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Theoretical predictions for inclusive $B \to X_u \tau\bar\nu$ decay

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With the expected large increase in data sets, previously not measured decays will be studied at Belle II. We derive standard model predictions for the $B \to X_u \tau\bar\nu$ decay rate and distributions. The region in the lepton energy spectrum where higher-dimension operators in the local OPE need to be resummed into the b-quark light-cone distribution function is a significantly greater fraction of the phase space than for massless leptons. The finite τ mass has the novel effect of shifting and squeezing how the distribution function enters the lepton energy spectrum. We also derive new predictions for the τ polarization.

I. INTRODUCTION

The more than 3σ deviation of the measured $B \to D^{(*)}\tau\bar\nu$ rates [1–9] from the standard model (SM) predictions motivates the study of all possible semileptonic decays with τ leptons in the final state, both experimentally and theoretically. Comparisons of measured spectra and rates to different hadronic final states can give information on the structure of contributing four-fermion operators. Comparisons of $b \to c\ell\bar\nu$ and $b \to u\ell\bar\nu$ decays give constraints on the flavor structure of beyond standard model scenarios at play.

In this paper we study the inclusive decay $B \to X_u \tau\bar\nu$, which has been much less explored theoretically. Precise predictions for this decay are naturally interesting as a signal channel to measure in the future. In the near term, reliably modelling this decay as a background is interesting both to SM measurements and analyses aimed at more precisely measuring $R(D^{(*)})$ and clarifying the current tension with the SM. The Belle Collaboration set the first bound on a $b \to u\tau\bar\nu$ mediated decay, $\mathcal{B}(B \to \pi\tau\bar\nu) < 2.5 \times 10^{-4}$ [10], at a level several times higher than SM predictions, and recent theoretical studies [11–13] also focused on exclusive decays.

Inclusive semileptonic decays of hadrons containing a heavy quark allow for a systematic expansion of nonperturbative effects in powers of $\Lambda_{\rm QCD}/m_Q$ [14]. The inclusive decay rates computed in the $m_Q \gg \Lambda_{\rm QCD}$ limit coincide with the free-quark decay rate, while corrections of order $\Lambda_{\rm QCD}/m_Q$ vanish [14, 15]. The leading nonperturbative corrections are of order $\Lambda_{\rm QCD}^2/m_Q^2$ and depend on only two hadronic quantities, λ_1 and λ_2, which describe certain forward matrix elements of local dimension-five operators. These corrections have been computed for a number of processes [16–22]. For $B \to X_u \tau\bar\nu$ decay, expressions for the total rate and leptonic q^2 spectra are straightforward to derive by taking the $m_q \to 0$ limit of the $B \to X_c \tau\bar\nu$ results [22, 23], but this limit is singular for the lepton energy spectrum and has not been given in the literature. Similarly, the perturbative $\mathcal{O}(\alpha_s)$ corrections to the total $B \to X_u \tau\bar\nu$ semileptonic decay rate [24], the dilepton q^2 spectrum [25], and the doubly differential $\mathrm{d}\Gamma/\mathrm{d}q^2\mathrm{d}y$ spectrum [26, 27] are known analytically. However, no closed form expressions have thus far been derived for the $\mathcal{O}(\alpha_s)$ corrections to the τ lepton energy spectrum. We present the results of the local OPE to $\mathcal{O}(\Lambda_{\rm QCD}^2/m_b^2,\, \alpha_s)$ in Sec. II.

Phase space regions in inclusive $B \to X_u e\bar\nu$ decay, when kinematic cuts restrict the invariant mass of the hadronic final state to be small (i.e., $m_X < m_D$), are relevant for the determination of $|V_{ub}|$. Decay rates in such regions are subject to large corrections, both perturbative and nonperturbative. In the region near maximal lepton energy the OPE breaks down and a resummation of the series of leading nonperturbative corrections is required [28, 29]. The lepton energy spectrum in a region of width $\Delta E_\ell \sim \Lambda_{\rm QCD}$ near the endpoint is determined by a nonperturbative b-quark distribution function in the B meson. Similarly, the local OPE for $B \to X_u \tau\bar\nu$ breaks down near the endpoint of the τ energy spectrum; however, since in $B \to X_u \tau\bar\nu$ decay, $m_\tau < E_\tau < (m_B^2 + m_\tau^2)/(2m_B)$ amounts to $1.78\,\mathrm{GeV} < E_\tau < 2.94\,\mathrm{GeV}$, the distribution function is important over a much greater fraction of the available phase space than in $B \to X_u e\bar\nu$, where $0 < E_e < m_B/2$. We consider the effects of the b-quark distribution function in Sec. III and explore its effect on the spectrum. Since the distribution of the measurable τ decay products (e.g., the charged lepton energy) are sensitive to the τ polarization, we also present results for decays to each polarization state.

To appreciate the mass suppressions in the decay rates, simply using the $\mathcal{O}(\Lambda_{\rm QCD}^2/m_b^2)$ [20–22] and $\mathcal{O}(\alpha_s)$ contributions [22, 24] in the $1S$ scheme [30–32], one finds [31]

$$\frac{\Gamma(B \to X_u \ell\bar\nu)}{\Gamma(B \to X_u \tau\bar\nu)} = 2.97\,, \qquad \frac{\Gamma(B \to X_c \ell\bar\nu)}{\Gamma(B \to X_c \tau\bar\nu)} = 4.50\,, \qquad (1)$$

Thus, the suppression of the rate due to finite m_τ is less strong in $b \to u$ than in $b \to c$ decays. Correspondingly, the suppression due to finite m_c is clearly greater in $B \to \tau$ than in $B \to e$ semileptonic decays,

$$\frac{\Gamma(B \to X_u \tau\bar\nu)}{\Gamma(B \to X_c \tau\bar\nu)}\, \frac{|V_{cb}|^2}{|V_{ub}|^2} = 3.13\,,$$
$$\frac{\Gamma(B \to X_u \ell\bar\nu)}{\Gamma(B \to X_c \ell\bar\nu)}\, \frac{|V_{cb}|^2}{|V_{ub}|^2} = 1.83\,. \qquad (2)$$

II. LOCAL OPE RESULTS

A. Nonperturbative Corrections

The inclusive $B \to X_q\, \ell\bar\nu$ decay ($q = u, c\,;\ \ell = e, \mu, \tau$) has been considered to order $1/m_b^2$ in the heavy quark expansion [20–22], including effects of the finite lepton mass. For $m_q = 0$ the lepton energy spectrum becomes singular, and the limit must be taken with care. We find for $B \to X_u \tau\bar\nu$ decay,[1]

$$\begin{aligned}\frac{1}{\Gamma_u}\frac{\mathrm{d}\Gamma}{\mathrm{d}y} &= 2\sqrt{y^2-4\rho_\tau}\,\bigg[3y - 2y^2 - 4\rho_\tau + 3\rho_\tau y + \frac{\lambda_2}{m_b^2}\,6y \\ &\quad + \frac{\lambda_1+3\lambda_2}{3m_b^2}\,(5y^2-14\rho_\tau)\bigg]\,\theta(1+\rho_\tau-y) \\ &\quad - \bigg[\frac{\lambda_1}{3m_b^2}\,(1+\rho_\tau) + \frac{\lambda_2}{m_b^2}\,(11-5\rho_\tau)\bigg] \\ &\quad \times (1-\rho_\tau)^3\,\delta(1+\rho_\tau-y) \\ &\quad - \frac{\lambda_1}{3m_b^2}\,(1-\rho_\tau)^5\,\delta'(1+\rho_\tau-y)\,, \end{aligned} \qquad (3)$$

where we use the dimensionless variables

$$y = \frac{2E_\tau}{m_b}\,, \qquad \hat q^2 = \frac{q^2}{m_b^2}\,, \qquad \rho_\tau = \frac{m_\tau^2}{m_b^2}\,, \qquad (4)$$

and

$$\Gamma_u = \frac{|V_{ub}|^2\, G_F^2\, m_b^5}{192\,\pi^3}\,. \qquad (5)$$

This agrees with the more complicated expression given in Ref. [21]. Here λ_1 and λ_2 are matrix elements in the heavy quark effective theory (HQET), defined by

$$\begin{aligned}\frac{1}{2m_B}\langle B|\,\bar b_v\,(iD)^2\,b_v|B\rangle &= 2\,\lambda_1\,, \\ \frac{1}{2m_B}\langle B|\,\frac{g_s}{2}\,\bar b_v\,\sigma_{\mu\nu}G^{\mu\nu}\,b_v|B\rangle &= 6\,\lambda_2\,, \end{aligned} \qquad (6)$$

and b_v is the heavy b-quark field of HQET [34] with velocity v.

The τ can have spin up ($s = +$) or spin down ($s = -$) relative to the direction of its three-momentum, and it is convenient to decompose the corresponding decay rates as

$$\Gamma(B \to X_u\,\tau(s=\pm)\,\bar\nu) = \frac{1}{2}\,\Gamma \pm \tilde\Gamma\,. \qquad (7)$$

The rate, summed over the tau polarizations, is given by Γ, while the average tau polarization is $A_{\rm pol} = 2\tilde\Gamma/\Gamma$. The τ polarization gives complementary sensitivity to BSM physics [35]. We obtain for its lepton energy dependence,

$$\begin{aligned}\frac{1}{\Gamma_u}\frac{\mathrm{d}\tilde\Gamma}{\mathrm{d}y} &= -(y^2-4\rho_\tau)\bigg[3 - 2y + \rho_\tau \\ &\quad + \frac{6\lambda_2}{m_b^2} + \frac{\lambda_1+3\lambda_2}{3m_b^2}5y\bigg]\,\theta(1+\rho_\tau-y) \\ &\quad + \bigg[\frac{\lambda_1}{6m_b^2}\,(1-3\rho_\tau) + \frac{\lambda_2}{2m_b^2}\,(11-5\rho_\tau)\bigg] \\ &\quad \times (1-\rho_\tau)^3\,\delta(1+\rho_\tau-y) \\ &\quad + \frac{\lambda_1}{6m_b^2}\,(1-\rho_\tau)^5\,\delta'(1+\rho_\tau-y)\,. \end{aligned} \qquad (8)$$

Note that for $\rho_\tau = 0$, $-2\,\mathrm{d}\tilde\Gamma = \mathrm{d}\Gamma$, since the massless lepton is purely left-handed. Angular momentum conservation in $B \to X_u\tau\bar\nu$ implies that the τ polarization is fully left-handed at maximal E_τ. This holds at the parton level to all orders in α_s, and our results indeed satisfy it at order α_s^0 and order α_s^1; i.e., $\Gamma/2 = -\tilde\Gamma$ at $y = 1+\rho_\tau$. However, the power-suppressed terms that start at order $\Lambda_{\rm QCD}^2/m_b^2$ incorporate nonperturbative corrections between the E_τ endpoint at the parton level and at the hadron level. As a result, the physical rate at maximal E_τ vanishes (it is nonzero at the parton level). In a small region very close to the endpoint the most singular terms of the form $\lambda_1\,\delta'(1+\rho_\tau-y)$ are the most important, and these also obey the $\Gamma/2 = -\tilde\Gamma$ relation.

For $\mathrm{d}\Gamma/\mathrm{d}\hat q^2$, the $m_q \to 0$ limit the of $B \to X_c\tau\bar\nu$ expression is smooth, which gives the known result [23],

$$\begin{aligned}\frac{1}{\Gamma_u}\frac{\mathrm{d}\Gamma}{\mathrm{d}\hat q^2} &= \frac{(\hat q^2-\rho_\tau)^2}{\hat q^6}\bigg\{\bigg(1+\frac{\lambda_1}{2m_b^2}\bigg)2(1-\hat q^2)^2 \\ &\quad \times \Big[\hat q^2(1+2\hat q^2) + \rho_\tau(2+\hat q^2)\Big] \\ &\quad + \frac{3\lambda_2}{m_b^2}\Big[\hat q^2(1-15\hat q^4+10\hat q^6) + \rho_\tau(2-3\hat q^2+5\hat q^6)\Big]\bigg\}. \end{aligned} \qquad (9)$$

Integrating over phase space, the $B \to X_u\tau\bar\nu$ rate is

$$\begin{aligned}\frac{\Gamma}{\Gamma_u} &= \bigg(1+\frac{\lambda_1}{2m_b^2}\bigg)\big(1-8\rho_\tau+8\rho_\tau^3-\rho_\tau^4-12\rho_\tau^2\ln\rho_\tau\big) \\ &\quad - \frac{3\lambda_2}{2m_b^2}\big(3-8\rho_\tau+24\rho_\tau^2-24\rho_\tau^3+5\rho_\tau^4+12\rho_\tau^2\ln\rho_\tau\big)\,, \end{aligned} \qquad (10)$$

and the polarization is given by

$$\begin{aligned}\frac{\tilde\Gamma}{\Gamma_u} &= -\frac{(1-\hat m_\tau)^3}{2}\bigg[\frac{(1-\hat m_\tau)^2}{3}\,\big(3+15\hat m_\tau+5\hat m_\tau^2+\hat m_\tau^3\big) \\ &\quad + \frac{\lambda_1}{6m_b^2}\,(1+\hat m_\tau)^3\,(3+\hat m_\tau^2) \\ &\quad - \frac{\lambda_2}{2m_b^2}\big(9+27\hat m_\tau+70\hat m_\tau^2+10\hat m_\tau^3-15\hat m_\tau^4-5\hat m_\tau^5\big)\bigg], \end{aligned} \qquad (11)$$

where $\hat m_\tau = \sqrt{\rho_\tau}$.

[1] The results in this section apply, with obvious changes of hadron masses and matrix elements, to inclusive $B_c \to X_c\tau\bar\nu$ decay, just like exclusive B_c decays can be calculated using HQET methods [33]. Treating charm as a heavy quark, the B_c has a size parametrically smaller than $\Lambda_{\rm QCD}$, and the b quark distribution function in B_c is in principle calculable in NRQCD. This decay might be observable in the tera-Z phase of a future e^+e^- collider.

B. Perturbative Corrections

Analytic results for the doubly differential $\mathrm{d}\Gamma/\mathrm{d}q^2\mathrm{d}y$ spectra (including the τ polarization dependence) were given in Refs. [26, 27] [2], but only numerical results were presented for the τ energy spectrum. Integrating the doubly differential spectra over q^2 gives the charged lepton energy spectra for both unpolarized and polarized τ leptons. In the unpolarized case, writing

$$\frac{1}{\Gamma_0}\frac{\mathrm{d}\Gamma_\tau}{\mathrm{d}y} = \Big[F_0(y) - \frac{\alpha_s C_F}{2\pi}F_1(y)\Big]\,\theta(1+\rho_\tau - y)\,, \tag{12}$$

where $C_F = 4/3$, we find

$$F_0(y) = 2\sqrt{y^2-4\rho_\tau}\,\big[(3-2y)y + \rho_\tau(3y-4)\big]\,, \tag{13}$$

and

$$\begin{aligned} F_1(y) = {} & F_0(y)\,\big[\mathrm{Li}_2(\tau_+) + \mathrm{Li}_2(\tau_-) + 4Y_p^2\big] + \big(6y^2 - 4y^3 + 6\rho_\tau y^2 - 12\rho_\tau^2\big)\,\big[\mathrm{Li}_2(\tau_+) - \mathrm{Li}_2(\tau_-)\big] \\ & - 2Y_p\left(\frac{5\rho_\tau^3}{3} + \rho_\tau(7y^2-6y+7) - 6y^3 + 10y^2 + \rho_\tau^2(4y-23) + 6y - \frac{41}{3}\right) \\ & + \sqrt{y^2-4\rho_\tau}\left(-\frac{34y^2}{3} + \rho_\tau\left(15y - \frac{74}{3}\right) + 24y - 6\right)\ln(1-y+\rho_\tau) \\ & + \sqrt{y^2-4\rho_\tau}\left(21\rho_\tau^2 + \frac{1}{6}\left[\left(86-16\pi^2\right)y^2 + \left(24\pi^2-153\right)y + 82\right] + \frac{\rho_\tau}{6}\left(24\pi^2 y - 167y - 32\pi^2 + 64\right)\right), \end{aligned} \tag{14}$$

where $Y_p = \frac{1}{2}\ln[(1-\tau_+)/(1-\tau_-)]$ is the rapidity of all decay products (combined) against which the τ recoils, and

$$\tau_\pm = \frac{1}{2}\left(y \pm \sqrt{y^2-4\rho_\tau}\right). \tag{15}$$

Similarly, defining the polarization dependence of the lepton energy spectrum as

$$\frac{\mathrm{d}\Gamma_\tau^\pm}{\mathrm{d}y} = \frac{1}{2}\frac{\mathrm{d}\Gamma_\tau}{\mathrm{d}y} \pm \frac{\mathrm{d}\tilde\Gamma_\tau}{\mathrm{d}y}\,, \tag{16}$$

we write the polarization dependence of the rate to produce a τ lepton as

$$\frac{1}{\Gamma_0}\frac{\mathrm{d}\tilde\Gamma_\tau}{\mathrm{d}y} = \Big[\tilde F_0(y) - \frac{\alpha_s C_F}{2\pi}\tilde F_1(y)\Big]\,\theta(1+\rho_\tau - y)\,. \tag{17}$$

At tree level,

$$\tilde F_0(y) = \left(y^2 - 4\rho_\tau\right)\left(2y - 3 - \rho_\tau\right), \tag{18}$$

while at one loop,

$$\begin{aligned} \tilde F_1(y) = {} & \tilde F_0(y)\,\big[\mathrm{Li}_2(\tau_+) + \mathrm{Li}_2(\tau_-) + 4Y_p^2\big] + \frac{12\rho_\tau^2 - \rho_\tau\left(y^3+6y^2-6y\right) + 2y^4 - 3y^3}{\sqrt{y^2-4\rho_\tau}}\,\big[\mathrm{Li}_2(\tau_+) - \mathrm{Li}_2(\tau_-)\big] \\ & - \frac{Y_p}{3\sqrt{y^2-4\rho_\tau}}\Big[\rho_\tau\left(y^3 - 210y^2 + 405y - 260\right) + \left(18y^4 - 12y^3 - 36y^2 + 41y\right) + \rho_\tau^3(y-24) + \rho_\tau^2(93y-12)\Big] \\ & + \left[\frac{34\rho_\tau^2}{3} + \rho_\tau\left(-\frac{23y^2}{6} - \frac{53y}{3} + 30\right) + \frac{17y^3}{3} - 9y^2\right]\ln(1-y+\rho_\tau) + \frac{\rho_\tau^3}{3} + \rho_\tau^2\left(-\frac{11y}{6} + \frac{8\pi^2}{3} - 21\right) \\ & + \frac{\rho_\tau}{12}\left(-8\pi^2y^2 + 149y^2 - 64\pi^2 y - 48y + 96\pi^2 + 32\right) + \frac{1}{12}\left(16\pi^2y^3 - 86y^3 - 24\pi^2y^2 + 153y^2 - 82y\right). \end{aligned} \tag{19}$$

III. THE LEPTON ENERGY ENDPOINT REGION

Near the endpoint of the lepton energy spectrum $y \sim 1+\rho_\tau$, a class of higher-order terms in the local OPE in Eq. (3) is no longer suppressed, and instead the differential rate is given by a nonlocal OPE in terms of the light-cone momentum distribution function of the b quark [28, 29, 36–39].

This endpoint region has been extensively studied in the context of massless leptons. It is straightforward to extend this to nonzero τ mass. At the parton level the

[2] We corrected some typos in the $m_c \to 0$ limit in these references.

lepton energy endpoint is determined by the θ function

$$\theta\big((p_b - p_\tau)^2\big) = \theta\left(m_b^2 + m_\tau^2 - 2\, p_\tau \cdot p_b\right). \qquad (20)$$

Writing

$$p_\tau^\mu = \frac{m_b}{2}\left(\tau_-\, n^\mu + \tau_+\, \bar{n}^\mu\right), \qquad (21)$$

where $\tau_\pm$ are given in Eq. (15), defines the light-like vectors n^μ and $\bar{n}^\mu = 2v^\mu - n^\mu$. Taking $p_b^\mu = m_b v^\mu + k^\mu$, expanding in powers of k^μ/m_b and applying the HQET onshell condition $k \cdot v = 0$, the θ function becomes

$$\theta\left(1 + \rho_\tau - y + \frac{k \cdot n}{m_b}\sqrt{y^2 - 4\rho_\tau} + \mathcal{O}(k^2)\right). \qquad (22)$$

Over most of the spectrum, the $\mathcal{O}(k \cdot n)$ term may be neglected at leading order in $1/m_b$ and we recover the OPE result in Eq. (3). However, when E_τ is near the partonic endpoint, i.e., $1 + \rho_\tau - y = \mathcal{O}(\Lambda_{\rm QCD}/m_b)$, $p_b - p_\tau$ approaches a light-like vector in the n direction. In this region the $\mathcal{O}(k \cdot n)$ term is the same order as the leading term, and so must be included in the leading-order expression. Defining

$$\Delta \equiv \frac{1 + \rho_\tau - y}{1 - \rho_\tau}, \qquad (23)$$

taking $\Delta \sim \mathcal{O}(\Lambda_{\rm QCD}/m_b)$, and expanding (22) in powers of Δ then gives

$$\theta\left(\Delta + \frac{k \cdot n}{m_b}\right) + \mathcal{O}(\Lambda_{\rm QCD}/m_b). \qquad (24)$$

Comparing with the $\rho_\tau \to 0$ limit, the nonzero τ mass shifts the endpoint of the lepton spectrum and squeezes it by a factor of $1 - \rho_\tau$. This is also reflected by the fact that the lepton energy endpoint changes between the parton- and hadron-level kinematics, at leading order, by $(1 - \rho_\tau)\,\bar{\Lambda}/2$, where $m_B = m_b + \bar{\Lambda} + \mathcal{O}(\Lambda_{\rm QCD}^2/m_b)$.

At the hadron level, matrix elements of the θ function may be expressed as an integral over the light-cone momentum distribution function of the b quark in the B meson,

$$f(\omega, \mu) = \frac{1}{2m_B}\langle B|\bar{b}_v \delta(\omega + iD \cdot n) b_v|B\rangle. \qquad (25)$$

Following [40], it is convenient to define the nonperturbative function $F(k)$ via the convolution

$$f(\omega, \mu) = \int \mathrm{d}k\, C_0(\omega - k, \mu)\, F(k), \qquad (26)$$

where, at one loop [41],

$$C_0(\omega, \mu) = \delta(\omega) - \frac{\alpha_s C_F}{4\pi}\left(\frac{\pi^2}{6}\delta(\omega) + \frac{4}{\mu}\left[\frac{\mu}{\omega}\right]_+ + \frac{8}{\mu}\left[\frac{\ln\frac{\omega}{\mu}}{\omega/\mu}\right]_+\right). \qquad (27)$$

The convolution (26) factors out the perturbative corrections to the parton-level matrix element of $f(\omega)$. With this definition, $F(k)$ is a nonperturbative function with support from $k = -\bar{\Lambda}$ to $k = \infty$, whose moments are related to the matrix elements of local operators. The τ energy spectrum may then be written in the endpoint region as the convolution

$$\frac{1}{\Gamma_\tau}\frac{\mathrm{d}\Gamma_\tau}{\mathrm{d}y} = \int \mathrm{d}\omega\, G_\tau\left(\Delta - \frac{\omega}{m_b}\right) F(\omega) + \mathcal{O}(\Delta, \Lambda_{\rm QCD}/m_b), \qquad (28)$$

where $G_\tau(x)$ is obtained by expanding the parton level perturbative results (14) in the limit $\Delta \to 0$,

$$G_\tau(x) = \theta(x)\left\{1 - \frac{\alpha_s C_F}{2\pi}\left[\ln^2 x + \left(\frac{31}{6} - 2\ln(1 - \rho_\tau)\right)\ln x + C(\rho_\tau)\right]\right\}, \qquad (29)$$

and $C(\rho_\tau) = \pi^2 + 5/4 + \rho_\tau(\pi^2 - 6) + \mathcal{O}(\rho_\tau^2)$. Note that in Eq. (29) the m_τ dependent terms at $\mathcal{O}(\alpha_s)$ are small corrections: $C(\rho_\tau)/C(0)$ is within 5% of unity, and the $2\ln(1 - \rho_\tau)$ term is less than a 6% correction relative to the "31/6" term. $G_\tau(x)$ therefore has very weak ρ_τ dependence: none at tree level, and only about $5\% \times \alpha_s C_F/(2\pi)$ at one loop. The large difference between the shapes arises almost entirely from the kinematic rescaling in Eq. (23). SCET techniques may be used to sum logarithms of Δ in this expression (as in Refs. [41] and [40]), but this is beyond the scope of this paper or the accuracy we desire.

The expression (28) is only valid in the region $\Delta \sim \Lambda_{\rm QCD}/m_b$; in order to have an expression which smoothly interpolates with the local OPE away from the endpoint, it is convenient instead to incorporate distribution function effects by redefining the b-quark mass $m_b \to m_b' = m_b + k \cdot n$ [28, 36]. Writing $p_b^\mu = m_b' v^\mu + k'^\mu$, where $k'^\mu = k^\mu - k \cdot n\, v^\mu$, the residual momentum k'^μ satisfies $k' \cdot n = 0$, and so the effects of nonzero $k \cdot n$ are automatically incorporated into the leading-order spectrum with this mass definition. The τ energy spectrum in the endpoint region may then be written as the convolution

$$\frac{\mathrm{d}\Gamma_\tau}{\mathrm{d}E_\tau} = 2\int \mathrm{d}\omega\, \frac{1}{m_b}\frac{\mathrm{d}\Gamma_\tau}{\mathrm{d}y}(y', \rho_\tau')\, F(\omega), \qquad (30)$$

where we have defined the scaled variables

$$y' \equiv \frac{2E_\tau}{m_b - \omega}, \qquad \rho_\tau' \equiv \frac{m_\tau^2}{(m_b - \omega)^2}, \qquad (31)$$

and $\mathrm{d}\Gamma/\mathrm{d}y$ is the parton level spectrum in Eq. (12). An analogous formula holds for the polarized spectrum Eq. (17). For simplicity, we have written the prefactor in Eq. (30) as $1/m_b$, not $1/(m_b - \omega)$, since the difference is higher order everywhere in the spectrum. In this form, Eq. (30) includes subleading terms suppressed by powers of Δ in the endpoint region, but which are leading order when Δ is not small, so are required to reproduce the local OPE away from the endpoint.

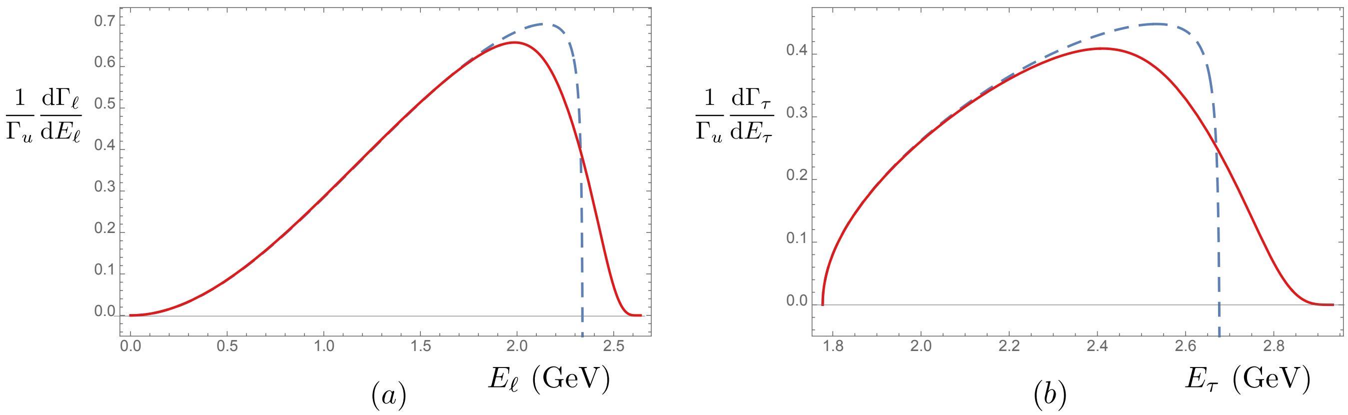

FIG. 1. The $B \to X_u \ell \bar{\nu}$ lepton energy spectrum for (a) $\ell = e$, μ and (b) $\ell = \tau$ in the parton model (blue, dashed), and incorporating the leading order b-quark distribution function (red, solid).

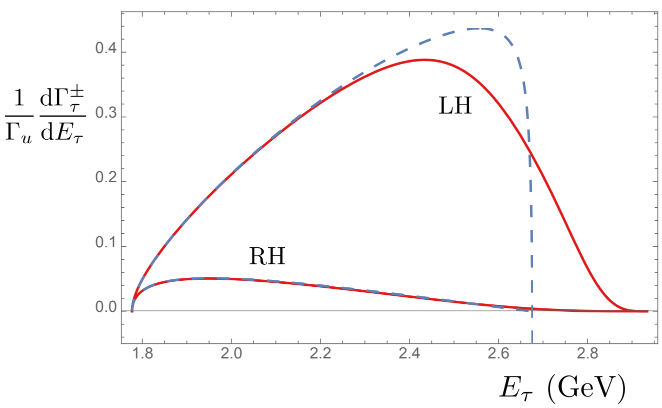

FIG. 2. The τ energy dependence of its polarization in $B \to X_u \tau \bar{\nu}$ in the parton model (blue, dashed), and incorporating the leading order b-quark distribution function (red, solid).

$F(k)$ has been extracted from the measured $B \to X_s \gamma$ spectra by the SIMBA collaboration [42]. At leading order in $\Lambda_{\rm QCD}/m_b$, it can be used to make predictions for $B \to X_u \ell \bar{\nu}$ decays. Figure 1 shows the $B \to X_u \ell \bar{\nu}$ lepton spectra for $\ell = e$ and $\ell = \tau$ in the parton model and including the effects of the b-quark distribution function. It is clear from this plot that the distribution function is indeed important in a greater fraction of the τ energy spectrum than in the massless lepton channels; the fraction of the lepton energy spectrum where the distribution function is important is enhanced by $(1-\rho_\tau)/(1-\sqrt{\rho_\tau})^2 \sim 2.2$. Figure 2 shows the E_τ spectra separately for left- and right-handed τ leptons in $B \to X_u \tau \bar{\nu}$. The average τ polarization, including order α_s and $\Lambda_{\rm QCD}^2/m_b^2$ corrections, is $2\tilde{\Gamma}/\Gamma = -0.77$.

IV. CONCLUSIONS

We presented theoretical predictions for inclusive $B \to X_u \tau \bar{\nu}$ decay. We derived previously unknown results at order $\Lambda_{\rm QCD}^2/m_b^2$ and analytic expressions for the order α_s corrections for the τ energy spectrum and polarization. We also incorporated the effects of the b-quark light-cone distribution function to the case of nonzero lepton mass. Due to the suppressed kinematic range, the b-quark distribution function is more important in determining the lepton energy spectrum in $B \to X_u \tau \bar{\nu}$ than in $B \to X_u e \bar{\nu}$ decay.

It will probably take many ab^{-1} of data at Belle II to have sensitivity to $B \to X_u \tau \bar{\nu}$. While it is clearly a challenging decay to measure, the rate according to Eqs. (1) and (2) is only about 3 times smaller than $B \to X_u e \bar{\nu}$, and about $|V_{cb}|^2/(3\,|V_{ub}|^2)$ times smaller than $B \to X_c \tau \bar{\nu}$. One may, for example, try to utilize the fact that electrons or muons from the τ decay with maximal allowed energies correspond to the most energetic τ leptons. We hope that Belle II will be able to make measurements of this decay.

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