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Quantum corrections to spinning strings in $AdS_5 \times S^5$ and Bethe ansatz: a comparative study

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Abstract

We analyze quantum corrections to rigid spinning strings in $AdS_5 \times S^5$. The one-loop worldsheet quantum correction to the string energy is compared to the finite-size correction from the quantum string Bethe ansatz. Expanding the summands of the string theory energy shift in the parameter $1/\mathcal{J}^2$ and subsequently resumming them yields a divergent result. However, upon zeta-function regularisation these results agree with the Bethe ansatz in the first three orders. We also perform an analogous computation in the limit of large winding number, which results in a disagreement with the string Bethe ansatz prediction. A similar mismatch is observed numerically. We comment on the possible origin of this discrepancy.

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Contents

1 Introduction

Understanding the quantum spectrum of string theory in $AdS_5 \times S^5$ is an important open problem. Solving this problem will open up venues for testing the ideas of gauge/string duality in the genuine stringy regime. It is becoming more and more clear that progress in quantizing strings on $AdS_5 \times S^5$ is impossible without serious input from the dual $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM). One idea that has proved extremely useful on the gauge theory side and could potentially be applied to AdS strings, is to compute the spectrum using a Bethe ansatz. The Bethe ansatz is the standard approach to quantize integrable systems [1] and it is believed that both planar $\mathcal{N} = 4$ SYM and string theory in $AdS_5 \times S^5$ are integrable.

As was observed first at one loop [2, 3] and then at higher orders in perturbation theory [4, 5, 6], the planar dilatation operator of $\mathcal{N} = 4$ SYM can be identified with a Hamiltonian of an integrable spin chain[1]. The integrability on the string theory side arises because the classical world-sheet sigma-model admits a Lax representation. For the bosonic reduction this almost immediately follows [10] from the integrability of the $O(n)$ model [11]. The Lax pair for the full supersymmetric sigma-model in $AdS_5 \times S^5$ [12] was constructed in [13].

Because the classical equations of motion of the AdS string are integrable, their solutions can be parameterized by the spectral data of the Lax operator. By reformulating the standard solution of the spectral problem [14] it was shown in [15] that the spectral density for the string moving on the $\mathbb{R} \times S^3$ subspace of $AdS_5 \times S^5$ satisfies an integral equation that strikingly resembles the large-volume (thermodynamic) limit of the quantum Bethe equations for the spectrum of the dilatation operator in the dual gauge theory. These results were extended to other sectors [16, 17, 18, 19] and eventually to the most general solution including world-sheet fermions [20]. Of course the classical approximation in the sigma-model is accurate only at strong 't Hooft coupling (*i.e.* weak worldsheet coupling). In addition, the Noether charges of the string have to be large. In order to quantize the string one needs to "undo" the thermodynamic limit and turn the integral equations for the sigma-model into discrete, quantum string Bethe equations. Such a discretization was first proposed for the $\mathfrak{su}(2)$ subsector [21], then for other rank-one sectors [22] and subsequently for the complete set of Bethe equations with the $\mathfrak{psu}(2,2|4)$ symmetry [23]. The quantum string Bethe equations work remarkably well in several tractable limits: they have the right classical limit (by construction), reproduce the leading quantum corrections for the BMN states and yield the correct energies of massive states in the strict strong-coupling limit.

There are very few explicit calculations for quantum strings in $AdS_5 \times S^5$. One major example is string quantization in the plane-wave limit [24] which leads to a solvable string theory [25] and can be understood as quantization around the simplest point-like solution of the string spinning on S^5 [26]. The curvature corrections [27] to the string states in this background (BMN states) were calculated in [28]. Frolov-Tseytlin solutions [29, 30] generalize this setup to macroscopic strings and it is possible to quantize fluctuations around these solutions in some

[1]Although the dilatation operator is not integrable beyond leading order in the $1/N$ expansion [4], the planar integrability is still useful in the study of decays of semiclassical strings [7] and in the computation of three-point functions [8]. We should also mention that the classical equations of motion of $\mathcal{N} = 4$ SYM admit a Lax representation [9], but we do not know if this property has anything to do with the quantum integrability of the planar dilatation operator.

cases [31, 32, 33, 34]. For these solutions, the classical string energies can be compared to the anomalous dimensions in the gauge theory (see [30, 35] for review), because the 't Hooft coupling λ combines with the R-charge J into the BMN coupling $1/\mathcal{J}^2 \equiv \lambda/J^2$, which can be small even if the 't Hooft coupling is large, provided that the R-charge is large enough. In particular, the string action reduces to the effectve action of the spin chain in the limit of large $\mathcal{J}$ [36]. Generically, one finds that string theory and SYM agree up to two loops and start to disagree at three loops. For the quantum corrections the comparison has only been done at the one-loop level [37, 38]. It would be interesting to understand what happens at higher orders of perturbation theory.

Our goal is to compare quantum corrections to macroscopic strings with the quantum string Bethe ansatz at higher loops [21, 22, 23]. The conjectured quantum string Bethe equations were rigorously tested at infinite λ, but they can potentially receive $1/\sqrt{\lambda}$ corrections [21]. Comparison of the quantum string Bethe ansatz to the direct quantum string calculation provides an explicit check of whether such corrections are present at $O(1/\sqrt{\lambda})$ or not. Furthermore, the string Bethe equations are known to exactly reproduce the first two orders of the SYM perturbation theory independently of J [39], and we can just expand the energies computed from them in the 't Hooft coupling to find the two loop anomalous dimensions in SYM. In this way we can extend the analysis of [37, 38] to two loops.

Let us briefly review the classical string configurations that we shall study. The one-loop quantum corrections were computed for two classes of string solutions – for circular strings rotating in S^5 with two independent angular momenta [31, 32] and for circular strings spinning in AdS_3 and rotating around S^5 [33]. The first case is plagued by instabilities [29, 31] and for this reason we shall concentrate on strings moving in $AdS_3 \times S^1 \subset AdS_5 \times S^5$ [40] (throughout the paper, we shall adopt the conventions of [33]). The relevant part of the $AdS_5 \times S^5$ metric in global coordinates is

$$ds^2 = -\cosh^2 \rho\, dt^2 + d\rho^2 + \sinh^2 \rho\, d\theta^2 + d\phi^2, \tag{1.1}$$

where the first three terms are the metric of AdS_3 and ϕ is the angle of a big circle in S^5. The circular string solution has the following form

$$\rho = \text{ const}\,, \qquad t = \kappa\tau, \qquad \theta = \sqrt{\kappa^2 + k^2}\,\tau + k\sigma, \qquad \phi = \sqrt{\kappa^2 + k^2}\,\tau + m\sigma, \tag{1.2}$$

where

$$\begin{aligned} r_1^2 \equiv \sinh^2 \rho &= \frac{\mathcal{S}}{\sqrt{\kappa^2 + k^2}}\,, && (1.3)\\ \mathcal{E} &= \frac{\kappa\mathcal{S}}{\sqrt{\kappa^2 + k^2}} + \kappa, && (1.4)\\ 2\kappa\mathcal{E} - \kappa^2 &= 2\sqrt{\kappa^2 + k^2}\,\mathcal{S} + \mathcal{J}^2 + m^2\,, && (1.5)\\ k\mathcal{S} + m\mathcal{J} &= 0. && (1.6) \end{aligned}$$

Global charges of the string (the energy E, the spin S, and the angular momentum J) combine with the string tension into the following "dimensionless" ratios, which stay finite in the classical ($\lambda \to \infty$, $J \to \infty$, $S \to \infty$) limit [30]:

$$\mathcal{E} = \frac{E}{\sqrt{\lambda}}, \qquad \mathcal{S} = \frac{S}{\sqrt{\lambda}}, \qquad \mathcal{J} = \frac{J}{\sqrt{\lambda}}\,. \tag{1.7}$$

Thus $1/\sqrt{\lambda}$ or $1/J$ can be used interchangeably as the loop counting parameters in the sigma-model. In addition, at any given order in $1/J$ one can further expand in the BMN coupling $1/\mathcal{J}^2 = \lambda/J^2$. In this way one recovers the two-loop perturbative SYM results.

In section 2 we review the string theory computation and evaluate the energy shift, at leading order in $1/J$ and at the first three orders in $1/\mathcal{J}^2$. Although the exact energy shift is finite, individual terms of the $1/\mathcal{J}$ expansion diverge. To render the results finite we use a particular prescription, the zeta-function regularization.

In section 3 we compute the energy shift from the quantum string Bethe ansatz, again perturbatively in $1/\mathcal{J}$. Unlike in the string theory calculation, the $1/\mathcal{J}$ expansion is manifestly finite. However, the resulting expressions agree with the zeta-regularized string energy shift at third order in perturbation theory.

In section 4 we calculate the energy shift in the non-perturbative regime (*i.e.*, small $\mathcal{J}$) of large winding number. The energy shift is finite on both sides in this case. We find a clear discrepancy between the Bethe ansatz and the string calculation. In section 5, we present numerical results which support the analytical evidence for the discrepancy.

We discuss our results in section 6. Various technical details are collected in the appendices.

2 Quantum corrections in string theory

2.1 Energy shift

The semiclassical string quantization of [33] yields the following correction to the classical energy (1.4)

$$\delta E^{string} = \delta E^{(0)} + \delta E^{osc} . \tag{2.1}$$

Here the zero-mode contribution is given by

$$\delta E^{(0)} = \frac{1}{2\kappa}\left(4\nu + 2\kappa + 2\sqrt{\kappa^2 + (1+r_1^2)k^2} - 8\sqrt{c^2+a^2}\right) . \tag{2.2}$$

The oscillator part has the following form

$$\begin{aligned}\delta E^{osc} =& \frac{1}{\kappa}\sum_{n=1}^{\infty}\Bigg(4\sqrt{n^2+\nu^2} + 2\sqrt{n^2+\kappa^2} - 4\sqrt{(n+\gamma)^2+\alpha^2} - 4\sqrt{(n-\gamma)^2+\alpha^2} \\ &+ \frac{1}{2}\sum_{I=1}^{4} sign(C_I^{(n)})\omega_{I,n}\Bigg) ,\end{aligned} \tag{2.3}$$

where the last term is the contribution of the $\mathfrak{sl}(2)$-modes, which are the four solutions of the quartic equation

$$(\omega^2 - n^2)^2 + 4r_1^2\kappa^2\omega^2 - 4(1+r_1^2)\left(\sqrt{\kappa^2+k^2}\,\omega - kn\right)^2 = 0. \tag{2.4}$$

The first line corresponds to the transverse and fermionic modes. The various parameters are defined as

$$
\begin{aligned}
\nu &= \sqrt{\mathcal{J}^2 - m^2} \\
\alpha &= \sqrt{\frac{\kappa^2+\nu^2}{2}} \\
r_1^2 &= \frac{\kappa^2 - 2m^2 - \nu^2}{2k^2} = -\frac{m}{k}\frac{\mathcal{J}}{\sqrt{\kappa^2+k^2}} \\
\gamma &= \frac{1}{2}\kappa\left(1+\frac{2k^2(1+r_1^2)}{\kappa^2-\nu^2}\right)\sqrt{\frac{\kappa^2-\nu^2-2k^2r_1^2}{2(\kappa^2+k^2)}}\,.
\end{aligned}
\tag{2.5}
$$

The sign factors are determined from

$$
C_I^{(n)} = (\omega_{I,n}^2 - n^2)\prod_{J\neq I}(\omega_I - \omega_J). \tag{2.6}
$$

It is possible to perform a partial summation of the series (2.3). The series is absolutely convergent, because the summand decreases as $1/n^2$ at $n \to \infty$. Therefore one can sum each frequency separately by regularizing the divergences; one adds and subtracts terms of the form $c_1 n + c_2/n$ before separating various frequencies. This does not change the result, because each partial sum is again absolutely convergent. The basic sum is

$$
\sum_{n=1}^{\infty}\left[\sqrt{(n+\gamma)^2+\alpha^2}+\sqrt{(n-\gamma)^2+\alpha^2}-2n-\frac{\alpha^2}{n}\right] = \gamma^2 - \sqrt{\gamma^2+\alpha^2} + F(\{\gamma\},\alpha), \tag{2.7}
$$

where $\{\gamma\}$ denotes the fractional part of γ and the function $F(\beta,\alpha)$ is defined by the following integral representation

$$
F(\beta,\alpha) \equiv \sqrt{\alpha^2+\beta^2} - \beta^2 + \alpha^2\int_0^\infty \frac{d\xi}{\mathrm{e}^{\xi}-1}\left(\frac{2J_1(\alpha\xi)}{\alpha\xi}\cosh\beta\xi - 1\right). \tag{2.8}
$$

Using this result we find

$$
\begin{aligned}
\delta E^{osc} \;=\; & \frac{1}{\kappa}\Bigg[2F(0,\nu)+F(0,\kappa)-4F(\{\gamma\},\alpha)-2\nu-\kappa-4\gamma^2+4\sqrt{\gamma^2+\alpha^2} \\
& +\frac{1}{2}\sum_{n=1}^{\infty}\sum_{I=1}^{4}\left(sign C_I^{(n)}\,\omega_{I,n}-n-\frac{\kappa^2}{2n}\right)\Bigg]\,.
\end{aligned}
\tag{2.9}
$$

The last sum can be seen to absolutely converge if we use the asymptotic values of the frequencies $\omega_{I,n}$ from [33]. The asymptotic expansion of $F(\beta,\alpha)$ in $1/\alpha$ terminates at the second order:

$$
F(\beta,\alpha) = -\alpha^2\ln\left(\frac{\mathrm{e}^{C-1/2}}{2}\alpha\right)+\frac{1}{6}+O\left(\mathrm{e}^{-\alpha}\right), \tag{2.10}
$$

where $C = 0.5772\ldots$ is the Euler constant. The dependence on the fractional part of γ is therefore non-perturbative in $1/\alpha$ and thus in $1/\mathcal{J}$. In particular it will not be seen in the numerical calculations in sec. 5 which will be done for sufficiently large values of $\mathcal{J}$.

2.2 Perturbative expansion

It is hard to find a useful integral representation for the $\mathfrak{sl}(2)$ modes because of the sign factors in (2.9). In computing the perturbative $1/\mathcal{J}$ expansion of the string energy shift we shall follow a more straightforward approach of evaluating the sum by first expanding all the frequencies in $1/\mathcal{J}$ and then computing the sum order by order in $1/\mathcal{J}$. As was already observed in [32] this procedure is not so harmless, because the sum is not uniformly convergent and modes with $n \sim \mathcal{J}^2$ can give a finite contribution. This is reflected in superficial divergences which arise starting from second order in $1/\mathcal{J}^2$. We shall ignore these problems and will use zeta-function regularization to sum the divergent series. This approach might not look well motivated but we shall find a surprising agreement of this naive summation prescription with the Bethe ansatz to third order in $1/\mathcal{J}^2$, which gives us a hint that this prescription may be the correct way to compute the energy correction on the string theory side.

Using the pertubative expressions for the mode frequencies, which are given in appendix B, we can write the pertubative expression for the energy shift δE in powers of $1/\mathcal{J}^2$

$$\delta E^{string} = \sum_{p=1}^{\infty} \frac{\delta E_p^{string}}{\mathcal{J}^{2p}} . \tag{2.11}$$

It is given by

$$\delta E_1^{string} = \frac{1}{2}m(k-m) + \frac{1}{2}\sum_{n=1}^{\infty} 2(k-m)m - n^2 + n\sqrt{n^2+4m(m-k)}, \tag{2.12}$$

$$\begin{aligned}\delta E_2^{string} = & -\frac{1}{8}m(k-m)(4k^2-11km+3m^2) \\ & +\sum_{n=1}^{\infty}\frac{1}{8}\{-2(k-m)m(4k^2-11km+3m^2)+2(3k^2-10km+5m^2)n^2+n^4\} \\ & -\frac{n(-4(k-m)m(5k^2-15km+6m^2)+2(k-3m)(3k-2m)n^2+n^4)}{8\sqrt{n^2+4m(m-k)}},\end{aligned} \tag{2.13}$$

$$\begin{aligned}\delta E_3^{string} = & \frac{1}{16}(k-m)m(8k^4-52k^3m+89k^2m^2-42km^3+5m^4) \\ & +\sum_{n=1}^{\infty}\frac{1}{16}\{2(k-m)m(8k^4-52k^3m+89k^2m^2-42km^3+5m^4) \\ & -(15k^4-128k^3m+279k^2m^2-202km^3+44m^4)n^2 \\ & -(15k^2-38km+19m^2)n^4-n^6\} \\ & +\frac{1}{16(n^2+4m(m-k))^{3/2}} \\ & \times\{4(k-m)^2m^2(45k^4-324k^3m+621k^2m^2-370km^3+60m^4)n \\ & -2(k-m)m(53k^4-481k^3m+1083k^2m^2-815km^3+192m^4)n^3 \\ & +(15k^4-218k^3m+603k^2m^2-556km^3+164m^4)n^5 \\ & +(15k^2-44km+25m^2)n^7+n^9\}.\end{aligned} \tag{2.14}$$

We shall compare this expression to the energy shift calculated using the Bethe ansatz in the next section.

3 Bethe ansatz

3.1 Classical limit

Classical solutions for the string moving in $AdS_3 \times S^1$ are uniquely specified by the spectral data of the Lax operator. One can introduce the spectral density $\rho(x)$ defined on a set of intervals $C_I = (a_I, b_I)$. The spectral density satisfies a singular integral equation [16]

$$2\fint dy\, \frac{\rho(y)}{x-y} = 2\pi k_I - 2\pi\left(\frac{\mathcal{J}+m}{x-1} + \frac{\mathcal{J}-m}{x+1}\right), \qquad x \in C_I. \tag{3.1}$$

This can be called the classical Bethe equation, as such type of equations arise in the thermodynamic limit of quantum Bethe equations.

In addition, the density obeys a set of normalization conditions

$$\int dx\, \frac{\rho(x)}{x} = -2\pi m, \tag{3.2}$$

$$\int dx\, \frac{\rho(x)}{x^2} = 2\pi(\mathcal{E} - \mathcal{S} - \mathcal{J}), \tag{3.3}$$

$$\int dx\, \rho(x) = 2\pi(\mathcal{E} + \mathcal{S} - \mathcal{J}). \tag{3.4}$$

Here $2\pi m$ is the total world-sheet momentum which must be quantized because of the periodic boundary conditions on the world-sheet coordinates.

We shall consider the simplest solutions of (3.1) with only one cut $C = (a, b)$ which corresponds to the circular string (1.2). There is only one mode number k in this case. This simplification is crucial and allows us to rewrite the integral equation (3.1) as an algebraic equation for the resolvant

$$G(x) = \int dy\, \frac{\rho(y)}{x-y}. \tag{3.5}$$

The normalization conditions for the density (3.2)–(3.4) become boundary conditions for $G(x)$

$$G(0) = 2\pi m, \tag{3.6}$$

$$G'(0) = -2\pi(\mathcal{E} - \mathcal{S} - \mathcal{J}), \tag{3.7}$$

$$\lim_{z\to\infty} zG(z) = 2\pi(\mathcal{E} + \mathcal{S} - \mathcal{J}). \tag{3.8}$$

Multiplying both sides of (3.1) by $\rho(x)/(z-x)$ and integrating over x we find

$$G^2(z) - 2\pi\left(k - 2\,\frac{\mathcal{J}z+m}{z^2-1}\right)G(z) - 2\pi\left(\frac{\mathcal{J}+m}{z-1}\,G(1) + \frac{\mathcal{J}-m}{z+1}\,G(-1)\right) = 0. \tag{3.9}$$

The boundary conditions (3.6)–(3.8) can be used to eliminate $G(\pm 1)$ from this equation. Expanding (3.9) at $z = 0$ and $z = \infty$ we get

$$k\mathcal{S} + m\mathcal{J} = 0, \tag{3.10}$$

in accord with [40], and

$$(\mathcal{J} \pm m)\, G(\pm 1) = -\pi k(\mathcal{E} + \mathcal{S} - \mathcal{J}) \pm \pi m(k+m). \tag{3.11}$$

The condition (3.10) imposes rationality on the spins and requires the integers k and m to have opposite signs. We shall assume for definiteness that $m > 0$ and $k < 0$.

Plugging (3.11) back into (3.9) we get

$$G^2(z) - 2\pi \left(k - 2\,\frac{\mathcal{J} z + m}{z^2 - 1}\right) G(z) + \frac{4\pi^2}{z^2 - 1}\left[k(\mathcal{E} + \mathcal{S} - \mathcal{J})z - m(k+m)\right] = 0. \tag{3.12}$$

The solution of this quadratic equation is

$$G(z) = \pi \left(k - 2\,\frac{\mathcal{J} z + m}{z^2 - 1}\right) + \frac{\pi\sqrt{P(z)}}{z^2 - 1}\,, \tag{3.13}$$

where

$$P(z) = k^2 z^4 - 4k(\mathcal{E} + \mathcal{S})z^3 + 2(2\mathcal{J}^2 + 2m^2 - k^2)z^2 + 4k(\mathcal{E} - \mathcal{S})z + k^2. \tag{3.14}$$

The resolvant determines the density through the discontinuity on the cut

$$G(x + i0) - G(x - i0) = 2\pi i \rho(x), \qquad x \in C, \tag{3.15}$$

and we find

$$\rho(x) = \frac{\sqrt{-P(x)}}{x^2 - 1}\,. \tag{3.16}$$

We need one extra condition to express the energy in terms of the spin and the angular momentum. This condition cannot arise from equation (3.9). Instead one should look more closely at the structure of the density $\rho(x)$. For general values of the energy, the angular momentum and the spin, the density is real on two cuts, whereas we have assumed that the solution has only one cut. This can be made consistent by requiring that the discriminant of the quartic polynomial (3.14) is zero, then $P(z)$ has one double root (fig.1)

$$P(c) = 0, \qquad P'(c) = 0\,. \tag{3.17}$$

These two equations determine the dependence of the energy on the angular momenta, $\mathcal{E} = \mathcal{E}(\mathcal{S}, \mathcal{J})$, in a parametric form and are equivalent to (1.4), (1.5) upon the identification

$$\kappa = -\frac{k}{2}\left(\frac{1}{c} - c\right). \tag{3.18}$$

3.2 Quantum corrections

If the integral equation (3.1) is interpreted as the classical limit of some Bethe equations[2], the density $\rho(x)$ has the meaning of an asymptotic distribution of Bethe roots in the limit when

[2]Bethe ansatz only works for integrable systems, so here we must assume quantum intergrability of the world-sheet sigma-model. There are indeed some indications that integrability is not destroyed by quantum corrections [41].

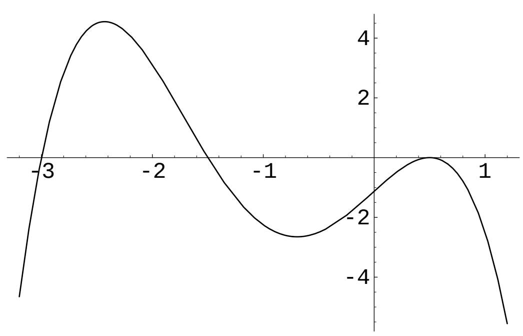

Figure 1: Graph of the quartic polynomial $-P(z)$ (the ordering of the zeroes is $a < b < c$).

their number (naturally identified with the spin S of the quantum string state) becomes infinite

$$\rho(x) = \frac{4\pi}{\sqrt{\lambda}} \sum_{k=1}^{S} \frac{x_k^2}{x_k^2 - 1}\, \delta(x - x_k). \tag{3.19}$$

The normalization factor $2\pi/\sqrt{\lambda}$ is the coupling constant of the world-sheet sigma-model. The classical (weak-coupling) limit corresponds to $\lambda \to \infty$. Because S scales with $\sqrt{\lambda}$ according to (1.7), the classical limit coincides with the thermodynamic limit, in which the number of roots becomes infinite.

Our starting point are the quantum Bethe equations proposed in [22, 23][3]

$$\left(\frac{x_k^+}{x_k^-}\right)^J = \prod_{j\neq k} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \, \frac{1 - \frac{1}{x_k^- x_j^+}}{1 - \frac{1}{x_k^+ x_j^-}} \left(\frac{1 - \frac{1}{x_k^- x_j^+}}{1 - \frac{1}{x_k^+ x_j^+}} \, \frac{1 - \frac{1}{x_k^+ x_j^-}}{1 - \frac{1}{x_k^- x_j^-}} \right)^{\frac{i\sqrt{\lambda}(u_k - u_j)}{2\pi}}, \tag{3.20}$$

where[4]

$$u_k = x_k + \frac{1}{x_k} \tag{3.21}$$

and

$$x_k^\pm + \frac{1}{x_k^\pm} = u_k \pm \frac{2\pi i}{\sqrt{\lambda}}. \tag{3.22}$$

These equations reduce to (3.1) in the thermodynamic limit when $\sqrt{\lambda}, J, S \to \infty$. Our goal will be to compute the leading-order quantum correction to the classical Bethe equations.

It might seem that (3.20) can only give rise to even powers of $1/\sqrt{\lambda}$, since the equations are invariant under $\sqrt{\lambda} \to -\sqrt{\lambda}$. Nevertheless the odd powers of $1/\sqrt{\lambda}$ arise in the expansion and the leading quantum correction is $O(1/\sqrt{\lambda})$ for the following reason. The Bethe roots x_k

[3]Although the quantum string can fluctuate in all directions in $AdS_5 \times S^5$, the quantum string Bethe equations have the same number of degrees of freedom as in the pure $\mathfrak{sl}(2)$ sector. On the gauge theory side different sectors do not talk to each other because operators with different quantum numbers do not mix [42], but it is not a priori clear why various sectors can be separated on the string theory side (see [43] for a more detailed discussion of this issue).

[4]Our notation differs from that of [23] by a rescaling of x_k and u_k: $x_k \to x_k\sqrt{\lambda}/4\pi$, $u_k \to u_k\sqrt{\lambda}/4\pi$.

condense into cuts in the thermodynamic limit such that the distance between nearby roots goes to zero. But the simultaneous limit of $\lambda \to \infty$ and $x_{k+1} - x_k \to 0$ is singular in the Bethe equations and this singularity gives rise to a local anomaly [44]. The anomaly cancels at the leading order [45], but contributes to the $1/\sqrt{\lambda}$ quantum correction [37, 38]. We shall calculate the anomaly directly from the Bethe equations (3.20). The calculations are rather complicated and the details are given in appendix A. The resulting equation for the resolvant differs from (3.12) by a correction term

$$G^2(z) - 2\pi\left(k - 2\,\frac{\mathcal{J}z + m}{z^2-1}\right)G(z) + \frac{4\pi^2}{z^2-1}\Big[k(\mathcal{E}+\mathcal{S}-\mathcal{J})z - m(k+m)\Big]$$
$$+\frac{4\pi}{\sqrt{\lambda}}\,\frac{z^2}{z^2-1}\int dx\;\frac{\rho'(x)\pi\rho(x)\coth\pi\rho(x)}{z-x} = 0. \tag{3.23}$$

Solving this quadratic equation we find a density which is of the form (3.16), where the function $P(z)$ obtains a correction

$$\delta P(z) = \frac{4\pi}{\sqrt{\lambda}}\frac{z^2(1-z^2)}{\pi^2}\int dx\;\frac{\rho'(x)\pi\rho(x)\coth\pi\rho(x)}{z-x}\,. \tag{3.24}$$

The energy can be found as before, from the requirement that there is only one cut present

$$P(c+\delta c) + \delta P(c+\delta c) = 0, \qquad P'(c+\delta c) + \delta P'(c+\delta c) = 0. \tag{3.25}$$

Expanding the first equation to linear order we get

$$\frac{\partial P(c)}{\partial \mathcal{E}}\,\delta\mathcal{E} + \frac{\partial P(c)}{\partial c}\,\delta c + \delta P(c) = 0. \tag{3.26}$$

Taking into account that $\partial P(c)/\partial c = 0$ we find

$$\delta\mathcal{E} = -\frac{\delta P(c)}{\partial P(c)/\partial\mathcal{E}}\,. \tag{3.27}$$

For $\partial P/\partial\mathcal{E}$ we get from (3.14)

$$\frac{\partial P(c)}{\partial\mathcal{E}} = -4kc(c^2-1). \tag{3.28}$$

Rescaling back to the physical energy we obtain

$$\delta E^{Bethe} = \frac{c}{\pi k}\int dx\;\frac{\rho'(x)\pi\rho(x)\coth\pi\rho(x)}{x-c}\,. \tag{3.29}$$

We can also introduce

$$\tilde{\rho}(x) = \frac{1}{\pi}\int_0^{\pi\rho(x)} d\xi\xi\coth\xi\,. \tag{3.30}$$

Then integration by parts in (3.29) yields

$$\delta E^{Bethe} = \frac{c}{\pi k}\int dx\frac{\tilde{\rho}(x)}{(x-c)^2}\,. \tag{3.31}$$

Let us see how the one-loop SYM result [37, 38] is recovered. From (3.17), (3.14) we find that $c = -k/(2\mathcal{J})$ at large $\mathcal{J}$. Inserting this into (3.31) and rescaling $x \to 4\pi\mathcal{J}x$, we get for the energy shift at the leading order in $1/\mathcal{J}$

$$\delta E_1^{Bethe} = -\frac{1}{8\pi^2\mathcal{J}^2}\int dx\ \frac{\tilde{\rho}(x)}{x^2}\,, \tag{3.32}$$

in agreement with [37].

To perturbatively evaluate the integral (3.29), we shall need to expand various parameters characterizing the classical string configuration in a power series in $1/\mathcal{J}$. In particular, we need to find the zeroes of the quartic polynomial $P(x)$. Recall that $P(x)$ defined in (3.14) can be factorized as

$$P(x) = (x-a)(x-b)(x-c)^2\,, \tag{3.33}$$

For our sign choice ($m > 0$, $k < 0$), the roots are ordered as $a < b < c$.

The zeroes a, b, c admit an expansion in $\frac{1}{\mathcal{J}}$. Solving (3.17) perturbatively in $1/\mathcal{J}$ we get

$$\begin{aligned} c &= -\frac{k}{2\mathcal{J}} + \frac{k}{8\mathcal{J}^3}(2m^2 - 4mk + k^2) \\ &\quad + \frac{k}{16\mathcal{J}^5}(-3m^4 + 16m^3k - 23m^2k^2 + 10mk^3 - k^4) + O\left(\frac{1}{\mathcal{J}^7}\right), \qquad (3.34) \\ \mathcal{E} &= \left(1 - \frac{m}{k}\right)\mathcal{J} + \frac{1}{2\mathcal{J}}\,m(m-k) - \frac{1}{8\mathcal{J}^3}\,m(m-k)(m^2 - 3mk + k^2) \\ &\quad + \frac{1}{16\mathcal{J}^5}\,m(m-k)(m^4 - 7m^3k + 13m^2k^2 - 7mk^3 + k^4) + O\left(\frac{1}{\mathcal{J}^7}\right). \qquad (3.35) \end{aligned}$$

The expression (3.35) agrees with the perturbative expansion of the classical string energy computed in [33].

3.3 Mode expansion

Our starting point is (3.29), which can be written as a contour integral, because the integrand has a square-root branch cut along the contour of integration. If we introduce the function

$$f(z) = \frac{\sqrt{P(z)}}{z^2 - 1}\,, \tag{3.36}$$

the energy shift becomes

$$\delta E^{Bethe} = \frac{c}{k}\oint_{\mathcal{C}_{ab}} \frac{dz}{2\pi i}\frac{f'(z)f(z)\cot(\pi f(z))}{z - c}\,, \tag{3.37}$$

where the integration contour $\mathcal{C}_{ab}$ encircles the cut clockwise. We can use the following series representation for $\cot \pi f(z)$

$$\cot(\xi) = \frac{1}{\xi} + 2\xi\sum_{n=1}^{\infty}\frac{1}{\xi^2 - n^2\pi^2}\,. \tag{3.38}$$

Inserting this into the contour integral we obtain

$$\delta E^{Bethe} = \frac{c}{k}\oint_{\mathcal{C}_{ab}} dz \frac{f'(z)}{(z-c)} + \frac{2c}{k}\sum_{n=1}^{\infty}\oint_{\mathcal{C}_{ab}} dz \frac{f'(z)f^2(z)}{(z-c)(f^2(z)-n^2)}\,. \tag{3.39}$$

The only singularities of the integrands outside the contour of integration are poles and the integrals can be calculated by evaluating the residues. The integrand in the first term has poles at $z = c$ and $z = \pm 1$. The poles of the second term are at $z = \pm 1$ and at $z = z_n$, where the z_n's are solutions of

$$f(z_n) = \pm n\,, \qquad n \in \mathbb{N}\,. \tag{3.40}$$

Squaring this equation we find that z_n's are the roots of the quartic equation

$$P(z) = n^2(z^2-1)^2. \tag{3.41}$$

It can be shown that the fluctuation energies around the classical solution are determined by the same equation, in accord with the general relationship between fluctuations [46] and finite-size corrections for Bethe ansatz [48]. The residues at $z = \pm 1$ are rather complicated, but the residues at $z = z_n$ are easy to evaluate

$$\mathrm{Res}_{z=z_n} = \frac{c}{k}\left(\frac{n\epsilon_n}{z_n - c}\right). \tag{3.42}$$

The sign ϵ_n of the residue is the same as the sign in the equation $f(z_n) = \pm n$ and can be determined by analyzing (3.41) with the help of (3.33)

$$\epsilon_n = \begin{cases} +1 & \text{for} \quad z \in [-\infty, a] \cup [-1, c] \cup [1, \infty] \\ -1 & \text{for} \quad z \in [b, -1] \cup [c, 1]\,. \end{cases} \tag{3.43}$$

3.4 Perturbative expansion and comparison to string theory

We have evaluated the residues in (3.39) perturbatively in $1/\mathcal{J}$. The calculations are lengthy and are given in appendix C. We also checked that the first two orders are reproduced by a direct expansion of the integral (3.31). Unlike the string sum over modes, its Bethe counterpart is manifestly finite at each order of the perturbative expansion. This might indicate that our method of computing the series over string modes breaks down at two loops (see also the discussion in [32]). However, if we compare the zeta-regularized sum (2.12), (2.13) and (2.14) with the Bethe ansatz, we find complete agreement! We checked this up to the third order

$$\delta E_p^{Bethe} = \delta E_p^{string}, \qquad p = 1, 2, 3. \tag{3.44}$$

The agreement at the first two orders implies that the string energy shifts agree with the finite-size corrections to the anomalous dimensions at two loops in the SYM theory. At three loops, the string Bethe ansatz that was our starting point, differs from the gauge Bethe ansatz [47] which computes the anomalous dimensions.

The agreement between the Bethe ansatz and the direct string calculation is rather spectacular. The initial expressions look too complicated for this to be a pure accident. Nevertheless, the string and the Bethe calculation have a different status. The Bethe ansatz energy shift is automatically finite order by order in $1/\mathcal{J}$. On the string side we encountered divergences despite the complete, unexpanded energy shift being finite. No doubt, there should be a better way to approach the weak-coupling (large $\mathcal{J}$) limit on the string side.

4 Limit of large winding number

Because of the divergences in the naive $1/\mathcal{J}$ expansion of the string sum, it would be desirable to do an independent test which avoids the convergence issues mentioned earlier. One option is to evaluate the energy shifts numerically. This is done in the next section. Here we consider a particular regime, the limit of large winding number ($|k| \gg 1$), in which the energy shifts can be calculated analytically[5]. In this limit $\mathcal{J}$, $\mathcal{E}$ and m stay finite, but the spin goes to zero: $\mathcal{S} \ll 1$. The string remains macroscopic in this limit, since it winds the big circle of S^5, but in AdS_5 the string shrinks to zero size (*cf.* (1.3)). We will have to assume that $\mathcal{J}/|k| \ll 1$, which means that there is no overlap with the perturbative regime we have discussed so far. In fact, the energy shift turns out to depend on $1/\mathcal{J} = \sqrt{\lambda}/J$ rather than $1/\mathcal{J}^2$ in the large-k limit, and it is not possible to compare string quantum corrections to perturbative SYM theory in this regime.

The details of the string calculation are given in appendix D. The result is

$$\delta E = \frac{2F\left(0,\sqrt{\mathcal{J}^2-m^2}\right)+2F\left(0,\mathcal{J}+m\right)-4F\left(\{\frac{|k|}{2}\},\sqrt{\mathcal{J}(\mathcal{J}+m)}\right)}{\mathcal{J}+m} + \sqrt{m\mathcal{J}}+(\mathcal{J}+m)\ln\frac{\sqrt{\mathcal{J}+m}}{\sqrt{\mathcal{J}}+\sqrt{m}} - m\,, \qquad (4.1)$$

where the function $F(\beta,\alpha)$ is defined in (2.8). A peculiar property of this result is the dependence on the fractional part of $k/2$, which means that the large-k limit of the string energy shift depends on whether the winding number k is even or odd. This effect probably arises because of the k-dependent field redefinition of the world-sheet fermions which was used to find the spectrum of fluctuations [31, 32, 33]. This kind of irregularity does not arise in the Bethe ansatz, and also in the zeta-regularized large-$\mathcal{J}$ expansion.

[5]In the narrow sense, we are just comparing two mathematical expressions – the string one-loop corrections (2.1)–(2.3) and the finite-size correction from the Bethe ansatz (3.29). Each is a well-defined function of the parameters k, m and $\mathcal{J}$. If the two expressions agree (or disagree), they must agree (disagree) at all values of the parameters, in particular if one of the parameters (k in this case) takes its extreme value. From this point of view the limit of large k is just a simplifying assumption that allows us to calculate δE^{String} and δE^{Bethe} explicitly in some corner of the parameter space. On the other hand, not only the classical energy of the string, but also the quantum correction to it stays finite in the large-k limit. This probably means that the limit of large winding (or small spin) is well-defined for this type of string solutions and it would be very interesting to study this limit further. The winding number in that, more general setting should be much larger than the rescaled quantities $\mathcal{E}$ and $\mathcal{J}$, but should be much smaller than $\sqrt{\lambda}$ (and thus E and J) in order not to interfere with the loop expansion of the sigma-model.

4.1 Bethe ansatz calculation

We begin with the classical limit. To take the large-k limit it is convenient to rewrite (3.14) in the two equivalent forms

$$P(x) = k^2(x^2-1)^2 - 4k\mathcal{E}x(x^2-1) + 4m\mathcal{J}x(x\pm 1)^2 + 4(\mathcal{J}\mp m)^2x^2. \tag{4.2}$$

The first two terms blow up in the $k\to\infty$ limit unless x is close to 1 or -1. The roots of P, a, b and c, thus lie in the vicinity of ± 1. Changing the variables to

$$x = \pm 1 + \frac{v}{k}, \tag{4.3}$$

and taking the limit $k\to\infty$, we get

$$P(x) = 4v^2 - 8\mathcal{E}v + 4(\mathcal{J}\pm m)^2, \qquad \text{at} \qquad x\to\pm 1. \tag{4.4}$$

Thus two of the roots of $P(x)$ lie near 1 and two lie near -1. The double root should lie at $x\approx 1$, from which we find

$$\mathcal{E} = \mathcal{J} + m \tag{4.5}$$

and

$$c = 1 - \frac{\mathcal{E}}{|k|}. \tag{4.6}$$

Solving (4.4) near $x=-1$, we find the endpoints of the cut

$$\left\{ \begin{array}{c} b \\ a \end{array} \right\} = -1 - \frac{\left(\sqrt{\mathcal{J}}\pm\sqrt{m}\right)^2}{|k|}. \tag{4.7}$$

We see that the cut shrinks to a very small size, whereas the density according to (3.2)-(3.4) is still normalized to $O(1)$. Thus the density is highly peaked near -1. Indeed, from (3.16) and (4.4) we find

$$\rho(x) = \frac{|k|}{v}\sqrt{2\left(\mathcal{J}+m\right)v - v^2 - \left(\mathcal{J}-m\right)^2}. \tag{4.8}$$

The integral (3.31) can be easily evaluated in the $k\to\infty$ limit. Because the density is large, $\cosh\xi$ in (3.30) can be approximated by 1, and thus

$$\tilde{\rho} = \frac{\pi}{2}\rho^2, \qquad \text{at} \qquad \rho\to\infty. \tag{4.9}$$

We thus get from (3.31)

$$\delta E^{Bethe} = \frac{1}{8k}\int dx\,\rho^2(x). \tag{4.10}$$

Using $dx = dv/|k|$ and the explicit expression (4.8) for the density, we find

$$\delta E^{Bethe} = \sqrt{m\mathcal{J}} - \frac{\mathcal{J}+m}{2}\ln\frac{\sqrt{\mathcal{J}}+\sqrt{m}}{\sqrt{\mathcal{J}}-\sqrt{m}}. \tag{4.11}$$

This clearly disagrees with the string theory calculation (4.1), in particular the Bethe ansatz result has a regular dependence on k. We shall see this discrepancy also in the numerical calculations. Let us also note that even though the explicit computation in this section was done in the simplifying large k limit, the deviations between the Bethe ansatz and the string theory computation are also observed numerically for finite values of the parameter k (see figure 3 in the next section).

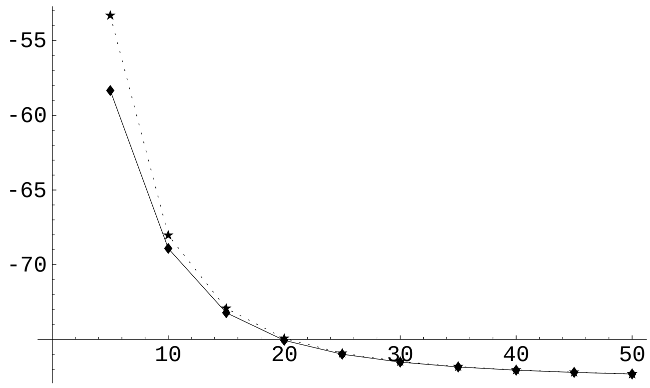

Figure 2: Energy shifts $(\delta E)\times\mathcal{J}^2$ for $\mathcal{J}=5...50$, $m=3$, $k=-2$, Bethe vs. semi-classical string.

5 Numerical evaluation of energy shifts

In this section we numerically compare corrections to the energy of the circular string obtained by the semiclassical quantization (2.3) and the one deduced from the proposed quantum string Bethe equation (3.29). Both evaluations of the sums are done for various values of the parameters.

We first consider the large-$\mathcal{J}$ limit. From figure 2 we see that both functions have the same leading order behaviour, in agreement with the earlier analytic results. Next, we try to extract the coefficients of the $1/\mathcal{J}^2$ expansion of the energy shift numerically. In practice, numerically computing higher order effects is hard, since it requires a high numerical precision and stability.

Yet, by using high precision numerical evaluations let us try to extract the first subleading $(1/\mathcal{J}^2)$ correction from the exact semiclassical expression (2.3) and compare it with the zeta-function regularized result (2.13). Subtracting the analytic one-loop piece (2.12) from the numerical expression for the semiclassical energy shift (2.3) leads to very unstable numerical results, given in table 5.1.

m=3.0 , k=-2 .

$\mathcal{J}$	50	100	150	200	250
$(\delta E^{string}-\delta E_1)\times\mathcal{J}^2$	1041	620	-82	-1066	-2329
$\mathcal{J}$	300	350	400	450	500
$(\delta E^{string}-\delta E_1)\times\mathcal{J}^2$	-3871	-5693	- 7794	-10174	- 12831

(5.1)

This should be compared to the zeta-function regularized two-loop result (2.13) for the same values of m and k which gives

$$\delta E_2 = 393.375\,. \tag{5.2}$$

The numerical stability is greatly improved, if instead of subtracting the analytic one-loop result (2.12), we use the asymptotic numerical value for the energy shift (obtained for $\mathcal{J}=10^3$)

$$\delta E^{string}_{asymptot} = -77.781\,. \tag{5.3}$$

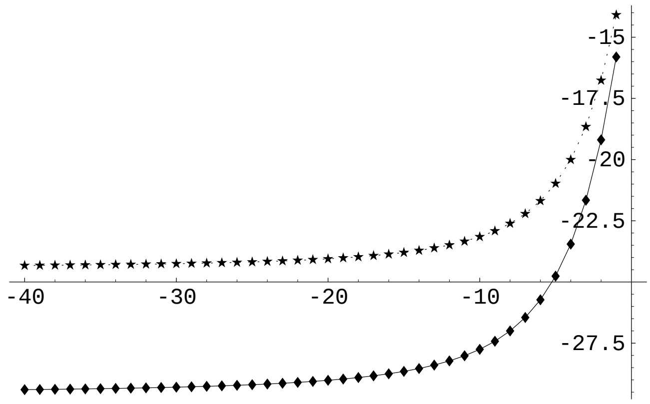

Figure 3: Energy shifts $(\delta E) \times \mathcal{J}^2$ for $\mathcal{J} = 3$, $m = 2$, $-k = (40...1)$. The upper curve is the string calculation. The lower curve is the prediction of the Bethe ansatz.

The results are given in table (5.4). We see that it is much less fluctuating compared to the result in table (5.1). The deviations from the constant value, may be attributed to higher orders in $1/\mathcal{J}^2$ and insufficient numerical precision. The average value from the table (5.4) is different from the regularized two-loop result (5.2), but the numerics is rather unstable and we cannot draw any definite conclusions at this point because of insufficient numerical accuracy.

m=3.0 , k=-2 ,

$\mathcal{J}$	50	100	150	200	250
$(\delta E^{string} - \delta E^{asymptot}) \times \mathcal{J}^2$	1170	1167	1147	1120	1087
$\mathcal{J}$	300	350	400	450	500
$(\delta E^{string} - \delta E^{asymptot}) \times \mathcal{J}^2$	1048	1004	952	896	835

(5.4)

We get much better accuracy if we look at a finite value of $\mathcal{J}$ and vary k at fixed m and $\mathcal{J}$. We shall take $\mathcal{J} = 3$ and $m = 2$ and vary k from -40 to -1. The results are given in figure 3. The upper curve is the semiclassical string computation, the lower curve is computed from the Bethe ansatz[6]. We see that both the semiclassical and the Bethe energy shifts tend asymptotically to constant but different values, which are in good numerical agreement with the analytic calculations in the previous section. Here our numerical precision is sufficient to discriminate the two results.

6 Conclusions

We have compared quantum correction to the energy of macroscopic rigid strings in $AdS_5 \times S^5$ with the finite-size corrections to the quantum string Bethe ansatz. Taken at face value, the two results disagree, but an interpretation of this discrepancy is unclear to us. If we do the string calculation in a more naive way by first expanding fluctuation frequencies in $1/\mathcal{J}$ and

[6]By that we mean numerical integration in (3.29). Direct numerical solution of the discrete Bethe equations with subsequent extrapolation to the thermodynamic limit requires substantially more involved calculations.

then summing the series over string modes, the straightforward zeta-regularized expansion in $1/\mathcal{J}^2$ agrees with the Bethe ansatz to the first three orders. Perhaps the sum over frequencies on the string side should be redefined such that it automatically reproduces zeta-regularized $1/\mathcal{J}$ expansion. The methods used to evaluate related sums in the context of plane-wave string theory [49] can be helpful to implement such zeta-function prescription. On the other hand the sum is finite and well-defined as it stands and there are no apparent regularization ambiguities.

Another possible explanation of the discrepancy is that the string Bethe equations receive non-trivial $1/\sqrt{\lambda}$ corrections. We cannot discriminate between these two possibilities at present. Studying other classes of string solutions will be certainly helpful to resolve this puzzle. We should first of all mention stable circular strings on S^5 which were analyzed both in string theory [31] and using the Bethe ansatz [50].

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Appendix A Calculation of anomaly

In this appendix the anomaly term is derived from the quantum string Bethe equations (3.20). The following integral representation turns out to be useful

$$\ln \frac{f(x_1^+,\ldots,x_S^+;x_1^-,\ldots,x_S^-)}{f(x_1,\ldots,x_S;x_1,\ldots,x_S)} = i\int_0^{\frac{2\pi}{\sqrt{\lambda}}} d\varepsilon\ \frac{1}{f}\sum_{k=1}^{S}\left(\frac{x_k^{+\,2}}{x_k^{+\,2}-1}\,\frac{\partial f}{\partial x_k^+} - \frac{x_k^{-\,2}}{x_k^{-\,2}-1}\,\frac{\partial f}{\partial x_k^-}\right), \qquad \text{(A.1)}$$

where f is an arbitrary function and

$$x_k^\pm + \frac{1}{x_k^\pm} = u_k \pm i\varepsilon\,, \qquad \text{(A.2)}$$

under the integral ($x_k^\pm$ on the left-hand-side is defined in (3.22)). This representation singles out a particular branch of the logarithm, so when we write the Bethe equations (3.20) in the logarithmic form, we should introduce an arbitrary phase which parameterizes different branches of the logarithm

$$\int_0^{\frac{2\pi}{\sqrt{\lambda}}} d\varepsilon \left\{ J\left(\frac{x_k^+}{x_k^{+\,2}-1}+\frac{x_k^-}{x_k^{-\,2}-1}\right)+\sum_{j\neq k}\left[\left(\frac{x_k^{+\,2}}{x_k^{+\,2}-1}+\frac{x_j^{-\,2}}{x_j^{-\,2}-1}\right)\frac{1}{x_k^+-x_j^-}+\right.\right.$$
$$\left.\left(\frac{x_k^{-\,2}}{x_k^{-\,2}-1}+\frac{x_j^{+\,2}}{x_j^{+\,2}-1}\right)\frac{1}{x_k^--x_j^+}\right]-\sum_{j\neq k}\left[\frac{x_j^+}{(x_j^{+\,2}-1)(x_k^-x_j^+-1)}+\frac{x_j^-}{(x_j^{-\,2}-1)(x_k^+x_j^--1)}\right.$$
$$\left.-\frac{x_k^+}{(x_k^{+\,2}-1)(x_k^+x_j^--1)}-\frac{x_k^-}{(x_k^{-\,2}-1)(x_k^-x_j^+-1)}\right]$$
$$-\frac{i\sqrt{\lambda}}{2\pi}\sum_{j\neq k}(u_k-u_j)\left[\frac{x_k^{+\,2}(x_j^+-x_j^-)}{(x_k^{+\,2}-1)(x_k^+x_j^--1)(x_k^+x_j^+-1)}+\frac{x_k^{-\,2}(x_j^+-x_j^-)}{(x_k^{-\,2}-1)(x_k^-x_j^+-1)(x_k^-x_j^--1)}\right.$$
$$\left.\left.+\frac{x_j^{+\,2}(x_k^+-x_k^-)}{(x_j^{+\,2}-1)(x_k^+x_j^+-1)(x_k^-x_j^+-1)}+\frac{x_j^{-\,2}(x_k^+-x_k^-)}{(x_j^{-\,2}-1)(x_k^+x_j^--1)(x_k^-x_j^--1)}\right]\right\}=2\pi k. \qquad (A.3)$$

An important property of this terrible-looking equation is the symmetry with respect to $\varepsilon \to -\varepsilon$, which means that the direct strong-coupling expansion starts from order $O(1/\lambda)$. The only source of $1/\sqrt{\lambda}$ corrections is the first sum over j, in which terms with $j \sim k$ become singular in the $\varepsilon \to 0$ limit. The contribution of these terms is the anomaly. In the remaining terms we can take the limit $\varepsilon \to 0$ directly

$$\frac{4\pi\mathcal{J}x_k}{x_k^2-1}+\frac{4\pi}{\sqrt{\lambda}}\sum_{jk}\frac{x_k-x_j}{(x_k^2-1)(x_j^2-1)}+\int_0^{\frac{2\pi}{\sqrt{\lambda}}}d\varepsilon\sum_{j\neq k}\left[\left(\frac{x_k^{+\,2}}{x_k^{+\,2}-1}+\frac{x_j^{-\,2}}{x_j^{-\,2}-1}\right)\frac{1}{x_k^+-x_j^-}+\right.$$
$$\left.\left(\frac{x_k^{-\,2}}{x_k^{-\,2}-1}+\frac{x_j^{+\,2}}{x_j^{+\,2}-1}\right)\frac{1}{x_k^--x_j^+}\right]=2\pi k, \qquad (A.4)$$

where we have used the equality

$$u_k-u_j=\frac{(x_k-x_j)(x_kx_j-1)}{x_kx_j}.$$

The next step is to multiply both sides of (A.4) by $1/(z-x_k)$ and sum over k. Because of the anti-symmetry in k and j, in the double sums $1/(z-x_k)$ can be replaced by

$$\frac{1}{z-x_k}\to\frac{1}{2}\left(\frac{1}{z-x_k}-\frac{1}{z-x_j}\right)=\frac{x_k-x_j}{2(z-x_k)(z-x_j)}.$$

Now we can disentangle the "normal" contribution of $j-k\sim\sqrt{\lambda}$ from the local "anomalous" contribution of $j-k\ll\sqrt{\lambda}$. In the latter case

$$x_{j+n}\approx x_j+\frac{4\pi x_j^2 n}{\sqrt{\lambda}(x_j^2-1)\rho(x_j)}, \qquad (A.5)$$

according to the definition of the density in (3.19). Also,

$$x_{j+n}^{\pm} \approx x_j + \frac{x_j^2}{x_j^2-1}\left(\frac{4\pi n}{\sqrt{\lambda}\rho(x_j)} \pm i\varepsilon\right) \tag{A.6}$$

and

$$\frac{x_{j+n}-x_j}{x_{j+n}^{\pm}-x_j^{\mp}} - 1 = \mp\frac{2i\varepsilon}{\frac{4\pi n}{\sqrt{\lambda}\rho(x_j)} \pm 2i\varepsilon}\,.$$

Separating the long-distance contributions from the short-distance ones we find, after some calculations

$$G^2(z) - 2\pi\left(k - 2\,\frac{\mathcal{J}z+m}{z^2-1}\right)G(z) + \frac{4\pi^2}{z^2-1}\left[k(\mathcal{E}-\mathcal{S}-\mathcal{J})z - 2m\mathcal{J}z - m(k+m)\right]$$
$$-\frac{4\pi}{\sqrt{\lambda}}\,\frac{z^2}{z^2-1}\sum_j \frac{2x_j^2}{(x_j^2-1)(z-x_j)^2}\int_0^{\frac{2\pi}{\sqrt{\lambda}}} d\varepsilon \sum_{n=-\infty}^{n=+\infty}\frac{\varepsilon^2}{\frac{4\pi^2n^2}{\lambda\rho^2(x_j)}+\varepsilon^2} = 0, \tag{A.7}$$

where

$$G(z) = \frac{4\pi}{\sqrt{\lambda}}\sum_k \frac{x_k^2}{x_k^2-1}\,\frac{1}{z-x_k}\,. \tag{A.8}$$

The asymptotics of (A.7) at $z\to\infty$ shows that the condition (3.10) does not receive quantum corrections. Performing the summation in the anomaly term and changing the integration variable to $\xi = \sqrt{\lambda}\rho\varepsilon/2$ we finally get

$$G^2(z) - 2\pi\left(k - 2\,\frac{\mathcal{J}z+m}{z^2-1}\right)G(z) + \frac{4\pi^2}{z^2-1}\left[k(\mathcal{E}+\mathcal{S}-\mathcal{J})z - m(k+m)\right]$$
$$-\frac{4\pi}{\sqrt{\lambda}}\,\frac{z^2}{z^2-1}\int dx\,\frac{\tilde{\rho}(x)}{z-x} = 0, \tag{A.9}$$

where $\tilde{\rho}(x)$ is defined in (3.30). The form of the anomaly used in the main text is obtained after integrating by parts and taking into account that

$$\tilde{\rho}' = \rho'\pi\rho\coth\pi\rho\,. \tag{A.10}$$

Appendix B Details of string theory computation

B.1 Contribution of $\mathfrak{sl}(2)$ modes

The main difficulty in evaluating the energy from the string theory is the sum over the roots of the quartic polynomial (2.4)

$$\delta E^{\mathfrak{sl}(2)} = \sum_I sign(C_I^{(n)})\omega_{I,n}\,. \tag{B.1}$$

The quartic equation is equivalently given by

$$\omega^4 + a_2\omega^2 + a_1\omega + a_0 = 0\,, \tag{B.2}$$

where

$$a_2 = -4k^2 - 2n^2 - 4k^2 r_1^2 - 4\kappa^2$$
$$a_1 = 8kn\sqrt{k^2+\kappa^2}(1+r_1^2) \qquad \text{(B.3)}$$
$$a_0 = n^4 - 4k^2 n^2 (1+r_1^2)\,.$$

In particular, the absence of the cubic term implies $\sum_{I=1}^4 \omega_{I,n} = 0$. The roots can be written as

$$\omega_{1/2,n} = \frac{1}{2}\left(R_n \pm D_n\right)$$
$$\omega_{3/4,n} = \frac{1}{2}\left(-R_n \pm F_n\right)\,, \qquad \text{(B.4)}$$

where

$$R_n = \sqrt{y_1 - a_2}$$
$$D_n = \sqrt{-R_n^2 - 2a_2 - \frac{2a_1}{R_n}} \qquad \text{(B.5)}$$
$$F_n = \sqrt{-R_n^2 - 2a_2 + \frac{2a_1}{R_n}}\,,$$

and y_1 is a real root of the discriminant cubic equation

$$y^3 - a_2 y^2 - 4a_0 y + 4a_2 a_0 - a_1^2 = 0\,. \qquad \text{(B.6)}$$

That is

$$y_1 = \frac{1}{3}a_2 + \left(M + \sqrt{M^2+S^3}\right)^{1/3} + \left(M - \sqrt{M^2+S^3}\right)^{1/3}\,, \qquad \text{(B.7)}$$

where

$$S = \frac{1}{9}(-12a_0 - a_2^2)$$
$$M = \frac{1}{54}(27a_1^2 - 72a_0 a_2 + 2a_2^3)\,. \qquad \text{(B.8)}$$

Furthermore, we need to address the issue of the signs in front of the frequencies. If we take all square roots with positive sign, it is clear that for a generic n and $\mathcal{J}$ there are two possibilities for the relative ordering of the frequencies ω_I

$$\text{I}: \quad \omega_4 < \omega_3 < \omega_2 < \omega_1 \qquad \text{(B.9)}$$
$$\text{II}: \quad \omega_4 < \omega_2 < \omega_3 < \omega_1\,. \qquad \text{(B.10)}$$

In order to discriminate these, consider the large $\mathcal{J} \gg n$ limit. The asymptotics are $\omega_1 \sim -\omega_4 \sim 2\mathcal{J}$ and so $(\omega^2 - n^2) > 0$. Hence,

$$sign(C_{1,B}^{(n)}) = +1\,, \qquad sign(C_{4,B}^{(n)}) = -1\,. \qquad \text{(B.11)}$$

On the other hand, in the same limit we have $\omega_2 \sim -\omega_3 \sim n/2\mathcal{J}$ and thus $(\omega^2 - n^2) < 0$, wherefore

$$sign(C_{2,B}^{(n)}) = -1\,, \qquad sign(C_{3,B}^{(n)}) = +1\,. \qquad \text{(B.12)}$$

Hence, in the large $\mathcal{J}$ limit the eigenvalues are ordered as in the first case in (B.9). Note that the ordering of ω_n^I as a function of n keeping $\mathcal{J}$ fixed does not change, *i.e.*, the roots do not "cross" (see figure 5).

Using (B.11) and (B.12) the expression for $\delta E^{\mathfrak{sl}(2)}$ in the large $\mathcal{J}$ limit can be simplified to

$$\delta E^{\mathfrak{sl}(2)} = \sum_n (-\omega_4 + \omega_3 - \omega_2 + \omega_1) = 2\sum_n (\omega_1 + \omega_3) = \sum_n D_n + F_n \,. \tag{B.13}$$

In summary, to compute $\delta E^{\mathfrak{sl}(2)}$ one only needs to determine the sum over the combination $D_n + F_n$.

B.2 Perturbative expansion of modes

The combination of $\mathfrak{sl}(2)$ modes, $D_n + F_n$, has the following expansion in $1/\mathcal{J}$

$$\begin{aligned}
\frac{\delta E^{\mathfrak{sl}(2)}}{2\kappa} = \sum_n &\left(\frac{2k(k-m) + n^2 + n\sqrt{4m(m-k) + n^2}}{2}\right)\frac{1}{\mathcal{J}^2} \\
&+ \Bigg(- \frac{-4m(k-m)(5k^2 - 15km + 6m^2) + 2(k-3m)(3k-2m)n^3 + n^5}{8\sqrt{n^2 + 4m(m-k)}} \\
&+ \frac{1}{8}(-2k(k-m)(k^2 - 11km + 6m^2) - 2(3k^2 - 10km + 5m^2)n^2 - n^4)\Bigg)\frac{1}{\mathcal{J}^4} \\
&+ \Bigg(\frac{1}{16}\{2k(k-m)(k^4 - 23k^3m + 86k^2m^2 - 71km^3 + 15m^4) \\
&+ (15k^4 - 128k^3m + 279k^2m^2 - 202km^3 + 44m^4)n^2 + (15k^2 - 38km + 19m^2)n^4 + n^6\} \\
&+ \frac{1}{16(n^2 + 4m(m-k))^{3/2}}\{ + 4(k-m)^2m^2(45k^4 - 324k^3m + 621k^2m^2 - 370km^3 + 60m^4)n \\
&- 2(k-m)m(53k^4 - 481k^3m + 1083k^2m^2 - 815km^3 + 192m^4)n^3 \\
&+ (15k^4 - 218k^3m + 603k^2m^2 - 556km^3 + 164m^4)n^5 + (15k^2 - 44km + 25m^2)n^7 + n^9\}\Bigg)\frac{1}{\mathcal{J}^6} \,.
\end{aligned} \tag{B.14}$$

The other terms, *i.e.*, the transverse and fermionic terms, are as follows

$$\begin{aligned}
&\delta E - \frac{\delta E^{\mathfrak{sl}(2)}}{2\kappa} \\
&= \sum_n \left(-(k-m)^2 - n^2\right)\frac{1}{\mathcal{J}^2} \\
&+ \frac{1}{16}\big((k-m)^2(k^2 - 42km - 7m^2) + 8(3k^2 - 10km + 5m^2)n^2 + 4n^4\big)\frac{1}{\mathcal{J}^4} \\
&+ \frac{1}{128}\big(- (k-m)^2(k^4 - 232k^3m + 962k^2m^2 - 80km^3 - 11m^4) \\
&- 4(15k^4 - 260k^3m + 594k^2m^2 - 340km^3 + 23m^4)n^2 - 16(15k^2 - 38km + 19m^2)n^4 - 16n^6\big)\frac{1}{\mathcal{J}^6} \,.
\end{aligned} \tag{B.15}$$

Note the three-loop term, where the expression at order n^2 has a different structure from the one in (B.14).

Furthermore, expanding the zero mode part of the energy shift (2.2) in $1/\mathcal{J}$ we obtain

$$E^{(0)} = \frac{1}{2}m(k-m)\frac{1}{\mathcal{J}^2} - \frac{1}{32}(3k-7m)(k-m)(k+m)^2\frac{1}{\mathcal{J}^4} + \frac{1}{256}(k-m)\left(15k^5 - 135k^4m + 182k^3m^2 - 94k^2m^3 + 171km^4 - 11m^5\right)\frac{1}{\mathcal{J}^6} + O\left(\frac{1}{\mathcal{J}^8}\right). \tag{B.16}$$

We shall now combine these terms and obtain the energy shifts up to third order in perturbation theory.

B.3 First and Second order

The first and second order terms in the $1/\mathcal{J}^2$ expansion of the energy shift (2.1) are

$$\begin{aligned}\delta E_1^{osc} &= \sum_n \frac{2(k-m)m - n^2 + n\sqrt{n^2+4m(m-k)}}{2}\\ \delta E_2^{osc} &= \sum_n -\frac{n(-4m\,(k-m)\,(5k^2-15km+6m^2) + 2\,(k-3m)\,(3k-2m)\,n^2 + n^4)}{8\sqrt{n^2+4m(m-k)}}\\ &\quad + \frac{1}{16}((k-m)\,(k+m)^2\,(-3k+7m) + 4\left(3k^2-10km+5m^2\right)n^2 + 2n^4).\end{aligned} \tag{B.17}$$

The large n behaviour of the summand in δE_1^{osc} is $1/n^2$, which ensures that the energy shift at first order is finite. In the second order term the summand has asymptotics

$$(\delta E_2^{osc})_n = -\frac{1}{16}(k-m)^2(3k^2-14km+19m^2) + O\left(\frac{1}{n^2}\right). \tag{B.18}$$

Thus, there is an anomalous pieces, which needs to be regularized. Applying zeta-function regularization the regularized energy reads

$$\begin{aligned}(\delta E_2^{osc})_{reg} &= \frac{1}{32}(k-m)^2(3k^2-14km+19m^2)\\ &+ \sum_n \Bigg\{\frac{n(-4m\,(k-m)\,(5k^2-15km+6m^2) + 2\,(k-3m)\,(3k-2m)\,n^2 + n^4)}{8\sqrt{n^2+4m(m-k)}}\\ &\quad + \frac{1}{8}(-2\,(k-m)\,m\left(4k^2-11km+3m^2\right) + 2\left(3k^2-10km+5m^2\right)n^2 + n^4)\Bigg\}.\end{aligned} \tag{B.19}$$

Combining the zero-mode energy shift with the oscillator contribution, we obtain in summary that at order $1/\mathcal{J}^2$ and $1/\mathcal{J}^4$ the shift is

$$\begin{aligned}
\delta E_1^{string} &= \frac{1}{2}m(k-m) + \sum_n \frac{2(k-m)m - n^2 + n\sqrt{n^2+4m(m-k)}}{2} \\
\delta E_2^{string} &= -\frac{1}{8}m(k-m)(4k^2-11km+3m^2) \\
&+ \sum_n \Bigg\{ -\frac{n(-4m\,(k-m)\,(5k^2-15km+6m^2) + 2\,(k-3m)\,(3k-2m)\,n^2 + n^4)}{8\sqrt{n^2+4m(m-k)}} \\
&\quad + \frac{1}{8}(-2\,(k-m)\,m\left(4k^2-11km+3m^2\right) + 2\left(3k^2-10km+5m^2\right)n^2 + n^4) \Bigg\}\,.
\end{aligned} \tag{B.20}$$

B.4 Third order

Further expanding the string theory result for the contributions of the oscillators to the energy up to third order, *i.e.*, order $1/\mathcal{J}^6$, yields

$$\begin{aligned}
\delta E_3^{osc} =& \sum_n \frac{1}{128}\{15k^6 - 150k^5m + 317k^4m^2 - 276k^3m^3 + 265k^2m^4 - 182km^5 + 11m^6 \\
&+ 4(15k^4 + 4k^3m - 36k^2m^2 - 64km^3 + 65m^4)n^2 - 8(15k^2 - 38km + 19m^2)n^4 - 8n^6\} \\
&+ \frac{1}{16(n^2+4m(m-k))^{3/2}}\{4(k-m)^2m^2(45k^4 - 324k^3m + 621k^2m^2 - 370km^3 + 60m^4)n \\
&- 2(k-m)m(53k^4 - 481k^3m + 1083k^2m^2 - 815km^3 + 192m^4)n^3 \\
&+ (15k^4 - 218k^3m + 603k^2m^2 - 556km^3 + 164m^4)n^5 + (15k^2 - 44km + 25m^2)n^7 + n^9\}\,.
\end{aligned} \tag{B.21}$$

The sum is again divergent as the large n behaviour of the summand in (B.21) is

$$\begin{aligned}
(\delta E_3^{osc})_n =& \frac{9}{32}(k-m)^2(5k^2 - 18km + 17m^2)n^2 \\
&+ \frac{1}{128}(k-m)^2(15k^4 - 248k^3m + 766k^2m^2 - 752km^3 + 91m^4) + O\left(\frac{1}{n^2}\right)\,.
\end{aligned} \tag{B.22}$$

We again apply zeta-function regularization. In the present case, we need to evaluate the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ at $s = -2, 0$. The values can be calculated by writing the zeta-function as

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+1)^{-s}\,, \tag{B.23}$$

and evaluating the inner sum first. This results for $k > 1$ in

$$\zeta(-k+1) = -\frac{B_k}{k}\,, \tag{B.24}$$

where B_k are the Bernoulli numbers. Now $B_3 = 0$ and therefore only $\zeta(0)$ gives a non-vanishing contribution, namely $\zeta(0) = -1/2$. The regularized contribution from the oscillators to the zero modes is thus

$$(\delta E_3^{osc})_{reg} = \frac{1}{256}(k-m)^2\left(15k^4 - 248k^3m + 766k^2m^2 - 752km^3 + 91m^4\right) + \sum_n \cdots\,, \quad \text{(B.25)}$$

where the dots indicate the non-zero mode contributions, with the terms in (B.22) subtracted.

Combining all terms, we arrive at the third order energy shift as computed from the string theory side

$$\begin{aligned}
\delta E_3^{string} &= \frac{1}{16}(k-m)m(8k^4 - 52k^3m + 89k^2m^2 - 42km^3 + 5m^4) \\
&+ \sum_n \frac{1}{16}\{2(k-m)m(8k^4 - 52k^3m + 89k^2m^2 - 42km^3 + 5m^4) \\
&- (15k^4 - 128k^3m + 279k^2m^2 - 202km^3 + 44m^4)n^2 - (15k^2 - 38km + 19m^2)n^4 - n^6\} \\
&+ \frac{1}{16(n^2+4m(m-k))^{3/2}}\{4(k-m)^2m^2(45k^4 - 324k^3m + 621k^2m^2 - 370km^3 + 60m^4)n \\
&- 2(k-m)m(53k^4 - 481k^3m + 1083k^2m^2 - 815km^3 + 192m^4)n^3 \\
&+ (15k^4 - 218k^3m + 603k^2m^2 - 556km^3 + 164m^4)n^5 + (15k^2 - 44km + 25m^2)n^7 + n^9\}\,.
\end{aligned} \quad \text{(B.26)}$$

We shall see subsequently, that this regularized energy shift agrees with the prediction from the Bethe ansatz.

Appendix C Details of Bethe ansatz computation

C.1 Zero-modes

The zero mode integral is

$$\delta E^{(0)} = \frac{c}{k}\oint_{\mathcal{C}_{ab}} dz \frac{f'(z)}{(z-c)}\,. \quad \text{(C.1)}$$

By deforming the contour to infinity, we pick up the residues at $z = c$ and $z = \pm 1$.

Combining these residues and subsequently expanding them in $1/\mathcal{J}$ by making use of (3.34), yields

$$\begin{aligned}
\delta E^{(0)} &= \frac{1}{2}m(k-m)\frac{1}{\mathcal{J}^2} \\
&- \frac{1}{8}m(k-m)(4k^2 - 11km + 3m^2)\frac{1}{\mathcal{J}^4} \\
&+ \frac{1}{16}(k-m)m(8k^4 - 52k^3m + 89k^2m^2 - 42km^3 + 5m^4)\frac{1}{\mathcal{J}^6} + O\left(\frac{1}{\mathcal{J}^8}\right)\,.
\end{aligned} \quad \text{(C.2)}$$

Comparison to the string theory result, which were computed in the previous section shows that up to third order in the $1/\mathcal{J}^2$ perturbation expansion, the zero-mode terms (C.2) agree with the ones of the zeta-function regularized expressions on the string side.

C.2 Non-zero modes

The non-zero mode contributions come from the sum in (3.38) and are

$$\delta E^{osc} = \sum_{n=1}^{\infty} \delta E^{(n)} = \frac{2c}{k} \sum_{n=1}^{\infty} \oint_{\mathcal{C}_{ab}} dz \frac{f'(z) f^2(z)}{(z-c)(f^2(z)-n^2)} \,. \tag{C.3}$$

Again, deforming the contour to infinity, we pick up (possibly non-trivial) residues at $z=c$, $z=\infty$, $z=\pm 1$ as well as $z=z_n$, where z_n were defined in (3.40).

The residues at $z=c$ and $z=\infty$ vanish. The residue at $z=z_n$ was evaluated in (3.42). In order to expand this in $1/\mathcal{J}$, one first needs to solve (3.41) perturbatively for z_n (note that there are two roots z_n each for positive n and for negative n).

The expansion of (3.42) yields up to third order

$$\begin{aligned}
&\mathrm{Res}_{z=z_n} \\
&= \frac{1}{2}\left\{ 2k(m-k) - n^2 + n\sqrt{n^2+4m(m-k)} \right\} \frac{1}{\mathcal{J}^2} \\
&+ \frac{1}{8}\Bigg\{ -2k(m-k)(k^2-11km+6m^2) + 2(3k^2-10km+5m^2)n^2 + n^4 \\
&\qquad - \frac{-4(k-m)m(5k^2-15km+6m^2)n + 2(k-3m)(3k-2m)n^3 + n^5}{\sqrt{n^2+4m(m-k)}} \Bigg\} \frac{1}{\mathcal{J}^4} \\
&+ \frac{1}{32}\Bigg\{ -9k^6 + 184k^5m - 848k^4m^2 + 1380k^3m^3 - 934k^2m^4 + 252km^5 - 20m^6 \\
&\qquad - 2(15k^4 - 128k^3m + 279k^2m^2 - 202km^3 + 44m^4)n^2 - 2(15k^2 - 38km + 19m^2)n^4 - 2n^6 \\
&\qquad + \frac{2}{(n^2+4m(m-k))^{3/2}} \Big(4(k-m)^2m^2(45k^4 - 324k^3m + 621k^2m^2 - 370km^3 + 60m^4)n \\
&\qquad - 2(k-m)m(53k^4 - 481k^3m + 1083k^2m^2 - 815km^3 + 192m^4)n^3 \\
&\qquad + (15k^4 - 218k^3m + 603k^2m^2 - 556km^3 + 164m^4)n^5 \\
&\qquad + (15k^2 - 44km + 25m^2)n^7 + n^9 \Big) \Bigg\} \frac{1}{\mathcal{J}^6} \\
&+ O\left(\frac{1}{\mathcal{J}^8}\right) .
\end{aligned} \tag{C.4}$$

Finally, there are the residues at $z=\pm 1$, which contribute to the n-independent terms of the summands $\delta E^{(n)}$

$$\begin{aligned}
&\mathrm{Res}_{z=1} + \mathrm{Res}_{z=-1} \\
&\quad = (k^2-m^2)\frac{1}{\mathcal{J}^2} - \frac{1}{4}(k-m)(k+m)(k^2-8km+3m^2)\frac{1}{\mathcal{J}^4} \\
&\quad - \frac{1}{32}k(-9k^5 + 152k^4m - 608k^3m^2 + 816k^2m^3 - 410km^4 + 64m^5)\frac{1}{\mathcal{J}^6} + O\left(\frac{1}{\mathcal{J}^8}\right) .
\end{aligned} \tag{C.5}$$

Putting the residues in (C.4) and (C.5) together we obtain

$$\begin{aligned}
&\delta E^{(n)} \\
&= \frac{1}{2}\Big\{ -2m(m-k) - n^2 + n\sqrt{n^2+4m(m-k)}\Big\}\frac{1}{\mathcal{J}^2} \\
&+ \frac{1}{8}\Big\{ -2(k-m)m(4k^2-11km+3m^2) + 2(3k^2-10km+5m^2)n^2 + n^4 \\
&\quad - \frac{-4(k-m)m(5k^2-15km+6m^2)n + 2(k-3m)(3k-2m)n^3 + n^5}{\sqrt{n^2+4m(m-k)}}\Big\}\frac{1}{\mathcal{J}^4} \\
&+ \frac{1}{32}\Big\{ 4(k-m)m(8k^4-52k^3m+89k^2m^2-42km^3+5m^4) \\
&\quad - 2(15k^4-128k^3m+279k^2m^2-202km^3+44m^4)n^2 - 2(15k^2-38km+19m^2)n^4 - 2n^6 \\
&\quad + \frac{2}{(n^2+4m(m-k))^{3/2}}\Big(4(k-m)^2m^2(45k^4-324k^3m+621k^2m^2-370km^3+60m^4)n \\
&\quad - 2(k-m)m(53k^4-481k^3m+1083k^2m^2-815km^3+192m^4)n^3 \\
&\quad + (15k^4-218k^3m+603k^2m^2-556km^3+164m^4)n^5 \\
&\quad + (15k^2-44km+25m^2)n^7 + n^9\Big)\Big\}\frac{1}{\mathcal{J}^6} \\
&+ O\left(\frac{1}{\mathcal{J}^8}\right).
\end{aligned} \tag{C.6}$$

The complete energy shift is then

$$\delta E = \delta E^{(0)} + \sum_{n=1}^{\infty} \delta E^{(n)}, \tag{C.7}$$

where the various terms are written out in (C.2) and (C.6).

In summary, the Bethe result agrees with the string results (B.20), (B.26) including order $1/\mathcal{J}^6$.

Appendix D Details of the large k string computation

We evaluate the energy shift δE^{string} in the large k limit, for fixed m and $\mathcal{J}$. Again, the problematic part in the computation are the ω-dependent terms, for which we are forced to use approximations for finding the roots in different regions of the parameters.

Note that first expanding the summands in (2.2) and (2.3) $1/k$ before summing them yields divergent expressions. However, unlike the divergences that occured in the $1/\mathcal{J}$ expansion at second and third order in perturbation theory, these divergences cannot be removed, using standard regularisation procedures such as zeta-function regularisation as they contain logarithmic divergences. The origin of this divergence is the irregular dependence on k of the resummed expression (2.9).

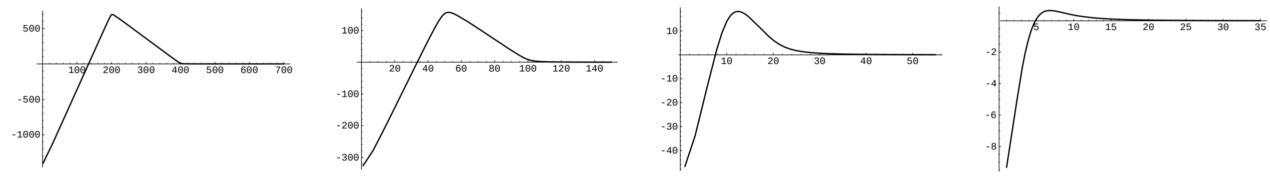

Figure 4: Profiles of the summands for $k = 400$, $k = 100$, $k = 20$ and $k = 5$, respectively, with ($\mathcal{J} = 3$, $m = 2$).

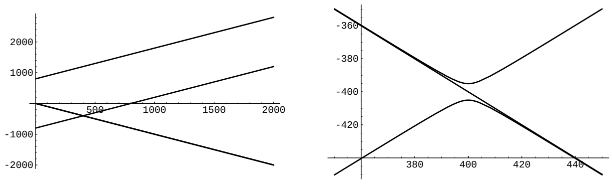

Figure 5: Profiles of the ω frequencies for $k = 400$, ($\mathcal{J} = 3$, $m = 2$). The plot on the right hand side zooms into the plot on the left hand side.

In order to ascertain what kind of function we are summing, it is useful to numerically plot the summands. This is done in figure 3 for various, mainly large, values of k. Solving (B.2) in the limit $n \sim |k| \to \infty$ we find, up to $O(1/k^2)$ corrections

$$\omega_{n\,1,2} = n \pm 2|k| \pm \frac{1}{|k|}\left[m\mathcal{J} + \frac{n \pm 2|k|}{n \pm |k|}\,\frac{(\mathcal{J}+m)^2}{2}\right], \tag{D.1}$$

$$\omega_{n\,3,4} = -n - \frac{(\mathcal{J}+m)n}{2|k|(n^2-k^2)}\left[(\mathcal{J}+m)|k| \pm \sqrt{(\mathcal{J}-m)^2 n^2 + 4m\mathcal{J}k^2}\right]. \tag{D.2}$$

These expressions approximate the frequencies well enough in the entire range of n, except for $n - |k| = O(1)$, where $1/k$ corrections to ω_2 and ω_3 blow up. Solving (B.2) in that region we find

$$\omega_1 = 3|k|, \tag{D.3}$$

$$\omega_4 = -n, \tag{D.4}$$

$$\omega_\pm = -|k| \pm \sqrt{(n-|k|)^2 + (\mathcal{J}+m)^2}. \tag{D.5}$$

Comparing (D.5) to (D.1), (D.2) we see that ω_+ asymptotes ω_2 at $n \gg |k|$ and ω_3 at $n \ll |k|$, while ω_- asymptotes ω_3 at $n \gg |k|$ and ω_2 at $n \ll |k|$. Thus ω_2 and ω_3 interchange at $n = |k|$ by passing through the singularity.

Computing the sign factors from (2.6) we get

$$\begin{array}{lllll}
n < |k| : & sign C_1^{(n)} = 1, & sign C_2^{(n)} = -1, & sign C_3^{(n)} = 1, & sign C_4^{(n)} = -1, \\
n > |k| : & sign C_1^{(n)} = 1, & sign C_2^{(n)} = 1, & sign C_3^{(n)} = -1, & sign C_4^{(n)} = -1, \\
n - |k| \sim 1 : & sign C_1^{(n)} = 1, & sign C_4^{(n)} = -1, & sign C_\pm^{(n)} = \pm 1 . &
\end{array} \tag{D.6}$$

We are now ready to compute the sum over modes. To do that we divide the range of summation into three parts

$$\begin{aligned} &(\mathrm{I}) \quad 1 \leq n \leq |k| - s - 1 \\ &(\mathrm{II}) \quad |k| - s \leq n \leq |k| + s \\ &(\mathrm{III}) \quad |k| + s + 1 \leq n \,, \end{aligned} \tag{D.7}$$

where $1 \ll s \ll |k|$. In the regions (I) and (III) the summation of $O(1/k)$ terms can be replaced by an integration over $x = n/|k|$

$$\begin{aligned} &\sum_{n=1}^{\infty}\sum_{I=1}^{4}\left(signC_I^{(n)}\omega_{n,I} - n - \frac{\kappa^2}{2n}\right) = 4\sum_{n=1}^{|k|-s-1}(|k| - n) - 2(\mathcal{J}+m)^2\sum_{n=1}^{|k|-s-1}\frac{1}{n} \\ &+ \int_0^{1-s/|k|} dx\left[2m\mathcal{J} + (\mathcal{J}+m)\frac{(2-x^2)(\mathcal{J}+m) + x\sqrt{(\mathcal{J}-m)^2x^2 + 4m\mathcal{J}}}{1-x^2}\right] \\ &+ \sum_{l=-s}^{s}\left[2\sqrt{l^2 + (\mathcal{J}+m)^2} - 3l\right] + 2(\mathcal{J}+m)^2\int_{1-s/|k|}^{\infty} dx\left(\frac{x}{x^2-1} - \frac{1}{x}\right) \\ &= 2\left[k^2 - |k| + (\mathcal{J}+m)\sqrt{m\mathcal{J}} + m\mathcal{J} + F(0,\mathcal{J}+m) + (\mathcal{J}+m)^2\ln\frac{\sqrt{\mathcal{J}+m}}{\sqrt{\mathcal{J}}+\sqrt{m}}\right] \end{aligned} \tag{D.8}$$

Combining this with the expansion of the zero modes

$$\delta E^{(0)} = -\frac{|k|}{\mathcal{J}+m} + \left(1 + 2\sqrt{\frac{\mathcal{J}-m}{\mathcal{J}+m}}\right) - \frac{7\mathcal{J}^2 + 10\mathcal{J}m + m^2}{2\,|k|(\mathcal{J}+m)} + \mathcal{O}(\frac{1}{k^2})\,, \tag{D.9}$$

and (2.9) we obtain

$$\begin{aligned} \delta E \;=\; & \frac{k^2 - 4\gamma^2 + m\mathcal{J} + 2F\left(0,\sqrt{\mathcal{J}^2-m^2}\right) + 2F\left(0,\mathcal{J}+m\right) - 4F\left(\{\frac{|k|}{2}\},\sqrt{\mathcal{J}(\mathcal{J}+m)}\right)}{\mathcal{J}+m} \\ & + \sqrt{m\mathcal{J}} + (\mathcal{J}+m)\ln\frac{\sqrt{\mathcal{J}+m}}{\sqrt{\mathcal{J}}+\sqrt{m}}\,. \end{aligned} \tag{D.10}$$

Since $\gamma = |k|/2 + O(1/k)$, this expression has a finite $k \to \infty$ limit, as was observed numerically. In order to determine the asymptotic values of the constant one needs the expression for γ with an $O(1/k)$ accuracy

$$\gamma = \frac{|k|}{2} + \frac{m(2\mathcal{J}+m)}{4|k|} + O\left(\frac{1}{k^3}\right)\,, \tag{D.11}$$

which implies

$$\begin{aligned} \delta E \;=\; & \frac{2F\left(0,\sqrt{\mathcal{J}^2-m^2}\right) + 2F\left(0,\mathcal{J}+m\right) - 4F\left(\{\frac{|k|}{2}\},\sqrt{\mathcal{J}(\mathcal{J}+m)}\right)}{\mathcal{J}+m} \\ & + \sqrt{m\mathcal{J}} + (\mathcal{J}+m)\ln\frac{\sqrt{\mathcal{J}+m}}{\sqrt{\mathcal{J}}+\sqrt{m}} - m\,. \end{aligned} \tag{D.12}$$

For large enough α, the function $F(\alpha,\beta)$ can be approximated as in (2.10), and thus the previous sum can be further simplified to

$$\begin{aligned}\delta E &= -\frac{1}{2}(\mathcal{J}+m)\ln(\mathcal{J}+m)-(\mathcal{J}-m)\ln(\mathcal{J}-m)+2\mathcal{J}\ln\mathcal{J}+\\ &+\sqrt{m\mathcal{J}}-(\mathcal{J}+m)\ln(\sqrt{\mathcal{J}}+\sqrt{m})\,-m\,+O\left(\frac{1}{\alpha}\right). \qquad \text{(D.13)}\end{aligned}$$

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