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THE MESON SPECTRUM OF LATTICE QCD IN FIRST ORDER STRONG COUPLING APPROXIMATION

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The meson spectrum of lattice QCD in first order strong coupling approximation

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Abstract:

We calculate the meson spectrum in the strong coupling approximation up to the first order in β. We get 22 new particles which appear for the first time only at this order. These are interpreted as excited states. By defining "irreducible meson fields", the 'Susskind-flavours' are discussed and are physically interpreted as quark-flavours.

Wir berechnen das Mesonenspektrum der QCD in der starken Kopplungsapproximation bis zur ersten Ordnung in β. Wir erhalten 22 neue Teilchen, welche erst in dieser Ordnung erscheinen. Diese werden als angeregte Zustände identifiziert. Weiterhin definieren wir irreduzible Mesonenfelder und diskutieren mit ihrer Hilfe die 'Susskind-flavours' die wir physikalisch als Quark-flavours interpretieren.

For those, who carry love in their hearts.

Content

1 Introduction

The calculation of the hadron spectrum in the framework of the lattice approximation of QCD [1],[2] is an important problem of the actual research in theoretical particle physics. However, there are many difficulties which prevent an immediate significant calculation of meson and hadron masses. One of these problems is the spectrum doubling of lattice fermions [1]. The naive lattice approximation of the (Euclidean) Dirac equation describes, in the continuum, 16 species of Dirac particles [3]. Thus it is not immediately clear, what is a consistent lattice description of a single Dirac particle. K.Wilson [4] tried to solve this question by adding additional terms to the action in such a way that only one Dirac particle remains in the continuum limit. However, this procedure destroys important symmetries. In an other approach one describes the 16 Dirac particles of the naive lattice Dirac field by four independent 'staggered' fermion fields [5]. In this procedure one considers one component per lattice site instead of the four Dirac components. Restriction to one staggered fermion field formally reduces the number of Dirac particles by a factor of four. In the calculation of meson masses which we perform in this work, the quarks are described by staggered fermion fields.

From a different point of view, this lattice Fermion problem has been investigated by P. Becher and H. Joos [6]. Their approach is based on the Dirac-Kähler equation [7], a generalization of the Dirac equation. This formulation uses geometrical concepts extensively. The Dirac particles are described by differential forms. The DeRham isomorphism between differential forms and 'lattice fields' (cochains) is used to formulate the Dirac-Kähler equation for free particles on the lattice. It can be shown that free lattice Dirac-Kähler fields are equivalent to staggered fermion fields. However, in the Dirac-Kähler formulation the appearance of four flavours becomes immediately clear. These four flavours appear on the lattice, because they are already described by the differential forms in the continuum. The formulation of the gauge interaction of Dirac-Kähler forms is not so straight forward as the free case. There is a difference between Susskind formulation of interacting staggered Fermions and the interaction of DK-fields as first proposed by Becher and Joos ('coarse interaction') [8].There are physical differences between these schemes [9]. The Susskind interaction seems to have advantages because it is more symmetric. It is also a natural possibility in the Dirac-Kähler framework ('fine interaction').

All these different schemes can be used to describe quarks in the lattice approximation of the QCD. In this work we investigate the QCD with Dirac-Kähler fermions. Its continuum action is

$$S = \int \left\{ \frac{1}{2g^2} \mathrm{Tr}(F,F)_0 + (\overline{\Phi}, (d_A - \delta_A + m)\Phi)_0 \right\} \tag{1}$$

with F the field form of the gluon , ϕ an inhomogeneous differential form describing Dirac-Kähler fermions, d_A and δ_A the covariant exterior differential and codifferential operator. (More explicite definitions are given in Sect.(1.1). Because of the close connection of this formulation to geometry, we call the theory based on this Lagrangian 'geometric QCD'. The advantage of this theory is its straight forward lattice approximation. We use the Susskind coupling, because it preserves a maximum of the geometric continuum symmetry. Formally, Euclidean geometric QCD describes Dirac fields related by flavour transformations of an SU(4) symmetry group. Therefore its physical interpretation is not immediately clear. In the lattice approximation the flavour symmetry gets restricted to a discrete symmetry. It is not clear, if the complete symmetry gets restored in the continuum limit of the interacting theory .(There is symmetry in the free case). Only dynamical calculations can reveal the possible physical content of geometric QCD. Here we shall calculate the meson spectrum to the first order of the strong coupling constant expansion, using a resummed hopping parameter expansion. The resumming technique we use was suggested by O. Martin [10], and explicitly given to the zeroth order in β for naive fermions, together with some hints for a first order calculation. We explicitly developed the first order scheme and combined it with a consistent use of the symmetry of the problem. This procedure, together with the physical intuition, allows us to interprete the states appearing in the zeroth order calculation as ground states. The first order gives us, in addition to the non-excited states, excited states also, which we don't have in zeroth order. Our calculation has to be compared with other strong coupling calculations [11] ,[12] and numerical investigations [13] of the non-excited states. The calculation of the masses of the excited states are new.

The large number of mesons which have been discovered - more than 20 explicitly calculated particles and at least 32 expected to be exist at the second order approximation- are mainly distinguished by different quantum numbers like spin, parity, charge parity, and flavour quantum numbers. The relation of these continuum quantum numbers to lattice quantum numbers is essential for the physical interpretation of the lattice calculation. The lattice quantum numbers are defined by the irreducible representations of the lattice symmetry group of the staggered fermions. These were investigated by M.F.L. Goltermann [14], and in a more systematic and complete way by H. Joos and M. Schaefer [15]. We use these results in our calculation.

In Section 2 we explain the Dirac-Kähler equation, its symmetry on the continuum and on the lattice. The resummed hopping parameter will be explained in section 3, where in the fourth section we introduce the multi-link operator, and calculate the zero order states. In section 5 we produce the first order calculation , and in the last section we find the physical discussion.

2 Lattice Approximation of Geometric QCD

2.1 From the continuum to the lattice

As we mentioned in the introduction, there is a straightforward way to get from a field theory in differential forms to it's lattice approximation. For this, we give now the more explicit definitions of quantities appearing in the expression for the action of the geometric QCD, (Eq. (1)), together with the basic formulas of the differential calculus used . Afterwards we transcribe these formulas to the lattice and formulate, in this spirit, the action of the geometric lattice QCD.

2.1.1 The geometric calculus in the continuum.

On the (Euclidean) space-time manifold we consider inhomogenous differential forms

$$\begin{aligned}\Phi = \; & \varphi^0(x) + \varphi_\mu dx^\mu + \tfrac{1}{2!}\varphi_{\mu\nu}(x) dx^\mu \wedge dx^\nu + \tfrac{1}{3!}\varphi_{\mu\nu\rho}(x) dx^\mu \wedge dx^\nu \wedge dx^\rho \\ & + \varphi_{1234} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \equiv \textstyle\sum_H \Phi(x,H) dx^H\end{aligned} \tag{2}$$

Usually we sum over the same lower and upper indices. Often we use a multi-index notation with $H = (\mu_1, \mu_2, ...\mu_h), \mu_1 < \mu_2 < ... < \mu_h$, and $\Phi(x,H) \equiv \varphi_{\mu_1\mu_2...\mu_h}$ like $\varphi(x)^0 = \Phi(x,\emptyset), \varphi_{12} = \Phi(x,12)$ etc. On the linear space of differential forms we define some operations by their action on the basis elements dx^H.

(a) The main morphism $\mathcal{A}$ and antimorphism $\mathcal{B}$ are :

$$\mathcal{A} dx^H = (-1)^h dx^H \tag{3}$$

with the property

$$\mathcal{A}(\Phi \wedge \Psi) = \mathcal{A}\Phi \wedge \mathcal{A}\Psi$$

and

$$\mathcal{B} dx^H = (-1)^{\frac{h(h-1)}{2}} dx^H \tag{4}$$

with the property

$$\mathcal{B}(\Phi \wedge \Psi) = \mathcal{B}\Psi \wedge \mathcal{B}\Phi$$

(b) Wedge left multiplication $d^\mu \Phi$:

$$d^\mu . dx^H \equiv dx^\mu \wedge dx^H = \hat{\rho}_{\mu,H} dx^{H \cup \mu}$$

This operation is anti-commuting

$$d^\mu . d^\nu + d^\nu . d^\mu = 0 \tag{5}$$

A similar operation could be defined by wedge right multiplication.

(c) The anti-derivation $e_\mu \neg \Phi \equiv \delta_\mu \Phi$

$$\delta_\mu dx^H = \check{\rho}_{\mu,H} dx^{H/\mu}, \qquad \check{\rho}_{\mu,H} = \pm 1, 0$$

is defined by

$$\delta_\mu(\Phi \wedge \Psi) = \delta_\mu \Phi \wedge \Psi + \mathcal{A}\Phi \wedge \delta_\mu \Psi, \qquad \delta_\mu dx^\nu = g^{\mu\nu}, \qquad \delta_\mu 1 = 0, \qquad g^{\mu\nu} = \left\{ \begin{array}{ll} 0 & \mu \neq 0 \\ 1 & \mu = \nu \end{array} \right\}$$

This operation is also anti-commuting

$$\delta_\mu \delta_\nu + \delta_\nu \delta_\mu = 0 \tag{6}$$

There is a "canonical anticommutation relation" between $d_\mu$andδ_μ

$$\delta_\nu d_\mu + d_\mu \delta_\nu = g^{\mu\nu} \tag{7}$$

(d) Clifford left multiplication $dx^\mu \vee \Phi$

$$dx^\mu \vee dx^H \equiv (\delta_\mu + d_\mu) dx^H \tag{8}$$

In particular we get

$$dx^\mu \vee dx^\nu \equiv (\delta_\mu + d_\mu) dx^\nu = g^{\mu\nu} + dx^\mu \wedge dx^\nu$$

It was emphasized by E. Kähler [7] that Eq. (8) defines a destributive and assiociative product in the space of differential forms. For the basis elements dx^K it takes the form :

$$dx^K \vee dx^L = \check{\rho}_{KL} dx^{K \triangle L}, \qquad \check{\rho}_{KL} = \pm 1 \tag{9}$$

(e) Differential operators: With help of the partial differential operator ∂_μ acting on the "coefficients" $\varphi(x, H)$ of the differential forms and with the operations (b)-(d) one defines the following differential operators acting on the forms Φ

Exterior differentiation: $d \equiv d^\mu \partial_\mu$

Codifferentiation : $\delta \equiv \delta^\mu \partial_\mu$

Dirac-Kähler operator :

$$d - \delta = dx^\mu \vee \partial_\mu = (d^\mu + \delta^\mu)\partial_\mu \tag{10}$$

It follows from (5), (6) and (7) that

$$(d - \delta)^2 = -(d\delta + \delta d) = \partial_\mu \partial^\mu = \Box \tag{11}$$

This means that $d - \delta$ is a "square root" of the Laplacian $\Box$, similar to the Dirac operator in the Dirac equation.

It follows immediately from Eq. (5), (6) and (7) that the operator $d_\mu + \delta_\mu$ satisfies

$$\{d_\mu + \delta_\mu, d_\nu + \delta_\nu\} = 2g_{\mu\nu} \tag{12}$$

which is identical to the defining relations of the algebra of the Dirac-matrices: $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$. This means that $d_\mu + \delta_\mu$ defines a representation of the Dirac matrix algebra with the differential forms as representation space. In order to find the irreducible invariant subspaces of this representation, we construct the following matrix differential form :

$$Z = 2^{-d/2} \sum_H (\gamma^H)^T \mathcal{B} d^H. \tag{13}$$

$\gamma^H = \gamma^{\mu_1} \ldots \gamma^{\mu_h}$. Acting by $dx^\mu \vee$ on Z we get:

$$dx^\mu \vee Z = \gamma^{\mu T} Z, \qquad Z \vee dx^\mu = Z \gamma^{\mu T} \tag{14}$$

This equation shows that the columns of Z span an irreducible representation space of the algebra defined by Eq. (12). In this basis $\{z_a^b\}$. the differential forms decompose into invariant subspaces. The component $\psi_a^b(x)$ with respect to this basis is defined by

$$\Phi = \sum_H \phi(x, H) dx^H = \sum_{a,b} \psi_a^b(x) Z_a^b, \tag{15}$$

where b denotes the columns, a the rows. From (14) and (15) we see that $\phi_a^b(x)$, with fixed b, are the components of a vector in the irreducible subspace. With help of the completeness, and orthogonality relations of the γ-matrices, we get, from Eq. (15)

$$\phi(x, H) = 2^{-d/2} trace(\gamma^{H\dagger} \psi(x, H)), \qquad \psi(x) = \sum_H \phi(x, H) \gamma^H, \tag{16}$$

With help of the equations (8), (14), and (16),we can establish the relation between the Dirac-Kähler equation and the Dirac equation. If the Dirac-Kähler equation

$$(d - \delta + m)\Phi = (dx^\mu \vee \partial_\mu + m)\Phi = 0 \tag{17}$$

is imposed on diffenential forms, then we get the Dirac equation for a vector $\{\psi^b\}$ of the invariant subspace:

$$(\gamma^\mu \partial_\mu + m)\psi^b(x) = 0. \tag{18}$$

The "vector" $\psi^b(x)$ is a Dirac spinor.

We add to the geometric calculus a last definition :

(f) A scalar product for two differential forms is defined as

$$(\Phi, \Psi)(x) = \sum_H \phi^*(x, H)\psi(x, H) = 2^{-d/2} \sum_{a,b} \phi_a^{*b}(x)\psi_a^b(x), \tag{19}$$

This is sometimes considered as a volume form by the definition

$$(\Phi, \Psi)_0 = (\Phi, \Psi) dx^{1234} : \tag{20}$$

Now we have collected all the definitions and relations necessary for the understanding of the action of geometric QCD mentioned in the introduction. First we give the free action

$$S_q = \frac{1}{4} \int (\overline{\Phi}, (d - \delta + m)\Phi)_0 = \sum_b \int dx^{1234} \overline{\psi}^b(x)(\gamma^\mu \partial_\mu + m)\psi^b(x), \tag{21}$$

where the second form follows from the first by transformations of the type Eq. (16) and :

$$\overline{\psi}(x)_a^{(b)} = \sum_H \overline{\phi}(x, H)\gamma_{ab}^{H\dagger}, \tag{22}$$

The DK equation in the presence of a gluon gauge field follows from the free DKE (17) by the substitution of the covariant derivative $D_\mu = \partial_\mu - iA_\mu$ for ∂_μ.Here : $A_\mu = g\frac{\lambda_a}{2} A_\mu^a(x)$,a the gluon index , λ_a the Gell-Mann SU(3)-matrices $\Phi = \{\Phi_\alpha = \sum_H \Phi_\alpha(x, H) dx^H\}$, α color index. Then the DKE with interaction becomes

$$(d_A - \delta_A + m)\Phi = dx^\mu \vee D_\mu \Phi + m\Phi = 0 \tag{23}$$

The exterior differentiation $d_A = d^\mu D_\mu$ and codifferentiation $\delta = -\delta^\mu D_\mu$ are covariant extensions of Eq. (11). From the gauge field 1-form $A = A_\mu dx^\mu$ one defines the 2-form field $F = dA + [A, A]$ .By adding the action of the gluon field, it is then straightforward to get the action of the geometric QCD:

$$S = \int \left\{ \frac{1}{2g^2} \mathrm{Tr}(F, F)_0 + (\overline{\Phi}_j (d_A - \delta_A + m)\Phi)_0 \right\} \tag{24}$$

From the above discussion of the relation between the forms ϕ and the Dirac fields $\psi^b(x)$, which leads to the two expressions for the free actions Eq. (21), one sees that geometric QCD describes formally the gluonic interactions of 4 species of quarks. The label b, which distinguishes these 4 fermion fields, is sometimes called 'f', the flavour index.

2.1.2 The Lattice Formulation

In the following we discuss the lattice approximation of geometric QCD. First we describe the calculus with lattice quantities in close analogy to the geometric calculus in the continuum. Differential forms are approximated on the lattice by "cochains" Φ. Because of their physical meaning, we call these (general) 'lattice fields':

$$\Phi = \sum_{x,H} \Phi(x, H) d^{x,H}, \qquad d^{x,H}([x', H']) = \delta^x_{x'} \delta^H_{H'}. \tag{25}$$

Here we use the notation of lattice quantities explained in Fig.1 . $[x, H]$ denotes a h-dimensional lattice cell located at x ; f.i. $[x, \emptyset]$ is the lattice point x, $[x, 1]$ the link from x to $x + e_1$ (e_μ unit lattice vector),$[x, 12]$ denotes the plaquette with the corners $x, x + e_1, x + e_1 + e_2, x + e_2$,etc. The simple "chains" C are defined as

$$C = \sum_{x',H'} \alpha(x', H')[x', H'] \qquad \alpha(x', H') = \pm 1 \qquad \textit{depending on orientation.} \tag{26}$$

The basis element $d^{x,H}$ of the cochains is defined as linear functional on the chains by

$$d^{x,H}([x', H']) = \delta^x_{x'} \delta^H_{H'} a^h \tag{27}$$

where a is the lattice constant, which we put equal to one most of the time. The meaning of the lattice approximation of a continuous differential form, by a cochain of type Eq. (25), becomes clear by the following remark. Suppose $\alpha(x, H)$ in Eq. (26) is a lattice approximation of the characteristic function of a h-dimensional region g in spacetime, then we have by the usual definition of the integral

$$\lim_{a \to 0} \Phi(C) = \int_G \Phi \tag{28}$$

with Φ according to (2), and $\Phi(x, H) \equiv \phi(x, H)$ restricted to lattice points x in (25). (For the precise details we have to refer to the literature [20]). Based on the mapping ("De Rham mapping") of differential forms on cochains: (a=1)

$$\phi(x, H) = \Phi([x, H]) = \int_{[x,H]} \Phi \tag{29}$$

we may associate with the operations on differential forms, the following operations on lattice fields,([6],[18]):
(a) The main morphism $\mathcal{A}$ and antimorphism $\mathcal{B}$.

$$\mathcal{A} d^{x,H} = (-1)^h d^{x,H}, \qquad \mathcal{B} dx^H = (-1)^{\frac{h(h-1)}{2}} dx^H \tag{30}$$

(b) The wedge multiplication

$$d^{x,\mu} \wedge d^{y,H} = \hat{\rho}_{\mu,H} \delta^{x+e_\mu, y} d^{x,\mu \cup H},$$

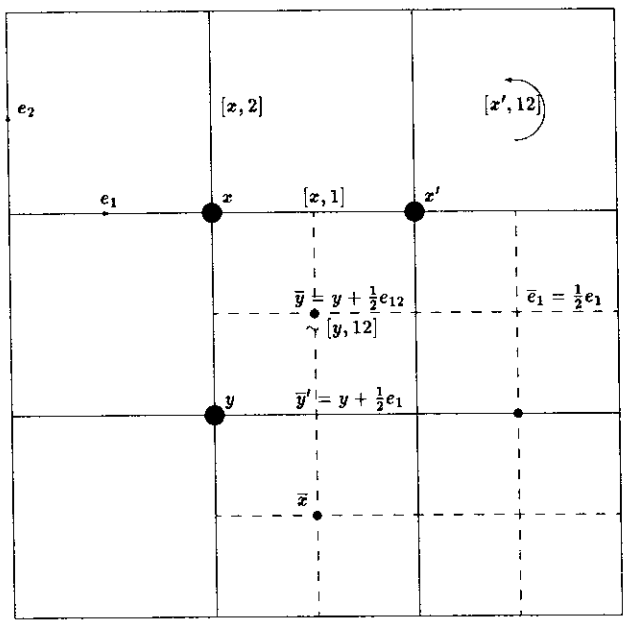


Figure 1: Illustration of the Lattice Notions

or in general

$$d^{x,K} \wedge d^{y,H} = \hat{\rho}_{K,H}\delta^{x+e_K,y}d^{x,K\cup H}, \qquad with \quad e_K = \sum_{\mu\in K} e_\mu \tag{31}$$

The wedge product is the "cup product" in a cubical cell complex.
(c) The operation $\epsilon\neg$ on the lattice

$$\epsilon^{\mu,x}\neg d^{y,H} = \hat{\rho}_{\mu,H}d^{y,H-\mu}\delta^{x+e_\mu,y}$$

or in general

$$\epsilon^{K,x}\neg d^{y,H} = \hat{\rho}_{K,H}d^{y,H/K}\delta^{x+e_k,y} \tag{32}$$

(d) Clifford left multiplication:

$$d^{x,\mu} \vee d^{y,H} \equiv (d^{x,\mu} \wedge +\epsilon^{\mu,x}\neg)d^{y,H} = \hat{\rho}_{\mu,H}\delta^{x+e_\mu,y}d^{x+e_\mu\cap H,\mu\triangle H}$$

or in general, as a non-associative product

$$d^{x,H} \vee d^{y,K} = \check{\rho}_{H,K}\delta^{x+e_H,y}d^{x+e_\Lambda,H\triangle K}, \qquad \Lambda = H\cap K, \qquad H\triangle K = H\cup K - \Lambda. \tag{33}$$

(e) Boundary, Coboundary and Translation operators. As the lattice analogue of the exterior differentiation d follows from the de Rham mapping and "Stokes" theorem

$$\Phi\prime(\mathcal{C}) \equiv \int_{\mathcal{C}} d\Phi = \int_{\triangle\mathcal{C}} \Phi \equiv \Phi(\triangle\mathcal{C}) = \check{\triangle}\Phi(\mathcal{C}) \tag{34}$$

$\triangle$ is our notation of the boundary operator acting on chains, $\check{\triangle}$ the "dual" boundary acting on lattice fields (cochains). In a similar way we get as a lattice analogue of codifferentiation δ the dual coboundary operator $\check{\nabla}\Phi(\mathcal{C}) = \Phi(\nabla\mathcal{C})$, ∇ coboundary operator acting on chains. With help of the translation operator T_{e_μ} and the difference operators $\partial_\mu^\pm$:

$$T_{e_\mu}d^{x,H} = d^{x-e_\mu,H}, \qquad \partial_\mu^+ = T_{e_\mu} - 1, \qquad \partial_\mu^- = 1 - T_{-e_\mu} \tag{35}$$

one gets formulas for the dual boundary, dual coboundary, and the lattice Dirac Kähler operator:

$$\begin{aligned} \check{\triangle} = \sum_\mu \check{\triangle}^\mu\partial_\mu^+ \equiv \sum_\mu(\sum_x d^{x,\mu}) \wedge \partial_\mu^+ \qquad \check{\nabla} &= \sum_\mu \check{\nabla}^\mu\partial_\mu^- \equiv \sum_\mu(\sum_x e^{x,\mu}\neg)\partial_\mu^- \\ \check{\triangle} - \check{\nabla} = \sum_x d^{x,\mu} \vee \partial_\mu^- &\equiv d^\mu \vee \partial_\mu^- \end{aligned} \tag{36}$$

which are very similar to Eq. (10). The operators $\{\check{\triangle}^\mu - \check{\nabla}^\mu\}$, which act as Clifford left multiplication by the constant differential form $d^\mu = \sum_x d^{x,\mu}$, satisfy the anticommutation relations

$$\{\check{\triangle}^\mu - \check{\nabla}^\mu, \check{\triangle}^\nu - \check{\nabla}^\nu\} = 2g^{\mu,\nu}.T_\mu \tag{37}$$

This shows that the lattice algebra of the Dirac γ-matrices is connected with translations. The boundary and coboundary operators satisfy $(\check{\triangle}^\mu)^2 = (\check{\nabla}^\mu)^2 = 0$. It follows from Eqs. (37,36,35), that the lattice Laplace operator is :

$$(\check{\triangle} - \check{\nabla})^2 == -(\check{\triangle}\check{\nabla} + \check{\nabla}\check{\triangle}) == \sum_{\mu,\nu} -(\check{\triangle}^\mu\check{\nabla}^\nu + \check{\nabla}^\nu\check{\triangle}^\mu) = \partial_\mu^-\partial^{\mu+} \tag{38}$$

The Dirac Kähler equation on the lattice is now given as

$$(\check{\triangle} - \check{\nabla} + m)\Phi = 0. \tag{39}$$

where $\Phi, \check{\triangle}, \check{\nabla}$ are defined above. If we multiply Eq. (39) by $(\check{\triangle} - \check{\nabla} - m)$ from the left side and use Eq. (38), we get that Φ is a solution of the correct lattice Klein-Gordon equation, known from

lattice approximation of scalar particles. No additional zero modes are introduced by the lattice approximation of the Dirac-Kähler operator - in contrast to the "naive" lattice approximation of Dirac operator. There is the species doubling inherent of the Dirac-Kähler forms, which describe in the continuum four Dirac fields. Thus we have formally 16 components $\phi(x,H)$ at each lattice site. L. Susskind [5] introduced a useful formalism, where fermions are described by only one component per lattice site. For this we subdivide the lattice into a fine lattice by adding all the central points of the lattice elements to the new lattice (see Fig.1). The lattice points of the fine lattice $\overline{x}$ are therefore related to those of the old lattice by

$$\overline{x} = x + \frac{1}{2}e_H, \qquad e_H = \sum_{\mu\in H} e_\mu \tag{40}$$

This equation also associates with each point of the fine lattice a multi-index $\overline{x} \to H(\overline{x})$. The definition of the so called "staggered" fermion fields ("Susskind fermions") consists now in the identification

$$\phi(x,H) = \phi(\overline{x},H(\overline{x})) = \chi(\overline{x})$$

The relation of the staggered fermion field to the "naive" Dirac field is given by (7,21)

$$\psi_a(\overline{x}) = \gamma_{ai}^H(\overline{x})\chi(\overline{x}) \tag{41}$$

If $\phi(x,H)$ belongs to the solution of the DKE (39), then $\psi_a(x)$, for fixed i, satisfies the naive Dirac equation:

$$(\gamma^\mu\overline{\partial}_\mu + m)\psi^b(x) = 0 \qquad \overline{\partial}_\mu = \partial_\mu^+ + \partial_\mu^- \tag{42}$$

Now we can formulate the lattice action of geometric QCD. With the inner product on the lattice $(\phi,\psi)(x) = \sum_H \phi^*(x,H)\psi(x,H)$, we get the lattice analogy of Eq. (21)

$$S_q = \sum_{x,H}\{\overline{\Phi}(x,H)((\check{\triangle} - \check{\nabla} + m)\Phi)(x,H)\} \tag{43}$$

With help of definition (40) this action could be written as the following (with lattice constant a of the fine lattice)

$$S_q = \sum_{x,H}\overline{\Phi}(x,H)\Big[\sum_{\mu\in H}\frac{\hat{\rho}_{\mu,H}}{2}\partial_\mu^+\Phi(x,H\setminus\{\mu\}) + \sum_{\mu\notin H}\frac{\hat{\rho}_{\mu,H}}{2}\partial_\mu^-\Phi(x,H\cup\{\mu\}) + m\Phi(x,H)\Big]$$

Using Eq (40) we get

$$S_q = \sum_{\overline{x}}\left\{\sum_\mu\frac{\hat{\rho}_{\mu H}}{2}[\overline{\chi}(\overline{x})\chi(\overline{x}+\overline{e}_\mu) - \overline{\chi}(\overline{x}+\overline{e}_\mu)\chi(\overline{x})] + m\overline{\chi}(\overline{x})\chi(\overline{x})\right\} \tag{44}$$

finally we introduced lattice gauge fields $U(\overline{x},\mu)$ on the fine lattice. With these we get the lattice action of geometric QCD

$$\begin{aligned} S &= S_W(u) + S_{DK} \\ S_W &= -\beta\sum_{\mathbf{P}\in\Gamma}(U(\mathbf{P}) + U^{-1}(\mathbf{P}) - 2) \\ U(\mathbf{P}) &= U(\overline{x},\mu\nu) = U^{-1}(\overline{x},\nu)U^{-1}(\overline{x}+\overline{e}_\nu,\mu)U(\overline{x}+\overline{e}_\mu,\nu)U(\overline{x},\mu) \sim \exp(i\mathbf{F}_{\mu\nu}(\overline{x})) \\ S_q &= \sum_{\overline{x}}\left\{\sum_\mu\frac{\hat{\rho}_{\mu H}}{2}[\overline{\chi}(\overline{x})U(\overline{x}),\mu)\chi(\overline{x}+\overline{e}_\mu) - \overline{\chi}(\overline{x}+\overline{e}_\mu)U^{-1}(\overline{x},\mu)\chi(\overline{x})] + m\overline{\chi}(\overline{x})\chi(\overline{x})\right\} \end{aligned} \tag{45, 46}$$

as the sum of the lattice approximation of the action of the gauge field $S_W(U)$ (due to K. Wilson), and the gauge invariant extension of the action of the staggered fermion fields, i.e. the lattice approximation of the Dirac Kähler fields, Eq. (44). This action is invariant under the gauge transformations on the fine lattice

$$\begin{aligned} \chi(\overline{x}) &\to g(\overline{x})\chi(\overline{x}), & \overline{\chi}(\overline{x}) &\to \overline{\chi}(\overline{x})g^{-1}(\overline{x}) \\ U(\overline{x},\mu) &\to g(\overline{x})U(\overline{x},\mu)g^{-1}(\overline{x}+\overline{e}_\mu) & \overline{e}_\mu &= \frac{1}{2}e_\mu \end{aligned} \tag{47}$$

The lattice approximation Eqs. (45,46) of the action of geometric QCD Eqs. (24) or (1) is the starting point of our dynamical calculations.

2.2 The Symmetry of Geometric Lattice QCD

The symmetry groups of the action of geometric QCD in the continuum Eq. (24) or Eq. (1) and on the lattice Eqs. (45,46) play a distinct role in our strong coupling calculation of the meson spectrum. For the physical interpretation of our results, the relation between these groups is important. In a geometrically transparent way these groups and their representations were discussed by M. Gockeler, H. Joos, and M. Schäfer [8],[15]; compare also ref.[21]. We quote a short resume of their results: The continuum geometric QCD describes four degenerate Dirac fields. Therefore infinitesimal spinorial Euclidean transformations, and SU(4)-flavour transformations

$$\begin{aligned} (M_{\mu\nu}\Phi)^b_a(x) &= (x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi^b_a(x) + \frac{1}{2}(\gamma_\mu\gamma_\nu)_{ac}\Phi^b_c(x), \quad \mu \neq \nu, \\ (P_\mu\Phi)^b_a(x) &= \partial_\mu\Phi^b_a(x), \\ (T^i\Phi)^b_a(x) &= \frac{1}{2}\lambda^{i\,b}_c\Phi^c_a(x), \quad i = 1,\dots 15. \end{aligned} \tag{48}$$

generate a symmetry group $S\mathcal{E} \times SU(4)$. λ^i denote the 15 Gell-Mann matrices for SU(4). There are additional symmetries, like space reflection $(\Pi_s\Phi)^b_a(x) = \gamma^4_{a\,c}\Phi^b_c(\Pi_s x)$, $(\Pi_s x) = (-x^1,-x^2,-x^3,x^4)$, charge conjugation, general phase transformations and chiral transformations for masses m=0. To some of these transformations we come back only later. In order to understand the lattice restriction of $S\mathcal{E} \times SU(4)$ from a geometric point of view, we have to consider the relation between the transformations of the Dirac components and that of the forms. Let us first regard the structure of the flavour transformation. The solutions of the DKE, (17)

$$(d - \delta + m)\Phi = (dx^\mu \vee \partial_\mu + m)\Phi. \tag{49}$$

are invariant under Clifford multiplication with a constant differential form from the right. This follows from the associativity of the Clifford product:

$$((d - \delta + m)\Phi) \vee c(u) = ((dx^\mu \vee \partial_\mu + m)\Phi) \vee c(u) = 0 \tag{50}$$

If we write the constant forms c(u) in Dirac components as defined in Eq. (15):

$$c(u) = \sum_{a,b} u^b_a Z^b_a \tag{51}$$

then it follows from Eq. (14) that the Dirac components ψ^b_a of Φ transform with the matrix u^c_b :

$$\Phi \vee c(u) \to \sum_b \psi^b_a(x)u^c_b \tag{52}$$

In the case where u^c_b is an unitary matrix of SU(4), this is the finite form of the infinitesimal flavour transformations in Eq (48) $U = e^{\frac{i}{2}c_i\lambda^i}$. The finite flavour subgroup generated by the basis elements $\pm dx^H$ of the differential forms

$$e\Phi \vee dx^H \to \pm\sum_b \psi^b_a(x)\gamma^{Hc}_b \tag{53}$$

plays an important role in the lattice restriction of the flavour group. Following the suggestion of H. Joos [17] , we call this group of 32 elements "Kent-group". The operations d, δ are invariant operations under the group O(4) of orthogonal transformations of Euclidean space-time. This means

$$U(R)(d - \delta)U^{-1}(R) = d - \delta \tag{54}$$

where U(R) is the representation of the rotations $R \in O(4)$in the space of differential forms. Formula (54) simply means the possibility that one can interchange the calculation of the boundary and coboundary with the transition to a new orthogonal basis. From this we know that a transformed differential form, which is a solution of the DKE is also a solution. By infinitesimal rotation of the space-time $\delta^G_{\mu\nu}$ in the plane (μ,ν) the form transforms infinitesimally as :

$$\delta^G_{\mu\nu}\Phi = (x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi + \frac{1}{2}(S_{\mu\nu} \vee \Phi - \Phi \vee S_{\mu\nu}). \tag{55}$$

We have to distinguish these geometrical rotations of the differential forms, from the spinorial rotations of the Dirac components described by $M_{\mu,\nu}$ in Eq. (48). With help of Eq. (14),we may express this spinorial rotations $\delta^s_{\mu,\nu} \to M_{\mu,\nu}$ of the differential forms as

$$\delta^s_{\mu\nu}\Phi = (x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi + \frac{1}{2}S_{\mu\nu} \vee \Phi, \qquad S_{\mu\nu} = dx_\mu \wedge dx_\nu. \tag{56}$$

In both expressions (55) and (56) $(x_\mu\partial_\nu - x_\nu\partial_\mu)$ expresses the "orbital" part of the 4 dimensional angular momentum. However in the geometric transformations we have, in addition to the spin part of the angular momentum $S_{\mu\nu} \vee \Phi$, also a flavour part-$\Phi \vee S_{\mu,\nu}$. The geometric rotations are composed of a spinor rotation and a flavour transformation.

Summarizing, we can express an element g of the symmetry group of the DKE as a direct product $g = (f) \otimes (a,s) \equiv (f,a,s)$ with $(f) \in \mathcal{F}$ (flavourgroup), $(a,s) \in S\mathcal{E}$, that is the "spinorial Euclidean group generated by translations (a,1) and spinor transformations (0.s) .Thus $\mathcal{G} \simeq \mathcal{F} \times S\mathcal{E}$ (up to a factor Z_2). We may decompose any element g into geometric transformations (s,a,s) with the generators $\delta^G_{\mu,\nu}$ Eq. (55) and another flavour transformation $(\overline{f}) = fs^{-1}$,

$$g = (\overline{f}) \circ (s,a,s) = (\overline{f}s,a,s) = [\overline{f},a,R(s)]. \tag{57}$$

In this new notation, the group multiplication gets the form

$$\begin{aligned} (f,a,s) \circ (f',a',s') &= (ff', R(s)a' + a, ss'), \\ [f,a,R(s)] \circ [f',a',R(s')] &= [fsf's^{-1}, R(s)a' + a, R(s)R(s')]. \end{aligned} \tag{58}$$

The symmetry group of the DKE on the lattice as a restriction of the continuum symmetry was extensively studied by H. Joos and M. Schäfer . Before we quote their proposition on this subject, we make some explanatory remarks. First we remark that the restriction of the translations and geometric rotations to a symmetry of the lattice approximation of the DKE (39) is obvious. The continuum translations are restricted to the lattice translations

$$\mathcal{T} \supset \mathcal{T}_L == \{[1,a,1]\}\{a \mid a = b(n^1,n^2,n^3,n^4), n^i \in Z\}$$

The lattice constant 'b' of the coarse lattice Γ we set most of the time equal to 1. The lattice restriction of the geometric rotations follows also geometrically. The geometric rotations are restricted

to the symmetry group W_4 of the 4-dimensional cube. W_4 is generated by the rotations $R_{\mu\nu}$ in the (μ,ν)-plane by $\pi/2$, and by a space reflection Π_s. The symmetry of the DKE on the lattice under these transformations follows immediately from Eq. (34). It is a little more difficult to find the lattice restriction of the flavour transformations. Since the Clifford product defined in Eq. (33) is not associative, the argument given in the continuum, Eq. (50) , can not be applied directly. However the Clifford product defined in Eq. (33) satifies [8]:

$$(d^{x-e_{H\circ L},H} \vee d^{y,K}) \vee d^{z-e_{H\circ K},L} = d^{x,H} \vee (d^{y,K} \vee d^{z,L}). \tag{59}$$

Therefore we can conclude similar as in Eq. (50), with help of Eq. (59), that

$$\epsilon d^K \Phi : \Phi \to \epsilon\Phi \vee (d^K)^{-1},$$

is a symmetry transformation of the DKE. A direct calculation leads to $(d^K)^2 = [-e_K]$, i.e. the flavour transformations generate translations. The group $\mathcal{FT}_L$ generated by the ϵd^K contains $\mathcal{T}_L$ as a normal subgroup with the factor group $\mathcal{F}_L = \mathcal{FT}_L/\mathcal{T}_L \simeq \mathcal{K}_4$, the Kent group mentioned in Eq. (53)

After this preparations we can quote the proposition on the lattice restriction of the symmetry group of the DKE by Joos and Schäfer:

PROPOSITION 2.1. The lattice restriction of $\mathcal{G}$ is

$$\mathcal{G}_L = \{[\epsilon d^K, -\frac{1}{2}e_K + a, R] \mid a \in \mathcal{T}_L, R \in W_4\}$$

with the composition law

$$[\epsilon d^K, -\frac{1}{2}e_K + a, R] \circ [\epsilon' d^L, -\frac{1}{2}e_L + a', R']$$

$$= [\epsilon\epsilon' \rho(R, R\circ L)\check{p}_{K,R\circ L} d^{K\triangle R\circ L}, -\frac{1}{2}(e_K + Re_L) + Ra' + a, RR']. \tag{60}$$

It is a symmetry group of the free DKE if it acts on staggered fermion fields according to

$$\begin{aligned}
([a]\chi)(y,H(y)) &= \chi(y - a^\mu e_\mu, H(y)), \qquad H(y - a^\mu e_\mu) = H(y),\\
([R]\chi)(y,H(y)) &= \rho(R,H(y))\chi(R^{-1}y, H(R^{-1}y)), \qquad R \in W_4,\\
(\epsilon d^K \chi)(y,H(y)) &= \epsilon\check{p}_{H(y),K}\chi(y + \frac{1}{2}e_K, H(y + \frac{1}{2}e_K)).
\end{aligned} \tag{61}$$

where the sign $\rho(R,H)$ is the same as the sign in the transformation of the basis differentials of the continuum: $Rdx^H = R^{-1} \vee dx^H \vee R = \rho(R,H)dx^{R^{-1}\circ H}$.

A direct calculation shows that also the action $S(\overline{\chi},\chi,U)$, (Eq. (46)) with gauge interaction, is invariant, if $U(x,\mu)$ is formed as link fields under translations and geometric rotations, and as

$$(\epsilon d^K U)(x,\mu) = U(x + \frac{1}{2}e_K, \mu) \tag{62}$$

under flavour transformations.

2.3 The Representations of the Lattice Fermion Group $\mathcal{G}_L$

The representations of $\mathcal{G}_L$ and some of its subgroups play an important role in our investigations. The irreducible representations of $\mathcal{G}_L$ give the quantum numbers of the lattice states. Thus we shall base the discussion of the particle content of the different strong coupling approximations on this representation theory. By this method we get systematic relations between the lattice particle states and the continuum states. Further, as usual in quantum mechanics, the use of irreducible representations of

symmetry groups allows a simplification of our meson mass calculations. The representation theory of $\mathcal{G}_L$ was extensively studied by H. Joos and M. Schäfer and others [14]. We shall summarize their results and collect the formulas which we explicitly use in our calculations. According to a proposition by Joos and Schäfer, the irreducible (unitary) representations of $\mathcal{G}_L$ are determined by a 'momentum star' St_j, a 'flavor orbit' Θ_k and a 'reduced spin' σ. The representations $\mathcal{G}_L \to U(g)$ related to mesons ('representations with even flavour orbits') have, in an appropriate basis

$$\left| \begin{matrix} j, & k, & \sigma \\ p, & L, & n \end{matrix} \right\rangle$$

the following form:

$$U(a) \left| \begin{matrix} j, & k, & \sigma \\ p, & L, & n \end{matrix} \right\rangle = e^{i(p,a)} \left| \begin{matrix} j, & k, & \sigma \\ p, & L, & n \end{matrix} \right\rangle \tag{63}$$

$$U(R) \left| \begin{matrix} j, & k, & \sigma \\ p, & L, & n \end{matrix} \right\rangle = \sum_{n'} \left| \begin{matrix} j, & k, & \sigma \\ Rp, & \omega(R,p)\circ L, & n' \end{matrix} \right\rangle D^\sigma_{n'n}(X(R,L,p)), \tag{64}$$

$$U(\epsilon d^K) \left| \begin{matrix} j, & k, & \sigma \\ p, & L, & n \end{matrix} \right\rangle = e^{i(p,\frac{1}{2}e_K)} e^{i\pi(e_L,e_K)} \left| \begin{matrix} j, & k, & \sigma \\ p, & L, & n \end{matrix} \right\rangle \tag{65}$$

We shall explain these formulas for the case we use later.

The momentum p varies over the Brillouin zone of the coarse lattice: $-\frac{\pi}{b} \le p_\mu \le \frac{\pi}{b}$. In an irreducible representation of $\mathcal{G}_L$, p is restricted to a set $\{Rp | R \in W_4\}$, called momentum star St_j. As an example we consider the St_4 of 8 momenta:

$$St_4 : (P_\mu = (0,0,0,\pm E), (0,0,\pm E,0), (0,\pm E,0,0), (\pm E,0,0,0). \tag{66}$$

The representations of the flavour translations(Eq. (65)) are 1-dimensional and characterized by the vector e_L. The e_L transforms under rotations: $Re_L \to e_{RL}$. For the characterization of an irreducible representation of $\mathcal{G}_L$ we may choose a reference momentum $\bar{p} \in St_j$, for instance in our case $\bar{p} = (0,0,0,E)$. The group of rotations, which leaves $\bar{p}$ invariant transform the e_L of a flavour orbit into each other. In our case this group is W_3, the e_L of the different flavour orbits Θ_k are the singlet orbits

$$\{(0,0,0,0,)\}, \{(0,0,0,1)\}, \{(1,1,1,0)\}, \{(1,1,1,1)\} \tag{67}$$

and the triplet orbits:

$$\begin{aligned}
&\{(1,0,0,0), (0,1,0,0), \underline{(0,0,1,0)}\}, \quad \{(0,1,1,0), (1,0,1,0), \underline{(1,1,0,0)}\},\\
&\{(1,0,0,1), (0,1,0,1), \underline{(0,0,1,1,)}\}, \quad \{(0,1,1,1), (1,0,1,1), \underline{(1,1,0,1)}\}.
\end{aligned} \tag{68}$$

The group of rotations, which leaves $\bar{p}$ and a representative $\bar{e}_L$ (underlined in Eq. (68) invariant is called the reduced spin group $S_{j,k}$. The irreducible representations $D^\sigma_{n'n}(R)$ of this group determine the reduced spin σ. In our case of St_4 the reduced spin group of the singlet orbits is W_3. That of the triplet orbits is the group $D_4 \times Z_2$. The 3-dimensional cubic group W_3 has 10 irreducible representations:

$$(1^\pm)_{W_3}, (1'^\pm)_{W_3}, (2^\pm)_{W_3}, (3^\pm)_{W_3}, (3'^\pm)_{W_3} \tag{69}$$

The integer denotes the dimension of the representation, $\pm$ refers to the parity. D_4 is the symmetry group of the square in the (1,2) plane, Z_2 the reflection of the 3 axis. $D_4 \times Z_2$ has also 10 irreducible representations:

$$(1^\pm)_{D_4}, (1'^\pm)_{D_4}, (1''^\pm)_{D_4}, (1'''^\pm)_{D_4}, (2^\pm)_{D_4} \tag{70}$$

These representations are explicitely given (see the Appendix). $\omega(R,p)$, and $X(s,L)$ are Wigner type rotations defined in [15]. If we consider only states refering to $\bar{p}$ and $\bar{e}_L$ (as we do in our calculations), and rotations R of the reduced spin group, then $\omega(R,p) = x(s,L) = R$.

3 The Resummed Hopping Parameter Expansion

In this chapter we shall explain the method of the "resummed hopping parameter expansion". For details we refer to O. Martin [10]. Until now coarse lattice site labels are denoted by unbarred variables. From now on, all site labels ,with or without a bar , refer only to the fine lattice , unless otherwise specified . We have derived in Chapter 2 the gauge invariant action of interacting staggered lattice fermions

$$S_q = \sum_{\overline{x}} \left\{ \sum_{\mu} \tilde{\rho}_{\mu H(\overline{x})} \frac{1}{2} [\overline{\chi}(\overline{x}) U(\overline{x},\mu) \chi(\overline{x}+\overline{e}_\mu) - \overline{\chi}(\overline{x}+\overline{e}_\mu) U^{-1}(\overline{x},\mu) \chi(\overline{x})] + m \overline{\chi}(\overline{x}) \chi(\overline{x}) \right\}$$
$$\equiv m \sum_{\overline{x}_2, \overline{x}_1} \overline{\chi}(\overline{x}_2) (\overline{x}_2 | Q[U] | \overline{x}_1) \chi(\overline{x}_1) \qquad (71)$$

which has $\mathcal{G}_L$ as symmetry group. The sum is over the sites of the lattice, and the directions $\pm\mu = \pm 1, \pm 2, \pm 3, \pm 4$; $\tilde{\rho}_{-\mu,H(\overline{x})} = \tilde{\rho}_{\mu,H(\overline{x})}$ by definition. Here we introduced for later use the abreviation $Q_{\overline{x}_2 \overline{x}_1} \equiv (\overline{x}_2 | Q[U] | \overline{x}_1)$

$$Q_{\overline{x}_2 \overline{x}_1} = m[\delta_{\overline{x}_2, \overline{x}_1} + \tilde{Q}_{\overline{x}_2 \overline{x}_1}]$$
$$\tilde{Q}_{\overline{x}_2 \overline{x}_1} = \frac{1}{2m} \sum_{\overline{x}, \pm\mu} \tilde{\rho}_{\mu, H(\overline{x})} \delta_{\overline{x}_1, \overline{x}} \delta_{\overline{x}_2, \overline{x}+\overline{e}_\mu} U(\overline{x}, \mu) \qquad (72)$$

In order to calculate the correlation function $\langle : \mathcal{M}(x)\mathcal{M}(y) : \rangle = \langle \mathcal{M}(x)\mathcal{M}(y) \rangle - \langle \mathcal{M}(x) \rangle \langle \mathcal{M}(y) \rangle$ of the local meson field $\mathcal{M} = \frac{1}{\sqrt{3}} \sum_\alpha \overline{\chi}_\alpha(x) \chi_\alpha(x)$, we use the path integral formula:

$$\langle \overline{\chi}\chi(\overline{x}) \overline{\chi}\chi(\overline{y}) \rangle = \frac{1}{Z} \int\int \mathcal{D}[U] \mathcal{D}[\chi, \overline{\chi}] \overline{\chi}(\overline{x}) \chi(\overline{x}) \overline{\chi}(\overline{y}) \chi(\overline{y}) e^{-(S_g + m\overline{\chi}(1+\tilde{Q}[U])\chi)}, \qquad (73)$$

$$\langle \overline{\chi}(\overline{x}) \chi(\overline{x}) \rangle = \frac{1}{Z} \int\int \mathcal{D}[U] \mathcal{D}[\chi, \overline{\chi}] \overline{\chi}(\overline{x}) \chi(\overline{x}) e^{-(S_g + m\overline{\chi}(1+\tilde{Q}[U])\chi)}, \qquad (74)$$

Here S_g denotes the pure gauge action:

$$S_g = -\beta \sum_{\square} (U(\square) + U^{-1}(\square) - 2) \qquad (75)$$

where $U(\square)$ is the parallel transport around a plaquette. The sum in Eq. (75) goes over all plaquettes of the lattice Γ. Integrating Eq. (73) over the "Grassmann variables" $\overline{\chi}(x), \chi(x)$ we get

$$\langle : \mathcal{M}(\overline{x}), \mathcal{M}(\overline{x}) : \rangle = \frac{1}{Z} \int \mathcal{D}[U] \frac{1}{m^2} (\overline{x}|Q^{-1}|\overline{y})(\overline{y}|Q^{-1}|\overline{x}) \det mQ \exp^{-S_g}. \qquad (76)$$

where Q is given by Eq. (72). The normalization factor is

$$Z = \int\int \mathcal{D}[U] \mathcal{D}[\chi, \overline{\chi}] e^{-(S_g + m\overline{\chi}(1+\tilde{Q}[U])\chi)} = \int\int \mathcal{D}[U] \det(mQ) e^{-S_g} \qquad (77)$$

The following calculation will be performed effectively in the quenched approximation where

$$\det mQ = 1 \qquad (78)$$

Substituting Eq. (78) into Eq. (77), we get Z=1 in the zeroth and first orders in β. One can see this by expanding e^{-S_g} and using the orthogonality relation for matrix elements $U^X_{\alpha\beta}(g)$ of irreducible group representations of dimension $\dim[U^X]$:

$$\int_{SU(3)} d\mu(g) U_{\alpha,\beta}(g) U^*_{\alpha',\beta'}(g) = \frac{1}{\dim U} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'} \qquad (79)$$

Substituting Eq. (78) in Eq. (76) we get in the same approximation the following expression:

$$\langle : \mathcal{M}(\overline{x}), \mathcal{M}(\overline{x}) : \rangle = \int \mathcal{D}[U] \frac{1}{m^2} (\overline{x}|Q^{-1}|\overline{y})(\overline{y}|Q^{-1}|\overline{x}) e^{-S_g}. \qquad (80)$$

for the correlation function of the local meson fields. This is the main formula for the following calculations. Since Q is of the form $Q = m(1+\tilde{Q})$ we can expand Q^{-1} in a geometric series:

$$Q^{-1}_{\overline{x}_2 \overline{x}_1}[U] = \frac{1}{m} \sum_{n=0}^{\infty} (-1)^n \tilde{Q}^n \qquad (81)$$

From Eq. (72), we see that $\langle \overline{x}_2 | \tilde{Q} | \overline{x}_1 \rangle \neq 0$ only if $\overline{x}_1, \overline{x}_2$ are neighbouring points. Therefore we may describe $\langle \overline{x}_2 | \tilde{Q}^n | \overline{x}_1 \rangle$ with help of a sum over paths $\mathcal{C}^n_{\overline{x}_2 \overline{x}_1}$ of length n from $\overline{x}_1$ to $\overline{x}_2$, which are generated by the links l_μ connecting such neighbouring points. The contribution to the matrix element of $\tilde{Q}^n$ associated with such a path is the product of the factors associated with a single link according to Eq. (72). This leads to the well known hopping parameter formula for the quark propagator with interaction

$$\langle \overline{x}_2 | (\widetilde{Q})^n | \overline{x}_1 \rangle = \frac{1}{m} \sum_{n=0}^{\infty} \left(\frac{-1}{2m} \right)^n \sum_{\mathcal{C}^n_{\overline{x}_2, \overline{x}_1}} \prod_{l_\mu \in \mathcal{C}^n} \check{\rho}(l_\mu) U(l_\mu), \qquad (82)$$

Thus we see that the contribution to the coefficient of the n^{th} power of the hopping parameter $\kappa = \frac{-1}{2m}$ associated with the path $\mathcal{C}^n$ is the parallel transport $U(C)$ along the path, modified by the factors: $\check{\rho}(l_\mu) = \tilde{\rho}_{\mu,H(x)}$ for the links $l_\mu = [\overline{x}, \mu], \mu > 0, \check{\rho}(-l_\mu) = -\check{\rho}(l_\mu)$ .These factors are characteristic for the propagation of staggered fermions. Now we insert the expression Eq. (82) into Eq. (80), and perform the integration over the gauge fields $U(l_\mu)$ in strong coupling approximation. First, we consider the zero order in β, i.e. we set $e^{-S_g} = 1$. The integration in Eq. (80) over products $U(\mathcal{C}^n_{xy}) U(\mathcal{C} l^{n'}_{yx})$ of parallel transports associated with the quark and antiquarkline respectively. Because of the orthogonality relation Eq. (79) we get a non-zero contribution only if, for each link l_μ, the link $l_{-\mu}$ is contained with the same multiplicity in $\mathcal{C}^n_{xy} \cup \mathcal{C} l_{yx}{}^{n'}$. This implies that the area enclosed by $\mathcal{C}^n_{xy} \cup \mathcal{C} l^{n'}_{yx} = \overline{\mathcal{C}}$ is zero, and that $\prod_{l_\mu \in \overline{\mathcal{C}}} U(l_\mu) = 1$. We call this type of quark anti-quark paths $\overline{\mathcal{C}}$:" $q\overline{q}$-treegraphs". Because of $\check{\rho}(l_\mu)\check{\rho}(-l_\mu) = -1$, we get for such graphs $\overline{\mathcal{C}}$ with 2N links:

$$\prod_{l_\mu \in \mathcal{C}^n_{xy}} \check{\rho}(l_\mu) \prod_{l'_\nu \in \mathcal{C}^n_{yx}} \check{\rho}(l'_\nu) = (-1)^N, \qquad 2N = n + n'$$

These considerations lead to the following evaluation of Eq. (80) in zero order of β :

$$\langle : \mathcal{M}(\overline{x}) \mathcal{M}(\overline{y}) : \rangle = \sum_{N}^{\infty} \left(\frac{-1}{4\overline{m}^2} \right)^N T_{\overline{x}\overline{y}}(N). \qquad (83)$$

where $T_{\overline{x}\overline{y}}(N)$ is the number of $q\overline{q}$ zero order graphs of "length" N connecting $\overline{x}$ and $\overline{y}$. In order to evaluate the expression, Eq. (83), we have to discuss the structure of $q\overline{q}$-zero order graphs in more detail. Let us consider a simple quark path $\mathcal{C}^n$ from y to x, which is described by an ordered set of unit vectors $\mathcal{C}^n = \{e_{\mu_1}, ...e_{\mu_n}\}$. The first step goes from y to $y + e_{\mu_1}$,the second from $y + e_{\mu_1}$ to $y_1 + e_{\mu_1} + e_{\mu_2}$ and so on. Along the path $\mathcal{C}^n$ we may return to y a last time with the step e_{μ_i} in such a way that the subpath $C^n_i = \{e_{\mu_1}, ...e_{\mu_i}\}$ incloses zero area. Then we call $C^n_i = D_0$ a dressing of $\mathcal{C}^n$ at y of zero order. We continue to the point $y + e_{\mu_{i+1}}$ and determine there the dressing of zero order in the same manner. Continuing in this way, we decompose the quarkpath $\mathcal{C}^n$ into dressings D_i and remainders (f_{μ_i}) i.e. $\mathcal{C}^n = D_0 f_{\mu_1} D_1 f_{\mu_2} ... D_m f_{\mu_m}$. The path without dressing $\Im = \{f_{\mu_1}, ..., f_{\mu_m}\}$ is called a quark trunk. It has important property that

$$f_{\mu_i} \neq -f_{\mu_{i+1}} \qquad (84)$$

This means geometrically that $\mathfrak{S}$ is not backtracking. The representation of a quark line by a quark trunk and dressings allows an important reordering of the sum in Eq. (83). One may consider a $q\bar{q}$ zero order graph as consisting of a non-backtracking quark line with dressings and a strictly parallel non-backtracking anti quark line with its own dressings. Therefore we may bring Eq. (83) in the form

$$\langle : \mathcal{M}(x)\mathcal{M}(y) : \rangle = \frac{1}{m^2}\sum_{n=0}^{\infty}(-1)^n \frac{B_{xy}(n)}{(2m)^{2n}} \left\{ \sum_r^{\infty} d_r \left(\frac{-1}{4m^2}\right)^{2r} \right\}^{2n} \tag{85}$$

the sum $\sum_r d_r(\frac{-1}{2m^2})^{2r} \equiv 2m\alpha^{-1}$ is the contribution of all dressings of zero order at a given point of a non-backtracking quark, or antiquark line $B_{xy}(n)$. Eq. (85) shows that one can restrict the summation over all paths to a summation over all non-backtracking paths, if one "renormalizes" the hopping parameter $\frac{1}{2m}$ to $\frac{1}{\alpha}$. Following Martin, we calculate now the contribution of the dressing with the technique of generating functions . For this we define

$$W(z) = \sum_{r=0}^{\infty} \bar{d}_r z^r \tag{86}$$

where $\bar{d}_r$ the number of closed paths from x to x enclosing zero area. If we would calculate $\langle \overline{\chi}(x)\chi(x) \rangle$ according to Eq. (74) by the hopping parameter method combined with zero order strong coupling approximation, the result would be $\langle \overline{\chi}(x)\chi(x) \rangle = W(-\frac{1}{4m^2})$. Therefore the behaviour of this function for $m^2 \to 0$, decides on the spontaneous chiral symmetry breaking in this model [10]. We shortly indicate the calculation of W(z). A closed path $\mathcal{C} = \{f_1, ..f_i, f_{i+1}, ..f_{2r}\}$ starting from x, enclosing zero area, is called tree graph of lenght r. It is called an irreducible tree graph (ITG), if no subgraph $\mathcal{C}\prime = \{f_1, ...f_i\}, i < 2r$ exists, which ends in x. Let I(n) be the number of irreducible tree graphs of length n and K(z) the corresponding generating function

$$K(z) = \sum_{n=1}^{\infty} I(n) z^n \tag{87}$$

The number $\bar{d}_r$ of all tree graphs of length r is then

$$\bar{d}_r = \sum_{n_1+n_2+...n_i=r} I(n_1)I(n_2)...I(n_i) \tag{88}$$

It follows from Eqs. (86) to (88) that

$$W(z) = \sum_{r=0}^{\infty} \bar{d}_r z^r = \sum_{m=0}^{\infty} K^m(z) = \frac{1}{1-K(z)} \tag{89}$$

In order to calculate K(z) we remark that one can compose an ITG of lenght n from a first double line of length 1, and an arbitrary tree graph beginning at $x + f_1$, of length n-1. Because of irreducibility no first step of the irreducible components of this arbitrary graph is allowed in direction $-f^1$. Since in d dimensions we have 2d direction, this consideration leads to the recursion formula

$$I(n) = I(1) \sum_{n_1+n_2+...n_p=n-1} I(n_1)I(n_2)...I(n_p) \left(\frac{2d-1}{2d}\right)^p \qquad I(1) = 2d, I(0) = 0 \tag{90}$$

Multiplying both sides by z^n, and summing over n we get the equation

$$K(z) = 2d.z \sum_{l=0}^{\infty} \left(\frac{2d-1}{2d}\right)^l K^l(z) = \frac{2dz}{1-(\frac{2d-1}{2d})K(z)} \tag{91}$$

Solving this for K(z) leads to

$$K(z) = d\frac{(1-\sqrt{1-(2d-1)4z})}{(2d-1)} \tag{92}$$

and hence we can get W(z) from Eq. (89). With help of K(z) we calculate the renormalized hopping parameter α. We remark that the dressing at a site is composed by ITG's. In order to avoid over-counting of the paths one has to take into account the nonback tracking conditions. This leads, as in Eq. (90) to,

$$d_r = \sum_{n_1+n_2+...n_p=r} I(n_1)I(n_2)...I(n_p) \left(\frac{2d-1}{2d}\right)^p \tag{93}$$

Thus we get the renormalized hopping parameter α:

$$2m\alpha^{-1} = \sum_r^{\infty} d_r z^r = \sum_r K^r(z) \left(\frac{2d-1}{2d}\right)^r = \frac{1}{1-(\frac{2d-1}{2d})K(z)} \tag{94}$$

with $z = -\frac{1}{4m^2}$, or with help of (92)

$$\alpha = m + \sqrt{m^2 + 2d - 1} \tag{95}$$

and therefore from Eq. (85)

$$\langle : \mathcal{M}(x)\mathcal{M}(y) : \rangle = \frac{1}{m^2}\sum_{n=0}^{\infty} \left(\frac{-1}{\alpha^2}\right)^n B_{xy}(n). \tag{96}$$

For the final evaluation of Eq. (96) we generate the paths between x and y = 0 step wise. Classifying the paths of length n by the last direction μ we get for their number $B_{x0;\mu}(n) \equiv B_{x,\mu}(n)$ the recursion relation

$$B_{\mu,x}(n+1) = \sum_{\nu \neq -\mu} B_{\nu,x-e_\nu}(n). \tag{97}$$

We do not sum over $\mu = -\nu$, because the paths are not backtracking. Now we do the Fourier transformation of Eq. (96) and Eq. (97). For the Fourier formations $\tilde{B}_{p,\mu}(n)$ of $B_{x,\mu}(n)$, follows from Eq. (97) the recursion relation

$$\tilde{B}_{p,\mu}(n+1) = \sum_{\nu \neq -\mu} e^{i(p,e_\nu)} \tilde{B}_{p,\nu}(n) \equiv \sum_{\nu} M_{\mu\nu}(p)\tilde{B}_{\nu,p}(n) \tag{98}$$

with

$$M_{\mu\nu} = e^{i(p,e\nu)}(1-\delta_{\mu,-\nu}). \tag{99}$$

Inserting Eq. (98) in the Fourier transformation of Eq. (96) leads to

$$\begin{aligned} \langle : \mathcal{M}(x)\mathcal{M}(0) : \rangle &= \frac{1}{m^2}\int dp e^{-i(p,x)} \sum_{n=0}^{\infty} \left(\frac{-1}{\alpha^2}\right)^n \xi^T M^n(p)\xi \\ &= \frac{1}{m^2}\int dp e^{-i(p,x)} \xi^T \left(\frac{1}{1+\frac{1}{\alpha^2}M(p)}\right)\xi \end{aligned} \tag{100}$$

The matrix M(p) is an operator acting in the 8 (=2d)-dimensional space of the "step vectors" spanned by the basis $\xi_\mu^T = (0, 0, .. \overset{\mu}{1}, 0, ..)$. It follows from Eq. (96), that we have to count paths, which start in all directions at y=0 and end from all directions at x, therefore we have in Eq. (100): $\xi = \sum_\mu \xi_\mu$. The poles of the Fourier transformation of the propagator determine the "masses" of the mesons. These are determined by the equation

$$\det(M(p) + \alpha^2 I) = 0 \tag{101}$$

We discuss this "mass spectrum" for the different meson quantum numbers following from the representation theory of the symmetry group of staggered fermions. The method of the "resummed hopping parameter expansion" which we explained here in the zero order of β, will be extended to first order in section 5.1 .

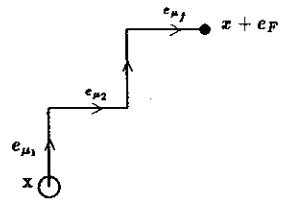


Figure 2: The Multi-link Operator

4 The Propagator for Muliti-link Meson Operators in Zero-Order Approximation

4.1 Multi-link operators

It is the aim of this Section to investigate meson field operators composite of quark and antiquark fields, which allow the description of the whole variety of meson quantum numbers associated with the symmetry group $\mathcal{G}_L$ of lattice fermions. (Compare Sect. 2.3). It is seen from Eqs. (16), (41) that the 16 spinor components with different flavours in the continuum are mapped on 1-component lattice fields defined on 16 different sites of the fine lattice. This is one reason why we need for our purpose gauge invariant bilocal products of quark fields, which we call multi-link fields. Such fields are also necessary for the description of rotationally excited meson states.

We define the multi-link meson operator $\mathcal{M}^{L,\mathcal{F}}(x)$ as

$$\begin{aligned}\mathcal{M}^{L,\mathcal{F}}(x) &= e^{i\pi(e_L+\tilde{f}e_{1234}+e_F,x)}\overline{\chi}(x)\check{\rho}_{H(x),F}U(\mathcal{F})\chi(x+e_F)\\ &\equiv \zeta^{L,\mathcal{F}}(x)\overline{\chi}(x)U(\mathcal{F})\chi(x+e_F) \qquad \zeta^{L,\mathcal{F}}(x)=\pm 1\end{aligned} \tag{102}$$

$\mathcal{F}$ is a path from x to $x+e_f$, $\mathcal{F}=(x,\hat{\mu}_1,..\hat{\mu}_f)$, $\hat{\mu}=e_\mu, e_f=\sum_i\hat{\mu}_i$ with a notation explained in Fig.2. We call $\mathcal{F}$ the multi-link-path. $U(\mathcal{F})$ is the parallel transport from x to $x+e_f$ along the multi-link-path $\mathcal{F}$.

$$U(\mathcal{F})=U(x,\mu_1)U(x+\hat{\mu}_1,\mu_2)...U(x+e_F-\hat{\mu_f},\mu_f) \tag{103}$$

F is an ordered multi index corresponding to the point $\overline{x}=e_f$,as defined in Eq. (40): $F=F(e_f), e_F=\sum_{\mu\in F}\hat{\mu}$. e_L is a vector defining the irreducible representation of the flavourgroup (see Eq. (67,68)). A special case is $e_{1234}=(1,1,1,1)$. $\tilde{f}=0$ or 1 for an even or odd number of links of $\mathcal{F}$, respectively.

The multi-link operators are transformed under the different symmetry transformations in the following way:

a) Gauge transformations. Under gauge transformations the fermions transform as :

$$\chi(x+e_f)\to g(x+e_f)\chi(x+e_f) \qquad \overline{\chi}(x)\to\overline{\chi}(x)g^{-1}(x) \tag{104}$$

and the gauge fields $U(x,\mu)$ as $U(x,\mu)\to g(x)U(x,\mu)g^{-1}(x+\hat{\mu})$. It follows that $U(\mathcal{F})$ transforms like

$$U(\mathcal{F})\to g(x)U(\mathcal{F})g^{-1}(x+e_f) \tag{105}$$

and hence that $\mathcal{M}^{L,\mathcal{F}}$ is gauge invariant.

b) Flavour transformations. We shall prove that

$$\left(\epsilon d^k\mathcal{M}^{L,\mathcal{F}}\right)(x)=e^{i\pi(e_L,e_K)}\mathcal{M}^{L,\mathcal{F}}(x+e_K) \tag{106}$$

For the proof we show first that the formula

$$\check{\rho}_{H(x+e_K),F}\check{\rho}_{H(x),K}\check{\rho}_{H(x+e_F),K}\check{\rho}_{H(x),F}=e^{i\pi(\tilde{f}e_{1234}+e_f,e_K)} \tag{107}$$

is correct. From the definition of $\check{\rho}_{H,K}$ in Eq .(9) and the associative law of Clifford multiplication it followsthat

$$\check{\rho}_{H,K}\check{\rho}_{H\triangle K,L}=\check{\rho}_{H,K\triangle L}\check{\rho}_{K,L} \tag{108}$$

In a similar way we get from the antimorphism $\mathcal{B}$ in Eq. (4)

$$\check{\rho}_{H,K}\check{\rho}_{K,H}=(-1)^{hk+r} \tag{109}$$

with h = ord$H$, k= ord$K$, r = ord$(H\triangle K)$. We note that: $H(x+e_f)=H\triangle F$ and calculate

$$\begin{aligned}\check{\rho}_{H(x),F}\check{\rho}_{H(x+e_f),K}\check{\rho}_{H(x),K}\check{\rho}_{H(x+e_K),F} &= \check{\rho}_{H,K\triangle F}\check{\rho}_{F,K}\check{\rho}_{H,K}\check{\rho}_{H\triangle K,F}=\\ \check{\rho}_{F,K}\check{\rho}_{H,K\triangle F}\check{\rho}_{H,K\triangle F}\check{\rho}_{K,F}=\check{\rho}_{F,K}\check{\rho}_{K,F} &= (-1)^{fk-ord(F\triangle K)}e^{i\pi(\tilde{f}e_{1234}+e_F,e_K)}\end{aligned} \tag{110}$$

With help of formula Eq. (107), we prove Eq. (106) by direct calculation starting from the definition of the flavour transformation of the χ fields in the proposition, Eq. (61) :

$$\begin{aligned}\left(\epsilon d^k\mathcal{M}^{L,\mathcal{F}}\right)(x) &= e^{i\pi(e_L+\tilde{f}e_{1234}+e_F,x)}\overline{\chi}(x+e_k)U(x+e_k,\mathcal{F})\chi(x+e_f+e_k)\check{\rho}_{H(x),k}\check{\rho}_{H(x+e_f),K}\check{\rho}_{H(x),F}\\ &= e^{i\pi(e_L,e_K)}\mathcal{M}^{L,\mathcal{F}}(x+e_k)\end{aligned}$$

c) Geometric Rotations. Under the rotations of the lattice the multi-link operator is transformed as

$$[\mathcal{M}^{L,\mathcal{F}}](x)=\rho(R,F)\mathcal{M}^{R^{-1}L,R^{-1}\mathcal{F}}(R^{-1}x) \tag{111}$$

In order to proof this from transformations of the $\chi(x)$ (Eq. (61)) we need the formula

$$\rho(R,H(x-e_F))=\check{\rho}_{H(x),F}\check{\rho}_{H(R^{-1}x),R^{-1}F}\rho(R,H(x))\rho(R,F) \tag{112}$$

which follows immediately from

$$\begin{aligned}R^{-1}\left(dx^H\vee dx^F\right)R &= R^{-1}\left(\check{\rho}_{H(x),F}dx^{H\triangle F}\right)R=\rho(R,H(x-F))\check{\rho}_{H(x),F}dx^{R^{-1}(H\triangle F)}\\ &= \left(R^{-1}dx^HR\vee R^{-1}dx^FR\right)=\rho(R,H(x))\rho(R,F)\check{\rho}_{H(R^{-1}x),R^{-1}F}dx^{(R^{-1}H)\triangle(R^{-1}F)}\end{aligned}$$

The transformation law Eq. (111) follows then by direct calculation as in (b).

d) Charge Conjugation. The action of the staggered fermions, Eq. (46), is invariant under charge conjugation C defined the following way

$$C: \qquad \chi^C(x)=\check{\rho}_{H(x),1234}\overline{\chi}^T(x) \qquad \overline{\chi}^C(x)=-\check{\rho}_{H(x),1234}\chi^T(x) \qquad U^C(x,\mu)=U^*(x,\mu) \tag{113}$$

The transposition $\chi\to\chi^T$ refers to the colour index, $U^*(x,\mu)$ is the complex conjugate of $U(x,\mu)$. This transformation corresponds to charge conjugation in the continuum. This transformation of the basic fields implies for the composite meson operators

$$\left(\mathcal{M}^{L,\mathcal{F}}\right)^C(x)=(-1)^{\frac{f(f+1)}{2}}e^{i\pi(e_L,e_F)}\mathcal{M}^{L,-\mathcal{F}}(x+e_F) \tag{114}$$

where $-\mathcal{F}$ is the path from $x+e_f$ to x: $(x+e_f,-\hat{\mu}_f,-\hat{\mu}_{f-1},..,-\hat{\mu}_1)$.

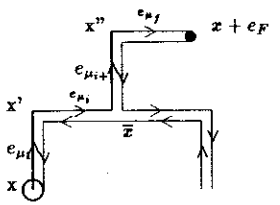


Figure 3: Screening at the Zero Order

4.2 The 2-point functions of multi-link operators

In the following we calculate the expectation value of $\mathcal{M}^{L,\mathcal{F}}(x)\mathcal{M}^{L,\mathcal{F}'}(y)$ in zero order of strong coupling approximation by the methods explained in Chapter 3. After fermion integration we get in quenched approximation a formula similar to eq. (80).

$$\left\langle :\mathcal{M}^{L,\mathcal{F}}(x)\mathcal{M}^{L,\mathcal{F}'}(y):\right\rangle = \int\int \mathcal{D}[U]\zeta^{L,\mathcal{F}}(x)\zeta^{L,\mathcal{F}'}(y)U(\mathcal{F})U(\mathcal{F}')\langle x|Q^{-1}|y\rangle\langle y|Q^{-1}|x\rangle \exp(-S_g). \tag{115}$$

We insert for $\langle x|Q^{-1}|y\rangle$ the hopping parameter expansion Eq. (82) and perform the D[U]-integration in zero order of β. In order to get non-vanishing contributions, a part of the quark and anti-quark lines must screen $U(\mathcal{F})$ and $U(\mathcal{F}')$ and the remaining parts must enclose zero area. We show a typical example of the screening of multi-link of fig.2 in Fig.3 . The summation over all zero order graphs from point x to point y can be performed essentially by the same procedure as in Chapter 3, with the result that we can represent $\left\langle :\mathcal{M}^{L,\mathcal{F}}(x)\mathcal{M}^{L,\mathcal{F}'}(0):\right\rangle$ in a way similar to Eq. (100):

$$\left\langle :\mathcal{M}^{L,\mathcal{F}}(x)\mathcal{M}^{L,\mathcal{F}'}(0):\right\rangle = \frac{1}{m^2}\int dp e^{-i(p,x)}\eta^\dagger(p,\mathcal{F},L)\frac{1}{1+\frac{1}{\alpha^2}M^L(p)}\eta(p,\mathcal{F}',L) \tag{116}$$

where $\eta^\dagger$ is the hermitian conjugate of η. In the following we have to discuss the dependence of $\eta^\dagger(p,\mathcal{F},L)$ on the multi-link operators and the relation of $M^L(p)$ to M(p).

First we discuss the different ways in which the zero order graphs screen the multi-link $\mathcal{F} = (x,\hat{\mu}_1,..\hat{\mu}_f)$. We call the non-backtracking line directed towards x, the quarkline. As it is explained in Fig.3 the quark line screens the part of $\mathcal{F}$ from $\bar{x}$ to x. The quark line meets $\mathcal{F}$ for the first time in $\bar{x}$. The non-back tracking anti-quark line is uniquely determined by the quark line. It screens $\mathcal{F}$ from $x + e_f$ to $\bar{x}$. We classify the paths by this point $\bar{x}$. In order to avoid overcounting, the quark line must come to $\bar{x}$ not along the direction $-\hat{\mu}_i$ (in the notation of the figure), because in this case it would meet $\mathcal{F}$ at the first time in x' and not in $\bar{x}$. By the same reason it can not come along $-\mu_{i+1}$. We have the same situation with respect to the multi-link path $\mathcal{F}'$ from 0 to $e_{f'}$, where the quark line leaves $\mathcal{F}'$ at $\bar{y}$, and screens it from $e_{f'}$ to $\bar{y}$. The remaining part of $\mathcal{F}$ is screened by the anti-quark line.

Now we consider the double lines from $\bar{y}$ to $\bar{x}$. The summation of these zero order graphs can be performed with help of Eq. (100) of Chapter 3. as explained there, the function

$$S_{\mu\nu}(x,y) = \frac{1}{m^2}\int dp e^{-i(p,x)}\xi^\dagger_\mu \frac{1}{1+\frac{1}{\alpha^2}M^L(p)}\xi_\nu \tag{117}$$

describes the summation over all zero order graphs beginning at $\bar{y}$ with first step $\hat{\nu}$, and come to $\bar{x}$ with last step $\hat{\mu}$. From the forbidden directions of the non-back tracking quark and anti-quark lines at the points $\bar{x}$ and $\bar{y}$ discussed above, follow restrictions on the initial and final steps of the trunks. These we describe by a sum $\eta^\dagger(\bar{x}),\eta(\bar{y})$ over all allowed step vectors:

$$\eta^\dagger(\bar{x}) = \sum_{\mu\neq-\mu_i,\mu_{i-1}} \xi^\dagger_\mu \qquad for\ \ \bar{x}\ \ interior\ \ point\ \ of\ \ \mathcal{F}$$
$$\eta^\dagger(x) = \sum_{\mu\neq-\mu_1}\xi^\dagger_\mu \qquad\qquad \eta^\dagger(x+e_f) = \sum_{\mu\neq\mu_f}\xi^\dagger_\mu \tag{118}$$

Therefore the summation over all allowed trunks from $\bar{y}$ to $\bar{x}$ may be denoted by $\eta^\dagger(\bar{x})S(\bar{x},\bar{y})\eta(\bar{y})$. The final expression for the multilink two point function now contains a factor $\alpha^f\prod_{l_\mu\in\mathcal{F}_{x,x+e_f}}\check{\rho}(l_\mu)$, which comes from the parts of the quark and anti-quark lines screening $\mathcal{F}$, as well as the factor $e^{i\pi(\hat{f}e_{1234}+e_f+e_L,x)}\check{\rho}_{H(x),F}$ given in the definition (Eq. (102)) of the multi-link operator. There are similar factors for $\mathcal{F}'$. Thus we get

$$\left\langle :\mathcal{M}^{L,\mathcal{F}}(x)\mathcal{M}^{L,\mathcal{F}'}(0):\right\rangle = \alpha^{f+f'}A(x)A'(0)e^{i\pi(e_L,x)}\sum_{\substack{\bar{x}\in\mathcal{F}\\ \bar{y}\in\mathcal{F}'}}\eta^\dagger(\bar{x})S(\bar{x},\bar{y})\eta(\bar{y}) \tag{119}$$

with $A(x,\mathcal{F}) = e^{i\pi(\hat{f}e_{1234}+e_f,x)}\check{\rho}_{H(x),F}\prod_{l_\mu\in\mathcal{F}_{x,x+e_f}}\check{\rho}(l_\mu)$. Before the evaluation of Eq. (119), we prove some important properties of the "screening factor" $A(x,\mathcal{F})$.
Lemma: (1) $A(x,\mathcal{F})$ independent of x: $A(x,\mathcal{F}) = A(\mathcal{F})$.
(2) $A(R^{-1}\mathcal{F}) = \rho(R,F)A(\mathcal{F})$, where $\rho(R,F)$ is defined in Eq. (61).
Proof:
Besides the vectors $e_\mu = (0,0,\overset{\mu}{1},0)$ we define the vectors $\underline{e}_\mu = \sum_{\nu\leq\mu}e_\nu$. One easily verifies the following formulas

$$(a)\ \ \check{\rho}_{H(x+\hat{\nu}),\mu} = e^{i\pi(\underline{e}_\nu-e_\nu,e_\mu)}\check{\rho}_{H(x),\mu} \qquad (b)\check{\rho}_{\mu,H(x+\hat{\nu})} = e^{i\pi(\underline{e}_\mu-e_\mu,e_\nu)}\check{\rho}_{\mu,H(x)}$$
$$(c)\ \ e^{i\pi(\underline{e}_\mu,e_\nu)+i\pi(\underline{e}_\nu,e_\mu)} = e^{i\pi(e_{1234}+e_\mu,e_\nu)} \tag{120}$$

With help of these we get the behaviour of the three factors of $A(x,\mathcal{F})$ under translation. From associativity, Eq. (108), we get

$$\check{\rho}_{H(x),F} = \check{\rho}_{H(x),\sigma_1}\check{\rho}_{H(x+\check{\sigma}_1),\sigma_2},...,\check{\rho}_{H(x+\check{\sigma}_1+...\check{\sigma}_{n-1}),\sigma_n} \qquad for \qquad F=(\sigma_1,...,\sigma_n)$$

and therefore by Eq. (120.a)

$$\check{\rho}_{H(x+\hat{\nu}),F} = e^{i\pi(\underline{e}_\nu-e_\nu,e_F)}\check{\rho}_{H(x),F}, \qquad e_F = \sum_{\mu_i\in F}e_{\mu_i} \tag{121}$$

For the second factor, $\prod_{l_\mu\in\mathcal{F}_{x,x+e_f}}\check{\rho}(l_\mu) = \check{\rho}_{\mu_1,H(x)}\check{\rho}_{\mu_2,H(x+\hat{\mu}_1)},...,\check{\rho}_{\mu_f,H(x+\hat{\mu}_1+...\hat{\mu}_{f-1})}$, with the convention $\rho_{H(x),-\mu} = -\rho_{H(x),\mu}$, we get with help of Eq. (120.b)

$$\prod_{l_\mu\in\mathcal{F}_{x+\hat{\nu},x+\hat{\nu}+e_f}}\check{\rho}(l_\mu) = e^{i\pi(\sum_{i=1}^f(\underline{e}_{\mu_i}-e_{\mu_i},e_\nu)}\prod_{l_\mu\in\mathcal{F}_{x,x+e_f}}\check{\rho}(l_\mu) \tag{122}$$

For the third factor we get immediately $e^{i\pi(\hat{f}e_{1234}+e_f,x+\hat{\nu})} = e^{i\pi(\hat{f}e_{1234}+e_f,\nu)}e^{i\pi(\hat{f}e_{1234}+e_f,x)}$. Multiplying the phases which the three factors of $A(x,\mathcal{F})$ get due to translation and using $\sum_{\sigma_i}\equiv\sum_{\mu_i}(mod2)$ and Eq. (120.c), proves the translation invariance of $A(x,\mathcal{F})$.

In order to prove the second part of the lemma we put x=0. Then we have the following

$$A(\mathcal{F}) = (-1)^{n(\mathcal{F})}\prod_{i=1}^f\check{\rho}_{|\mu_i|,H(\mu_1+...\mu_{i-1})} \tag{123}$$

$n(\mathcal{F})$ is the number of negativ μ_i. For the evaluation of $A(R^{-1}\mathcal{F})$ we use Eq. (112), which leads to

$$A(R^{-1}\mathcal{F}) = (-1)^{n(\mathcal{F})+n(R^{-1}\mathcal{F})}\prod_{i=1}^{f}\rho(R,|\mu_i|)\rho(R,|\mu_1|\triangle|\mu_2|...,\triangle|\mu_f|)A(\mathcal{F})$$

We remark that $\rho(R,|\mu_1|\triangle|\mu_2|...,\triangle|\mu_f|) = \rho(R,F)$, and that the product $\prod_{i=1}^{f}\rho(R,|\mu_i|)$ is the product of minus signs generated by the rotation R, i.e. $(-1)^{n(\mathcal{F})+n(R^{-1}\mathcal{F})}$. This completes the proof of the Lemma. In the future we restrict $\mathcal{F}'$ to such multi-links which are generated from $\mathcal{F}$ by rotation: $\mathcal{F}' = R^{-1}\mathcal{F}$. Then we can use the lemma to evaluate the factor $A^{\mathcal{F}}(x)A^{\mathcal{F}'}(0)$ as function of $\rho(R,F)$ and $\rho(R,F')$. We rewrite Eq. (119):

$$\left\langle :\mathcal{M}^{L,\mathcal{F}}(x)\mathcal{M}^{L,\mathcal{F}'}(0):\right\rangle = \alpha^{2f}\rho(R,F)\rho(R,F')\sum_{\overline{x}\overline{y}}\frac{1}{m^2}\int d\overline{p}\exp -i(\overline{p},x)\tilde{\eta}^{\dagger}(\overline{x})\frac{1}{1+\frac{1}{\alpha^2}M^L(p)}\tilde{\eta}(\overline{y}) \tag{124}$$

$$\tilde{\eta}^{\dagger}(\overline{x}) = \eta^{\dagger}(\overline{x})e^{-i\pi(p+e_L,\triangle x)} \qquad ,\tilde{\eta}(\overline{y}) = \eta(\overline{y})e^{+i\pi(p+e_L,\triangle y)} \qquad \triangle x = \overline{x}-x,$$

with $\eta^{\dagger}(\overline{x}),\eta(\overline{y})$ defined in Eq. (118). From Eq. (97) we get for $S_{\mu\nu}(x,y)$:

$$-\frac{1}{\alpha^2}\sum_{\rho\neq-\mu}S_{\rho\nu}(x-\hat{\mu},y) = S_{\mu\nu}(x,y) \tag{125}$$

and hence for $\overline{S}_{\mu\nu}(x,y) = e^{i\pi(x-y,e_L)}S_{\mu\nu}(x,y)$:

$$-\frac{1}{\alpha^2}\sum_{\rho\neq-\mu}\overline{S}_{\rho\nu}(x-\hat{\mu},y) = e^{i\pi(e_\mu,e_L)}\overline{S}_{\mu\nu}(x,y) \tag{126}$$

This formula helps us to perform the summations over μ (following from Eq. (118), $\overline{x} and \overline{y}$ in Eq. (124). If we do the sum over μ at the point $\overline{x} = x+e_f$, Eq. (126) shows that this results in the substitution $:\tilde{\eta}^{\dagger}(x+e_f) \longrightarrow -\xi_{-\mu_f}e^{-i\pi(p+e_L,e_f-\hat{\mu}_f)}$. Adding and subtracting this term and summing over μ at the point $\overline{x} = x+e_f-\hat{\mu}_f$, with help of Eq. (126), leads to the substitution: $\tilde{\eta}^{\dagger}(x+e_\mu-\hat{\mu}_f) \longrightarrow -\xi_{-\mu_{f-1}}e^{-i\pi(p+e_L,e_f-\hat{\mu}_f-\hat{\mu}_{f-1})}$. Following this procedure and shifting the integration variable $\overline{p}$ to $\overline{p}+e_L$, we get our main result which we anticipated in Eq. (115) and which we formulate as a proposition.

PROPOSITION: The 2-point function of the multi-link operator $\mathcal{M}^{L,\mathcal{F}}(x)$, Eq. (102), is, in zero order strong coupling approximation,

$$\left\langle :\mathcal{M}^{L,\mathcal{F}}(x)\mathcal{M}^{L,\mathcal{F}'}(0):\right\rangle = \alpha^{2f}\rho(R,F)\frac{1}{m^2}\int dp\exp -i(p,x)\eta^{\dagger}(p,\mathcal{F})\frac{1}{1+\frac{1}{\alpha^2}M^L(p)}\eta(p,\mathcal{F}') \tag{127}$$

where

$$\mathcal{F}' = R^{-1}\mathcal{F}, \qquad \eta^{\dagger}(p,\mathcal{F}) = -2\sum_{\mu_i\in\mathcal{F}}\xi^{\dagger}_{-\mu_i}e^{-i\pi\left(\overline{p},\sum_{i=1}^{i-1}\hat{\mu}_i\right)}+\sum_{all\nu}\xi^{\dagger}_{\nu}$$

$$\eta(p,\mathcal{F}') = -2\sum_{\mu_i\in\mathcal{F}'}\xi_{\mu'_i}e^{i\pi\left(\overline{p},\sum_{i=1}^{i-1}\hat{\mu'}_i\right)}+\sum_{all\nu}\xi_{\nu}$$

and $M^L(p) = M(\overline{p}) = M(p+e_L)$ defined in Eq. (99).

4.3 The group theoretical analysis of the particle content and the spectrum at $\beta = 0$

Now we determine the quantum numbers of the particle states described by the propagator of the multi-link operators calculated above. For this we construct "irreducible field operators" with transformation

properties which match the quantum numbers of the particles. These quantum numbers of the particles on the lattice are described in Sec. 2.3 by the irreducible representations of the symmetry group $\mathcal{G}_L$. They are the "momentum star", the "flavour orbit" and the "reduced spin". We consider in our calculation mainly particle states belonging to the momentum star St_4 with a reference momentum $\overline{p} = (0,0,0,iE)$. In this case E can be interpreted most directly as energy in the "rest system", i.e. as mass in the continuum limit. We expect from continuity of the propagator in the momentum p that the consideration of other stars does not lead to qualitatively different physical results. In Eq. (106) we showed how the multilink operator transforms under flavour transformations. This transformation rule (after Fourier transformation) is the same as that of a state vector of the irreducible representations of $\mathcal{G}_L$ described in Eq. (65). We have already defined the multilink fields in such a way that their flavour transformation property matches those of the particle states! This means that the fields "carry" the same flavour quantum number L as the particles.

In order to match also the reduced spin σ, the following meson operators are considered

$$\mathcal{M}^{L,\sigma}_{m,m'}(x) = \sum_{s\in S_{4,L}}D^{\sigma}_{m,m'}(s^{-1})\rho(s,F)\mathcal{M}^{L,S^{-1}\mathcal{F}} \tag{128}$$

where $S_{4,L}$ is the group of rotations of W_3 which leave L invariant: $S_{4,L} = \{s|s^{-1}L = L, s\in W_3\}$. $D^{\sigma}_{m,m'}(s)$ denotes an irreducible representation of $S_{4,L}$. The transformation law of $\mathcal{M}^{L,\sigma}_{m,m'}(x)$ under the rotations of $S_{4,L}$ follows from Eq. (111) by the usual calculation:

$$\left([s]\mathcal{M}^{L,\sigma}_{m,m'}(x)\right) = \sum_{\overline{m}}D^{\sigma}_{m,\overline{m}}(s)\mathcal{M}^{L,\sigma}_{\overline{m},m'}(s^{-1}x) \tag{129}$$

In the momentum star St_4 which we consider, $S_{4,\overline{L}}$ is the reduced spin group of the reference flavour $\overline{L}$ defined in Sec. 2.3. The Fourier transformed field $\tilde{\mathcal{M}}^{L,\sigma}_{m,m'} = \sum_x e^{ipx}\mathcal{M}^{L,\sigma}_{m,m'}(x), p = p_j = (0,0,0,iE)$ transforms according to Eq. (129) as

$$\left([s]\tilde{\mathcal{M}}^{L,\sigma}_{m,m'}(p_j)\right) = \sum_{\overline{m}}D^{\sigma}_{m,\overline{m}}(s)\tilde{\mathcal{M}}^{L,\sigma}_{\overline{m},m'}(p_j) \tag{130}$$

This means, precisely like the corresponding state vector with momentum p_j in the irreducible representation of $\mathcal{G}_L$ with reduced spin σ as described in Eq. (63).

According to the particle description by quantum fields, a pole in the 2-point function of $\tilde{\mathcal{M}}^{L,\sigma}_{m,m'}(p_j)$ indicates the existence of a particle with the lattice quantum numbers: flavour L and reduced spin σ. Thus we calculate this 2-point function with help of the expression Eq. (127). Inserting the definition Eq. (128) into Eq. (127) we get for the propagator of $\mathcal{M}^{L,\sigma}_{m,n}(x)$ in momentum space

$$\left\langle :\mathcal{M}^{L,\sigma}(p)\mathcal{M}^{L,\sigma}(0):\right\rangle \approx (R\eta(p,\mathcal{F}))^{\dagger}\frac{1}{1+\frac{1}{\alpha^2}M^L(p)}R\eta(p,\mathcal{F}') \tag{131}$$

with

$$R\eta(p,\mathcal{F}') = \sum_{s\in S_{4,L}}D^{\sigma}_{m,n}(s^{-1})\left(-2\sum_{\mu_i\in\mathcal{F}'}\xi_{-s\mu'_i}e^{i\pi\left(\overline{p},\sum_{i=1}^{i-1}s\hat{\mu'}_i\right)}+\sum_{all\nu}\xi_{s\nu}\right) \tag{132}$$

Because of $\rho^2(S,F) = 1$, the $\rho(S,F)$ in Eq. (127) is compensated by the $\rho(S,F)$ in the definition of the meson field Eq. (102) . With $s\mu_i$ we denote the permutation of the step directions μ under the rotations S. $U(s)\xi_\nu = \xi_{s\nu}$ defines a representation $s \to U(s)$ of $S_{4,L}$ in the step space. As usual $\sum_s D^{\sigma}_{mn}(s^{-1})\xi_{s\mu}$, for fixed n, projects on an invariant subspace in which U(s) is represented as the irreducible representation $D^{\sigma}(s)$. Since $\overline{p} = p_j+e_L$ is invariant under $s\in S_{4,L}$, the expression $e^{i\pi\left(\overline{p},\sum s\hat{\mu}_i\right)}$ is independent of s and the matrix $M^L_{\mu\nu}(p) = e^{i\overline{\pi}\mu}(1-\delta_{\mu,-\nu})$ commutes with U(s). Combining these two facts, namely that $M^L_{\mu,\nu}(p)$ decomposes into blocks with respect to the quantum number σ and

#	$\vec{L}$	σ	$\cosh E$	$(j^{\pi C},(n)_{SU(4)})$
1	$(1,1,1,1)$	$(1^+)_{W_3}$	$M_0(\alpha_0)$	$(0^{--},15)$
2	$(1,1,0,1)$	$(1^+)_{D_4}$	$M_0(\alpha_0)+2$	$(1^{-\cdots},15)$
3	$(0,0,1,1)$	$(1^+)_{D_4}$	$M_0(\alpha_0)+4$	$(1^{-\cdots},15)$
4	$(0,0,0,1)$	$(1^+)_{W_3}$	$M_0(\alpha_0)+6$	$(0^{-+},15)$

Table 1: Quantum Numbers of the Zero Order Mesons

that the vectors (Rη),Eq. (132), belong to a subspace with quantum number σ, leads to the result that the propagator of the meson field $\mathcal{M}^{L,\sigma}(x)$ has a pole, only if $M^L(p)$ has an eigenvalue $-\frac{1}{\alpha^2}$ for an eigenvector lying in an invariant subspace with the quantum number σ, i.e. in an invariant subspace belonging to the representation $D^\sigma(s)$ in the step space of the reduced spin group $S_{4,L}$. An explicite calculation leads to the following decomposition of the representation U(s) in the step space:

$$U(s) \simeq 3(1^+)_{W_3} + (2^+)_{W_3} + (3'^+)_{W_3} \tag{133}$$

for the reduced spin group W_3 associated with the singlet flavour orbits, Eq. (67), and to

$$U(s) \simeq 4(1^+)_{D_4} + (1^-)_{D_4} + (1'''^+)_{D_4} + (2^-)_{D_4} \tag{134}$$

for the reduced spin group $D_4 \times Z_2$ of the triplet flavour orbits, Eq. (68), where $M_0 = \frac{\alpha^4-6\alpha^2+7}{2\alpha^2}$. At the zero order $\alpha = \alpha_0 = m + \sqrt{m^2+2d-1}$. It is easy to see, by substituting α in M_0 for $m = 0$, that the state # 1 having $\cosh E = 1$ is just the expected Goldstone boson, which comes out as result of the spontaneous chiral symmetry breaking[10].

5 The Calculations in First Order

5.1 The 2-point function in first order of β

In this chapter we extend our considerations, which were restricted up to now to zero order strong coupling approximation, to include the terms in next order. First we consider the modifications of the resummed hopping parameter expansion explained for zero order in Chapter 3. For this we evaluate Eq. (80), after inserting Eq. (82), without setting $e^{-S_g} \equiv e^{-\beta\bar{S}_g}$ equal to one , as it was done in zero order. We get terms linear in β, if a combination of quark and antiquark paths of the hopping parameter expansion (Eq. (82)) encloses a single plaquette (see Fig.4) . In this case the integration over the lattice gluon fields , using the orthogonality relations ,Eq. (79) , gives a result different from zero, because the parallel transports along the $q\bar{q}$-lines around the single plaquette get "compensated" ("screened") by the plaquette terms of the action (Eq. (45)) in the expansion $e^{-\beta S_g} \sim 1 - \beta S_g$. The evaluation of Eq. (80) leads to an expression of type Eq. (83):

$$\langle : \mathcal{M}(x)\mathcal{M}(y) : \rangle = \frac{1}{m^2}\sum_{N'}^{\infty}\left(\frac{-1}{4m^2}\right)^{N'} T^{(1)}_{x,y}(N',\beta). \tag{135}$$

Here $T^{(1)}_{x,y}(N',\beta)$ is the weighted number of $q\bar{q}$ zeroth and first order graphs of "length" N' connecting x and y. The expression of the propagator, Eq. (135), also has to include some of the higher order terms. For the summation in Eq. (135) we proceed like in Chapt. 3 by separating the $q\bar{q}$-graphs in "dressings" of the q lines, ($\bar{q}$-lines) and trunks. The first order correction to the "renormalized" hopping parameter α^{-1}, Eq. (95), follows from the inclusion of a "plaquette" in the tree graphs of the dressing. Examples of such graphs are the same as those shown in fig.4 but include only one plaquette. O. Martin [10] has calculated α up to first order by methods similar to those explained in Chapter 3.

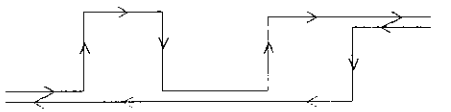
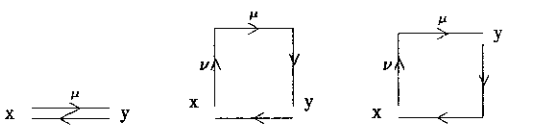


Figure 4: A First Order $q\bar{q}$ Path



(a): Step $\langle\mu\rangle$ (b): Step $\langle\nu\mu-\nu\rangle$ (c): Step $\langle\mu\nu\rangle$

Figure 5: The First Order Steps

His result is

$$\alpha^2 = \alpha_0^2 + \frac{16(d-1)^3}{\alpha_0^2(\alpha_0^2+2d-1)}\beta' \equiv a_0 + \beta D, \qquad ,\alpha_0 = m + \sqrt{m^2+2d-1} \tag{136}$$

with $\beta' = \frac{\beta}{N_{colour}} \equiv \frac{\beta}{3}$ for QCD .
After the renormalization of the hopping parameter we can represent Eq. (135) in the form

$$\langle : \mathcal{M}(x)\mathcal{M}(y) : \rangle = \frac{1}{m^2}\sum_{N}^{\infty}\left(\frac{-1}{\alpha^2}\right)^N B^{(1)}_{x,y}\left(N,\frac{\beta'}{\alpha_0^2}\right). \tag{137}$$

where $B^{(1)}_{x,y}(N,\frac{\beta'}{\alpha_0^2})$ refers to trunks generated by N steps. In first order calculation there are three types of basis steps denoted by $e_\mu, e_{\nu,\mu,-\nu}, e_{\mu,\nu}$,or $\langle\mu\rangle, \langle\nu,\mu,-\nu\rangle, \langle\mu,\nu\rangle$ (see Fig.5) . The first two types shift x to $x+\mu$, the last type shifts x to $x+e_\mu+e_\nu$. Any path from 0 to x which is a combination of these steps we call a first order trunk. As in Chapt. 3 we classify the trunks by their last steps. In four dimension we have 8 different steps of the first type. Because of $\mu \neq \pm\nu$ there are 48 different steps of the second type. The last type has 24 different steps because it is symmetric in the indices μ,ν. This makes all together 8+24+48=80 different (first order) steps. We represent the paths of N steps as a vector with components refering to this classification

$$\vec{B}_{x,0}\left(n,\frac{\beta'}{\alpha_0^2}\right) = \left(B^{(n)}_{\mu}, B^{(n)}_{\mu\nu}, B^{(n)}_{\mu\nu-\mu}\right) \tag{138}$$

Its components are the weighted numbers of the paths of the corresponding type. A step enclosing a plaquette contributes a factor $\frac{\beta'}{\alpha_0^2}$ to the weight. The factor β' results from the integration over the

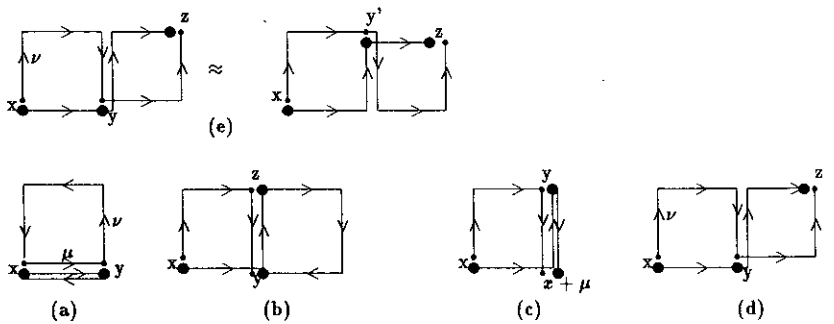


Figure 6: The non-backtracking Conditions

lattice gauge field. The factor $\frac{1}{\alpha_\delta^2}$ follows from the fact that a step with plaquette containes four links instead of two as in zero order. The equation in (137) is the sum of these components. We need to find a recursion formula which relates $B_{s'}^{(n)}$ to $B_s^{(n-1)}$

$$B_{x,s}^{(n)} = \sum_{s'} M_{ss'} B_{x-e_{s'},s'}^{(n-1)} \tag{139}$$

where s runs over the index of the 80-dimensional step space, and also denotes the shift by the steps. In our method of organizing the summation over all paths in dressings and summation over trunks, one of the main problems is to avoid overcounting. This leads in first order to the following conditions for consecutive steps described by the step matrix.

1. Step $\langle\mu\rangle$ cannot be followed by step $\langle-\mu\rangle$ or by step $\langle\nu-\mu-\nu\rangle$. (Fig.6.a)
2. A step : $\langle\mu,\nu\rangle$ cannot be followed by steps : $\langle-\mu,-\nu\rangle$, $\langle-\mu\rangle$, $\langle-\nu\rangle$, (Fig.6.b) nor by steps $\langle\mu',-\mu,-\mu'\rangle$, $\langle\mu',-\nu,-\mu'\rangle$ (Fig.6.c).
3. If we have the steps $\langle\mu\nu\rangle$ or $\langle\mu\nu-\mu\rangle$, we have a factor 2 in the corresponding step matrix element, because the quark and antiquark lines of the trunk can be interchanged (Fig.5.b,Fig.5.c). However if the step $\langle\nu,\mu,-\nu\rangle$ is followed by a step $\langle\mu',\nu\rangle$ we have only a factor 1, because the interchange would lead to backtracking (Fig.6.d).
4. Since we have the equivalence of the two consecutive steps $\langle\nu,\mu\rangle\oplus\langle-\nu,\mu,\nu\rangle$ with $\langle\nu,\mu,-\nu\rangle\oplus\langle\mu,\nu\rangle$, we have to omit one of these (Fig.6.e). From these conditions we construct the step matrix $M_{ss'}$ up to first order in β'. We give the matrix $M_{ss'}$ explicitly for the Fourier transform $\tilde{B}_{p,s}^{(n)}$:

$$\tilde{B}_{p,s}^{(n)} = \sum_{s'} M_{ss'}(p)\tilde{B}_{p,s'}^{(n-1)} \tag{140}$$

For the shift $\langle\mu\rangle$ and $\langle\nu,\mu,-\nu\rangle$ we get then a factor $e^{ip_\mu}, p_{-\mu} = -p_\mu$; for a step $\langle\mu\nu\rangle$ a factor $e^{i(p_\mu+p_\nu)}$. Thus with the summation convention for repeated indices the matrix $M_{ss'}$ is given by:

$$\begin{aligned}
\tilde{B}_{\mu'}^{(n)} &= e^{ip_{\mu'}}\left\{(1-\delta_{\mu'-\mu})\tilde{B}_{\mu}^{(n-1)} + (1-\delta_{\mu'-\mu})(1-\delta_{\mu'-\nu})\tilde{B}_{\mu\nu}^{(n-1)} + (1-\delta_{\mu'-\nu})(1-\delta_{\mu'\mu})\tilde{B}_{\mu\nu-\mu}^{(n-1)}\right\} \\
\tilde{B}_{\mu'\nu'}^{(n)} &= \beta'' e^{ip_{\mu'}+ip_{\nu'}}\left\{2(1-\delta_{-\mu'\mu})(1-\delta_{-\nu'\mu})\tilde{B}_{\mu}^{(n-1)}\right. \\
&+ \left. 2(1-\delta_{\mu'-\mu})(1-\delta_{\mu'-\nu})(1-\delta_{\nu'-\mu})(1-\delta_{\nu'-\nu})\tilde{B}_{\mu\nu}^{(n-1)}\right. \\
&+ \left.(1-\delta_{\mu'-\nu})(1-\delta_{\nu'-\nu})(2-\delta_{\nu'\mu}-\delta_{\mu'\mu})\tilde{B}_{\mu\nu-\mu}^{(n-1)}\right\} \\
\tilde{B}_{\mu'\nu'-\mu'}^{(n)} &= \beta'' e^{ip_{\nu'}}\left\{2(1-\delta_{-\nu'\mu})(1-\delta_{\mu'\mu})\tilde{B}_{\mu}^{(n-1)} + 2(1-\delta_{\nu'-\nu})(1-\delta_{\nu'-\mu})\tilde{B}_{\mu\nu}^{(n-1)}\right. \\
&+ \left.(1-\delta_{\nu'-\nu})(1-\delta_{\nu'\mu})(2-\delta_{\mu'\mu})\tilde{B}_{\mu\nu-\mu}^{(n-1)}\right\}
\end{aligned} \tag{141}$$

The Kronecker symbols δ in this equation guarantee the fullfilment of the above conditions 1 to 4. The factor $\beta'' = \frac{\beta'}{\alpha_\delta^2}$ associated with first order steps was explained above. The recursion relation Eq. (140) and the explicitly written matrix Eq (141) allow us the summation in Eq. (137), with the result

$$\begin{aligned}
\langle:\mathcal{M}(x)\mathcal{M}(0):\rangle &= \frac{1}{m^2}\int dp e^{-i(p,x)}\sum_{n=0}^{\infty}\left(\frac{-1}{\alpha^2}\right)^n \xi^T M^n(p)\xi \\
&= \frac{1}{m^2}\int dp e^{-i(p,x)}\xi^T\left(\frac{1}{1+\frac{1}{\alpha^2}M(p)}\right)\xi
\end{aligned} \tag{142}$$

In the case of the propagator for local fields we have for the initial and final step vector $\xi : \xi_s \equiv 1, s = \langle\mu\rangle, \langle\mu,\nu,-\mu\rangle, \langle\mu,\nu\rangle$. We discuss the more general cases connected with multi-link operators of different flavours in the following sections. Finally we want to emphasize again that the step matrix above generates trunks related to higher orders in β'. However the different plaquettes which may appear in the same trunk are separated.

Now we analyze the symmetry properties of $M^L(p) = M(p+e_L)$. From the four dimensional W_4 symmetry of the lattice it follows that under four dimensional rotations of the lattice any vector of the step space $\vec{B}^n(p) = \left(\tilde{B}_\mu^{(n)}(p), \tilde{B}_{\mu\nu}^{(n)}(p), \tilde{B}_{\mu\nu-\mu}^{(n)}(p)\right)$ transforms as :

$$[s]\vec{B}^n(p) \Longrightarrow \left(\tilde{B}_{s^{-1}\mu}^{(n)}(s^{-1}p), \tilde{B}_{s^{-1}\mu s^{-1}\nu}^{(n)}(s^{-1}p), \tilde{B}_{s^{-1}\mu s^{-1}\nu-s^{-1}\mu}^{(n)}(s^{-1}p)\right) \tag{143}$$

where $s \in W_4$,W_4-four rotational group and $p = (\vec{p}, p_4)$.

In the center mass system ($\vec{p} = 0$), we see from (143) that the matrix $M^L(p)$is invariant under the spacial rotations of W_3, which means:

$$D(s')M = MD(s') \tag{144}$$

if $s' \in W_3, s'\vec{p} = 0, s'e_L = e_L$ and $D(s')$ is the representation of the group W_3 in the step space . This representation is defined on the step basic $e_{\mu..\nu}$ in the following way:

$$D(s')\left(e_\mu, e_{\mu\nu}, e_{\mu\nu-\mu}\right) = \left(e_{s'^{-1}\mu}, e_{s'^{-1}\mu s'^{-1}\nu}, e_{s'^{-1}\mu s'^{-1}\nu-s'^{-1}\mu}\right) \tag{145}$$

Now we will use the known group theoretical techniques for choosing a new basis of the step space in such a way, that the representation in the step space decomposes into invariant subspaces of $D(s')$. According to Schur's Lemma the matrix M^L decomposes then into block matrices,which correspond to the irreducible representations of W_3 [19]. The decomposition into these invariant subspaces corresponds to a representation of $D(s')$ defined in (145) by a direct sum of irreducible representations of W_3 given in the Appendix .

$$D(s) \simeq 11(1^+)_{W_3} + 10(3^-)_{W_3} + 10(2^+)_{W_3} + (1'^+)_{W_3} + (3^+)_{W_3} + 3(3'^-)_{W_3} + 2(3'^+)_{W_3} \tag{146}$$

The same procedure can be applied for the symmtry group $S_{4,\overline{L}}$, which holds for the triplet flavours with representative element of the flavour orbit $e_{\overline{L}}$. In this case the $D(s')$ decomposes into irreducible representations of $S_{4,\overline{L}} \equiv D_4 \times Z_2$ as:

$$U(s) \simeq 21(1^+)_{D_4} + 10(1^-)_{D_4} + 11(1'''^+)_{D_4} + 13(2^-)_{D_4} + (1'^+)_{D_4} + 3(1'''^-)_{D_4} + 3(2^+)_{D_4} + 2(1''^+)_{D_4} \tag{147}$$

In order to get the pole of the meson propagator we solve Eq. (101) for every one of these matrices. For the evaluation of the determinants of the matrices which have dimensions up to 21 , we used partly the 'Reduce3 Program' on the computer. The results are given in the Tables.1,2,3 . For $\bar{p} = (0,0,0,E)$, the sub-matrices $M^{\bar{L}}(\bar{p})$, in the subspaces belonging to the last four representations in Eq. (146) and in Eq.(147), do not depend on the energy E, therefore they do not give any poles. Surely they give poles in the second order calculations at β. So there are poles in the subspaces of the representations $(1^+)_{W_3}, (3^-)_{W_3}, (2^+)_{W_3}, (1^+)_{D_4}, (1^-)_{D_4}, (1'^+)_{D_4}, (2^-)_{D_4}$. It will turn out that these representations determine the reduced spin σ in Tables 1,3 . For some of the $\bar{L}$, the poles are at $coshE > 0$; for the rest,the poles are at $\cosh E < 0$. The first ones are listed in Tables 1,3 . The latter ones have no physical meaning, they belong to $\bar{L}$'s with the opposite fourth component.

The considerations up to now are purely formal. These poles still have not yet a physical interpretation because they are not associated with qantum numbers of precisely defined expectation values of field operators. These will be constructed in the following section from the one link-operators , which seem to be enough to describe the physics at the first order approximation.

5.2 The first order approximation of the one link-field operator

For the one link-field operator :

$$\mathcal{M}^{L,\mu}(x) = e^{i\pi(e_L+e_{1234}+e_\mu,x)}\overline{\chi}(x)\tilde{\rho}_{H(x),\nu}U(x,\hat{\mu})\chi(x+e_\mu) \tag{148}$$

we calculate the two point function $\left\langle : \mathcal{M}^{L,\mu}(x)\mathcal{M}^{L,\mu'}(0) : \right\rangle$ up to the first order of β. As we mentioned before , the quark line starts from the point $e_{\mu'}$ and goes to the point x, the anti-quark line goes from $x + e_\mu$ to 0. The two lines are non-backtracking and can include at most the area of one plaquette. Because of the links $U(x,\hat{\mu})$, and $U(0,\hat{\mu'})$ in the expressions of $\mathcal{M}^{L,\mu}(x)$ and $\mathcal{M}^{L,\mu'}(x)$, the two lines (quark and anti-quark lines) also have to screen these gauge fields up to the first order of β . We show in Fig.7 the possible screenings up to first order. In this figure we see that the trunk until the point y is of the same form as we have already discussed above . Explicitly , it could be characterized by its last step before y, which could appear in 80 different forms. The possible steps after y are then restricted, by the conditions (1-4) and by the form of the screening which follows. In Fig.7 the possible screenings are given together with the steps $B_{\mu\nu}$ etc. after the point y. The factor 2 in Fig.(a) comes from the interchanging of two lines . In the Fig. (a) the path from x to $x + e_\mu$ and also the contribution of figure (d) are included in the renormalization of the hopping parameter up to the first order.

There are signs related to trunks with given steps after y and the screenings described in Fig.7 above. These signs are composed by the sign factors in the definition of the link operator (148) and by the $\check{\rho}(l_\mu)$ of the links of the quark lines. For convenience we represent this sign by factor A(x) given in (119) as:

$$A(x,\mu) = e^{i\pi(e_{1234}+e_\mu,x)}\check{\rho}_{H(x),\mu}\check{\rho}_{\mu,H(x)} \tag{149}$$

times the signs given in Fig.7.

In chapter 4 we gave the transformation properties of $A(x,\mathcal{F})$ for general multi-links. Now we write the analog of Eq (124) for the one link field operators. We get:

$$\left\langle : \mathcal{M}^{L,\mu}(x)\mathcal{M}^{L,\mu'}(0) : \right\rangle = C\rho(R,\mu)\rho(R,\mu')\int d\bar{p}\exp -i(\bar{p},x)\bar{\eta}^T(\bar{x})\frac{1}{1+\frac{1}{\alpha^2}M^L(p)}\bar{\eta}(\bar{y}) \tag{150}$$

where $\bar{\eta} = \sum_s \eta_s$, the sum is taken over the last steps s shown in Fig.7. $M^L(p)$ is the 80 x 80 matrix defined in (141).

The expression $S_{ss'}$ defined as :

$$S_{ss'} = \eta_s^T\frac{1}{1+\frac{1}{\alpha^2}M^L(p)}\eta_{s'} \tag{151}$$

is the Fourier representation of the weighted number of paths from 0 to x with initial step $\langle s\rangle$ and final step $\langle s'\rangle$.

From Fig.1 we get explicitly the expression of $\bar{\eta}$ as:

$$\bar{\eta}(\mu) = \sum_s \eta_s + 2\eta_{-\mu} + \sum_\nu \left(\eta_{\mu\nu-\mu} + \eta_{-\mu\nu} + \frac{3}{2}\eta_{\nu-\mu-\nu}\right). \tag{152}$$

Substituting (152) in (150) we can calculate the two point function of the meson field operator, Eq.(114), $\tilde{\mathcal{M}}^{L,\sigma}_{m,m'}(p_j)$ of the lattice quantum numbers σ and L as:

$$\left\langle : \mathcal{M}^{L,\sigma}(p)\mathcal{M}^{L,\sigma}(0) : \right\rangle \approx (\bar{\eta}^\sigma(\mu))^T\frac{1}{1+\frac{1}{\alpha^2}M^L(p)}\bar{\eta}^\sigma(\mu') \tag{153}$$

Here $\bar{\eta}^\sigma(\mu)$ is defined similar to Eq. (132) as

$$\bar{\eta}^\sigma(\mu) = \sum_{s\in S_{4,L}} D^\sigma_{m,n}(s^{-1})\bar{\eta}(s^{-1}\mu) \tag{154}$$

From the physics point of view it is the most important step of our calculation to determine the quantum numbers $\bar{L},\sigma$ for which these meson propagators have poles. We use the decomposition of the matrix $M^{\bar{L}}(p)$ into submatrices acting in subspaces of given symmetry described by representations D^σ as described in Sect.5.1 and in the Appendix . By the group theoretical structure of $\bar{\eta}^\sigma$,Eq. (154), one sees that the vector in the step space $\bar{\eta}^\sigma$ has σ-symmetry. It follows then from the orthogonality of vectors with different symmetries, that according to Eq. (153), the meson fields $\mathcal{M}^{L,\sigma}(p)$ couple only to poles which lay in a subspace of the step space with the same symmetry σ and L. The position of the poles is determined by the solution of Eq. (101).

In the Tables 1,3 , we summarize the masses, i.e.cosh E as function of the 'renormalized quark mass' α (Eq. (136)) and of β' together with the lattice quantum numbers σ and $\bar{L}$. The correspondence to the continumm quantum numbers $(j^{\pi C},(n)_{SU(4)})$, also included in the tables, we discuss in the next Section. For the states which appear already in zero order the quantum numbers are given in Table 1, Sect.4.3. The first order corrections to $\cosh E$ can be calculated from the following equation:

$$4C_2\ cosh^2E + 2C_1\cosh E + C_0 - 2C_2 = 0$$

The different C_0, C_1, C_2 are defined for every one of these states in Table 2. The $a = \alpha^2, a_0 = \alpha_0^2$ are defined in Eq. (136). The solution of the last equation in the first order of β' generaly is given by:

$$\cosh E = -\frac{\overline{C}_0}{2\overline{C}_1} - \frac{\beta'}{\overline{C}_1}(\dot{C}_1 y_0 + \frac{\dot{C}_0}{2} + 2\overline{C}_2(y_0^2 - 1/2)) \qquad y_0 = -\frac{\overline{C}_0}{2\overline{C}_1} \tag{155}$$

where we redefined the values :

$$C_2 = \beta'\overline{C}_2 \qquad C_1 = \overline{C}_1 + \beta' C_1' \qquad C_0 = \overline{C}_0 + \beta' C_0'$$

and

$$\dot{C}_1 = \frac{\partial\overline{C}_1}{\partial a}D + C_1' \qquad \dot{C}_0 = \frac{\partial\overline{C}_0}{\partial a}D + C_0'$$

The states # 5-14 given in Table 3 appear for the first time in the first order calculation.

#	$\cosh E$
1	$C_2 = (a_0^2 + a_0)\beta'$ $C_1 = -a^4 - a^3 + \beta'(-a_0^3 - 2a_0^2 + 60a_0 - 7)$ $C_0 = a^5 - 5a^4 + a^3 + 7a^2 + \beta'(-29a_0^3 - 17a_0^2 + 45a_0 + 41)$
2	$C_2 = \beta'(2a_0^3 + 4a_0^2 + 2a_0)$ $C_1 = a^5 - a^3 + \beta'(6a_0^4 + 27a_0^3 + 12a_0^2 + 17a_0 - 6)$ $C_0 = -a^6 + 2a^5 - 6a^4 - 2a^3 + 7a^2 + \beta'(17a_0^4 - 50a_0^3 - 94a_0^2 - 98a_0 - 3)$
3	$C_2 = \beta'(2a_0^3 + 4a_0^2 + 2a_0)$ $C_1 = a^5 - a^3 + \beta'(14a_0^4 + 21a_0^3 - 12a_0^2 - 25a_0 + 2)$ $C_0 = -a^6 - 2a^5 - 6a^4 + 2a^3 + 7a^2 + \beta'(-9a_0^4 - 122a_0^3 + 6a_0^2 + 282a_0 + 19)$
4	$C_2 = \beta'(-3a_0^2 - a_0)$ $C_1 = -a^4 + a^3 + \beta'(-21a_0^3 + 30a_0^2 + 68a_0 - 17)$ $C_0 = a^5 + 5a^4 + a^3 - 7a^2 + \beta'(49a_0^3 - 5a_0^2 - 361a_0 + 61)$

Table 2: First order correction of the zero order Mesons

#	$\overline{L}$	σ	$\cosh E$	$(j^{\pi^C}, (n)_{SU(4)})$
5	$(0,0,0,1)$	$(3^-)_{W_3}$	$\frac{6a_0-8}{a_0-3} + \frac{a^2+a}{2\beta'(a_0-3)}$	$(1^{+-}, 15)$
6	$(1,1,1,0)$	$(3^-)_{W_3}$	$\frac{-2}{a_0-1} + \frac{a^2-a}{6\beta'(a_0-1)}$	$(1^{-+}, 15)$
7	$(0,0,0,0)$	$(2^+)_{W_3}$	$-\frac{5a_0+3}{3a_0+1} + \frac{a^2-a}{2\beta'(3a_0+1)}$	$(2^{++}, 1)$
8	$(1,1,1,1)$	$(2^+)_{W_3}$	$\frac{3a_0-1}{a_0+1} + \frac{a^2+a}{2\beta'(a_0+1)}$	$(2^{--}, 15)$
9	$(0,0,1,0)$	$(1^-)_{D_4}$	$\frac{a_0-7}{3(a_0-1)} + \frac{a^2-a}{6\beta'(a_0-1)}$	$(0^-, 15)$
10	$(1,1,0,1)$	$(1^-)_{D_4}$	$\frac{a^2+a}{2\beta'(a_0-3)} - \frac{7a_0+5}{(a_0-3)}$	$(0^+, 15)$
11	$(0,0,1,0)$	$(1'''^+)_{D_4}$	$\frac{a^2-a}{2\beta'(3a_0+1)} - \frac{7a_0-9}{2(3a_0+1)}$	$(2^+, 15)$
12	$(1,1,0,1)$	$(1'''^+)_{D_4}$	$\frac{11a_0+5}{2(a_0+1)} + \frac{a^2+a}{2\beta'(a_0+1)}$	$(2^-, 15)$
13	$(0,0,1,1)$	$(2^+)_{D_4}$	$\frac{a^2+a}{2\beta'(a_0-3)} + \frac{4a_0-2}{(a_0-3)}$	$(1^+, 15)$
14	$(1,1,0,0)$	$(2^+)_{D_4}$	$-\frac{4a_0+2}{3(a_0-1)} + \frac{a^2-a}{6\beta'(a_0-1)}$	$(1^-, 15)$

Table 3: Meson states appearing in First Order

6 Physical Discussion

In the following we analyze the results which we got from our calculations. These are summarized in the two Tables (1) and (3). The table (1) contains the 4 multiplets of states which appear already in the zero order calculation [11],[10]. It gives cosh E, E energy, as a function of $a = \alpha^2$ (136) , β', together with the lattice quantum numbers L and σ. E and m are expressed in units of the inverse lattice constant of the 'fine' lattice. The first order corrections to the energy of these states are given in Eq. (155) supplemented by Table 2.It is possible to see after lengthy calculation , that the state #1 still has zero mass for $m = 0$, as it has at the zero order. This justifies our interpretation of #1 as the Goldstone boson of the spontaneous chiral symmetry breaking [10]. It is also simple to note that most of the mesons get mass different from zero for zero quark mass. This gives us a hint to getting mass by totally dynamical effects . The Table (3) contains additional 10 multiplets of states which appear for the first time in the first order calculation. The multiplets # 5-8 are singlets, those from # 9-14 are triplets. They represent alltogether 22 particles. The masses of these multiplets as a function of β' for different quark masses m are also shown in the Fig.9 . These curves show clearly how the masses of the states # 5 - 14 go to infinity like $-\log\beta$ for $\beta \to 0$. There is an intuitive picture of this feature of the strong coupling approximation on the lattice. The quark antiquark pathes of the first order, Fig.4 , describe a relative motion of the quarks over one lattice distance. This relative motion admits states with more complex lattice quantum numbers, like those we find for the states # 5-14 in Table 3. In analogy to conventional quantum mechanical states, where higher angular momenta, radial quantum numbers etc. are characteristic for excited states, we may regard these states heuristically as 'excited lattice states'. This picture is in agreement with the usual interpretation of the dynamics of lattice QCD by a confining potential. In strong coupling approximation the string constant of the linear potential is $\kappa \sim -log\beta$. In such a potential the energy difference between ground state and excited states goes to infinity, if $\kappa \to \infty$ like in our case. However, the significance of this dynamical picture can be seen only from a detailed comparison between continuum and lattice physics.

There is a group theoretical method for relating lattice quantum numbers to continuum quantum numbers, i.e. to the quantum numbers of physical particles. We have described in Sect.2.2 how the symmetry group of the lattice $\mathcal{G}_L$ is a subgroup of the symmetry group G of the DKE in the continuum. The quantum numbers of the 'physical particles' described by geometric QCD, like spin, parity, SU(4)-multiplets are determined by the irreducible representations of $\mathcal{G}$: $(j^{\pi^C}, (n)_{SU(4)})$. Therefore the lattice approximations of these particles should have lattice quantum numbers $\overline{L}, \sigma$ etc. given by irreducible representations of $\mathcal{G}_L$, which are contained in the restrictions of the continuum representations to $\mathcal{G}_L$: $(j^{\pi^C}, (n)_{SU(4)})|_L$. The lattice quantum numbers related in this manner to the physical quantum numbers first were calculated by M.F.L. Golterman [14], and later in a more systematic way by W. Neudenberger, who uses the same notation as we do. The Table 4 is due to Neudenberger [22] It contains the representations with St_4,(see Sect.2.3), which are contained in continuum representations with low spins. From this we can read off immediately the continuum quantum numbers of lowest spin which could be associated with the 14 lattice multiplets of Table 1 and Table 3. These quantum numbers were added to these Tables.

The charge parity C needs some additional remarks. For local meson operators, and for the 1-link operators, C can be calculated easily from Eq. (113) . However, triplet states contain states of different C, like in the continuum iso-spin multiplets contain different C states, therefore C is not given for these multiplets .

The main question in the physical discussion of our results is of course, how well these represent the states of an SU(4) quark model. For this we compare the spectrum on the lattice up to the first order with that of an usual SU(4) quark model. This is shown in Fig.8 . We see, there is only partial agreement between the quantum numbers of the ground state and the first excited states. This illustrates also the limited justification of our picture of interpreting the states of Table 3 as excited states. For a comparison of the masses we have chosen the following procedure. For the physical states we have chosen the energy of the SU(2)-flavour triplets. In order to get the energy values of the

$\bar{p}$	$S_{j,4}$	$\bar{L}$	0^+	0^-	1^+	1^-
$(0,0,0,E)$	$Z_2 \times D_4$	$(1,0,0,0)$	$(1')^+$	$(1)^-$	$(1)^+,(2)^-$	$(2)^+,(1')^-$
		$(0,1,1,0)$	$(1')^+$	$(1)^-$	$(1)^+,(2)^-$	$(2)^+,(1')^-$
		$(1,0,0,1)$	$(1)^-$	$(1')^+$	$(2)^+,(1')^-$	$(1)^+,(2)^-$
		$(0,1,1,1)$	$(1)^-$	$(1')^+$	$(2)^+,(1')^-$	$(1)^+,(2)^-$
	W_3	$(0,0,0,1)$	$(1)^-$	$(1)^+$	$(3)^-$	$(3)^+$
		$(1,1,1,1)$	$(1)^-$	$(1)^+$	$(3)^-$	$(3)^+$
		$(1,1,1,0)$	$(1)^+$	$(1)^-$	$(3)^+$	$(3)^-$

2^+	2^-	3^+	3^-
$(1')^+,(1'')^+,(1''')^+,(2)^-$	$(1)^-,(1'')^-,(1''')^-,(2)^+$	$(1)^+,(1'')^+,(1''')^+,2(2)^-$	$(1')^-,(1'')^-,(1''')^-,2(2)^+$
$(1')^+,(1'')^+,(1''')^+,(2)^-$	$(1)^-,(1'')^-,(1''')^-,(2)^+$	$(1)^+,(1'')^+,(1''')^+,2(2)^-$	$(1')^-,(1'')^-,(1''')^-,2(2)^+$
$(1)^-,(1'')^-,(1''')^-,(2)^+$	$(1')^+,(1'')^+,(1''')^+,(2)^-$	$(1')^-,(1'')^-,(1''')^-,2(2)^+$	$(1)^+,(1'')^+,(1''')^+,2(2)^-$
$(1)^-,(1'')^-,(1''')^-,(2)^+$	$(1')^+,(1'')^+,(1''')^+,(2)^-$	$(1')^-,(1'')^-,(1''')^-,2(2)^+$	$(1)^+,(1'')^+,(1''')^+,2(2)^-$
$(2)^-,(3')^-$	$(2)^+,(3')^+$	$(1')^-,(3)^-,(3')^-$	$(1')^+,(3)^+,(3')^+$
$(2)^-,(3')^-$	$(2)^+,(3')^+$	$(1')^-,(3)^-,(3')^-$	$(1')^+,(3)^+,(3')^+$
$(2)^+,(3')^+$	$(2)^-,(3')^-$	$(1')^+,(3)^+,(3')^+$	$(1')^-,(3)^-,(3')^-$

Table 4: The Lattice quantum numbers and their corresponding continuum numbers

lattice states of Fig.8 for quark masses m = 0, $\beta' = 0.15$, we have to choose for the lattice constant of the fine lattice b =0.4f .

In comparison with the physical quark model spectrum there are many lattice states missing. Of course for a significant comparison with the continuum, i.e. for $\beta \to \infty$ one has to go to higher order calculations. From this point of view, we can expect that new states will appear at the second order calculation. This we mentioned already in Section 5 together with formulas (146) (147). There the last four representations do not depend on E in first order. But it is easy to see that they depend on E in second order of β. So one can expect additional 16 multiplets, 8 of these are singlets, the other 8 are triplets. There are additional 32 particles in this order. Of course it is difficult to calculate the energy of these states explicitly, because the step matrix becomes of the order 1000 x 1000. The appearence of these additional states suggests the following conclusions. As we have seen, the construction of meson fields as irreducible representations of the lattice symmetry group from one link fields give us also the particles, which begin to appear at first order in β. Similarly it should be possible to check by the same procedure, that the two link meson field operators give us also those particles, which begin to appear at the second order of β. From these facts we conjecture that one may get in higher approximations on the lattice all the physical flavour states, which are described by geometric QCD. In this spirit we believe that we have explained in this paper the first steps of a non-perturbative treatment of bound states in a strong coupling regime.

A Definition of the Reduced Spin

For reference momenta $\bar{p} = (0,0,0,p_4)$ and singlet flavour orbits the reduced spin σ is defined by the irreduceble representations of the rotation- reflection group W_3. This group is generated by the rotations by $\pi/2$ in the (12)-and (23)-plane: R^{12}, R^{23} and the space reflection Π. There are altogether 10 irreducible representations defined in the following way:

$$
\begin{aligned}
(1^{\pm})_{W_3} &: \quad R^{12} \mapsto 1 \qquad R^{23} \mapsto 1 \qquad \Pi \mapsto \pm 1 \\
(1'^{\pm})_{W_3} &: \quad R^{12} \mapsto -1 \qquad R^{23} \mapsto -1 \qquad \Pi \mapsto \pm 1 \\
(2^{\pm})_{W_3} &: \quad R^{12} \mapsto \tfrac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \quad R^{23} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Pi \mapsto \pm 1 \\
(3^{\pm})_{W_3} &: \quad R^{12} \mapsto \begin{pmatrix} 0 & -1 & 01 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R^{23} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \Pi \mapsto \pm 1
\end{aligned}
\tag{156}
$$

$(3'^{\pm})_{W_3} \simeq (3^{\pm})_{W_3} \otimes (1'^{+})_{W_3}$

For the same $\bar{p}$ and triplet flavour orbits the reduced spin group is $D_4 \times Z_2$. As asubgroup of W_3 it is generated by

$$
A \mapsto \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Pi_3 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
$$

The 10 irreducible representations of $D_4 \times Z_2$ are given by

$$
\begin{aligned}
(1^{\pm})_{D_4} &: \quad A \mapsto 1 \qquad B \mapsto 1 \qquad \Pi_3 \mapsto \pm 1 \\
(1'^{\pm})_{D_4} &: \quad A \mapsto 1 \qquad B \mapsto -1 \qquad \Pi_3 \mapsto \pm 1 \\
(1''^{\pm})_{D_4} &: \quad A \mapsto -1 \qquad B \mapsto 1 \qquad \Pi_3 \mapsto \pm 1 \\
(1'''^{\pm})_{D_4} &: \quad A \mapsto -1 \qquad B \mapsto -1 \qquad \Pi_3 \mapsto \pm 1 \\
(2^{\pm})_{D_4} &: \quad A \mapsto \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad B \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Pi_3 \mapsto \pm 1
\end{aligned}
\tag{157}
$$

A Symmetric Vectors in the Step Space

In order to define the invariant subspaces of the step space spanned by the basic vectors $e_{\mu..\nu}$, we give first the following definitions:

$$\begin{aligned}
\overset{\pm}{F}{}^1_k &= e_k \pm e_{-k}, & \overset{\pm}{F}{}^2_k &= e_{0k} \pm e_{0-k}, & \overset{\pm}{F}{}^3_k &= e_{-0k} \pm e_{-0-k},\\
\overset{\pm}{T}{}^1_k &= e_{0k-0} \pm e_{0-k-0}, & \overset{\pm}{T}{}^2_k &= e_{-0k0} \pm e_{-0-k0},\\
\overset{\pm}{T}{}^3_k &= e_{k0-k} \pm e_{-k0k}, & \overset{\pm}{T}{}^4_k &= e_{k-0-k} \pm e_{-k-0k},\\
\overset{\pm}{E}{}_{lk} &= e_{lk-l} \pm e_{-l-kl}, & \overset{\pm}{E}{}_{\{lk\}} &= \overset{\pm}{E}{}_{lk} + \overset{\pm}{E}{}_{kl}\\
\overset{\pm}{E}{}_{[lk]} &= \overset{\pm}{E}{}_{lk} - \overset{\pm}{E}{}_{kl} & \overset{\pm}{e}{}_{lk} &= e_{lk} \pm e_{-l-k}
\end{aligned} \tag{158}$$

Now we define the vectors $e_{\pm 0}$, Υ^n, $\Upsilon^{n,\overset{+}{T}}$ which transform under the trivial representation $(1)_{W_3}$:

$$\begin{aligned}
e_0, e_{-0}, \qquad & \Upsilon^n = \textstyle\sum_{k>0} \overset{+}{F}{}^n_k, \quad n = 1,2,3; & & \Upsilon^0 = \sum_{\substack{l>0\\ l<0}} \overset{+}{e}{}_{kl};\\
& \Upsilon^{n,\overset{+}{T}} = \textstyle\sum_{k>0} \overset{+}{T}{}^n_k \quad n = 1,2,3,4; & & \Upsilon^{\overset{+}{E}{}_{\{lk\}}}_{+} = \sum_{\substack{l>0\\ l<0}} \overset{+}{E}{}_{\{lk\}}
\end{aligned} \tag{159}$$

For the representation $(3)_{W_3}$,where the basic vectors are ω^i_k, $k = 1,2,3$ (corresponding to three dimensional space vectors) the base vectors of the corresponding step subspace are :

$$\begin{aligned}
&\omega^a_k = \bar{F}^n_k, \quad n = 1,2,3; & &\omega^b_k = \textstyle\sum_l \bar{e}_{lk} & &\omega^c_k = \bar{T}^{n'}_k, \quad n' = 1,2,3,4;\\
&\omega^d_k = \textstyle\sum_l \bar{E}_{\{lk\}} & &\omega^e_k = \textstyle\sum_l \bar{E}_{[lk]}.
\end{aligned} \tag{160}$$

In the above formulas we sum over negative and positive values of l. We also denote the spacial variables by l and k, so that they have the values $l,k = \pm1, \pm2, \pm3$ the time variable we denote by 0.

For the $(2)_{W_3}$ representation we have the two dimensional basic denoted by Θ_1, Θ_2 ,the vectors of the step subspace are:

$$\begin{aligned}
&\Theta^n_1 = -\overset{+}{F}{}^n_2 + \overset{+}{F}{}^n_3, & &\Theta^n_2 = 2\overset{+}{F}{}^n_1 - \overset{+}{F}{}^n_2 - \overset{+}{F}{}^n_3, & &n = 1,2,3;\\
&\Theta^{n',T^+}_1 = -\overset{+}{T}{}^{n'}_2 + \overset{+}{T}{}^{n'}_3, & &\Theta^{n',T^+}_2 = 2\overset{+}{T}{}^{n'}_1 - \overset{+}{T}{}^{n'}_2 - \overset{+}{T}{}^{n'}_3, & &n = 1,2,3,4;\\
&\Theta^{\overset{+}{E}{}_{\{lk\}}}_1 = \textstyle\sum_l(\overset{+}{E}{}_{\{2l\}} - \overset{+}{E}{}_{\{3l\}}) & &\Theta^{\overset{+}{E}{}_{\{lk\}}}_2 = \textstyle\sum_l(\overset{+}{E}{}_{\{2l\}} + \overset{+}{E}{}_{\{3l\}} - \overset{+}{E}{}_{\{1l\}})\\
&\Theta^{\overset{+}{e}{}_{\{lk\}}}_1 = \textstyle\sum_l(\overset{+}{e}{}_{\{2l\}} - \overset{+}{e}{}_{\{3l\}}), & &\Theta^{\overset{+}{e}{}_{\{lk\}}}_2 = \textstyle\sum_l(\overset{+}{e}{}_{\{2l\}} + \overset{+}{e}{}_{\{3l\}} - \overset{+}{e}{}_{\{1l\}})\\
&\Theta^{\overset{+}{E}{}_{[lk]}}_1 = \textstyle\sum_l(\overset{+}{E}{}_{[3l]} - \overset{+}{E}{}_{[2l]}), & &\Theta^{\overset{+}{E}{}_{[lk]}}_2 = \textstyle\sum_l(2\overset{+}{E}{}_{[3l]} - \overset{+}{E}{}_{[2l]} - \overset{+}{E}{}_{[1l]})
\end{aligned} \tag{161}$$

We define the vectors of the $(1^+)_{D_4}$ representation space in the following way:

$$\begin{aligned}
&e_0, e_{-0}, & &\Upsilon^n_1 = \textstyle\sum_{k=1,2} \overset{+}{F}{}^n_k & &\Upsilon^n_2 = \overset{+}{F}{}^n_3, \quad n = 1,2,3; & &\Upsilon^0_1 = \textstyle\sum_{\substack{l>0\\ l<0}} \overset{+}{e}{}_{3l};\\
&\Upsilon^0_2 = \textstyle\sum_{l=\pm2} \overset{+}{e}{}_{1l}; & &\Upsilon^{n,\overset{+}{T}}_1 = \textstyle\sum_{k=1,2} \overset{+}{T}{}^n_k & &\Upsilon^{n,\overset{+}{T}}_2 = \overset{+}{T}{}^n_3 \quad n = 1,2,3,4;\\
&\Upsilon^1_{\overset{+}{E}{}_{\{lk\}}} = \sum_{\substack{l>0\\ l<0}} \overset{+}{E}{}_{\{l3\}} & &\Upsilon^2_{\overset{+}{E}{}_{\{lk\}}} = \textstyle\sum_{l=\pm2} \overset{+}{E}{}_{\{l1\}} & &\omega^+ = \textstyle\sum_l \overset{+}{E}{}_{[l3]}
\end{aligned} \tag{162}$$

The vectors of the $(1^-)_{D_4}$ representation are:

$$\begin{aligned}
&\Upsilon^n = \bar{F}^n_3 \quad n = 1,2,3; & &\Upsilon^0 = \textstyle\sum_{\substack{l>0\\ l<0}} \bar{e}_{3l};\\
&\Upsilon^{n,\bar{T}} = \bar{T}^n_3 \quad n = 1,2,3,4; & &\Upsilon^{\bar{E}{}_{\{lk\}}}_{-} = \textstyle\sum_{\substack{l>0\\ l<0}} \bar{E}_{\{l3\}} \qquad \omega^- = \textstyle\sum_l \bar{E}_{[l3]}
\end{aligned} \tag{163}$$

The vectors of the $(1^{III+})_{D_4}$ representation are:

$$\begin{aligned}
&\Upsilon^n = \overset{+}{F}{}^n_1 - \overset{+}{F}{}^n_2 \quad n = 1,2,3 & &\Upsilon^0 = \textstyle\sum_{l=\pm1} \overset{+}{e}{}_{3l} - \sum_{l=\pm1} \overset{+}{e}{}_{3l}\\
&\Upsilon^{n,\overset{+}{T}} = \overset{+}{T}{}^n_1 - \overset{+}{T}{}^n_2 \quad n = 1,2,3,4; & &\Upsilon^{\overset{+}{E}{}_{\{lk\}}}_{+} = \textstyle\sum_{l=\pm1} \overset{+}{E}{}_{\{l3\}} - \sum_{l=\pm1} \overset{+}{E}{}_{\{l3\}}\\
&\omega^+ = \textstyle\sum_{l=\pm1} \overset{+}{E}{}_{[l3]} - \sum_{l=\pm2} \overset{+}{E}{}_{[l3]}
\end{aligned} \tag{164}$$

For the $(2^+)_{D_4}$ representation we have a two dimensional basis denoted by (Θ_1, Θ_2) .The vectors of the step subspace are:

$$\begin{aligned}
&\Theta^n_1 = \bar{F}^n_2 + i\bar{F}^n_1, & &\Theta^n_2 = \bar{F}^n_1 + i\bar{F}^n_2 \; n = 1,2,3;\\
&\Theta^{n',T^-}_1 = \bar{T}^{n'}_2 + i\bar{T}^{n'}_1, & &\Theta^{n',T^-}_2 = \bar{T}^{n'}_1 + i\bar{T}^{n'}_2 \quad .n = 1,2,3,4;\\
&\Theta^{\overset{+}{E}{}_{\{lk\}}}_1 = (\overset{+}{E}{}_{\{32\}} - \overset{+}{E}{}_{\{3-2\}}) + i(\overset{+}{E}{}_{\{31\}} - \overset{+}{E}{}_{\{3-1\}}), & &\Theta^{\overset{+}{E}{}_{\{lk\}}}_2 = (\overset{+}{E}{}_{\{31\}} - \overset{+}{E}{}_{\{3-1\}}) + i(\overset{+}{E}{}_{\{32\}} - \overset{+}{E}{}_{\{3-2\}}),\\
&\Theta^{\overset{+}{E}{}_{[lk]}}_1 = (\overset{+}{E}{}_{[32]} - \overset{+}{E}{}_{[3-2]}) + i(\overset{+}{E}{}_{[31]} - \overset{+}{E}{}_{[3-1]}), & &\Theta^{\overset{+}{E}{}_{[lk]}}_2 = (\overset{+}{E}{}_{[31]} - \overset{+}{E}{}_{[3-1]}) + i(\overset{+}{E}{}_{[32]} - \overset{+}{E}{}_{[3-2]})\\
&\Theta^{\overset{+}{e}{}_{lk}}_1 = (\overset{+}{e}{}_{\{32\}} - \overset{+}{e}{}_{\{3-2\}}) + i(\overset{+}{e}{}_{\{31\}} - \overset{+}{e}{}_{\{3-1\}}), & &\Theta^{\overset{+}{e}{}_{\{lk\}}}_2 = (\overset{+}{e}{}_{\{31\}} - \overset{+}{e}{}_{\{3-1\}}) + i(\overset{+}{e}{}_{\{32\}} - \overset{+}{e}{}_{\{3-2\}})
\end{aligned} \tag{165}$$

These are all base vectors of the invariant subspaces , which have a physical pole in $\cosh E$.

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We suggest to call this group $\mathcal{K}_4$ which does not have a generally accepted name (H.W.Braden) in honour of this workshop: 'Kent Group'.

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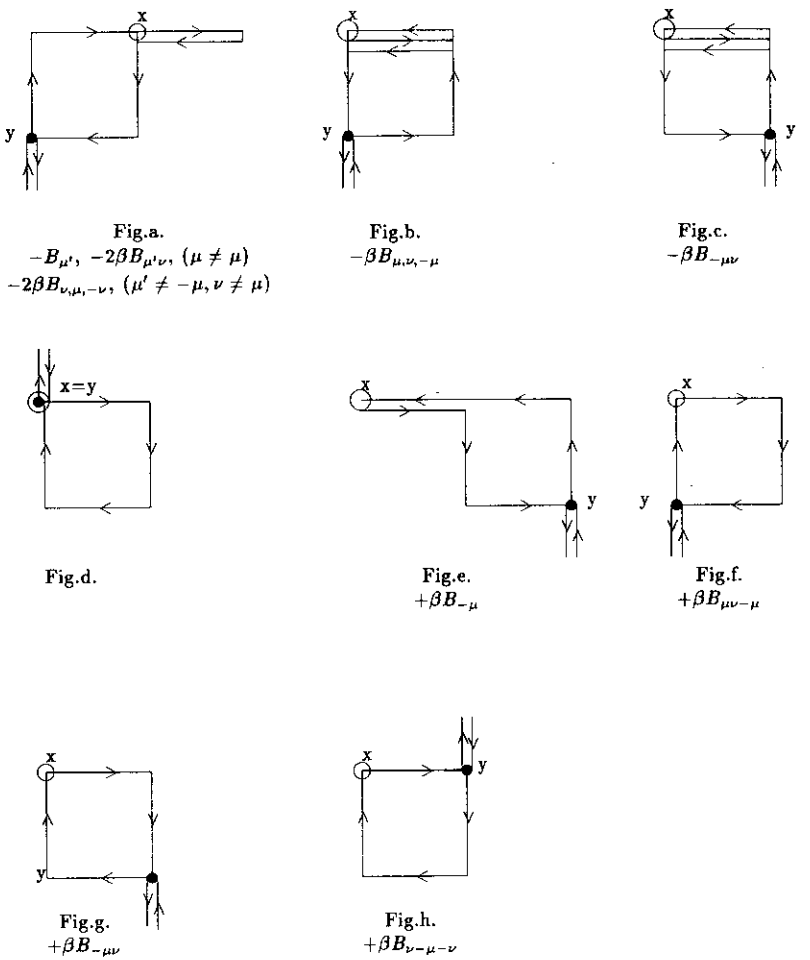


Fig.a. $-B_{\mu'}$, $-2\beta B_{\mu'\nu}$, $(\mu \neq \mu)$ $-2\beta B_{\nu,\mu,-\nu}$, $(\mu' \neq -\mu, \nu \neq \mu)$

Fig.b. $-\beta B_{\mu,\nu,-\mu}$

Fig.c. $-\beta B_{-\mu\nu}$

Fig.d.

Fig.e. $+\beta B_{-\mu}$

Fig.f. $+\beta B_{\mu\nu-\mu}$

Fig.g. $+\beta B_{-\mu\nu}$

Fig.h. $+\beta B_{\nu-\mu-\nu}$

x

Figure 7: The Screening Pathes in First Order

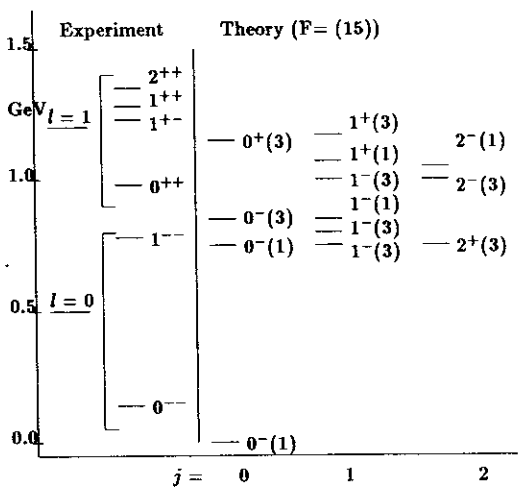


Figure 8: Comparison of the Physical Meson Spectrum with the First Order Calculation

Figure 9: Curves of the masses of the mesons, depending on β, with their lattice quantum numbers.In every Figure we find five curves for the quark masses $m = 0, 0.25, 0.5, 1, 2, 3$. They could be distinguished by their increasing values near $\beta = 0$ as function of the quark mass.

Fig.a.
$\overline{L} = (1,1,1,1)$
$\sigma = (1^+)_{W_3}$

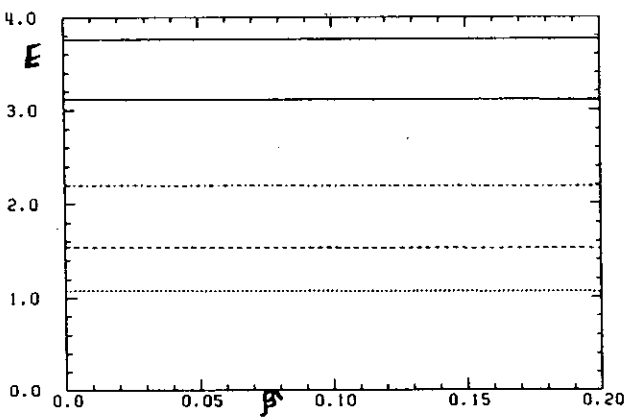


Fig.b.
$\overline{L} = (1,1,0,1)$
$\sigma = (1^+)_{D_4}$

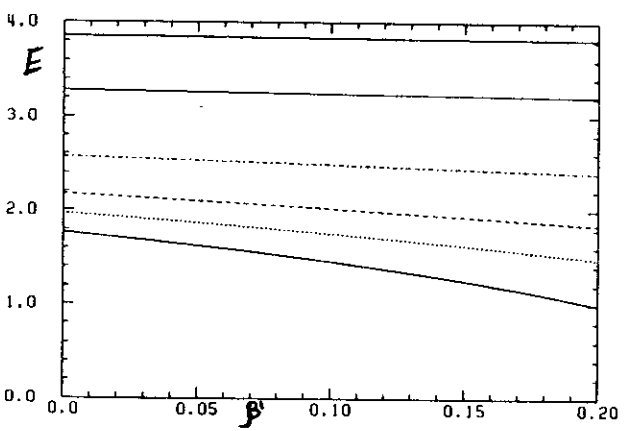


Fig.c.
$\bar{L} = (0,0,1,1)$
$\sigma = (1^+)_{D_s}$

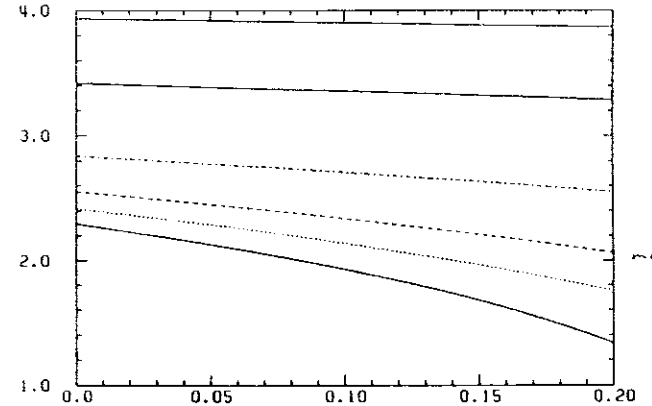


Fig.d.
$\bar{L} = (0,0,0,1)$
$\sigma = (1^+)_{W_s}$

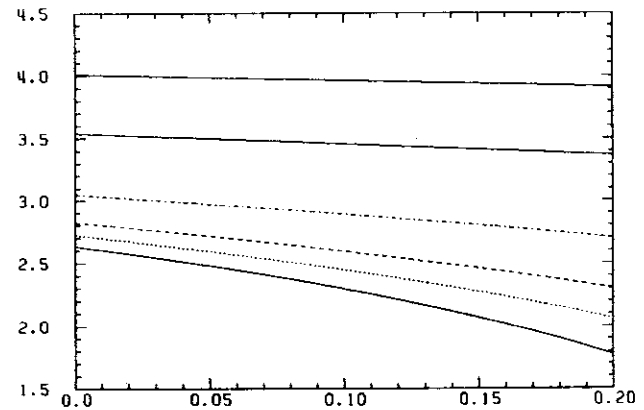


Fig.e.
$\bar{L} = (0,0,0,1)$
$\sigma = (3^-)_{W_s}$

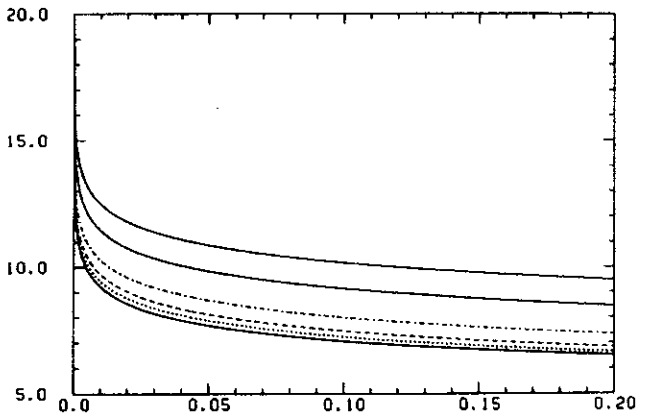


Fig.f.
$\bar{L} = (1,1,1,0)$
$\sigma = (3^-)_{W_s}$

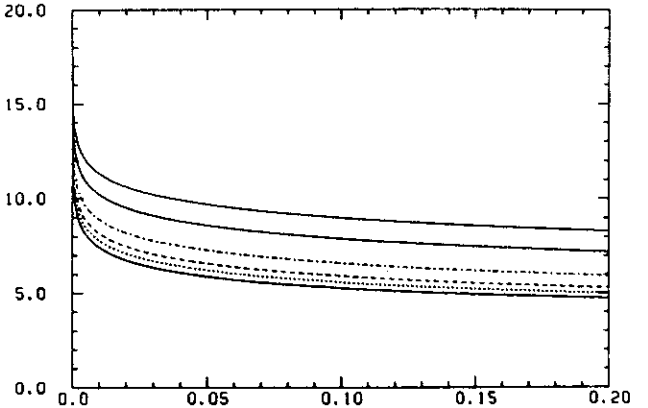


Fig.g.
$\bar{L} = (0,0,0,0)$
$\sigma = (2^+)_{W_s}$

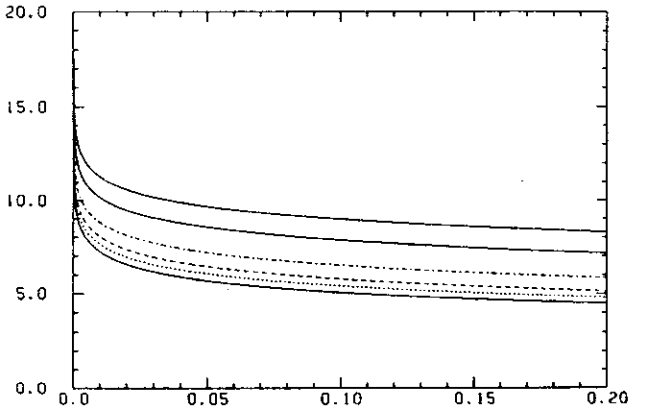


Fig.h.
$\bar{L} = (1,1,1,1)$
$\sigma = (2^+)_{W_s}$

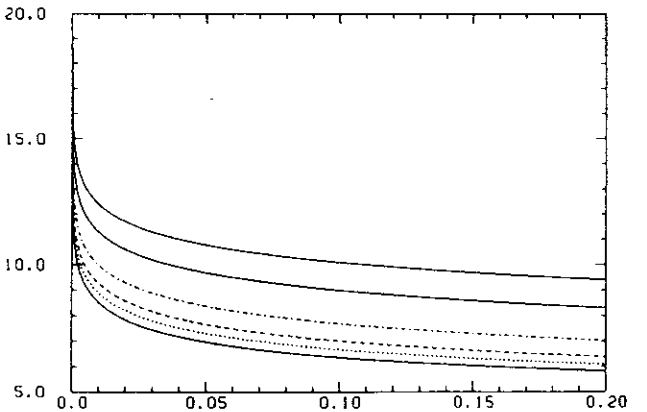


Fig.i.
$\overline{L} = (0,0,1,0)$
$\sigma = (1^-)_{D_4}$

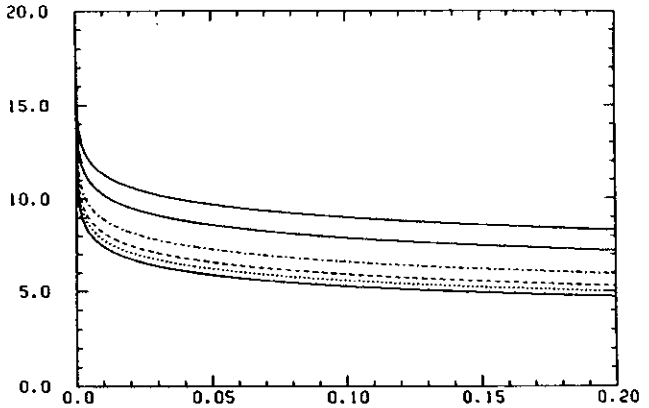


Fig.j.
$\overline{L} = (1,1,0,1)$
$\sigma = (1^-)_{D_4}$

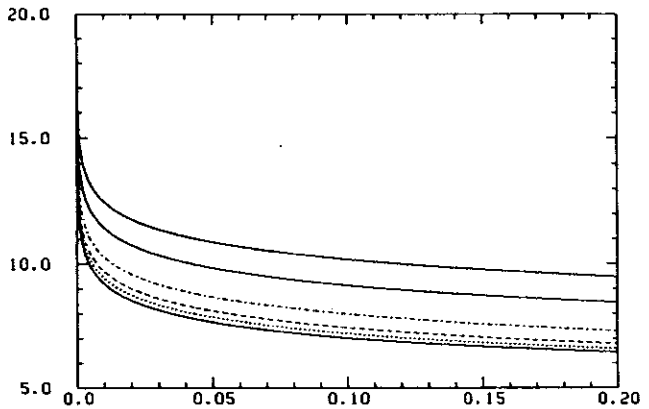


Fig.k.
$\overline{L} = (0,0,1,0)$
$\sigma = (1'^{+})_{D_4}$

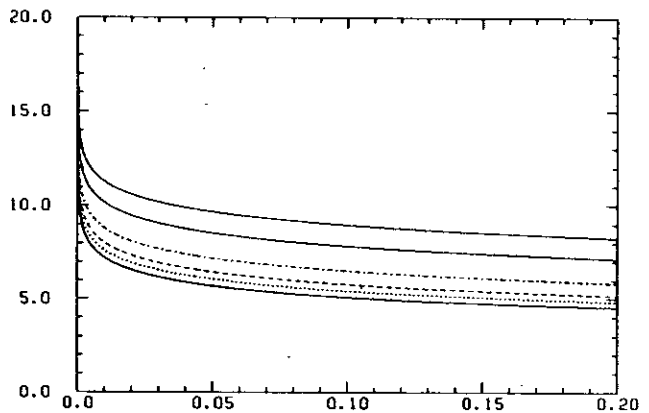


Fig.l.
$\overline{L} = (1,1,0,1)$
$\sigma = (1'^{+})_{D_4}$

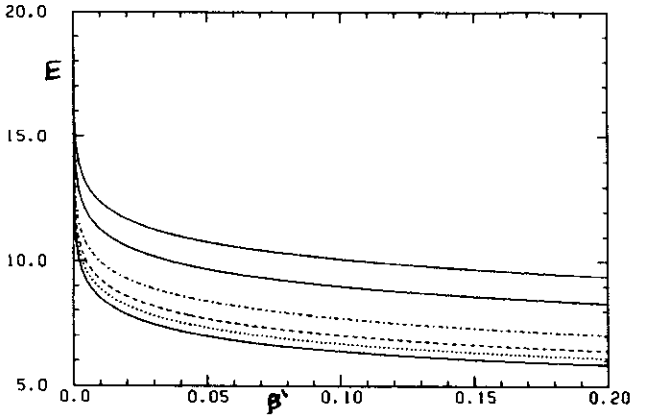


Fig.m.
$\overline{L} = (0,0,1,1)$
$\sigma = (2^+)_{D_4}$

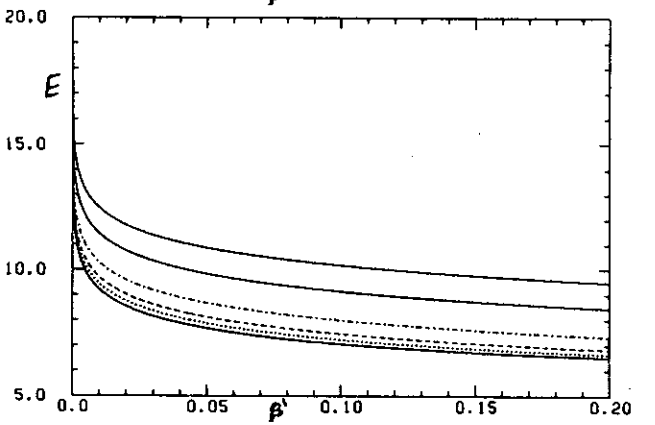
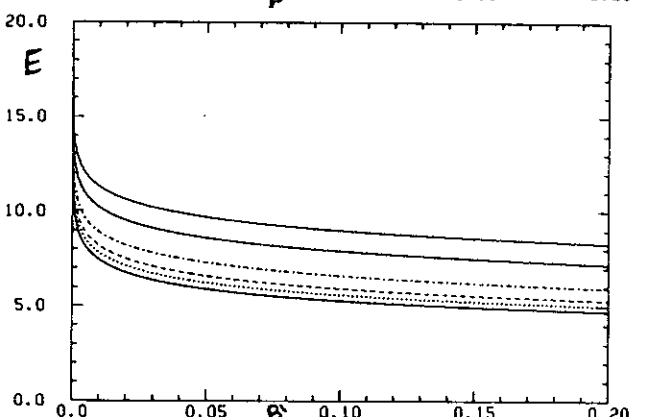


Fig.n.
$\overline{L} = (1,1,0,0)$
$\sigma = (2^+)_{D_4}$



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