theta_R\cos\varphi_R\right), \tag{C.124}$$

$$z_2 = -2k_2p = 2p^0\left(k_2^0 - |\vec{k}_2|\cos\theta_2\cos\theta_R - |\vec{k}_2|\sin\theta_2\sin\theta_R\cos\varphi_R\right). \tag{C.125}$$

The invariant V_1

$$V_1 = -2p_1p = 2p^0\left(p_1^0 - |\vec{p}_1|\cos\theta_R\right) \tag{C.126}$$

looks formally the same in both R-frames.

Finally, relations (C.126), (C.125) allow to perform $\int dV_1 dz$ in term of $\int d\cos\theta_R d\varphi_R$

$$\int_{V_1^{\min}}^{V_1^{\max}} dV_1 \int_{z_1^{\min}}^{z_1^{\max}} \frac{dz_1}{\pi\sqrt{R_z}} \equiv \frac{V_2}{\tau}\int_{-1}^{+1} d\cos\theta_R \int_0^{2\pi} d\varphi_R. \tag{C.127}$$

This property was widely used when we discussed the treatment of the infrared divergent part.

D Analytic approach. Tables of integrals

D.1 First and second analytic integrations. The R-integrals

$$\left[\mathcal{A}\right]_R = \int_{V_1^{\min}}^{V_1^{\max}} dV_1 \int_{z_1^{\min}}^{z_1^{\max}} \frac{dz_1}{\pi\sqrt{R_z}}\mathcal{A} \tag{D.128}$$

$$1)\quad \Big[1\Big]_R = \frac{V_2}{\tau}\sqrt{\lambda_\mu} \qquad 2)\quad \Big[V_1\Big]_R = \frac{1}{2}S_{p1}V_2^2\sqrt{\lambda_\mu}$$

$$3)\quad \left[\frac{1}{V_2}\right]_R = \mathrm{L}_{p1} \qquad 4)\quad \left[\frac{m_e^2}{V_1^2}\right]_R = \frac{1}{V_2}\sqrt{\lambda_\mu}$$

$$5)\quad \left[\frac{1}{Q_e^2}\right]_R = \mathrm{L}_{Q_e} \qquad 6)\quad \left[\frac{m_e^2}{Q_e^4}\right]_R = \frac{V_2}{Q_\mu^4}\sqrt{\lambda_\mu}$$

$$7)\quad \left[\frac{1}{z_1}\right]_R = \mathrm{L}_{k1} \qquad 8)\quad \left[\frac{1}{z_2}\right]_R = \mathrm{L}_{k2}$$

$$9)\quad \left[\frac{m_\mu^2}{z_1^2}\right]_R = \frac{1}{V_2} \qquad 10)\quad \left[\frac{m_\mu^2}{z_2^2}\right]_R = \frac{1}{V_2}$$

$$11)\quad \left[\frac{1}{z_1 z_2}\right]_R = \frac{2}{V_2}\mathrm{L}_m$$

$$12)\quad \left[\frac{m_\mu^2 V_1}{z_1^2}\right]_R = \frac{1}{\lambda_{k1}}\left[SS_{k1} - 2m_\mu^2 S_{p1} + m_\mu^2(S_{p1}S_{k1} - 2S\tau)\left[\frac{1}{z_1}\right]_R\right]$$

$$13)\quad \left[\frac{V_1}{z_1}\right]_R = \frac{S_{p1}S_{k1} - 2S\tau}{\lambda_{k1}\lambda_\mu}\Big[1\Big]_R + \left(SS_{k1} - 2m_\mu^2 S_{p1}\right)\frac{V_2}{\lambda_{k1}}\left[\frac{1}{z_1}\right]_R$$

$$14)\quad \left[\frac{V_1}{z_2}\right]_R = \frac{S_{p1}S_{k2} - 2S_1\tau}{\lambda_{k2}\lambda_\mu}\Big[1\Big]_R + \left[S_1S_{k2} - 2m_\mu^2 S_{p1}\right]\frac{V_2}{\lambda_{k2}}\left[\frac{1}{z_2}\right]_R$$

$$15)\quad \left[\frac{1}{V_1 z_1}\right]_R = \frac{1}{V_2}\mathrm{L}_S \qquad 16)\quad \left[\frac{1}{V_1 z_2}\right]_R = \frac{1}{V_2}\mathrm{L}_{S_1}$$

$$17)\quad \left[\frac{1}{Q_e^2 z_1}\right]_R = \frac{1}{Q_\mu^2}\mathrm{L}_{S_1} \qquad 18)\quad \left[\frac{1}{Q_e^2 z_2}\right]_R = \frac{1}{Q_\mu^2}\mathrm{L}_S$$

$$19)\quad \left[\frac{1}{Q_e^4 z_1}\right]_R = \frac{V_2 N_{k_1}}{m_e^2 Q_\mu^6 \lambda_l} + \frac{1}{Q_\mu^2 \lambda_l}\left[S_1 S_{k1} - 2m_\mu^2\left(\frac{S y_\mu V_2}{Q_\mu^2} + 2m_e^2\right)\right]\left[\frac{1}{Q_e^2 z_1}\right]_R$$

$$20)\quad \left[\frac{1}{Q_e^4 z_2}\right]_R = \frac{V_2 N_{k_2}}{m_e^2 Q_\mu^6 \lambda_S} + \frac{1}{Q_\mu^2 \lambda_S}\left[S S_{k2} - 2m_\mu^2\left(\frac{S y_\mu V_2}{Q_\mu^2} + 2m_e^2\right)\right]\left[\frac{1}{Q_e^2 z_2}\right]_R$$

$$21)\quad \left[\frac{1}{Q_e^2 z_1^2}\right]_R = \frac{N_{k_1}}{Q_\mu^2 \lambda_l}\left[\frac{1}{Q_e^2 z_1}\right]_R + \frac{1}{Q_\mu^2 \lambda_l}\left(\frac{S_1 V_2 + \lambda_l}{m_\mu^2 V_2} - \frac{2 S y_\mu}{Q_\mu^2}\right)$$

$$22)\quad \left[\frac{1}{Q_e^2 z_2^2}\right]_R = \frac{N_{k_2}}{Q_\mu^2 \lambda_S}\left[\frac{1}{Q_e^2 z_2}\right]_R + \frac{1}{Q_\mu^2 \lambda_S}\left(\frac{-S V_2 + \lambda_S}{m_\mu^2 V_2} - \frac{2 S y_\mu}{Q_\mu^2}\right)$$

$$\begin{aligned}
23)\quad \left[\frac{1}{Q_e^4 z_1^2}\right]_R = & \frac{1}{Q_\mu^2 \lambda_l}\left[\frac{S V_2}{Q_\mu^2} - S_1 + 2m_e^2\right. \\
& \left. + \frac{3 N_{k_1}}{Q_\mu^2 \lambda_l}\left(\lambda_l + S_1 V_2 - \frac{2 m_\mu^2 S y_\mu V_2}{Q_\mu^2}\right)\right]\left[\frac{1}{Q_e^2 z_1}\right]_R \\
& + \frac{1}{m_\mu^2 Q_\mu^4}\left(\frac{2S_1 + V_2 - 4m_\mu^2}{\lambda_l} + \frac{1}{V_2}\right) + \frac{V_2 \lambda_\mu}{m_e^2 Q_\mu^8 \lambda_l} \\
& + \frac{12 V_2}{Q_\mu^8 \lambda_l^2}\left(m_\mu^2 \lambda_\mu + m_e^2 Q_\mu^4 - S S_1 Q_\mu^2\right)
\end{aligned}$$

$$\begin{aligned}
24)\quad \left[\frac{1}{Q_e^4 z_2^2}\right]_R = & \frac{1}{Q_\mu^2 \lambda_S}\left[\frac{S_1 V_2}{Q_\mu^2} - S - 2m_e^2\right. \\
& \left. + \frac{3 N_{k_2}}{Q_\mu^2 \lambda_S}\left(\lambda_S - S V_2 - \frac{2 m_\mu^2 S y_\mu V_2}{Q_\mu^2}\right)\right]\left[\frac{1}{Q_e^2 z_2}\right]_R \\
& + \frac{1}{m_\mu^2 Q_\mu^4}\left(\frac{-2S + V_2 - 4m_\mu^2}{\lambda_S} + \frac{1}{V_2}\right) + \frac{V_2 \lambda_\mu}{m_e^2 Q_\mu^8 \lambda_S} \\
& + \frac{12 V_2}{Q_\mu^8 \lambda_S^2}\left(m_\mu^2 \lambda_\mu + m_e^2 Q_\mu^4 - S S_1 Q_\mu^2\right)
\end{aligned}$$

with

$$\begin{aligned}
S_{k1} &= S_1 + V_2, \qquad S_{k2} = S - V_2, \qquad S_{p1} = S y_\mu + 2m_e^2, \\
\mathrm{L}_{k1} &= \frac{1}{\sqrt{\lambda_{k1}}}\ln\frac{(S_{k1} + \sqrt{\lambda_{k1}})^2}{m_\mu^2 \tau}, \qquad \mathrm{L}_{k2} = \frac{1}{\sqrt{\lambda_{k2}}}\ln\frac{(S_{k2} + \sqrt{\lambda_{k2}})^2}{4 m_\mu^2 \tau},
\end{aligned}$$

$$
\begin{aligned}
\mathrm{L}_{p1} &= \ln\frac{(Sy_\mu+\sqrt{\lambda}_\mu+2m_e^2)^2}{4m_e^2\tau}, & \mathrm{L}_{Q_e} &= \ln\frac{V_2(Sy_\mu+\sqrt{\lambda}_\mu)+2m_e^2Q_\mu^2}{V_2(Sy_\mu-\sqrt{\lambda}_\mu)+2m_e^2Q_\mu^2},\\
\mathrm{L}_m &= \frac{1}{\sqrt{\lambda_m}}\ln\frac{\sqrt{\lambda}_m+Q_\mu^2}{\sqrt{\lambda}_m-Q_\mu^2}, & &\\
\mathrm{L}_S &= \frac{1}{\sqrt{\lambda_S}}\ln\frac{(S+\sqrt{\lambda_S})^2}{4m_e^2m_\mu^2}, & \mathrm{L}_{S_1} &= \frac{1}{\sqrt{\lambda_l}}\ln\frac{(S_1+\sqrt{\lambda_l})^2}{4m_e^2m_\mu^2},\\
N_{k_1} &= SS_1y_\mu-2m_e^2Q_\mu^2, & N_{k_2} &= S^2y_\mu+2m_e^2Q_\mu^2,\\
S_1 &= S(1-y_\mu), & \tau &= V_2+m_e^2,\\
\lambda_{k1} &= S_{k1}-4m_\mu^2\tau, & \lambda_{k2} &= S_{k2}-4m_\mu^2\tau,\\
\lambda_m &= Q_\mu^4+4m_\mu^2Q_\mu^2. & & \qquad\qquad (D.129)
\end{aligned}
$$

In (D.129) we defined the objects L_{k1}, L_{k2}, L_{p1}, L_{Q_e}, L_m, L_S, L_{S_1}, which will be used as integrands in the foregoing table. On top of them, we will also use two differences

$$
\mathrm{D}_{k1} = \mathrm{L}_{k1}-\mathrm{L}_S, \qquad \mathrm{D}_{k2} = \mathrm{L}_{k2}-\mathrm{L}_{S_1}. \qquad (D.130)
$$

D.2 The third analytic integration. The V_2-integrals

$$
\Big[A\Big]_V = \int_0^{V_2^{\max}} dV_2 A
$$

1) $$\Big[1\Big]_V = Sy_\mu - Q_\mu^{2\,\max}$$

2) $$\left[\frac{1}{Q_\mu^2}\right]_V = -\ln\frac{Q_\mu^{2\,\max}}{Sy_\mu}$$

3) $$\left[\frac{1}{Q_\mu^4}\right]_V = \frac{1}{S}\left(\frac{S}{Q_\mu^{2\,\max}}-\frac{1}{y_\mu}\right)$$

4) $$\left[\frac{1}{Q_\mu^6}\right]_V = \frac{1}{2S^2}\left(\frac{S^2}{Q_\mu^{4\,\max}}-\frac{1}{y_\mu^2}\right)$$

5) $$\left[\frac{1}{Q_\mu^4}\right]_V = \frac{1}{3S^3}\left(\frac{S^3}{Q_\mu^{6\,\max}}-\frac{1}{y_\mu^3}\right)$$

6) $$\Big[\mathrm{L}_m\Big]_V = \frac{1}{2}\left(\ln^2\frac{t^{\max}+1}{t^{\max}-1}-\ln^2\frac{t^{\min}+1}{t^{\min}-1}\right)$$

7) $$\left[\frac{\mathrm{L}_m}{Q_\mu^2}\right]_V = \frac{1}{2m_\mu^2}\left(t^{\min}\ln\frac{t^{\min}+1}{t^{\min}-1}-t^{\max}\ln\frac{t^{\max}+1}{t^{\max}-1}-\ln\frac{Q_\mu^{2\,\max}}{Sy_\mu}\right)$$

$$8)\quad \left[\frac{\mathrm{L}_m}{Q_\mu^4}\right]_V = -\frac{1}{12m_\mu^4}\Bigg[\left(\frac{2r_\mu}{y_\mu}-1\right)t^{\max}\ \ln\frac{t^{\max}+1}{t^{\max}-1}$$

$$-\left(\frac{2Sr_\mu}{Q_\mu^{2\,\max}}-1\right)t^{\min}\ \ln\frac{t^{\min}+1}{t^{\min}-1}-\ln\frac{Q_\mu^{2\,\max}}{Sy_\mu}-\frac{2Sr_\mu}{Q_\mu^{2\,\max}}+\frac{2r_\mu}{y_\mu}\Bigg]$$

$$9.1)\quad \left[\frac{1}{\lambda_1}\right]_V = \frac{1}{\sqrt{\mathrm{D}_{k_1}}}\left(\arctan\frac{\mathrm{B}_{k_1}}{\sqrt{\mathrm{D}_{k_1}}}-\arctan\frac{\mathrm{B}_{k_1}-V_2^{\max}}{\sqrt{\mathrm{D}_{k_1}}}\right)$$

$$9.2)\quad \left[\frac{1}{\lambda_1}\right]_V = \frac{1}{2\sqrt{\mathrm{D}_{k_1}}}\left(\ln\frac{\mathrm{B}_{k_1}-\sqrt{\mathrm{D}_{k_1}}}{\mathrm{B}_{k_1}+\sqrt{\mathrm{D}_{k_1}}}-\ln\frac{\mathrm{B}_{k_1}-V_2^{\max}-\sqrt{\mathrm{D}_{k_1}}}{\mathrm{B}_{k_1}-V_2^{\max}+\sqrt{\mathrm{D}_{k_1}}}\right)$$

$$10)\quad \left[\frac{1}{\lambda_2}\right]_V = \frac{1}{2\sqrt{\mathrm{D}_{k_2}}}\left(\ln\frac{\mathrm{B}_{k_2}-\sqrt{\mathrm{D}_{k_2}}}{\mathrm{B}_{k_2}+\sqrt{\mathrm{D}_{k_2}}}-\ln\frac{\mathrm{B}_{k_2}-V_2^{\max}-\sqrt{\mathrm{D}_{k_2}}}{\mathrm{B}_{k_2}-V_2^{\max}+\sqrt{\mathrm{D}_{k_2}}}\right)$$

$$11)\quad \left[\frac{V_2}{\lambda_1}\right]_V = \frac{1}{2}\ln\left[\frac{(S_1+V_2^{\max})^2-4m_\mu^2V_2^{\max}}{S_1^2}\right]+\mathrm{B}_{k_1}\left[\frac{1}{\lambda_1}\right]_V$$

$$12)\quad \left[\frac{V_2}{\lambda_2}\right]_V = \frac{1}{2}\ln\left[\frac{(S-V_2^{\max})^2-4m_\mu^2V_2^{\max}}{S^2}\right]+\mathrm{B}_{k_2}\left[\frac{1}{\lambda_2}\right]_V$$

$$13)\quad \left[\mathrm{L}_{k1}\right]_V = -\ln\left|\frac{t_1^{\max}+1}{t_1^{\max}-1}\right|\ln\left|\frac{t_1^{\max}-1}{2r_{\mu_1}}\right|-\mathrm{Li}_2\left[\frac{2(1+r_{\mu_1})}{t_1^{\max}+1}\right]+\mathrm{Li}_2\left(\frac{2r_{\mu_1}}{t_1^{\max}-1}\right)$$

$$14)\quad \left[\mathrm{L}_{k2}\right]_V = \ln\frac{t_2^{\max}+1}{t_2^{\max}-1}\ln\frac{t_2^{\max}-1}{2r_\mu}+\mathrm{Li}_2\left[\frac{2(1+r_\mu)}{t_2^{\max}+1}\right]-\mathrm{Li}_2\left(\frac{2r_\mu}{t_2^{\max}-1}\right)$$

$$15)\quad \left[\frac{1}{\lambda_1}\mathrm{L}_{k1}\right]_V = -\frac{2}{S_1^2}\Bigg[\frac{-t_1^{\max}}{\lambda_{k1}^{\max}}\ln\left|\frac{2r_{\mu_1}(t_1^{\max}+1)}{(t_1^{\max}-1)(t_1^{\max}-t_1^0)}\right|$$

$$+\frac{1}{2(t_1^0+1)}\ln\left|\frac{(t_1^{\max}-t_1^0)^2(t_1^{\max}-1)}{(t_1^{\max}+1)\lambda_{k1}^{\max}}\right|$$

$$+\frac{1}{(t_1^0)^2-1}\ln\left|\frac{(t_1^{\max}-t_1^0)(t_1^{\max}-1)}{\lambda_{k1}^{\max}}\right|$$

$$-\left|\frac{1}{t_{11}-t_{12}}\ln\left(\frac{t_1^{\max}-t_{11}}{t_1^{\max}-t_{12}}\right)+\frac{1}{t_{11}-t_{12}}\ln\left(\frac{t_1^0-t_{11}}{t_1^0-t_{12}}\right)\right|\Bigg]$$

$$16)\quad \left[\frac{1}{\lambda_2}\mathrm{L}_{k2}\right]_V = \frac{2}{S^2}\Bigg[-\frac{t_2^{\max}}{\lambda_{k2}^{\max}}\ln\left|\frac{2r_\mu(t_2^{\max}+1)}{(t_2^{\max}-1)(t_2^{\max}-t_2^0)}\right|$$

$$+\frac{1}{2(t_2^0+1)}\ln\left|\frac{(t_2^{\max}-t_2^0)^2(t_2^{\max}-1)}{(t_2^{\max}+1)\lambda_{k2}^{\max}}\right|$$

$$+\frac{1}{(t_2^0)^2-1}\ln\left|\frac{(t_2^{\max}-t_2^0)(t_2^{\max}-1)}{\lambda_{k2}^{\max}}\right| - \frac{1}{t_{21}-t_{22}}\ln\left|\frac{t_2^{\max}-t_{21}}{t_2^{\max}-t_{22}}\right|\Bigg]$$

$$17)\quad \left[\frac{V_2}{\lambda_1}\mathrm{L}_{k1}\right]_V = \frac{2}{S_1}\Bigg[-\frac{1}{\lambda_{k1}^{\max}}\ln\left|\frac{2r_{\mu_1}(t_1^{\max}+1)}{(t_1^{\max}-1)(t_1^{\max}-t_1^0)}\right|$$

$$+\frac{1}{(t_1^0)^2-1}\ln\left|\frac{(t_1^{\max}-t_1^0)(t_1^{\max}-1)}{\lambda_{k1}^{\max}}\right| + \frac{1}{2(t_1^0+1)}\ln\left|\frac{(t_1^{\max})^2-1}{\lambda_{k1}^{\max}}\right|$$

$$+\left|\frac{1}{t_{11}-t_{12}}\ln\left(\frac{t_1^{\max}-t_{11}}{t_1^{\max}-t_{12}}\right) + \frac{1}{t_{11}-t_{12}}\ln\left(\frac{t_1^0-t_{11}}{t_1^0-t_{12}}\right)\right|\Bigg]$$

$$18)\quad \left[\frac{V_2}{\lambda_2}\mathrm{L}_{k2}\right]_V = \frac{2}{S}\Bigg[-\frac{1}{\lambda_{k2}^{\max}}\ln\left|\frac{2r_\mu(t_2^{\max}+1)}{(t_2^{\max}-1)(t_2^{\max}-t_2^0)}\right|$$

$$+\frac{1}{(t_2^0)^2-1}\ln\left|\frac{(t_2^{\max}-t_2^0)(t_2^{\max}-1)}{\lambda_{k2}^{\max}}\right|$$

$$+\frac{1}{2(t_2^0+1)}\ln\left|\frac{(t_2^{\max})^2-1}{\lambda_{k2}^{\max}}\right| + \frac{1}{t_{21}-t_{22}}\ln\left|\frac{t_2^{\max}-t_{21}}{t_2^{\max}-t_{22}}\right|\Bigg]$$

$$19)\quad \left[\frac{1}{Q_\mu^2}\mathrm{L}_{k1}\right]_V = -\frac{1}{d_1}\Big[F_{L_k}(t_{11},t_{12},t_1^0,t_1^{\max}) - F_{L_k}(t_{11},t_{12},t_1^0,t_1^0)\Big]$$

$$20)\quad \left[\frac{1}{Q_\mu^2}\mathrm{L}_{k2}\right]_V = \frac{1}{d_2}\Big[F_{L_k}(t_{21},t_{22},t_2^0,t_2^{\max}) - F_{L_k}(t_{21},t_{22},t_2^0,t_2^0)\Big]$$

$$21)\quad \left[\frac{1}{\tau^2}\right]_V = \frac{1}{m_e^2}$$

$$22)\quad \left[\frac{1}{\tau}\right]_V = L_+$$

$$23)\quad \Big[\mathrm{L}_{p1}\Big]_V = (L_-+1)\Big[1\Big]_V$$

$$24)\quad \left[\frac{1}{Q_\mu^2}\mathrm{L}_{p1}\right]_V = L_-\left[\frac{1}{Q_\mu^2}\right]_V + \mathrm{Li}_2\left(\frac{V_2^{\max}}{Sy_\mu}\right)$$

$$25)\quad \left[\frac{1}{Q_\mu^4}\mathrm{L}_{p1}\right]_V = L_-\left[\frac{1}{Q_\mu^4}\right]_V + \frac{1}{Sy_\mu}\left[\frac{1}{Q_\mu^2}\right]_V$$

$$26)\quad \left[\mathrm{L}_{Q_e}\right]_V = (L_+ + 1)\left[1\right]_V + 2\,(V_2^{\max} - Sy_\mu)\left[\frac{1}{Q_\mu^2}\right]_V$$

$$27)\quad \left[\frac{1}{V_2}\mathrm{L}_{Q_e}\right]_V = \frac{1}{2}L_+^2 + F_1 + 2\mathrm{Li}_2\left(\frac{V_2^{\max}}{Sy_\mu}\right)$$

$$28)\quad \left[\frac{1}{V_2}\right]_V = \ln\frac{V_2^{\max}}{V_2^{\min}}$$

$$29)\quad \left[\frac{1}{V_2}\mathrm{L}_{p1}\right]_V = 2\ln\frac{Sy_\mu}{m_e^2}\left[\frac{1}{V_2}\right]_V - \frac{1}{2}L_+^2 - F_1$$

$$30)\quad \left[\frac{1}{V_2}\mathrm{D}_{k1}\right]_V = \frac{1}{S_1}\left[\frac{1}{2}\ln\left|\frac{{S_1}^2}{m_e^2 m_\mu^2}\right|\ln\left|\frac{{S_1}^2 m_e^2}{(V_2^{\max})^2 m_\mu^2}\right| - F_1 - \frac{1}{2}\ln^2\left|\frac{-S_1(t_1^{\max}-t_1^0)}{2(-S_1+m_\mu^2)}\right|\right.$$
$$\left. +\mathrm{Li}_2\left(\frac{t_1^{\max}-t_1^0}{1-t_1^0}\right) - \mathrm{Li}_2\left(\frac{t_1^{\max}-t_1^0}{-1-t_1^0}\right)\right]$$

$$31)\quad \left[\frac{1}{V_2}\mathrm{D}_{k2}\right]_V = \frac{1}{S}\left[\frac{1}{2}\ln\frac{S^2}{m_e^2 m_\mu^2}\ln\frac{S^2 m_e^2}{(V_2^{\max})^2 m_\mu^2} - \frac{1}{2}\ln^2\frac{S(t_2^{\max}-t_2^0)}{2(S+m_\mu^2)} - F_1\right]$$
$$+\mathrm{Li}_2\left(\frac{t_2^{\max}-t_2^0}{1-t_2^0}\right) - \mathrm{Li}_2\left(\frac{t_2^{\max}-t_2^0}{-1-t_2^0}\right)$$

$$32)\quad \left[\frac{1}{V_2}\lambda_m\right]_V = \frac{1}{\sqrt{\lambda_m^0}}\left\{\ln\frac{\sqrt{\lambda_m^0}+Sy_\mu}{\sqrt{\lambda_m^0}-Sy_\mu}\left[\ln\frac{(t_a^{\max}-t_a^0)(\lambda_m^0)^{3/2}}{(V_2^{\max})^2 m_\mu^4} + \ln\frac{V_2^{\max}}{V_2^{\min}}\right]\right.$$
$$\left. +\mathrm{Li}_2\left(\frac{t_a^{\max}-t_a^0}{1-t_a^0}\right) - \mathrm{Li}_2\left(\frac{t_a^{\max}-t_a^0}{-1-t_a^0}\right)\right\}$$

with

$$Q_\mu^{2\,\min} = \frac{m_\mu^2 y_\mu^2}{1-y_\mu},$$

$$V_2^{\max} = Sy_\mu - \frac{m_\mu^2 y_\mu^2}{1-y_\mu}, \qquad V_2^{\min} = 2m_e\bar{\omega},$$

$$r_\mu = \frac{m_\mu^2}{S}, \qquad r_{\mu_1} = -\frac{m_\mu^2}{S_1},$$

$$Q_\mu^{2\,\max} = Sy_\mu - V_2^{\max},$$

$$t^{\max} = \sqrt{1+\frac{4m_\mu^2}{Sy_\mu}}, \qquad t^{\min} = \sqrt{1+\frac{4m_\mu^2}{Q_\mu^{2\,\max}}},$$

$$\mathrm{D}_{k_1} = 4m_\mu^2[S(1-y_\mu)-m_\mu^2], \qquad \mathrm{D}_{k_2} = 4m_\mu^2(S+m_\mu^2),$$

$$
\begin{aligned}
B_{k_1} &= -S_1 + 2m_\mu^2, & B_{k_2} &= S + 2m_\mu^2, \\
L_+ &= \ln \frac{Sy_\mu}{m_e^2} + \ln \frac{V_2^{\max}}{Sy_\mu}, & L_- &= \ln \frac{Sy_\mu}{m_e^2} - \ln \frac{V_2^{\max}}{Sy_\mu},
\end{aligned}
\tag{D.131}
$$

and the additional notation

$$
\begin{aligned}
t_1^0 &= 1 + 2r_{\mu_1}, \\
t_{11} &= 1 + 2\left[r_{\mu_1} + \sqrt{x_{\mu 1}(1 + x_{\mu 1})}\right], \qquad t_{12} = 1 + 2\left[r_{\mu_1} - \sqrt{x_{\mu 1}(1 + x_{\mu 1})}\right], \\
\lambda_{k1}^{\max} &= (t_1^{\max} - 1)^2 - 4r_{\mu_1} t_1^{\max}, \\
t_2^0 &= 1 + 2r_\mu, \\
t_{21} &= 1 + 2r_\mu + 2\sqrt{r_\mu(1 + r_\mu)}, \qquad t_{22} = 1 + 2r_\mu - 2\sqrt{r_\mu(1 + r_\mu)}, \\
\lambda_{k2}^{\max} &= (t_2^{\max} - 1)^2 - 4r_\mu t_2^{\max}, \\
t_{11} &= \frac{-S_1 + \sqrt{S_{d1}}}{Sy_\mu}, \qquad t_{12} = \frac{-S_1 - \sqrt{S_{d1}}}{Sy_\mu}, \\
t_1^{\max} &= \frac{\sqrt{(S_1 + V_2^{\max})^2 - 4m_\mu^2 V_2^{\max}} - S_1}{V_2^{\max}}, \\
t_2^{\max} &= \frac{S - \sqrt{(S - V_2^{\max})^2 - 4m_\mu^2 V_2^{\max}}}{V_2^{\max}}, \\
t_1^{\min} &= \frac{2S_1 + V_2^{\min} - 4m_\mu^2}{\sqrt{(S_1 + V_2^{\min})^2 - 4m_\mu^2 V_2^{\min}} + S_1}, \\
t_2^{\min} &= \frac{2S - V_2^{\min} + 4m_\mu^2}{S + \sqrt{(S - V_2^{\min})^2 - 4m_\mu^2 V_2^{\min}}}, \\
\lambda_m^0 &= S^2 y_\mu^2 + 4m_\mu^2 S y_\mu, \\
t_a^0 &= \frac{Sy_\mu + 2m_\mu^2}{\sqrt{\lambda_m^0}}, \\
t_a^{\min} &= \frac{2(Sy_\mu + 2m_\mu^2) - V_2^{\min}}{,} \sqrt{\lambda_m^0} + \sqrt{\lambda_m^0 - 2V_2^{\min}(Sy_\mu + 2m_\mu^2) + (V_2^{\min})^2}, \\
t_a^{\max} &= \frac{\sqrt{\lambda_m^0} - \sqrt{\lambda_m^0 - 2V_2^{\max}(Sy_\mu + 2m_\mu^2) + (V_2^{\max})^2}}{V_2^{\max}}, \\
F_{L_k} &= \ln \frac{(t_0 - 1)(t_{11} + 1)}{(t_{11} - 1)(t_{11} - t_0)} \ln(t_a - t_{11}) - \ln \frac{(t_0 - 1)(t_{12} + 1)}{(t_{12} - 1)(t_{12} - t_0)} \ln(t_a - t_{12}) \\
&\quad - \mathrm{Li}_2\left(\frac{t_a - t_{11}}{-1 - t_{11}}\right) + \mathrm{Li}_2\left(\frac{t_a - t_{12}}{-1 - t_{12}}\right) + \mathrm{Li}_2\left(\frac{t_a - t_{11}}{1 - t_{11}}\right) \\
&\quad - \mathrm{Li}_2\left(\frac{t_a - t_{12}}{1 - t_{12}}\right) + \mathrm{Li}_2\left(\frac{t_a - t_{11}}{t_0 - t_{11}}\right) - \mathrm{Li}_2\left(\frac{t_a - t_{12}}{t_0 - t_{12}}\right).
\end{aligned}
\tag{D.132}
$$

E Soft contribution to $d\sigma^{\rm IR}$

In (3.52) the expression for $d\sigma^{\rm IR,soft}$ reads:

$$d\sigma^{\rm IR,soft} = \frac{2^7\pi^3\alpha^3}{\lambda_S}\mathcal{B}\left(\mathtt{Q}_\mu^2 F_{\mu\mu}^{\rm IR} + \mathtt{Q}_\mu\mathtt{Q}_e F_{\mu e}^{\rm IR} + \mathtt{Q}_e^2 F_{ee}^{\rm IR}\right)\theta(\varepsilon - p^0)d\Gamma_3. \tag{E.133}$$

We substitute now the phase space $d\Gamma_3$, (3.40), which in the soft photon limit can be factorized

$$d\Gamma_3 = (2\pi)^4\frac{d^3\vec{k}_2}{(2\pi)^3 2k_2^0}\frac{d^3\vec{p}_2}{(2\pi)^3 2p_2^0}\frac{d^3\vec{p}}{(2\pi)^3 2p^0}\delta(k_1+p_1-k_2-p_2) = d\Gamma_2\frac{d^3\vec{p}}{(2\pi)^3 2p^0}. \tag{E.134}$$

Using definition (2.14) and relation (3.48), we straightforwardly derive from (E.133)

$$d\sigma^{\rm IR,soft} \Rightarrow d\sigma^{\rm BORN}\ \frac{\alpha}{\pi}\ \delta^{\rm IR,soft}, \tag{E.135}$$

where(see also [9]) :

$$\delta^{\rm IR,soft} = \frac{2}{\pi}\int\frac{d^3\vec{p}}{2p^0}\ F^{\rm IR}\ \theta(\varepsilon - p^0). \tag{E.136}$$

Then, the function $F^{\rm IR}$ takes the unique form

$$\begin{aligned} F^{\rm IR} &= \mathtt{Q}_\mu^2\left(\frac{k_1}{2k_1p} - \frac{k_2}{2k_2p}\right)^2 + 2\mathtt{Q}_\mu\mathtt{Q}_e\left(\frac{k_1}{2k_1p} - \frac{k_2}{2k_2p}\right)\left(\frac{p_1}{2p_1p} - \frac{p_2}{2p_2p}\right) \\ &+\mathtt{Q}_e^2\left(\frac{p_1}{2p_1p} - \frac{p_2}{2p_2p}\right)^2 && \text{(E.137)} \\ &= \mathtt{Q}_\mu^2 F_{\mu\mu}^{\rm IR} + \mathtt{Q}_\mu\mathtt{Q}_e F_{\mu e}^{\rm IR} + \mathtt{Q}_e^2 F_{ee}^{\rm IR}. && \text{(E.138)} \end{aligned}$$

The symbol $\Rightarrow$ in (E.135) means that instead of z-integrated IR-factors (3.49)-(3.51), we use them in a completely differential form (E.137); they depend on two photonic angles in the R-frame, ϑ_R, φ_R, and on the photonic energy, p^0.

Since the energy of emitted soft photons is limited within a narrow interval

$$0 \le p^0 \le \varepsilon \tag{E.139}$$

it is always possible to choose ε small enough to ensure the phase space of soft photons be a sphere non-limited by experimental cuts. In other words, the phase space of soft photons is *isotropic* which allows to choose the z-axis along a convenient direction, differently for every term in (E.137). while performing angular integrations in (E.136). Therefore, the invariant variables V_1, V_2, z_1 and z_2 might be expressed in terms of only *the polar angle* $\cos\vartheta_R \equiv \xi$ and made independent of the *azimuthal angle* φ_R

$$\begin{aligned} V_2 &= -2p_2p = 2p_2^0p^0 = 2m_ep^0, && \text{(E.140)} \\ V_1 &= -2p_1p = 2p^0(p_1^0 - |\vec{p_1}|\xi), && \text{(E.141)} \\ z_1 &= -2k_1p = 2p^0(k_1^0 - |\vec{k_1}|\xi), && \text{(E.142)} \\ z_2 &= -2k_2p = 2p^0(k_2^0 - |\vec{k_2}|\xi). && \text{(E.143)} \end{aligned}$$

In the R-frame

$$\vec{p_2} + \vec{p} = 0 \tag{E.144}$$

and in soft limit $p \to 0$ R-frame degenerates to the rest frame of $\vec{p_2}$

$$p_2^0 = m_e. \tag{E.145}$$

This is why V_2 in (E.140) is angular independent.

We will use the dimensional regularization for the infrared divergences and rewrite the photonic phase space as follows (we took also $2p^0$ out of every invariant variable in (E.140)-(E.143))

$$\frac{2}{\pi}\int\frac{d^3\vec{p}}{(2p^0)^3} \Rightarrow \frac{4\pi}{(2\sqrt{\pi})^n\Gamma(n/2-1)}\int_0^{\varepsilon}\frac{(p^0)^{n-5}dp^0}{\mu^{n-4}}\int_0^{2\pi}d\varphi_R\int_0^{\pi}(\sin\vartheta_R)^{n-3}d\vartheta_R. \tag{E.146}$$

Here γ is the Euler constant and μ is an arbitrary parameter with a dimension of mass.

Now integration in (E.136) over p^0 is performed straightforwardly:

$$\delta^{\mathrm{IR,soft}} = \frac{1}{2}\int\limits_{-1}^{+1}d\xi\left[P^{\mathrm{IR}} + \ln\frac{\varepsilon}{\mu} + \frac{1}{2}\ln(1-\xi^2)\right]\mathcal{F}^{\mathrm{IR}}. \tag{E.147}$$

In (E.147)

$$\mathcal{F}^{\mathrm{IR}} = (2p^0)^2 F^{\mathrm{IR}} \tag{E.148}$$

and the typical pole term

$$P^{\mathrm{IR}} = \frac{1}{n-4} + \frac{1}{2}\gamma + \ln\frac{1}{2\sqrt{\pi}} \tag{E.149}$$

represents the infrared divergences at $n = 4$.

From (E.140)–(E.143) entering (E.138) and from (E.147) we see that only those integrals over ξ may occur, which are presented in Appendix D.2 of [9]). Writing $\mathcal{F}^{\mathrm{IR}}$ in a form similar to (E.138), we will calculate the three contributions $\delta_{\mu\mu}^{\mathrm{IR,soft}}$, $\delta_{\mu e}^{\mathrm{IR,soft}}$ and $\delta_{ee}^{\mathrm{IR,soft}}$ separately. Before this, however, we present a collection of formulae in the R-frame in the soft photon limit.

E.1 R-frame kinematics for the calculation of the *soft* photon contribution

The R-frame is defined by

$$\vec{p_2} + \vec{p} = 0 \tag{E.150}$$

or

$$\vec{Q} = \vec{p_1} + \vec{k_1} - \vec{k_2} = 0. \tag{E.151}$$

Since the R-frame is isotropic, there is no need to fix its z-axis along a given direction, say along $\vec{p_1}$[5]

$$p_1 = \left(0, 0, |\vec{p_1}|, p_1^0\right). \tag{E.152}$$

It might be equally chosen along $\vec{k}_1$,

$$k_1 = \left(0, 0, |\vec{k_1}|, k_1^0\right), \tag{E.153}$$

or along $\vec{k}_2$, then

$$k_2 = \left(0, 0, |\vec{k_2}|, k_2^0\right), \tag{E.154}$$

or along any linear combination of any vectors, say $\vec{k}_\alpha = \alpha\vec{k}_1 + (1-\alpha)\vec{k}_2$, then

$$k_\alpha = \left(0, 0, |\alpha\vec{k}_1 + (1-\alpha)\vec{k}_2|, \alpha k_1^0 + (1-\alpha)k_2^0\right). \tag{E.155}$$

So, we have indeed many R-frames, which differ one from another by a spatial rotation and when we write an arbitrarily oriented photonic 4-momentum as

$$p = p^0(\sin\vartheta_R \sin\varphi_R, \sin\vartheta_R\cos\varphi_R, \cos\vartheta_R, 1), \tag{E.156}$$

one should understand that in every R-frame one has its own angles ϑ_R, φ_R which vary within *the same limits* – covering the full solid angle. In this way, we arrive at equations (E.140)-(E.143) for invariants V_2, V_1, z_1, z_2 with formally one parameter ξ.

In the expression (E.147), which has to be integrated over ξ with (E.148) and (E.137), enter the energies $p_1^0, k_{1,2}^0$ and moduli $|\vec{p}_1|, |\vec{k}_{1,2}|$ (see (E.140)-(E.143)).

All the energies and momenta moduli depend only on three invariants: S, y and V_2. We make no distinction between y and y_μ in the soft photon kinematics. In the soft photon problem, we neglect the small invariant V_2 as compared to the others. In this limit, the table (C.117) reduces to

$$\begin{aligned}
|\vec{k_1}| &= \frac{\sqrt{\lambda_l}}{2m_e}, & k_1^0 &= \frac{S_1}{2m_e},\\
|\vec{k_2}| &= \frac{\sqrt{\lambda_S}}{2m_e}, & k_2^0 &= \frac{S}{2m_e},\\
|\vec{p_1}| &= \frac{\sqrt{\lambda_e^0}}{2m_e}, & p_1^0 &= \frac{Sy + 2m_e^2}{2m_e},\\
|\vec{p_2}| &= \frac{V_2}{2\sqrt{\tau}}, & p_2^0 &= m_e,\\
|\vec{p}| &= |\vec{p_2}|, & p^0 &= \frac{V_2}{2m_e}.
\end{aligned} \tag{E.157}$$

[5]In this subsection all 4-momentum coordinates are understood in the R-frame.

E.2 Muonic current

In *muonic* current we are dealing with

$$F^{\rm IR}_{\mu\mu} = \left(\frac{k_1}{2k_1p}\right)^2 + \left(\frac{k_2}{2k_2p}\right)^2 - \frac{2k_1k_2}{(2k_1p)(2k_2p)}, \tag{E.158}$$

see (E.137). By applying the Feynman parameterization for the last term, we have

$$F^{\rm IR}_{\mu\mu} = -\frac{m_e^2}{(2k_1p)^2} - \frac{m_e^2}{(2k_2p)^2} - \int_0^1 d\alpha \frac{2k_1k_2}{(2k_\alpha p)^2}, \tag{E.159}$$

where a new 4-vector was introduced as discussed in (E.155):

$$k_\alpha = \alpha k_1 + (1-\alpha)k_2. \tag{E.160}$$

Therefore, using all soft machinery described above we can write $\delta^{\rm IR,soft}$ as follows

$$\delta^{\rm IR,soft}_{\mu\mu} = \mathtt{Q}^2_\mu\left\{\left(P^{\rm IR} + \ln\frac{\varepsilon}{\mu}\right)\frac{1}{2}\int\limits_0^1 d\alpha \int\limits_{-1}^1 d\xi\, \mathcal{F}^{\rm IR}_{\mu\mu} + \frac{1}{4}\int\limits_0^1 d\alpha \int\limits_{-1}^1 d\xi \ln(1-\xi^2)\, \mathcal{F}^{\rm IR}_{\mu\mu}\right\}, \tag{E.161}$$

see (E.147), with

$$\mathcal{F}^{\rm IR}_{\mu\mu} = -\frac{m_\mu^2}{{k_1^0}^2}\frac{1}{(1-\beta_1\xi)^2} - \frac{m_\mu^2}{{k_2^0}^2}\frac{1}{(1-\beta_2\xi)^2} + \frac{Sy+2m_\mu^2}{{k_\alpha^0}^2}\frac{1}{(1-\beta_\alpha\xi)^2}. \tag{E.162}$$

On passing, we used

$$-2k_\alpha p = 2p^0(k_\alpha^0 - |\vec{k}_\alpha|\xi) = 2p^0k_\alpha^0(1-\beta_\alpha\xi), \tag{E.163}$$
$$z_1 = -2k_1p = 2p^0(k_1^0 - |\vec{k}_1|\xi) = 2p^0k_1^0(1-\beta_1\xi), \tag{E.164}$$
$$z_2 = -2k_2p = 2p^0(k_2^0 - |\vec{k}_2|\xi) = 2p^0k_2^0(1-\beta_2\xi), \tag{E.165}$$

with three velocities

$$\beta_1 = \frac{|\vec{k}_1|}{k_1^0} = \frac{\sqrt{\lambda_l}}{S_1}, \tag{E.166}$$
$$\beta_2 = \frac{|\vec{k}_2|}{k_2^0} = \frac{\sqrt{\lambda_S}}{S}, \tag{E.167}$$
$$\beta_\alpha = \frac{|\vec{k}_\alpha|}{k_\alpha^0}. \tag{E.168}$$

We also have from (E.160) the following relations:

$$k_\alpha^0 = \frac{S\alpha + S_1(1-\alpha)}{2m_e},$$
$$-k_\alpha^2 = m_\mu^2 + \alpha(1-\alpha)Sy. \tag{E.169}$$

Therefore

$$\beta_\alpha = \frac{\sqrt{[S\alpha + S_1(1-\alpha)]^2 - 4m_e^2\left[m_\mu^2 + \alpha(1-\alpha)Sy\right]}}{S\alpha + S_1(1-\alpha)}. \tag{E.170}$$

Using the table of integrals from Appendix D.2 of [9] we obtain

$$\begin{aligned}\delta^{\rm IR,soft}_{\mu\mu} &= \mathtt{Q}_\mu^2 \int\limits_0^1 d\alpha \Biggl\{ \left(P^{\rm IR} + \ln\frac{2\varepsilon}{\mu}\right)\left[-\frac{m_\mu^2}{k_1^{0\,2}}\frac{1}{(1-\beta_1^2)} - \frac{m_\mu^2}{k_2^{0\,2}}\frac{1}{(1-\beta_2^2)} + \frac{Sy+2m_\mu^2}{k_\alpha^{0\,2}}\frac{1}{(1-\beta_\alpha^2)}\right] \\ &+ \frac{1}{2\beta_1}\frac{m_\mu^2}{k_1^{0\,2}}\frac{1}{(1-\beta_1^2)}\ln\frac{1+\beta_1}{1-\beta_1} + \frac{1}{2\beta_2}\frac{m_\mu^2}{k_2^{0\,2}}\frac{1}{(1-\beta_2^2)}\ln\frac{1+\beta_2}{1-\beta_2} \\ &- \frac{1}{2\beta_\alpha}\frac{Sy+2m_\mu^2}{k_\alpha^{0\,2}}\frac{1}{(1-\beta_\alpha^2)}\ln\frac{1+\beta_\alpha}{1-\beta_\alpha}\Biggr\}. \end{aligned} \tag{E.171}$$

We have from (E.166)–(E.168)

$$(k_1^0)^2(1-\beta_1^2) = -k_1^2 = m_\mu^2, \tag{E.172}$$

$$(k_2^0)^2(1-\beta_2^2) = -k_2^2 = m_\mu^2, \tag{E.173}$$

$$(k_\alpha^0)^2(1-\beta_\alpha^2) = -k_\alpha^2. \tag{E.174}$$

And the expression (E.161) becomes

$$\begin{aligned}\delta^{\rm IR,soft}_{\mu\mu} &= \mathtt{Q}_\mu^2 \Biggl\{ \left(P^{\rm IR} + \ln\frac{2\varepsilon}{\mu}\right)\left[-2 + \left(Sy + 2m_\mu^2\right)\int\limits_0^1 \frac{d\alpha}{m_\mu^2 + \alpha(1-\alpha)Sy}\right] \\ &+ \frac{1}{2\beta_1}\ln\frac{1+\beta_1}{1-\beta_1} + \frac{1}{2\beta_2}\ln\frac{1+\beta_2}{1-\beta_2} \\ &+ \frac{Sy+2m_\mu^2}{2}\int\limits_0^1 \frac{d\alpha}{\beta_\alpha[m_\mu^2 + \alpha(1-\alpha)Sy]}\ln\frac{1-\beta_\alpha}{1+\beta_\alpha}\Biggr\}. \end{aligned} \tag{E.175}$$

Now we use the URA in the electron mass

$$\frac{1}{2\beta_1}\ln\frac{1+\beta_1}{1-\beta_1} \approx = \ln\frac{S_1}{m_e m_\mu}, \tag{E.176}$$

$$\frac{1}{2\beta_2}\ln\frac{1+\beta_2}{1-\beta_2} \approx = \ln\frac{S}{m_e m_\mu}. \tag{E.177}$$

The first integral we calculated precisely in [9]

$$\left(Sy + 2m_\mu^2\right)\int\limits_0^1 \frac{d\alpha}{[m_\mu^2 + \alpha(1-\alpha)Sy]} = \frac{1+\beta^2}{\beta}L_\beta \tag{E.178}$$

with

$$\begin{aligned}\beta &= \sqrt{1 + \frac{4m_\mu^2}{Sy}}, \\ L_\beta &= \ln\frac{\beta+1}{\beta-1}. \end{aligned} \tag{E.179}$$

With the second integral the situation is more complicated. Defining it as in [9],

$$S_\Phi \equiv \frac{1}{2}\left(Sy + 2m_\mu^2\right)\int\limits_0^1 \frac{d\alpha}{\beta_\alpha[m_\mu^2 + \alpha(1-\alpha)Sy]}\ln\frac{1-\beta_\alpha}{1+\beta_\alpha}, \tag{E.180}$$

we cannot take over the analogous result from [9] since there that integral was calculated in the URA in the leptonic mass m. Here we need result exact in m_μ (see also [5]). Making use of URA in m_e, with the aid of (E.170) we get

$$S_\Phi \approx \left(Sy + 2m_\mu^2\right)\int\limits_0^1 \frac{d\alpha}{m_\mu^2 + \alpha(1-\alpha)Sy}\ln\frac{m_e^2\left[m_\mu^2 + \alpha(1-\alpha)Sy\right]}{\left[S\alpha + S_1(1-\alpha)\right]^2}. \tag{E.181}$$

The expression for S_Φ simplifies drastically and can be calculated straightforwardly

$$\begin{aligned} S_\Phi &= \frac{Sy + 2m_\mu^2}{\sqrt{\lambda_m^0}}\Biggl\{\ln\frac{m_\mu^2(Sy + 4m_\mu^2)}{S^2(1-y)(1-y\alpha_1)(1-y\alpha_2)}\ln\frac{\alpha_2}{(-\alpha_1)} \\ &-\frac{1}{2}\ln^2\left[\frac{\alpha_2}{(-\alpha_1)(1-y)}\right] + \ln(1-y)\ln\frac{(1-y)(1-y\alpha_1)}{(1-y\alpha_2)} - \mathrm{Li}_2(1) \\ &+\mathrm{Li}_2\left[\frac{(-\alpha_1)}{(1-y)\alpha_2}\right] + \mathrm{Li}_2\left[\frac{(1-y)(-\alpha_1)}{\alpha_2}\right] - \mathrm{Li}_2\left[\left(\frac{-\alpha_1}{\alpha_2}\right)^2\right]\Biggr\}. \end{aligned} \tag{E.182}$$

with

$$\alpha_{1,2} = \frac{1 \mp \beta}{2}. \tag{E.183}$$

Collecting all terms together, we finally have

$$\delta_{\mu\mu}^{\mathrm{IR,soft}} = \mathsf{Q}_\mu^2\Biggl\{2\left(P^{\mathrm{IR}} + \ln\frac{2\varepsilon}{\mu}\right)\left(\frac{1+\beta^2}{2\beta}L_\beta - 1\right) + \ln\frac{S^2(1-y)}{m_e^2 m_\mu^2} + S_\Phi\Biggr\}. \tag{E.184}$$

From the virtual photon correction to the muon vertex we have the contribution

$$\begin{aligned} \delta_{\mu\mu}^{\mathrm{vert}} &= \mathsf{Q}_\mu^2\Biggl\{-2\left(P^{\mathrm{IR}} + \ln\frac{m_\mu}{\mu}\right)\left(\frac{1+\beta^2}{2\beta}L_\beta - 1\right) + \frac{3}{2}\beta L_\beta - 2 \\ &-\frac{1+\beta^2}{2\beta}\left[L_\beta\ln\frac{4\beta^2}{\beta^2-1} + \mathrm{Li}_2\left(\frac{1+\beta}{1-\beta}\right) - \mathrm{Li}_2\left(\frac{1-\beta}{1+\beta}\right)\right]\Biggr\}. \end{aligned} \tag{E.185}$$

The infrared divergence and the scale parameter μ cancel exactly in the sum of these two contributions. The sum reads

$$\begin{aligned} \delta_{\mu\mu}^{\mathrm{VR}} &= \delta_{\mu\mu}^{\mathrm{IR,soft}} + \delta_{\mu\mu}^{\mathrm{vert}} \\ &= \mathsf{Q}_\mu^2\Biggl\{\left[\frac{1+\beta^2}{2\beta}L_\beta - 1\right]\left(2\ln\frac{V_2^{\min}}{S} - \ln(1-y)\right) + \frac{3}{2}\beta L_\beta - 2 \\ &+\frac{1+\beta^2}{2\beta}\Biggl[L_\beta\ln\frac{1-y}{(1-y\alpha_1)(1-y\alpha_2)} - \frac{1}{2}\ln^2(1-y) \\ &+\ln(1-y)\ln\frac{(1-y)(1-y\alpha_1)}{(1-y\alpha_2)} + \mathrm{Li}_2\left[\frac{(-\alpha_1)}{(1-y)\alpha_2}\right] \\ &+\mathrm{Li}_2\left[\frac{(1-y)(-\alpha_1)}{\alpha_2}\right] - 2\mathrm{Li}_2\left(\frac{(-\alpha_1)}{\alpha_2}\right)\Biggr]\Biggr\}. \end{aligned} \tag{E.186}$$

E.3 Electronic current

The correction $\delta_{ee}^{\rm IR,soft}$ is very easy to calculate.

$$F_{ee}^{\rm IR} = \left(\frac{p_1}{2p_1p}\right)^2 + \left(\frac{p_2}{2p_2p}\right)^2 - \frac{2p_1p_2}{(2p_1p)(2p_2p)}. \tag{E.187}$$

Since $2p_2p$ is independent of $\xi = \cos\vartheta_R$, the α-parameterization is not needed here and $\delta_{ee}^{\rm IR,soft}$ reads

$$\delta_{ee}^{\rm IR,soft} = \mathtt{Q}_e^2\left[\left(P^{\rm IR} + \ln\frac{\varepsilon}{\mu}\right)\frac{1}{2}\int\limits_{-1}^{+1} d\xi \mathcal{F}_{ee}^{\rm IR} + \frac{1}{4}\int\limits_{-1}^{+1} d\xi \ln(1-\xi^2)\mathcal{F}_{ee}^{\rm IR}\right]. \tag{E.188}$$

Introducing

$$\beta = \frac{|\vec{p_1}|}{p_1^0} = \frac{\sqrt{S^2y^2 + 4m_e^2 Sy}}{Sy + 2m_e^2}, \tag{E.189}$$

we receive

$$\mathcal{F}_{ee}^{\rm IR} = -\frac{m_e^2}{p_1^{0\,2}}\frac{1}{(1-\beta\xi)^2} - 1 + \frac{2m_e p_1^0}{m_e p_1^0}\frac{1}{(1-\beta\xi)} = -1 + \frac{2}{1-\beta\xi} - \frac{m_e^2}{p_1^{0\,2}}\frac{1}{(1-\beta\xi)^2}. \tag{E.190}$$

Using the table of integrals from Appendix D.2 of [9] we derive

$$\delta_{ee}^{\rm IR,soft} = \mathtt{Q}_e^2\Biggl\{\left(P^{\rm IR} + \ln\frac{2\varepsilon}{\mu}\right)\left[-1 + \frac{1}{\beta}\ln\frac{1+\beta}{1-\beta} - \frac{m_e^2}{p_1^{0\,2}}\frac{1}{(1-\beta^2)}\right] \tag{E.191}$$
$$+1 + \frac{1}{2\beta}\left[\mathrm{Li}_2\left(\frac{2\beta}{\beta-1}\right) - \mathrm{Li}_2\left(\frac{2\beta}{\beta+1}\right)\right] + \frac{m_e^2}{p_1^{0\,2}}\frac{1}{2\beta}\frac{1}{(1-\beta^2)}\ln\frac{1+\beta}{1-\beta}\Biggr\}.$$

In the URA in m_e^2 we obtain

$$\frac{1}{\beta}\ln\frac{1+\beta}{1-\beta} \approx 2\ln\frac{Sy}{m_e^2} \tag{E.192}$$

and

$$\mathrm{Li}_2\left(\frac{2\beta}{\beta-1}\right) - \mathrm{Li}_2\left(\frac{2\beta}{\beta+1}\right) \approx -2\mathrm{Li}_2(1) - 2\ln^2\frac{Sy}{m_e^2}. \tag{E.193}$$

So, we can write

$$\delta_{ee}^{\rm IR,soft} = 2Q_e^2\Biggl\{\left(P^{\rm IR} + \ln\frac{2\varepsilon}{\mu}\right)\left(-1 + \ln\frac{Sy}{m_e^2}\right) + 1 + \ln\frac{Sy}{m_e^2} - \ln^2\frac{Sy}{m_e^2} - \mathrm{Li}_2(1)\Biggr\}. \tag{E.194}$$

The corresponding virtual photon correction to the electron vertex has the following form:

$$\delta_{ee}^{\rm vert} = Q_e^2\Biggl\{2\left(P^{\rm IR} + \ln\frac{m_e}{\mu}\right)\left(1 - \ln\frac{Sy}{m_e^2}\right) - 2 + \frac{3}{2}\ln\frac{Sy}{m_e^2} - \frac{1}{2}\ln^2\frac{Sy}{m_e^2} + \mathrm{Li}_2(1)\Biggr\}. \tag{E.195}$$

The complete answer is:

$$\begin{aligned}\delta_{ee}^{\rm VR} &= \delta_{ee}^{\rm IR,soft} + \delta_{ee}^{\rm vert} \\ &= Q_e^2\left[\left(\ln\frac{Sy}{m_e^2} - 1\right)\left(\ln\frac{Sy}{m_e^2} + 2\ln\frac{V_2^{\rm min}}{m_e^2}\right) - 1 + \frac{3}{2}\ln\frac{Sy}{m_e^2} - \frac{1}{2}\ln^2\frac{Sy}{m_e^2}\right].\end{aligned} \tag{E.196}$$

E.4 μe interference

In the μe interference the expression for $F^{\rm IR}_{\mu e}$ reads

$$F^{\rm IR}_{\mu e} = \frac{2k_1p_1}{(2k_1p)(2p_1p)} - \frac{2k_1p_2}{(2k_1p)(2p_2p)} - \frac{2k_2p_1}{(2k_2p)(2p_1p)} + \frac{2k_2p_2}{(2k_2p)(2p_2p)}. \tag{E.197}$$

We introduce the α parameterization with the aid of two new 4-vectors

$$k_{1\alpha} = k_1\alpha + p_1(1-\alpha), \tag{E.198}$$

$$k_{2\alpha} = k_2\alpha + p_2(1-\alpha), \tag{E.199}$$

resulting in

$$F^{\rm IR}_{\mu e} = \frac{S_1}{2k_1^0p^0(1-\beta_1\xi)} - \frac{S}{2k_2^0p^0(1-\beta_2\xi)} + \int\limits_0^1 d\alpha \left[\frac{S_1}{2(k_{2\alpha}p)^2} - \frac{S}{2(k_{1\alpha}p)^2}\right], \tag{E.200}$$

and $\delta^{\rm IR,soft}_{\mu e}$ becomes

$$\delta^{\rm IR,soft}_{\mu e} = \mathsf{Q}_\mu \mathsf{Q}_e \left[\left(P^{\rm IR} + \ln\frac{\varepsilon}{\mu}\right)\frac{1}{2}\int\limits_0^1 d\alpha \int\limits_{-1}^{+1} d\xi \mathcal{F}^{\rm IR}_{\mu e} + \frac{1}{4}\int\limits_0^1 d\alpha \int\limits_{-1}^{+1} d\xi \ln(1-\xi^2)\mathcal{F}^{\rm IR}_{\mu e}\right] \tag{E.201}$$

with

$$\mathcal{F}^{\rm IR}_{\mu e} = \frac{2}{1-\beta_1\xi} - \frac{2}{1-\beta_2\xi} + \frac{S_1}{(k^0_{2\alpha})^2(1-\beta_{2\alpha}\xi)^2} - \frac{S}{(k^0_{1\alpha})^2(1-\beta_{1\alpha}\xi)^2}. \tag{E.202}$$

Here

$$\begin{aligned}
-k^2_{1\alpha} &= m_\mu^2\alpha^2 + m_e^2(1-\alpha)^2 - 2k_1p_1\alpha(1-\alpha) \\
&= m_\mu^2\alpha^2 + m_e^2(1-\alpha)^2 + S\alpha(1-\alpha), \\
k^0_{1\alpha} &= \frac{S_1\alpha + (Sy+2m_e^2)(1-\alpha)}{2m_e},
\end{aligned} \tag{E.203}$$

$$\begin{aligned}
-k^2_{2\alpha} &= m_\mu^2\alpha^2 + m_e^2(1-\alpha)^2 - 2k_2p_1\alpha(1-\alpha) \\
&= m_\mu^2\alpha^2 + m_e^2(1-\alpha)^2 + S_1\alpha(1-\alpha), \\
k^0_{2\alpha} &= \frac{S\alpha + (Sy+2m_e^2)(1-\alpha)}{2m_e}.
\end{aligned} \tag{E.204}$$

$$\begin{aligned}
\beta_{1\alpha} &= \frac{|k_{1\alpha}|}{k^0_{1\alpha}} \\
&= \frac{\sqrt{\left[S_1\alpha + (Sy+2m_e^2)(1-\alpha)\right]^2 - 4m_e^2\left[m_\mu^2\alpha^2 + m_e^2(1-\alpha)^2 + S\alpha(1-\alpha)\right]}}{S_1\alpha + (Sy+2m_e^2)(1-\alpha)},
\end{aligned} \tag{E.205}$$

$$\begin{aligned}
\beta_{2\alpha} &= \frac{\left|\vec{k}_{1\alpha}\right|}{k^0_{1\alpha}} \\
&= \frac{\sqrt{\left[S\alpha + (Sy+2m_e^2)(1-\alpha)\right]^2 - 4m_e^2\left[m_\mu^2\alpha^2 + m_e^2(1-\alpha)^2 + S_1\alpha(1-\alpha)\right]}}{S\alpha + (Sy+2m_e^2)(1-\alpha)}.
\end{aligned} \tag{E.206}$$

Using the table of integrals from Appendix D.2 of [9] we derive

$$\delta_{\mu e}^{\rm IR,soft} = \mathtt{Q}_\mu \mathtt{Q}_e \Biggl\{ \left(P^{\rm IR} + \ln\frac{2\varepsilon}{\mu} \right) \left[\frac{1}{2\beta_1}\frac{1+\beta_1}{1-\beta_1} - \frac{1}{2\beta_2}\frac{1+\beta_2}{1-\beta_2} - S\int\limits_0^1 \frac{d\alpha}{-k_{1\alpha}^2} + S_1 \int\limits_0^1 \frac{d\alpha}{-k_{2\alpha}^2} \right]$$
$$-\frac{2}{\beta_2}\left[{\rm Li}_2\left(\frac{2\beta_2}{\beta_2-1}\right) - {\rm Li}_2\left(\frac{2\beta_2}{\beta_2+1}\right)\right] + \frac{2}{\beta_1}\left[{\rm Li}_2\left(\frac{2\beta_1}{\beta_1-1}\right) - {\rm Li}_2\left(\frac{2\beta_1}{\beta_1+1}\right)\right]$$
$$-\frac{S}{2}\int\limits_0^1 \frac{d\alpha}{-k_{1\alpha}^2\beta_{1\alpha}} \ln\frac{1-\beta_{1\alpha}}{1+\beta_{1\alpha}} + \frac{S_1}{2}\int\limits_0^1 \frac{d\alpha}{-k_{2\alpha}^2\beta_{2\alpha}} \ln\frac{1-\beta_{2\alpha}}{1+\beta_{2\alpha}} \Biggr\}. \qquad \text{(E.207)}$$

In the URA in m_e^2

$$S\int_0^1 \frac{d\alpha}{-k_{1\alpha}^2} \approx \ln\frac{S^2}{m_\mu^2 m_e^2}, \qquad \text{(E.208)}$$

$$S_1\int_0^1 \frac{d\alpha}{-k_{2\alpha}^2} \approx \ln\frac{S_1^2}{m_\mu^2 m_e^2}, \qquad \text{(E.209)}$$

$$\frac{S}{2}\int \frac{d\alpha}{-k_{1\alpha}^2\beta_{1\alpha}} \ln\frac{1-\beta_{1\alpha}}{1+\beta_{1\alpha}} \approx S_\Phi(S,S_1), \qquad \text{(E.210)}$$

$$\frac{S}{2}\int \frac{d\alpha}{-k_{2\alpha}^2\beta_{2\alpha}} \ln\frac{1-\beta_{2\alpha}}{1+\beta_{2\alpha}} \approx S_\Phi(S_1,S). \qquad \text{(E.211)}$$

Where we introduced the generic function $S_\Phi(I,\hat{I})$

$$S_\Phi(I,\hat{I}) = \frac{I}{2}\int\limits_0^1 \frac{d\alpha}{m_\mu^2\alpha^2 + m_e^2(1-\alpha)^2 + I\alpha(1-\alpha)} \ln\frac{m_e^2\left[m_\mu^2\alpha^2 + m_e^2(1-\alpha)^2 + I\alpha(1-\alpha)\right]}{\left[\hat{I}\alpha + (Sy+2m_e^2)(1-\alpha)\right]^2}$$
$$= \frac{1}{2}\ln\frac{m_e^2 I}{S^2y^2}\ln\frac{I^2}{m_e^2 m_\mu^2} - \frac{1}{4}\ln^2\frac{m_\mu^2}{I} - \frac{1}{4}\ln^2\frac{m_e^2}{I} - {\rm Li}_2\left(1-\frac{Sym_\mu^2}{I\hat{I}}\right)$$
$$+\ln\frac{\hat{I}}{Sy}\ln\frac{m_\mu^2}{I} - \frac{1}{2}\ln^2\frac{\hat{I}}{Sy}. \qquad \text{(E.212)}$$

The expression (E.207) reduces in the URA in m_e to

$$\delta_{\mu e}^{\rm IR,soft} = \left(P^{\rm IR} + \ln\frac{2\varepsilon}{\mu}\right) 4\ln(1-y) + 2\ln(1-y)\left(\ln\frac{m_e^2}{S} - \ln y\right). \qquad \text{(E.213)}$$

In the next Appendix on the box contribution we derive

$$\delta_{\mu e}^{\rm box} = \mathtt{Q}_e \mathtt{Q}_\mu \Biggl\{ \left[\left(-P^{\rm IR} - \frac{1}{2}\ln\frac{Q^2}{\mu^2} \right) 4\ln(1-y) \right] + B_{\mu e}^{\rm fin} \Biggr\}, \qquad \text{(E.214)}$$

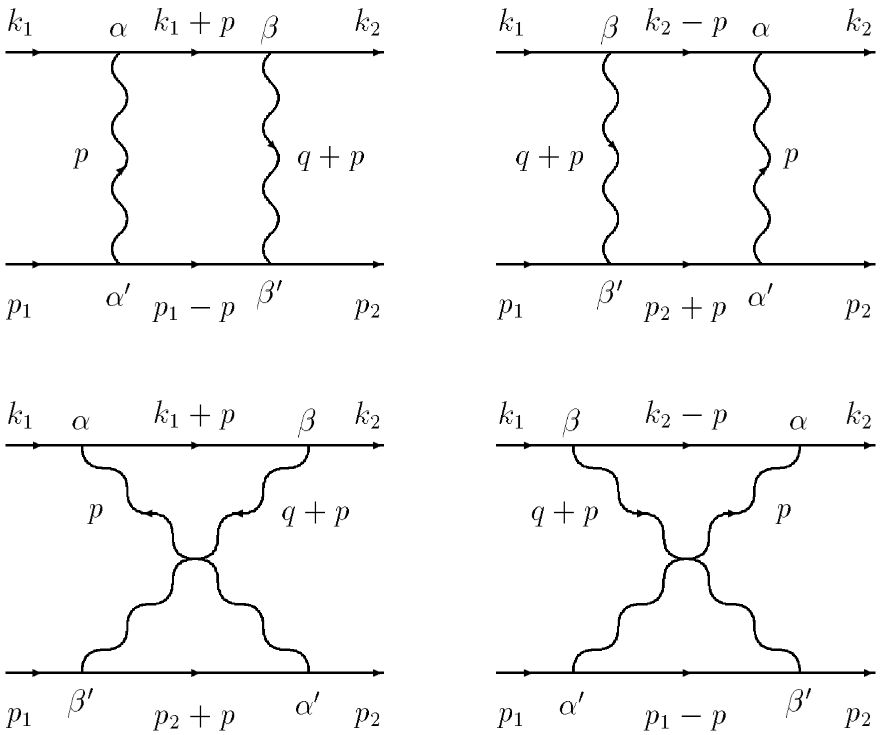
and the following short final expression is obtained

$$\delta_{\mu e}^{\rm VR} = \delta_{\mu e}^{\rm IR,soft} + \delta_{\mu e}^{\rm box} = \mathtt{Q}_e \mathtt{Q}_\mu \Biggl\{ 4\ln(1-y)\ln\frac{V_2^{\rm min}}{Sy} + B_{\mu e}^{\rm fin} \Biggr\}. \qquad \text{(E.215)}$$

The finite contribution from box diagram, $B_{\mu e}^{\rm fin}$, is presented in Appendix F.

F Two photon exchange contribution

The two-photon exchange contribution is described by the two box diagrams: *direct box* and *crossed box*. For the sake of symmetry, it is convenient to duplicate the number of diagrams and to deal with four diagrams shown in the figure below:



This is why the two-photon exchange matrix element is given by an expression, which contains an extra factor 1/2:

$$
\begin{aligned}
M^{\text{BOX}} = & \frac{1}{2}\int \frac{d^4p}{p^2(p+q)^2} \\
& \times \bar{u}(k_2)\left[\frac{\gamma_\beta(2k_{1\alpha}+\hat{p}\gamma_\alpha)}{p^2+2pk_1}+\frac{(2k_{2\alpha}-\gamma_\alpha\hat{p})\gamma_\beta}{p^2-2pk_2}\right]u(k_1) \\
& \times \bar{u}(p_2)\left[\frac{(2p_{2\alpha}+\gamma_\alpha\hat{p})\gamma_\beta}{p^2+2pp_2}+\frac{\gamma_\beta(2p_{1\alpha}-\hat{p}\gamma_\alpha)}{p^2-2pp_1}\right]u(p_1). \qquad \text{(F.216)}
\end{aligned}
$$

For later use we introduce the short hand notation for propagators

$$
\begin{aligned}
\prod(k_1) &= p^2+2pk_1, \\
\prod(k_2) &= p^2-2pk_2, \\
\prod(p_1) &= p^2-2pp_1, \\
\prod(p_2) &= p^2+2pp_2. \qquad \text{(F.217)}
\end{aligned}
$$

Now we separate the infrared divergent (IRD) part of the two-photon exchange contribution. There are two IRD-poles: 1) one is located at $p \to 0$ and 2) another one – at $(p+q) \to 0$.

At $p \to 0$, one has

$$\frac{1}{2}\frac{1}{p^2q^2}\left[\frac{4k_1p_2}{\prod(k_1)\prod(p_2)}+\frac{4k_1p_1}{\prod(k_1)\prod(p_1)}+\frac{4k_2p_2}{\prod(k_2)\prod(p_2)}+\frac{4k_2p_1}{\prod(k_2)\prod(p_1)}\right]\gamma_\beta\otimes\gamma_\beta, \quad (F.218)$$

While at $p \to q$, one has

$$\frac{1}{2}\frac{1}{q^2(p+q)^2}\left[\frac{4k_1p_2}{\prod(k_1)\prod(p_2)}+\frac{4k_1p_1}{\prod(k_1)\prod(p_1)}+\frac{4k_2p_2}{\prod(k_2)\prod(p_2)}+\frac{4k_2p_1}{\prod(k_2)\prod(p_1)}\right]\gamma_\beta\otimes\gamma_\beta. \quad (F.219)$$

For the latter term, we perform the substitution $p = p^{'} - q$ and observe that at this substitution the propagators transform as follows

$$\begin{aligned}\prod(k_1) &\longrightarrow \prod(-k_2),\\ \prod(k_2) &\longrightarrow \prod(-k_1),\\ \prod(p_1) &\longrightarrow \prod(-p_2),\\ \prod(p_2) &\longrightarrow \prod(-p_1).\end{aligned} \quad (F.220)$$

With one more substitution $p^{'} \to -p$, and using equalities $k_2.p_2 = k_1.p_1,\ k_2.p_1 = k_1.p_2$, we see that the second pole gives exactly the same contribution as the first one. Therefore, the full IRD-part of the two-photon exchange reads:

$$\frac{1}{q^2}\frac{1}{p^2}\left[\frac{4k_1p_2}{\prod(k_1)\prod(p_2)}+\frac{4k_1p_1}{\prod(k_1)\prod(p_1)}+\frac{4k_2p_2}{\prod(k_2)\prod(p_2)}+\frac{4k_2p_1}{\prod(k_2)\prod(p_1)}\right]\gamma_\beta\otimes\gamma_\beta. \quad (F.221)$$

Now we add and subtract the IRD-part of the two-photon exchange and when adding it we use (F.221), while when subtracting it we use the sum of (F.218) and (F.219). Using the identity

$$\frac{1}{p^2(p+q)^2}-\frac{1}{p^2q^2}-\frac{1}{q^2(p+q)^2}\equiv\frac{-2p.(p+q)}{q^2p^2(p+q)^2}, \quad (F.222)$$

we arrive at the final expression for the two-photon exchange contribution before integration over d^4p.

$$\begin{aligned}M^{\text{BOX}} \sim\ & \frac{4}{q^2p^2}\left[\frac{-S}{\prod(k_1)\prod(p_1)}+\frac{-S_1}{\prod(k_1)\prod(p_2)}\right]\gamma_\beta\otimes\gamma_\beta\\ &-\frac{4p(p+q)}{p^2(p+q)^2q^2}\left[\frac{-S}{\prod(k_1)\prod(p_1)}+\frac{-S_1}{\prod(k_1)\prod(p_2)}\right]\gamma_\beta\otimes\gamma_\beta\\ &+\frac{1}{p^2(p+q)^2}\left[\frac{k_{1\alpha}}{\prod(k_1)}+\frac{k_{2\alpha}}{\prod(k_2)}\right]\gamma_\beta\otimes\left[\frac{\gamma_\alpha\hat{p}\gamma_\beta}{\prod(p_2)}-\frac{\gamma_\beta\hat{p}\gamma_\alpha}{\prod(p_1)}\right]\\ &+\frac{1}{p^2(p+q)^2}\left[\frac{\gamma_\beta\hat{p}\gamma_\alpha}{\prod(k_1)}-\frac{\gamma_\alpha\hat{p}\gamma_\beta}{\prod(k_2)}\right]\otimes\gamma_\beta\left[\frac{p_{2\alpha}}{\prod(p_2)}+\frac{p_{1\alpha}}{\prod(p_1)}\right]\\ &+\frac{1}{2p^2(p+q)^2}\left[\frac{\gamma_\beta\hat{p}\gamma_\alpha}{\prod(k_1)}-\frac{\gamma_\alpha\hat{p}\gamma_\beta}{\prod(k_2)}\right]\otimes\left[\frac{\gamma_\alpha\hat{p}\gamma_\beta}{\prod(p_2)}-\frac{\gamma_\beta\hat{p}\gamma_\alpha}{\prod(p_1)}\right].\end{aligned} \quad (F.223)$$

The first raw of this formula represents the IRD-contribution, the second raw stands for the IR-free scalar, the next two represent the IR-free vector, and finally the last one is an IR-free tensor.

For the IR-divergent part we introduce the well-known presentation

$$\begin{aligned} J_3^{\mathrm{IR}}(k_1,p_1) &= \frac{16\pi^2}{i}\int\frac{d^4p}{(2\pi)^4p^2\prod(k_1)\prod(p_1)} \\ &= P_{\mathrm{IR}}(\mu)\int_0^1\frac{dy}{(-K_y^2)}+\frac{1}{2}\int_0^1\frac{dy}{(-K_y^2)}\ln\frac{(-K_y^2)}{q^2}+\mathcal{O}(n-4). \end{aligned} \quad (\text{F.224})$$

where

$$-K_y^2 = m_\mu^2y^2+m_e^2(1-y)^2-Sy(1-y). \quad (\text{F.225})$$

We will use the short hand notations

$$\begin{aligned} P_S^{\mathrm{IR}} &\equiv P^{\mathrm{IR}}(S) = -P_{\mathrm{IR}}(\mu)\int_0^1\frac{dy}{(-K_y^2)}, \\ K_S &\equiv K(S) = \int\frac{dy}{(-K_y^2)}\ln\frac{(-K_y^2)}{q^2}, \end{aligned} \quad (\text{F.226})$$

Then all needed IRD-integrals are

$$J_3^{\mathrm{IR}}(k_1,p_1) = J_3^{\mathrm{IR}}(k_2,p_2) = -P_S^{\mathrm{IR}}+\frac{K_S}{2}, \quad (\text{F.227})$$

$$J_3^{\mathrm{IR}}(k_1,p_2) = J_3^{\mathrm{IR}}(k_2,p_1) = P_{S_1}^{\mathrm{IR}}+\frac{K_{S_1}}{2}. \quad (\text{F.228})$$

with

$$\begin{aligned} P^{\mathrm{IR}}(S_1) &= -P^{\mathrm{IR}}(-S_1), \\ K_{S_1} &= K(-S_1). \end{aligned} \quad (\text{F.229})$$

For the IR-finite scalars

$$J_4^{\mathrm{IR}}(k_ip_j) = \frac{16\pi^2}{i}\int\frac{2p.(p+q)d^4p}{(2\pi)^4p^2(p+q)^2\prod(k_i)\prod(p_j)}, \quad (\text{F.230})$$

we derived

$$J_4(k_1,p_1) = J_4(k_2,p_2) = K_S, \quad (\text{F.231})$$

$$J_4(k_1,p_2) = J_4(k_2,p_1) = K_{S_1}, \quad (\text{F.232})$$

F.1 Vectorial integrals

We begin with the reduction of vector integrals. First we define the coefficients of the vector reduction

$$\begin{aligned} V_\mu^{ij} &= \frac{16\pi^2}{i}\int\frac{d^4p\;p_\mu}{(2\pi)^4p^2(p+q)^2\prod(k_i)\prod(p_j)} \\ &= (V_q)^{ij}q_\mu+(V_k)^{ij}(k_i)_\mu+(V_p)^{ij}(p_j)_\mu, \end{aligned} \quad (\text{F.233})$$

The system of linear equations for these coefficients in terms of scalar integrals, derived by contraction of (F.233) with q_μ, $(k_i)_\mu$, $(p_j)_\mu$, reads

$$
\begin{aligned}
&2q^2V_q^{ij} + 2S_{k_i}q\cdot k_iV_k^{ij} + 2S_{p_j}q\cdot p_jV_p^{ij} - J_4(k_i,p_j) + 2J_3^{\mathrm{IR}}(k_i,p_j) = 0\\
&2S_{k_i}q\cdot k_iV_q^{ij} + 2S_{k_i}{}^2k_i^2V_k^{ij} + 2S_{k_i}S_{p_j}k_i\cdot p_jV_p^{ij} - J_3(p_j) + J_3^{\mathrm{IR}}(k_i,p_j) = 0,\\
&2S_{p_j}q\cdot p_jV_q^{ij} + 2S_{k_i}S_{p_j}k_i\cdot p_jV_k^{ij} + 2S_{p_j}{}^2p_j^2V_p^{ij} - J_3(k_i) + J_3^{\mathrm{IR}}(k_i,p_j) = 0. \quad \text{(F.234)}
\end{aligned}
$$

Here

$$
\begin{aligned}
S_{k_1} &= -1, \qquad S_{k_2} = +1,\\
S_{p_1} &= +1, \qquad S_{p_2} = -1. \quad \text{(F.235)}
\end{aligned}
$$

Two additional types of scalar integrals

$$
J_3\left(k_i\right) = \frac{16\pi^2}{i}\int\frac{d^4p}{(2\pi)^4p^2(p+q)^2\prod(k_i)}, \quad \text{(F.236)}
$$

and

$$
J_3\left(p_j\right) = \frac{16\pi^2}{i}\int\frac{d^4p}{(2\pi)^4p^2(p+q)^2\prod(p_j)}, \quad \text{(F.237)}
$$

are introduced. They safisfy the equalities

$$
J_3\left(k_1\right) = J_3\left(k_2\right) \quad \text{(F.238)}
$$

and

$$
J_3\left(p_1\right) = J_3\left(p_2\right), \quad \text{(F.239)}
$$

and may be described by one generic formula:

$$
J_3\left(q_i\right) = \int_0^1\frac{dy}{-q^2y - q_i^2(1-y) + (q-q_i)^2y(1-y)}\ln\frac{(q-q_i)^2y(1-y) - q_i^2(1-y)}{q^2y}. \quad \text{(F.240)}
$$

with $q_i = (k_1,\ -k_2,\ p_2,\ -p_1)$.

The solution of the system (F.234) is presented in subsection F.4 of this Appendix.

F.2 Tensorial integrals

Define the coefficients of tensor reduction

$$
\begin{aligned}
T_{\mu\nu}^{ij} &= \frac{16\pi^2}{i}\int\frac{d^4p\ p_\mu p_\nu}{(2\pi)^4p^2(p+q)^2\prod(k_i)\prod(p_j)}\\
&= T_0\delta_{\mu\nu} + T_{qq}^{ij}q_\mu q_\nu + T_{kk}^{ij}(k_i)_\mu(k_j)_\nu + T_{qp}^{ij}(p_i)_\mu(p_j)_\nu\\
&+T_{qk}^{ij}\left[q_\mu(k_i)_\nu + q_\mu(k_i)_\mu\right] + T_{qp}^{ij}\left[q_\mu(p_j)_\nu + q_\mu(p_j)_\mu\right]\\
&+T_{kp}^{ij}\left[(k_i)_\mu(p_j)_\nu + (k_i)_\nu(p_j)_\mu\right]. \quad \text{(F.241)}
\end{aligned}
$$

The system of equations for the coefficients T^{ij} was derived in the following way:

- the first equation was received by multiplying (F.241) with $\delta_{\mu\nu}$

$$J_3^{\rm IR}(k_i p_j) = 4T_0^{ij} + q^2 T_{qq}^{ij} + (k_i)^2 T_{kk}^{ij} + (p_j)^2 T_{pp}^{ij} + q^2 T_{qk}^{ij} + q^2 T_{qp}^{ij} + I_{ij} T_{kp}^{ij}, \qquad \text{(F.242)}$$

- the next three equations were received by multiplying (F.241) with q_ν

$$\begin{aligned}
&\int \frac{d^4p\,[2pq + q^2 + p^2 - q^2 - p^2]\,p_\mu}{(2\pi)^4 p^2 (p+q)^2 \prod(k_i) \prod(p_j)} \\
&= \int \frac{d^4p}{(2\pi^4)} \frac{p^\mu}{p^2 \prod(k_i) \prod(p_j)} - \int \frac{d^4p}{(2\pi^4)} \frac{p^\mu}{(p+q)^2 \prod(k_i) \prod(p_j)} \\
&-q^2 \int \frac{d^4p}{(2\pi^4)} \frac{p^\mu}{p^2 (p+q)^2 \prod(k_i) \prod(p_j)} \\
&= 2q_\mu T_0^{ij} + 2q^2 q_\mu T_{qq}^{ij} + q^2 T_{kk}^{ij} (k_i)_\mu + q^2 T_{pp}^{ij} (p_j)_\mu \\
&+T_{qk}^{ij} \left[q_\mu q^2 + 2q^2 (k_i)_\mu\right] + T_{qp}^{ij} \left[q_\mu q^2 + 2q^2 (p_j)_\mu\right] \\
&+q^2 T_{kp}^{ij} \left[(k_i)_\mu + (p_j)_\mu\right], \qquad \text{(F.243)}
\end{aligned}$$

- another six equations were received by multiplying (F.241) with $(k_i)_\nu$ and $(p_j)_\nu$.

The latter nine equations are

$$\begin{aligned}
&q^2 T_{kk}^{ij} + 2q^2 T_{qk}^{ij} + q^2 T_{kp}^{ij} + q^2 V_k(k_i, p_j) = 0 \\
&2k_i^2 T_{kk}^{ij} + q^2 T_{qk}^{ij} + 2S_{k_i} S_{p_j} k_i \cdot p_j T_{kp}^{ij} + V_{1k}(k_i, p_j) + q^2 V_q(k_i, p_j) + J_3^{\rm IR}(k_i, p_j) = 0 \\
&2S_{k_i} S_{p_j} k_i \cdot p_j T_{kk}^{ij} + q^2 T_{qk}^{ij} + 2p_j^2 T_{kp}^{ij} - V_{2k}(k_i, p_j) + V_{1k}(k_i, p_j) = 0 \\
&q^2 T_{pp}^{ij} + 2q^2 T_{qp}^{ij} + q^2 T_{kp}^{ij} + q^2 V_p(k_i, p_j) = 0 \\
&2S_{k_i} S_{p_j} k_i \cdot p_j T_{pp}^{ij} + q^2 T_{qp}^{ij} + 2k_i^2 T_{kp}^{ij} - V_{3p}(k_i, p_j) + V_{1p}(k_i, p_j) = 0 \\
&2p_j \cdot p_j T_{pp}^{ij} + q^2 T_{qp}^{ij} + 2S_{k_i} S_{p_j} k_i \cdot p_j T_{kp}^{ij} + V_{1p}(k_i, p_j) + J_3{}^{IR}(k_i, p_j) + q^2 V_q(k_i, p_j) = 0 \\
&2q^2 T_{qq}^{ij} + q^2 T_{qk}^{ij} + q^2 T_{qp}^{ij} - V_{1k}(k_i, p_j) - V_{1p}(k_i, p_j) + 2q^2 V_q(k_i, p_j) = 0 \\
&q^2 T_{qq}^{ij} + 2k_i^2 T_{qk}^{ij} + 2S_{k_i} S_{p_j} k_i \cdot p_j T_{qp}^{ij} \\
&-V_{3q}(k_i, p_j) - V_{1k}(k_i, p_j) - V_{1p}(k_i, p_j) - J_3{}^{\rm IR}(k_i, p_j) = 0 \\
&q^2 T_{qq}^{ij} + 2S_{k_i} S_{p_j} k_i \cdot p_j T_{qk}^{ij} + 2p_j{}^2 T_{qp}^{ij} \\
&-V_{2q}(k_i, p_j) - V_{1k}(k_i, p_j) - V_{1p}(k_i, p_j) - J_3^{\rm IR}(k_i, p_j) = 0 \qquad \text{(F.244)}
\end{aligned}$$

Together with the tenth equation (F.242), they can be solved and the answer may be expressed in terms V^{ij} and additional vectors arising from various pinches of the box diagrams.

F.3 The box contribution to the cross-section

Substituting all these solutions to the box amplitude and computing its interference with Born amplitude, we derive the contributions to the unpolarized and polarized cross-section parths from the two-photon exchange diagrams to the differential cross-section of elastic μe scattering

$$\frac{d\sigma_{\rm SIURA}^{\rm BOX}}{dy} \left[-4P^{IR}(\mu) \ln(1-y) \frac{d\sigma^{\rm BORN}}{dy} + \mathcal{B}_{unp} + P_e P_\mu \mathcal{B}_{pol}\right], \qquad \text{(F.245)}$$

where unpolarized and polarized non-factorized contributions $\mathcal{B}_{unp}$ and $\mathcal{B}_{pol}$ are:

$$\begin{aligned}
\mathcal{B}_{unp} &= \left(1-\frac{2}{y}+\frac{2R_\mu}{y}\right)(SK_S+S_1K_{S_1})+4\left[-1+\frac{2}{y}-2R_\mu\left(1+\frac{1}{y_{\mu_1}}\right)\right] \\
&+2\left(\frac{y_{\mu_1}}{y}-2R_\mu\right)l_S+2\left(\frac{1}{y}-\frac{R_\mu}{y_{\mu_1}}\right)l_{S_1} \\
&+4R_\mu\left[-1-\frac{1}{y_{\mu_1}}+\frac{2(2-y)}{y(y+4R_\mu)}\right]\ln\frac{Q^2}{m_\mu^2}+2\left(-1+\frac{2}{y}\right)S(K_S-yJ_P) \\
&+2\left[-2+y-4R_\mu\left(\frac{y_{\mu_1}}{y}+\frac{1+2R_\mu}{y+4R_\mu}\right)\right]SJ_K, \\
\mathcal{B}_{pol} &= \left(-1+\frac{2}{y}-2R_\mu\right)(SK_S+S_1K_{S_1}) \\
&+4\left[-1+2R_\mu\left(-1+\frac{2}{y}+\frac{1-R_\mu}{y_{\mu_1}}-R_\mu\right)\right] \\
&+2\left[\frac{y_{\mu_1}}{y}(1+2R_\mu)-2R_\mu^2\right]l_S+2\left[\frac{2R_\mu(1-R_\mu)}{y_{\mu_1}}-\frac{1-2R_\mu}{y}\right]l_{S_1} \\
&+4R_\mu\left(\frac{2}{y}+\frac{1-R_\mu}{y_{\mu_1}}-\frac{2(1+2R_\mu)}{y+4R_\mu}-R_\mu\right)\ln\frac{Q^2}{m_\mu^2} \\
&+4\frac{R_\mu y(1+2R_\mu)}{y+4R_\mu}SJ_K, \qquad \text{(F.246)}
\end{aligned}$$

with

$$\begin{aligned}
R_\mu &= \frac{m_\mu^2}{S}, \\
Y &= \frac{1}{1+R_\mu}. \qquad \text{(F.247)}
\end{aligned}$$

There are two useful combinations:

$$S_1K_{S_1}+SK_S = -\ln\frac{Q^4}{SS_1}\ln\frac{S_1}{S}-2\mathrm{Li}_2\left(1-\frac{m_\mu^2}{S_1}\right)+2\mathrm{Li}_2\left(1+\frac{m_\mu^2}{S}\right)+\pi^2, \qquad \text{(F.248)}$$

$$S(K_S-yJ_P) = -\ln^2\frac{Q^2}{S}+\frac{1}{2}\ln^2\frac{Q^2}{m_\mu^2}+2\mathrm{Li}_2\left(1+\frac{m_\mu^2}{S}\right)+\frac{\pi^2}{3}. \qquad \text{(F.249)}$$

Here

$$J_K = J(m_\mu^2) = \frac{1}{\sqrt{\lambda_m^0}}\left[\ln\frac{m_\mu^2}{Q^2}\ln y_{\mu_2}-2\mathrm{Li}_2(1-y_{\mu_2})-\frac{1}{2}\ln^2 y_{\mu_2}+\pi^2\right], \qquad \text{(F.250)}$$

$$J_P = J(m_e^2). \qquad \text{(F.251)}$$

and

$$y_{\mu_1} = \frac{Q^2+2m_\mu^2+\sqrt{\lambda_m^0}}{2m_\mu^2}, \qquad \text{(F.252)}$$

$$y_{\mu_2} = \frac{1}{y_{\mu_1}}. \qquad \text{(F.253)}$$

The two-photon exchange contribution simplifies drastically in the DOURA. We list the answer because of its elegance.

$$\frac{d\sigma^{\text{BOX}}_{\text{DOURA}}}{dy} = \frac{\alpha^3}{S}\mathtt{Q}_\mu\mathtt{Q}_e\Biggl\{-P^{IR}(\mu)\left[\frac{2}{y^2}+(1+P_eP_\mu)\left(1-\frac{2}{y}\right)\right]8\ln(1-y)$$
$$+(1-P_eP_\mu)\Biggl[\left(1-\frac{2}{y}\right)\left(\ln^2(1-y)-2\ln(1-y)\ln y+\pi^2\right)$$
$$+\frac{2}{y}\ln(1-y)-2\ln y\Biggr]$$
$$+2\left(1-\frac{2}{y}\right)\ln^2(1-y)-4\frac{1-y}{y}\ln(1-y)\Biggr\}. \qquad \text{(F.254)}$$

G Description of www-Figures

In March 1996, when the `FORTRAN` code μela was completed, a lot of figures for the radiative corrections to the polarized cross-sections and the polarization asymmetry were produced and put to a home-page of the theory group of DESY-Zeuthen, `http://www.ifh.de/~bardin`.

In this Appendix, we give a short guide to these figures.

All the cross-sections, radiative corrections and asymmetries are shown as function of y_μ. The units are: cross-sections in microbarns, radiative corrections in percent, asymmetries in absolute, dimensionless units varying from 0 to 1.
Some legends:

- parallel, antiparallel means mutual longitudinal polarization orientations;
- P_e and P_μ are modules of polarization degrees, always written in figures;
- **All corrections** mean that all 12 contributions (1.4) are taken into account;
- **AN** means Analytic calculations;
- when **AN** is not written, the numerical calculations are meant.

The δ_{y_μ} is defined by

$$\delta_{y_\mu} = \frac{\dfrac{d\sigma^{\text{QED}}}{dy_\mu}}{\dfrac{d\sigma^{\text{BORN}}}{dy_\mu}} - 1,\ (\%). \qquad \text{(G.255)}$$

SIURA-series of 43 figures contain following plots:

- 1-3 Born cross-section and Born asymmetry;
- 4-5 Results of completely integrated analytic calculations without cuts;
- 6-7 Results of completely numerical calculations without cuts;

- 8-17 Illustrate effects of some cuts separately, muon angular cut being treated both analytically and numerically;

- 18-19 are our main result: All corrections, All cuts (four indeed);

- 20-27 illustrate four cases: $\mu + e$ corrections, μ corrections, e corrections and μe interference corrections, correspondingly, without cuts;

- 28-35 the same, but with all cuts;

- 36-43 is a series of figures with 'realistic' P_e and P_μ. It is quite senceless for the asymmetry, where P_e and P_μ cancel. We only 'gain' an instability due to cancellation of nearly equal numbers.

References

[1] SMC collaboration, D. Adams et al., Phys. Lett. **B396** (1997) 338; Phys. Rev. **D56** (1997) 5330.

[2] F. Fainstein, Note SMC/95/04 and references therein.

[3] A.I. Nikischov, Sov. J. Exp. Theor. Phys. Lett. **9** (1960) 757.

[4] P. van Nieuwenhuizen, Nucl. Phys. **B28** (1971) 429.

[5] D. Bardin and N. Shumeiko, Nucl.Phys. **B127** (1977) 242.

[6] T.V. Kukhto, N.M. Shumeiko and S.I. Timoshin, J. Phys. G: Nucl. Phys. **13** (1987) 725.

[7] D. Bardin and L. Kalinovskaya, μela, version 1.00, March 1996. The source code is available from `http://www.ifh.de/~bardin`.

[8] S.M. Bilenky, Introduction to the Physics of Electroweak Interaction (Pergamon Press, 1982).

[9] A. Akhundov, D. Bardin, L. Kalinovskaya, T. Riemann, Fortschritte d. Physik **44** (1996) 5, 373–482.

[10] A. Arbuzov at al, `hep-ph9602262`.

[11] A. Arbuzov, D. Bardin, A. Leike, Modern Physics Letters A, Vol. 7, No. 22 (1992) 2029-2038.

[12] A. Arbuzov, D. Bardin, J. Blümlein, L. Kalinovskaya, T. Riemann, Comput. Phys. Comm. **94** (1996) 128-184.

[13] S. Eidelman and F. Jegerlehner, Z. Physik **C67** (1995) 585;
F. Jegerlehner, Fortran subroutine `hadr5`.