Laurent series expansion of a class of massive scalar one-loop integrals up to $\mathcal{O}(\varepsilon^2)$ in terms of multiple polylogarithms.

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In a recent paper we have presented results for a set of massive scalar one-loop master integrals needed in the NNLO parton model description of the hadroproduction of heavy flavors. The one–loop integrals were evaluated in $n = 4 - 2\varepsilon$ dimension and the results were presented in terms of a Laurent series expansion up to $\mathcal{O}(\varepsilon^2)$. We found that some of the ε^2 coefficients contain a new class of functions which we termed the L functions. The L functions are defined in terms of one–dimensional integrals involving products of logarithm and dilogarithm functions. In this paper we derive a complete set of algebraic relations that allow one to convert the L functions of our previous approach to a sum of classical and multiple polylogarithms. Using these results we are now able to present the ε^2 coefficients of the one-loop master integrals in terms of classical and multiple polylogarithms.

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I. INTRODUCTION

Recently, we have calculated the complete set of massive one-loop master integrals [1] needed in the calculation of the next-to-next-to-leading order (NNLO) parton model corrections to the hadroproduction of heavy flavors [2]. We used Feynman parametrization to evaluate the one-loop master integrals in $n = 4 - 2\varepsilon$ dimensions. We obtained the coefficients of the Laurent series expansion of the relevant scalar integrals in terms of the parameter ε up to $\mathcal{O}(\varepsilon^2)$ as needed for the NNLO calculation. We found that the real parts of some of the ε^2 coefficients contain a new class of functions which can be written in terms of one–dimensional integral representations involving products of log and dilog functions. These so–called single and triple index L functions cannot be expressed in terms of classical polylogaritms but can be seen to belong to a generalization of the classical polylogarithms which are called multiple polylogarithms.

Functions analogous to the triple index functions $L_{\sigma_1\sigma_2\sigma_3}$ also arise in the approach of [3] when one analytically continues their $\mathcal{O}(\varepsilon^2)$ integral representation for a general vertex function. Methods differing from ours have been used for the derivation of master N-point integrals such as the differential equations method [4] or the nested sum method [5]. Depending on the number of scales involved, the results include multiple polylogarithms [6] and/or harmonic [7] or two-dimensional harmonic [8] polylogarithms. The latter function all are subsets of multiple polylogarithms. Presenting our results in terms of multiple polylogarithms will facilitate a comparison with the results of possible rederivations of the scalar one–loop integrals using other methods. It is very likely that future results of multiloop calculations will be presented in terms of multiple polylogarithms or their subclasses. Alongside with this the necessary tools will be developed to deal with multiple polylogarithms, be it analytically or numerically. In fact, recently a computer code has been written for the numerical evaluation of the multiple polylogarithms [9]. It is therefore timely that we express the results of [1] also in terms of multiple polylogarithms.

It is a purpose of this paper to show that the single and triple index L functions introduced in [1] can all be related to multiple polylogarithms. This is done in explicit form. We are thus able to present our results for the scalar massive one–loop master integrals in terms of multiple polylogarithms and classical polylogarithms [10]. In Sec. II we recapitulate material on the definition of the single and triple index L functions as they arise in the approach of [1]. Simple symmetry relations allow one to restrict the discussion to the triple index L functions L_{-++} and L_{+++}, and to the single index L function L_+. In Sec. II we also recapitulate the definition of multiple polylogarithms. In the subsequent sections we will write down the formulas needed to transform the L functions to multiple polylogarithms for general arguments. The general formulas are not always applicable when the arguments take special values as they do in the massive one-loop calculation. For these special values one must carefully discuss the limiting behavior of the general formulas. In Sec. III A we

derive the general formula which relates the L_{-++} functions to the set of multiple polylogarithms. Section III B considers special cases of the general relation. Similarly, Sec. IV A gives general relations which allow one to express the L_{+++} functions in terms of multiple polylogarithms. In Sec. IV B we discuss special cases for the arguments of the L_{+++} functions. Sections V A and V B repeat the discussion for the single index L_+ functions. Finally, Sec. VI presents our conclusions.

As remarked on before, the L functions appear only in the real parts of some of the $\mathcal{O}(\varepsilon^2)$ coefficient functions of the masive one–loop integrals. In the notation of [1] these are the three–point coefficient functions $\mathrm{Re}\,C_1^{(2)}$, $\mathrm{Re}\,C_2^{(2)}$ and $\mathrm{Re}\,C_5^{(2)}$, and the four–point coefficient functions $\mathrm{Re}\,D_1^{(2)}$, $\mathrm{Re}\,D_2^{(2)}$ and $\mathrm{Re}\,D_3^{(2)}$. For the sake of brevity we have decided to present multiple polylogarithm results in this paper only for the four–point coefficient function $\mathrm{Re}\,D_1^{(2)}$. This result is listed in the Appendix. The corresponding results for the other five coefficient functions are readily available in electronic form [11].

II. BASIC FEATURES

In order to make the paper self–contained, we write down a number of basic definitions for the L functions and the multiple polylogarithms in this section, as well as some symmetry properties and domains of definitions for the single and triple index L functions. These will be of help when presenting the subsequent material.

The definition for the L functions is as follows [1]:

$$L_{\sigma_1\sigma_2\sigma_3}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \int_0^1 dy \frac{\ln(\alpha_1 + \sigma_1 y)\ln(\alpha_2 + \sigma_2 y)\ln(\alpha_3 + \sigma_3 y)}{\alpha_4 + y} \tag{1}$$

and

$$L_{\sigma_1}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \int_0^1 dy \frac{\ln(\alpha_1 + \sigma_1 y)\mathrm{Li}_2(\alpha_2 + \alpha_3 y)}{\alpha_4 + y}. \tag{2}$$

Here the σ_i $(i = 1, 2, 3)$ take the values ± 1 and the α_j's are either integers $\{1, 0, -1\}$ or else kinematical variables. We want to emphasize that the numerical evaluation of the L functions is straightforward.

The L functions possess simple symmetry properties as follows. One notices that a change of the integration variable $y \to 1 - y$ results in the identity

$$L_{\sigma_1}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = -L_{-\sigma_1}(\alpha_1 + \sigma_1, \alpha_2 + \alpha_3, -\alpha_3, -\alpha_4 - 1) \tag{3}$$

which implies that L_- can always be related to L_+, and vice versa. We have thus written our results for the three-point and four-point functions in [1] only in terms of the L_+ functions.

Turning to the triple index L function one notices that $L_{\sigma_1\sigma_2\sigma_3}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is symmetric under permutations of any two pairs of indices and arguments $\{\sigma_i, \alpha_i\}$ and

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$\{\sigma_j, \alpha_j\}$ for $(i \neq j)$. The same change of variables as above $y \to 1 - y$ results in

$$L_{\sigma_1 \sigma_2 \sigma_3}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = -L_{-\sigma_1 - \sigma_2 - \sigma_3}(\alpha_1 + \sigma_1, \alpha_2 + \sigma_2, \alpha_3 + \sigma_3, -\alpha_4 - 1). \qquad (4)$$

Therefore, from the eight functions L_{---}, L_{--+}, L_{-+-}, L_{+--}, L_{-++}, L_{+-+}, L_{++-}, and L_{+++} only two are independent. We have chosen to write our results in terms of L_{-++} and L_{+++}.

The domains of definition of the functions L_{+++}, L_{-++}, and L_+ that follow from the requirement that these functions take real values can be read off from the defining relations Eqs. (1) and (2) considering the arguments of the log and dilog functions in the integrands, as well as from ensuring that the denominator of Eqs. (1) and (2) does not change sign on the integration path. One has

$$
\begin{array}{llll}
L_{+++}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) & : & \alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0, \alpha_4 < -1 & \text{or} \quad \alpha_4 > 0; \\
L_{-++}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) & : & \alpha_1 > 1, \alpha_2 > 0, \alpha_3 > 0, \alpha_4 < -1 & \text{or} \quad \alpha_4 > 0; \\
L_+(\alpha_1, \alpha_2, \alpha_3, \alpha_4) & : & \alpha_1 > 0, \alpha_2 \leq 1, \alpha_2 + \alpha_3 \leq 1, \alpha_3 \neq 0, \alpha_4 < -1 & \text{or} \quad \alpha_4 > 0.
\end{array}
\qquad (5)
$$

Looking at the definition of the triple index L function in (1) one concludes that the boundary points $\alpha_1 = 0$ and/or $\alpha_2 = 0$ and/or $\alpha_3 = 0$ can be included in the domain of the definition for L_{+++}. The same holds true for $\alpha_1 = 1$ and/or $\alpha_2 = 0$ and/or $\alpha_3 = 0$ for L_{-++}. Also, from the definition of the single index function L_+ in (2) one concludes that the boundary point $\alpha_1 = 0$ can be added to its domain of definition.

The points $\alpha_4 = \{-1, 0\}$ can also be included in the domain if the values taken by the other parameters α_i guarantee the convergence of the integral. We mention that for all of our purposes the conditions (5), with the boundary points included, are satisfied, e.g., our results for the integrals are real. Nevertheless, it is of course always possible to analytically continue the parameters to the complex plane.

There are some further relations for the L functions which result from applying integration-by-parts identities. They are not listed here but can be found in Appendix C of [1]. They have been used to reduce the set of L functions occurring in the master integrals to a subset of L functions having real values in physical phase space [1].

Multiple polylogarithms are defined as a limit of Z sums [6], e.g.,

$$Li_{m_k, \ldots, m_1}(x_k, \ldots, x_1) = \lim_{n_1 \to \infty} \sum_{n_1 > n_2 \ldots > n_k > 0} \frac{x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}}{n_1^{m_1} n_2^{m_2} \ldots n_k^{m_k}}. \qquad (6)$$

The number $w = m_1 + \ldots + m_k$ is called the weight and k is called the depth of the multiple polylogarithm. The power series (6) is convergent for $|x_i| < 1$, and can be analytically continued via the iterated integral representation:

$$Li_{m_k, \ldots, m_1}(x_k, \ldots, x_1) = \int_0^{x_1 x_2 \ldots x_k} \left(\frac{dt}{t} \circ\right)^{m_1 - 1} \frac{dt}{x_2 x_3 \ldots x_k - t} \circ$$

$$\left(\frac{dt}{t} \circ\right)^{m_2 - 1} \frac{dt}{x_3 \ldots x_k - t} \circ \ldots \circ \left(\frac{dt}{t} \circ\right)^{m_k - 1} \frac{dt}{1 - t}, \qquad (7)$$

where the following notation is used for the iterated integrals:

$$\int\limits_0^\lambda \frac{dt}{a_n - t} \circ ... \circ \frac{dt}{a_1 - t} = \int\limits_0^\lambda \frac{dt_n}{a_n - t_n} \int\limits_0^{t_n} \frac{dt_{n-1}}{a_{n-1} - t_{n-1}} \times ... \times \int\limits_0^{t_2} \frac{dt_1}{a_1 - t_1}. \qquad (8)$$

III. TRANSFORMATION OF L_{-++} TO MULTIPLE POLYLOGARITHMS

In this section we will show that all our L_{-++} functions can be expressed in terms of multiple polylogarithms.

A. General case for the L_{-++} function

We begin with the L_{-++} function Eq. (1),

$$L_{-++}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \int\limits_0^1 dy \frac{\ln(\alpha_1 - y)\ln(\alpha_2 + y)\ln(\alpha_3 + y)}{\alpha_4 + y}. \qquad (9)$$

After changing the integration variable $y = \alpha_1 t$ one gets

$$\int\limits_0^{1/\alpha_1} dt \frac{\ln(\alpha_1 - \alpha_1 t)\ln(\alpha_2 + \alpha_1 t)\ln(\alpha_3 + \alpha_1 t)}{\frac{\alpha_4}{\alpha_1} + t} = \int\limits_0^{1/\alpha_1} dt \frac{\ln\alpha_1 \ln(\alpha_2 + \alpha_1 t)\ln(\alpha_3 + \alpha_1 t)}{\frac{\alpha_4}{\alpha_1} + t}$$

$$+ \int\limits_0^{1/\alpha_1} dt \frac{\ln(1 - t)[\ln\alpha_1 + \ln(\frac{\alpha_2}{\alpha_1} + t)][\ln\alpha_1 + \ln(\frac{\alpha_3}{\alpha_1} + t)]}{\frac{\alpha_4}{\alpha_1} + t} =$$

$$\ln\alpha_1 \int\limits_0^1 dy \frac{\ln(\alpha_2 + y)\ln(\alpha_3 + y)}{\alpha_4 + y} + \ln^2\alpha_1 \int\limits_0^{1/\alpha_1} dt \frac{\ln(1 - t)}{\frac{\alpha_4}{\alpha_1} + t}$$

$$+ \ln\alpha_1 \int\limits_0^{1/\alpha_1} dt \frac{\ln(1 - t)\ln(\frac{\alpha_2}{\alpha_1} + t)}{\frac{\alpha_4}{\alpha_1} + t} + \ln\alpha_1 \int\limits_0^{1/\alpha_1} dt \frac{\ln(1 - t)\ln(\frac{\alpha_3}{\alpha_1} + t)}{\frac{\alpha_4}{\alpha_1} + t} \qquad (10)$$

$$+ \int\limits_0^{1/\alpha_1} dt \frac{\ln(1 - t)\ln(\frac{\alpha_2}{\alpha_1} + t)\ln(\frac{\alpha_3}{\alpha_1} + t)}{\frac{\alpha_4}{\alpha_1} + t}.$$

With the help of (7) the integral in the second term of the last equation of (10) can be written as

$$\int\limits_0^{1/\alpha_1} dt \frac{\ln(1 - t)}{\frac{\alpha_4}{\alpha_1} + t} = \int\limits_0^{1/\alpha_1} \frac{dt_1}{-\frac{\alpha_4}{\alpha_1} - t_1} \int\limits_0^{t_1} \frac{dt_2}{1 - t_2} = Li_{1,1}\left(-\frac{\alpha_4}{\alpha_1}, -\frac{1}{\alpha_4}\right). \qquad (11)$$

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The third and fourth terms of the last equation in (10) contain integrals of the form

$$\int_0^{t_m} dt \frac{\ln(1-t)\ln(\beta_1+t)}{\beta_2+t} \; . \tag{12}$$

To express such integrals in terms of multiple polylogarithms one proceeds as follows:

$$-Li_{1,1,1}\left(-\beta_2,\frac{\beta_1}{\beta_2},\frac{-t_m}{\beta_1}\right) = \int_0^{t_m} \frac{dt_2}{\beta_1+t_2} \int_0^{t_2} dt_1 \frac{\ln(1-t_1)}{\beta_2+t_1} = \int_0^{t_m} dt_1 \frac{\ln(1-t_1)}{\beta_2+t_1} \int_{t_1}^{t_m} \frac{dt_2}{\beta_1+t_2} =$$

$$\ln(\beta_1+t_m) \int_0^{t_m} dt_1 \frac{\ln(1-t_1)}{\beta_2+t_1} - \int_0^{t_m} dt_1 \frac{\ln(1-t_1)\ln(\beta_1+t_1)}{\beta_2+t_1} = \tag{13}$$

$$\ln(\beta_1+t_m) Li_{1,1}\left(-\beta_2,\frac{-t_m}{\beta_2}\right) - \int_0^{t_m} dt_1 \frac{\ln(1-t_1)\ln(\beta_1+t_1)}{\beta_2+t_1} \; .$$

In the first line of (13) we have changed the order of integration in the two-dimensional integral. We shall frequently use this trick further on. From Eq. (13) one immediately concludes that

$$\int_0^{t_m} dt \frac{\ln(1-t)\ln(\beta_1+t)}{\beta_2+t} = Li_{1,1,1}\left(-\beta_2,\frac{\beta_1}{\beta_2},\frac{-t_m}{\beta_1}\right) + \ln(\beta_1+t_m) Li_{1,1}\left(-\beta_2,\frac{-t_m}{\beta_2}\right) . \tag{14}$$

Let us now turn to the more involved integral [first term of Eq. (10)]:

$$\int_0^1 dy \frac{\ln(\alpha_2+y)\ln(\alpha_3+y)}{\alpha_4+y} \stackrel{y\to-\alpha_2 t}{=} -\int_0^{-1/\alpha_2} dt \frac{[\ln\alpha_2+\ln(1-t)]\ln(\alpha_3-\alpha_2 t)}{\frac{\alpha_4}{\alpha_2}-t} =$$

$$-\ln\alpha_2 \int_0^{-1/\alpha_2} dt \frac{\ln(\alpha_3-\alpha_2 t)}{\frac{\alpha_4}{\alpha_2}-t} - \int_0^{-1/\alpha_2} dt \frac{\ln(1-t)[\ln\alpha_2+\ln(\frac{\alpha_3}{\alpha_2}-t)]}{\frac{\alpha_4}{\alpha_2}-t} = \tag{15}$$

$$+\ln\alpha_2 \int_0^1 dy \frac{\ln(\alpha_3+y)}{\alpha_4+y} - \ln\alpha_2 \int_0^{-1/\alpha_2} dt \frac{\ln(1-t)}{\frac{\alpha_4}{\alpha_2}-t} - \int_0^{-1/\alpha_2} dt \frac{\ln(1-t)\ln(\frac{\alpha_3}{\alpha_2}-t)}{\frac{\alpha_4}{\alpha_2}-t} \; .$$

The integral in the first term can be expressed as

$$\int_0^1 dy \frac{\ln(\alpha_3+y)}{\alpha_4+y} \stackrel{y\to-\alpha_3 t}{=} -\int_0^{-1/\alpha_3} dt \frac{[\ln\alpha_3+\ln(1-t)]}{\frac{\alpha_4}{\alpha_3}-t} =$$

$$\ln\alpha_3 \ln\left(\frac{\alpha_4+1}{\alpha_4}\right) + Li_{1,1}\left(\frac{\alpha_4}{\alpha_3},-\frac{1}{\alpha_3}\right) \; . \tag{16}$$

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The integral in the second term can be written as

$$\int\limits_0^{-1/\alpha_2} dt \frac{\ln(1-t)}{\frac{\alpha_4}{\alpha_2} - t} = -Li_{1,1}\left(\frac{\alpha_4}{\alpha_2}, -\frac{1}{\alpha_2}\right) . \tag{17}$$

The third term from the last line of Eq. (15) has a form which is an analogue of the integral (12) and can be calculated in a similar way,

$$\int\limits_0^{t_m} dt \frac{\ln(1-t)\ln(\beta_1-t)}{\beta_2-t} = Li_{1,1,1}\left(\beta_2, \frac{\beta_1}{\beta_2}, \frac{t_m}{\beta_1}\right) + \ln(\beta_1 - t_m) Li_{1,1}\left(\beta_2, \frac{t_m}{\beta_2}\right) . \tag{18}$$

Combining the Eqs. (16), (17) and (18) we arrive at the result for Eq. (15),

$$\int\limits_0^1 dy \frac{\ln(\alpha_2+y)\ln(\alpha_3+y)}{\alpha_4+y} = Li_{1,1,1}\left(\frac{\alpha_4}{\alpha_2}, \frac{\alpha_3}{\alpha_4}, -\frac{1}{\alpha_3}\right) + \ln\alpha_2 Li_{1,1}\left(\frac{\alpha_4}{\alpha_3}, -\frac{1}{\alpha_4}\right)$$

$$+ \ln(1+\alpha_3) Li_{1,1}\left(\frac{\alpha_4}{\alpha_2}, -\frac{1}{\alpha_4}\right) + \ln\alpha_2 \ln\alpha_3 \ln\left(\frac{\alpha_4+1}{\alpha_4}\right) . \tag{19}$$

Because the initial integrand is symmetric under the exchange of the parameters α_2 and α_3, the rhs of (19) can be rewritten in a symmetric form if desired.

We are now left with the fifth term in (10). The fifth term is an integral of the type

$$\int\limits_0^{t_m} dt \frac{\ln(1-t)\ln(\gamma_1+t)\ln(\gamma_2+t)}{\gamma_3+t} . \tag{20}$$

In order to express such integrals in terms of multiple polylogarithms one can perform the following chain of transformations resulting in a multiple polylogarithm of weight four:

$$-Li_{1,1,1,1}\left(-\gamma_3, \frac{\gamma_2}{\gamma_3}, \frac{\gamma_1}{\gamma_2}, \frac{-t_m}{\gamma_1}\right) = \int\limits_0^{t_m} \frac{dt_4}{\gamma_1+t_4} \int\limits_0^{t_4} \frac{dt_3}{\gamma_2+t_3} \int\limits_0^{t_3} \frac{dt_2}{\gamma_3+t_2} \int\limits_0^{t_2} \frac{dt_1}{1-t_1} = \tag{21}$$

$$-\int\limits_0^{t_m} \frac{dt_4}{\gamma_1+t_4} \int\limits_0^{t_4} \frac{dt_3}{\gamma_2+t_3} \int\limits_0^{t_3} dt_2 \frac{\ln(1-t_2)}{\gamma_3+t_2} = -\int\limits_0^{t_m} \frac{dt_4}{\gamma_1+t_4} \int\limits_0^{t_4} dt_2 \frac{\ln(1-t_2)}{\gamma_3+t_2} \int\limits_{t_2}^{t_4} \frac{dt_3}{\gamma_2+t_3} =$$

$$-\int\limits_0^{t_m} dt_4 \frac{\ln(\gamma_2+t_4)}{\gamma_1+t_4} \int\limits_0^{t_4} dt_2 \frac{\ln(1-t_2)}{\gamma_3+t_2} + \int\limits_0^{t_m} \frac{dt_4}{\gamma_1+t_4} \int\limits_0^{t_4} dt_2 \frac{\ln(1-t_2)\ln(\gamma_2+t_2)}{\gamma_3+t_2} =$$

$$-I'(t_m) + \int\limits_0^{t_m} dt_2 \frac{\ln(1-t_2)\ln(\gamma_2+t_2)}{\gamma_3+t_2} \int\limits_{t_2}^{t_m} \frac{dt_4}{\gamma_1+t_4} =$$

$$-I'(t_m) + I''(t_m) - \int\limits_0^{t_m} dt_2 \frac{\ln(\gamma_1+t_2)\ln(\gamma_2+t_2)\ln(1-t_2)}{\gamma_3+t_2} ,$$

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where we have introduced the notation

$$I'(t_m) = \int\limits_0^{t_m} dt_4 \frac{\ln(\gamma_2 + t_4)}{\gamma_1 + t_4} \int\limits_0^{t_4} dt_2 \frac{\ln(1 - t_2)}{\gamma_3 + t_2}, \tag{22}$$

$$I''(t_m) = \ln(\gamma_1 + t_m) \int\limits_0^{t_m} dt_2 \frac{\ln(1 - t_2)\ln(\gamma_2 + t_2)}{\gamma_3 + t_2}.$$

The third term on the last line of (21) is exactly the integral of the required type Eq. (20).

The integral in $I''(t_m)$ has the form of (14). For the integral $I'(t_m)$ we write

$$I'(t_m) = \int\limits_0^{t_m} dt_4 \frac{\ln(\gamma_2 + t_4)}{\gamma_1 + t_4} \int\limits_0^{t_4} dt_2 \frac{\ln(1 - t_2)}{\gamma_3 + t_2} = \int\limits_0^{t_m} dt_4 \frac{\ln(\gamma_2 + t_4)}{\gamma_1 + t_4} Li_{1,1}\left(-\gamma_3, \frac{-t_4}{\gamma_3}\right). \tag{23}$$

On the other, hand one has

$$Li_{1,1,1,1}\left(-\gamma_3, \frac{\gamma_1}{\gamma_3}, \frac{\gamma_2}{\gamma_1}, \frac{-t_m}{\gamma_2}\right) = \int\limits_0^{t_m} \frac{dt_2}{\gamma_2 + t_2} \int\limits_0^{t_2} \frac{dt_1}{\gamma_1 + t_1} Li_{1,1}\left(-\gamma_3, \frac{-t_1}{\gamma_3}\right) = \tag{24}$$

$$\int\limits_0^{t_m} \frac{dt_1}{\gamma_1 + t_1} Li_{1,1}\left(-\gamma_3, \frac{-t_1}{\gamma_3}\right) \int\limits_{t_1}^{t_m} \frac{dt_2}{\gamma_2 + t_2} =$$

$$\ln(\gamma_2 + t_m) \int\limits_0^{t_m} \frac{dt_1}{\gamma_1 + t_1} Li_{1,1}\left(-\gamma_3, \frac{-t_1}{\gamma_3}\right) - \int\limits_0^{t_m} dt_1 \frac{\ln(\gamma_2 + t_1)}{\gamma_1 + t_1} Li_{1,1}\left(-\gamma_3, \frac{-t_1}{\gamma_3}\right) =$$

$$- \ln(\gamma_2 + t_m) Li_{1,1,1}\left(-\gamma_3, \frac{\gamma_1}{\gamma_3}, \frac{-t_m}{\gamma_1}\right) - I'(t_m).$$

One then concludes that

$$I'(t_m) = -Li_{1,1,1,1}\left(-\gamma_3, \frac{\gamma_1}{\gamma_3}, \frac{\gamma_2}{\gamma_1}, \frac{-t_m}{\gamma_2}\right) - \ln(\gamma_2 + t_m) Li_{1,1,1}\left(-\gamma_3, \frac{\gamma_1}{\gamma_3}, \frac{-t_m}{\gamma_1}\right). \tag{25}$$

Finally, substituting $I'(t_m)$ and $I''(t_m)$ into Eq. (21) we write down the result for the integral of the required type Eq. (20),

$$\int\limits_0^{t_m} dt \frac{\ln(1 - t)\ln(\gamma_1 + t)\ln(\gamma_2 + t)}{\gamma_3 + t} = \ln(\gamma_1 + t_m)\ln(\gamma_2 + t_m) Li_{1,1}\left(-\gamma_3, \frac{-t_m}{\gamma_3}\right) +$$

$$\ln(\gamma_2 + t_m) Li_{1,1,1}\left(-\gamma_3, \frac{\gamma_1}{\gamma_3}, \frac{-t_m}{\gamma_1}\right) + \ln(\gamma_1 + t_m) Li_{1,1,1}\left(-\gamma_3, \frac{\gamma_2}{\gamma_3}, \frac{-t_m}{\gamma_2}\right) \tag{26}$$

$$+ Li_{1,1,1,1}\left(-\gamma_3, \frac{\gamma_2}{\gamma_3}, \frac{\gamma_1}{\gamma_2}, \frac{-t_m}{\gamma_1}\right) + Li_{1,1,1,1}\left(-\gamma_3, \frac{\gamma_1}{\gamma_3}, \frac{\gamma_2}{\gamma_1}, \frac{-t_m}{\gamma_2}\right).$$

We are now in the position to collect all required contributions to express the L_{-++} function in terms of multiple polylogarithms. Taking into account Eqs. (11), (14), (19), and (26) we obtain

$$L_{-++}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = Li_{1,1,1,1}\left(-\frac{\alpha_4}{\alpha_1}, \frac{\alpha_2}{\alpha_4}, \frac{\alpha_3}{\alpha_2}, -\frac{1}{\alpha_3}\right)$$

$$+Li_{1,1,1,1}\left(-\frac{\alpha_4}{\alpha_1}, \frac{\alpha_3}{\alpha_4}, \frac{\alpha_2}{\alpha_3}, -\frac{1}{\alpha_2}\right) + \ln\alpha_1 Li_{1,1,1}\left(\frac{\alpha_4}{\alpha_2}, \frac{\alpha_3}{\alpha_4}, -\frac{1}{\alpha_3}\right)$$

$$+\ln(1+\alpha_2)Li_{1,1,1}\left(-\frac{\alpha_4}{\alpha_1}, \frac{\alpha_3}{\alpha_4}, -\frac{1}{\alpha_3}\right) + \ln(1+\alpha_3)Li_{1,1,1}\left(-\frac{\alpha_4}{\alpha_1}, \frac{\alpha_2}{\alpha_4}, -\frac{1}{\alpha_2}\right)$$

$$+\ln\alpha_1 \ln\alpha_2 Li_{1,1}\left(\frac{\alpha_4}{\alpha_3}, -\frac{1}{\alpha_4}\right) + \ln\alpha_1 \ln(1+\alpha_3)Li_{1,1}\left(\frac{\alpha_4}{\alpha_2}, -\frac{1}{\alpha_4}\right) \qquad (27)$$

$$+\ln(1+\alpha_2)\ln(1+\alpha_3)Li_{1,1}\left(-\frac{\alpha_4}{\alpha_1}, -\frac{1}{\alpha_4}\right) + \ln\alpha_1 \ln\alpha_2 \ln\alpha_3 \ln\left(\frac{\alpha_4+1}{\alpha_4}\right).$$

Some remarks are in order at this place. The final formula (27) contains multiple polylogarithms up to weight four. All multiple polylogarithms up to weight three can be expressed in terms of logarithms and clasical polylogarithms Li_2 and Li_3. This fact is used by us when we reexpress our results for the massive scalar integrals in terms of multiple polylogarithms, i.e., our final results will contain only multiple polylogarithms of weight four. For the variables α_i the conditions (5) are assumed. But in the results for the massive scalar integrals there are also cases when $\alpha_1 = 1$ and/or $\alpha_2 = 0$ and/or $\alpha_3 = 0$ and/or $\alpha_4 = \{-1, 0\}$. In such cases the general formula (27) is no longer valid and these cases must be studied separately.

B. Special cases for the L_{-++} function

In the Laurent series expansion of the massive scalar one-loop integrals one encounters special values of the arguments α_i for which the general formula Eq. (27) no longer applies. This is quite obvious from the list of special cases discussed in the following.

1. $\alpha_1 = 1$, $\alpha_4 = 0$

In such case one can make use of Eq. (26). One should find the limit of the expression on the right-hand side for $t_m = 1, \gamma_3 \to 0$. One obtains

$$\int_0^1 dt \frac{\ln(1-t)\ln(\gamma_1+t)\ln(\gamma_2+t)}{t} =$$

$$\lim_{\gamma_3 \to 0}\left\{ \ln(\gamma_1+1)\ln(\gamma_2+1)\int_0^1 \frac{dt_2}{-\gamma_3-t_2}\int_0^{t_2}\frac{dt_1}{1-t_1} + \ln(\gamma_2+1)\int_0^1 \frac{dt_3}{-\gamma_1-t_3}\int_0^{t_3}\frac{dt_2}{-\gamma_3-t_2}\int_0^{t_2}\frac{dt_1}{1-t_1}\right.$$

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$$+ \ln(\gamma_1 + 1) \int\limits_0^1 \frac{dt_3}{-\gamma_2 - t_3} \int\limits_0^{t_3} \frac{dt_2}{-\gamma_3 - t_2} \int\limits_0^{t_2} \frac{dt_1}{1 - t_1} + \int\limits_0^1 \frac{dt_4}{-\gamma_1 - t_3} \int\limits_0^{t_4} \frac{dt_3}{-\gamma_2 - t_3} \int\limits_0^{t_3} \frac{dt_2}{-\gamma_3 - t_2} \int\limits_0^{t_2} \frac{dt_1}{1 - t_1}$$

$$+ \int\limits_0^1 \frac{dt_4}{-\gamma_2 - t_3} \int\limits_0^{t_4} \frac{dt_3}{-\gamma_1 - t_3} \int\limits_0^{t_3} \frac{dt_2}{-\gamma_3 - t_2} \int\limits_0^{t_2} \frac{dt_1}{1 - t_1} \Big\} = - \ln(\gamma_1 + 1) \ln(\gamma_2 + 1) \int\limits_0^1 \frac{dt_2}{t_2} \int\limits_0^{t_2} \frac{dt_1}{1 - t_1}$$

$$- \ln(\gamma_2 + 1) \int\limits_0^1 \frac{dt_3}{-\gamma_1 - t_3} \int\limits_0^{t_3} \frac{dt_2}{t_2} \int\limits_0^{t_2} \frac{dt_1}{1 - t_1} - \ln(\gamma_1 + 1) \int\limits_0^1 \frac{dt_3}{-\gamma_2 - t_3} \int\limits_0^{t_3} \frac{dt_2}{t_2} \int\limits_0^{t_2} \frac{dt_1}{1 - t_1}$$

$$- \int\limits_0^1 \frac{dt_4}{-\gamma_1 - t_3} \int\limits_0^{t_4} \frac{dt_3}{-\gamma_2 - t_3} \int\limits_0^{t_3} \frac{dt_2}{t_2} \int\limits_0^{t_2} \frac{dt_1}{1 - t_1} - \int\limits_0^1 \frac{dt_4}{-\gamma_2 - t_3} \int\limits_0^{t_4} \frac{dt_3}{-\gamma_1 - t_3} \int\limits_0^{t_3} \frac{dt_2}{t_2} \int\limits_0^{t_2} \frac{dt_1}{1 - t_1} \; .$$

$$(28)$$

In order to get the expression under the sign of the limit in Eq. (28) one applies the definition (7) for the multiple polylogarithms in Eq. (26). Using the same definition for the final multidimensional integrals in (28) and making the change $\gamma_1 \to \alpha_2, \gamma_2 \to \alpha_3$ one finally arrives at the result for the case $\alpha_1 = 1$ and $\alpha_4 = 0$,

$$L_{-++} (1, \alpha_2, \alpha_3, 0) = -Li_{2,1,1} \left(-\alpha_3, \frac{\alpha_2}{\alpha_3}, -\frac{1}{\alpha_2} \right) - Li_{2,1,1} \left(-\alpha_2, \frac{\alpha_3}{\alpha_2}, -\frac{1}{\alpha_3} \right).$$

$$- \ln(\alpha_3 + 1) Li_{2,1} \left(-\alpha_2, -\frac{1}{\alpha_2} \right) - \ln(\alpha_2 + 1) Li_{2,1} \left(-\alpha_3, -\frac{1}{\alpha_3} \right)$$

$$- \ln(\alpha_2 + 1) \ln(\alpha_3 + 1) \zeta(2). \qquad (29)$$

2. $\alpha_1 = 1, \; \alpha_2 = \alpha_3 = 0$

For these values of the parameters α_i one has an integral of the very simple form

$$L_{-++}(1, 0, 0, \alpha_4) = \int\limits_0^1 dy \frac{\ln(1 - y) \ln^2 y}{\alpha_4 + y} \; .$$

After a change of variable $y \to 1 - t$ one gets

$$\int\limits_0^1 dt \frac{\ln t \ln^2(1 - t)}{\alpha_4 + 1 - t} = - \int\limits_0^1 dt_1 \frac{\ln^2(1 - t_1)}{\alpha_4 + 1 - t_1} \int\limits_{t_1}^1 \frac{dt_2}{t_2} =$$

$$- \int\limits_0^1 \frac{dt_2}{t_2} \int\limits_0^{t_2} dt_1 \frac{\ln^2(1 - t_1)}{\alpha_4 + 1 - t_1} = -2 \int\limits_0^1 \frac{dt_2}{t_2} \int\limits_0^{t_2} \frac{dt_1}{\alpha_4 + 1 - t_1} \int\limits_0^{t_1} \frac{dt_3}{1 - t_3} \int\limits_0^{t_3} \frac{dt_4}{1 - t_4} \; . \qquad (30)$$

Applying the definition (7) one obtains

$$L_{-++}(1, 0, 0, \alpha_4) = -2 Li_{1,1,2} \left(1, \alpha_4 + 1, \frac{1}{\alpha_4 + 1} \right). \qquad (31)$$

3. $\alpha_1 = 1$, $\alpha_2 = 0$ (and $\alpha_4 = -1$)

We shall again find the limit of the rhs of (26) for $t_m = 1$ and $\gamma_1 \to 0$. The first and the third terms are equal to 0 because of the limit $\lim_{\gamma_1 \to 0} \ln(\gamma_1 + 1) = 0$. The other terms transform into

$$\lim_{\gamma_1 \to 0} Li_{1,1,1}\left(-\gamma_3, \frac{\gamma_1}{\gamma_3}, \frac{-1}{\gamma_1}\right) = -Li_{1,2}\left(-\gamma_3, -\frac{1}{\gamma_3}\right),$$

$$\lim_{\gamma_1 \to 0} Li_{1,1,1,1}\left(-\gamma_3, \frac{\gamma_2}{\gamma_3}, \frac{\gamma_1}{\gamma_2}, \frac{-t_m}{\gamma_1}\right) = -Li_{1,1,2}\left(-\gamma_3, \frac{\gamma_2}{\gamma_3}, \frac{-1}{\gamma_2}\right),$$

$$\lim_{\gamma_1 \to 0} Li_{1,1,1,1}\left(-\gamma_3, \frac{\gamma_1}{\gamma_3}, \frac{\gamma_2}{\gamma_1}, \frac{-t_m}{\gamma_2}\right) = -Li_{1,2,1}\left(-\gamma_3, \frac{\gamma_2}{\gamma_3}, \frac{-1}{\gamma_2}\right).$$

Finally we write

$$L_{-++}(1, 0, \alpha_3, \alpha_4) = -Li_{1,1,2}\left(-\alpha_4, \frac{\alpha_3}{\alpha_4}, \frac{-1}{\alpha_3}\right) - Li_{1,2,1}\left(-\alpha_4, \frac{\alpha_3}{\alpha_4}, \frac{-1}{\alpha_3}\right)$$
$$- \ln(\alpha_3 + 1) Li_{1,2}\left(-\alpha_4, -\frac{1}{\alpha_4}\right). \qquad (32)$$

For the special case $\alpha_4 = -1$ one gets

$$L_{-++}(1, 0, \alpha_3, -1) = -Li_{1,1,2}\left(1, -\alpha_3, \frac{-1}{\alpha_3}\right) - Li_{1,2,1}\left(1, -\alpha_3, \frac{-1}{\alpha_3}\right)$$
$$- \ln(\alpha_3 + 1)\, \zeta(3). \qquad (33)$$

4. $\alpha_2 = \alpha_3 = 0$ (and $\alpha_4 = -1$)

In this case one proceeds along the following lines:

$$L_{-++}(\alpha_1, 0, 0, \alpha_4) = \int_0^1 dy \frac{\ln(\alpha_1 - y)\ln^2 y}{\alpha_4 + y} \overset{y \to 1-t}{=} \int_0^1 dt \frac{\ln(\alpha_1 - 1 + t)\ln^2(1 - t)}{\alpha_4 + 1 - t} =$$

$$-\int_0^1 dt_1 \frac{\ln^2(1 - t_1)}{-\alpha_4 - 1 + t_1} \int_{-\alpha_1+2}^{t_1} \frac{dt_2}{\alpha_1 - 1 + t_2} = -\int_0^1 dt_1 \frac{\ln^2(1 - t_1)}{-\alpha_4 - 1 + t_1} \left\{ \int_1^{t_1} + \int_{-\alpha_1+2}^1 \right\} \frac{dt_2}{\alpha_1 - 1 + t_2} =$$

$$\int_0^1 \frac{dt_2}{\alpha_1 - 1 + t_2} \int_0^{t_2} dt_1 \frac{\ln^2(1 - t_1)}{-\alpha_4 - 1 + t_1} - \ln \alpha_1 \int_0^1 dt_1 \frac{\ln^2(1 - t_1)}{-\alpha_4 - 1 + t_1} =$$

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$$2\int_0^1 \frac{dt_2}{\alpha_1 - 1 + t_2}\int_0^{t_2}\frac{dt_1}{-\alpha_4 - 1 + t_1}\int_0^{t_1}\frac{dt_3}{1 - t_3}\int_0^{t_3}\frac{dt_4}{1 - t_4} - 2\ln\alpha_1 \mathrm{Li}_3\left(-\frac{1}{\alpha_4}\right).$$

(34)

Using the definition (7) we arrive at the result

$$L_{-++}(\alpha_1, 0, 0, \alpha_4) = 2Li_{1,1,1,1}\left(1, \alpha_4 + 1, \frac{1 - \alpha_1}{\alpha_4 + 1}, \frac{1}{1 - \alpha_1}\right) - 2\ln\alpha_1 \mathrm{Li}_3\left(-\frac{1}{\alpha_4}\right). \quad (35)$$

For the case $\alpha_4 = -1$ one obtains

$$L_{-++}(\alpha_1, 0, 0, -1) = -2Li_{1,2,1}\left(1, 1 - \alpha_1, \frac{1}{1 - \alpha_1}\right) - 2\ln\alpha_1\,\zeta(3). \quad (36)$$

5. $\alpha_2 = 0$ (and $\alpha_4 = -1$)

For this integral we change the integration variable $y \to 1 - t$,

$$\int_0^1 dy \frac{\ln(\alpha_1 - y)\ln y \ln(\alpha_3 + y)}{\alpha_4 + y} = \int_0^1 dt \frac{\ln(1 - t)\ln(\alpha_1 - 1 + t)\ln(\alpha_3 + 1 - t)}{\alpha_4 + 1 - t} =$$

$$\int_0^1 dt \frac{\ln(1 - t)\ln(\gamma_1 + t)\ln(\gamma_2 - t)}{\gamma_3 - t}. \quad (37)$$

One notes that the last integral is an analogue of the integral in Eq. (26). The calculation proceeds in a similar way,

$$-Li_{1,1,1,1}\left(\gamma_3, \frac{\gamma_2}{\gamma_3}, -\frac{\gamma_1}{\gamma_2}, -\frac{1}{\gamma_1}\right) = \int_0^1 \frac{dt_4}{\gamma_1 + t_4}\int_0^{t_4}\frac{dt_3}{\gamma_2 - t_3}\int_0^{t_3}\frac{dt_2}{\gamma_3 - t_2}\int_0^{t_2}\frac{dt_1}{1 - t_1} =$$

$$-\int_0^{t_m}\frac{dt_4}{\gamma_1 + t_4}\int_0^{t_4}\frac{dt_3}{\gamma_2 - t_3}\int_0^{t_3}dt_2\frac{\ln(1 - t_2)}{\gamma_3 - t_2} = -\int_0^{t_m}\frac{dt_4}{\gamma_1 + t_4}\int_0^{t_4}dt_2\frac{\ln(1 - t_2)}{\gamma_3 - t_2}\int_{t_2}^{t_4}\frac{dt_3}{\gamma_2 - t_3} =$$

$$\int_0^1 dt_4\frac{\ln(\gamma_2 - t_4)}{\gamma_1 + t_4}\int_0^{t_4}dt_2\frac{\ln(1 - t_2)}{\gamma_3 - t_2} - \int_0^1\frac{dt_4}{\gamma_1 + t_4}\int_0^{t_4}dt_2\frac{\ln(1 - t_2)\ln(\gamma_2 - t_2)}{\gamma_3 - t_2} =$$

$$Y'(1) - \int_0^1 dt_2\frac{\ln(1 - t_2)\ln(\gamma_2 - t_2)}{\gamma_3 - t_2}\int_{t_2}^1\frac{dt_4}{\gamma_1 + t_4} =$$

$$Y'(1) - \ln(\gamma_1 + 1)\int_0^1 dt_2\frac{\ln(1 - t_2)\ln(\gamma_2 - t_2)}{\gamma_3 - t_2} + \int_0^1 dt_2\frac{\ln(1 - t_2)\ln(\gamma_1 + t_2)\ln(\gamma_2 - t_2)}{\gamma_3 - t_2} =$$

$$= Y'(1) - Y''(1) + \int_0^1 dt_2 \frac{\ln(1 - t_2)\ln(\gamma_1 + t_2)\ln(\gamma_2 - t_2)}{\gamma_3 - t_2} \,,$$

$$(38)$$

where we have introduced the notation

$$Y'(t_m) = \int_0^{t_m} dt_4 \frac{\ln(\gamma_2 - t_4)}{\gamma_1 + t_4} \int_0^{t_4} dt_2 \frac{\ln(1 - t_2)}{\gamma_3 - t_2},$$

$$Y''(t_m) = \ln(\gamma_1 + t_m) \int_0^{t_m} dt_2 \frac{\ln(1 - t_2)\ln(\gamma_2 - t_2)}{\gamma_3 - t_2}. \qquad (39)$$

The last term in (38) is the required integral. The expansion of the integral $Y'(t_m)$ in terms of multiple polylogarithms is similar to the evaluation of $I'(t_m)$ in Eq. (22). The result of the calculation is

$$Y'(t_m) = Li_{1,1,1,1}\left(\gamma_3, -\frac{\gamma_1}{\gamma_3}, -\frac{\gamma_2}{\gamma_1}, \frac{t_m}{\gamma_2}\right) + \ln(\gamma_2 - t_m)Li_{1,1,1}\left(\gamma_3, -\frac{\gamma_1}{\gamma_3}, -\frac{t_m}{\gamma_1}\right) . \quad (40)$$

For the calculation of $Y''(t_m)$ one can make use of (18). Finally using Eqs. (38) and (40) one arrives at the result

$$\int_0^1 dt \frac{\ln(1 - t)\ln(\gamma_1 + t)\ln(\gamma_2 - t)}{\gamma_3 - t} = -Li_{1,1,1,1}\left(\gamma_3, -\frac{\gamma_1}{\gamma_3}, -\frac{\gamma_2}{\gamma_1}, \frac{1}{\gamma_2}\right)$$

$$-Li_{1,1,1,1}\left(\gamma_3, \frac{\gamma_2}{\gamma_3}, -\frac{\gamma_1}{\gamma_2}, -\frac{1}{\gamma_1}\right) - \ln(\gamma_1 + 1)Li_{1,1,1}\left(\gamma_3, \frac{\gamma_2}{\gamma_3}, \frac{1}{\gamma_2}\right) \qquad (41)$$

$$-\ln(\gamma_2 - 1)Li_{1,1,1}\left(\gamma_3, -\frac{\gamma_1}{\gamma_3}, -\frac{1}{\gamma_1}\right) - \ln(\gamma_1 + 1)\ln(\gamma_2 - 1)Li_{1,1}\left(\gamma_3, \frac{1}{\gamma_3}\right) .$$

To obtain the formula for the L function with $\alpha_2 = 0$ we must only change γ_1, γ_2, and γ_3 to $\alpha_1 - 1$, $\alpha_3 + 1$, and $\alpha_4 + 1$ according to Eq. (37):

$$L_{-++}(\alpha_1, 0, \alpha_3, \alpha_4) = \int_0^1 dy \frac{\ln(\alpha_1 - y)\ln y \ln(\alpha_3 + y)}{\alpha_4 + y} =$$

$$-Li_{1,1,1,1}\left(1 + \alpha_4, \frac{1 - \alpha_1}{1 + \alpha_4}, \frac{1 + \alpha_3}{1 - \alpha_1}, \frac{1}{1 + \alpha_3}\right) - Li_{1,1,1,1}\left(1 + \alpha_4, \frac{1 + \alpha_3}{1 + \alpha_4}, \frac{1 - \alpha_1}{1 + \alpha_3}, \frac{1}{1 - \alpha_1}\right)$$

$$-\ln \alpha_1 Li_{1,1,1}\left(1 + \alpha_4, \frac{1 + \alpha_3}{1 + \alpha_4}, \frac{1}{1 + \alpha_3}\right) - \ln \alpha_3 Li_{1,1,1}\left(1 + \alpha_4, \frac{1 - \alpha_1}{1 + \alpha_4}, \frac{1}{1 - \alpha_1}\right)$$

$$-\ln \alpha_1 \ln \alpha_3 Li_{1,1}\left(1 + \alpha_4, \frac{1}{1 + \alpha_4}\right) . \quad (42)$$

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For the case $\alpha_4 = -1$ we calculate the limit of the rhs of (42) for $\alpha_4 \to -1$ and obtain

$$L_{-++}(\alpha_1, 0, \alpha_3, -1) = Li_{2,1}\left(1 - \alpha_1, \frac{1 + \alpha_3}{1 - \alpha_1}, \frac{1}{1 + \alpha_3}\right)$$

$$+ Li_{2,1}\left(1 + \alpha_3, \frac{1 - \alpha_1}{1 + \alpha_3}, \frac{1}{1 - \alpha_1}\right) + \ln\alpha_1 Li_{2,1}\left(1 + \alpha_3, \frac{1}{1 + \alpha_3}\right)$$

$$+ \ln\alpha_3 Li_{2,1}\left(1 - \alpha_1, \frac{1}{1 - \alpha_1}\right) + \ln\alpha_1 \ln\alpha_3\, \zeta(2) . \tag{43}$$

IV. TRANSFORMATION OF L_{+++} TO MULTIPLE POLYLOGARITHMS

In this section we will show that all our L_{+++} functions can be expressed in terms of multiple polylogarithms.

A. General case for the L_{+++} function

We now proceed with the transformation of the triple index function L_{+++},

$$L_{+++}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \int_0^1 dy \frac{\ln(\alpha_1 + y)\ln(\alpha_2 + y)\ln(\alpha_3 + y)}{\alpha_4 + y} . \tag{44}$$

After changing the integration variable $y = -\alpha_1 t$ we obtain

$$- \int_0^{-1/\alpha_1} dt \frac{\ln(\alpha_1 - \alpha_1 t)\ln(\alpha_2 - \alpha_1 t)\ln(\alpha_3 - \alpha_1 t)}{\frac{\alpha_4}{\alpha_1} - t} = - \int_0^{-1/\alpha_1} dt \frac{\ln\alpha_1 \ln(\alpha_2 - \alpha_1 t)\ln(\alpha_3 + \alpha_1 t)}{\frac{\alpha_4}{\alpha_1} - t}$$

$$- \int_0^{-1/\alpha_1} dt \frac{\ln(1 - t)[\ln\alpha_1 + \ln(\frac{\alpha_2}{\alpha_1} - t)][\ln\alpha_1 + \ln(\frac{\alpha_3}{\alpha_1} - t)]}{\frac{\alpha_4}{\alpha_1} - t} =$$

$$\ln\alpha_1 \int_0^1 dy \frac{\ln(\alpha_2 + y)\ln(\alpha_3 + y)}{\alpha_4 + y} - \ln^2\alpha_1 \int_0^{-1/\alpha_1} dt \frac{\ln(1 - t)}{\frac{\alpha_4}{\alpha_1} - t}$$

$$- \ln\alpha_1 \int_0^{-1/\alpha_1} dt \frac{\ln(1 - t)\ln(\frac{\alpha_2}{\alpha_1} - t)}{\frac{\alpha_4}{\alpha_1} - t} - \ln\alpha_1 \int_0^{-1/\alpha_1} dt \frac{\ln(1 - t)\ln(\frac{\alpha_3}{\alpha_1} - t)}{\frac{\alpha_4}{\alpha_1} - t}$$

$$- \int_0^{-1/\alpha_1} dt \frac{\ln(1 - t)\ln(\frac{\alpha_2}{\alpha_1} - t)\ln(\frac{\alpha_3}{\alpha_1} - t)}{\frac{\alpha_4}{\alpha_1} - t} . \tag{45}$$

The first integral on the rhs of (45) has been calculated in Eq. (19). For the second integral one makes use of the formula (17) (the only change is $\alpha_2 \to \alpha_1$). For the

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evaluation of the third and fourth integrals one uses Eq. (18). We are left with the most complicated fifth integral. Let us consider an integral of the type

$$\int\limits_0^{t_m} dt \, \frac{\ln(1-t)\ln(\gamma_1-t)\ln(\gamma_2-t)}{\gamma_3-t} \, . \tag{46}$$

This integral is an analogue of the integral in Eq. (26). The calculation proceeds in a similar way. One obtains the result

$$\int\limits_0^{t_m} dt \, \frac{\ln(1-t)\ln(\gamma_1-t)\ln(\gamma_2-t)}{\gamma_3-t} = -\ln(\gamma_1-t_m)\ln(\gamma_2-t_m)Li_{1,1}\left(\gamma_3,\frac{t_m}{\gamma_3}\right)$$

$$-\ln(\gamma_2-t_m)Li_{1,1,1}\left(\gamma_3,\frac{\gamma_1}{\gamma_3},\frac{t_m}{\gamma_1}\right) -\ln(\gamma_1-t_m)Li_{1,1,1}\left(\gamma_3,\frac{\gamma_2}{\gamma_3},\frac{t_m}{\gamma_2}\right) \tag{47}$$

$$-Li_{1,1,1,1}\left(\gamma_3,\frac{\gamma_2}{\gamma_3},\frac{\gamma_1}{\gamma_2},\frac{t_m}{\gamma_1}\right) - Li_{1,1,1,1}\left(\gamma_3,\frac{\gamma_1}{\gamma_3},\frac{\gamma_2}{\gamma_1},\frac{t_m}{\gamma_2}\right) \, .$$

Taking into account everything mentioned above for Eq. (45) we arrive at the final result for the L_{+++} function,

$$L_{+++}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = Li_{1,1,1,1}\left(\frac{\alpha_4}{\alpha_1},\frac{\alpha_2}{\alpha_4},\frac{\alpha_3}{\alpha_2},-\frac{1}{\alpha_3}\right)$$

$$+Li_{1,1,1,1}\left(\frac{\alpha_4}{\alpha_1},\frac{\alpha_3}{\alpha_4},\frac{\alpha_2}{\alpha_3},-\frac{1}{\alpha_2}\right) + \ln\alpha_1 Li_{1,1,1}\left(\frac{\alpha_4}{\alpha_2},\frac{\alpha_3}{\alpha_4},-\frac{1}{\alpha_3}\right)$$

$$+\ln(1+\alpha_2)Li_{1,1,1}\left(\frac{\alpha_4}{\alpha_1},\frac{\alpha_3}{\alpha_4},-\frac{1}{\alpha_3}\right) + \ln(1+\alpha_3)Li_{1,1,1}\left(\frac{\alpha_4}{\alpha_1},\frac{\alpha_2}{\alpha_4},-\frac{1}{\alpha_2}\right)$$

$$+\ln\alpha_1\ln\alpha_2 Li_{1,1}\left(\frac{\alpha_4}{\alpha_3},-\frac{1}{\alpha_4}\right) + \ln\alpha_1\ln(1+\alpha_3)Li_{1,1}\left(\frac{\alpha_4}{\alpha_2},-\frac{1}{\alpha_4}\right) \tag{48}$$

$$+\ln(1+\alpha_2)\ln(1+\alpha_3)Li_{1,1}\left(\frac{\alpha_4}{\alpha_1},-\frac{1}{\alpha_4}\right) + \ln\alpha_1\ln\alpha_2\ln\alpha_3\ln\left(\frac{\alpha_4+1}{\alpha_4}\right) \, .$$

For this equation the conditions (5) are assumed. We emphasize that the arguments of L_{+++} functions occuring in the actual calculation of the massive scalar one–loop integrals are not of the most general type as assumed in the derivation of (48). We have nevertheless included a discussion of the general case because Eq. (48) may be useful in other applications. In the results for the massive scalar integrals one has only the special cases where $\alpha_1 = \alpha_2$ or $\alpha_1 = \alpha_3$ as well as the cases $\alpha_1 = 0$ and/or $\alpha_2 = 0$ and/or $\alpha_3 = 0$ and/or $\alpha_4 = \{-1,0\}$. If some α's coincide with each other Eq. (48) becomes simpler. In this case one can also make use of symmetry properties to obtain simpler relations between the L_{+++} functions and multiple polylogarithms. For the cases $\alpha_1 = 0$ and/or $\alpha_2 = 0$ and/or $\alpha_3 = 0$ and/or $\alpha_4 = \{-1,0\}$ the general

formula (48) is no longer valid and these cases must be studied separately.

B. Special cases for the L_{+++} function

In the Laurent series expansion of the massive scalar one-loop integrals the following special cases for the α_i are present.

1. $\alpha_1 = \alpha_2$ or $\alpha_1 = \alpha_3$

As it was stated in Sec. II the L_{+++} function is symmetric under the permutations $\alpha_i \leftrightarrow \alpha_j$. Therefore, it suffices to consider the case $\alpha_1 = \alpha_2$.

We must evaluate the integral

$$L_{+++}(\alpha_1, \alpha_1, \alpha_3, \alpha_4) = \int_0^1 dy \frac{\ln^2(\alpha_1 + y) \ln(\alpha_3 + y)}{\alpha_4 + y} . \tag{49}$$

This integral can be expressed in different ways. First of all one can directly use Eq. (48) replacing α_2 by α_1. The second possibility is to use symmetry properties. One takes into account the rhs of Eq. (48) and notes that the part with multiple polylogarithms of weight four is symmetric under the exchange $\alpha_2 \leftrightarrow \alpha_3$. It allows one to reduce the number of the multiple polylogarithms from two to one. First we apply Eq. (48) for the case $\alpha_2 = \alpha_3$ replacing α_3 by α_2. Second we change $\alpha_1 \to \alpha_3$ and $\alpha_2 \to \alpha_1$. After these transformations one obtains the following result:

$$L_{+++}(\alpha_1, \alpha_1, \alpha_3, \alpha_4) = \int_0^1 dy \frac{\ln^2(\alpha_1 + y) \ln(\alpha_3 + y)}{\alpha_4 + y} =$$

$$+ 2 Li_{1,1,1,1}\left(\frac{\alpha_4}{\alpha_3}, \frac{\alpha_1}{\alpha_4}, 1, -\frac{1}{\alpha_1}\right) + \ln\alpha_3\, Li_{1,1,1}\left(\frac{\alpha_4}{\alpha_1}, \frac{\alpha_1}{\alpha_4}, -\frac{1}{\alpha_1}\right)$$

$$+ 2\ln(1+\alpha_1) Li_{1,1,1}\left(\frac{\alpha_4}{\alpha_3}, \frac{\alpha_1}{\alpha_4}, -\frac{1}{\alpha_1}\right) + \ln\alpha_3\left[\ln(\alpha_1+1) + \ln\alpha_1\right] Li_{1,1}\left(\frac{\alpha_4}{\alpha_1}, -\frac{1}{\alpha_4}\right) \tag{50}$$

$$+ \ln^2(1+\alpha_1) Li_{1,1}\left(\frac{\alpha_4}{\alpha_3}, -\frac{1}{\alpha_4}\right) + \ln^2\alpha_1 \ln\alpha_3 \ln\left(\frac{\alpha_4+1}{\alpha_4}\right) .$$

There is also the third possibility to express $L_{+++}(\alpha_1, \alpha_1, \alpha_3, \alpha_4)$ in terms of multiple polylogarithms:

$$\int_0^1 dy \frac{\ln^2(\alpha_1 + y) \ln(\alpha_3 + y)}{\alpha_4 + y} \overset{y \to -\alpha_1 t}{=} -\int_0^{-1/\alpha_1} dt \frac{\ln^2(\alpha_1 - \alpha_1 t) \ln(\alpha_3 - \alpha_1 t)}{\frac{\alpha_4}{\alpha_1} - t} =$$

$$- \int_0^{-1/\alpha_1} dt \frac{\left[\ln^2\alpha_1 + 2\ln\alpha_1 \ln(1-t) + \ln^2(1-t)\right]\left[\ln\alpha_1 + \ln(\frac{\alpha_3}{\alpha_1} - t)\right]}{\frac{\alpha_4}{\alpha_1} - t} =$$

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$$+\ln^3\alpha_1\int\limits_0^1\frac{dy}{\alpha_4+y}+\ln^2\alpha_1\int\limits_0^1 dy\frac{\ln(\alpha_3+y)}{\alpha_4+y}-2\ln^2\alpha_1\int\limits_0^{-1/\alpha_1}dt\frac{\ln(1-t)}{\frac{\alpha_4}{\alpha_1}-t}$$

$$-\ln\alpha_1\int\limits_0^{-1/\alpha_1}dt\frac{\ln^2(1-t)}{\frac{\alpha_4}{\alpha_1}-t}-2\ln\alpha_1\int\limits_0^{-1/\alpha_1}dt\frac{\ln(1-t)\ln\left(\frac{\alpha_3}{\alpha_1}-t\right)}{\frac{\alpha_4}{\alpha_1}-t}$$

$$-\int\limits_0^{-1/\alpha_1}dt\frac{\ln^2(1-t)\ln\left(\frac{\alpha_3}{\alpha_1}-t\right)}{\frac{\alpha_4}{\alpha_1}-t}\,.\qquad(51)$$

The first term can be integrated immediately. For the second and third term one uses Eq. (16) and Eq. (17), respectively. The integral of the fourth term can be rewritten as

$$\int\limits_0^{-1/\alpha_1}dt\frac{\ln^2(1-t)}{\frac{\alpha_4}{\alpha_1}-t}=2\int\limits_0^{-1/\alpha_1}\frac{dt_1}{\frac{\alpha_4}{\alpha_1}-t_1}\int\limits_0^{t_1}\frac{dt_2}{1-t_2}\int\limits_0^{t_2}\frac{dt_3}{1-t_3}=2Li_{1,1,1}\left(1,\frac{\alpha_4}{\alpha_1},\frac{-1}{\alpha_4}\right)\,.\qquad(52)$$

The fifth term is calculable with Eq. (14). To integrate the last term one first evaluates the following integral:

$$\int\limits_0^{t_m}dt\frac{\ln^2(1-t)\ln(\beta_1-t)}{\beta_2-t}=-\int\limits_0^{t_m}dt_1\frac{\ln^2(1-t_1)}{\beta_2-t_1}\left\{\int\limits_{t_m}^{t_1}+\int\limits_{\beta_1-1}^{t_m}\right\}\frac{dt_2}{\beta_1-t_2}=$$

$$\int\limits_0^{t_m}\frac{dt_2}{\beta_1-t_2}\int\limits_0^{t_2}dt_1\frac{\ln^2(1-t_1)}{\beta_2-t_1}+\ln(\beta_1-t_m)\int\limits_0^{t_m}dt\frac{\ln^2(1-t)}{\beta_2-t}=\qquad(53)$$

$$2Li_{1,1,1,1}\left(1,\beta_2,\frac{\beta_1}{\beta_2},\frac{t_m}{\beta_1}\right)+2\ln(\beta_1-t_m)Li_{1,1,1}\left(1,\beta_2,\frac{t_m}{\beta_2}\right)\,.$$

Then to calculate the last term of Eq. (51) one only has to change $\beta_1,\beta_2,$ and t_m by the corresponding combinations of α_i. Finally we arrive at the result for the $L_{+++}(\alpha_1,\alpha_1,\alpha_3,\alpha_4)$ function,

$$L_{+++}(\alpha_1,\alpha_1,\alpha_3,\alpha_4)=-2Li_{1,1,1,1}\left(1,\frac{\alpha_4}{\alpha_1},\frac{\alpha_3}{\alpha_4},-\frac{1}{\alpha_3}\right)-2\ln(\alpha_3+1)Li_{1,1,1}\left(1,\frac{\alpha_4}{\alpha_1},-\frac{1}{\alpha_4}\right)$$

$$+2\ln\alpha_1 Li_{1,1,1}\left(\frac{\alpha_4}{\alpha_1},\frac{\alpha_3}{\alpha_4},-\frac{1}{\alpha_3}\right)+2\ln\alpha_1\ln(\alpha_3+1)Li_{1,1}\left(\frac{\alpha_4}{\alpha_1},-\frac{1}{\alpha_4}\right)$$

$$+\ln^2\alpha_1 Li_{1,1}\left(\frac{\alpha_4}{\alpha_3},-\frac{1}{\alpha_4}\right)+\ln^2\alpha_1\ln\alpha_3\ln\left(\frac{\alpha_4+1}{\alpha_4}\right)\,.\qquad(54)$$

This is the third possibility to express $L_{+++}(\alpha_1,\alpha_1,\alpha_3,\alpha_4)$ function in terms of multiple polylogaritms. Each of the Eqs. (50) and (54) contains only one multiple polylogarithm of weight four and they are both equally acceptable from this point of view.

One has a free choice to apply any of these equations for the required L functions. The situation with the $L_{+++}(\alpha_1, \alpha_1, \alpha_3, \alpha_4)$ function is an example of the statement that the expansion of the L functions in terms of multiple polylogarithms is not unique.

2. $\alpha_1 = 0$ (or $\alpha_2 = 0$ or $\alpha_3 = 0$)

For this integral we change the integration variable $y \to 1 - t$,

$$L_{+++}(0, \alpha_2, \alpha_3, \alpha_4) = \int_0^1 dy \frac{\ln y \ln(\alpha_2 + y) \ln(\alpha_3 + y)}{\alpha_4 + y} =$$

$$\int_0^1 dt \frac{\ln(1 - t) \ln(\alpha_2 + 1 - t) \ln(\alpha_3 + 1 - t)}{\alpha_4 + 1 - t} \tag{55}$$

and using Eq. (48) we arrive at the result

$$L_{+++}(0, \alpha_2, \alpha_3, \alpha_4) = -Li_{1,1,1,1}\left(1 + \alpha_4, \frac{1 + \alpha_2}{1 + \alpha_4}, \frac{1 + \alpha_3}{1 + \alpha_2}, \frac{1}{1 + \alpha_3}\right)$$

$$-Li_{1,1,1,1}\left(1 + \alpha_4, \frac{1 + \alpha_3}{1 + \alpha_4}, \frac{1 + \alpha_2}{1 + \alpha_3}, \frac{1}{1 + \alpha_2}\right) - \ln \alpha_2 Li_{1,1,1}\left(1 + \alpha_4, \frac{1 + \alpha_3}{1 + \alpha_4}, \frac{1}{1 + \alpha_3}\right)$$

$$- \ln \alpha_3 Li_{1,1,1}\left(1 + \alpha_4, \frac{1 + \alpha_2}{1 + \alpha_4}, \frac{1}{1 + \alpha_2}\right) - \ln \alpha_2 \ln \alpha_3 Li_{1,1}\left(1 + \alpha_4, \frac{1}{1 + \alpha_4}\right). \tag{56}$$

3. $\alpha_1 = \alpha_2 = 0$

To calculate this integral we again change the integration variable $y \to 1 - t$,

$$L_{+++}(0, 0, \alpha_3, \alpha_4) = \int_0^1 dy \frac{\ln^2 y \ln(\alpha_3 + y)}{\alpha_4 + y} =$$

$$\int_0^1 dt \frac{\ln^2(1 - t) \ln(\alpha_3 + 1 - t)}{\alpha_4 + 1 - t}. \tag{57}$$

For the last integral we use Eq. (53). An additional simplification can be done if one notes that

$$Li_{1,1,1}\left(1, \alpha_4 + 1, \frac{1}{\alpha_4 + 1}\right) = -\mathrm{Li}_3\left(-\frac{1}{\alpha_4}\right). \tag{58}$$

Finally one has

$$L_{+++}(0, 0, \alpha_3, \alpha_4) = 2Li_{1,1,1,1}\left(1, \alpha_4 + 1, \frac{\alpha_3 + 1}{\alpha_4 + 1}, \frac{1}{\alpha_3 + 1}\right) - 2 \ln \alpha_3 \mathrm{Li}_3\left(-\frac{1}{\alpha_4}\right). \tag{59}$$

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4. $\alpha_1 = \alpha_2 = 0$, $\alpha_4 = -1$ *(or $\alpha_2 = \alpha_3 = 0$, $\alpha_4 = -1$)*

In this case one should calculate the limit of the rhs of (59) for $t_m = 1$ and $\alpha_4 \to -1$. After this procedure one obtains

$$L_{+++}(0,0,\alpha_3,-1) = -2Li_{1,2,1}\left(1,\alpha_3+1,\frac{1}{\alpha_3+1}\right) - 2\ln\alpha_3\zeta(3). \qquad (60)$$

For the case $\alpha_2 = \alpha_3 = 0$ and $\alpha_4 = -1$ one can use the same formula. The only change is $\alpha_3 \to \alpha_1$.

5. $\alpha_1 = 0$, $\alpha_4 = -1$

To obtain the solution for these values of the α_i we must find the limit of the rhs of (56) for $\alpha_4 \to -1$. After taking the limit one arrives at the result

$$L_{+++}(0,\alpha_2,\alpha_3,-1) = +Li_{2,1,1}\left(1+\alpha_2,\frac{1+\alpha_3}{1+\alpha_2},\frac{1}{1+\alpha_3}\right)$$

$$+Li_{2,1,1}\left(1+\alpha_3,\frac{1+\alpha_2}{1+\alpha_3},\frac{1}{1+\alpha_2}\right) + \ln\alpha_2 Li_{2,1}\left(1+\alpha_3,\frac{1}{1+\alpha_3}\right)$$

$$+\ln\alpha_3 Li_{2,1}\left(1+\alpha_2,\frac{1}{1+\alpha_2}\right) + \ln\alpha_2\ln\alpha_3\zeta(2). \qquad (61)$$

V. TRANSFORMATION OF L_+ TO MULTIPLE POLYLOGARITHMS

In this section we will show that all our L_+ functions can be expressed in terms of multiple polylogarithms.

A. General case for the L_+ function

Here we derive the general formula for the single index L_+ function Eq. (2),

$$L_+(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \int\limits_0^1 dy\,\frac{\ln(\alpha_1+y)}{\alpha_4+y}Li_2(\alpha_2+\alpha_3 y). \qquad (62)$$

After changing the integration variable $y \to (t-\alpha_2)/\alpha_3$ one gets

$$L_+ = \int\limits_{\alpha_2}^{\alpha_2+\alpha_3} \frac{dt}{\alpha_3}\,\frac{\ln(\alpha_1+\frac{t-\alpha_2}{\alpha_3})}{\alpha_4+\frac{t-\alpha_2}{\alpha_3}}Li_2(t) = \int\limits_{\alpha_2}^{\alpha_2+\alpha_3} dt\,\frac{-\ln\alpha_3 + \ln(\alpha_1\alpha_3-\alpha_2+t)}{\alpha_3\alpha_4-\alpha_2+t}Li_2(t). \qquad (63)$$

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The integration interval can be split into two pieces, $[\alpha_2, 0]$ and $[0, \alpha_2 + \alpha_3]$. One can then write L_+ as a sum of four terms,

$$L_+ = -\ln \alpha_3 \left\{ \int_0^{\alpha_2+\alpha_3} - \int_0^{\alpha_2} \right\} \frac{dt}{\gamma + t} \mathrm{Li}_2(t) + \left\{ \int_0^{\alpha_2+\alpha_3} - \int_0^{\alpha_2} \right\} dt \frac{\ln(\alpha + t)}{\gamma + t} \mathrm{Li}_2(t), \quad (64)$$

where we have introduced the notation

$$\alpha = \alpha_1 \alpha_3 - \alpha_2, \qquad \gamma = \alpha_3 \alpha_4 - \alpha_2. \quad (65)$$

Looking at Eq. (64) it is clear that there are only two different types of integrals to be dealt with,

$$\int_0^{t_m} \frac{dt}{\gamma + t} \mathrm{Li}_2(t) \qquad \text{and} \qquad \int_0^{t_m} dt \frac{\ln(\alpha + t)}{\gamma + t} \mathrm{Li}_2(t). \quad (66)$$

The upper limits are $t_m = \alpha_2 + \alpha_3$ or $t_m = \alpha_2$. The first integral can be evaluated analytically in terms of standard logarithms and classical polylogarithms up to Li_3. However, the same integral can also be expressed in terms of multiple polylogarithms via the integral representation (7), e.g.,

$$\int_0^{t_m} \frac{dt}{\gamma + t} \mathrm{Li}_2(t) = \int_0^{t_m} \frac{dt_1}{\gamma + t_1} \int_0^{t_1} \frac{dt_2}{t_2} \int_0^{t_2} \frac{dt_3}{1 - t_3} = -Li_{2,1}\left(-\gamma, \frac{-t_m}{\gamma}\right). \quad (67)$$

We now deal with the second integral in (66). Consider the following multiple polylogarithm of weight four:

$$Li_{2,1,1}\left(-\gamma, \frac{\alpha}{\gamma}, \frac{t_m}{-\alpha}\right) = \int_0^{t_m} \frac{dt_2}{-\alpha - t_2} \int_0^{t_2} \frac{dt_1}{-\gamma - t_1} \mathrm{Li}_2(t_1) = \int_0^{t_m} \frac{dt_1}{\gamma + t_1} \mathrm{Li}_2(t_1) \int_{t_1}^{t_m} \frac{dt_2}{\alpha + t_2} =$$

$$\int_0^{t_m} \frac{dt_1}{\gamma + t_1} \mathrm{Li}_2(t_1) \ln(\alpha + t_m) - \int_0^{t_m} \frac{dt_1}{\gamma + t_1} \mathrm{Li}_2(t_1) \ln(\alpha + t_1). \quad (68)$$

In the first step we have used the usual trick to change the order of integration. As already noted before [see Eq. (67)] the first term on the second line can be expressed through a multiple polylogarithm of weight three. Thus one has

$$\int_0^{t_m} dt \frac{\ln(\alpha + t)}{\gamma + t} \mathrm{Li}_2(t) = -Li_{2,1,1}\left(-\gamma, \frac{\alpha}{\gamma}, \frac{t_m}{-\alpha}\right) - Li_{2,1}\left(-\gamma, \frac{-t_m}{\gamma}\right) \ln(\alpha + t_m). \quad (69)$$

Finally, substituting Eqs. (67) and (69) into Eq. (64) we arrive at the desired relation

$$L_+(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = Li_{2,1,1}\left(\alpha_2 - \alpha_3\alpha_4, \frac{\alpha_2 - \alpha_1\alpha_3}{\alpha_2 - \alpha_3\alpha_4}, \frac{\alpha_2}{\alpha_2 - \alpha_1\alpha_3}\right)$$

$$-Li_{2,1,1}\left(\alpha_2 - \alpha_3\alpha_4, \frac{\alpha_2 - \alpha_1\alpha_3}{\alpha_2 - \alpha_3\alpha_4}, \frac{\alpha_2 + \alpha_3}{\alpha_2 - \alpha_1\alpha_3}\right) + \ln \alpha_1 Li_{2,1}\left(\alpha_2 - \alpha_3\alpha_4, \frac{\alpha_2}{\alpha_2 - \alpha_3\alpha_4}\right) \quad (70)$$

$$-\ln(\alpha_1 + 1) Li_{2,1}\left(\alpha_2 - \alpha_3\alpha_4, \frac{\alpha_2 + \alpha_3}{\alpha_2 - \alpha_3\alpha_4}\right).$$

We should note that, similar to Eq. (27), the conditions (5) are assumed for the variables α_i. Also, one cannot directly use Eq. (70) if $\alpha_2 - \alpha_3\alpha_4 = 0$ or $\alpha_2 - \alpha_1\alpha_3 = 0$. However, in the results for the massive scalar integrals precisely these special cases appear, as well as the cases where $\alpha_1 = 0$ and/or $\alpha_2 = 0$ and/or $\alpha_3 = 0$ and/or $\alpha_4 = \{-1, 0\}$. In such cases the general formula (70) is no longer valid and these cases must be studied separately.

B. Special cases for the L_+ function

In the Laurent series expansion of the massive scalar one-loop integrals the following special cases appear for the arguments of the L_+ functions:

1. $\alpha_2 - \alpha_3\alpha_4 = 0$ (or $\alpha_2 - \alpha_1\alpha_3 = 0$)

In this case one must find the limit of the rhs of Eq. (70) for $\alpha_2 \to \alpha_3\alpha_4$. First we rewrite the rhs of Eq. (70) in terms of multidimensional integrals via the definition (7). Second we replace α_2 by $\alpha_3\alpha_4$. We finally again use the definition (7) to obtain the result

$$L_+(\alpha_1, \alpha_3\alpha_4, \alpha_3, \alpha_4) = -Li_{3,1}\left(\alpha_3(\alpha_4 - \alpha_1), \frac{\alpha_4}{\alpha_4 - \alpha_1}\right)$$

$$+Li_{3,1}\left(\alpha_3(\alpha_4 - \alpha_1), \frac{\alpha_4 + 1}{\alpha_4 - \alpha_1}\right) - \ln\alpha_1 Li_3(\alpha_3\alpha_4) + \ln(\alpha_1 + 1)Li_3(\alpha_3(\alpha_4 + 1)). \quad (71)$$

When $\alpha_2 - \alpha_1\alpha_3 = 0$ one must find the limit of the rhs of Eq. (70) for $\alpha_2 \to \alpha_1\alpha_3$. We again rewrite the rhs of Eq. (70) in terms of multidimensional integrals. We then replace α_2 by $\alpha_1\alpha_3$ and use the definition (7). We arrive at the result

$$L_+(\alpha_1, \alpha_1\alpha_3, \alpha_3, \alpha_4) = -Li_{2,2}\left(\alpha_3(\alpha_1 - \alpha_4), \frac{\alpha_1}{\alpha_1 - \alpha_4}\right)$$

$$+Li_{2,2}\left(\alpha_3(\alpha_1 - \alpha_4), \frac{\alpha_1 + 1}{\alpha_1 - \alpha_4}\right) + \ln\alpha_1 Li_{2,1}\left(\alpha_3(\alpha_1 - \alpha_4), \frac{\alpha_1}{\alpha_1 - \alpha_4}\right) \quad (72)$$

$$-\ln(\alpha_1 + 1)Li_{2,1}\left(\alpha_3(\alpha_1 - \alpha_4), \frac{\alpha_1 + 1}{\alpha_1 - \alpha_4}\right).$$

2. $\alpha_1 = 0$

Unfortunately in this case one cannot use Eq. (70) for $\alpha_1 = 0$ because one is immediately faced with the problem of a logarithmic infinity. One must find another algorithm to express the $L_+(0, \alpha_2, \alpha_3, \alpha_4)$ function in terms of multiple polylogarithms.

After changing the integration variable $y \to 1 - t$ one gets

$$\int\limits_0^1 dy \, \frac{\ln y}{\alpha_4 + y} \mathrm{Li}_2(\alpha_2 + \alpha_3 y) = \int\limits_0^1 dt \, \frac{\ln(1-t)}{\alpha_4 + 1 - t} \mathrm{Li}_2(\alpha_2 + \alpha_3 - \alpha_3 t) =$$

$$\int\limits_0^1 dt_1 \frac{\ln(1-t_1)}{\alpha_4 + 1 - t_1} \int\limits_{\alpha_2/\alpha_3+1}^{t_1} dt_2 \frac{\ln(1 - \alpha_2 - \alpha_3 + \alpha_3 t_2)}{\frac{\alpha_2}{\alpha_3} + 1 - t_2} =$$

$$\int\limits_0^1 dt_1 \frac{\ln(1-t_1)}{\alpha_4 + 1 - t_1} \left\{ \int\limits_1^{t_1} + \int\limits_{\alpha_2/\alpha_3+1}^1 \right\} dt_2 \frac{\ln(1 - \alpha_2 - \alpha_3 + \alpha_3 t_2)}{\frac{\alpha_2}{\alpha_3} + 1 - t_2} = \quad (73)$$

$$- \int\limits_0^1 dt_2 \frac{\ln(1 - \alpha_2 - \alpha_3 + \alpha_3 t_2)}{\frac{\alpha_2}{\alpha_3} + 1 - t_2} \int\limits_0^{t_2} dt_1 \frac{\ln(1-t_1)}{\alpha_4 + 1 - t_1} - \mathrm{Li}_2(\alpha_2) Li_{1,1}\left(\alpha_4 + 1, \frac{1}{\alpha_4 + 1}\right).$$

The last integral is an analogue of $I'(t_m)$ in Eq. (22). First one notes that

$$\int\limits_0^{t_2} dt_1 \frac{\ln(1-t_1)}{\alpha_4 + 1 - t_1} = -Li_{1,1}\left(\alpha_4 + 1, \frac{t_1}{\alpha_4 + 1}\right). \quad (74)$$

Then one considers the following chain of transformations:

$$\int\limits_0^1 \frac{dt_2}{1 - \alpha_2 - \alpha_3 + \alpha_3 t_2} \int\limits_0^{t_2} \frac{dt_1}{\frac{\alpha_2}{\alpha_3} + 1 - t_1} Li_{1,1}\left(\alpha_4 + 1, \frac{t_1}{\alpha_4 + 1}\right) =$$

$$\int\limits_0^1 \frac{dt_1}{\frac{\alpha_2}{\alpha_3} + 1 - t_1} Li_{1,1}\left(\alpha_4 + 1, \frac{t_1}{\alpha_4 + 1}\right) \int\limits_{t_1}^1 \frac{dt_2}{1 - \alpha_2 - \alpha_3 + \alpha_3 t_2} =$$

$$\frac{1}{\alpha_3} \ln(1 - \alpha_2) \int\limits_0^1 \frac{dt_1}{\frac{\alpha_2}{\alpha_3} + 1 - t_1} Li_{1,1}\left(\alpha_4 + 1, \frac{t_1}{\alpha_4 + 1}\right) \quad (75)$$

$$- \frac{1}{\alpha_3} \int\limits_0^1 dt_1 \frac{\ln(1 - \alpha_2 - \alpha_3 + \alpha_3 t_1)}{\frac{\alpha_2}{\alpha_3} + 1 - t_1} Li_{1,1}\left(\alpha_4 + 1, \frac{t_1}{\alpha_4 + 1}\right)$$

Using Eq. (74) we see that the last integral is exactly the integral required in Eq. (73). The initial integral of Eq. (75) and the first integral of the rhs of Eq. (75) can be expressed in terms of multiple polylogarithms due to the definition (7). Finally for the $L_+(0, \alpha_2, \alpha_3, \alpha_4)$ function we obtain

$$L_+(0, \alpha_2, \alpha_3, \alpha_4) = Li_{1,1,1,1}\left(\alpha_4 + 1, \frac{\alpha_2 + \alpha_3}{\alpha_3(\alpha_4 + 1)}, \frac{\alpha_2 + \alpha_3 - 1}{\alpha_2 + \alpha_3}, \frac{\alpha_3}{\alpha_2 + \alpha_3 - 1}\right) \quad (76)$$

$$+ \ln(1 - \alpha_2) Li_{1,1,1}\left(\alpha_4 + 1, \frac{\alpha_2 + \alpha_3}{\alpha_3(\alpha_4 + 1)}, \frac{\alpha_3}{\alpha_2 + \alpha_3}\right) - \mathrm{Li}_2(\alpha_2) Li_{1,1}\left(\alpha_4 + 1, \frac{1}{\alpha_4 + 1}\right).$$

3. $\alpha_1 = 0$, $\alpha_4 = -1$

For these values of the α_i one uses Eq. (76) to calculate the limit of the rhs for $\alpha_4 \to -1$. One arrives at the result

$$L_+(0, \alpha_2, \alpha_3, -1) = -Li_{2,1,1}\left(\frac{\alpha_2 + \alpha_3}{\alpha_3}, \frac{\alpha_2 + \alpha_3 - 1}{\alpha_2 + \alpha_3}, \frac{\alpha_3}{\alpha_2 + \alpha_3 - 1}\right) \tag{77}$$
$$- \ln(1 - \alpha_2) Li_{2,1}\left(\frac{\alpha_2 + \alpha_3}{\alpha_3}, \frac{\alpha_3}{\alpha_2 + \alpha_3}\right) + Li_2(\alpha_2)\zeta(2).$$

4. $\alpha_1 = 0$, $\alpha_2 + \alpha_3 = 1$ *(and $\alpha_4 = -1$)*

If one takes a look at Eq. (76) one realizes that there is a problem if $\alpha_2 + \alpha_4 = 1$. To express the L_+ function for this configuration of the α_i the limit of the rhs of (76) for $\alpha_2 \to 1 - \alpha_3$ must be found. The result is

$$L_+(0, 1 - \alpha_3, \alpha_3, \alpha_4) = -Li_{1,1,2}\left(\alpha_4 + 1, \frac{1}{\alpha_3(\alpha_4 + 1)}, \alpha_3\right) \tag{78}$$
$$+ \ln \alpha_3 Li_{1,1,1}\left(\alpha_4 + 1, \frac{1}{\alpha_3(\alpha_4 + 1)}, \alpha_3\right) - Li_2(1 - \alpha_3) Li_{1,1}\left(\alpha_4 + 1, \frac{1}{\alpha_4 + 1}\right).$$

For the case $\alpha_1 = 0$, $\alpha_2 + \alpha_3 = 1$, and $\alpha_4 = -1$ one must find in addition the limit for $\alpha_4 \to -1$. One arrives at the result

$$L_+(0, 1 - \alpha_3, \alpha_3, -1) = Li_{2,2}\left(\frac{1}{\alpha_3}, \alpha_3\right) - \ln \alpha_3 Li_{2,1}\left(\frac{1}{\alpha_3}, \alpha_3\right) + \zeta(2) Li_2(1 - \alpha_3). \tag{79}$$

5. $\alpha_1 = 0$, $\alpha_2 = -\alpha_3$

To obtain the result for this case one must calculate the limit of the rhs of (76) for $\alpha_3 \to -\alpha_2$. After taking the limit one has

$$L_+(0, \alpha_2, -\alpha_2, \alpha_4) = -Li_{1,1,2}\left(\frac{\alpha_2}{\alpha_2 - 1}, -\alpha_4, -\frac{1}{\alpha_4}\right)$$
$$+ \ln(1 - \alpha_2) Li_{1,2}\left(-\alpha_4, -\frac{1}{\alpha_4}\right) + Li_2(\alpha_2) Li_2\left(-\frac{1}{\alpha_4}\right). \tag{80}$$

6. $\alpha_1 = 0,\ \alpha_2 = 0$

For this case one can directly use Eq. (76),

$$L_+(0,0,\alpha_3,\alpha_4) = Li_{1,1,1,1}\left(\alpha_4 + 1, \frac{1}{\alpha_4 + 1}, \frac{\alpha_3 - 1}{\alpha_3}, \frac{\alpha_3}{\alpha_3 - 1}\right). \tag{81}$$

But there is also another very simple possibility. We first change the integration variable $y \to t/\alpha_3$,

$$\int_0^1 dy\, \frac{\ln y}{\alpha_4 + y} \mathrm{Li}_2(\alpha_3 y) = \int_0^{\alpha_3} dt\, \frac{\ln(t/\alpha_3)}{\alpha_3\alpha_4 + t}\mathrm{Li}_2(t) = \int_0^{\alpha_3} \frac{dt_1}{\alpha_3\alpha_4 + t_1}\mathrm{Li}_2(t_1)\int_{\alpha_3}^{t_1} \frac{dt_2}{t_2} =$$

$$-\int_0^{\alpha_3} \frac{dt_2}{t_2}\int_0^{t_2} \frac{dt_1}{\alpha_3\alpha_4 + t_1}\mathrm{Li}_2(t_1) = \int_0^{\alpha_3} \frac{dt_2}{t_2}\int_0^{t_2} \frac{dt_1}{-\alpha_3\alpha_4 + t_1}\int_0^{t_1} \frac{dt_3}{t_3}\int_0^{t_3} \frac{dt_4}{1 - t_4}. \tag{82}$$

Now using the definition (7) we obtain the result

$$L_+(0,0,\alpha_3,\alpha_4) = Li_{2,2}\left(-\alpha_3\alpha_4, \frac{-1}{\alpha_4}\right). \tag{83}$$

The reader has a free choice to use either formula (81) or (83). Both equations contain multiple polylogarithm of weight four. The depth of the multiple polylogarithm in Eq. (83) is two against four in Eq. (81). For $\alpha_4 = -1$ Eq. (83) can be directly used. However, in the case of Eq. (81) one must first calculate the limit for $\alpha_4 \to -1$.

7. $\alpha_1 = 0,\ \alpha_2 = 1$

Unfortunately, in this case one cannot use Eq. (76) because of the term $\ln(1 - \alpha_2)$. To express this L_+ function in terms of multiple polylogarithms we first make use of a standard relation between dilogs with arguments x and $1 - x$ for the function Li_2 under the sign of the integral:

$$\int_0^1 dy\frac{\ln y}{\alpha_4 + y}\mathrm{Li}_2(1 + \alpha_3 y) = \int_0^1 dy\frac{\ln y}{\alpha_4 + y}[\zeta(2) - \ln(-\alpha_3 y)\ln(1 + \alpha_3 y) - \mathrm{Li}_2(-\alpha_3 y)] =$$

$$\zeta(2)\int_0^1 dy\, \frac{\ln y}{\alpha_4 + y} - \int_0^1 dy\frac{\ln y}{\alpha_4 + y}\mathrm{Li}_2(-\alpha_3 y) - \int_0^1 dy\frac{\ln y[\ln(-\alpha_3) + \ln y]\ln(1 + \alpha_3 y)}{\alpha_4 + y} =$$

$$\zeta(2)\mathrm{Li}_2\left(-\frac{1}{\alpha_4}\right) - Li_{1,1,1,1}\left(\alpha_4 + 1, \frac{1}{\alpha_4 + 1}, \frac{\alpha_3 + 1}{\alpha_3}, \frac{\alpha_3}{\alpha_3 + 1}\right) \tag{84}$$

$$-\ln(-\alpha_3)\int_0^1 dy\, \frac{\ln y\ln(1 + \alpha_3 y)}{\alpha_4 + y} - \int_0^1 dy\frac{\ln^2 y\ln(1 + \alpha_3 y)}{\alpha_4 + y},$$

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where the $Li_{1,1,1,1}$ function was obtained with the help of Eq. (81). To obtain the last integral in Eq. (84) one proceeds as follows:

$$\int\limits_0^1 dy \frac{\ln^2 y \ln(1+\alpha_3 y)}{\alpha_4+y} \overset{y\to 1-t}{=} \int\limits_0^1 dt \frac{\ln^2(1-t)\ln(1+\alpha_3-\alpha_3 t)}{\alpha_4+1-t} =$$

$$\int\limits_0^1 dt_1 \frac{\ln^2(1-t_1)}{\alpha_4+1-t_1} \int\limits_1^{t_1} \frac{-\alpha_3 dt_2}{1+\alpha_3-\alpha_3 t_2} = \int\limits_0^1 \frac{dt_2}{\frac{1}{\alpha_3}+1-t_2} \int\limits_0^{t_2} dt_1 \frac{\ln^2(1-t_1)}{\alpha_4+1-t_1} =$$

$$2\int\limits_0^1 \frac{dt_2}{\frac{1}{\alpha_3}+1-t_2} \int\limits_0^{t_2} \frac{dt_1}{\alpha_4+1-t_1} \int\limits_0^{t_1} \frac{dt_3}{1-t_3} \int\limits_0^{t_3} \frac{dt_4}{1-t_4} = \tag{85}$$

$$2Li_{1,1,1,1}\left(1,\alpha_4+1,\frac{\alpha_3+1}{\alpha_3(\alpha_4+1)},\frac{\alpha_3}{\alpha_3+1}\right).$$

Similarly one can evaluate the remaining integral

$$\int\limits_0^1 dy \frac{\ln y \ln(1+\alpha_3 y)}{\alpha_4+y} = -Li_{1,1,1}\left(\alpha_4+1,\frac{\alpha_3+1}{\alpha_3(\alpha_4+1)},\frac{\alpha_3}{\alpha_3+1}\right). \tag{86}$$

Now combining Eqs. (84), (85), and (86) one arrives at the result

$$L_+(0,1,\alpha_3,\alpha_4) = -2Li_{1,1,1,1}\left(1,\alpha_4+1,\frac{\alpha_3+1}{\alpha_3(\alpha_4+1)},\frac{\alpha_3}{\alpha_3+1}\right)$$

$$-Li_{1,1,1,1}\left(\alpha_4+1,\frac{1}{\alpha_4+1},\frac{\alpha_3+1}{\alpha_3},\frac{\alpha_3}{\alpha_3+1}\right) \tag{87}$$

$$+\ln(-\alpha_3)Li_{1,1,1}\left(\alpha_4+1,\frac{\alpha_3+1}{\alpha_3(\alpha_4+1)},\frac{\alpha_3}{\alpha_3+1}\right) + \zeta(2)\mathrm{Li}_2\left(-\frac{1}{\alpha_4}\right).$$

8. $\alpha_1 = 0$, $\alpha_2 = -\alpha_3 = 1$

For these values of the α_i we must find the limit of the rhs of Eq. (87) for $\alpha_3 \to -1$. After taking the limit we obtain

$$L_+(0,1,-1,\alpha_4) = Li_{1,1,2}\left(\alpha_4+1,\frac{1}{\alpha_4+1},1\right)$$

$$+2Li_{1,1,2}\left(1,\alpha_4+1,\frac{1}{\alpha_4+1}\right) + \zeta(2)\mathrm{Li}_2\left(-\frac{1}{\alpha_4}\right). \tag{88}$$

VI. CONCLUSIONS

We have presented all the necessary relations to transform the L functions [as defined in Eqs. (1) and (2)] that occur in our $\mathcal{O}(\varepsilon^2)$ results [1] for the Laurent series

expansion of massive scalar one-loop integrals to multiple polylogarithms. We have used these relations to transform our results on massive one-loop integrals involving L functions to corresponding results involving multiple polylogarithms. The multiple polylogarithms results are readily available in electronic form [11].

Despite of the fact that the relations between the L functions and the multiple polylogarithms have been derived having the massive scalar one-loop integrals in mind they can also be used in a more general setting. In fact, any definite integral given by

$$\int\limits_{A}^{B} \frac{\ln(a_1 + b_1 x)\ln(a_2 + b_2 x)\ln(a_3 + b_3 x)dx}{a_4 + b_4 x} \text{ or } \int\limits_{A}^{B} \frac{\ln(a_1 + b_1 x)\text{Li}_2(a_2 + b_2 x)dx}{a_3 + b_3 x}$$

can be written in terms of multiple polylogarithms with the help of the relations presented in this paper. It is worthwhile to mention that all the equations presented in the present paper have been also checked numerically.

We have found several examples where the representation of the L functions in terms of multiple polylogarithms is not unique. This reflects the fact that multiple polylogarithms obey quasishuffle and shuffle Hopf algebras and hence satisfy numerous identities as is the case for the classical polylogarithms. More information about identities between multiple polylogarithms can be found, e.g., in [5] and [9], and references therein.

For future parton model applications of our results numerical efficiency is an important issue. We are presently writing numerical C++ codes to compare the numerical efficiency of the two representations in terms of L functions and multiple polylogarithms.

ACKNOWLEDGMENTS

APPENDIX

In this Appendix we consider as an example the real part of the $\mathcal{O}(\varepsilon^2)$ coefficient $\text{Re } D_1^{(2)}$ of the Laurent series expansion of the massive box D_1 with three massive propagators. Using the rules written down in the main text of this paper we have expressed the corresponding results of [1] involving L functions in terms of multiple

polylogarithms. The L function structure of $\operatorname{Re} D_1^{(2)}$ in [1] is sufficiently rich to provide an illustration of the corresponding complexity in terms of multiple polylogarithms when transforming to the latter representation. We mention that all multiple polylogarithms up to weight three have been reexpressed in terms of classical polylogarithms. We then used automatic program codes to simplify the classical polylogarithms as much as possible, as was also done in [1].

We use the notation and the conventions of [1]. In brief, we use the Mandelstam-type variables

$$s \equiv (p_1 + p_2)^2, \quad t \equiv T - m^2 \equiv (p_1 - p_3)^2 - m^2, \quad u \equiv U - m^2 \equiv (p_2 - p_3)^2 - m^2 \quad \text{(A1)}$$

for the $2 \to 2$ partonic process $a(p_1) + b(p_2) \to Q(p_3) + \overline{Q}(p_4)$ with $p_1^2 = p_2^2 = 0$ and $p_3^2 = p_4^2 = m^2$. We also introduce the abbreviations $\left(\beta = \sqrt{1 - 4m^2/s}\right)$

$$
\begin{aligned}
z_3 &\equiv (s + 2t + s\beta)/2, &\qquad z_4 &\equiv (s + 2t - s\beta)/2, \\
z_5 &\equiv (2m^2 + t + t\beta)/2, &\qquad z_6 &\equiv (2m^2 + t - t\beta)/2, \\
l_s &\equiv \ln \frac{s}{m^2}, \quad l_t \equiv \ln \frac{-t}{m^2}, &\quad l_T &\equiv \ln \frac{-T}{m^2}, \quad l_x \equiv \ln x, \\
l_\beta &\equiv \ln \beta, \quad l_{z3} \equiv \ln \frac{z_3}{m^2}, &\quad l_{z4} &\equiv \ln \frac{-z_4}{m^2}.
\end{aligned}
\quad \text{(A2)}
$$

One finds

$$
\begin{aligned}
\operatorname{Re} D_1^{(2)} &= \frac{1}{st\beta}\Bigg[\frac{1}{192}\Big\{ -109l_s^4 + 240l_t^4 + 32l_T^3 l_x + 264l_T^2 l_x^2 - 200l_T l_x^3 - 177l_x^4 - \quad \text{(A3)} \\
& 96l_T^2 l_x l_{z3} + 192l_T l_x^2 l_{z3} + 12l_x^3 l_{z3} + 96l_T l_x l_{z3}^2 - 32l_x l_{z3}^3 - 480l_T^2 l_x l_{z4} + \\
& 24l_T l_x^2 l_{z4} + 180l_x^3 l_{z4} - 144l_x^2 l_{z3} l_{z4} + 480l_T l_x l_{z4}^2 - 336l_x^2 l_{z4}^2 - 96l_x l_{z3} l_{z4}^2 + \\
& 320l_x l_{z4}^3 - 168l_{z4}^4 + 480l_T^2 l_x l_\beta - 480l_T l_x^2 l_\beta - 40l_x^3 l_\beta - 192l_T l_x l_{z3} l_\beta + \\
& 336l_x^2 l_{z3} l_\beta + 96l_x l_{z3}^2 l_\beta - 384l_T l_x l_{z4} l_\beta + 336l_x^2 l_{z4} l_\beta + 192l_x l_{z3} l_{z4} l_\beta + \\
& 96l_x l_{z4}^2 l_\beta + 192l_{z4}^3 l_\beta + 96l_T l_x l_\beta^2 + 24l_x^2 l_\beta^2 - 96l_x l_{z3} l_\beta^2 - 96l_{z4}^2 l_\beta^2 + 32l_x l_\beta^3 + \\
& 32l_\beta^4 - 32l_t^3(9l_T + 20l_x + 25l_{z3} + l_{z4} + 13l_\beta) - 4l_s^3(8l_T - 36l_x - 9l_{z3} - \\
& 43l_{z4} + 94l_\beta) - 6l_s^2 \left(52l_t^2 - 28l_T^2 + 15l_x^2 + 26l_x l_{z3} + 46l_x l_{z4} + 24l_{z3} l_{z4} + \right. \\
& 32l_{z4}^2 - 4l_T(9l_x + 8l_{z3} - 15l_{z4} - 4l_\beta) - 60l_x l_\beta - 24l_{z3} l_\beta - 56l_{z4} l_\beta + 76l_\beta^2 + \\
& l_t(-44l_T + 60l_x - 8l_{z3} - 60l_{z4} + 8l_\beta)) - 24l_t^2 \left(4l_T^2 + 15l_x^2 - 8l_T(2l_x + \right. \\
& 2l_{z3} + 4l_{z4} - 3l_\beta) + 4l_x(3l_{z3} - 8l_{z4} + 13l_\beta) + 2(-4l_{z3}^2 + l_{z4}^2 + 12l_{z3} l_{z4} - \\
& 12l_{z3} l_\beta - 8l_{z4} l_\beta + 10l_\beta^2)) + 8l_t \left(8l_T^3 - 31l_x^3 + l_x^2(-6l_{z3} + 33l_{z4} + 6l_\beta) + \right. \\
& 6l_x(3l_{z3}^2 - 22l_{z4}^2 - 2l_{z3}(5l_{z4} - 9l_\beta) + 20l_{z4} l_\beta - 10l_\beta^2) + 4(-2l_{z3}^3 + 6l_{z3}^2 l_{z4} + \\
& 16l_{z4}^3 + 9l_{z3}(l_{z4} - l_\beta)^2 - 18l_{z4}^2 l_\beta + 9l_{z4} l_\beta^2 + 2l_\beta^3) + 6l_T^2(3l_x - 4l_{z3} + 4l_{z4} - \\
& 4l_\beta) - 3l_T(5l_x^2 - 4l_x(5l_{z3} - 4l_{z4}) - 8(l_{z3}^2 - 2l_{z3} l_{z4} - 3l_{z4}^2 + 4l_{z4} l_\beta + l_\beta^2))) -
\end{aligned}
$$

$$4l_s \left(80l_t^3 + 8l_T^3 + 18l_x^3 - 27l_x^2 l_{z3} - 8l_{z3}^3 - 165l_x^2 l_{z4} - 72l_x l_{z3} l_{z4} + 48l_x l_{z4}^2 - \right.$$

$$24l_{z3} l_{z4}^2 - 64l_{z4}^3 + 18l_x^2 l_\beta + 120l_x l_{z3} l_\beta + 24l_{z3}^2 l_\beta + 72l_x l_{z4} l_\beta + 48l_{z3} l_{z4} l_\beta +$$

$$72l_{z4}^2 l_\beta - 84l_x l_\beta^2 - 24l_{z3} l_\beta^2 - 48l_{z4} l_\beta^2 + 40l_\beta^3 - 12l_T^2(13l_x + 2l_{z3} - 6l_{z4} +$$

$$6l_\beta) + 12l_t^2(8l_T + 5l_x - 26l_{z3} - 10l_{z4} + 12l_\beta) - 12l_t(5l_T^2 + 6l_x^2 - 7l_{z3}^2 -$$

$$10l_{z3} l_{z4} - 16l_{z4}^2 + l_x(14l_{z3} + 27l_{z4} - 4l_\beta) + 10l_{z3} l_\beta + 16l_{z4} l_\beta + 2l_\beta^2 +$$

$$l_T(-5l_x - 2l_{z3} + 4l_{z4} + 8l_\beta)) + 12l_T(11l_x^2 + l_x(8l_{z3} + 7l_{z4} - 4l_\beta) + 2l_{z3}^2 -$$

$$\left. 6l_{z4}^2 - 4l_{z3} l_\beta + 8l_{z4} l_\beta + 2l_\beta^2)\right) \Big\} + \left(3l_s^2/4 - 7l_t^2/2 + l_T l_x + 3l_x^2/4 + 5l_x l_{z3} + \right.$$

$$5l_x l_{z4} + l_{z4}^2/2 + l_t(2l_T - l_x + 10l_{z3} + 2l_{z4} - 15l_\beta) - 8l_x l_\beta + 2l_{z4} l_\beta - 11l_\beta^2 -$$

$$\left. l_s(7l_t + l_T + 8l_x + 5l_{z3} + l_{z4} + 6l_\beta)\right) \zeta(2) + (-3l_s + 2l_x)\zeta(3) - 35\zeta(4)/4 -$$

$$2\text{Li}_2^2 \left(\frac{m^2}{z_5}\right) + 2\text{Li}_2^2 \left(\frac{-t(1-\beta)}{2m^2}\right) + \frac{1}{8}\text{Li}_2 \left(\frac{m^2}{z_5}\right) \left(-11l_s^2 - 4l_t^2 + 25l_x^2 + \right.$$

$$8l_x l_{z4} - 24l_{z4}^2 + l_s(4l_t + 26l_x + 24l_{z4} - 8l_\beta) + 24l_x l_\beta + 16l_{z4} l_\beta - 8l_\beta^2 -$$

$$\left. 4l_t(9l_x - 4l_{z4} + 4l_\beta)\right) + \text{Li}_2 \left(\frac{m^2 x}{-T}\right) \left(11l_s^2/8 - l_t^2 + 2l_t(l_x + l_{z4} - 2l_\beta) + \right.$$

$$\left. l_s(-l_t + 7l_x/4 - l_{z4} + 2l_\beta) + l_x(19l_x - 24l_{z4} + 16l_\beta)/8\right) + \frac{1}{8}\text{Li}_2(-x) \times$$

$$\left(-15l_s^2 + 4l_t^2 + l_x(-39l_x + 32l_{z4} - 16l_\beta) + 2l_s(10l_t + 5l_x - 8l_\beta) + \right.$$

$$\left. l_t(-44l_x + 32l_\beta)\right) + \text{Li}_2 \left(\frac{z_3}{z_4}\right) \left(-2\text{Li}_2 \left(\frac{m^2}{z_5}\right) + \frac{1}{4} \left(-5l_s^2 + 20l_t^2 - 7l_x^2 - \right.\right.$$

$$4l_x l_{z3} + 12l_x l_{z4} - 4l_{z4}^2 + l_s(-2l_t + 4(2l_x + l_{z3} + l_{z4} - l_\beta)) - 4l_x l_\beta +$$

$$\left.\left. l_t(-22l_x - 8l_{z3} + 8l_\beta) - 8\zeta(2)\right)\right) + \text{Li}_2(x) \left(2\text{Li}_2 \left(\frac{m^2}{z_5}\right) + \right.$$

$$2\text{Li}_2 \left(\frac{-t(1-\beta)}{2m^2}\right) - 7l_s^2/4 - 3l_t^2/2 - 11l_x^2/4 + l_x l_{z3} + 3l_x l_{z4} + l_{z4}^2 -$$

$$\left. 2l_x l_\beta + l_t(-4l_x + 2l_{z3} + 4l_\beta) + l_s(4l_t + l_x/2 - l_{z3} - l_{z4} - 2l_\beta) - 6\zeta(2)\right) +$$

$$\text{Li}_2 \left(\frac{T}{m^2}\right) \left(-4\text{Li}_2(x) - 4\text{Li}_2 \left(\frac{z_3}{z_4}\right) - 4\text{Li}_2 \left(\frac{m^2}{z_5}\right) + 4\text{Li}_2 \left(\frac{-t(1-\beta)}{2m^2}\right) + \right.$$

$$l_s^2/8 - 3l_t^2/2 + l_x^2/8 + 2l_x l_{z3} - 4l_{z4}^2 + l_t(3l_x/2 + 4l_{z3} + 4l_{z4} - 4l_\beta) -$$

$$\left. 2l_x l_\beta + 4l_{z4} l_\beta - 2l_\beta^2 - \frac{l_s}{4}(18l_t + 7l_x + 8l_{z3} - 16l_{z4} + 8l_\beta) - 4\zeta(2)\right) +$$

$$\text{Li}_2 \left(\frac{T}{z_3}\right) \left(-\frac{l_s^2}{8} - 2l_t^2 + l_s(\frac{9}{4}l_x - l_{z4}) - 2l_t(l_x - l_{z4}) - l_x(\frac{17}{8}l_x - l_{z4})\right) +$$

$$\text{Li}_2 \left(\frac{-t(1-\beta)}{2m^2}\right) \left(-2\text{Li}_2 \left(\frac{z_3}{z_4}\right) - 7l_t^2 - l_x^2/2 + 2l_x l_{z3} - l_x l_{z4} - l_{z4}^2 + \right.$$

$$l_s(l_t + 3l_x/2 - 2l_{z3} + l_{z4} - l_\beta) + 3l_x l_\beta + 2l_{z4} l_\beta - l_\beta^2 - l_t(l_x - 4l_{z3} - 2l_{z4} +$$

$$2l_\beta) + 12\zeta(2)\Big) + \text{Li}_3\left(\frac{-1+\beta}{2\beta}\right)(4l_s - 7l_t) + 5\text{Li}_3\left(\frac{z_5}{t\beta}\right)l_t + \text{Li}_3\left(\frac{m^2}{z_5}\right) \times$$

$$(5l_s - 6l_t - 11l_x)/2 - \text{Li}_3\left(\frac{z_3}{t}\right)(4l_t + 6l_x) + \text{Li}_3\left(\frac{z_6}{m^2}\right)(3l_s/2 - 5l_t -$$

$$7l_x/2) + 4\text{Li}_3\left(\frac{z_4}{t}\right)(l_s - l_t - 2l_x) + \text{Li}_3\left(-\frac{m^2 x z_3}{sT\beta}\right)(l_s - 2l_t - l_x) +$$

$$\text{Li}_3\left(\frac{-x^2}{1-x^2}\right)(l_s - l_t - l_x) + \text{Li}_3\left(\frac{z_3}{s\beta}\right)(l_s + 5l_t - l_x) + 2\text{Li}_3\left(\frac{m^2}{-t}\right)l_x +$$

$$\text{Li}_3\left(\frac{z_5}{T}\right)\left(-\frac{5}{2}l_s - 3l_t - \frac{5}{2}l_x + 4l_{z4} - 4l_\beta\right) + \text{Li}_3\left(\frac{-t(1-\beta)}{2z_5}\right) \times$$

$$\left(-\frac{5}{2}l_s - l_t - \frac{3}{2}l_x + 4l_{z4} - 4l_\beta\right) + \text{Li}_3\left(\frac{-2z_6}{t(1+\beta)}\right)\left(-\frac{3}{2}l_s - 2l_t + 2l_{z4} -\right.$$

$$\frac{l_x}{2} - 2l_\beta\Big) + \text{Li}_3\left(\frac{z_3}{z_4}\right)\left(\frac{l_s}{2} - l_t - \frac{l_x}{2} + 2l_{z4} - 2l_\beta\right) + 2\text{Li}_3(-x)(3l_s +$$

$$2l_t + 2l_x - 4l_{z4} + 4l_\beta) + \text{Li}_3\left(\frac{z_6}{z_5}\right)\left(\frac{l_s}{2} + 4l_t - \frac{l_x}{2} - 2l_{z4} + 2l_\beta\right) +$$

$$\text{Li}_3\left(\frac{2z_6}{m^2(1+\beta)}\right)\left(\frac{3}{2}l_s + 2l_t + \frac{l_x}{2} - 2l_{z4} + 2l_\beta\right) + \text{Li}_3\left(\frac{z_4}{T}\right)(3l_s + 3l_t +$$

$$3l_x - 4l_{z4} + 4l_\beta) + \text{Li}_3\left(\frac{T}{z_3}\right)\left(\frac{l_s}{2} + l_t + \frac{7}{2}l_x - 2l_{z4} + 2l_\beta\right) +$$

$$\text{Li}_3\left(\frac{m^2(1-\beta)}{2z_5}\right)\left(\frac{3}{2}l_s + 3l_t + \frac{5}{2}l_x - 4l_{z4} + 4l_\beta\right) +$$

$$\text{Li}_3(x)\left(\frac{15}{2}l_s - 2l_t + \frac{5}{2}l_x - 6l_{z4} + 6l_\beta\right) + \text{Li}_3\left(\frac{T}{z_6}\right)(-2l_s +$$

$$l_t - 2l_x + 2l_{z4} - 2l_\beta) + 2\text{Li}_4(x) - 4\text{Li}_4\left(\frac{z_3}{t}\right) + 4\text{Li}_4\left(\frac{z_4}{t}\right) - \text{Li}_4\left(\frac{z_4}{T}\right) -$$

$$\text{Li}_4\left(\frac{Tz_4}{D}\right) + \text{Li}_4\left(\frac{z_5}{T}\right) + 2\text{Li}_4\left(\frac{s(1-\beta)}{-2t}\right) + 3\text{Li}_4\left(\frac{s(1-\beta)}{2z_4}\right) +$$

$$\text{Li}_4\left(\frac{T}{z_6}\right) + 4\text{Li}_4\left(\frac{-1+\beta}{2\beta}\right) + 2\text{Li}_4\left(\frac{-2t}{s(1+\beta)}\right) + 4\text{Li}_4\left(\frac{2\beta}{1+\beta}\right) +$$

$$3\text{Li}_4\left(\frac{2z_3}{s(1+\beta)}\right) + 2Li_{3,1}\left(-\frac{m^2 x z_3}{sT\beta}, \frac{-T}{m^2 x}\right) -$$

$$2Li_{3,1}\left(-\frac{m^2 x z_3}{sT\beta}, \frac{2T}{t(1-\beta)}\right) - 6Li_{1,2,1}\left(1, \frac{s(1-\beta)}{2z_4}, \frac{z_5}{m^2}\right) +$$

$$6Li_{1,2,1}\left(1, \frac{s(1+\beta)}{2z_3}, \frac{z_6}{m^2}\right) - 2Li_{2,1,1}\left(1, \frac{z_4}{z_3}, \frac{z_3}{t}\right) -$$

$$2Li_{2,1,1}\left(\frac{m^2}{T}, \frac{T}{z_5}, \frac{z_5}{m^2}\right) + 2Li_{2,1,1}\left(\frac{m^2}{T}, \frac{T}{z_6}, \frac{z_6}{m^2}\right) -$$

$$2Li_{2,1,1}\left(\frac{z_3}{z_4}, \frac{z_4}{z_3}, \frac{z_3}{t}\right) - 2Li_{2,1,1}\left(\frac{m^2}{z_5}, 1, \frac{z_5}{m^2}\right) - 2Li_{2,1,1}\left(\frac{m^2}{z_5}, \frac{z_5}{T}, \frac{T}{m^2}\right) +$$

$$2Li_{2,1,1}\left(\frac{s(1-\beta)}{2z_4}, \frac{z_5}{z_6}, \frac{z_6}{m^2}\right) + 2Li_{2,1,1}\left(-\frac{m^2 x z_3}{sT\beta}, -\frac{sT\beta}{m^2 x z_3}, \frac{z_3}{s\beta}\right) -$$

$$2Li_{2,1,1}\left(-\frac{m^2 x z_3}{sT\beta}, -\frac{sT\beta}{m^2 x z_3}, -\frac{z_6}{t\beta}\right) + 2Li_{2,1,1}\left(\frac{s(1+\beta)}{2z_3}, 1, \frac{z_6}{m^2}\right) +$$

$$2Li_{2,1,1}\left(\frac{s(1+\beta)}{2z_3}, \frac{z_6}{T}, \frac{T}{m^2}\right) - 2Li_{2,1,1}\left(\frac{s(1+\beta)}{2z_3}, \frac{z_6}{z_5}, \frac{z_5}{m^2}\right) +$$

$$Li_{1,1,1,1}\left(1, \frac{T}{z_6}, \frac{z_6}{z_5}, \frac{z_5}{m^2}\right) - Li_{1,1,1,1}\left(1, \frac{s(1-\beta)}{2z_4}, \frac{z_5}{T}, \frac{T}{m^2}\right) +$$

$$3Li_{1,1,1,1}\left(1, \frac{s(1-\beta)}{2z_4}, \frac{z_5}{z_6}, \frac{z_6}{m^2}\right) + Li_{1,1,1,1}\left(1, \frac{s(1+\beta)}{2z_3}, \frac{z_6}{T}, \frac{T}{m^2}\right) -$$

$$3Li_{1,1,1,1}\left(1, \frac{s(1+\beta)}{2z_3}, \frac{z_6}{z_5}, \frac{z_5}{m^2}\right) - 2Li_{1,1,1,1}\left(\frac{t}{T}, 1, \frac{T}{z_3}, \frac{z_3}{t}\right) +$$

$$2Li_{1,1,1,1}\left(\frac{t}{T}, 1, \frac{T}{z_4}, \frac{z_4}{t}\right) + 2Li_{1,1,1,1}\left(\frac{t}{T}, \frac{T}{z_3}, 1, \frac{z_3}{t}\right) -$$

$$2Li_{1,1,1,1}\left(\frac{t}{T}, \frac{T}{z_3}, \frac{z_3}{T}, \frac{T}{t}\right) - 2Li_{1,1,1,1}\left(\frac{t}{T}, \frac{T}{z_3}, \frac{z_3}{z_4}, \frac{z_4}{t}\right) -$$

$$2Li_{1,1,1,1}\left(\frac{t}{T}, \frac{T}{z_4}, 1, \frac{z_4}{t}\right) + 2Li_{1,1,1,1}\left(\frac{t}{T}, \frac{T}{z_4}, \frac{z_4}{T}, \frac{T}{t}\right) +$$

$$2Li_{1,1,1,1}\left(\frac{t}{T}, \frac{T}{z_4}, \frac{z_4}{z_3}, \frac{z_3}{t}\right) + 2Li_{1,1,1,1}\left(\frac{t}{z_3}, 1, \frac{z_3}{T}, \frac{T}{t}\right) -$$

$$2Li_{1,1,1,1}\left(\frac{t}{z_3}, \frac{z_3}{T}, 1, \frac{T}{t}\right) + 2Li_{1,1,1,1}\left(\frac{t}{z_3}, \frac{z_3}{T}, \frac{T}{z_3}, \frac{z_3}{t}\right) -$$

$$2Li_{1,1,1,1}\left(\frac{t}{z_3}, \frac{z_3}{T}, \frac{T}{z_4}, \frac{z_4}{t}\right) - 2Li_{1,1,1,1}\left(\frac{t}{z_3}, \frac{z_3}{z_4}, \frac{z_4}{T}, \frac{T}{t}\right) -$$

$$2Li_{1,1,1,1}\left(\frac{t}{z_4}, 1, \frac{z_4}{T}, \frac{T}{t}\right) + 2Li_{1,1,1,1}\left(\frac{t}{z_4}, \frac{z_4}{T}, 1, \frac{T}{t}\right) +$$

$$2Li_{1,1,1,1}\left(\frac{t}{z_4}, \frac{z_4}{T}, \frac{T}{z_3}, \frac{z_3}{t}\right) - 2Li_{1,1,1,1}\left(\frac{t}{z_4}, \frac{z_4}{T}, \frac{T}{z_4}, \frac{z_4}{t}\right) +$$

$$2Li_{1,1,1,1}\left(\frac{t}{z_4}, \frac{z_4}{z_3}, \frac{z_3}{T}, \frac{T}{t}\right) - 2Li_{1,1,1,1}\left(\frac{T}{z_4}, \frac{z_4}{z_3}, 1, \frac{z_3}{t}\right) +$$

$$Li_{1,1,1,1}\left(\frac{z_4}{T}, \frac{T}{z_3}, 1, \frac{z_3}{t}\right) - Li_{1,1,1,1}\left(\frac{m^2}{z_5}, 1, \frac{z_5}{T}, \frac{T}{m^2}\right) +$$

$$Li_{1,1,1,1}\left(\frac{m^2}{z_5}, \frac{z_5}{T}, 1, \frac{T}{m^2}\right) - Li_{1,1,1,1}\left(\frac{m^2}{z_5}, \frac{z_5}{T}, \frac{T}{z_5}, \frac{z_5}{m^2}\right) -$$

$$Li_{1,1,1,1}\left(\frac{m^2}{z_6}, \frac{z_6}{T}, 1, \frac{T}{m^2}\right) - Li_{1,1,1,1}\left(\frac{m^2}{z_6}, \frac{z_6}{T}, \frac{T}{z_5}, \frac{z_5}{m^2}\right) -$$

$$Li_{1,1,1,1}\left(\frac{m^2}{z_6}, \frac{z_6}{z_5}, 1, \frac{z_5}{m^2}\right) - Li_{1,1,1,1}\left(\frac{m^2}{z_6}, \frac{z_6}{z_5}, \frac{z_5}{T}, \frac{T}{m^2}\right) -$$

$$Li_{1,1,1,1}\left(\frac{z_6}{T}, \frac{T}{z_5}, 1, \frac{z_5}{m^2}\right) + Li_{1,1,1,1}\left(\frac{z_6}{z_5}, \frac{z_5}{T}, 1, \frac{T}{m^2}\right) -$$

$$3Li_{1,1,1,1}\left(\frac{s(1-\beta)}{2z_4}, 1, \frac{z_5}{z_6}, \frac{z_6}{m^2}\right) + Li_{1,1,1,1}\left(\frac{s(1-\beta)}{2z_4}, \frac{z_5}{T}, \frac{T}{z_6}, \frac{z_6}{m^2}\right) -$$

$$3Li_{1,1,1,1}\left(\frac{s(1-\beta)}{2z_4}, \frac{z_5}{z_6}, 1, \frac{z_6}{m^2}\right) + Li_{1,1,1,1}\left(\frac{s(1-\beta)}{2z_4}, \frac{z_5}{z_6}, \frac{z_6}{T}, \frac{T}{m^2}\right) -$$

$$3Li_{1,1,1,1}\left(\frac{s(1-\beta)}{2z_4}, \frac{z_5}{z_6}, \frac{z_6}{z_5}, \frac{z_5}{m^2}\right) + Li_{1,1,1,1}\left(\frac{s(1+\beta)}{2z_3}, 1, \frac{z_6}{T}, \frac{T}{m^2}\right) -$$

$$Li_{1,1,1,1}\left(\frac{s(1+\beta)}{2z_3}, 1, \frac{z_6}{z_5}, \frac{z_5}{m^2}\right) + Li_{1,1,1,1}\left(\frac{s(1+\beta)}{2z_3}, \frac{z_6}{T}, \frac{T}{z_6}, \frac{z_6}{m^2}\right) -$$

$$Li_{1,1,1,1}\left(\frac{s(1+\beta)}{2z_3}, \frac{z_6}{z_5}, \frac{z_5}{z_6}, \frac{z_6}{m^2}\right) - i\pi(l_s - 2l_t - l_x)\left(\frac{l_s^2}{2} + l_s l_T + l_s l_x + \right.$$

$$\frac{l_x^2}{2} - l_T l_{z3} + \frac{l_{z3}^2}{2} - l_s l_{z4} + \frac{l_{z4}^2}{2} + l_s l_\beta + l_T l_\beta + l_x l_\beta - l_{z4} l_\beta + \frac{l_\beta^2}{2} +$$

$$\left. 2Li_2(-x) + 2Li_2(x) - Li_2\left(\frac{m^2 x}{-T}\right) + Li_2\left(\frac{T}{z_3}\right) + Li_2\left(\frac{z_3}{z_4}\right) + \zeta(2)\right)\right].$$

At the very end of the expression one finds an explicit imaginary part. Since the whole expression must be real this clearly indicates that the same imaginary contribution with opposite sign must be contained in multiple polylogarithms, e.g., some of them are sitting on branch cuts. This is in fact true for the multiple polylogarithms

$$Li_{3,1}\left(-\frac{m^2 x z_3}{sT\beta}, \frac{-T}{m^2 x}\right), \qquad Li_{3,1}\left(-\frac{m^2 x z_3}{sT\beta}, \frac{2T}{t(1-\beta)}\right),$$

$$Li_{2,1,1}\left(-\frac{m^2 x z_3}{sT\beta}, -\frac{sT\beta}{m^2 x z_3}, \frac{z_3}{s\beta}\right), \qquad Li_{2,1,1}\left(-\frac{m^2 x z_3}{sT\beta}, -\frac{sT\beta}{m^2 x z_3}, -\frac{z_6}{t\beta}\right).$$

Indeed, one finds that the imaginary contributions cancel out when one numerically evaluates the result.

As regards the length the representations of $\text{Re}\,D_1^{(2)}$ in terms of L functions in [1] and in terms of multiple polylogarithms are of similar size. The representation in terms of L functions contains 43 different L function expressions against 59 different multiple polylogarithm expressions.

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[11] See EPAPS Document No. E-JMAPAQ-47-224604 for all the relevant results for the one-loop scalar integrals from [1] expressed in terms of multiple polylogarithms in MATHEMATICA format. This document can be reached through a direct link in the online article's HTML reference section or via the EPAPS homepage (http://www.aip.org/pubservs/epaps.html)