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The sigma-model with Wilson lattice fermions

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Abstract

The σ-model with Wilson-fermions is considered in one-loop lattice perturbation theory and in the hopping parameter expansion at large bare couplings. Chiral symmetry restoration in the large cut-off limit of perturbation theory is only possible if asymmetric counterterms are added to the lattice action. In the hopping parameter expansion at infinitely large bare Yukawa-coupling or at zero bare fermion mass dynamical parity doubling of the fermion occurs.

1 Introduction

The σ-model [1] has different important applications in elementary particle theory: in quantum chromodynamics it gives an effective description of the low-energy pion-nucleon system, whereas in the standard elecroweak model it serves as a basis for the Higgs-sector. In fact, the physics of a heavy Higgs-boson and heavy fermion doublet would be described in the standard electroweak model to a good approximation by the σ-model, because for such a system (if it would exist) the scalar quartic coupling and the Yukawa-coupling would be strong and the weak $SU(2) \otimes U(1)$ gauge interaction would be a small perturbation.

A particularly interesting aspect of the σ-model is the $SU(2)_L \otimes SU(2)_R$ chiral symmetry for vanishing fermion mass. In the non-perturbative lattice regularization scheme, where it is natural to start with a massive theory and to obtain the massless fermion as a limiting case, the chiral symmetry is not only broken by the non-vanishing mass, but also by the regularization at the cut-off scale. The reason is *fermion doubling* on the lattice, which always occurs for finite lattice spacing if some rather mild assumptions are fulfilled [2]. In the case of free fermions the superficial fermion species ("fermion doublers") can be kept at the cut-off scale, therefore they are removed from the spectrum in the continuum limit. This is achieved by adding a higher dimensional "Wilson-term" to the lattice action [3]. The Wilson-term, however, breaks the chiral symmetry explicitly also for zero (bare) fermion mass. The important question is, whether the fermion doublers can also be removed from the physical spectrum in the continuum limit of the interacting theory and whether at the same time the chiral symmetry is realized?

There is an important point which has to be mentioned in this respect. In the σ-model both couplings are not asymptotically free and the only known fix point of the Callan-Symanzik renormalization group equations is an infrared fix point (IRFP) at vanishing renormalized couplings [4]. Therefore, the continuum limit of the σ-model is most probably trivial

(non-interacting). Questions about the continuum limit have to be reformulated in such a way that they refer to a quasi-continuum situation with very large but finite ratio of the cut-off (Λ) to the physical mass scale (m). In this case the interaction can be non-zero, but there are upper limits for the renormalized couplings which are going to zero as an inverse power of the logarithm of the cut-off.

The physically relevant region of the bare parameter space where the lattice artifacts of order m/Λ are negligible is called the *scaling region*. This region is in the vicinity of *multicritical points* (C) where all the relevant physical masses are zero in lattice units. In the σ-model the physical particles are the fermion and the σ- and π-scalar bosons. The expected critical structure is qualitatively shown by Fig. 1 in the plane of the two mass parameters in the lattice action (called *hopping parameters*): κ for the scalars and K for the fermion. The bare quartic coupling (λ) and bare Yukawa-coupling (G) are fixed in the figure at some arbitrary values. The relevant points in the plane are below the line $C_\psi C C'_\psi$ where the fermion mass in lattice units vanishes. The mass of the σ- and π-boson vanishes, respectively, along the curves $C_\sigma C$ and $C_\pi C$. Some of these lines may shrink to a point or they may also coincide with each other. For instance, $C_\pi C$ may coincide with $C_\sigma C$ and/or with $C'_\psi C$. The vacuum expectation value of the σ-field (v) is zero in some part of the plane (called *symmetric phase*) and non-zero in the other part (called *phase with spontaneously broken symmetry*). The question of the realization of the chiral symmetry arises along the line $C_\psi C C_\pi$, which can be called *chiral subspace*. (Note that the chiral subspace has one dimension less than the whole bare parameter space.) The standard assumption is that on $C_\psi C$ the vacuum expectation value vanishes ($v = 0$) and the unbroken chiral symmetry is realized by a massless fermion and massive degenerate σ- and π-bosons. Along $C_\pi C$ the vacuum expectation value is assumed to be non-zero in physical units (that is $v/m_\sigma \neq 0$) and the spontaneously broken chiral symmetry is supposed to be realized by a massive fermion and massive σ-boson together with three massless π's as the Goldstone-bosons. If the couplings have only the trivial IRFP at zero ($\lambda_R = G_R = 0$), then the quasi-continuum theories in the scaling region near C are equivalent for arbitrary values of the bare couplings (λ, G). In the present paper this will always be assumed.

The question is, what happens with the fermion doublers in the scaling region near the critical point C? In particular, one would like to show that it is possible to make the masses of the fermion doublers much larger than the physical scale set by the σ-boson mass. Since the renormalized couplings can be non-zero there, the answer to this question requires the knowledge of the spectrum of an interacting 4-dimensional quantum field theory. Of course, to give a definitive answer is very difficult. For the formulation of the standard electroweak model it is required that, according to the above standard assumption about the realization of the chiral symmetry, the physical spectrum of the quasi-continuum theory in the chiral subspace of the broken phase consists of a (possibly light) fermion doublet, an isoscalar scalar Higgs-boson and three zero mass Goldstone bosons, which will be "eaten" by the gauge bosons, once the chiral symmetry is appropriately gauged.

In the present paper I shall consider lattice perturbation theory in the σ-model with Wilson fermions (Section II). In addition, the double hopping parameter expansion in powers of κ and K near $\kappa = K = 0$ will be investigated (Sections III and IV). The aim is to obtain information concerning the above questions about chiral symmetry and the physical spectrum in the quasi-continuum limit of the interacting theory. The summary and conclusions will be discussed in Section V.

2 Lattice perturbation theory

2.1 Lattice action

The scalar field in the σ-model can be considered either as a doublet under the global $\mathrm{SU(2)_L} \otimes \mathrm{SU(2)_R}$ symmetry or as a four-vector under O(4), which is locally equivalent to $\mathrm{SU(2)_L} \otimes \mathrm{SU(2)_R}$. In the O(4)-notation the lattice field is ϕ_{Sx}, ($S = 0, 1, 2, 3$; x = lattice point). The doublet scalar field ϕ^A_x, ($A = 1, 2$) can be extended to a $2 \otimes 2$ matrix φ_x as

$$\varphi_x \equiv \begin{pmatrix} \tilde{\phi}^1_x & \phi^1_x \\ \tilde{\phi}^2_x & \phi^2_x \end{pmatrix} \equiv \sigma_x + i\tau_s \pi_{sx} \tag{1}$$

Here the other doublet field $\tilde{\phi}$ is defined as $\tilde{\phi}^A_x \equiv \epsilon_{AB}\phi^{*B}_x$, with the unit antisymmetric matrix ϵ_{AB}. The isospin is the vector-like diagonal SU(2) subgroup of $\mathrm{SU(2)_L} \otimes \mathrm{SU(2)_R}$, therefore the field component $\sigma_x \equiv \phi_{0x}$ is isoscalar, the components $\pi_{sx} \equiv \phi_{sx}$; ($s = 1, 2, 3$) are isovector. In Eq. (1) τ_s denotes the isospin Pauli-matrix, and over the repeated isospin index s an automatic summation is understood. This summation convention will be applied in this paper for isospin- ($s = 1, 2, 3$) and O(4)-indices ($S = 0, 1, 2, 3$). The transformation properties of the fields φ and ϕ with respect to $\mathrm{SU(2)_L} \otimes \mathrm{SU(2)_R}$, respectively O(4) are:

$$\varphi'_x = U_L^{-1}\varphi_x U_R \qquad\qquad \phi'_{Sx} = O^{-1}_{ST}\phi_{Tx} \tag{2}$$

The connection between the group elements of $\mathrm{SU(2)_L} \otimes \mathrm{SU(2)_R}$ and O(4) is given by

$$O^{-1}_{ST} = \frac{1}{2} Tr\left(\tau_S U_L^{-1} \tau_T^+ U_R\right) = \frac{1}{2} Tr\left(\tau_S^+ U_R^{-1} \tau_T U_L\right) = O_{TS} \tag{3}$$

where τ_S stands for $\tau_S \equiv (i, \tau_s)$.

The fermion doublet fields will be denoted by ψ_x and $\bar{\psi}_x$. Using the projections on the left- and right-handed components $P_L \equiv \frac{1}{2}(1 + \gamma_5)$, respectively $P_R \equiv \frac{1}{2}(1 - \gamma_5)$, the transformation properties of the fermion fields are:

$$\psi'_x = [U_L^{-1}P_L + U_R^{-1}P_R]\psi_x \qquad\qquad \bar{\psi}'_x = \bar{\psi}_x[P_R U_L + P_L U_R] \tag{4}$$

From Eqs. (3-4) follows that the fermion bilinears

$$\bar{\psi}_x \Gamma_S \psi_x \qquad (S = 0, 1, 2, 3) \tag{5}$$

with $\Gamma_S \equiv (1, -i\gamma_5\tau_s)$ transform as a four-vector with respect to O(4).

Using these fields, the lattice fermion action of the σ-model can be written as

$$S = \sum_x \Big\{ \mu\phi_{Sx}\phi_{Sx} + \lambda(\phi_{Sx}\phi_{Sx})^2 - \kappa\sum_\mu \phi_{Sx+\hat{\mu}}\phi_{Sx}$$

$$+ M(\bar{\psi}_x\psi_x) + G\phi_{Sx}(\bar{\psi}_x\Gamma_S\psi_x) - K\sum_\mu(\bar{\psi}_{x+\hat{\mu}}[r + \gamma_\mu]\psi_x)\Big\} \tag{6}$$

The summation $\sum_\mu$ over the neighbours is performed here over both positive and negative directions. The normalization of the fields is left arbitrary here. It can be chosen according to convenience: in perturbation theory the simplest choice is $\kappa = K = \frac{1}{2}$, whereas in numerical studies a frequent convention is $\mu = 1 - 2\lambda$ and $M = 1$. The Wilson-parameter $r > 0$ is required for removing the degeneracy of the fermion doublers. It is expected to be an irrelevant parameter in the sense that it should not influence the physical content of the model in the scaling region. Taking into account the freedom of field normalizations, the number of independent relevant bare parameters in the above action is four.

2.2 1-loop effective potential of the scalar field

The field normalizations are chosen in perturbation theory in such a way that $\kappa = K = \frac{1}{2}$. The propagator of the scalar field Δ^ϕ_{xy} and of the fermion field Δ^ψ_{xy} is on a periodic finite lattice given by

$$\Delta^\phi_{xy} = \frac{1}{N}\sum_k e^{-i(k,x-y)}\frac{1}{\mu_0^2 + \hat{k}^2} \qquad\qquad \Delta^\psi_{xy} = \frac{1}{N}\sum_k e^{-i(k,x-y)}\frac{M_{rk} - i\gamma\cdot\bar{k}}{M_{rk}^2 + \bar{k}^2} \tag{7}$$

Here $\sum_k$ denotes a summation over the Brillouin-zone of momenta and N is the number of lattice points. The bare mass parameters in the propagators are defined (for general field normalization) as

$$\mu_0^2 \equiv \frac{\mu}{\kappa} - 8 \tag{8}$$

and

$$M_{rk} \equiv M_0 + \frac{r}{2}\hat{k}^2 \qquad\qquad M_0 \equiv \frac{M}{2K} - 4r \tag{9}$$

A reminescent notation of the trigonometric functions appearing in lattice perturbation theory is:

$$\hat{k}_\mu \equiv 2\sin\frac{k_\mu}{2} \qquad \bar{k}_\mu \equiv \sin k_\mu \qquad \bar{\bar{k}}_\mu \equiv \frac{1}{2}\sin(2k_\mu) \tag{10}$$

On an infinite lattice the summation over the Brillouin-zone is replaced by an integral:

$$\frac{1}{N}\sum_k \to \frac{1}{(2\pi)^4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d^4k \tag{11}$$

The fermion propagator mass M_{rk} in Eq. (9) shows that, due to the Wilson-term proportional to r, the fermion doublers at the corners of the Brillouin-zone with $k_\mu = \pi$ get an additional mass contribution which is finite in lattice units. (In the present paper usually always lattice units are used, in other words the lattice spacing is $a = 1$.) In the continuum limit when the physical masses go to zero in lattice units, the ratio of the fermion doubler masses to the physical masses tends to infinity, at least in the propagator. If this remains true after the inclusion of the interaction then, as anticipated, the fermion doublers are decoupled from the physical spectrum.

Important properties of the interacting system can be expressed in terms of the *effective action of the scalar field* $\Gamma[\phi]$. In the 1-loop approximation $\Gamma[\phi]$ is given by

$$\Gamma[\phi] = S_\phi[\phi] + \frac{1}{2}Tr\log\left(D[\phi]\Delta^\phi\right) - Tr\log\left(M[\phi]\Delta^\psi\right) \tag{12}$$

Here S_ϕ denotes the pure scalar part of the action in Eq. (6), D is the second derivative matrix of the action with respect to the scalar fields and M is the fermion matrix standing between $\bar{\psi}$ and ψ in the fermionic part of the action.

The *effective potential* $V_{eff}(\phi)$ is defined by the effective action for x-independent fields. In the 1-loop approximation it is

$$V_{eff}(\phi) \equiv \frac{1}{N}\Gamma[\phi_{Sx} = \phi_S \equiv (\sigma, \pi_s)] = \frac{\mu_0^2}{2}(\sigma^2 + \pi^2) + \lambda(\sigma^2 + \pi^2)^2$$

$$+ \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{2l}(4\lambda)^l(3^l + 3)(\sigma^2 + \pi^2)^l\frac{1}{N}\sum_k(\mu_0^2 + \hat{k}^2)^{-l}$$

$$+\sum_{l=1}^{\infty}\frac{(-1)^l}{l}G^l\frac{1}{N}\sum_k(M_{rk}^2+\bar{k}^2)^{-l}Tr\left\{\left[(\sigma-i\gamma_5\tau_s\pi_s)(M_{rk}-i\gamma\cdot\bar{k})\right]^l\right\} \tag{13}$$

The lowest order terms are in detail

$$\begin{aligned}V_{eff}(\sigma,\pi) &= \frac{\mu_0^2}{2}(\sigma^2+\pi^2)+\lambda(\sigma^2+\pi^2)^2\\
&+\frac{1}{N}\sum_k\Bigg\{\frac{12\lambda}{\mu_0^2+\hat{k}^2}(\sigma^2+\pi^2)-\frac{48\lambda^2}{(\mu_0^2+\hat{k}^2)^2}(\sigma^2+\pi^2)^2+\cdots\\
&-\frac{8GM_{rk}}{M_{rk}^2+\bar{k}^2}\sigma-\frac{4G^2}{M_{rk}^2+\bar{k}^2}(\sigma^2+\pi^2)+\frac{8G^2M_{rk}^2}{(M_{rk}^2+\bar{k}^2)^2}\sigma^2\\
&+\frac{8G^3M_{rk}}{(M_{rk}^2+\bar{k}^2)^2}\sigma(\sigma^2+\pi^2)-\frac{32G^3M_{rk}^3}{3(M_{rk}^2+\bar{k}^2)^3}\sigma^3\\
&+\frac{2G^4}{(M_{rk}^2+\bar{k}^2)^2}(\sigma^2+\pi^2)^2-\frac{16G^4M_{rk}^2}{(M_{rk}^2+\bar{k}^2)^3}\sigma^2(\sigma^2+\pi^2)+\frac{16G^4M_{rk}^4}{(M_{rk}^2+\bar{k}^2)^4}\sigma^4+\cdots\Bigg\}\end{aligned} \tag{14}$$

The 1-loop effective potential of the scalar field shows the explicit chiral symmetry breaking due to the fermion propagator mass M_{rk}. Because of the Wilson-parameter, the symmetry breaking terms are non-zero also for a vanishing bare fermion mass ($M_0 = 0$). In order to find the symmetric points of the model the symmetry breaking has to be compensated by appropriate counterterms. In other words, one has to find the set of points in the bare parameter space where all the symmetry breaking terms vanish. It is clear from Eq. (14) that this is only possible if additional symmetry breaking bare parameters are introduced in the action. The symmetric points have to be localized in this higher dimensional space. The simplest symmetry breaking counterterm is linear in the σ-field. Such an "external magnetic field" is required in the symmetric phase in order that the point $\sigma = 0$ be a minimum of the effective action. (In the broken phase with non-zero vacuum expectation value the situation is more involved. This will be discussed in the next subsection.) At 1-loop the coefficient of the linear term $\epsilon\sigma_x$ in the action (6) has to be (for $\kappa = K = \frac{1}{2}$):

$$\epsilon = \frac{1}{N}\sum_k\frac{8GM_{rk}}{M_{rk}^2+\bar{k}^2} \tag{15}$$

This is finite in lattice units, therefore the external magnetic field is cubically divergent in physical units.

The other non-symmetric terms appearing in Eq. (14) have to be compensated in a similar way by explicit counterterms in the action. (The higher dimensional terms in Eq. (13) need not be compensated because they are expected to be negligible in the continuum limit.) The necessity of extending the parameter space to higher dimension becomes also apparent if one tries to find the critical points where the masses vanish. According to Eq.(14) the critical value of the bare parameter μ_0^2 for vanishing π-mass and fermion-mass is (see also the expressions for the renormalized masses given in the next subsection):

$$\mu_{0cr}^2(\pi) = \frac{1}{N}\sum_k\left\{-\frac{24\lambda}{\hat{k}^2}+\frac{8G^2}{\frac{1}{4}r^2(\hat{k}^2)^2+\bar{k}^2}\right\} \tag{16}$$

The same condition for the σ-mass is, however:

$$\mu_{0cr}^2(\sigma) = \frac{1}{N}\sum_k\left\{-\frac{24\lambda}{\hat{k}^2}+\frac{8G^2}{\frac{1}{4}r^2(\hat{k}^2)^2+\bar{k}^2}-\frac{4G^2r^2(\hat{k}^2)^2}{[\frac{1}{4}r^2(\hat{k}^2)^2+\bar{k}^2]^2}\right\} \tag{17}$$

Therefore, in the space of symmetrical bare parameters the critical points where the π-mass and σ-mass vanish are distinct. In order to find a common critical point one has to go in a higher dimensional space with non-symmetric couplings. In addition, this common critical point has to be on the boundary of the broken phase where the vacuum expectation value of the σ-field changes from zero to non-zero. This will be discussed in the next subsection.

2.3 Renormalized quantities

In the limit of a very large cut-off the bare perturbation theory is not useful because of the appearing logarithmic divergences. The physical quantities have to be expressed by power series in the renormalized couplings. In this way the renormalized perturbation theory is obtained which gives a good approximation, provided that the renormalized couplings are small and the physical scales involved are not far away from each other. The divergent field normalizations have to be factored out from the Green's functions. For instance, for the σ-field the normalization factor $Z_{R\sigma}$ can be defined from the effective action (12) by the small momentum behaviour of the function

$$Z(k) \equiv \frac{1}{N}\sum_{xy}e^{i(k,x-y)}\left.\frac{\partial^2\Gamma}{\partial\sigma_x\partial\sigma_y}\right|_{\sigma_z=\pi_{sz}=0} \tag{18}$$

The definition is

$$Z_{R\sigma}^{-1} \equiv \frac{1}{8}\sum_{\mu>0}\left.\frac{\partial^2}{\partial k_\mu\partial k_\mu}\right|_{k=0}Z(k) \tag{19}$$

In 1-loop order this gives:

$$\begin{aligned}Z_{R\sigma} &= 1-\frac{G^2}{N}\sum_k\Big\{(M_{rk}^2+\bar{k}^2)^{-2}\left[8+\frac{3r}{2}M_{rk}(8-\hat{k}^2)+2r^2\bar{k}^2\right]\\
&-(M_{rk}^2+\bar{k}^2)^{-3}\left[4\bar{\bar{k}}^2+8r^2M_{rk}^2(\bar{k}^2+\bar{\bar{k}}^2)+16rM_{rk}\bar{k}\cdot\bar{\bar{k}}+2M_{rk}^2(8-\bar{k}^2)+2rM_{rk}^3(8-\hat{k}^2)\right]\\
&+(M_{rk}^2+\bar{k}^2)^{-4}16M_{rk}^2\left[\bar{\bar{k}}^2+2rM_{rk}\bar{k}\cdot\bar{\bar{k}}+r^2M_{rk}\bar{\bar{k}}^2\right]\Big\}\\
&\longrightarrow 1-\frac{G^2}{N}\sum_k\left\{\frac{8}{(M_{rk}^2+\bar{k}^2)^2}-\frac{4\bar{\bar{k}}^2}{(M_{rk}^2+\bar{k}^2)^3}\right\}+\cdots\quad .\end{aligned} \tag{20}$$

The last line shows the logarithmically divergent terms explicitly. A similar expression can be obtained for $Z_{R\pi}$, too.

The field normalization factor $Z_{R\psi}$ for the fermion can be defined together with the renormalized fermion mass M_R by the low momentum behaviour of the inverse fermion propagator:

$$\Gamma_\psi(k) = Z_{R\psi}^{-1}\left(M_R+ik\cdot\gamma+O(k^2)\right) \tag{21}$$

The 1-loop result for $Z_{R\psi}$ is

$$Z_{R\psi} = 1 - \frac{2G^2}{N} \sum_k (\mu_0^2 + \hat{k}^2)^{-2} (M_{rk}^2 + \bar{k}^2)^{-1} \bar{k}^2 \tag{22}$$

The 1-loop renormalized fermion mass M_R is given by

$$M_R Z_{R\psi}^{-1} = M_0 + \frac{2G^2}{N} \sum_k (\mu_0^2 + \hat{k}^2)^{-1} (M_{rk}^2 + \bar{k}^2)^{-1} M_{rk} \tag{23}$$

The renormalized mass and the renormalized couplings of the scalar fields can be directly obtained from the effective potential (14). In what follows only the mass and couplings of the σ-field will be explicitly given, but very similar expressions can also be obtained for the corresponding quantities with the π-field. The renormalized σ-mass-squared $\mu_{R\sigma}^2$ is given by

$$\mu_{R\sigma}^2 Z_{R\sigma}^{-1} \equiv \left.\frac{\partial^2 V_{eff}}{\partial \sigma^2}\right|_{\sigma=\pi=0} = \mu_0^2 + \frac{1}{N} \sum_k \left\{ \frac{24\lambda}{\mu_0^2 + \hat{k}^2} - \frac{8G^2}{M_{rk}^2 + \bar{k}^2} + \frac{16 G^2 M_{rk}^2}{(M_{rk}^2 + \bar{k}^2)^2} \right\} \tag{24}$$

The 1-loop renormalized quartic coupling of the σ-field can be obtained from

$$\lambda_{R\sigma} Z_{R\sigma}^{-2} \equiv \left.\frac{1}{24} \frac{\partial^4 V_{eff}}{\partial \sigma^4}\right|_{\sigma=\pi=0}$$

$$= \lambda + \frac{1}{N} \sum_k \left\{ \frac{-48\lambda^2}{(\mu_0^2 + \hat{k}^2)^2} + \frac{2G^4}{(M_{rk}^2 + \bar{k}^2)^2} - \frac{16 G^4 M_{rk}^2}{(M_{rk}^2 + \bar{k}^2)^3} + \frac{16 G^4 M_{rk}^4}{(M_{rk}^2 + \bar{k}^2)^4} \right\} \tag{25}$$

The renormalized Yukawa-coupling $G_{R\sigma}$ of the σ-field to the fermion can be defined by the value of the $\sigma\psi\psi$ vertex function at zero external momenta. The 1-loop result is given by

$$G_{R\sigma} Z_{R\psi}^{-1} Z_{R\sigma}^{-\frac{1}{2}} \equiv \Gamma_{\sigma\psi\psi}(0,0,0) = G + \frac{2G^3}{N} \sum_k (\mu_0^2 + \hat{k}^2)^{-1} (M_{rk}^2 + \bar{k}^2)^{-2} (\bar{k}^2 - M_{rk}^2) \tag{26}$$

Up to now it was assumed that the vacuum expectation value of the σ-field is zero. This is true in the symmetric phase, but there is also a phase with broken symmetry, where the vacuum expectation value is non-zero. If the coefficients of the asymmetric counterterms in the action are considered to be functions of the symmetric ones, the expected critical structure can still be visualized by Fig. 1. One way to determine the vacuum expectation value in perturbation theory is to consider a generic situation when the σ-field is

$$\sigma_x = \sigma_{0x} + v \tag{27}$$

where v is the vacuum expectation value. v can be considered as an unknown function of the bare parameters to be determined later. Substituting Eq.(27) into the action (6) one obtains another action in terms of the shifted field σ_{0x}, which has by assumption no vacuum expectation value. The scalar part of the action becomes more complicated (e. g. $\sigma\pi\pi$ and $\sigma\sigma\sigma$ couplings appear: see in Ref. [5]), but the form of the fermion part does not change. The only difference in the fermion part compared to Eq. (6) is that the bare fermion mass M is replaced by $M + Gv$. The effective potential and the renormalized quantities can be calculated in the same way as before. The results are quite similar to the previously given ones, only some additional terms appear which are proportional to

$$w^2 \equiv 4\lambda v^2 \tag{28}$$

(Note that in the perturbation series λ and G^2 are considered on equal footing.) The value of the vacuum expectation value v is determined by the requirement that the effective potential has a minimum at $\sigma_0 = 0$. This equation at the 1-loop order is

$$0 = v \left\{ \mu_0^2 + w^2 + \frac{1}{N} \sum_k \left[\frac{12\lambda}{\mu_\sigma^2 + \hat{k}^2} + \frac{12\lambda}{\mu_\pi^2 + \hat{k}^2} \right] \right\} + \epsilon - \frac{1}{N} \sum_k \frac{8 G M_{rk}}{M_{rk}^2 + \bar{k}^2} \tag{29}$$

Here the contribution of the term linear in the σ-field (ϵ) is already included. The bare mass squares of the scalar fields are:

$$\mu_\sigma^2 = \mu_0^2 + 12\lambda v^2 \qquad\qquad \mu_\pi^2 = \mu_0^2 + 4\lambda v^2 \tag{30}$$

In the symmetric phase ($v = 0$) the parameter ϵ is given at 1-loop by Eq. (15). In order to reproduce the expected singularity structure near the multicritical point C, it is necessary to add to ϵ a term proportional to v. A convenient choice is

$$\epsilon = \frac{1}{N} \sum_k \left\{ \frac{8 G M_{rk}}{M_{rk}^2 + \bar{k}^2} - \frac{8 G^2 v}{M_{rk}^2 + \bar{k}^2} \right\} \tag{31}$$

In this case Eq. (29) has two solutions: $v = 0$ and the one given by

$$w^2 = -\mu_0^2 - \frac{1}{N} \sum_k \left\{ \frac{12\lambda}{\mu_\sigma^2 + \hat{k}^2} + \frac{12\lambda}{\mu_\pi^2 + \hat{k}^2} - \frac{8G^2}{M_{rk}^2 + \bar{k}^2} \right\} \tag{32}$$

The two solutions correspond to two local minima of the effective potential. The absolute minimum in the symmetric phase is $v = 0$, in the spontaneously broken phase $v \neq 0$. For the choice in Eq. (31) the non-zero solution tends to zero in lattice units on the critical line $C_\pi C$ where the π-boson mass is zero (see Fig.1).

Note that by the choice in Eq. (31) it is achieved that the vacuum expectation value is zero in the whole region left to the line $C_\pi C$ in Fig. 1 and non-zero right to $C_\pi C$. As it was discussed in the introduction, from the point of view of the realization of chiral symmetry it is enough that v is zero on the zero mass fermion line $C_\psi C$ and non-zero, in physical units, on the line $C_\pi C$. In the above case v is zero in lattice units on $C_\pi C$ too, but v/m_σ has a non-zero limit if this line is approached from the right.

2.4 Scaling behaviour

The physically interesting region in the bare parameter space is the scaling region where the cut-off dependence is negligible. In this region it is important to know the *curves of constant physics (CCP's)*, where the dimensionless physical quantities are constant and only the lattice spacing (cut-off) changes. The general procedure for obtaining equations for these curves in perturbation theory is described in detail in Ref. [5].

In order to define the CCP's in the bare parameter space one has to choose a set of independent physical quantities $(F_1, \ldots, F_4)$ and a set of bare parameters $(g_1, \ldots, g_4)$. In the present case it is assumed that there are 4 independent relevant parameters. These can be taken as

$$g_1 = \lambda; \qquad g_2 = G; \qquad g_3 = \mu_0^2; \qquad g_4 = M_0^2$$

$$F_1 = \lambda_{R\sigma}; \qquad F_2 = G_{R\sigma}; \qquad F_3 = \frac{\mu_{R\sigma}^2}{M_R^2}; \qquad F_4 = M_R^2 \tag{33}$$

If the last physical quantity F_4 is the *reference quantity* parametrizing the curves, then the differential equations for the functions $\{g_1(F_4),\ldots,g_4(F_4)\}$ are given by

$$\frac{dg_i(F_4)}{dF_4}=\left.\frac{\det_3^{[4,i]}\left(\frac{\partial F}{\partial g}\right)}{\det_4\left(\frac{\partial F}{\partial g}\right)}\right|_{g_4=g_4(g_1,g_2,g_3,F_4)} \qquad (i=1,\ldots,4) \tag{34}$$

where $\det_4(\cdots)$ is the 4×4 determinant of the derivative matrix and $\det_3^{[i,k]}(\cdots)$ denotes its 3×3 subdeterminants.

From the above expressions for the renormalized quantities the right hand sides can be easily worked out. The obtained formulas include also small scale breaking corrections which go to zero in the continuum limit as powers of the lattice spacing. The last two equations ($i=3,4$) give also the dependence of the mass parameters along the CCP's. Here only the first two equations determining the behaviour of the couplings λ and G will be considered in some detail. The scale breaking corrections will also be neglected. In this case only the logarithmically divergent terms have to be considered. The usual parameter along the CCP's is the logarithm of an inverse mass in lattice units, in our case $\tau\equiv\log M_R^{-1}$. The equations for $i=1,2$ in (34) can be written with this parameter as

$$\frac{d\lambda(\tau)}{d\tau}=\beta_\lambda(\lambda,G)\equiv-2M_R^2\frac{d\lambda}{dM_R^2}$$

$$\frac{dG(\tau)}{d\tau}=\beta_G(\lambda,G)\equiv-2M_R^2\frac{dG}{dM_R^2} \tag{35}$$

Using Eq. (34) and the general structure of the perturbation series it can be shown that the Callan-Symanzik β-functions on the right hand sides are given by

$$\beta_\lambda(\lambda,G)=2\left(M_0^2\frac{\partial\lambda_{R\sigma}}{\partial M_0^2}+\mu_0^2\frac{\partial\lambda_{R\sigma}}{\partial\mu_0^2}\right)_{M_0=M_R}$$

$$\beta_G(\lambda,G)=2\left(M_0^2\frac{\partial G_{R\sigma}}{\partial M_0^2}+\mu_0^2\frac{\partial G_{R\sigma}}{\partial\mu_0^2}\right)_{M_0=M_R} \tag{36}$$

The 1-loop result, neglecting scale breaking corrections, is

$$\beta_\lambda(\lambda,G)=\frac{1}{16\pi^2}\left(96\lambda^2+16G^2\lambda-4G^4\right) \qquad \beta_G(\lambda,G)=\frac{1}{16\pi^2}\cdot 4G^3 \tag{37}$$

It can be shown that the logarithmically divergent terms remain unchanged if in the definition of the renormalized couplings in Eqs. (25-26) the σ-fields are replaced by π-fields. Therefore the universal 1-loop terms in the β-function are O(4)-symmetric.

According to Eqs. (35-37), for decreasing lattice spacing the couplings (λ,G) increase along the CCP's. It can be shown using the last two equations in (34) that in the (κ,K)-plane in Fig. 1 the point corresponding to a given CCP moves closer and closer to the multicritical point C. The triviality of the continuum limit implies that no CCP with non-zero renormalized couplings can reach C, because the lattice spacing is zero on C. Therefore the CCP's have to go to $(\lambda=\infty,\ G=\infty)$. Of course, the perturbative form (37) of the β-functions is only valid if λ and G are small.

Up to now the change of the bare parameters was considered for fixed renormalized quantities. It is also possible to fix the bare couplings (λ,G) and ask how the renormalized couplings change in the vicinity of the multicritical point C. This behaviour is determined by another set of β-functions. In the 1-loop approximation, however, the functional forms are the same as Eq. (37). The derivation of these scaling equations is similar to the case of the ϕ^4 theory [6,7]. The only difference is that in the σ-model there are two mass parameters, therefore the ratio of two renormalized masses can, in general, also be fixed (as F_3 in Eq. (33)). In the broken phase one way of exploiting this freedom is to stay in the chiral subspace where the fermion-scalar mass ratio is a function of the two renormalized couplings.

Near the multicritical point C, where the renormalized couplings are small, the β-functions for the renormalized couplings are given to a good approximation by perturbation theory. On the boundary to the region with larger couplings initial conditions are needed. These can be obtained in the symmetric phase by a high order hopping parameter expansion, similarly to the pure ϕ^4 theory [7]. The hopping parameter expansion in models with fermions can also give an independent non-perturbative information about the physical spectrum, in particular concerning the fermion doublers. The question of the spectrum has to be cleared before going on with a perturbative investigation of the realization of chiral symmetry in the scaling region.

3 Hopping parameter expansion

3.1 General formulas

In this Section the general formalism of the hopping parameter expansion at $(\kappa=K=0)$ will be considered. The generating function of the connected Green's functions is defined as

$$W[J,\eta,\bar\eta]=\log\frac{I[J,\eta,\bar\eta]}{I[0,0,0]} \tag{38}$$

where I is an integral over the field variables:

$$I[J,\eta,\bar\eta]=\int[d\phi d\bar\psi d\psi]\exp\left\{-S+\sum_x\left[J_{Sx}\phi_{Sx}+(\bar\psi_x\eta_x)-(\bar\eta_x\psi_x)\right]\right\} \tag{39}$$

In the double expansion according to the powers of κ and K the Green's functions at an arbitrary point of the bare parameter space (λ,G,κ,K) are expressed by the Green's functions at $(\lambda,G,\kappa=0,K=0)$. In the generating function of the connected Green's functions at $(\kappa=K=0)$ the integral in Eq. (39) is:

$$I[J,\eta,\bar\eta]_0\equiv\int[d\phi d\bar\psi d\psi]\exp\Big\{\sum_x\Big[-\mu\phi_{Sx}\phi_{Sx}-\lambda(\phi_{Sx}\phi_{Sx})^2$$
$$-M(\bar\psi_x\psi_x)-G\phi_{Sx}(\bar\psi_x\Gamma_S\psi_x)+J_{Sx}\phi_{Sx}+(\bar\psi_x\eta_x)-(\bar\eta_x\psi_x)\Big]\Big\} \tag{40}$$

This can be factorized into a product of integrals over the field variables of a single lattice point.

In order to write down the *master formula* for the hopping parameter expansion of connected Green's functions, let us introduce a shorthand notation for index repetitions:

$$(f.)^n_\nu\equiv f_{\nu_1}f_{\nu_2}\cdots f_{\nu_n}$$

$$\sum_{[\nu]_n}(f.)^n_\nu \equiv \sum_{\nu_1\cdots\nu_n} f_{\nu_1} f_{\nu_2}\cdots f_{\nu_n} \tag{41}$$

Omitting a $(J\eta\tilde{\eta})$-independent factor in I, the master formula is:

$$\log I[J,\eta,\bar{\eta}] = \sum_{[x\mu]_m[y\nu]_n[SX]_M[aY]_N[bZ]_N} \frac{K^m\kappa^n}{m!n!M!(N!)^2}\cdot$$

$$(J.)^M_{SX}(\tilde{\eta}.)^N_{aY}(\eta.)^N_{bZ}\left\langle(\phi.)^M_{SX}(\tilde{\psi}.)^N_{bZ}(\psi.)^N_{aY}\left[\tilde{\psi}_{x+\hat{\mu}}(r+\gamma_\mu)\psi_x\right]^m_{x\mu}\left[\phi_{Ty+\hat{\nu}}\phi_{Ty}\right]^n_{y\nu}\right\rangle^{mc}_0 \tag{42}$$

Here the index positions for the repetitions of more complicated expressions (in square brackets) are explicitly indicated. The notation $<\cdots>^{mc}_0$ stands for a connected expectation value in the $(\kappa = K = 0)$ point, where in the definition of connectedness the hopping terms in the square brackets are considered as single entities. Such connected expectation values of monomial functions of the field variables can be expressed by the ordinary connected expectation values, where every field variable is considered to be a different entity. Namely, the connected expectation value of monomials $<\cdots>^{mc}$ is a sum of products of ordinary connected expectation values $<\cdots>^c$. The sum is over the different partitions of the set of points representing the field variables $(\phi_x, \bar{\psi}_x, \psi_x)$ in such a way, that the subsets representing the ordinary connected expectation values are "held together" if the points belonging to the same monomial are considered to be connected. In the point $(\kappa = K = 0)$ the connected expectation values $<\cdots>^c_0$ are non-zero only if all the points are at the same lattice site. As a consequence, every term in Eq. (42) gives rise to a sum of terms, which can graphically be represented as connected clusters of points, where every cluster is at some lattice site and the different clusters are held together by the links corresponding to the hopping terms. (Several clusters can occupate the same lattice site.) The contribution of a graph depends only on the topology of the graph but not on its embedding in the lattice. Collecting together graphs with the same topology one obtains the ***linked cluster expansion*** [8].

In order to work out the contribution of a linked cluster expansion graph it is necessary to evaluate the single site integral appearing in Eq. (40). The Grassmann-integral in the general case is:

$$\int d\bar{\psi}d\psi \exp\left\{-M(\bar{\psi}\psi) - G\sigma(\bar{\psi}\psi) - G\pi_s(\bar{\psi}\Gamma_s\psi) + (\bar{\psi}\eta) - (\bar{\eta}\psi)\right\}$$

$$= \left[(M+G\sigma)^2 + G^2\pi^2\right]^4 \exp\left\{-\left[(M+G\sigma)(\bar{\eta}\eta) - G\pi_s(\bar{\eta}\Gamma_s\eta)\right]\left[(M+G\sigma)^2 + G^2\pi^2\right]^{-1}\right\} \tag{43}$$

In the case of a large bare Yukawa-coupling or small bare fermion mass it is convenient to use the freedom in the normalization of the fermion fields and put the coefficient of the coupling term equal to unity. Using the notation $F \equiv M/G$ one obtains

$$\int d\bar{\psi}d\psi \exp\left\{-F(\bar{\psi}\psi) - \sigma(\bar{\psi}\psi) - \pi_s(\bar{\psi}\Gamma_s\psi) + (\bar{\psi}\eta) - (\bar{\eta}\psi)\right\}$$

$$= \left[(F+\sigma)^2 + \pi^2\right]^4 \exp\left\{-\left[(F+\sigma)(\bar{\eta}\eta) - \pi_s(\bar{\eta}\Gamma_s\eta)\right]\left[(F+\sigma)^2 + \pi^2\right]^{-1}\right\} \tag{44}$$

In the case of $G = \infty$ or $M = 0$ the chiral invariant combination $(\sigma^2 + \pi^2)$ appears on the right hand side. If, in addition, the bare quartic coupling is also infinite then the length of the scalar field is frozen to $\sigma^2 + \pi^2 = 1$, and the result of the Grassmann integration is

$$\exp\left\{-\left[\sigma(\bar{\eta}\eta) - \pi_s(\bar{\eta}\Gamma_s\eta)\right]\right\} = \exp\left\{-\phi_S(\bar{\eta}\Gamma^+_S\eta)\right\} \tag{45}$$

Substituting back the result of the Grassmann-integration, the integral over the scalar fields in the general case is:

$$\mathcal{I}_{\lambda F}(J,j) \equiv \int_{-\infty}^{+\infty} d\sigma d^3\pi \exp\left\{-(\sigma^2+\pi^2) - \lambda(\sigma^2+\pi^2-1)^2 + J_0\sigma + J_s\pi_s\right.$$

$$\left.+\left[(F+\sigma)j_0 + \pi_s j_s\right]\left[(F+\sigma)^2+\pi^2\right]^{-1}\right\}\cdot\left[(F+\sigma)^2+\pi^2\right]^4 \tag{46}$$

Here the second form in Eq. (44) was taken, the normalization of the scalar fields was fixed by $\mu = 1 - 2\lambda$, and the fermion bilinear current was introduced as

$$j_S \equiv -(\bar{\eta}\Gamma^+_S\eta) \tag{47}$$

The generating function of the connected expectation values at $(\kappa = K = 0)$ is given by the integral $\mathcal{I}_{\lambda F}$ as

$$W[J,\bar{\eta},\eta]_0 \equiv \sum_x \log\frac{\mathcal{I}_{\lambda F}(J_x,j_x)}{\mathcal{I}_{\lambda F}(0,0)} \tag{48}$$

As it was noted before, the field normalization in Eq. (44) and the use of the bare parameter F is convenient for small F. If F is large, the original parametrization with M and G is better. We are, however, mainly interested in the chiral limit $M \to 0$ where, for any finite G, also $F \to 0$.

3.2 The $F = 0$ limit

It is possible to give a series representation for the integral $\mathcal{I}_{\lambda F}(J,j)$ in general, but here only the important special case $F = 0$ will be considered in detail. In this case the integral over the scalar fields can be expressed by the parabolic cylinder function $D_n(z)$ [9]:

$$\frac{\mathcal{I}_{\lambda 0}(J,j)}{\mathcal{I}_{\lambda 0}(0,0)} = \sum_{n_1,n_2,n_3,n,N=0}^{\infty} \theta_{8-n_2-2n_3}\theta_{n+8}\delta_{n,2n_1-2n_3}\delta_{N,n_1+n_2+n_3}\frac{1}{n_1!n_2!n_3!4^N N!(N+1)!}$$

$$\cdot(J_SJ_S)^{n_1}(2J_Tj_T)^{n_2}(j_Uj_U)^{n_3}\frac{\Gamma(6+\frac{n}{2})}{\Gamma(6)(\sqrt{2\lambda})^{\frac{n}{2}}}\frac{D_{-6-\frac{n}{2}}\left(\frac{1}{\sqrt{2\lambda}}-\sqrt{2\lambda}\right)}{D_{-6}\left(\frac{1}{\sqrt{2\lambda}}-\sqrt{2\lambda}\right)} \tag{49}$$

Here, besides the Cronecker δ's, also θ_n was used, which is defined as $\theta_n = 1$ for $n \geq 0$ and zero otherwise.

In the limit of very large bare quartic coupling $(\lambda \to \infty)$ the asymptotic behaviour is

$$\frac{\Gamma(6+\frac{n}{2})}{\Gamma(6)(\sqrt{2\lambda})^{\frac{n}{2}}}\frac{D_{-6-\frac{n}{2}}\left(\frac{1}{\sqrt{2\lambda}}-\sqrt{2\lambda}\right)}{D_{-6}\left(\frac{1}{\sqrt{2\lambda}}-\sqrt{2\lambda}\right)} \longrightarrow 1 + \frac{n(n+14)}{16\lambda} + O(\lambda^{-2}) \tag{50}$$

therefore Eq. (49) at $\lambda = \infty$ is simplified to

$$\frac{\mathcal{I}_{\infty 0}(J,j)}{\mathcal{I}_{\infty 0}(0,0)} = \sum_{N=0}^{\infty}\frac{\left[(J+j)_S(J+j)_S\right]^N}{4^N N!(N+1)!} \tag{51}$$

In order to obtain the generating function of the connected 1-site expectation values, the expansion of the logarithm of $\mathcal{I}(Jj)/\mathcal{I}(00)$ is needed. For instance, in the simple case of $(\lambda = \infty, F = 0)$ let us define

$$\log\frac{\mathcal{I}_{\infty 0}(J,j)}{\mathcal{I}_{\infty 0}(0,0)} \equiv \sum_{N=0}^{\infty}\frac{C_{2N}}{(2N)!}\left[\frac{1}{4}(J+j)_S(J+j)_S\right]^N \tag{52}$$

The coefficients C_{2N} are given by

$$C_{2N} = \sum_{m,n_1,n_2,\cdots=1}^{\infty} \delta_{N,n_1+2n_2+3n_3+\cdots}\,\delta_{m,n_1+n_2+n_3+\cdots}\,\frac{(2N)!(-1)^{m-1}(m-1)!}{n_1!n_2!n_3!\cdots(1!2!)^{n_1}(2!3!)^{n_2}(3!4!)^{n_3}\cdots} \quad (53)$$

The first few of them are:

$$C_2 = 1;\ \ C_4 = -1;\ \ C_6 = 5;\ \ C_8 = -56;\ \ C_{10} = 1092;\ \ C_{12} = -32670;\ \ C_{14} = 1387815 \quad (54)$$

In the case of $(\lambda = \infty, F = 0)$ the general formula for the connected 1-site expectation value is:

$$\left\langle \phi_{S_1x}\cdots\phi_{S_kx}\bar{\psi}_{b_1x}\psi_{a_1x}\cdots\bar{\psi}_{b_lx}\psi_{a_lx}\right\rangle_0^c =$$

$$= C_{k+l}\frac{k!l!}{2^{k+l}(k+l-1)!!(k+l)!}\sum_{k_1,k_2,k_3=0}^{\infty}\delta_{k,2k_1+k_3}\delta_{l,2k_2+k_3}\frac{2^{k_3}(k_1+k_2+k_3)!}{k_1!k_2!k_3!}$$

$$\cdot \sum_{(k+l)\{(S_iS_j)\}}\prod_{(S_iS_j)}\delta_{S_i,S_j}\sum_{\pi(l)}\sigma_\pi\Gamma^+_{S_{k+1};a_1b_{\pi(1)}}\cdots\Gamma^+_{S_{k+l};a_lb_{\pi(l)}} \quad (55)$$

Here $\sum_{(k+l)\{(S_iS_j)\}}$ is a summation over all different pairings of the O(4)-indices $S_1, S_2, \cdots, S_{k+l}$ and $\sum_{\pi(l)}$ means summation over the permutations $\{\pi(1), \pi(2), \cdots, \pi(l)\}$ of $\{1, 2, \cdots, l\}$ with parity σ_π.

From Eq. (55) an important general property of the linked cluster expansion at $(\lambda = \infty, F = 0)$ follows. Namely, in the clusters (also called *vertices*) representing the connected expectation value $< \cdots >_0^c$, the sum of the number of scalars (π_x) and of the number of fermion pairs $(\bar{\psi}_x\psi_x)$ is always even. Using Eq. (49) it can also be shown that this is true for $F = 0$ at an arbitrary value of λ. This is a general consequence of the exact chiral symmetry of the vertices at $F = 0$. The chiral symmetry breaking due to the Wilson parameter in the fermion hopping term occurs only on the links. On the contrary, for $F \neq 0$, as it is shown by Eq. (46), the chiral symmetry is also broken in the vertices, therefore the sum of the number of scalars and of the number of fermion pairs can also be odd. An immediate consequence is that for $F = 0$ in the region of convergence of the hopping parameter expansion the expectation value of the scalar fields and of the fermion bilinears is zero:

$$\langle\phi_{Sx}\rangle = \langle\bar{\psi}_x\Gamma_S\psi_x\rangle = 0 \quad (56)$$

This follows from the fact that on a hypercubical lattice the number of links on a finite closed curve is even. Therefore, the contribution of a graph to the above expectation values can only be non-zero if the number of links (the order of the hopping parameter expansion) is infinite.

Another similar consequence is that within the convergence radius of the hopping parameter expansion at $(F = 0)$ the fermion propagator vanishes for even lattice distances of the initial and final points:

$$\langle\bar{\psi}_x\psi_y\rangle = 0 \quad \text{if} \quad |x-y| \quad \text{even} \quad (57)$$

In momentum space this means that the fermion propagator is the same at $(k_1+\pi, k_2+\pi, k_3+\pi, k_4+\pi)$ as at (k_1, k_2, k_3, k_4). The reason is that the sum of coordinate differences is even(odd) if the lattice distance defined by the minimum number of connecting links is even(odd). In particular, if the fermion propagator has a pole corresponding to a particle at

$(k_1 = k_2 = k_3 = k_4 = 0)$ then it has also a pole at $(k_1 = k_2 = k_3 = k_4 = \pi)$ corresponding to a particle with opposite intrinsic parity. (For the discussion of the quantum numbers of the additional pole see also the next Section.) This proves that:

In the convergence region of the hopping parameter expansion at an arbitrary non-zero value of the bare Yukawa-coupling ($G \neq 0$) in the chiral limit ($M = 0$), or at any finite bare mass M in the case of $G = \infty$, the fermion spectrum is parity-doubled.

Therefore, the removal of the lattice fermion doublers by the Wilson-term in the action cannot be completely succesful: out of the 16 lattice fermion species at least 2 is left in the physical spectrum. This is in conflict with perturbation theory, where it seems that all the fermion doublers are removed by the Wilson-term. An explanation may be, that perturbation theory gives a good approximation only for $(F \neq 0)$. This would also mean that the limits $G \to 0$ and $M \to 0$ cannot be interchanged. In view of this the conjectured critical structure in Fig. 1 has to be confronted with the hopping parameter expansion, too.

4 Random walk approximation to the hopping parameter expansion

The hopping parameter expansion is expected to give a good quantitative description of the physical properties of scalar-fermion models in regions of the symmetric phase where the correlation length is up to 2-5, provided that high enough orders are available. Based on the experience in pure ϕ^4 models [7] and taking into account the general structure of the linked cluster expansion graphs in the σ-model, the required order for $F = 0$ could be around 16-20-th. To work out such a high order is in principle possible but non-trivial. In order to obtain first a qualitative insight, in this Section the so called *random walk approximation* to the hopping parameter expansion will be considered, which was succesfully applied in the strong gauge coupling region for some problems in QCD [10]. In the present context this approximation is equivalent to a partial resummation of the series within specific classes of the linked cluster expansion graphs. In this Section only the case $(\lambda = \infty, F = 0)$ will be considered, but the qualitative structure is the same also for $F = 0$ at any other λ. In the (κ, K)-plane the line $\kappa = 0$ will be considered in detail, which is the simplest, because a pure fermionic description is possible. As it will be clear from the discussion below, the $\kappa = 0$ line corresponds to an interacting scalar-fermion theory. It is also the most interesting case because, according to the random walk approximation, the multicritical point C is at $\kappa = 0$.

The lattice action at $(\lambda = G = \infty)$ or at $(\lambda = \infty, M = 0)$, for an appropriate choice of the fermion field normalization factors, is

$$S = \sum_x \left\{\phi_{Sx}(\bar{\psi}_x\Gamma_S\psi_x) - \kappa\sum_\mu \phi_{Sx+\hat{\mu}}\phi_{Sx} - K\sum_\mu(\bar{\psi}_{x+\hat{\mu}}[r+\gamma_\mu]\psi_x)\right\} \quad (58)$$

One possibility is to perform first the fermion integration. The result is an effective scalar action S^ϕ_{eff} containing the logarithm of the determinant of the fermion matrix $M[\phi]$. This determinant can be evaluated by using

$$\det(M[\phi]) = \sqrt{\det(M[\phi])\cdot\det(M[\bar{\phi}])} \quad (59)$$

where $\bar{\phi}$ is defined as $\bar{\phi}_{Sx} \equiv (\sigma_x, -\pi_{sx})$. The effective scalar action can be written as

$$S^{\phi}_{eff} = -\kappa \sum_{x,\mu} \phi_{Sx+\hat{\mu}}\phi_{Sx} + \sum_{n=1}^{\infty} \frac{(-K)^n}{2n} Tr\{M_1[\phi] + KM_2\}^n \tag{60}$$

where the matrices $M_{1,2}$ are defined by

$$M_1[\phi]_{yx} \equiv \sum_{\mu} \delta_{y,x+\hat{\mu}} \left[\Gamma_S(\bar{\phi}_{Sy} + \phi_{Sx})r + \Gamma_S(\phi_{Sy} + \phi_{Sx})\gamma_\mu\right]$$

$$M_{2,yx} \equiv \sum_{\mu,\nu} \delta_{y,x+\hat{\mu}+\hat{\nu}}(r + \gamma_\mu)(r + \gamma_\nu) \tag{61}$$

S^{ϕ}_{eff} in (60) is valid for every κ and K in the ($\lambda = G = \infty$) plane. For $\kappa = 0$, instead of the fermion integral, one can also perform the integration over the scalar fields. The result is an effective fermion action. Using relations like Eqs. (51-52) one obtains

$$S^{\psi}_{eff} \equiv -K \sum_{x,\mu} (\bar{\psi}_{x+\hat{\mu}}[r + \gamma_\mu]\psi_x) - \sum_{x} \sum_{N=1}^{4} \frac{C_{2N}}{4^N(2N)!} \left[(\bar{\psi}_x\Gamma_S\psi_x)(\bar{\psi}_x\Gamma_S\psi_x)\right]^N \tag{62}$$

This action is equivalent to Eq. (60) for $\kappa = 0$. For the derivation of the hopping parameter expansion in powers of K the form in Eq. (62) is more convenient. The result is, of course, identical to Eqs. (42,55).

4.1 The boson propagator

Let us first consider the random walk approximation for the σ- and π-boson propagator at ($\lambda = G = \infty, \kappa = 0$). For $\kappa = 0$ the scalar fields do not propagate, they can only appear as external lines in the linked cluster expansion graphs. In general, the scalar Green's functions at $\kappa = 0$ can be obtained from purely fermionic ones by replacing the external scalar lines with fermion-antifermion pairs having the same quantum numbers. Therefore, instead of the scalar propagator, it is enough to consider the propagator of a fermion antifermion pair

$$G(y,x)_{cd,ab} \equiv \left\langle \psi_{cy}\bar{\psi}_{dy}\psi_{ax}\bar{\psi}_{bx}\right\rangle^c \equiv \frac{1}{N}\sum_k e^{-i(k,x-y)}\tilde{G}(k)_{cd,ab} \tag{63}$$

The space coordinates are denoted, as usual, by $x, y, \cdots$, whereas $a, b, c, d, \cdots$ stand for the components of the fermion fields, that is they summarize both isospin- and Dirac-indices: for instance, $a \equiv (A, \alpha)$ where A is the isospin- and α the Dirac-index.

The linked cluster graphs for the above fermion-antifermion pair propagator can be divided into *reducible* and *irreducible* ones. The reducible ones are those, which can be separated in two disconnected parts by cutting a fermion- and an antifermion line ending in the same vertex. The random walk approximation corresponds to summing up the chains of the simplest (or of a few simple) irreducible graphs by a recursion relation generating the chains. The simplest irreducible graph for the propagator of a fermion-antifermion pair is a pair of fermion-antifermion lines on a link connecting two neighbouring sites. A reducible chain of such graphs is illustrated by Fig. 2. The recursion relation summing up these chains is:

$$\tilde{G}(k)_{cd,ab} = -\frac{1}{4}\left[\Gamma^+_{S,cd}\Gamma^+_{S,ab} - \Gamma^+_{S,ad}\Gamma^+_{S,cb}\right]$$

$$-\frac{K^2}{4}\sum_{\mu} e^{-ik_\mu}\left\{\Gamma^+_{T,cd}\left[(r-\gamma_\mu)\Gamma^+_T(r+\gamma_\mu)\right]_{fe} - \left[(r-\gamma_\mu)\Gamma^+_T\right]_{fd}\left[\Gamma^+_T(r+\gamma_\mu)\right]_{ce}\right\}\tilde{G}(k)_{ef,ab} \tag{64}$$

Using the relations

$$Tr\{i\gamma_5\gamma_\mu\tau_r\Gamma^+_S\} = \Gamma^+_S i\gamma_5\gamma_\mu\tau_r\Gamma^+_S = 0 \tag{65}$$

it can be shown, that the subspace of the fermion-antifermion quantum numbers corresponding to the σ- and π-bosons is invariant and diagonal with respect to the equation in (64). Therefore, defining the projection to the (σ, π) quantum numbers by

$$G(y,x)_{TS} \equiv -\Gamma_{T,dc}G(y,x)_{cd,ab}\Gamma_{S,ba} \equiv \frac{1}{N}\sum_k e^{-i(k,x-y)}\tilde{G}(k)_{TS} \tag{66}$$

the solution of Eq. (64) in the (σ, π)-sector has the form:

$$\tilde{G}(k)_{TS} = \delta_{T,0}\delta_{S,0}\tilde{G}_\sigma(k) + \delta_{T,S}(1 - \delta_{T,0}\delta_{S,0})\tilde{G}_\pi(k) \tag{67}$$

Substituting Eqs. (66-67) into Eq. (64) one obtains for the σ-, respectively, π-boson propagator

$$\tilde{G}_\sigma(k) = 20\left[1 - \frac{5}{2}K^2(1 - r^2)(8 - \hat{k}^2)\right]^{-1}$$

$$\tilde{G}_\pi(k) = 20\left[1 - \frac{5}{2}K^2(1 + r^2)(8 - \hat{k}^2)\right]^{-1} \tag{68}$$

For vanishing Wilson-parameter $r = 0$ there is no difference between G_σ and G_π, in accordance with the exact O(4) chiral symmetry. The critical values of the hopping parameter K, where the propagators have a zero energy pole in lattice units, are:

$$K^2_{cr}(\sigma) = \frac{1}{20(1 - r^2)} \qquad K^2_{cr}(\pi) = \frac{1}{20(1 + r^2)} \tag{69}$$

If $r \neq 0$, the critical hopping parameter for the π-boson is always smaller than for the σ-boson. This situation can be changed, however, if more complicated irreducible graphs are taken into account. (See the discussion below.)

4.2 The fermion propagator

As it was shown in the previous Section, in the convergence region of the hopping parameter expansion the fermion is parity doubled. In the random walk approximation this is manifested by the fact that the simplest reducible chain for the fermion propagator has irreducible elements consisting out of two links. As it is shown by Fig. 3, the simplest fermion propagation proceeds via an oscillation between a fermion and a fermion-antifermion-fermion state. Defining the fermion propagator as

$$G(y,x)_{ba} \equiv \left\langle \bar{\psi}_{by}\psi_{ax}\right\rangle^c \equiv \frac{1}{N}\sum_k e^{-i(k,x-y)}\tilde{G}(k)_{ba} \tag{70}$$

the recursion relation summing up the graphs in Fig. 2 is in momentum space given by

$$\tilde{G}(k)_{ba} = \frac{5K^3}{4}\sum_\mu e^{-ik_\mu}\left[r(2r^2+1) - (r^2+2)\gamma_\mu\right]_{ba}$$

$$+\frac{5K^4}{4}\sum_{\mu\nu}\epsilon^{-i(k_\mu+k_\nu)}\left[r(2r^2+1)-(r^2+2)\gamma_\nu\right]_{bc}(r+\gamma_\mu)_{cd}\tilde{G}(k)_{da} \tag{71}$$

This has the solution:

$$\tilde{G}(k)_{ba} = \frac{5K^3}{4}\left[(2r^2+1)r(8-\hat{k}^2)+2(r^2+2)i\gamma\cdot\bar{k}\right]_{bc}$$
$$\cdot\left\{1-\frac{5K^4}{4}\left[(2r^2+1)r^2(8-\hat{k}^2)^2+4(r^2+2)\bar{k}^2\right]+\frac{5K^4}{2}r(1-r^2)(8-\hat{k}^2)i\gamma\cdot\bar{k}\right\}_{ca}$$
$$\cdot\left\{\left[1-\frac{5K^4}{4}[(2r^2+1)r^2(8-\hat{k}^2)^2+4(r^2+2)\bar{k}^2]\right]^2+\frac{25K^8}{4}r^2(1-r^2)^2(8-\hat{k}^2)^2\bar{k}^2\right\}^{-1} \tag{72}$$

The critical hopping parameter where this fermion propagator has a zero energy pole is given by

$$K^4_{cr}(f) = \frac{1}{80r^2(2r^2+1)} \tag{73}$$

This is also larger than $K_{cr}(\pi)$ in Eq. (69). For instance, in the case of $r = 1$ we have $K_{cr}(\pi) = 0.1581\ldots$ and $K_{cr}(f) = 0.2541\ldots$.

The doubling of the fermion spectrum is explicitly displayed by the fermion propagator in Eq. (72): besides the pole at $k_1 = k_2 = k_3 = 0, k_4 = -iaE$ there is also a pole at $k_1 = k_2 = k_3 = \pi, k_4 = \pi + iaE$; $(aE \to 0)$. The quantum numbers of the two poles are best identified by considering the propagation of a specific fermion-antifermion-fermion composite state χ. The relevance of such states in the fermion propagator is already clear from Fig. 3. The appropriate composite fermion operators are:

$$\chi_x \equiv \frac{1}{10}\Gamma_S\psi_x(\bar{\psi}_x\Gamma_S\psi_x) \qquad \bar{\chi}_x \equiv \frac{1}{10}(\bar{\psi}_x\Gamma_S\psi_x)\bar{\psi}_x\Gamma_S \tag{74}$$

It follows from Eqs. (2-4) that the chiral transformation properties of χ and $\bar{\chi}$ are:

$$\chi'_x = [U_R^{-1}P_L + U_L^{-1}P_R]\chi_x \qquad \bar{\chi}'_x = \bar{\chi}_x[P_RU_R + P_LU_L] \tag{75}$$

This corresponds to the transformation properties of the fermion fields in Eq. (4), if the left- and right-handed components are interchanged. Therefore, χ and $\bar{\chi}$ describe composite *mirror fermions.* The composite mirror fermion operators also appear in the fermion effective action (62), which can be rewritten as

$$S^{\psi}_{eff} \equiv -K\sum_{x,\mu}(\bar{\psi}_{x+\hat{\mu}}[r+\gamma_\mu]\psi_x) - \sum_x\sum_{N=1}^{4}\frac{C_{2N}5^N}{(2N)!4^N}\left[(\bar{\chi}_x\psi_x)+(\bar{\psi}_x\chi_x)\right]^N \tag{76}$$

A large class of linked cluster graphs for the χ-propagator has the form illustrated by Fig. 4. In particular, in the random walk approximation the fermion propagator represented by the curly bracket in Fig. 4 has to be replaced by Fig. 3. Considering only the class of graphs in Fig. 4, the χ-propagator can be expressed by the ψ-propagator through

$$H(y,x)_{ab} \equiv \langle\bar{\chi}_{ay}\chi_{bx}\rangle^c = K^2\sum_{\mu\nu}(r+\gamma_\nu)_{ac}G(y-\hat{\nu},x+\hat{\mu})_{cd}(r+\gamma_\mu)_{db} \tag{77}$$

There are also off-diagonal propagators between the fields ψ and χ which can be obtained similarly. In the random walk approximation corresponding to Eq. (72) the whole 2⊗2 matrix

can be easily obtained. For brevity, let us here consider only the case $r = 1$. In momentum space the 2⊗2 (ψ,χ) propagator matrix is:

$$\frac{\frac{15}{4}K^3}{1-\frac{15}{4}K^4[(8-\hat{k}^2)^2+4\bar{k}^2]}\begin{pmatrix} 8-\hat{k}^2+2i\gamma\cdot\bar{k} & K[(8-\hat{k}^2)^2+4\bar{k}^2] \\ K[(8-\hat{k}^2)^2+4\bar{k}^2] & K^2[(8-\hat{k}^2)^2+4\bar{k}^2](8-\hat{k}^2-2i\gamma\cdot\bar{k}) \end{pmatrix} \tag{78}$$

The residua of the two poles at $k_1 = k_2 = k_3 = 0, k_4 = -iaE$ and $k_1 = k_2 = k_3 = \pi, k_4 = \pi + iaE$; $(aE \to 0)$ can be written as

$$const\cdot\begin{pmatrix} \pm 1 & 8K_{cr}(f) \\ 8K_{cr}(f) & \pm 64K_{cr}(f)^2 \end{pmatrix} \tag{79}$$

Therefore, appart from a field renormalization factor $8K_{cr}(f)$ for the χ-field, the eigenvector belonging to the non-zero eigenvalue at the first pole (upper sign in Eq. (79)) is: $(\psi+\chi)/\sqrt{2}$. The eigenvector belonging to the non-zero eigenvalue at the second pole (lower sign in Eq. (79)) is: $(\psi-\chi)/\sqrt{2}$. It is natural to define the parity operator in such a way that it interchanges ψ with χ. In this case the two poles correspond to two degenerate fermion states with opposite intrinsic parity.

4.3 Critical structure

Summing up the simplest irreducible graphs in the random walk approximation for the σ- and π-bosons and ψ- and χ-fermions results in different critical hopping parameters. The smallest critical hopping parameter is obtained for the π-boson: $K_{cr}(\pi)$. At $K = K_{cr}(\pi)$ the high order graphs of the type in Fig. 2 are not suppressed any more, they can be arbitrarily long. This has an important effect also on the propagators of the other particles. For instance, in the irreducible part of the fermion propagator in Fig. 3 the fermion-antifermion pair on the first link can be replaced by a long fermion-antifermion graph of the type in Fig. 2, if this pair has the quantum numbers of a π-boson (see Fig. 5). The sum of the contributions in Fig. 5 diverges at $K = K_{cr}(\pi)$, therefore the sum over the chains of such graphs in the fermion propagator is expected to diverge even earlier.

In order to see, how this works out in detail, let us write the recursion relation (71) summing up the reducible chains of graphs for the fermion propagator in momentum space as

$$\tilde{G}(k)_{ba} = K\tilde{G}^{(1)}(k)_{ba} + K^2\sum_{\mu}\epsilon^{-ik_\mu}\tilde{G}^{(1)}(k)_{bc}(r+\gamma_\mu)_{cd}\tilde{G}(k)_{da} \tag{80}$$

The factor $\tilde{G}^{(1)}$ contains the propagator of a bosonic fermion-antifermion pair between two neighbouring points. This can be written as

$$\tilde{G}^{(1)}(k)_{ba} = \sum_{\mu}\epsilon^{-ik_\mu}[-B_\sigma(r+\gamma_\mu)_{ba} + B_\pi(r-\gamma_\mu)_{ba}] \tag{81}$$

Here, for completeness, also the propagation of the σ-boson is included. B_σ and B_π are functions of the Wilson-parameter r and of the hopping parameter K:

$$B_\sigma \equiv B_\sigma(r,K) \equiv \frac{1}{80N}\sum_k\epsilon^{ik_\mu}\tilde{G}_\sigma(k) = \frac{1}{4}\sum_{j=0}^{\infty}h_{2j+1}\left[\frac{5}{2}K^2(1-r^2)\right]^{2j+1}$$

$$B_\pi \equiv B_\pi(r,K) \equiv \frac{3}{80N}\sum_k e^{ik_\mu}\tilde{G}_\pi(k) = \frac{3}{4}\sum_{j=0}^{\infty} h_{2j+1}\left[\frac{5}{2}K^2(1+r^2)\right]^{2j+1} \tag{82}$$

Here h_n is the number of paths of length n on the lattice between two neighbouring points:

$$h_n \equiv \frac{1}{N}\sum_k e^{ik_\mu}\sum_{\nu_1,\cdots,\nu_n} e^{-i(k_{\nu_1}+\cdots+k_{\nu_n})} = \sum_{\nu_1,\cdots,\nu_n}\delta_{\hat{\mu},\hat{\nu}_1+\cdots+\hat{\nu}_n} \tag{83}$$

The solution of Eq. (80) is:

$$\tilde{G}(k)_{ba} = K\left[r(8-\hat{k}^2)(B_\pi - B_\sigma) + 2i\gamma\cdot\bar{k}(B_\pi + B_\sigma)\right]_{bc}$$
$$\cdot\left\{1 - K^2\left[r(8-\hat{k}^2) - 2i\gamma\cdot\bar{k}\right]\left[r(8-\hat{k}^2)(B_\pi - B_\sigma) + 2i\gamma\cdot\bar{k}(B_\pi + B_\sigma)\right]\right\}^{-1}_{ca} \tag{84}$$

Replacing here $B_{\pi,\sigma}$ by the first ($j=0$) terms in Eq. (82), one obtains Eq.(72). The critical hopping parameter in Eq. (84) is the solution of the equation

$$K_{cr}(f)^2 = \frac{1}{64r^2[B_\pi(r,K_{cr}(f)) - B_\sigma(r,K_{cr}(f))]} \tag{85}$$

Again, if only the first terms in $B_{\pi,\sigma}$ are taken into account, then the previous result in Eq. (73) is reproduced.

The number of different curves on the four-dimensional hypercubical lattice connecting two neighbouring sites h_n grows with the length n asymptotically as 8^n. Therefore, the series for $B_{\sigma,\pi}$ in Eq. (82) diverge at $K = K_{cr}(\sigma,\pi)$. For $K \to K_{cr}(\pi)$ the combination $(B_\pi + B_\sigma)$ tends to $+\infty$ and, consequently, the solution of Eq. (85) for the critical value of the hopping parameter $K_{cr}(f)$ is somewhat smaller than $K_{cr}(\pi)$. These are, of course, still results of a partial resummation of the hopping parameter series. The diverging fermion propagators can, however, themselves included in the irreducible boson propagator graphs which, after resummation, make the critical hopping parameter for bosons again somewhat smaller than the critical hopping parameter for fermions. This procedure can be repeated many times. It is plausible that the final outcome is a common multicritical point C at some hopping parameter $K = K_{cr}$ smaller than $K_{cr}(\pi)$ in Eq. (69). At $(\lambda = G = \infty, \kappa = 0)$ the strongly interacting elementary fermion ψ produces fermionic and bosonic bound states, which can be thought of as being bound states of each other in a ***bootstrap*** manner. This suggest the critical structure of the lattice σ-model at $\lambda = G = \infty$ as shown in Fig. 6. The critical lines of π and σ and of the fermions are separated from each other at $\kappa \neq 0$, because of the additional graphs containing scalar internal lines. These are more effective for the bosons than for the fermions. On the line $K = 0$ the fermions do not propagate at all and the model is reduced to a pure O(4)-symmetric ϕ^4 model. Appart from the fermion parity doubling, Fig. 6 can be considered as a degenerate version of the expected critical structure in Fig. 1: the line $C_\psi C$ is reduced to a single point in Fig. 6.

5 Summary and discussion

The form of the effective potential and of the renormalized quantities in 1-loop perturbation theory shows, that in order to find the expected critical structure with chiral symmetry in the vicinity of the point with vanishing bare couplings one has to add asymmetric counterterms

to the action. However, this is only a necessary condition and the question of the restoration of the chiral symmetry near the multicritical points, where the physical masses vanish in lattice units, can only be answered if the spectrum of physical states is established. For the formulation of the standard electroweak model the usual assumption about the spectrum is, that in the σ-model the chiral symmetry is realized in the symmetric phase by a massless fermion (and massive degenerate σ- and π-bosons), and in the spontaneously broken phase by a massless triplet of Goldstone-bosons (and a massive fermion and σ-boson). In the lattice regularized σ-model with Wilson-fermions this is contradicted in the limit of infinitely strong bare Yukawa-coupling at any bare fermion mass, or in the limit of zero bare fermion mass at any non-zero bare Yukawa-coupling, by the general structure of the hopping parameter expansion at vanishing hopping parameters. As it was shown in Section 3, the structure of the hopping parameter expansion implies the parity doubling of the fermion spectrum. The parity partner of the original fermion can be considered as a remnant of the lattice fermion doubling. From this point of view the hopping parameter expansion is in conflict with lattice perturbation theory, where it seems that all the lattice fermion doublers are removed from the physical spectrum by the Wilson-term in the action. The clash of perturbation theory with the hopping parameter expansion in the massless fermion limit can indicate the non-commutativity of the zero fermion mass limit with the zero Yukawa-coupling limit.

The dynamics of the fermion parity doubling was investigated in detail in Section 4 within the random walk approximation to the hopping parameter expansion. It turned out, that the parity doubling is realized by a composite mirror fermion field. In an interesting limit of the model at infinite bare couplings and zero scalar hopping parameter the fermion propagation on the lattice proceeds by an oscillation between the elementary fermion and its composite mirror fermion partner. This mechanism of parity doubling seems to be at work also in other, more general, scalar-fermion models.

Of course, the solution of the spectrum problem in an interacting 4-dimensional quantum field theory is very difficult. The hopping parameter expansion can only be expected to reproduce the physical content of the model in a limited range of the symmetric phase, where the correlation lengths are not larger than, say, 2-5. It is, however, plausible that in the σ-model the Lüscher-Weisz procedure [7] is applicable, in the same way as in pure scalar ϕ^4 theory. Then, due to the triviality of the continuum limit, at these correlation lengths the renormalized couplings are already small enough for the application of the perturbative Callan-Symanzik renormalion group equations, and for the application of renormalized perturbation theory in general. The renormalization group equations can also be continued over the multicritical point into the scaling region of the phase with spontaneously broken symmetry. In the broken phase the fermion parity doublet is transformed into a mirror pair with, in general, different masses [11]. Even if in the present paper the hopping parameter expansion was considered in detail only in an approximation, and even if generally much less is known about the properties of scalar-fermion theories than about the pure scalar ϕ^4 theories, it seems rather plausible that in the σ-model with Wilson-fermions the physical spectrum consists out of the σ- and π-bosons and a mirror pair of fermions.

Where can one hope to avoide the mirror partners of the fermions in a non-perturbative lattice formulation? It is, in principle, possible that some other lattice fermion formulation does not have this dynamical fermion doubling property. It is also possible that a chiral symmetry without mirror fermions can only be realized in some other, more complicated, models. In this case one has to find, however, the connection of such models to the standard electroweak model. In the framework of the σ-model a possible attitude is to assume, that the model with a chirally asymmetric spectrum can be reached, if it exists, as some limit of the model with a mirror pair of fermions. In this case it is convenient to start with a formulation where the mirror fermions are included in the action at the level of elementary fields [11]. The important advantage of such a formulation is the possibility of a local chiral $SU(2)_L \otimes SU(2)_R$ symmetry which can be gauged in the way it is required in the standard model. In this case the question to be answered is, whether there exists some limit of the extended model where the mirror partners of the fermions are removed from the physical spectrum.

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Figure captions

Fig. 1. The expected critical structure in the (κ, K)-plane as explained in the text. The couplings (λ, G) are fixed at an arbitrary value. The scaling region is in the vicinity of the multicritical point C.

Fig. 2. A reducible chain of graphs for the propagator of a fermion-antifermion pair which is obtained by repeating the simplest irreducible graph.

Fig. 3. The simplest type of reducible graphs for the fermion propagator.

Fig. 4. The class of graphs for the propagator of the composite mirror fermion which can be reduced to the propagator of the elementary fermion, represented in the figure by the curly bracket.

Fig. 5. A more complicated irreducible part of the fermion propagator, where the propagation of the fermion-antifermion pair between neighbouring sites is given by a graph like in Fig. 2. A large effect near $K_{cr}(\pi)$ is given by this graph if the fermion-antifermion pair has the quantum numbers of a π-boson.

Fig. 6. The tentative critical structure in the (κ, K)-plane at $\lambda = G = \infty$ suggested by the random walk approximation to the hopping parameter expansion. C is the multicritical point and the critical lines where the σ- and π-boson masses or the mass of the fermion pair (ψ, χ) vanishes are indicated by the corresponding letters.

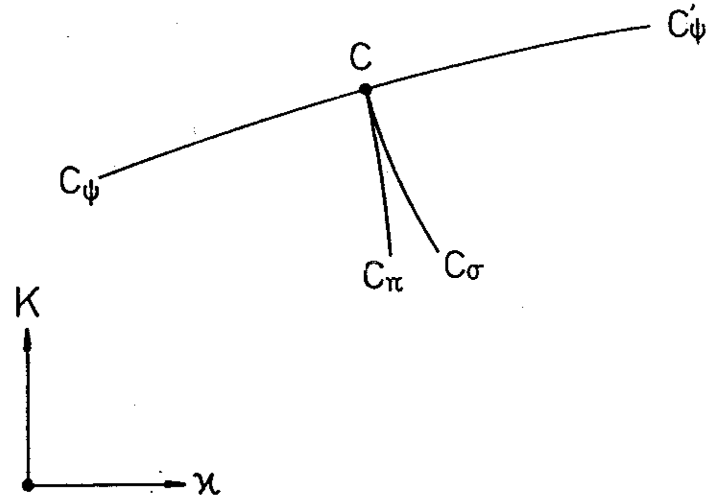

Fig 1

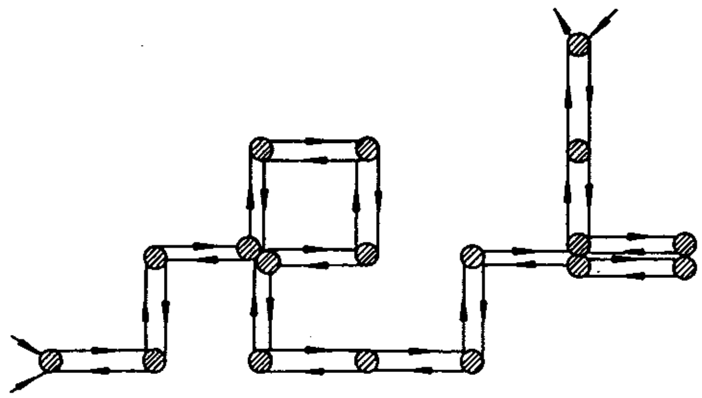

Fig. 2

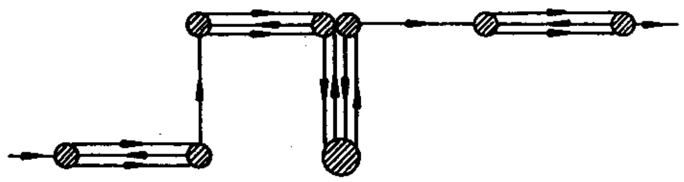

Fig.3

Fig.4

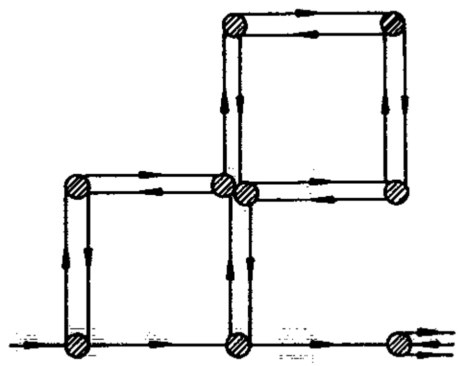
Fig.5

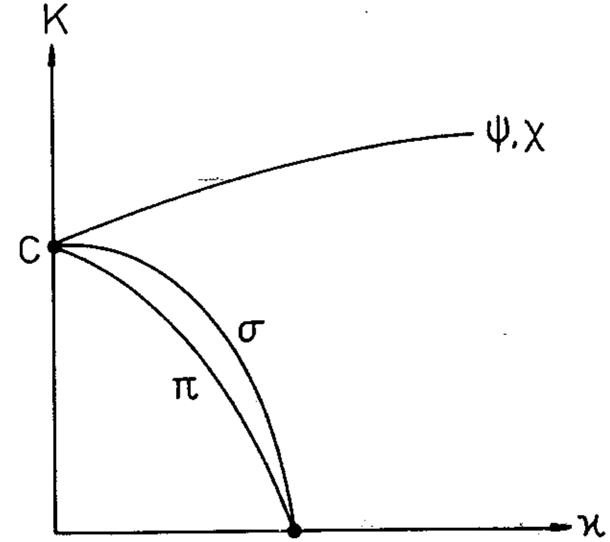

Fig 6