Desy 97-005

INTERACTING QUANTUM FIELDS IN CURVED SPACE: RENORMALIZABILITY OF φ^4 [1]

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Abstract. We present a perturbative construction of the φ^4 model on a smooth globally hyperbolic curved space-time. Our method relies on an adaptation of the Epstein and Glaser method of renormalization to curved space-times using techniques from microlocal analysis.

1. Introduction

Renormalization led to a well defined perturbation expansion of quantum field theory whose lowest order terms are in excellent agreement with experimental particle physics [1]. First, in the late 40's, quantum electrodynamic was renormalized by the method of Schwinger, Feynman, Tomonaga and Dyson, leading to truly remarkable predictions, e.g., on the magnetic moment of the electron. In the seventies, the renormalization program was extended to non-abelian gauge theories by the works of Faddeev-Popov, 't Hooft, Becchi-Rouet-Stora and others, and led to the present standard model of elementary particle physics. Attempts to include also gravity in the renormalization program failed; more recent proposals look for theories of a different kind like string theory which is hoped to describe all known forces.

Because of the large difference between the Planck scale (10^{-33}cm) and scales relevant for the present standard model ($10^{-5}-10^{-17}$cm) a reasonable approximation should be to consider gravity as a classical background field and therefore investigate quantum field theory on curved space-times. This Ansatz already led to interesting results, the most famous being the Hawking radiation of black-holes [2]. But a look through the literature (see, e.g., [3]) shows that predominantly free field theories were treated on curved backgrounds. To our knowledge, e.g., there is no serious attempt to discuss the influence of interaction on the Hawking effect. Most of the papers on interacting quantum

[1]To be published on the proceedings of the conference "Operator Algebras and Quantum Field Theory", held at Accademia Nazionale dei Lincei, Rome, Italy, July 1996.

field theory on curved spacetime deal with the Euclidean case and discuss the renormalization of certain diagrams. There seems to exist only one attempt to a general proof of renormalizability for $\lambda\varphi^4$, that given by Bunch [4]. However, also his attempt is confined to the rather special case of real analytic space-times which can be analytically continued to the Euclidean situation. It is interesting to note that the main technical tool of this paper is a kind of local Fourier transformation which is a particular case of the powerful techniques that we use in this paper.

The situation is then tricky for general (smooth) space-times with the (physical) Lorentzian signature. Here more or less *nothing* was done.

Why is the problem of renormalization so difficult on curved spacetime? The main problem is absence of translation invariance. So there is no notion of a vacuum, which is a central object in most treatments of quantum field theory; the spectrum condition (positivity of the energy operator), responsible for deep theorems like the Spin-Statistics Theorem, cannot be formulated. There is no general connection between the Riemannian and Lorentzian field theory, and the meaning of the functional integral for quantum field theory on curved spacetime is unclear. On the more technical side, a momentum space description is not possible, so the BPHZ method of renormalization [5] is not directly applicable. Also the popular method of dimensional renormalization seems to be restricted to the Euclidean situation [6].

On the other hand, physically motivated by the Equivalence Principle, a quick look at the possible ultraviolet divergences indicates that they are of the same nature as in flat space, so no obstruction for renormalization on curved spacetime is visible. Despite of the interest in its own right, renormalization on curved spacetime might also trigger a conceptual revisitation of renormalization theory on flat space in the light of the principle of locality.

We sketch on this paper only the main ideas. The complete proofs should be found in [15].

2. The Epstein and Glaser Method

A direct application of BPHZ or dimensional renormalization seems not to be possible for curved spacetime with Lorentzian signature. But there exists another general method developed by Epstein and Glaser [7] (also [8]) on the basis of ideas generally attributed to Bogoliubov, Stückelberg and their collaborators (see [9] and references therein). This method is local in spirit and is therefore our favorite candidate for renormalization on curved spacetime. A closer inspection shows that also in this method translation invariance plays an important role, both conceptually and technically, and it will require a lot of work to replace translation invariance by other structures. In the past there has been an attempt on Minkowski space for quantum electrodynamics with external time independent electro-magnetic fields done by Dosch and Müller [10]. This use of the Hadamard parametrix of the Dirac operator is already much in the spirit of a local formulation of perturbation theory; by the assumption of time independence of the external fields, however, translation invariance w.r.t. time still plays a crucial role in their approach. At this point one might get the impression that a combination of techniques from their paper and that of Bunch (see above) will provide a useful purely local perturbation theory. As a matter of fact, it will turn out that techniques from *microlocal analysis* [11] are ideally suited to carry through the program.

Let us describe the general strategy on the example of the φ^4 theory on a $d = 4$ dimensional globally hyperbolic space-time $(\mathcal{M}, g)$. We start from a quasi-free state ω of a free massive field φ, satisfying the Klein-Gordon equation of motion

$$(\Box_g + m^2)\varphi = 0$$

where $\Box_g$ is the d'Alembertian operator w.r.t. the metric g and where ω is supposed to satisfy the Hadamard condition [12]

$$\omega(\varphi(x)\varphi(y)) = \frac{u}{\sigma} + v \log \sigma + w$$

with $\sigma(x, y)$ denoting the square of the geodesic distance between x and y and u, v, w being smooth functions where u, v are determined only by geometry. The commutator of the field is

$$[\varphi(x), \varphi(y)] = iE(x, y)$$

where $E(x, y) = E_{ret}(x, y) - E_{adv}(x, y)$, and E_{ret}, resp. E_{adv} are retarded resp. advanced Green functions which are uniquely defined on globally hyperbolic space-times.

As it is known one defines the "S-matrix" by a formal power series in the "coupling constant"' which in the Epstein and Glaser scheme is a fixed test-function on space-time

$$S(\lambda) \equiv \mathbf{1} + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int T_n(x_1, \ldots, x_n)\lambda(x_1) \cdots \lambda(x_n) \; d\mu_1 \cdots d\mu_n$$

where $\lambda \in \mathcal{D}(\mathcal{M})$ and $d\mu$ is the natural invariant volume measure on $\mathcal{M}$ w.r.t. the fixed Lorentzian metric g. We remark that this definition is purely local thanks to the introduction of the space-time "coupling constant" λ. Eventually, this test function should be sent to a fixed value over all space-time, but this amounts to treat the infrared nature of the theory to which this paper is not addressed.

In the Epstein and Glaser scheme the natural objects to use for constructing the theory are the time-ordered products

$$T_n^{k_1,\ldots,k_n}(x_1, \ldots, x_n), \qquad k_i \leq 4, \qquad n \in \mathbb{N}$$

which are operator-valued distributions on the GNS-Hilbert space induced by ω. T_n^k is interpreted as the time-ordered product of the Wick's monomials $:\varphi^{k_1}(x_1):, \ldots, :\varphi^{k_n}(x_n):$. It is characterized by the following properties;

(P1) $T_1^k(x) =: \varphi^k(x):$,

(P2) $T_n^{k_1,\ldots,k_n}(x_1, \ldots, x_n)$, is symmetric under permutations of indices.

Crucial is the following causality property;

(P3) If none of the points $x_1, \ldots, x_l$ $(1 \leq l \leq n)$ lies in the past of the points $x_{l+1}, \ldots, x_n$, then the time-ordered product factorizes,

$$T_n^{k_1,\ldots,k_n}(x_1, \ldots, x_n) = T_l^{k_1,\ldots,k_l}(x_1, \ldots, x_l)\; T_{n-l}^{k_{l+1},\ldots,k_n}(x_{l+1}, \ldots, x_n).$$

3. Wick's Polynomials and Theorem 0

In the Epstein and Glaser scheme one requires in addition translation covariance and proves then that the T_n's have an expansion into Wick's products

(P4) For any integer n it holds

$$T_n^{k_1,\dots,k_n}(x_1,\dots,x_n) = \sum t_n^{l_1,\dots,l_n}(x_1,\dots,x_n) :\varphi^{k_1-l_1}(x_1)\cdots\varphi^{k_n-l_n}(x_n):$$

where now, the t_n's are translation invariant numerical distributions. It is crucial for the program that the Wick polynomials are operator-valued distributions and that they can be multiplied with translation invariant numerical distributions. This is the content of Epstein and Glaser Theorem 0.

On curved spacetime the first step to do is to prove the existence of Wick products as operator-valued distributions (even this was, to our knowledge, not done previous to our work [13]). Our construction relies on the finding of Radzikowski [14] that the wavefront set[2] of the two-point function of a Hadamard state is

$$\mathrm{WF}(\omega_2) = \{(x,k;x',-k') \in T^*\mathcal{M}^2 \setminus \{0\} \mid (x,k) \sim (x',k'),\ k \in \overline{V}_+\}$$

where the equivalence relation $\sim$ means that there exists a light-like geodesic from x to x' such that k is coparallel to the tangent vector to the geodesic and k' is its parallel transport from x_1 to x_2. The proof then uses Hörmander's Theorem [11] that distributions can be pointwise multiplied provided the convex combinations of their wave front sets do not meet elements of the zero section. In our case the convexity of the forward light-cone is crucial.

The time-ordered two-point function E_F (Feynman propagator) arising from ω_2 is given by

$$iE_F(x_1,x_2) = \omega_2(x_1,x_2) + E_{ret}(x_1,x_2).$$

It has wave front set as

$$\begin{aligned}&\mathrm{WF}(E_F) = \\ &\{(x,k;x',-k') \in T^*M^2 \setminus \{0\} \mid (x,k) \sim (x',k'), x \neq x', k \in \overline{V}_\pm \text{ if } x \in \mathcal{J}_\pm(x')\} \\ &\qquad \cup \{(x,k;x,-k), x \in \mathcal{M}, k \in T^*_x\mathcal{M} \setminus \{0\}\}\end{aligned}$$

where $\mathcal{J}_\pm(x')$ are the future/past, respectively, of x'.

The next step amounts to replace the EG axiom of translation invariance by something else. We therefore assume, as an Ansatz, the expansion (P4), but need some restriction on the distributions t_n which replaces the notion of translation invariance.

We impose, instead of translation invariance, a condition on the wavefront set which should be satisfied by time-ordered functions. We require

(P5) It holds

$$\mathrm{WF}(t_n) \subset \Gamma_n^{to}$$

[2]The wavefront set of a distribution f on $\mathcal{M}$ is defined as $\mathrm{WF}(f) = \{(x,k) \in T^*\mathcal{M}\setminus\{0\} | \phi \in \mathcal{D}(\mathcal{M}), \phi(x) \neq 0, \text{cone } C \ni k \Rightarrow \widehat{\phi f} \text{ does not decay rapidly in } C\}$. Hence, it is a closed conic set in $T^*\mathcal{M} \setminus \{0\}$.

where

$$\Gamma_n^{to} = \{(x_1, k_1; \ldots; x_n, k_n) \in T^*\mathcal{M}^n \setminus \{0\} \quad | \quad \exists \quad \text{a graph } G \text{ with vertices } \{1, \ldots, n\}, \text{ and an association of lines } l \text{ from vertex } i = s(l) \text{ to } j = r(l) \text{ of } G \text{ to future oriented lightlike geodesics } \gamma_l \text{ which connect } x_i \text{ and } x_j \text{ and a covariantly constant covector field } k_l \in \overline{V}_+ \text{ on } \gamma_l \text{ coparallel to } \dot{\gamma}_l \text{ such that } k_i = \sum_{m:s(m)=i} k_m(x_i) - \sum_{n:r(n)=i} k_n(x_i)\}$$

This may be motivated by the fact that for non coinciding points t_n can be expressed in terms of the usual Feynman graphs and for the set of coinciding points we have an infinitesimal remnant of translation invariance.

One then proves [15]: If $\mathrm{WF}(t_n) \subset \Gamma_n^{to}$ then

$$t_n(x_1, \ldots, x_n) :\varphi^{k_1}(x_1) \cdots \varphi^{k_n}(x_n):$$

is a well-defined operator-valued distribution (microlocal version of Theorem 0 of Epstein and Glaser).

4. Algebraic formulation of the Epstein and Glaser approach.

In the algebraic formulation of quantum field theory the basic object is a net of algebras

$$\mathcal{O} \to \mathcal{A}(\mathcal{O})$$

where $\mathcal{O}$ is a relatively compact region in $\mathcal{M}$ and $\mathcal{A}(\mathcal{O})$ is a von Neumann algebra of observables localized in $\mathcal{O}$ which in typical cases is known to be a hyperfinite type III_1 factor. The basic hypotheses the net has to satisfy are

A1. Isotony. If $\mathcal{O}_1 \subset \mathcal{O}_2$ then

$$\mathcal{A}(\mathcal{O}_1) \to \mathcal{A}(\mathcal{O}_2).$$

where $i_{\mathcal{O}_1,\mathcal{O}_2}$ is an injective unital homomorphism for which

$$i_{\mathcal{O}_3,\mathcal{O}_2} \circ i_{\mathcal{O}_2,\mathcal{O}_1} = i_{\mathcal{O}_3,\mathcal{O}_1} \qquad \text{if } \mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{O}_3.$$

A2. Locality. If $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{O}_3$ and $\mathcal{O}_1$ is spacelike separated from $\mathcal{O}_2$ then

$$i_{\mathcal{O}_3,\mathcal{O}_1}(\mathcal{A}(\mathcal{O}_1)) \subset i_{\mathcal{O}_3,\mathcal{O}_2}(\mathcal{A}(\mathcal{O}_2))'$$

where $\mathcal{A}'$ means the commutant.

One then proceeds to construct the C*-inductive limit of the net $\mathrm{C}^*(\mathcal{A})$. In the case of free field theories, the net construction can be made explicitely. In the interacting case, there exist constructions only in the very special case of two dimensional Minkowski spacetime mainly due to Glimm and Jaffe [20].

It seems to be less well-known that the Bogoliubov-Stückelberg method of S-matrices as functionals of spacetime dependent sources actually directly leads to a definition of local nets for interacting theories (unfortunately, at present, only in perturbation theory). Namely, let $\underline{g}, \underline{h}$ be finite families of test functions on $\mathcal{M}$, coupled to the various elements of the Borchers class of the free field, and consider the relative S-matrices

$$V(\underline{g}, \underline{h}) = S(\underline{g})^{-1} S(\underline{g} + \underline{h}).$$

From (P3) one finds the causality relation

Causality. $V(\underline{g}, \underline{h}_1 + \underline{h}_2) = V(\underline{g}, \underline{h}_1)V(\underline{g}, \underline{h}_2)$ whenever there are no future directed causal curves from $\operatorname{supp}\underline{h}_1$ to $\operatorname{supp}\underline{h}_2$.

The physical interpretation of this property is that given an interaction described by $\underline{g}$ the time evolution operator in the interaction picture w.r.t. any additional interaction has the usual factorization property. One of the main corollaries to this condition is that it implies locality. Indeed, if $\operatorname{supp}\underline{h}_1$ is spacelike separated from $\operatorname{supp}\underline{h}_2$ then this is equivalent to say that no causal curves connect the two and hence

$$V(\underline{g}, \underline{h}_1 + \underline{h}_2) = V(\underline{g}, \underline{h}_1)V(\underline{g}, \underline{h}_2) = V(\underline{g}, \underline{h}_2)V(\underline{g}, \underline{h}_1).$$

The local algebras of the interacting theory are now defined by

$$\mathcal{A}_{\underline{g}}(\mathcal{O}) \equiv \{V(\underline{g}, \underline{h}),\ \operatorname{supp}\underline{h} \subset \mathcal{O}\}'', \qquad \underline{g} \in \mathcal{D}(\mathcal{O}).$$

We want to extend the definition of interacting nets $\mathcal{A}_{\underline{g}}$ to $\mathcal{C}^\infty$-functions $\underline{g}$ with not necessarily compact support, e.g., for $\underline{g} = \mathit{constant}$. It is gratifying that this can be done without any infrared problem.

Let $\mathcal{O} \subset \mathcal{M}$ be open and relatively compact. We define the restriction of the net $\mathcal{A}_{\underline{g}}$ to $\mathcal{O}$ to be isomorphic to the net $\mathcal{A}_{\underline{g}\phi}$ where $\phi \in \mathcal{D}(\mathcal{M})$ with $\phi \equiv 1$ on a neighbourhood of $\mathcal{J}_-(\mathcal{O}) \cap \mathcal{J}_+(\mathcal{O})$. Since the net is, up to isomorphy, uniquely defined by its restriction to relatively compact open subsets of $\mathcal{M}$, we only have to show that $\mathcal{A}_{\underline{g}\phi} \upharpoonright_{\mathcal{O}}$ does not depend on the choice of ϕ. Indeed, let $\phi' \in \mathcal{D}(\mathcal{M})$ with $\phi' \equiv 1$ on a neighbourhood of $\mathcal{J}_-(\mathcal{O}) \cap \mathcal{J}_+(\mathcal{O})$. Then there exist $\phi_\pm \in \mathcal{D}(\mathcal{M})$ with $\phi' = \phi + \phi_- + \phi_+$ such that $\mathcal{J}_+(\operatorname{supp}\phi_+) \cap \mathcal{J}_-(\mathcal{O}) = \emptyset$ and $\mathcal{J}_-(\operatorname{supp}\phi_-) \cap \mathcal{J}_+(\mathcal{O}) = \emptyset$. Let $\underline{h} \in \mathcal{D}(\mathcal{O})$. Then, by the definition of V,

$$\begin{aligned} V(\underline{g}\phi', \underline{h}) &= V(\underline{g}(\phi + \phi_-), \underline{g}\phi_+)^{-1} V(\underline{g}(\phi + \phi_-), \underline{g}\phi_+ + \underline{h}) \\ &= V(\underline{g}(\phi + \phi_-), \underline{h}) \end{aligned}$$

where the last equality follows from causality. Hence the operator $V(\underline{g}\phi', \underline{h})$ does not depend on the interaction in the future of $\operatorname{supp}\underline{h}$. It depends, however, on the interaction in the past of $\operatorname{supp}\underline{h}$,

$$\begin{aligned} V(\underline{g}(\phi + \phi_-), \underline{h}) &= V(\underline{g}\phi, \underline{g}\phi_-)^{-1} V(\underline{g}\phi, \underline{g}\phi_- + \underline{h}) \\ &= \operatorname{Ad} V(\underline{g}\phi, \underline{g}\phi_-)^{-1}(V(\underline{g}\phi, \underline{h})) \end{aligned}$$

where again we used the definition of V and causality. But the dependence is through a unitary transformation which does not depend on $\underline{h} \in \mathcal{D}(\mathcal{O})$. Hence the nets $\mathcal{A}_{\underline{g}\phi} \upharpoonright_{\mathcal{O}}$ and $\mathcal{A}_{\underline{g}\phi'} \upharpoonright_{\mathcal{O}}$ are unitarily equivalent.

The interacting fields can be defined through the Bogoliubov formula

$$\varphi_{\underline{g}}(\underline{h}) = \frac{d}{d\lambda} V(\underline{g}, \lambda\underline{h}) \upharpoonright_{\lambda=0} .$$

They may be considered as operators which are affiliated to the local algebras $\mathcal{A}_{\underline{g}}(\mathcal{O})$. By the uniqueness above their local properties do not depend on the behaviour of $\underline{g}$ outside $\mathcal{O}$. In particular, one may expect that the wave front sets of their n-point functions for a generic class of states can be determined locally.

5. Inductive Construction up to the Diagonal

After these preparations we can mimick the argument of Epstein and Glaser (see also [16]) to construct T_n on $\mathcal{M}^n \setminus \Delta_n$, where Δ_n is the total diagonal in $\mathcal{M}^n$, provided T_l has been constructed for all $l < n$ and satisfies the causality condition (P3).

Let J be the set of all $\emptyset \neq I \subsetneq \{1, \ldots, n\}$. Let $\mathcal{C}_I = \{(x_1, \ldots, x_n) \in \mathcal{M}^n, x_i \notin \mathcal{J}_-(x_j), i \in I, j \in I^c\}$. On a globally hyperbolic space-time

$$\cup_I \mathcal{C}_I = \mathcal{M}^n \setminus \Delta_n.$$

In fact, if $x_i \neq x_j$ for some $i \neq j$, the points x_i and x_j can be separated by a Cauchy surface $\mathcal{S}$, containing none of the points x_k $k = 1, \ldots, n$, hence $I = \{k, x_k \in \mathcal{J}_+(\mathcal{S})\} \in \mathcal{J}$, and $(x_1, \ldots, x_n) \in \mathcal{C}_I$.

We set on $\mathcal{C}_I$

$$T_I^{k_1,\ldots,k_n}(x_1, \ldots, x_n) = T_{|I|}^{k_i, i \in I}(x_i, i \in I)\ T_{|I^c|}^{k_j, j \in I^c}(x_j, j \in I^c).$$

According to the induction hypothesis and the microlocal Theorem 0, this is a well-defined operator-valued distribution on $\mathcal{D}(\mathcal{C}_I)$. Different $\mathcal{C}_I$'s may overlap but one can show [15] that, due to the causality (P3) hypothesis valid for the lower order terms, for any $x \in \mathcal{C}_{I_1} \cap \mathcal{C}_{I_2}$ we have $T_{I_1}(x) = T_{I_2}(x)$.

Let now $\{f_I\}_{I \in J}$ be a smooth partition of unity of $\mathcal{M}^n \setminus \Delta_n$ subordinate to $\{\mathcal{C}_I\}_{I \in J}$. Then we define

$${}^0T_n = \sum_{I \in J} f_I\ T_I$$

as an operator-valued distribution on $\mathcal{M}^n \setminus \Delta_n$.

We convince ourselves that ${}^0T_n$ is independent of the choice of the partition of unity and symmetric under permutations of the arguments: Namely, let $\{f'_I\}_{I \in J}$ be another partition of unity. Let $x \in \mathcal{M}^n \setminus \Delta_n$, and let $\mathcal{K} = \{I \in J, x \in \mathcal{C}_I\}$. Then there exists a neighbourhood V of x such that $V \subset \cap_{I \in \mathcal{K}} \mathcal{C}_I$, and supp f_I and supp f'_I do not meet V for all $I \notin \mathcal{K}$. Then

$$\sum_{I \in J} (f_I - f'_I)\ T_I \restriction_V = \sum_{I \in \mathcal{K}} (f_I - f'_I)\ T_I \restriction_V .$$

But on V, T_I is independent of the choice of $I \in \mathcal{K}$. Since $\sum_{I \in \mathcal{K}} f_I = \sum_{I \in \mathcal{K}} f'_I = 1$ on V, we arrive at the conclusion. To prove symmetry we just observe that the permuted distribution ${}^0T_n^\pi(x_1 \ldots, x_n) = {}^0T_n(x_{\pi(1)}, \ldots, x_{\pi(n)})$ has the expansion

$${}^0T_n^\pi = \sum_{I \in J} f_I^\pi T_I^\pi = \sum_{I \in J} f_{\pi(I)}^\pi\ T_{\pi(I)}^\pi$$

where we used the fact that the set J is invariant under permutations, but $T_{\pi(I)}^\pi = T_I$ and $\{f_{\pi(I)}^\pi\}_{I \in J}$ is a partition of unity subordinate to $\{\mathcal{C}_I\}_{I \in J}$, so symmetry follows from the previous result on the independence of ${}^0T_n$ on the choice of the partition of unity.

6. The Microlocal Scaling Degree and the Extension of Distributions

We now want to extend ${}^0T_n$ to the whole $\mathcal{D}(\mathcal{M}^n)$. For this purpose we use the expansion of ${}^0T_n$ into Wick polynomials

$$ {}^0T_n^{k_1,\dots,k_n} = \sum {}^0t_n^{l_1,\dots,l_n} \ :\varphi^{k_1-l_1}\otimes\cdots\otimes\varphi^{k_n-l_n}: $$

where ${}^0t_n \in \mathcal{D}'(\mathcal{M}^n\setminus\Delta_n)$ and $\mathrm{WF}(\chi\ {}^0t_n)\subset\Gamma_n^{to}$ for any choice of the l_i and any smooth function $\chi\in\mathcal{C}^\infty(\mathcal{M}^n)$ such that $\operatorname{supp}\chi\subset\mathcal{M}^n\setminus\Delta_n$.

The extension to the diagonal of the tensor products of Wick's monomials proceeds as in our last paper [13]. Everything is therefore reduced to the extension of the numerical distributions ${}^0t_n$ which is performed in two steps. First ${}^0t_n$ is extended by continuity to the subspace of test-functions which vanish on Δ_n up to a certain order, and then a general test-function is projected into this subspace. It is this last step which corresponds to the method of counterterms in the classical procedure of perturbative renormalization. The extension of ${}^0t_n$ by continuity requires some topology on test-function space. The seminorms used by Epstein and Glaser in their paper are quite complicated, and their generalization to curved space-times appears to be rather involved. We found it preferable therefore to apply a different method already introduced by Steinmann [17], namely the concept of the scaling degree at a point of a time-ordered distribution. Its generalization to curved spacetime is very similar to the concept of the scaling limit as introduced by Haag-Narnhofer-Stein [18] and further developed by Fredenhagen and Haag [19]. Actually, what is really needed to implement correctly the inductive procedure is a more general concept than the scaling degree at a point. The requirement that the renormalized time-ordered distribution t_n should have wave front set in Γ_n^{to} drives us to consider a concept of scaling degree w.r.t. the diagonal Δ_n. This is in order to get more "uniformity" compared to the pointwise case. This uniformity may be seen as a kind of translation invariance over the diagonal. For simplicity, we first deal with the pointwise case and afterwards we comment on the necessary changes for the general one.

Let then $\mathcal{M}$ be a smooth manifold of dimension d and $x\in\mathcal{M}$. Choose a diffeomorphism α from some convex bounded neighbourhood V of the origin in $T_x\mathcal{M}$ onto some neighbourhood U of x such that $\alpha(x,0)=x$ and $d\alpha_0$ is the natural identification of the tangent space on the origin of $T_x\mathcal{M}$ with $T_x\mathcal{M}$ itself (one may take the exponential map for definiteness).

Let $f\in\mathcal{D}'(U)$. We define the scaled distribution $f_\lambda\in\mathcal{D}'(U)$ by

$$ f_\lambda\circ\alpha(\xi) = f\circ\alpha(\lambda\xi) \qquad \xi\in V, 1\geq\lambda\geq 0. $$

Note that f_λ is well defined on $U=\alpha(V)$ since by assumption $\lambda V\subset V$ for $0<\lambda\leq 1$. In case $f\in\mathcal{D}'(U\setminus\{x\})$ we use the above definition with $\xi\neq 0$ and obtain $f_\lambda\in\mathcal{D}'(U\setminus\{x\})$.

We say that f has scaling degree $\omega\in\mathbb{R}$ at x if ω is the smallest number such that $\forall\omega'>\omega$

$$ \lim_{\lambda\downarrow 0}\lambda^{\omega'} f_\lambda(\phi) = 0 $$

in the sense of distributions. For our analysis we need a somewhat stronger version of the scaling degree which controls also the wavefront sets of the distributions f_λ.

1. Definition. *$f \in \mathcal{D}'(U)$ has at x the* microlocal scaling degree *ω w.r.t. a closed cone $\Gamma_x \subset T^*_xU \setminus \{0\}$ if*

(*i*) *there exists a closed conic set $\Gamma \subset T^*U \setminus \{0\}$ with $\Gamma \cap T^*_xU \subset \Gamma_x$ such that $\mathrm{WF}(f)_\lambda \subset \Gamma$ for sufficiently small λ.*

(*ii*) *ω is the smallest number such that for all $\omega' > \omega$*

$$\lim_{\lambda \downarrow 0} \lambda^{\omega'} f_\lambda = 0$$

in the sense of the Hörmander pseudotopology on $\mathcal{D}'_\Gamma(U)$.

We recall that by the Hörmander pseudotopology it is meant the following [11]; given a sequence of distributions $u_i \in \mathcal{D}'_\Gamma(\mathcal{M}) \equiv \{v \in \mathcal{D}(\mathcal{M}) \mid \mathrm{WF}(v) \subset \Gamma\}$, we say that the sequence converges to u in the sense of Hörmander pseudotopology in $\mathcal{D}'_\Gamma(\mathcal{M})$ whenever the following two properties hold true:

(a) $u_i \to u$ weakly,

(b) for any properly supported pseudodifferential operator A such that $\mathrm{WF}(A) \cap \Gamma = \emptyset$, we have that $Au_i \to Au$ in the sense of $\mathcal{C}^\infty(\mathcal{M})$

where $\mathrm{WF}(A)$ is defined by the projection in $T^*\mathcal{M}$ of the wave front set of the Schwartz kernel associated to A.

For $f \in \mathcal{D}'(U \setminus \{x\})$ we cannot directly define the microlocal scaling degree. We require instead that for all $\chi \in \mathcal{C}^\infty(U)$ with $x \notin \operatorname{supp}\chi$ the sequence χf_λ (considered as a distribution on U) satisfies the two conditions of the Definition above.

The microlocal scaling degree (μsd) has nice properties. For f_1 and f_2 with μsd ω_1 and resp. ω_2 at x, w.r.t. Γ^1_x resp. Γ^2_x such that $\{0\} \notin (\Gamma^1_x + \Gamma^2_x)$, the wave front sets of $f_{1,\lambda}$ and $f_{2,\lambda}$ for sufficiently small λ satisfy the condition $(\mathrm{WF}(f_{1,\lambda}) \oplus \mathrm{WF}(f_{2,\lambda})) \cap \{0\} = \emptyset$, hence their product exists [11] and, because of the sequential continuity of the products in the Hörmander pseudotopology, have μsd $\omega \leq \omega_1 + \omega_2$ w.r.t. $\Gamma_x = \Gamma^1_x \cup \Gamma^2_x \cup (\Gamma^1_x + \Gamma^2_x)$.

We now want to show how to extend the distribution $f \in \mathcal{D}'(U \setminus \{x\})$ to all space. We first deal with the extension problem using the scaling degree. The next section contains the proof of the extension problem w.r.t. the μsd. There are two possible cases: when the scaling degree $\omega \geq d$ or otherwise $\omega < d$. We first study the second case.

2. Theorem. *If $f_0 \in \mathcal{D}'(U \setminus \{x\})$ has scaling degree $\omega < d$ at x then there exists a unique $f \in \mathcal{D}'(U)$ with $f(\phi) = f_0(\phi)$, $\phi \in \mathcal{D}(U \setminus \{x\})$ and the same scaling degree.*

Proof. Let $\vartheta \in \mathcal{D}(U)$ with $\vartheta \equiv 1$ on a neighbourhood of x. We define for $0 < \lambda < 1$

$$\vartheta_{\lambda^{-1}}(x) = \begin{cases} \vartheta(\alpha(\lambda^{-1}\alpha^{-1}(x))), & x \in \alpha(\lambda V) \\ 0, & \text{else.} \end{cases}$$

Then $1 - \vartheta_{\lambda^{-1}} \in \mathcal{C}^\infty(U)$ with $x \notin \operatorname{supp}(1 - \vartheta_{\lambda^{-1}})$. We want to define f as the limit

$$f \equiv \lim_{n \to \infty} f_0(1 - \vartheta_{2^n}).$$

Let $\phi \in \mathcal{D}(U)$. Then $f_0(1-\vartheta_{2^n})(\phi)$ is a Cauchy sequence. Namely, for $n > m$

$$\begin{aligned} f_0(1-\vartheta_{2^n})(\phi) - f_0(1-\vartheta_{2^m})(\phi) &= f_0(\vartheta_{2^m} - \vartheta_{2^n})(\phi) \\ &= \sum_{j=m}^{n-1} (\phi f_0)(\vartheta_{2^j} - \vartheta_{2^{(j+1)}}) \\ &= \sum_{j=m}^{n-1} (\phi f_0)_{2^{-j}}(\vartheta - \vartheta_2) 2^{-jd} \end{aligned}$$

where we used the definition of the scaled distribution as well as an identification of densities and functions by the use of a measure $d\mu$ with $\alpha^* d\mu = d\xi$, the latter denoting the Lebesgue measure on $T_x\mathcal{M}$.

As a smooth function ϕ has scaling degree equal to 0. Hence $\forall \omega' > \omega$ there is a constant c such that

$$|(\phi f_0)_{2^{-j}}(\vartheta - \vartheta_2)| \leq c\ 2^{j\omega'}.$$

We insert this estimate for $\omega < \omega' < d$ and obtain the desired result.

It remains to prove that f has the same scaling degree as f_0. Let $\omega < \omega' < d$. Let $\phi \in \mathcal{D}(U)$. Then

$$(1-\vartheta_{2^n})\phi_{\lambda^{-1}} = 0$$

if $n < n_\lambda$ for some $n_\lambda \in \mathbb{N}$ with $2^{-n_\lambda}\lambda^{-1} \to_{\lambda\to 0}$ const. $\neq 0$. Hence

$$\begin{aligned} \lambda^{\omega'} f_\lambda(\phi) &= \lambda^{(\omega'-d)} f(\phi_{\lambda^{-1}}) \\ &= \lim_{n\to\infty} \lambda^{(\omega'-d)} (f_0(1-\vartheta_{2^n}))(\phi_{\lambda^{-1}}) \\ &= \sum_{n \geq n_\lambda} \lambda^{(\omega'-d)} f_0((\vartheta_{2^n} - \vartheta_{2(n+1)})\phi_{\lambda^{-1}}) \\ &= \sum_{n \geq n_\lambda} \lambda^{(\omega'-d)} 2^{-nd} (f_0)_{2^{-n}}((\vartheta - \vartheta_2)\phi_{\lambda^{-1}2^{-n}}). \end{aligned}$$

The test-function has all Schwartz norms uniformly bounded in λ and $n \geq n_\lambda$. Hence

$$|(f_0)_{2^{-n}}((\vartheta - \vartheta_2)\phi_{\lambda^{-1}2^{-n}})| \leq c\ 2^{n\omega''}$$

for some $\omega < \omega'' < \omega'$. But then

$$\begin{aligned} |\lambda^{\omega'} f_\lambda(\phi)| &\leq \sum_{n \geq n_\lambda} \lambda^{(\omega'-d)} 2^{-n(d-\omega'')} \\ &\leq \lambda^{(\omega'-d)} \frac{2^{-n(d-\omega'')}}{1-2^{-(d-\omega'')}} \\ &\leq \lambda^{(\omega'-\omega'')} \frac{(\lambda^{-1}2^{-n_\lambda})^{(d-\omega'')}}{1-2^{-(d-\omega'')}} \to 0. \end{aligned}$$

The uniqueness is obvious since any other extension differs by a derivative of the delta function based at the point x which has scaling degree $\geq d$. ■

For the case when f_0 has scaling degree $\omega \geq d$ we deal here only with the preliminary step of extension on a subspace of test-functions $\mathcal{D}_\delta(U) \subset \mathcal{D}(U)$ whose derivatives at the point x vanish up to order $\delta = [\omega] - d$.

3. Theorem. *Let $f_0 \in \mathcal{D}'(U \setminus \{x\})$ have scaling degree $\omega \geq d$. Then the sequence $f_0((1-\vartheta_{2^n})\phi)$ with $\phi \in \mathcal{D}_\delta(U)$, $\delta = [\omega] - d$, converges and the limit defines a unique distribution $\overline{f}(\phi) \equiv \lim_{n\to\infty} f_0((1-\vartheta_{2^n})\phi)$ over $\mathcal{D}_\delta(U)$.*

Proof. (Sketch) The proof goes similar to the one of Theorem 2. The only change is that now ϕ has scaling degree $\leq -\delta - 1$ and the estimate would change as follows

$$|(f_0\phi)(\vartheta_{2^m} - \vartheta_{2^n})| \leq c \sum_{j=m}^{n-1} 2^{j(\omega'-\delta-1-d)}$$

hence, choosing ω' such that the exponent is negative we get the convergence. That $\overline{f}$ is a distribution follows from the Banach-Steinhaus Theorem applied to $\mathcal{D}_\delta(U)$ which is a closed subset of $\mathcal{D}(U)$. ■

In the application of this procedure to the n-th order of perturbation theory we want to scale only in the difference variables. On a curved space-time this might be done in the following way. We choose a map $\alpha : T\mathcal{M} \to \mathcal{M}$ such that $\alpha(x,0) = x$ and $d\alpha(x,\,\cdot\,)\restriction_0 = \mathrm{id}$ (for instance, the exponential map). We then define $\alpha_n : T\mathcal{M}^n \restriction_\Delta \to \mathcal{M}^n$ by

$$\alpha_n(x,\xi_1,\ldots,\xi_n) = (\alpha(x,\xi_1),\ldots,\alpha(x,\xi_n)).$$

We restrict α_n to the following sub-bundle which is isomorphic to the normal bundle of Δ_n

$$N\Delta_n = \{(x,\xi_1,\ldots,\xi_n) \in T\mathcal{M}^n \restriction_{\Delta_n}, \sum \xi_i = 0\}.$$

For a sufficiently small neighbourhood of the zero section in $N\Delta_n$ α_n restricted to it is a diffeomorphism onto some neighbourhood of Δ_n. We now express our time-ordered function as a distribution on $N\Delta_n$ and do the scaling w.r.t. the variables ξ. There is however a complication with this procedure; namely, in the inductive construction, the coordinates so obtained do not factorize, hence it is not obvious how the microlocal scaling degree of lower order terms determines the microlocal scaling degree at higher order. Much easier is the behaviour of the total scaling degree, w.r.t. all variables. Here it is easy to see that the scaling degree of the factors determines the scaling degree for tensor and pointwise products. We therefore prove a Lemma which states that the condition on the wave front set of time-ordered distributions implies that they can be restricted to the submanifolds

$$\mathcal{M}_x = \{\alpha_n(x,\xi_1,\ldots,\xi_n), \sum \xi_i = 0\}.$$

Again the result on the continuity of restrictions in the Hörmander pseudotopology gives us the desired information on the microlocal scaling degree.

4. Lemma. *Let $\mathrm{WF}({}^0t_n) \subset \Gamma_n^{to}$. Then $\forall x \in \mathcal{M}$ there exists $\phi \in \mathcal{D}(\mathcal{M}^n)$ with $\phi(x,\ldots,x) \neq 0$, such that $\phi^0 t_n$ can be restricted to $\mathcal{M}_x$.*

Proof. It suffices to show that the wavefront set of ${}^0t_n$ in a neighbourhood of $(x,\ldots,x)$ does not intersect the conormal bundle of the submanifold $\mathcal{M}_x \subset$

$\mathcal{M}^n$. But at the point $(x, \ldots, x)$ the elements $(x, k_1, \ldots, k_n)$ of the wavefront set, with $(k_1, \ldots, k_n) \neq 0$, satisfy $\sum k_i = 0$, hence the equation

$$\sum \langle k_i, \xi_i \rangle = 0$$

which characterizes a point in the conormal bundle of $\mathcal{M}_x$, cannot hold for all $(\xi_1, \ldots, \xi_n)$ with $\sum \xi_i = 0$.

Now the wavefront set intersected with the conormal bundle of $\mathcal{M}_x$ is a closed conic subset of $T^*\mathcal{M}^n$ which does not contain $T^*_{(x,\ldots,x)}\mathcal{M}^n$, hence does not contain also a conic neighbourhood of $T^*_{(x,\ldots,x)}\mathcal{M}^n$, but such a neighbourhood always contains a set T^*U where U is a neighbourhood of $(x, \ldots, x)$. If we choose ϕ with support in U we arrive at the desired conclusion. ∎

We now want to impose some condition on the smoothness of the above construction w.r.t. x which serves as a substitute for translation invariance. Let t be a distribution from $\mathcal{D}'(\mathcal{M}^n)$ whose wave front set is orthogonal to the tangent bundle of the diagonal, i.e.,

$$\langle \xi, k \rangle = 0, \quad x \in \Delta_n, \quad \xi \in T_x\Delta_n, \quad (x, k) \in \mathrm{WF}(t).$$

We set

$$\tilde{t}_\lambda(x, \alpha_n(x, \xi)) = t(\alpha_n(x, \lambda\xi)) \qquad (x, \xi) \in T\mathcal{M}^n \restriction_{\Delta_n} .$$

$\tilde{t}_\lambda \equiv (1 \otimes t)_\lambda$ is a distribution on a neighbourhood $\tilde{U}$ of the diagonal Δ_{n+1} in $\mathcal{M}^{n+1}$. We say that t has μsd ω at Δ_n if there is a closed conic set $\tilde{\Gamma} \subset T^*\tilde{U}$ with

$$\langle \xi, k \rangle = 0, \quad \forall \xi \in T_x\Delta_{n+1}, \quad (x, k) \in \tilde{\Gamma}$$

such that

(*i*) $\mathrm{WF}(\tilde{t}_\lambda) \subset \tilde{\Gamma}$, for all sufficiently small λ,

(*ii*) ω is the smallest number such that for all $\omega' > \omega$

$$\lambda^{\omega'} \tilde{t}_\lambda \to 0$$

in the sense of $\mathcal{D}'_{\tilde{\Gamma}}(\tilde{U})$.

We then restrict $\tilde{t}_\lambda$ to the submanifold $\cup_{x \in \mathcal{M}}(\{x\} \times \mathcal{M}_x) \subset \mathcal{M}^{n+1}$. This is possible by the argument in the proof of Lemma 4. Note that the submanifold $\cup_{x \in \mathcal{M}}(\{x\} \times \mathcal{M}_x)$ might be identified with the open set $U \equiv \cup_{x \in \mathcal{M}}\mathcal{M}_x \subset \mathcal{M}^n$.

Let t_λ denote the restriction of $\tilde{t}_\lambda$ to $\mathcal{D}(U)$. By the sequential continuity of the restriction operator we find that

$$\lambda^{\omega'} t_\lambda \to 0 \qquad \omega' > \omega$$

in the sense of $\mathcal{D}'_\Gamma(U)$ as $\lambda \to 0$.

We can now calculate the microlocal scaling degree of time-ordered distributions, given by pointwise and/or tensor products ot lower order time-ordered distributions.

For the Feynman propagator we find that the μsd is $d-2$ at every point of the diagonal, w.r.t. the $\mathrm{WF}(\Delta_F)$ on flat space.

If we assume that t_n^l has μsd ω_n^l at the diagonal in $\mathcal{M}^n$ w.r.t. Γ_n^{to} then

$$t_{I,L}^{l_1,l_2} = (t_{|I|}^{l_1} \otimes t_{|I^c|}^{l_2}) \prod_L \Delta_F(x_i, x_j)$$

has μsd at the diagonal equal to $\omega_{|I|}^{l_1} + \omega_{|I^c|}^{l_2} + (d-2)|L|$ w.r.t. Γ_n^{to}. Hence the μsd of ${}^0t_n^l$ is determined by the μsd at lower orders. We now have to study whether ${}^0t_n^l$ can be extended to all $\mathcal{D}(\mathcal{M}^n)$ such that the μsd is conserved.

7. Extension to the Diagonal

In the last Section we saw that the time-ordered functions ${}^0t_n \equiv {}^0t$, originally defined only on $\mathcal{D}(\mathcal{M}^n \setminus \Delta_n)$, can be extended to $\mathcal{D}(\mathcal{M}^n)$ or $\mathcal{D}_\delta(\mathcal{M}^n)$, where $\delta = [\omega] - (n-1)d$, whenever the μsd ω, which is computed in terms of the μsd of the time-ordered functions at lower orders, satisfies either $\omega < (n-1)d$ or $\omega \geq (n-1)d$. Note that the presence of the term $(n-1)d$ is related to our choice of the relative coordinates. We now want to remove the restriction in the second case by simply projecting arbitrary test-functions onto $\mathcal{D}_\delta(\mathcal{M}^n)$. It is this last step which corresponds in other renormalization schemes to the subtraction of infinite counterterms.

Here we do the projection in the following way. We choose a function w which is equal to 1 on a neighbourhood of Δ_n and with support in range(α_n), where the map α_n of the last Section is used in order to introduce relative coordinates. We set

$$(W\phi)(x_1,\ldots,x_n) = \phi(x_1,\ldots,x_n) - w(x_1,\ldots,x_n) \sum_{|\beta|\leq\delta} \frac{\xi^\beta}{\beta!} \partial_\xi^\beta (\phi \circ \alpha_n)(x, \xi = 0)$$

with $\alpha_n(x,\xi) = (\alpha(x,\xi_1),\ldots,\alpha(x,\xi_n))$, $\xi = (\xi_1,\ldots,\xi_n)$ and the usual multi-index notations for ξ^β and ∂_ξ^β and define, following the Theorems 2 and 3, $t(\phi) \equiv \overline{{}^0t}(W\phi)$.

If we would apply ${}^0t$ to the single terms in the definition of $W\phi$ the first term would correspond to the divergence and the second one to the counterterm. This can be made explicit by choosing a sequence of smooth functions ${}^kt$ on $\mathcal{M}^n$ which converges to t in the sense of distributions. Then

$$t = \lim_{k\to\infty} \left({}^kt - \sum_\beta \langle {}^kt, w\frac{\xi^\beta}{\beta!}\rangle \, \delta^{(\beta)}(\xi) \right).$$

Let us now reconsider the extension problem. If ${}^0t \in \mathcal{D}'(\mathcal{M}^n \setminus \Delta_n)$ we say that it has at Δ_n μsd ω w.r.t. Γ_n^{to} if, for all $\chi \in \mathcal{C}^\infty(\mathcal{M}^{n+1})$ with $(\mathcal{M}\times\Delta_n)\cap \operatorname{supp}\chi = \emptyset$, the two following properties hold true

(i) $\mathrm{WF}(\chi(1\otimes {}^0t)_\lambda) \subset \tilde{\Gamma}_n^{to} = \{(x,0;y,k), x\in\mathcal{M}, (y,k)\in\Gamma_n^{to}\}$,

(ii) $\lambda^{\omega'}\chi(1\otimes {}^0t)_\lambda \to 0$ in $\mathcal{D}'_{\tilde{\Gamma}_n^{to}}$.

Choosing $\vartheta \in \mathcal{C}^\infty(\mathcal{M}^n)$, $\vartheta \equiv 1$ on a neighbourhood of Δ_n with $\operatorname{supp}\vartheta \cap \mathcal{M}_x$ a compact set for any $x \in \Delta_n$, we recall that the extensions of ${}^0t$ are obtained by $t \equiv \lim_{n\to\infty}(1-\vartheta_{2^n})^0 t$ whenever $\omega < (n-1)d$ or by $t \equiv \lim_{n\to\infty}(1-\vartheta_{2^n})^0 t \circ W$ whenever $\omega \geq (n-1)d$, where $\vartheta_{2^n}(\alpha_n(x,\xi)) = \vartheta(\alpha_n(x, 2^n\xi))$ with $(x,\xi) \in N\Delta_n$.

We can prove that the sequences converge in $\mathcal{D}'_{\Gamma_n^{to}}(U)$ and keep the same μsd at Δ_n of their respective ${}^0t$'s. We recall that $U \equiv \cup_{x\in\mathcal{M}}\mathcal{M}_x$. The convergence in the sense of distributions is given by Theorems 2 and 3, the one in the smooth sense, after application of a suitable pseudodifferential operator, will be done in the following way. Note that we consider the proof only in the case when $\omega \geq (n-1)d$ the proof of the other trivially follows by simply choosing W equal to the identity operator.

5. Theorem. *If $W : \mathcal{D}(U) \to \mathcal{D}_\delta(U)$ is the restriction to U of the map W defined above, and $\delta = [\omega] - (n-1)d$, the expression*

$$ {}^0t(1-\vartheta_{2^m}) \circ W $$

converges to t as $m \to \infty$ in the Hörmander pseudotopology for $\mathcal{D}'_{\Gamma_n^{to}}(\mathcal{M}^n)$.

Proof. Since in Section 6 we already proved the convergence in the sense of distributions we need only to check convergence in the smooth sense of the sequence of smooth functions $AW^t\ {}^0t(1-\vartheta_{2^m})$ for the appropriate pseudodifferential operators A (see remark after Definition 1.). Note that the functions $1-\vartheta_{2^m}$ are such that their product with ${}^0t$ gives a distribution in $\mathcal{M}^n$. The argument for proving the convergence in the smooth sense goes similar to the proof of the convergence in distribution sense.

For a pseudodifferential operator with smooth kernel the result follows from the convergence in the sense of distributions. By subtracting from A an operator with smooth kernel, if necessary, we may assume that the kernel of A has support in a sufficiently small neighbourhood of the diagonal in $\mathcal{M}^n \times \mathcal{M}^n$. Since for $\phi \in \mathcal{D}(\mathcal{M}^n \setminus \Delta_n)$ we have

$$ {}^0t(1-\vartheta_{2^m})(W\phi) = {}^0t(\phi) $$

we may restrict the consideration to a sufficiently small neighbourhood of Δ_n.

In terms of local coordinates on $\mathcal{M}^n$, A may be described, in a neighbourhood of a point on the diagonal, in terms of its symbol

$$ (Au)(x,\xi) = \int\int \sigma_A(x,\xi;p,k)\hat{u}(p,k)dpdk $$

where σ_A is fast decreasing in a conical neighbourhood of $\Gamma_n^{to} \cap T^*U$. Since Γ_n^{to} contains the conormal bundle of Δ_n, the symbol σ_A decays fast in a cone around the point $p = 0$,

$$ |\sigma_A(x,\xi;p,k)| \leq c_N(1+|p|+|k|)^{-N} $$

where the constants c_N are independent of $(x,\xi) \in U$.

We now want to apply A to the distribution ${}^0t(1-\vartheta_{2^m})\circ W$. Using the Lagrange formula for the rest term in the Taylor expansion, we find for the action $u \to u\circ W$ the expression

$$u\circ W = u(1-w) + Pu$$

where the first term vanishes near the diagonal and may therefore be ignored, and where in local coordinates (x,ξ), the Fourier transformation of the second term is

$$\widehat{Pu}(p,k) = \int_0^1 dt \sum_{|\beta|=\delta+1} \widehat{uw\xi^\beta}(p,tk)k^\beta \frac{(1-t)^\delta}{\delta!}.$$

Therefore we obtain for sufficiently large m

$$|(A^0t(\vartheta_{2^m}-\vartheta_{2^{m+1}})\circ W)(x,\xi)| \le \text{const. } 2^{-m(n-1)d} \sum_\beta \int |\sigma_A(x,\xi;p,k)|$$
$$\times\, |\widehat{({}^0tw\xi^\beta)_{2^{-m}}(\vartheta-\vartheta_2)}(p,2^{-m}tk)||k|^\beta \; dp\; dk\; dt.$$

Because of the assumption on the μsd of ${}^0t$ we have the estimate

$$|\widehat{({}^0tw\xi^\beta)_{2^{-m}}(\vartheta-\vartheta_2)}(p,2^{-m}tk)| \le c_N\; (1+|p|+|2^{-m}tk|)^{-N}2^{m(\omega'-\delta-1)}$$

for every closed cone which does not contain $p=0$.

Since σ_A and $\widehat{({}^0tw\xi^\beta)_{2^{-m}}(\vartheta-\vartheta_2)}$ are polynomially bounded we conclude that the integrand is fast decreasing and we get

$$|(A^0t_n(\vartheta_{2^m}-\vartheta_{2^{m+1}})\circ W)(x,\xi)| \le \text{const. } 2^{m(\omega'-\delta-1-(n-1)d)}.$$

Since $\delta = [\omega]-(n-1)d$ then the exponent becomes equal to $\omega'-[\omega]-1$ which for a choice of a sufficiently small ω' is negative, hence the thesis follows. ■

It remains to compute the μsd of $t_n^l \equiv t$. We first check the scaling degree definition and we get

$$\begin{aligned}
\tilde t_\lambda(\phi) &= \tilde t(\phi^\lambda), \quad \text{where } \phi^\lambda(\xi) = \lambda^{-(n-1)d}\phi(\lambda^{-1}\xi)\\
&= {}^0\tilde t\left(\phi^\lambda - w\sum \frac{\xi^\beta}{\beta!}\,\partial_\xi^\beta\phi^\lambda(0)\right)\\
&= {}^0\tilde t\left(\phi^\lambda - w(\lambda^{-1}\,\cdot\,)\sum\frac{(\lambda^{-1}\xi)^\beta}{\beta!}\,\partial_\xi^\beta\phi(0)\;\lambda^{-(n-1)d}\right)\\
&\qquad\qquad + \sum {}^0\tilde t\big((w(\lambda^{-1}\,\cdot\,)-w)(\lambda^{-1}\xi)^\beta\big)\;\frac{\partial_\xi^\beta\phi(0)}{\beta!}\;\lambda^{-(n-1)d}\\
&= {}^0\tilde t_\lambda(W\phi) + \sum\frac{\partial_\xi^\beta\phi(0)}{\beta!}\;{}^0\tilde t_\lambda\big((w-w_\lambda)\xi^\beta\big).
\end{aligned}$$

The first term scales as assumed. The term of the sum over β can be analyzed in the following way. We write

$$w - w_\lambda = \int_\lambda^1 \frac{d}{d\mu}w_\mu d\mu.$$

We convince ourselves that the integral can be commuted with the application of the distribution. We find

$$\begin{aligned} {}^0\tilde{t}_\lambda\left((\frac{d}{d\mu}w_\mu)\xi^\beta\right) &= {}^0\tilde{t}_\lambda\left((\partial_i w)(\mu\xi)\xi^i\xi^\beta\right) \\ &= {}^0\tilde{t}_\lambda\left((\partial_i w)(\mu\xi)(\mu\xi)^i(\mu\xi)^\beta\right)\mu^{-|\beta|-1} \\ &= {}^0\tilde{t}_{\lambda/\mu}\left((\partial_i w)(\xi)\xi^i\xi^\beta\right)\mu^{-|\beta|-1-d(n-1)}. \end{aligned}$$

But by assumption

$$|{}^0\tilde{t}_{\lambda/\mu}\left((\partial_i w)\xi^i\xi^\beta\right)| \leq \text{const.}\left(\frac{\lambda}{\mu}\right)^{-\omega'} \qquad \forall \omega' > \omega,$$

hence

$$|{}^0\tilde{t}_\lambda\left((w - w_\lambda)\xi^\beta\right)| \leq \text{const.}\ \lambda^{-\omega'}\int_\lambda^1 d\mu\mu^{\omega'-|\beta|-1-(n-1)d}$$

$$\leq \text{const.}\ \lambda^{-\omega'} \ \cdot \begin{cases} \frac{1-\lambda^{\omega'-|\beta|-d(n-1)}}{\omega'-|\beta|-d(n-1)}, & \text{for } \omega' - |\beta| - d(n-1) > 0 \\ |\ln\lambda|, & \text{for } \omega' - |\beta| - d(n-1) = 0. \end{cases}$$

We conclude that we obtain the same formula for the singularity degree as in the well-known power counting rules. Indeed, we get that $|\beta| \leq \omega - (n-1)d$, ω being the infimum over all ω'. Now, ω can be computed since for $t_n^{k_1,\ldots,k_n}$ we have $\omega = \sum_i k_i(d-2)/2$ where $(d-2)/2$ is the canonical dimension of the scalar field as can be seen from its two point function on Minkowski space. Collecting the formulas we get, taking $k_i = 4$ for all $i = 1,\ldots,n$, that $|\beta| \leq 2n(d-2) - (n-1)d$ hence $|\beta| \leq n(d-4) + d$ which for $d = 4$ does not depend on n anymore. This implies that the renormalization prescription works well since the number of counterterms does not grow up with the induction step.

It remains to check the convergence in the smooth sense after application of a properly supported pseudodifferential operator whose "wave front set" is disjoint from Γ_n^{to}. We get from the formula above an identity for distributions as

$$\lambda^{\omega'}A\tilde{t}_\lambda = \lambda^{\omega'}A({}^0\tilde{t}_\lambda \circ W) + \lambda^{\omega'}\sum_\beta (-1)^{|\beta|}\ A({}^0\tilde{t}_\lambda((w - w_\lambda)\frac{\xi^\beta}{\beta!})\delta^{(\beta)}(\xi)).$$

The first term has been already discussed and scales as $\lambda^{\omega'-\delta-1-(n-1)d}$. For the second one, the above proof of the convergence in Hörmander pseudotopology can be redone almost word by word since the application of A gives a smooth function. One finally finds the same scaling behaviour as above by looking at the behaviour of the term ${}^0\tilde{t}_\lambda((w - w_\lambda)\xi^\beta)$ as $\lambda \to 0$. Hence we get convergence also in the smooth sense, provided the same choice of ω' is done consistently with the discussion done until now.

8. Conclusions

To summarize: The inductive procedure gives symmetric, renormalized time ordered distributions which satisfy the causality condition. Moreover, we have shown how these renormalized objects satisfy the microlocal requirements in terms of wave front set and microlocal scaling degree. We found that the criterion for renormalizability follows the same power counting rules as on Minkowski space. All that by purely local methods.

It is now important to remove the remaining ambiguity by fixing the finite renormalization. We hope to report elsewhere on this last attempt [15]. At this stage several questions arise:

(a) Do the interacting fields satisfy the μSC of [13]?
(b) How can the renormalization group be treated? (see, e.g., [21])
(c) Is there a corresponding Euclidean formulation [22]?
(d) Can the construction be extended to gauge theories?
(e) What are the gravitational corrections to quantum field effects?
(f) How does the interaction modify the Hawking radiation?

9. Acknowledgements

Special thanks are due to Martin Köhler for help during the early stage of this project. We also thank Raymond Stora for helpful suggestions and for communications of unpublished material on the Epstein and Glaser method. The first author wishes to thank the members of the II Institut für Theoretische Physik, Hamburg Universität, for warm hospitality during 1996. His research was supported by EEC under contract n. ERBFMBICT950355 of the Training and Mobility of Reaserchers Programme.

References

[1] S. Weinberg: *The quantum theory of fields*, vol.I-II, Cambridge University Press, 1995-1996.
[2] S. Hawking: *Particle creation by black holes*, Comm. Math. Physics. **43**, (1975) p.199.
[3] R. Wald: *Quantum field theory in curved spacetime and black hole thermodynamics*, Chicago University Press, 1994. Compare also; N. D. Birrel and P. C. W. Davies: *Quantum fields in curved space*, Cambridge University Press, 1982; and S. Fulling: *Aspects of quantum field theory in curved space-time*, Cambridge University Press, 1989.
[4] T. S. Bunch: *BPHZ Renormalization of $\lambda\phi^4$ field theory in curved spacetime*, Annals of Physics, **131**, (1981), p.118.
[5] C. Itzykson and J-B. Zuber: *Quantum field theory*, McGraw-Hill, 1980.
[6] M. Lüscher: *Dimensional regularisation in the presence of large background fields*, Annals of Physics, **142**, 1982, p.359.
[7] H. Epstein and V. Glaser: *The role of locality in perturbation theory*, Ann. Inst. Henri Poincaré-Section A, vol. XIX, n.3, (1973), p.211.
[8] G. Scharf: *Finite quantum electrodynamics: The causal approach*, Springer-Verlag, 1995, 2nd edition.
[9] N. N. Bogoliubov and D. V. Shirkov: *Introduction to the theory of quantized fiels*, John Wiley and Sons, 1976, 3rd edition.

[10] H. G. Dosch and V. F. Müller: *Renormalization of quantum electrodynamics in an arbitrary strong time independent external field*, Fortschritte der Physik, **23**, (1975), p.661.

[11] There are several books available on the subject, among them a selection (due to our uncomplete knowledge of the existing literature) might be;

(a) L. Hörmander: *The analysis of linear partial differential operators*, vol. I-IV, Springer-Verlag, 1983-1986.

(b) F. Treves: *Introduction to pseudodifferential and Fourier integral operators*, vol. I-II, Plenum Press, 1980.

(c) M. E. Taylor: *Partial differential equations*, vol. I-III, Springer-Verlag, 1996.

[12] B. Kay and R. Wald: *Theorems on the uniqueness and thermal properties of stationary, non singular, quasifree states on spacetimes with a bifurcate Killing horizon*, Phys. Rep., **207**, (1991), p.49.

[13] R. Brunetti, K. Fredenhagen and M. Köhler: *The microlocal spectrum condition and the Wick's polynomials of free fields*, Comm. Math. Physics, **180**, (1996), p.312.

[14] M. Radzikowski: *Micro-local approach to the Hadamard condition in quantum field theory on curved space-time*, Comm. Math. Physics, **179**, (1996), p.529; and, *A local-to-global singularity theorem for quantum field theory on curved space-time*, Comm. Math. Physics, **180**, (1996), p.1.

[15] R. Brunetti and K. Fredenhagen: *Microlocal analysis and interacting quantum field theory*, papers in preparation.

[16] R. Stora: *Differential algebras in Lagrangean field theory*, ETH Lectures, January-February 1993; and also, G. Popineau and R. Stora: *A pedagogical remark on the main theorem of perturbative renormalization theory*, unpublished preprint.

[17] O. Steinman: *Perturbation expansions in axiomatic field theory*, Lect. Notes in Physics, **11**, Springer-Verlag, 1971.

[18] R. Haag, H. Narnhofer and U. Stein: *On quantum field theory in gravitational background*, Comm. Math. Physics, **94**, (1984), p.219.

[19] K. Fredenhagen and R. Haag: *Generally covariant quantum field theory*, Comm. Math. Physics, **108**, (1987), p.91.

[20] J. Glimm and A. Jaffe: *Quantum physics. A functional integral point of view*, McGraw-Hill, 1981.

[21] I. L. Bukbinder, S. D. Odintsov, I. L. Shapiro: *Renormalization group approach to quantum field theory on curved space time*, Riv. Nuovo Cimento, **12**, 1989, n.10 p.1.

[22] W. Junker: work in progress.