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Resonance form factors from finite-volume correlation functions with the external field method

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Abstract: A novel method for the extraction of form factors of unstable particles on the lattice is proposed. The approach is based on the study of two-particle scattering in a static, spatially periodic external field by using a generalization of the Lüscher method in the presence of such a field. It is shown that the resonance form factor is given by the derivative of the resonance pole position in the complex plane with respect to the coupling constant to the external field. Unlike the standard approach, this proposal does not suffer from problems caused by the presence of the triangle diagram.

Contents

1 Introduction

The study of the form factors of unstable particles from lattice field theory provides plenty of information about the structure of these particles. This study is however complicated by a non-trivial mapping of the results of lattice calculations — performed in a finite volume — onto the relevant infinite-volume form factors[1].

Such a mapping is rather trivial in case of a stable particle. Namely, let $|p\rangle$ be an infinite-volume state, which describes a single particle moving with the on-shell momentum p^μ. The infinite-volume form factor $\langle p|J(0)|q\rangle$ is defined as the matrix element of some current $J(x)$ between one-particle states (in order to ease the notations, we consider the spinless particles and the scalar currents here). Due to the Lorentz-invariance, the form factor is a function of a single variable $t = (p-q)^2$. Furthermore, on the lattice, the spectrum contains the one-particle states $|p\rangle_L$. Here, L denotes the spatial extension of a cubic lattice, while, for simplicity, the time extension is assumed to be infinite. The finite-volume form factor is given by the matrix element between these states, $\langle p|J(0)|q\rangle_L$. Then, in the limit $L \to \infty$, one has

$$\langle p|J(0)|q\rangle_L = \langle p|J(0)|q\rangle + O(e^{-\mu L})\,, \tag{1.1}$$

where μ is a characteristic mass scale—typically a multiple of the mass of lightest particle in the system. Furthermore, recall that three-momenta in a finite volume are discretized, $\mathbf{p} = (2\pi/L)\,\mathbf{n}$ and $\mathbf{n} \in \mathbb{Z}^3$. Hence, in order to have a fixed $\mathbf{p}$ in infinite volume, one cannot keep $\mathbf{n}$ constant. One could, for example, choose a monotonic sequence of discrete values of $L \in \{L_i\}$, such that $\mathbf{p}$ and $\mathbf{q}$ are allowed finite-volume momenta. Equation (1.1) must be interpreted exactly in this sense.

The situation is far less trivial in case of unstable particles. First, a one-particle state describing a resonance does not exist in the infinite-volume spectrum. Let us, for simplicity, consider a situation in which the resonance emerges in the scattering of two identical spinless particles. In order to define the resonance form factor in the infinite volume, one has to start from the five-point Green function $\langle p_1, p_2; out|J(0)|q_1, q_2; in\rangle$. Defining the total momenta of the outgoing and incoming particle pair by $P = p_1 + p_2$ and $Q = q_1 + q_2$, respectively, it can be shown that, if a resonance in a given channel exists, this five-point function possesses a double pole in the complex plane,

$$\langle p_1, p_2; out|J(0)|q_1, q_2; in\rangle \sim \frac{1}{(M_R^2 - P^2)(M_R^2 - Q^2)}, \tag{1.2}$$

located on some unphysical Riemann sheet for the variables P^2, Q^2. The infinite-volume resonance form factor can be expressed through the residue at this double pole (for more details, see e.g, Ref. [1]). This (complex-valued) form factor is a function of a single variable $t = (P-Q)^2$ and, in case of the conserved currents, obeys the usual Ward identities — for example, it is properly normalized at $t = 0$.

In a finite volume, one can access the spectrum of a Hamiltonian having the quantum number of two particle states. Let us denote the eigenstates of the Hamiltonian by $|\alpha, \mathbf{P}\rangle_L$

[1] Discretization effects are neglected throughout this paper.

(the so-called scattering states). Here, $\mathbf{P}$ is the total three-momentum of two particles and α labels different states having the same $\mathbf{P}$. If one varies L while keeping the $\mathbf{P}$ constant, the energies $E_\alpha(\mathbf{P}, L)$ exhibit power-law corrections in L with respect to the sum of the energies of one-particle states. Furthermore, one can evaluate the matrix elements of a current $\langle \alpha, \mathbf{P}|J(0)|\beta, \mathbf{Q}\rangle_L$ on the lattice for any α, β. Interpreting an infinite-volume limit of such a matrix element, as well as performing the analytic continuation to the resonance pole is however a delicate task. As in case of a stable particle, the momenta $\mathbf{P}, \mathbf{Q}$ are discretized and the limit $L \to \infty$ has to be treated accordingly (namely, the pertinent integer vectors $\mathbf{n}, \mathbf{m}$ cannot be considered fixed, see the discussion above). Furthermore, even for a fixed $\mathbf{P}$, the eigenvalues $E_\alpha(\mathbf{P}, L)$ collapse toward the threshold, as $L \to \infty$. Therefore, in order to stay in the vicinity of a fixed infinite-volume center-of-mass (CM) energy E (or, equivalently, $E_\alpha(\mathbf{P}, L) \simeq \sqrt{E^2 + \mathbf{P}^2}$), one cannot treat α as fixed anymore[2]. Higher excited states should be considered in the limit $L \to \infty$ and fixed E.

After fixing carefully the kinematics, one might ask oneself, how the infinite-volume limit should be carried out in the matrix elements. For instance, it is well known that the corrections are no more exponentially suppressed for unstable particles. Even for a much simpler case of the finite-volume decay matrix element of an unstable state, this limit is not well defined mathematically and can be performed only after removing the factor that corresponds to the interactions of the decay products in the final state, the well-known the Lellouch-Lüscher factor [3]. This approach works perfectly for the transition form factors of resonances into stable states, as well as for timelike form factors of stable particles, see [4–13]. Recently, a three-particle analog of the Lellouch-Lüscher formula has been also derived [14, 15]. The situation, however, becomes more complicated in case of the resonance matrix elements which is studied in the present paper. The problem is that, even after explicitly removing the Lellouch-Lüscher factors that correspond to the unstable particles, the remaining expression still does not exhibit a regular behavior in L and, hence, the infinite-volume limit cannot be performed [1, 16–18]. Additional developments concerning the evaluation of the resonance matrix elements can be found in Refs. [2, 19, 20].

To summarize the findings of Refs. [1, 16–18], a consistent procedure for the analytic continuation of the obtained result into the complex plane, which is needed to define a resonance form factor rigorously, cannot be straightforwardly formulated for the whole finite-volume matrix element. The culprit is the so-called triangle diagram, in which one of the "constituent particles" of a resonance couples to the external current J, whereas the second acts as a spectator, see Fig. 1a (for simplicity, we consider the resonance emerging in two-particle scattering). Such a triangle diagram is more singular in the finite volume than a loop diagram with two propagators, which corresponds to the Lüscher zeta-function. In Refs. [1, 16–18] the problem was addressed in different frameworks, but from a very similar physics perspective. Schematically, the proper procedure could be described as follows. It is proposed to single out the contribution of the triangle diagram in a finite volume. The

[2]Note that in Ref. [2], the finite-volume matrix elements at a fixed α, β are considered. For instance, the ground-state matrix element can be expanded in $1/L$, which gives only access to the form factor at zero-momentum transfer. By contrast, matrix elements at fixed energy have irregular behavior as a function of L, "jumping" over the poles of the one-loop diagrams in a finite volume.

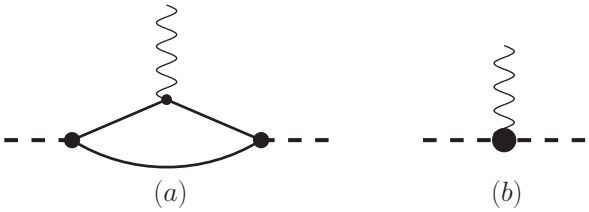

Figure 1. a) The triangle diagram which leads to an irregular behavior of the matrix element in a finite volume, and b) a local vertex that has a regular behavior. The dashed, single and wiggled lines denote the resonance, its constituents and the external current, respectively.

infinite-volume limit and the analytic continuation in the remainder of the amplitude can be performed without further ado. The triangle contribution, calculated analytically in the infinite volume and at the resonance pole, can be added back at the final stage. Even if the above procedure is absolutely consistent, the necessity of subtracting/adding the triangle diagram, to our taste, may turn the extraction of the resonance form factors into quite a challenging endeavor, with hard-to-control systematic errors.

On the other hand, the Feynman-Hellmann theorem [21, 22] has been successfully used to compute form factors of *stable* hadrons in a static, spatially periodic external field [23][3]. Moreover, the same method has been applied to the study of baryon structure functions and doubly virtual Compton scattering amplitude on the lattice [24–27]. In case of the form factor, one computes the two-point function *in an external field* in the Breit frame[4], and determines the mass of a particle in the external field. It can then be shown that the derivative of the particle mass with respect to the coupling constant to the external field gives, at leading order in this coupling constant, the form factor in the Breit frame.

It is natural to ask, whether one can generalize this method to the calculation of the form factors of unstable particles. The role of the particle mass in this case is played by the resonance pole position in the complex plane. In this paper we shall demonstrate that, at leading order, the derivative of the pole position with respect to the coupling constant to the external field gives the resonance form factor. It will also be shown that, in order to compute the resonance form factor, performing analytic continuation and finding a pole in the presence of the external field is even superfluous. It suffices to determine the local contribution to the form factor, see Fig. 1b, which can be extracted directly on the real axis. Then, the analytic continuation can be performed in the explicit expressions, evaluated in the infinite volume and in the absence of the external field. This does not cause any problems and hence, the problem related to the triangle diagram, does not show

[3]Note however that the study of the limit of zero-momentum transfer in this approach requires further scrutiny and is by no means trivial. The structure of the energy levels changes in this limit — the Landau levels emerge in the constant field. More discussion of this subtle issue is given in Ref. [24].

[4]Since the external field breaks translational invariance, the three-momentum is not conserved. The two-point function in the Breit frame is then defined as the one whose initial and final three-momenta $\mathbf{p}, \mathbf{q}$ are opposite in direction and have the same magnitude $|\mathbf{p}| = |\mathbf{q}| = \omega/2$, where $2\pi/\omega$ defines the period of the external field.

up in this approach. Note that the local contribution, unlike the matrix element itself, contains only exponentially suppressed corrections in a finite volume, which are easy to handle. Finally, note that applying the Feynman-Hellmann theorem to resonances is not new. In particular, this theorem has been used to define the sigma-terms for the resonances in Ref. [28]. In the present paper, the approach of Ref. [28] is generalized to the case of spatially periodic fields[5].

The material of the present paper is rather technical. In order to make the argument easier to follow, here we present a brief synopsis with the focus on the physics content.

i) In our derivation, we make use of the framework of the non-relativistic effective field theory (NREFT). As already mentioned, the contributions to the form factor fall into two classes. This is shown in Fig. 1, where the triangle diagram causes difficulties in the infinite-volume limit, and the local contribution does not. Note also that all ingredients needed to construct the triangle diagram are assumed to be known in advance: the external field coupling to a single constituent, and a resonance coupling to the constituents (described by the elastic phase shift at the energies close to the resonance mass). By contrast, the contact contribution is unknown and should be determined. Such a splitting can be naturally described in an NREFT framework, with a Lagrangian similar to the one given below in Eq. (2.1). Here, the coupling $C_R, \ldots$ determines the single particle form factor, the couplings C_0, C_2 describe the S-wave elastic scattering phase shift near threshold, and the quantity κ characterizes the lowest-order contact interaction. Higher-order terms are not displayed explicitly. At this order, determining κ on the lattice is equivalent to extracting the form factor, which can be straightforwardly calculated from the known analytic expression in the infinite volume that contains κ (as well as other constants) as free parameters.

ii) In order to describe the form factor, one has to inject a non-zero momentum transfer between the initial and final states. This can be achieved by placing a system in a spatially periodic external field, whose frequency is equal to the momentum transfer. The details are considered in Sect. 3, where the result of Ref. [23] concerning the determination of the form factor of a stable particle using the Feynman-Hellmann method has been re-derived and extended (see Sect. 3.2).

iii) The central result of the paper is the derivation of the generalized Lüscher equation in the periodic external field. Symbolically, this can be written as

$$\det\left(X^{-1} - \frac{1}{2}\,\Pi\right) = 0\,, \tag{1.3}$$

[5]Here, we mention in addition an application of the Feynman-Hellmann theorem to the calculation of the matrix element of a current between the two-body scattering states, which was carried out in Ref. [2]. However, as already mentioned, the results of that paper cannot be directly compared to ours. First, in our calculations, the CM energies of the incoming/outgoing pairs and not the labels α, β are fixed. Second, the Feynman-Hellmann theorem in Ref. [2] is used for the energies on the real axis and not in the complex plane. Lastly, the method of Ref. [2] is restricted to the scalar current which can be obtained through the differentiation of the Lagrangian over the particle mass, whereas the method, described in the present paper, can be applied to a generic current.

see Eqs. (4.8) and (4.9) for more details. Here, the matrix X^{-1} is a counterpart of the inverse K-matrix, $p\cot\delta(p)$, and the loop function Π corresponds to the Lüscher zeta-function Z_{00}, when the external periodic field is turned on. This equation enables one to extract the contact contribution κ from the two-particle energy spectrum in the external field, provided the single particle form factor and the phase shift have been computed in advance. The extraction of κ can be performed at real energies, an analytic continuation is not needed. The infinite-volume limit is trivial since, by definition, the contact terms can contain only exponentially suppressed corrections for the large box sizes.

iv) The Feynman-Hellmann theorem in quantum mechanics deals with the Hamiltonians, linearly depending on a parameter λ:

$$H = H_0 + \lambda\mathcal{O}\,, \tag{1.4}$$

where $\mathcal{O}$ is some operator. The eigenstates of the Hamiltonian, $|n(\lambda)\rangle$, and the eigenvalues, $E_n(\lambda)$, also depend on λ. The Feynman-Hellmann theorem states that

$$\frac{dE_n(\lambda)}{d\lambda} = \langle n(\lambda)|\mathcal{O}|n(\lambda)\rangle\,. \tag{1.5}$$

We generalize this result for unstable states. One can namely extract the resonance pole position P_R^0 in the complex energy plane, also for a non-vanishing external field, see Sect. 4.5. The derivative of P_R^0 with respect to the coupling to the external field, e (which plays the role of λ here), at $e = 0$ is proportional to the resonance form factor, evaluated at the (complex) resonance pole:

$$\left.\frac{dP_R^0(e)}{de}\right|_{e=0} \propto F\,. \tag{1.6}$$

For more details, see Sect. 4.6 and, in particular, Eq. (4.36).

The layout of the paper is as follows: In Sect. 2 we consider the problem exclusively in the infinite volume and give a consistent definition of the resonance form factor. Sect. 3 contains a collection of the formulae that describe the motion of a single spinless particle in a periodic external field. Here, we derive an exact expression for the one-particle propagator as well as the modified Lüscher zeta-function in the external field. Sect. 4 is directly dedicated to the extraction of the local contribution to the form factor. The proof of the Feynman-Hellman theorem for the resonance form factor within the NREFT framework is also described here. Finally, Sect. 5 contains the results of the numerical study of the quantization condition in an external field, which was carried out within a toy model. Note also that this paper provides a proof of principle only. For this reason, we have simplified the physical problem as much as possible. For example, we consider a non-relativistic case in detail, neglecting relativistic corrections whatsoever in the beginning. Moreover, to avoid clutter of indices, we restrict ourselves to the case of a single scalar field and neglect all partial waves other than the S-wave. All these effects can be taken into account in a rather straightforward fashion, see a very brief discussion in Sect. 4.7.

2 Resonance form factor in the infinite volume

Let us consider a scalar non-relativistic particle with mass m, moving in an external electromagnetic field $A^\mu(x)$. We shall further assume that only $A^0(x)$ is different from zero, and that it corresponds to the static field, i.e., $A^0(x) = A^0(\mathbf{x})$. The Lagrangian which describes particles in this field consists of an infinite tower of operators with increasing mass dimension that respect all symmetries, namely rotational invariance, the discrete symmetries, and gauge invariance. In the following, we shall restrict ourselves to at most two particles in the initial and final states. Hence, the operators in the Lagrangian should contain at most two fields ϕ and two conjugated fields. Furthermore, only terms up to first order in the coupling e will be included in the Lagrangian, since we are exclusively interested in the linear shift in the external field.

We shall start from the Lagrangian[6]

$$\begin{aligned}\mathcal{L} = \phi^\dagger\left(i\partial_t - m + eA^0 + \frac{eC_R}{6m^2}\triangle A^0 + \frac{\nabla^2}{2m}\right)\phi + C_0\phi^\dagger\phi^\dagger\phi\phi \\ + C_2\left(\phi^\dagger\phi^\dagger(\phi\overset{\leftrightarrow}{\nabla}^2\phi) + \text{h.c.}\right) + \frac{e\kappa}{4}\,\phi^\dagger\phi^\dagger\phi\phi\triangle A^0\,,\end{aligned} \tag{2.1}$$

where the Galilei-invariant derivative is defined as $a\overset{\leftrightarrow}{\nabla}b = \frac{1}{2}\,(a\nabla b - b\nabla a)$ and $\triangle$ denotes the Laplacian. Note that in the above Lagrangian we did not make an attempt to write down all possible terms up to a given order in the expansion in the inverse powers of m. Hence, the theory, defined by it, is only a model that nevertheless possesses all essential ingredients of the full theory. For the sake of clarity, we shall consider the proof on the basis of this model first, and address the general case very briefly only at the end[7].

The main aim of this section is to set up the framework for the evaluation of the resonance form factor in a theory described by the Lagrangian (2.1). The final result, given in Eq. (2.23), can be derived in few steps. We start from the two-particle scattering amplitude for the process $q_1+q_2 \to p_1+p_2$ at $e = 0$ (no external field). In the non-relativistic effective theory, this amplitude is given by a sum of bubble diagrams (we remind the reader that, for simplicity, we focus on S-wave scattering only):

$$T(\mathbf{p},\mathbf{q};\mathbf{P};P^0) = \frac{8\pi}{m}\left\{K(p,q) + K(p,q_0)\frac{iq_0}{1-iq_0K(q_0,q_0)}\,K(q_0,q)\right\}, \tag{2.2}$$

where $p = |\mathbf{p}|$, $q = |\mathbf{q}|$, and

$$q_0^2 = m\left(P^0 - 2m - \frac{\mathbf{P}^2}{4m}\right), \qquad K(p,q) = \frac{m}{8\pi}\left(4C_0 - 4C_2(p^2+q^2)\right). \tag{2.3}$$

[6] For a review of the non-relativistic effective theories for hadrons see, e.g., Ref. [29]

[7] A brief comment about gauge invariance is in order. The restrictions $A^0 = A^0(\mathbf{x})$ and $\mathbf{A} = 0$ do not leave room for gauge transformations except a trivial shift of A^0 by a constant. In order to arrive at the Lagrangian given in Eq. (2.1), one has first to write down the most general gauge-invariant Lagrangian for arbitrary A^μ, and choose a particular configuration of the external field afterwards. Note also that $\triangle A^0$ in Eq. (2.1) emerges from the gauge-invariant expression $-\nabla\mathbf{E}$, which reduces to $\triangle A^0$ for $\mathbf{A} = 0$.

Note that P^0 has an infinitesimal positive imaginary part $P^0 \to P^0 + i\varepsilon$ which, for brevity, is never shown explicitly. External particles are on mass shell: $p_i^0 = m + \mathbf{p}_i^2/(2m)$ and $q_i^0 = m + \mathbf{q}_i^2/(2m)$ for $i = 1, 2$. Furthermore, the center-of-mass and relative momenta are given by

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{q}_1 + \mathbf{q}_2\,, \qquad \mathbf{p} = \frac{\mathbf{p}_1 - \mathbf{p}_2}{2}\,, \qquad \mathbf{q} = \frac{\mathbf{q}_1 - \mathbf{q}_2}{2}\,. \tag{2.4}$$

On the energy shell, $\mathbf{p}^2 = \mathbf{q}^2 = q_0^2$ and the total energy P^0 is given by

$$P^0 = 2m + \frac{\mathbf{P}^2}{4m} + \frac{q_0^2}{m}\,. \tag{2.5}$$

The on-shell scattering amplitude takes the form

$$T(q_0) = \frac{8\pi/m}{K^{-1}(q_0, q_0) - iq_0} = \frac{8\pi/m}{-1/a + rq_0^2/2 + \cdots - iq_0}\,, \tag{2.6}$$

where

$$C_0 = -\frac{2\pi a}{m}\,, \qquad C_2 = \frac{\pi r a^2}{2m}\,, \tag{2.7}$$

and a, r denote the S-wave scattering length and effective range, respectively.

Let us now adjust the parameters a, r so that there is a low-lying resonance in the S-wave. In this case, the resonance pole position is determined from the equation:

$$-\frac{1}{a} + \frac{1}{2}\, r q_R^2 - \sqrt{-q_R^2} = 0\,. \tag{2.8}$$

The choice of the minus sign in front of the square root corresponds to the second Riemann sheet.

Suppose that a, r are chosen so that the above equation has a solution with $\mathrm{Re}\, q_R^2 > 0$, $\mathrm{Im}\, q_R^2 < 0$, with $|\mathrm{Im}\, q_R^2| \ll |\mathrm{Re}\, q_R^2| \ll m^2$. This solution corresponds to a low-lying resonance in the S wave. In moving frames, the complex resonance energy is then given by

$$P_R^0 = 2m + \frac{\mathbf{P}^2}{4m} + \frac{q_R^2}{m} = \mathrm{Re}\, P_R^0 - \frac{i}{2}\, \Gamma_R\,, \tag{2.9}$$

where Γ_R denotes the width of the resonance.

In the vicinity of the resonance pole, the two-body amplitude behaves as

$$\begin{aligned} T(q_0) &= \frac{8\pi/m}{K^{-1}(q_0, q_0) - \sqrt{-q_0^2}} \to \frac{Z}{q_0^2 - q_R^2} + \text{regular terms}\,, \\ Z &= \frac{8\pi/m}{\left[K^{-1}(q_R, q_R)\right]' - \left[\sqrt{-q_R^2}\right]'}\,, \end{aligned} \tag{2.10}$$

where primes indicate derivatives with respect to the variable q_0^2, and $q_0^2 = q_R^2$ is set at the end. In the following, Z will be referred to as the wave function renormalization constant of the resonance. It is, in general, a complex quantity.

Let us now turn the coupling to the external field on, and consider the two-point function of a particle in the external field up to $O(e)$:

$$\begin{aligned} S(\mathbf{p},\mathbf{q};p^0) =& i\int dt\, d^3\mathbf{x} d^3\mathbf{y}\, e^{ip^0 t - i\mathbf{p}\mathbf{x} + i\mathbf{q}\mathbf{y}} \langle 0|T\phi(\mathbf{x},t)\phi^\dagger(\mathbf{y},0)|0\rangle \\ =& \frac{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})}{m+\dfrac{\mathbf{p}^2}{2m}-p^0} + \frac{e\Gamma(\mathbf{p},\mathbf{q})\tilde{A}^0(\mathbf{p}-\mathbf{q})}{\left(m+\dfrac{\mathbf{p}^2}{2m}-p^0\right)\left(m+\dfrac{\mathbf{q}^2}{2m}-p^0\right)} + O(e^2)\,, \end{aligned} \tag{2.11}$$

where

$$\tilde{A}^0(\mathbf{p}-\mathbf{q}) = \int d^3\mathbf{x}\, e^{-i(\mathbf{p}-\mathbf{q})\mathbf{x}} A^0(\mathbf{x}) \tag{2.12}$$

is the Fourier-transform of the (static) scalar potential. Furthermore, the one-particle form factor $\Gamma(\mathbf{p},\mathbf{q};p^0)$ can be directly read off from the Lagrangian,

$$\Gamma(\mathbf{p},\mathbf{q}) = 1 - \frac{C_R}{6m^2}\,(\mathbf{p}-\mathbf{q})^2\,, \tag{2.13}$$

where the quantity C_R is related to the mean charge radius through $C_R = m^2\langle r^2\rangle$. Note also that the on-shell condition for the non-relativistic particles is $\mathbf{p}^2/(2m) = \mathbf{q}^2/(2m) = p^0 - m$.

Next, we turn to the definition of the resonance form factor. This quantity can be defined through the expansion of the (equal-time) four-point function in the external field, similarly to the one-particle form factor obtained through the expansion of the two-point function. This four-point function is defined as

$$\begin{aligned} \tilde{G}(\mathbf{p},\mathbf{P};\mathbf{q},\mathbf{Q};P^0) =& \; i\int dt\, d^3\mathbf{x}_1 d^3\mathbf{x}_2 d^3\mathbf{y}_1 d^3\mathbf{y}_2\, e^{iP^0 t - i\mathbf{p}_1\mathbf{x}_1 - i\mathbf{p}_2\mathbf{x}_2 + i\mathbf{q}_1\mathbf{y}_1 + i\mathbf{q}_2\mathbf{y}_2} \\ & \times \langle 0|T\phi(\mathbf{x}_1,t)\phi(\mathbf{x}_2,t)\phi^\dagger(\mathbf{y}_1,0)\phi^\dagger(\mathbf{y}_2,0)|0\rangle\,. \end{aligned} \tag{2.14}$$

In the absence of the external field, the equal-time four-point function can be related to the two-particle scattering amplitude, considered above. Writing $\tilde{G} = \tilde{G}_0 + e\tilde{G}_1 + O(e^2)$, we obtain

$$\begin{aligned} \tilde{G}_0(\mathbf{p},\mathbf{P};\mathbf{q},\mathbf{Q};P^0) =& \frac{(2\pi)^3\delta^3(\mathbf{P}-\mathbf{Q})\,(2\pi)^3\left(\delta^3(\mathbf{p}-\mathbf{q})+\delta^3(\mathbf{p}+\mathbf{q})\right)}{2m+\dfrac{\mathbf{P}^2}{4m}+\dfrac{\mathbf{p}^2}{m}-P_0} \\ & + \frac{(2\pi)^3\delta^3(\mathbf{P}-\mathbf{Q})T(\mathbf{p},\mathbf{q};\mathbf{P};P^0)}{\left(2m+\dfrac{\mathbf{P}^2}{4m}+\dfrac{\mathbf{p}^2}{m}-P_0\right)\left(2m+\dfrac{\mathbf{P}^2}{4m}+\dfrac{\mathbf{q}^2}{m}-P^0\right)}\,, \end{aligned} \tag{2.15}$$

where $T(\mathbf{p},\mathbf{q};\mathbf{P};P^0)$ is the two-body amplitude, introduced in Eqs. (2.2), (2.3), and $\mathbf{P}$, $\mathbf{Q}$ are the total three-momenta of the particle pairs in the initial and final states, respectively, see Eq. (2.4). The four-point function has a simple pole at $P^0 \to P^0_R$,

$$\tilde{G}_0(\mathbf{p},\mathbf{P};\mathbf{q},\mathbf{Q};P^0) \to (2\pi)^3\delta^3(\mathbf{P}-\mathbf{Q})\frac{\Psi(\mathbf{P},\mathbf{p})\bar{\Psi}(\mathbf{Q},\mathbf{q})}{P^0-P^0_R}\,, \tag{2.16}$$

where

$$\Psi(\mathbf{P},\mathbf{p}) = \frac{1}{2m+\dfrac{\mathbf{P}^2}{4m}+\dfrac{\mathbf{p}^2}{m}-P_0}\sqrt{\frac{Z}{m}}\frac{K(p,q_R)}{K(q_R,q_R)}\,,$$

$$\bar\Psi(\mathbf{Q},\mathbf{q}) = \sqrt{\frac{Z}{m}}\frac{K(q,q_R)}{K(q_R,q_R)}\frac{1}{2m+\dfrac{\mathbf{Q}^2}{4m}+\dfrac{\mathbf{q}^2}{m}-P_0}\,. \tag{2.17}$$

In analogy to the case of the two-particle bound states, we shall refer to the quantity Ψ as the "wave function of a resonance." Note that this does not have anything to do with the interpretation of a resonance as a true quantum-mechanical state described by this "wave function", but just represents a convenient brief name.

Next, it can be checked that, like true wave functions, the quantities $\Psi, \bar\Psi$ are normalized according to

$$\begin{aligned}\frac{1}{2!}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\bar\Psi(\mathbf{P},\mathbf{p})\Psi(\mathbf{P},\mathbf{p}) &= \frac{Z}{2mK^2(q_R,q_R)}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{K^2(p,q_R)}{\left(2m+\dfrac{\mathbf{P}^2}{4m}+\dfrac{\mathbf{p}^2}{m}-P_0\right)^2}\\ &= -\frac{Zm}{8\pi K^2(q_R,q_R)}\,\frac{d}{dq_0^2}\Big(K^2(q_0,q_R)\sqrt{-q_0^2}\Big)\Big|_{q_0^2=q_R^2}\\ &= -\frac{Zm}{8\pi K^2(q_R,q_R)}\left(K'(q_R,q_R)+K^2(q_R,q_R)[\sqrt{-q_R^2}]'\right)=1\,.\end{aligned} \tag{2.18}$$

Here, the factor $1/2!$ emerges from the Bose-symmetry. In the derivation, we have used the fact that the function $K(p,q)$ is real and symmetric (this, in its turn, stems from the hermiticity of the Hamiltonian) and, hence,

$$\frac{d}{dq_0^2}\,K(q_0,q_R)\Big|_{q_0^2=q_R^2} = \frac{1}{2}\Big[K(q_R,q_R)\Big]'\,. \tag{2.19}$$

Up to order e, the equal-time Green function takes the form

$$\begin{aligned}\tilde G_1(\mathbf{p},\mathbf{P};\mathbf{q},\mathbf{Q};P^0) = \frac{1}{(2!)^2}\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\,\frac{d^3\mathbf{P}'}{(2\pi)^3}\,\frac{d^3\mathbf{q}'}{(2\pi)^3}\,\frac{d^3\mathbf{Q}'}{(2\pi)^3}\,\tilde G_0(\mathbf{p},\mathbf{P};\mathbf{p}',\mathbf{P}';P^0)\\ \times\,\tilde\Gamma(\mathbf{p}',\mathbf{P}';\mathbf{q}',\mathbf{Q}')\tilde G_0(\mathbf{q}',\mathbf{Q}';\mathbf{q},\mathbf{Q};P^0)\,.\end{aligned} \tag{2.20}$$

At this order, the vertex $\tilde\Gamma$ is given by a sum of a finite number of diagrams, shown in Fig. 2:

$$\tilde\Gamma(\mathbf{p},\mathbf{P};\mathbf{q},\mathbf{Q}) = \bar\Gamma(\mathbf{p},\mathbf{P};\mathbf{q},\mathbf{Q})\tilde A^0(\mathbf{P}-\mathbf{Q})\,, \tag{2.21}$$

where

$$\begin{aligned}\bar\Gamma(\mathbf{p},\mathbf{P};\mathbf{q},\mathbf{Q}) &= -\kappa(\mathbf{P}-\mathbf{Q})^2\\ &+\left\{(2\pi)^3\delta^3\Big(\frac{(\mathbf{P}-\mathbf{Q})}{2}+(\mathbf{p}-\mathbf{q})\Big)\Gamma\Big(\frac{\mathbf{P}}{2}-\mathbf{p},\frac{\mathbf{Q}}{2}-\mathbf{q}\Big)+\begin{pmatrix}\mathbf{p}\to-\mathbf{p}\\ \mathbf{q}\to-\mathbf{q}\\ \mathbf{p}\to-\mathbf{p},\ \mathbf{q}\to-\mathbf{q}\end{pmatrix}\right\},\end{aligned} \tag{2.22}$$

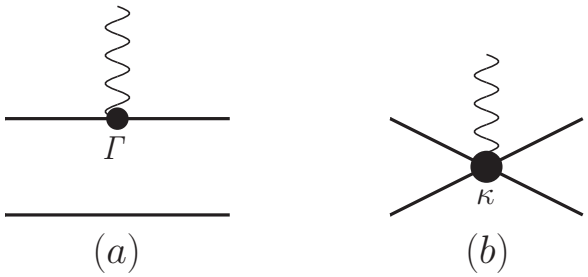

Figure 2. Diagrams contributing to the kernel $\tilde{\Gamma}$ which, convoluted with the wave functions $\bar{\Psi}, \Psi$, yields the resonance form factor, see Eq. (2.23). The troublesome triangle diagram (a) in the finite-volume form factor, mentioned in the Introduction, emerges from the disconnected part, where the photon is attached to one of the particles, whereas the second one acts as a spectator. The fully connected diagram (b), where the photon is emanated from the four-particle vertex, is unproblematic.

and Γ is the one-particle vertex, given in Eq. (2.13).

The quantity $\tilde{G}_1$ has a double pole in the variables P^0, Q^0 that is contained in the free Green functions $\tilde{G}_0$. The residue at the double pole defines the form factor of the resonance:

$$F(\mathbf{P},\mathbf{Q}) = \frac{1}{(2!)^2}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{q}}{(2\pi)^3}\,\bar{\Psi}(\mathbf{P},\mathbf{p})\bar{\Gamma}(\mathbf{p},\mathbf{P};\mathbf{q},\mathbf{Q})\Psi(\mathbf{Q},\mathbf{q})\,, \tag{2.23}$$

where the energy is fixed at the resonance pole. Using the normalization of the wave functions, given in Eq. (2.18), and the fact that, due to the Bose-symmetry, these wave functions are symmetric under $\mathbf{p}\to-\mathbf{p}$ and $\mathbf{q}\to-\mathbf{q}$, respectively, it can be immediately shown that the resonance form factor is properly normalized at zero momentum transfer, as required by the Ward identity (we remind the reader that the charge of the resonance is equal to $2e$). In Appendix D we provide the explicit form of Eq. (2.23) in dimensional regularization.

All parameters that are present in the Lagrangian (2.1) enter the expression (2.23) as well. Namely, the wave functions $\Psi, \bar{\Psi}$ contain the elastic two-particle scattering parameters C_0, C_2, whereas the kernel $\bar{\Gamma}$ depends on the parameter C_R that describes the single particle form factor, as well as the coupling κ, characterizing the contact term. There will be more couplings, if higher-order derivative terms, higher partial waves, etc., are included, but the general pattern is already clear. All these couplings should be determined on the lattice, on the same configurations. In order to determine the elastic scattering phase, related to C_0, C_2, one may use standard Lüscher approach for the two-body scattering at $e=0$. The value of C_R can be established by calculating the single particle form factor by using either the standard method or the Feynman-Hellmann theorem. At the order we are working, only a single constant κ remains unknown. Below it will be shown, how this constant can be fixed in the external field.

The framework that we considered in this Section is not new and represents a properly adapted version of the Mandelstam formalism [30, 31], which is used to define form factors

of *stable* particles. The purpose of such a detailed treatment was to set the stage for a similar calculation in a finite volume. In the following, it will be demonstrated that using the Feynman-Hellmann theorem in a periodic external field, one arrives exactly at the quantity defined by Eq. (2.23) in the infinite-volume limit.

3 Single particle in a periodic external field

3.1 Solutions of the Mathieu equation

Up to this point, the discussion was carried out for a generic static external field $A^0(\mathbf{x})$. In order to inject a momentum between the initial and final states on the lattice, it is convenient to consider a spatially periodic field

$$A^0(\mathbf{x}) = A_0 \cos(\boldsymbol{\omega}\mathbf{x})\,, \qquad \boldsymbol{\omega} = (0,0,\omega)\,. \tag{3.1}$$

Here, for convenience, we have chosen the vector $\boldsymbol{\omega}$ in the direction of the z-axis. Furthermore, we project all vectors onto the direction of $\boldsymbol{\omega}$: for instance, the position vector has the components $\mathbf{x} = (\mathbf{x}_\perp, x_\parallel)$, where $\mathbf{x}_\perp$, $x_\parallel$ denote the components perpendicular and parallel to the z-axis, respectively.

In this section, we shall derive a closed expression of the two-point function of the field ϕ in the external field. We are working here in a cubic box with a spatial elongation L (the time elongation is assumed to be infinite). Periodic boundary conditions are imposed in the spatial directions. As a result, the three-momenta of the particles as well as the frequency ω are quantized:

$$\mathbf{p} = \frac{2\pi}{L}\,\mathbf{n}\,, \quad \mathbf{n} \in \mathbb{Z}^3\,, \quad \text{and} \quad \omega = \frac{2\pi}{L}\,N\,, \quad N \in \mathbb{Z}\,. \tag{3.2}$$

Let us denote by $|1\rangle$ a state with a single particle in the periodic field. The matrix element of the field operator between the vacuum and the one-particle state defines the Schrödinger wave function

$$\Phi(\mathbf{x},t) = \langle 0|\phi(\mathbf{x},t)|1\rangle\,. \tag{3.3}$$

The wave function obeys a differential equation that can be obtained by using the equations of motion for the field $\phi(\mathbf{x},t)$:

$$\left(i\partial_t + e\Gamma A_0 \cos(\omega x_\parallel) - m + \frac{\nabla^2}{2m}\right)\Phi(\mathbf{x},t) = 0\,. \tag{3.4}$$

Here,

$$\Gamma = \Gamma(\boldsymbol{\omega}) = 1 - \frac{C_R}{6m^2}\,\omega^2 \tag{3.5}$$

is the single-particle form factor evaluated at the three-momentum transfer ω. Note that, after factorizing Eq. (3.4) by using an ansatz $\Phi(\mathbf{x},t) = e^{-iEt+i\mathbf{p}_\perp\mathbf{x}_\perp} f(x_\parallel)$, this equation

can be reduced to a so-called Mathieu equation for the function $f(x_\parallel)$. The (unnormalized) solutions of Eq. (3.4) that obey periodic boundary conditions are given by

$$\begin{aligned}\Phi(\mathbf{x},t) &= e^{-iEt+i\mathbf{p}_\perp\mathbf{x}_\perp}\mathrm{me}_{\nu_i+2n}(z,q)\,,\\ z &= \frac{\omega x_\parallel}{2}\,, \qquad q = -\frac{4meFA_0}{\omega^2}\,.\end{aligned} \tag{3.6}$$

where $\mathrm{me}_{\nu_i+2n}(z,q)$ denotes the Mathieu function and the index ν_i+2n, $n\in\mathbb{Z}$, $i=1,\ldots,N$ labels the eigenfunctions of the Mathieu differential equation corresponding to the eigenvalues $\lambda_{\nu_i+2n}(q)$ [32, 33]. Details are given in App. A.

The completeness condition for the solutions of the Mathieu equation takes the form:

$$\frac{1}{\pi}\sum_{i=1}^{N}\sum_{n=-\infty}^{\infty}\mathrm{me}_{\nu_i+2n}(z,q)\mathrm{me}_{\nu_i+2n}(-z',q) = N\sum_{k=-\infty}^{\infty}\delta(z-z'-\pi kN)\,, \tag{3.7}$$

with the ν_i as given in Eq. (A.8).

The propagator of the particle ϕ in the external field is defined as:

$$S(\mathbf{x},\mathbf{y};E) = i\int_{-\infty}^{+\infty}dt e^{iEt}\langle 0|T\phi(\mathbf{x},t)\phi^\dagger(\mathbf{y},0)|0\rangle\,, \tag{3.8}$$

and it is given by the sum over the eigenfunctions (spectral representation):

$$\begin{aligned}S(\mathbf{x},\mathbf{y};E) = \frac{1}{L^3}\sum_{\mathbf{p}_\perp}\sum_{i=1}^{N}\sum_{n=-\infty}^{\infty}&\frac{e^{i\mathbf{p}_\perp(\mathbf{x}_\perp-\mathbf{y}_\perp)}}{m+\dfrac{\mathbf{p}_\perp^2}{2m}+\dfrac{\omega^2}{8m}\lambda_{\nu_i+2n}(q)-E}\\ &\times\mathrm{me}_{\nu_i+2n}\left(\frac{\omega x_\parallel}{2},q\right)\mathrm{me}_{\nu_i+2n}\left(-\frac{\omega y_\parallel}{2},q\right).\end{aligned} \tag{3.9}$$

Indeed, it can be directly verified that

$$\left(E+eA_0\cos(\omega x_\parallel)-m+\frac{\nabla^2}{2m}\right)S(\mathbf{x},\mathbf{y};E) = -\sum_{\mathbf{m}\in\mathbb{Z}^3}\delta^3(\mathbf{x}-\mathbf{y}-\mathbf{m}L)\,. \tag{3.10}$$

The propagator can be expanded in powers of e (this corresponds to the Taylor expansion in the parameter q). Up to $O(e)$, the result takes the expected simple form:

$$\begin{aligned}S(\mathbf{p},\mathbf{q};E) &= \int^L d^3\mathbf{x}d^3\mathbf{y}e^{-i\mathbf{p}\mathbf{x}+i\mathbf{q}\mathbf{y}}S(\mathbf{x},\mathbf{y};E)\\ &= L^3\left\{\frac{\delta^3_{\mathbf{pq}}}{m+\dfrac{\mathbf{p}^2}{2m}-E}+\frac{1}{2}eA_0F\frac{\delta^3_{\mathbf{p}-\boldsymbol{\omega},\mathbf{q}}+\delta^3_{\mathbf{p}+\boldsymbol{\omega},\mathbf{q}}}{\left(m+\dfrac{\mathbf{p}^2}{2m}-E\right)\left(m+\dfrac{\mathbf{q}^2}{2m}-E\right)}\right\}+O(e^2)\,.\end{aligned} \tag{3.11}$$

The proof of this equation is given in Appendix B.

3.2 The energy shift in the periodic field

The spectrum of a particle in an external periodic field is determined by the poles of the propagator. Performing the Fourier transform using Eq. (A.9), the propagator can be rewritten in the following form:

$$S(\mathbf{p},\mathbf{q};E) = L^3\delta^2_{\mathbf{p}_\perp,\mathbf{q}_\perp}\sum_{i=1}^{N}\sum_{n=-\infty}^{\infty}\sum_{a,b=-\infty}^{\infty} C_{2a}^{\nu_i+2n}(q)C_{2b}^{\nu_i+2n}(q) \times \frac{\delta_{-p_\parallel,\frac{\omega}{2}(\nu_i+2n+2a)}\delta_{-q_\parallel,\frac{\omega}{2}(\nu_i+2n+2b)}}{m+\dfrac{\mathbf{q}_\perp^2}{2m}+\dfrac{\omega^2}{8m}\lambda_{\nu_i+2n}(q)-E}\,. \tag{3.12}$$

Here, the coefficients $C_{2a,2b}^{\nu_i+2n}(q)$ $(a,b\in\mathbb{Z})$ are the same as in Eq. (A.9), and their explicit form does not matter here. We see now that, instead of one pole, the propagator in the external field has a tower of poles. This was expected, because the periodic external field carries the momentum $\boldsymbol{\omega}$. Consequently, the three-momentum is not conserved in such a field, and $\mathbf{q}=\mathbf{p}+\ell\boldsymbol{\omega}$, where $\ell\in\mathbb{Z}$ is an integer. In addition, since the particle interacts with the field, the energies (pole positions) are slightly displaced from the non-interacting values corresponding to $\lambda_{\nu_i+2n}(0)=(\nu_i+2n)^2$ and are determined through the equation

$$E = m+\frac{\mathbf{p}_\perp^2}{2m}+\frac{\omega^2}{8m}\lambda_{\nu_i+2n}(q)\,. \tag{3.13}$$

The crucial point is that $\lambda_{\nu_i+2n}(q)=(\nu_i+2n)^2+O(q^2)$ for all values of ν_i+2n except $(\nu_i+2n)=\pm1$. In this case,

$$\lambda_1(q)=1+q+O(q^2)\,,\qquad \lambda_{-1}(q)=1-q+O(q^2)\,. \tag{3.14}$$

In the following, for simplicity, we shall take $\mathbf{p}_\perp=\mathbf{q}_\perp=0$ and determine the *lowest* eigenvalue in the sectors with different values of N. (Note that the integer number N characterizes the momentum transfer in the external field vertex, in units of $2\pi/L$.) The components $p_\parallel, q_\parallel$ are given by $p_\parallel=2\pi n_\parallel/L$ and $q_\parallel=2\pi n'_\parallel/L$. In the sectors with $N=1,2$ the argument goes as follows:

<u>$N=1$:</u>

In this case, $\nu_i=0$, and we have

$$-n_\parallel = n+a\,,\qquad -n'_\parallel = n+b\,. \tag{3.15}$$

For any choice of $n_\parallel, n'_\parallel$, we may find a,b so that $n=0$. Hence, the lowest eigenvalue is $\lambda_{\nu_i+2n}(q)=\lambda_0(q)=O(q^2)$.

<u>$N=2$:</u>

In this case, we have $\nu_i=0,1$ and

$$-n_\parallel = \nu_i+2n+2a\,,\qquad -n'_\parallel = \nu_i+2n+2b\,. \tag{3.16}$$

This means that $n_\parallel, n'_\parallel$ should be either odd or even. If both are chosen to be even, then $\nu_i = 0$ should be fulfilled and $n = 0$ is allowed. Then, the lowest eigenvalue is $\lambda_0(q)$. On the other hand, if $n_\parallel, n'_\parallel$ are odd, then $\nu_i = 1$ is picked up. Since n is integer, $\nu_i + 2n$ is odd and $\lambda_0(q)$ never appears. The lowest eigenvalues are then $\lambda_{\pm 1}(q)$. Assuming $A_0 > 0$ and $q > 0$, we get $\lambda_1(q) < \lambda_1(-q)$ and the lowest energy level will be at

$$E = m + \frac{\omega^2 \lambda_1(q)}{8m} = m + \frac{\omega^2}{8m} - \frac{1}{2} e A_0 \Gamma \,. \tag{3.17}$$

Hence, differentiating the pole shift with respect to e, one gets the particle form factor $\Gamma = \Gamma(\boldsymbol{\omega})$. This is exactly the case considered in Ref. [23]: one places the charged particle in the periodic external field with $\omega = 4\pi/L$, and considers the Breit frame $\mathbf{p} = -\mathbf{q} = -\boldsymbol{\omega}/2$. Then, the linear derivative of the shift of the lowest energy level with respect to the coupling to the external field yields the form factor at the momentum transfer $\boldsymbol{\omega}$. Hence, our result confirms and extends the findings of Ref. [23] to different incoming and outgoing momenta, as well as to the higher values of N.

4 Two-particle scattering in the periodic external field

4.1 Lüscher equation

In this section, we shall derive the counterpart of the Lüscher equation in the external field, which allows one to extract the contact coupling, κ, from the finite-volume energy spectrum. To this end, let us consider the two-point function of the composite fields $\phi^2, [\phi^\dagger]^2$:

$$D(\mathbf{P}, \mathbf{Q}; t) = i \int^L d^3\mathbf{x} d^3\mathbf{y} e^{-i\mathbf{P}\mathbf{x} + i\mathbf{Q}\mathbf{y}} \langle 0 | T\phi^2(\mathbf{x}, t) [\phi^\dagger(\mathbf{y}, 0)]^2 | 0 \rangle \,. \tag{4.1}$$

The diagrams that contribute to this quantity are shown in Fig. 3. These are reminiscent of the diagrams in the absence of an external field, with two differences: a) the particle propagators in these diagrams are the full ones that include the summation of all external field insertions in these propagators, and b) in addition to the conventional four-particle vertices, there are vertices with the external field attached (the pertinent operator comes with the coupling κ in the Lagrangian). Below, we shall study the implications of these modifications.

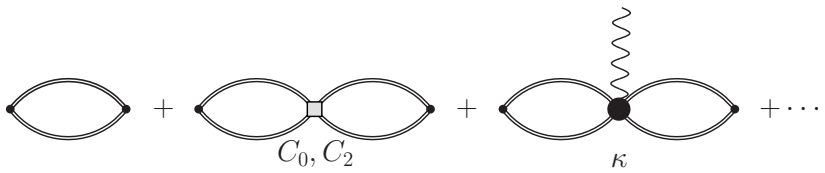

Figure 3. Two-point function of the composite field $\phi^2(x)$. Double lines denote the full one-particle propagator in the external field, see Fig. 4. The diagrams, in which the external field is attached to the four-particle vertex, are explicitly included.

Figure 4. Full one-particle propagator in the external field.

First, note that, since the three-momentum is not conserved in the presence of the external field, the two-point function is no more diagonal in the incoming/outgoing total three-momenta $\mathbf{Q}$ and $\mathbf{P}$. Instead of an overall factor $L^3\delta^3_{\mathbf{PQ}}$ it contains a tower of terms with $L^3\delta^3_{\mathbf{P}+\ell\boldsymbol{\omega},\mathbf{Q}}$ and $\ell \in \mathbb{Z}$, $\ell \neq 0$. However, since each momentum flip proceeds through the interaction with the external field and thus adds one power of the coupling e, the terms that multiply $L^3\delta^3_{\mathbf{P}+\ell\boldsymbol{\omega},\mathbf{Q}}$ start at $O(e^\ell)$. Hence, at a given order in e, the quantity $D(\mathbf{P},\mathbf{Q};E)$ is a band matrix with the indices $\mathbf{P},\mathbf{Q}$. Note also that here we do not attempt *a priori* to expand the particle propagator in powers of e, since such an expansion cannot be easily justified on the real axis of the energy.

Let us consider the elementary loop diagram in Fig. 3. If both four-particle vertices in such a diagram do not contain derivatives of the field ϕ, such a loop is given merely by a convolution of two propagators in the external field

$$\mathit{\Pi}(\mathbf{P},\mathbf{Q};E) = \int \frac{dp^0}{2\pi i} \int^L d^3\mathbf{x} d^3\mathbf{y}\, e^{-i\mathbf{P}\mathbf{x}+i\mathbf{Q}\mathbf{y}} S(\mathbf{x},\mathbf{y};p^0) S(\mathbf{x},\mathbf{y};E-p^0)\,. \tag{4.2}$$

Here, $S(\mathbf{x},\mathbf{y};E)$ is defined by Eq. (3.9).

The situation is slightly more complicated in case of vertices with derivatives, e.g., the vertex that is proportional to the coupling C_2 in Eq. (2.1). In the case with no external field and using dimensional regularization, it is possible to "pull out" the derivatives acting on the internal lines and transform them into the external momenta. The difference is an off-shell term, which cancels the denominator in the loop, leaving a low-energy polynomial that vanishes after integration in dimensional regularization. The above fact allows one to derive the Lüscher equation in a very simple manner.[8] When the external field is switched on, "pulling out" the derivatives leads to an extra term that depends on the external field.

[8]The final result is the same in all regularizations but the use of dimensional regularization makes the derivation particularly simple.

Indeed, using Eq. (4.2) together with Eq. (A.4), one can easily show that

$$\begin{aligned}
&\int^L d^3\mathbf{x}d^3\mathbf{y}\, e^{-i\mathbf{P}\mathbf{x}+i\mathbf{Q}\mathbf{y}}\Big\{S(\mathbf{x},\mathbf{y};p^0)(\overrightarrow{\nabla}_x-\overleftarrow{\nabla}_x)^2S(\mathbf{x},\mathbf{y};E-p^0)\Big\}\\
&=\int^L d^3\mathbf{x}d^3\mathbf{y}\, e^{-i\mathbf{P}\mathbf{x}+i\mathbf{Q}\mathbf{y}}\\
&\times\frac{1}{L^3}\sum_{\mathbf{p}_\perp}\sum_{i=1}^{N}\sum_{n=-\infty}^{\infty}\frac{e^{i\mathbf{p}_\perp(\mathbf{x}_\perp-\mathbf{y}_\perp)}\mathrm{me}_{\nu_i+2n}\Big(\frac{\omega x_\parallel}{2},q\Big)\mathrm{me}_{\nu_i+2n}\Big(-\frac{\omega y_\parallel}{2},q\Big)}{m+\frac{\mathbf{p}_\perp^2}{2m}+\frac{\omega^2}{8m}\lambda_{\nu_i+2n}(q)-p^0-i\varepsilon}\\
&\times\Big(-2\mathbf{p}_\perp^2-2\mathbf{q}_\perp^2+\mathbf{P}^2-\frac{\omega^2}{2}\left(\lambda_{\nu_i+2n}(q)+\lambda_{\nu_j+2m}(q)\right)-8me\Gamma A^0(\mathbf{x})\Big)\\
&\times\frac{1}{L^3}\sum_{\mathbf{q}_\perp}\sum_{j=1}^{N}\sum_{m=-\infty}^{\infty}\frac{e^{i\mathbf{q}_\perp(\mathbf{x}_\perp-\mathbf{y}_\perp)}\mathrm{me}_{\nu_j+2m}\Big(\frac{\omega x_\parallel}{2},q\Big)\mathrm{me}_{\nu_j+2m}\Big(-\frac{\omega y_\parallel}{2},q\Big)}{m+\frac{\mathbf{q}_\perp^2}{2m}+\frac{\omega^2}{8m}\lambda_{\nu_j+2m}(q)-E+p^0-i\varepsilon}\,.
\end{aligned}\tag{4.3}$$

The expression in the brackets, which is present in the numerator, can be rewritten as

$$\begin{aligned}
&-2\mathbf{p}_\perp^2-2\mathbf{q}_\perp^2+\mathbf{P}^2-\frac{\omega^2}{2}\left(\lambda_{\nu_i+2n}(q)+\lambda_{\nu_j+2m}(q)\right)-8me\Gamma A^0(\mathbf{x})\\
&=-4m\Big(m+\frac{\mathbf{p}_\perp^2}{2m}+\frac{\omega^2}{8m}\lambda_{\nu_i+2n}(q)-p^0\Big)-4m\Big(m+\frac{\mathbf{q}_\perp^2}{2m}+\frac{\omega^2}{8m}\lambda_{\nu_j+2m}(q)-E+p^0\Big)\\
&-4m\Big(E-2m-\frac{\mathbf{P}^2}{4m}\Big)-8me\Gamma A^0(\mathbf{x})\,.
\end{aligned}\tag{4.4}$$

The first two terms cancel with one of the denominators in Eq. (4.3). Using dimensional regularization, it can be argued that these two terms give a vanishing contribution to the integral. The third term corresponds to "pulling out" the derivatives on the internal lines. Only the last term is new and shows that, in case of a non-vanishing external field, there is an additional contribution. Physically, this corresponds to the four-particle vertex with the external field attached (topologically equivalent to the one that contains the coupling κ). In other words, pulling out the derivatives is equivalent to adjusting the coefficients of such terms[9].

Carrying out this procedure consistently in all loops, it is seen that the full two-point function D obeys the equation

$$\frac{1}{4}D_{\mathbf{P}\mathbf{Q}}(E)=\frac{1}{2}\Pi_{\mathbf{P}\mathbf{Q}}(E)+\frac{1}{L^6}\sum_{\mathbf{P}'\mathbf{Q}'}\frac{1}{2}\Pi_{\mathbf{P}\mathbf{P}'}(E)X_{\mathbf{P}'\mathbf{Q}'}(E)\frac{1}{4}D_{\mathbf{Q}'\mathbf{Q}}(E)\,.\tag{4.5}$$

Here, for convenience, we have used matrix notation, considering the momenta $\mathbf{P},\mathbf{Q},\ldots$ as the indices. The kernel X is given by

$$X_{\mathbf{P}\mathbf{Q}}(E)=L^3\delta^3_{\mathbf{P}\mathbf{Q}}X^{(0)}_{\mathbf{P}}(E)+\frac{e}{2}L^3(\delta^3_{\mathbf{P}+\boldsymbol{\omega},\mathbf{Q}}+\delta^3_{\mathbf{P}-\boldsymbol{\omega},\mathbf{Q}})X^{(1)}_{\mathbf{P}\mathbf{Q}}(E)+O(e^2)\,,\tag{4.6}$$

[9]Note, however, that this additional term does not carry the momenta $\mathbf{P},\mathbf{Q}$ and, in particular, does not vanish at $(\mathbf{P}-\mathbf{Q})^2=0$. This indicates that pulling all derivatives out of the loop, checking Ward identities as well as the normalization of the form factor at zero momentum transfer can become technically complicated, albeit gauge invariance still holds.

where

$$X^{(0)}_{\mathbf{P}}(E) = 4C_0 - 8mC_2\left(E - 2m - \frac{\mathbf{P}^2}{4m}\right) + \cdots ,$$

$$X^{(1)}_{\mathbf{PQ}}(E) = -\kappa\omega^2 A_0 - 16mC_2\Gamma A_0 + \cdots . \tag{4.7}$$

Note that the second term in $X^{(1)}$ emerges after pulling out the derivatives.

The derivation of the Lüscher equation is now straightforward. The energy levels are determined by the equation

$$\det \mathcal{M} = 0\,, \qquad \mathcal{M}_{\mathbf{PQ}}(E) = [X_{\mathbf{PQ}}(E)]^{-1} - \frac{1}{2}\,\Pi_{\mathbf{PQ}}(E)\,. \tag{4.8}$$

Furthermore, up to first order in e, the inverse of the matrix X is given by

$$[X_{\mathbf{PQ}}(E)]^{-1} = L^3\delta^3_{\mathbf{PQ}}k(\mathbf{P};E) - \frac{e}{2}\,L^3(\delta^3_{\mathbf{P}+\boldsymbol{\omega},\mathbf{Q}} + \delta^3_{\mathbf{P}-\boldsymbol{\omega},\mathbf{Q}})k(\mathbf{P};E)X^{(1)}_{\mathbf{PQ}}(E)k(\mathbf{Q};E) + O(e^2)\,, \tag{4.9}$$

where

$$k(\mathbf{P};E) = \left(4C_0 - 8mC_2\left(E - \frac{\mathbf{P}^2}{4m}\right) + \cdots\right)^{-1} = \frac{m}{8\pi}\left(-\frac{1}{a} + \frac{1}{2}\,r q_0^2(\mathbf{P};E) + \cdots\right),$$

$$q_0^2(\mathbf{P};E) = m\left(E - 2m - \frac{\mathbf{P}^2}{4m}\right). \tag{4.10}$$

As seen from the above equation, at leading order in e, the inverse of the kernel reduces to the well-known expression $q_0 \cot\delta(q_0)$. This was of course expected from the beginning. The $O(e)$ corrections to the kernel can be calculated perturbatively in a consistent manner. At this order, they are characterized by a single unknown effective coupling κ.

Furthermore, if $\mathbf{P} = -\mathbf{Q}$ (Breit frame), the non-diagonal term in Eq. (4.9) can be rewritten as

$$k(\mathbf{P};E)X^{(1)}_{\mathbf{PQ}}(E)k(-\mathbf{P};E) = -k^2(\mathbf{P};E)\kappa\omega^2 A_0 - \frac{m^2}{4\pi}\,\Gamma A_0\frac{dK^{-1}(q_0,q_0)}{dq_0^2} + \cdots$$
$$= -k^2(\mathbf{P};E)\kappa\omega^2 A_0 - \frac{m^2}{8\pi}\,\Gamma A_0 r + \cdots . \tag{4.11}$$

In other words, at this order, everything is expressed in terms of the effective-range parameters a, r and the coupling κ.

Equation (4.8) is one of our main results, namely the Lüscher equation in the presence of an external field. In contrast to the conventional Lüscher equation, which reduces to a single equation in the absence of partial-wave mixing, Eq. (4.8) results from the matrix equation that connects sectors with different momenta $\mathbf{P}, \mathbf{Q}$. This happens because three-momentum is not a conserved quantity in the case considered here.

In order to make the equations tractable, a truncation should be applied. Let us consider the Breit frame again, with $\mathbf{P} = -\mathbf{Q} = \boldsymbol{\omega}/2$. If $e = 0$, $\mathcal{M}$ is a diagonal matrix,

whose matrix elements at $\mathbf{P}=\mathbf{Q}=\pm\boldsymbol{\omega}/2$ linearly vanish at the energies that correspond to the finite-volume spectrum of a system in a frame moving with a momentum $\mathbf{P}$. Turning the external field on, it is seen that the energy levels split and continuously shift from these values[10]. It can be straightforwardly checked that, at order e, it suffices to consider a 2×2 matrix with $\mathbf{P}=\pm\boldsymbol{\omega}/2$ and $\mathbf{Q}=\pm\boldsymbol{\omega}/2$. Adding more rows and columns to this matrix shifts the spectrum in higher orders only.

It is important to realize that the only missing piece in our knowledge of the resonance form factor is the contact contribution, which is proportional to the constant κ in our example. Everything else is known: the form factor in the impulse approximation (this corresponds to the triangle diagram in Fig. 1a) is determined through the known form factors of individual particles. This way, the coupling κ can be extracted by fitting the data to the energy spectrum obtained from the Lüscher equation in the external field, Eq. (4.8). Unlike measured matrix elements of the external current, κ, by definition, may contain only exponentially suppressed contributions in a finite volume. Hence, this method allows one to circumvent the problem of the irregular L-dependence, mentioned in the Introduction[11].

4.2 The Lüscher zeta-function in the external field; perturbative expansion

We shall now provide an explicit expression for the loop function $\mathit{\Pi}$, defined in Eq. (4.2). Carrying out the integration over the transverse momenta and the energy, we get

$$\mathit{\Pi}(\mathbf{P},\mathbf{Q};E)=L^2\delta^2_{\mathbf{P}_\perp\mathbf{Q}_\perp}\bar{\mathit{\Pi}}(P_\parallel,Q_\parallel;\mathbf{P}_\perp;E)\,, \tag{4.12}$$

where

$$\bar{\mathit{\Pi}}(P_\parallel,Q_\parallel;\mathbf{P}_\perp;E)=\frac{1}{L^4}\sum_{\mathbf{p}_\perp}\sum_{i,j=1}^{N}\sum_{n,m=-\infty}^{\infty}\int_0^L dx_\parallel\int_0^L dy_\parallel\, D_{in,jm}(\mathbf{p}_\perp;\mathbf{P}_\perp;E)e^{-iP_\parallel x_\parallel+iQ_\parallel y_\parallel}$$

$$\times\,\mathrm{me}_{\nu_i+2n}\left(\frac{\omega x_\parallel}{2},q\right)\mathrm{me}_{\nu_i+2n}\left(-\frac{\omega y_\parallel}{2},q\right)\mathrm{me}_{\nu_j+2m}\left(\frac{\omega x_\parallel}{2},q\right)\mathrm{me}_{\nu_j+2m}\left(-\frac{\omega y_\parallel}{2},q\right), \tag{4.13}$$

and

$$D_{in,jm}(\mathbf{p}_\perp;\mathbf{P}_\perp;E)=\frac{1}{2m+\dfrac{\mathbf{p}_\perp^2}{2m}+\dfrac{(\mathbf{P}-\mathbf{p})_\perp^2}{2m}+\dfrac{\omega^2}{8m}\left(\lambda_{\nu_i+2n}(q)+\lambda_{\nu_j+2m}(q)\right)-E}\,. \tag{4.14}$$

Below, we shall consider the perturbative expansion of this expression in powers of the coupling e (or, equivalently, the quantity q). The reason for this is twofold. First, in

[10] In fact, as we shall see later, the structure of the spectrum at $e\neq 0$ is more complicated. There exist "fake" states which do not have a counterpart at $e=0$.

[11] This statement should be clarified by an example. Suppose that one calculates the finite-volume energy spectrum in an "exact" theory (be this QCD or relativistic EFT), and then extracts κ from this spectrum by using the NREFT setting described in this paper. The extracted quantity $\kappa=\kappa(L)$ will depend on L. We state that the difference $\kappa(L)-\kappa(\infty)=O(e^{-\mu L})$ (modulo a prefactor that may contain powers of L), where μ is some scale given by a multiple of the lightest mass in the system (here, the only available scale is the particle mass m itself). In this context, one might term this statement, which applies to all effective couplings in NREFT, as the finite-volume counterpart of the Appelquist-Carazzone decoupling theorem.

the standard method of evaluating the resonance form factor, the matrix element between the two-particle scattering states is calculated on the lattice. This corresponds to taking exactly $O(e)$ term in the perturbative expansion. Hence, expanding the result in e, one may establish a closer relation between the "standard" approach and the approach which is proposed in the present paper. The second reason is practical. The full expression of the Lüscher function in the external field is quite cumbersome and is not well suited for numerical evaluation. The expansion allows one to arrive at a much simpler expression. One should be however aware of pitfalls, see below.

In what follows, we shall display the result of the calculation of this quantity at first order in the parameter q. Note that the energy denominator depends on q as well since $\lambda_{\pm 1}(q) = 1 \pm q + O(q^2)$. Hence, the perturbative expansion fails at the energies where the pertinent denominators vanish at $O(q^0)$. For this reason, along with the "perturbative" result, we also present the "exact" one, obtained by expanding the numerator in powers of q but keeping the $O(q)$ terms in the denominator unexpanded. The implications of using the "perturbative" result instead of the "exact" one are also considered in detail.

The initial and final momenta $\mathbf{P}, \mathbf{Q}$ in the above equations are arbitrary. Below, we shall use the notation $P_\parallel = a\omega/2$, $Q_\parallel = b\omega/2$. For simplicity, we shall further restrict ourselves to the 2×2 matrix with $a, b = \pm 1$ (recall that $\mathbf{P}_\perp = \mathbf{Q}_\perp$). The detailed derivation can be found in Appendix C. The pertinent elements of the matrix $\mathit{\Pi}_{ab}$ are denoted as $\tilde{\mathit{\Pi}}_{11} = \tilde{\mathit{\Pi}}_{-1,-1} \doteq \mathit{\Pi}_0$ and $\tilde{\mathit{\Pi}}_{1,-1} = \tilde{\mathit{\Pi}}_{-1,1} \doteq \mathit{\Pi}_1$. The "exact" and "perturbative" results are denoted by $\Pi_{0,1}$ and $\Pi'_{0,1}$, respectively:

$$\mathit{\Pi}_0 = \mathit{\Pi}_0^{(1)} + \mathit{\Pi}_0^{(2)}\,, \qquad \mathit{\Pi}_1 = \mathit{\Pi}_1^{(1)} + \mathit{\Pi}_1^{(2)}\,, \tag{4.15}$$

where

$$\begin{aligned}
\mathit{\Pi}_0^{(1)} &= \frac{1}{L^3}\sum_{\mathbf{p}} \frac{1}{2m + \dfrac{\mathbf{p}^2}{2m} + \dfrac{(\mathbf{P}-\mathbf{p})^2}{2m} - E}\,,\\
\mathit{\Pi}_1^{(1)} &= -\frac{\omega^2 q}{4m}\,\frac{1}{L^3}\sum_{\mathbf{p}} \frac{1}{\left(2m + \dfrac{\mathbf{p}^2}{2m} + \dfrac{(\mathbf{P}-\mathbf{p})^2}{2m} - E\right)\left(2m + \dfrac{\mathbf{p}^2}{2m} + \dfrac{(\mathbf{Q}-\mathbf{p})^2}{2m} - E\right)}\,,
\end{aligned} \tag{4.16}$$

and

$$
\begin{aligned}
\Pi_0^{(2)} &= \frac{1}{L^3}\sum_{\mathbf{p}_\perp}\left\{\frac{1+\frac{q}{4}}{\frac{\omega^2}{8m}(1+q)+\frac{\omega^2}{2m}-W}+\frac{1-\frac{q}{4}}{\frac{\omega^2}{8m}(1+q)-W}\right.\\
&\quad\left.+\frac{1-\frac{q}{4}}{\frac{\omega^2}{8m}(1-q)+\frac{\omega^2}{2m}-W}+\frac{1+\frac{q}{4}}{\frac{\omega^2}{8m}(1-q)-W}-\frac{2}{\frac{\omega^2}{8m}+\frac{\omega^2}{2m}-W}-\frac{2}{\frac{\omega^2}{8m}-W}\right\},\\
\Pi_1^{(2)} &= \frac{1}{L^3}\sum_{\mathbf{p}_\perp}\left\{\frac{\frac{\omega^2 q}{4m}}{\left(\frac{\omega^2}{8m}-W\right)^2}+\left(\frac{1}{\frac{\omega^2}{8m}(1+q)-W}-\frac{1}{\frac{\omega^2}{8m}(1-q)-W}\right)\right.\\
&\quad+\frac{q}{4}\left(\frac{1}{\frac{\omega^2}{8m}(1+q)+\frac{\omega^2}{2m}-W}-\frac{2}{\frac{\omega^2}{8m}(1+q)-W}\right.\\
&\quad\left.\left.+\frac{1}{\frac{\omega^2}{8m}(1-q)+\frac{\omega^2}{2m}-W}-\frac{2}{\frac{\omega^2}{8m}(1-q)-W}-\frac{2}{\frac{\omega^2}{8m}+\frac{\omega^2}{2m}-W}+\frac{4}{\frac{\omega^2}{8m}-W}\right)\right\}.
\end{aligned}
\tag{4.17}
$$

Note also that the following notation is used:

$$
W = E - 2m - \frac{\mathbf{p}_\perp^2}{2m} - \frac{(\mathbf{P}-\mathbf{p})_\perp^2}{2m}\,. \tag{4.18}
$$

Some comments are in order now. As already said, the quantities $\Pi_{0,1}^{(1)}$ can be obtained straightforwardly by using the perturbative expansion of the one-particle propagator, see Eq. (3.11). Namely, $\Pi_0^{(1)}$ leads to the Lüscher zeta-function, and $\Pi_1^{(1)}$ is nothing but the triangle diagram (or, the so-called G-function, in the approach of Refs. [17, 18]). Hence, the relation to the "standard" approach is clearly visible. However, we already know that this expansion fails in the vicinity of the free particle poles. Indeed, instead of one simple pole at $q=0$, the one-particle propagator possesses two poles at $q\neq 0$, which are located symmetrically on both sides. Expanding the denominator in powers of q, one gets one double pole instead of two single poles, separated by a distance $2q$. This is schematically shown in Fig. 5. where the names "exact" and "perturbative" refer to $\Pi_{0,1}$ and $\Pi_{0,1}^{(1)}$, respectively. Furthermore, it is worth noting that, formally, $\Pi_{0,1}^{(2)}$ are at least of order q^2 and can be neglected. We have seen, however that such an approximation does not suffice in the vicinity of the singularities. Another observation is that, in the infinite-volume limit, which can be performed for energies below the two-particle threshold, the quantities $\Pi_{0,1}^{(2)}$ behave like L^{-1} for large L modulo exponential corrections.

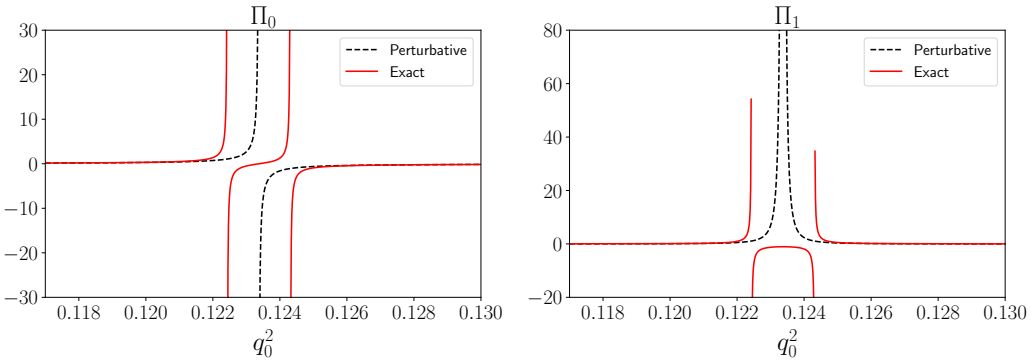

Figure 5. The diagonal and non-diagonal elements of the matrix $\mathit{\Pi}$, given in Eqs. (4.16) and (4.17). The quantity q_0^2 is given in Eq. (2.3). It is seen that the quantity $\mathit{\Pi}_1^{(1)}$ ("perturbative") develops a double pole instead of the two separated simple poles found in $\mathit{\Pi}_1$ ("exact"). Since this figure serves illustrative purpose only, we do not specify the values of the parameters in calculations. The units on the vertical axis are arbitrary.

4.3 "Exact" vs. "perturbative" solution

A very interesting question arises, namely, whether the solutions of the Lüscher equation in the external field are the same up to terms of order e^2, if one replaces $\mathit{\Pi}_{0,1}$ by $\mathit{\Pi}_{0,1}^{(1)}$ (we remind the reader that the difference between these quantities is *formally* of order e^2). The answer to this question is positive, and will be discussed below.

Let us take, for simplicity, $\mathbf{P}_\perp = 0$ and try to obtain a solution in the vicinity of $E = 2m + \omega^2/(8m)$. The quantities $\mathit{\Pi}_{0,1}$ exhibit here the following behavior

$$\mathit{\Pi}_{0,1} = \frac{c_{0,1}^+(q)}{2m + \dfrac{\omega^2}{8m}(1+q) - E} + \frac{c_{0,1}^-(q)}{2m + \dfrac{\omega^2}{8m}(1-q) - E} + \bar{\mathit{\Pi}}_{0,1}\,, \tag{4.19}$$

where $\bar{\mathit{\Pi}}_{0,1}$ is a smooth function of E in the vicinity of $E = 2m + \omega^2/(8m)$ and the coefficients

$$c_0^\pm = \frac{1}{L^3}\left(1 \mp \frac{q}{4}\right), \qquad c_1^\pm = \pm\frac{1}{L^3}\left(1 \mp \frac{q}{2}\right), \tag{4.20}$$

can be directly read off from Eqs. (4.16) and (4.17). Thus, the Lüscher equation in this case can be reduced to two algebraic equations of the type

$$\mathscr{G}_\pm^{-1}(E) \doteq \frac{c_+(q)}{2m + \dfrac{\omega^2}{8m}(1+q) - E} + \frac{c_-(q)}{2m + \dfrac{\omega^2}{8m}(1-q) - E} + f_\pm(E,q) = 0\,. \tag{4.21}$$

Here, $c_\pm = -\frac{1}{2}\left(c_0^\pm \pm c_1^\pm\right)$ and $f_\pm(E,q)$ are some smooth functions of their arguments which contains $\bar{\mathit{\Pi}}_{0,1}$ as well as the elements of the matrix X, Eq. (4.7). Hence, one could expand $f_\pm(E,q)$ in Taylor series in E in the vicinity of the unperturbed level and solve the obtained

equations iteratively. At lowest order, $f_\pm(E,q)$ are just constants and one gets quadratic equations for E which can be easily solved. Indeed, rewriting the above equation as

$$\begin{aligned}\mathscr{G}_\pm^{-1}(E) &= \frac{mc_+}{q_+^2-q_0^2}+\frac{mc_-}{q_-^2-q_0^2}+f_\pm=0\,,\\ q_\pm^2 &= \frac{\omega^2}{16}\,(1\pm 2q)\,,\end{aligned} \tag{4.22}$$

one gets two roots

$$\begin{aligned}q_0^2 &= q_1^2+\frac{m(c_++c_-)}{2f_\pm}\pm\frac{1}{2f_\pm}\sqrt{D}\,,\\ q_1^2 &= \frac{1}{2}\,(q_+^2+q_-^2)=\frac{\omega^2}{16}\,,\\ D &= \Big(2f_\pm q_1^2+m(c_++c_-)\Big)^2-4f_\pm(f_\pm q_+^2q_-^2+mc_+q_-^2+mc_-q_+^2)\,.\end{aligned} \tag{4.23}$$

Expanding this solution in powers of q one gets a pair of solutions that differ by the choice of sign in front of the square root

$$q_0^2=q_1^2+\frac{m(c_++c_-)}{2f_\pm}\pm\frac{m(c_++c_-)}{2f_\pm}\left(1-\frac{f_\pm(q_-^2-q_+^2)(c_+-c_-)}{m(c_++c_-)^2}\right)+\cdots\,. \tag{4.24}$$

Choosing the "+" sign, one may verify that one gets exactly the same result as first expanding Eq. (4.21) in powers of q and then solving it with respect to E. Higher orders in the expansion in E can be treated in the similar fashion. On the other hand, with the choice of the "−" sign, one arrives at the solution $q_0^2=q_1^2+O(q)$. As we shall see below, this corresponds to an unphysical solution.

Albeit the argument, given above, proves that the linear dependence of the energy levels is not altered by using perturbative expansion in $\Pi_{0,1}$, the situation for a finite e is not that clear. It cannot be excluded that the structure of the energy levels is qualitatively different, until e becomes sufficiently small (later, we shall demonstrate this explicitly). For this reason, using the exact solution for $\Pi_{0,1}$ in the fit is preferable.

4.4 Residua

As mentioned earlier, the perturbative expansion contains pitfalls. Here we shall consider one of these. Namely, it will be shown that the use of the *expanded* Lüscher equation might lead to the "fake" poles. The residua of these poles are however of order e^2 and could be thus neglected at the order one is working. In the actual fit to the lattice data, one should carefully identify such poles and exclude them from the analysis.

Below, for simplicity, we restrict ourselves to the case of the 2×2 matrix already considered above. The quantity $\mathscr{G}(E)$, defined in Eq. (4.21), contains single poles at $q_0^2=q_n^2$, which are given, in particular, by Eq. (4.24). In the vicinity of such a pole,

$$\mathscr{G}(E)=\frac{\mathscr{Z}_n}{q_n^2-q_0^2}+\text{regular}\,. \tag{4.25}$$

Differentiating both sides of the above equation with respect to q_0^2, one may easily ensure that the residuum at the pole, $\mathscr{Z}_n$, is given by the derivative of $\mathscr{G}_\pm^{-1}(E)$ at the pole:

$$\mathscr{Z}_n^{-1} = -\lim_{q_0^2 \to q_n^2} [\mathscr{G}_\pm^{-1}(E)]' \,. \tag{4.26}$$

On the other hand, from Eq. (4.22) one gets

$$[\mathscr{G}_\pm^{-1}(E)]' = \frac{mc_+}{(q_+^2 - q_0^2)^2} + \frac{mc_-}{(q_-^2 - q_0^2)^2} + f'_\pm \,. \tag{4.27}$$

Equation (4.24) describes two solutions, which differ by the choice of the sign in front of the last term. Taking into account the fact that $c_+ \pm c_- = O(1)$ and $q_\pm^2 - q_1^2 = O(q)$, one immediately gets that the quantity $\mathscr{Z}$ is of order 1 and order q^2 for the first and the second solution, respectively. This demonstrates that the second solution is an artifact of the approximations used, since the terms of order q^2 in the numerators have been systematically neglected.

4.5 Extracting the resonance pole

In the infinite volume momenta are no more quantized. However, the conservation of the three-momentum, $\mathbf{P} = \mathbf{Q} \pm \ell\boldsymbol{\omega}$, implies that the two-point function of the composite fields for a fixed $\boldsymbol{\omega}$ still obeys the matrix equation (4.5). In this equation, however, $\mathbf{P}, \mathbf{Q}, \boldsymbol{\omega}$ are no more restricted to integer multiples of $2\pi/L$. Furthermore, the kernel X is the same (modulo replacing the Kronecker deltas by the Dirac delta-functions), and the loop function $\mathit{\Pi}$ is replaced by its infinite-volume counterpart that amounts to replacing the sum over the loop momenta by an integral.

A crucial point is that we can use the perturbative expansion (3.11) in the coupling e. The reason is that, in order to find the position of the resonance pole, we are going to solve the infinite-volume analog of Eq. (4.5) in the complex plane, where the energy denominator, appearing in the loop, is not singular and the perturbative expansion is justified. However, as seen above, when solving the Lüscher equation on the real axis, the perturbative series diverges in the vicinity of the singularities. Hence, in a finite volume, it is preferable to work with the full expression of the Lüscher zeta-function in the external field[12].

Up to $O(e)$ terms, the loop function $\mathit{\Pi}$ in the infinite volume takes the form:

$$\begin{aligned} \mathit{\Pi}(\mathbf{P}, \mathbf{Q}; E) &= (2\pi)^3 \delta^3(\mathbf{P} - \mathbf{Q}) \mathit{\Pi}_0(\mathbf{P}; E) \\ &\quad + e A_0 \mathit{\Gamma} \, (2\pi)^3 \big[\delta^3(\mathbf{P} + \boldsymbol{\omega} - \mathbf{Q}) + \delta^3(\mathbf{P} + \boldsymbol{\omega} - \mathbf{Q})\big] \mathit{\Pi}_1(\mathbf{P}, \mathbf{Q}; E) + O(e^2) \,, \end{aligned} \tag{4.28}$$

where

$$\mathit{\Pi}_0(\mathbf{P}; E) = \int \frac{dp^0}{2\pi i} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \, \frac{1}{\left(m + \dfrac{\mathbf{p}^2}{2m} - p^0\right)\left(m + \dfrac{(\mathbf{P} - \mathbf{p})^2}{2m} - E + p^0\right)}$$

[12] One should stress here once more that one is forced to exclusively use perturbative expressions within the "standard" approach. From the discussion above it is however clear that both approaches are algebraically equivalent at $O(e)$ (as it should be).

$$= \frac{m}{4\pi}\left[-m\left(E-2m-\frac{\mathbf{P}^2}{4m}\right)\right]^{1/2},$$

$$\begin{aligned}
\mathit{\Pi}_1(\mathbf{P},\mathbf{Q};E) &= \int\frac{dp^0}{2\pi i}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{\left(m+\dfrac{\mathbf{p}^2}{2m}-p^0\right)\left(m+\dfrac{(\mathbf{P}-\mathbf{p})^2}{2m}-E+p^0\right)}\\
&\times\frac{1}{\left(m+\dfrac{(\mathbf{Q}-\mathbf{p})^2}{2m}-E+p^0\right)}\\
&= -\frac{m^2}{8\pi}\int_0^1 dx\,\frac{1}{\sqrt{m(2m+\mathbf{P}^2/(4m)-E)+\frac{1}{4}\,\omega^2x(1-x)}}\\
&= -\frac{m^2}{2\pi\omega}\,\arcsin\frac{\omega}{\sqrt{16m(2m+\mathbf{P}^2/(4m)-E)+\omega^2}}\,. \qquad (4.29)
\end{aligned}$$

Note that the sign convention in front of the square roots in the above expressions corresponds to the choice of the second Riemann sheet.

The procedure for determining the position of the pole on the second Riemann sheet is as follows. First, one uses the finite-volume energy levels in the Breit frame, $\mathbf{P}=-\mathbf{Q}=\pm\boldsymbol{\omega}/2$, to extract the parameters of the Lagrangian using the Lüscher equation with an external field, Eq. (4.5). In our case, there is a single unknown parameter κ. Next, solving the same equation in the infinite volume, using Eq. (4.29), with the extracted values of the couplings, one determines the position of the pole on the second sheet. In this manner, one could study the dependence of the pole position with e. It can be seen that a pair of poles emerges which move in opposite directions as e increases. At first order of e, they move with the same rate.

4.6 Relation to the resonance form factor

From the previous discussion, the infinite-volume two-point function in the external field possesses poles on the second Riemann sheet. In the vicinity of a pole, the residue factorizes. In the Breit frame, one has:

$$D(\mathbf{P},\mathbf{Q};E)\to(2\pi)^3\delta^3(\mathbf{P}-\mathbf{Q}\pm\boldsymbol{\omega})\frac{\Phi(\mathbf{P})\bar\Phi(-\mathbf{P})}{P^0(\mathbf{P})-P_R^0(\mathbf{P},e)}\,. \qquad (4.30)$$

Here, we have explicitly indicated the dependence on the parameter e.

Next, we shall differentiate both sides of the above equation with respect to e and set $e=0$ at the end (because we are interested only in the terms linear in e). The most singular term (a double pole) comes from differentiating the denominator. Hence,

$$\begin{aligned}
\frac{d}{de}\,D(\mathbf{P},\mathbf{Q};E)\Big|_{e=0} &\to (2\pi)^3\delta^3(\mathbf{P}-\mathbf{Q}\pm\boldsymbol{\omega})\frac{\Phi(\mathbf{P})\bar\Phi(-\mathbf{P})}{(P^0(\mathbf{P})-P_R^0(\mathbf{P},0))^2}\,\frac{dP_R^0(\mathbf{P},e)}{de}\Big|_{e=0}\\
&+\text{ less singular terms.} \qquad (4.31)
\end{aligned}$$

On the other hand, the quantity D can be identically rewritten as $D = DD^{-1}D$. Differentiating with respect to e, one gets $\frac{d}{de}\,D = -D\Big(\frac{d}{de}\,D^{-1}\Big)D$. The quantity D at $e = 0$ has a pole

$$D(\mathbf{P},\mathbf{Q};E) \to (2\pi)^3\delta^3(\mathbf{P}-\mathbf{Q})\frac{\Phi(\mathbf{P})\bar{\Phi}(\mathbf{P})}{P^0(\mathbf{P}) - P_R^0(\mathbf{P},0)}\,. \tag{4.32}$$

Taking into account the definition (2.14) and comparing with Eq. (2.16), one can straightforwardly read off the relation between the quantity $\Phi(\mathbf{P})$ and the wave function $\Psi(\mathbf{P},\mathbf{p})$, introduced in Sect. 2:

$$\Phi(\mathbf{P}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\Psi(\mathbf{P},\mathbf{p})\,. \tag{4.33}$$

An important remark is in order. In the limit $e = 0$, the three-momentum is conserved and, hence, one can establish the relation between Ψ and Φ only for $\mathbf{P} = \mathbf{Q}$. However, the fact that the residue at the pole factorizes, enables one to write down the residue for $\mathbf{P} \neq \mathbf{Q}$ as well. In simple cases like the one considered here, the factorization at the pole can be verified explicitly, carrying out the truncation in the $\mathbf{P},\mathbf{Q}$ space and inverting the resulting matrix.

Finally, using $D^{-1} = \frac{1}{2}\,\mathit{\Pi}^{-1} - \frac{1}{4}\,X$ and taking into account Eqs. (4.6) and (4.28), one gets

$$\begin{aligned}\frac{d}{de}\,D(\mathbf{P},\mathbf{Q};E) \to -(2\pi)^3\delta^3(\mathbf{P}-\mathbf{Q}\pm\boldsymbol{\omega})\frac{\Phi(\mathbf{P})\bar{\Phi}(\mathbf{P})\,\Phi(-\mathbf{P})\bar{\Phi}(-\mathbf{P})}{(P^0(\mathbf{P}) - P_R^0(\mathbf{P},0))^2}\\ \times\,\Big[-\frac{1}{2}\,A_0\mathit{\Pi}_0^{-1}(\mathbf{P},E_R)\mathit{\Gamma}\mathit{\Pi}_1(\mathbf{P},-\mathbf{P};E_R)\mathit{\Pi}_0^{-1}(-\mathbf{P};E_R) - \frac{1}{8}\,X^{(1)}_{\mathbf{P},-\mathbf{P}}(E_R)\Big]\,.\end{aligned} \tag{4.34}$$

Comparing Eqs. (4.31) and (4.34), we finally obtain:

$$\begin{aligned}&\frac{dP_R^0(\mathbf{P})}{de}\Big|_{e=0}\\ &= \frac{1}{2}\,\bar{\Phi}(\mathbf{P})\big[\mathit{\Pi}_0^{-1}(\mathbf{P},E_R)A_0\mathit{\Gamma}\mathit{\Pi}_1(\mathbf{P},-\mathbf{P};E_R)\mathit{\Pi}_0^{-1}(-\mathbf{P};E_R) + \frac{1}{4}\,X^{(1)}_{\mathbf{P},-\mathbf{P}}(E_R)\big]\Phi(-\mathbf{P})\,.\end{aligned} \tag{4.35}$$

In order to prove that this expression is the same as Eq. (2.23), let us first assume that $C_2 = 0$ and use $\tilde{A}^0(\mathbf{P}-\mathbf{Q}) = \frac{1}{2}\,(2\pi)^3(\delta^3(\mathbf{P}-\mathbf{Q}+\boldsymbol{\omega}) + \delta^3(\mathbf{P}-\mathbf{Q}-\boldsymbol{\omega})$. When $C_2 = 0$, the integration over the relative momenta in Eq. (2.23) is performed trivially, yielding Eq. (4.35) (note that Eq. (4.33) should be used to prove this relation). If $C_2 \neq 0$, in analogy to what was done before, one has to pull out the derivatives acting on the internal lines. Then, the expression for $X^{(1)}$ will be modified and one arrives again at Eq. (4.7). Equations (2.23) and (4.35) are also equivalent in this case. Finally, we arrive at our final result that looks remarkably simple:

$$\frac{1}{2}\,A_0F(\mathbf{P},-\mathbf{P}) = \frac{dP_R^0(\mathbf{P})}{de}\Big|_{e=0}\,. \tag{4.36}$$

In other words, in the Breit frame the resonance form factor is given by the derivative of the resonance pole position with respect to the coupling constant with the external field[13].

To summarize, all what is needed to compute the resonance form factor is the contact contribution (at the lowest order, this is parameterized by a single coupling constant, κ). The latter can be determined by fitting directly the energy levels in the external field[14]. The resonance form factor can be then calculated using Eqs. (2.22) and (2.23). Hence, extracting the resonance pole first and using then a Feynman-Hellmann theorem is even superfluous. However, the direct analogy with the Feynman-Hellmann theorem for the form factors of stable particles is still remarkable.

Extracting the contact contribution could, however, be complicated, since this contribution contains suppression factors. For example, from Eq. (4.11) it is seen that the contribution containing κ is multiplied by a factor $k^2(\mathbf{P}; E)$. In the case of a shallow and narrow resonance, this approximately equals to q_0^2. In addition, owing to gauge invariance, a factor ω^2 is present. This, however, is not an obstacle for the extraction of the form factor, since the same suppression factors also emerge in the expression of the latter in the infinite volume. In other words, if the quantity κ, determined on the lattice, is zero within the error bars, this simply means that the form factor at this accuracy is given only by the impulse approximation.

4.7 Relativistic corrections, higher partial waves and all that

In this section we briefly consider the generalization of the above approach to higher orders in the momentum expansion. This is needed, in particular, to render the approach applicable to the study of the problems where relativistic effects cannot be neglected. The inclusion of the higher-derivative interaction terms (an analog of the term with C_2), which also describe higher partial waves, as well as derivative four-particle interaction with the external field (similar to the coupling κ), proceeds relatively straightforwardly and will not be considered here. A single non-trivial piece is the modification of the Lagrangian in the single particle sector. As it is known, derivative insertions in the non-relativistic propagators should be summed up to all orders, in order to arrive at a correct dispersion relation. We shall try to do the same in presence of the external field below.

In general, writing down all terms in the one-particle sector is a complicated task (in higher orders) and can be carried out order by order in the expansion in the inverse mass. Matching should be performed in the same setting, order by order in the expansion. The situation simplifies dramatically, if we additionally restrict ourselves to terms that are linear in e. These should be matched to the relativistic form factor $F_\mu(p', p) = ie(p'_\mu + p_\mu)F(t)$, with $t = (p' - p)^2$. In this case, the form of the Lagrangian can be read off directly from the matching condition and takes the form (the differential operators act on everything

[13] As already mentioned, each pole $e = 0$ splits into two, moving in the opposite direction at equal speed, when the external field is turned on. Choosing another pole yields just a different sign in Eq. (4.36).

[14] For instance, it could be advantageous to fit the quantity $\Delta \doteq \langle E \rangle_{\phi^2} - 2\langle E \rangle_\phi$, calculated on the lattice in the presence of the external field. This quantity describes the energy shift of the two-particle state caused by the interactions between them and might be more sensitive to the small effects coming from contact interactions parameterized by κ.

right to them):

$$\mathcal{L} = \phi^\dagger \Big(i\partial_t - W + e\Gamma \frac{1}{\sqrt{2W}} \left(W A^0(\mathbf{x}) + A^0(\mathbf{x}) W \right) \frac{1}{\sqrt{2W}} \Big) \phi$$
$$+ \text{ terms with four fields}\,. \tag{4.37}$$

Here, $W = \sqrt{m^2 - \triangle}$ denotes the relativistic energy operator, and $\Gamma(\boldsymbol{\omega}) = F(-\omega^2)$. The equation for the one-particle wave function takes the form

$$\Big(i\partial_t - W + e\Gamma \frac{1}{\sqrt{2W}} \left(W A^0(\mathbf{x}) + A^0(\mathbf{x}) W \right) \frac{1}{\sqrt{2W}} \Big) \Phi(\mathbf{x}, t) = 0\,. \tag{4.38}$$

Using Eq. (A.1), this equation can be rewritten as

$$\Big(E - W_\perp + e\Gamma \frac{1}{\sqrt{2W_\perp}} \left(W_\perp A^0(\mathbf{x}) + A^0(\mathbf{x}) W_\perp \right) \frac{1}{\sqrt{2W_\perp}} \Big) \bar{\Phi}(z) = 0\,, \tag{4.39}$$

where

$$W_\perp = \sqrt{m^2 + \mathbf{p}_\perp^2 + \frac{4}{\omega^2} \frac{d^2}{dz^2}}\,. \tag{4.40}$$

Albeit Eq. (4.39) does not have the form of the Mathieu equation, at first order in e it can be reduced to it through the redefinition of the wave function:

$$\bar{\Phi}(z) = \sqrt{2W_\perp} \Big(1 - \frac{e\Gamma}{\sqrt{2W_\perp}} A^0(\mathbf{x}) \Big) \bar{\Phi}'(z)\,. \tag{4.41}$$

The equation for the transformed wave function can then be rewritten as:

$$\Big(E - W_\perp + e\Gamma A^0(\mathbf{x}) + O(e^2) \Big) \bar{\Phi}'(z) = 0\,, \tag{4.42}$$

or similarly,

$$\Big((E + e\Gamma A^0(\mathbf{x}))^2 - W_\perp^2 + O(e^2) \Big) \bar{\Phi}'(z)$$
$$= \Big(E^2 - m^2 - \mathbf{p}_\perp^2 + \frac{4}{\omega^2} \frac{d^2}{dz^2} + \frac{2Ee\Gamma}{\omega} \cos 2z + O(e^2) \Big) \Phi'(z) = 0\,. \tag{4.43}$$

This is an equation of the Mathieu type, where the non-relativistic dispersion law (as in Eq. (A.2)) is replaced by the relativistic expression $E^2 - m^2 + \mathbf{p}_\perp^2$. Note, however, that the parameter q in this equation depends on the eigenvalue E, so the solutions can be found only numerically with an iterative procedure. Once this is done, one can construct the eigenvectors, using Eq. (4.41). These eigenvectors, in turn, can be used to construct the one-particle propagators and to calculate the Lüscher zeta-function in the periodic external field. Since the primary aim of the present paper is the proof of principle, we shall not consider all these rather straightforward issues here, which form a separate piece of work for the future.

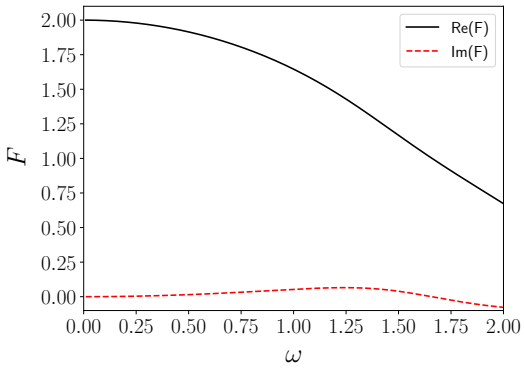

Figure 6. Real and imaginary parts of the resonance form factor.

5 Numerical implementation

In this section, we shall test our theoretical predictions numerically. Since this test serves an illustrative purpose only, we have not made an attempt to choose realistic values of the different parameters in the toy model. In particular, we choose $m = 1$ from the beginning and show everything in mass units. The values of other parameters are $a = -1.5$, $r = -9$, $\kappa = 10$ and $C_R = 0.9$. Without loss of generality, one may set $A_0 = 1$. With this choice of parameters, there exist a couple of poles on the second Riemann sheet located at $q_R^2 = 0.123 \pm i\,0.082$. The resonance form factor, evaluated with the help of Eqs. (2.22) and (2.23), is shown in Fig. 6. Note that, owing to the Ward identity, the form factor is normalized as $F(0) = 2$ at $\omega = 0$.

Furthermore, when $e \neq 0$, the pole in the complex plane splits into two that move in the opposite direction from the initial location. In Fig. 7 we plot the real and imaginary parts of these poles versus e. It is seen that at small values of e this dependence is almost linear and is determined by the Feynman-Hellmann theorem. For this example with $\omega = 1$, we obtain

$$F(\mathbf{P},\mathbf{Q}) = 1.6454 + i0.0535\,, \qquad 2\,\frac{dP_R^0}{de}\bigg|_{e=0} = 1.6455 + i0.0534\,. \tag{5.1}$$

The second number has been obtained by numerically differentiating the pole trajectory in the complex plane. The explicit expression of the form factor in this model is written down in Appendix D.

In Fig. 8, we display the spectrum in a finite volume at different values of e and for a fixed L (In order to discuss the qualitative behavior of the spectrum we used an arbitrarily chosen value $L = 20$). The structure of the levels turns out to be rather complicated. In the absence of field, there is a set of doubly degenerate energy levels (black filled dots) corresponding to states with momentum $\mathbf{P}$ and $-\mathbf{P}$, which are related by a time-reversal transformation. When $e \neq 0$, time-reversal invariance is broken and these two levels split symmetrically at $O(e)$. Moreover, there are additional energy levels which do not have a zero-field counterpart. This is attributed to the fact that, at $e \neq 0$, the poles in the functions

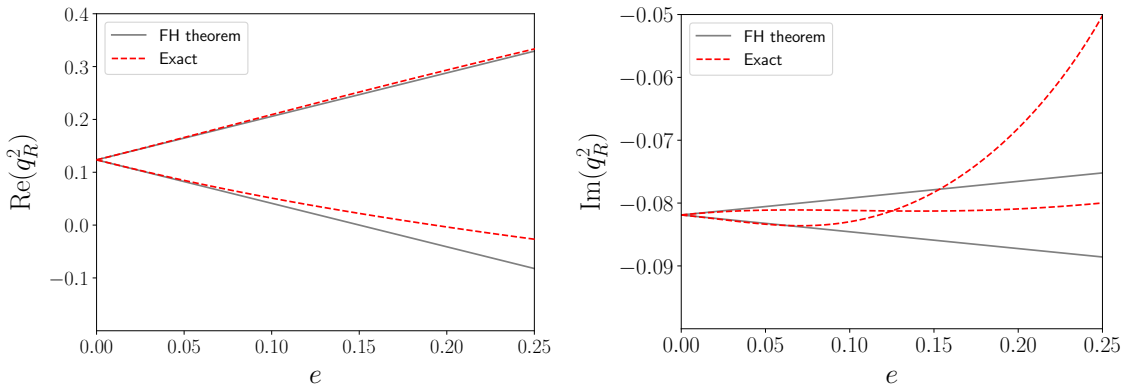

Figure 7. Verification of the Feynman-Hellmann theorem for the real and imaginary parts of the pole position in the complex plane at $\omega = 1$. Thin black lines depict the prediction of the theorem.

$\mathit{\Pi}_0$, $\mathit{\Pi}_1$ also split (see Fig. 5), and the determinant in the Lüscher equation can cross the real axis at more places, e.g., between the poles. However, it can be easily checked that these solutions correspond to the "artifacts" that were discussed in the previous section. Namely, the residua corresponding to these levels are of order e^2 and have a different sign as compared to the physical levels. We have also checked that the unphysical levels, in difference with the physical ones, are not stable if the dimension of the matrix in the quantization condition is increased. This fact further supports the conclusion that these levels emerge due to the approximations that were made during the derivation of the quantization condition. Hence, in the analysis of data, such unphysical levels should be merely discarded.

Furthermore, Fig. 8 nicely demonstrates the limitations of the use of the perturbative approach to the calculation of $\mathit{\Pi}_0$, $\mathit{\Pi}_1$. In the vicinity of $e = 0.015$, two "perturbative" levels merge and disappear (the determinant does not cross the real axis anymore), whereas the "exact" levels still exist. Note that this happens already at rather small values of e, for which other levels are very well described by the perturbative solution.

In Fig. 9, the difference between the physical and unphysical levels is clearly seen. Here, we plot the e-dependence of the residua, calculated using Eq. (4.26) (blue and red lines correspond to the two roots of the quantization condition that merge in the limit $e = 0$). For the physical levels, the residua converge to a non-zero limit and exhibit a linear dependence on e for small e. In contrast to this, the residua for the unphysical levels behave as e^2 and vanish for $e = 0$. This agrees with our theoretical findings.

In addition to this, in Fig. 10 we show the L-dependence of the energy levels ("exact" solutions only). Finally, note that, with our choice of parameters, the contribution from the contact interaction, parameterized by the coupling κ, is negligibly small. This could be expected, since the first term in Eq. (4.11) is much smaller that the second. However, as already discussed, this cannot pose an obstacle for the calculation of the resonance form factor, the goal we are after.

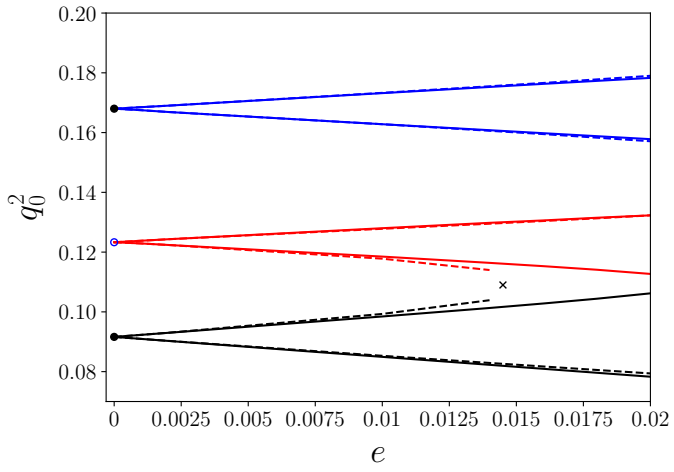

Figure 8. Qualitative structure of the energy levels in the external field. The levels at $e = 0$ are denoted by black filled dots. These are split when $e \neq 0$. There are additional levels (red curves) that do not have counterparts at $e = 0$. On the figure, they emanate from the empty blue dot. The solid and dotted lines denote the "exact" and "perturbative" solutions, respectively, depending on the use of the "exact" and "perturbative" expressions for the loop function. The approximate location where two "perturbative" levels merge and disappear is marked by a cross.

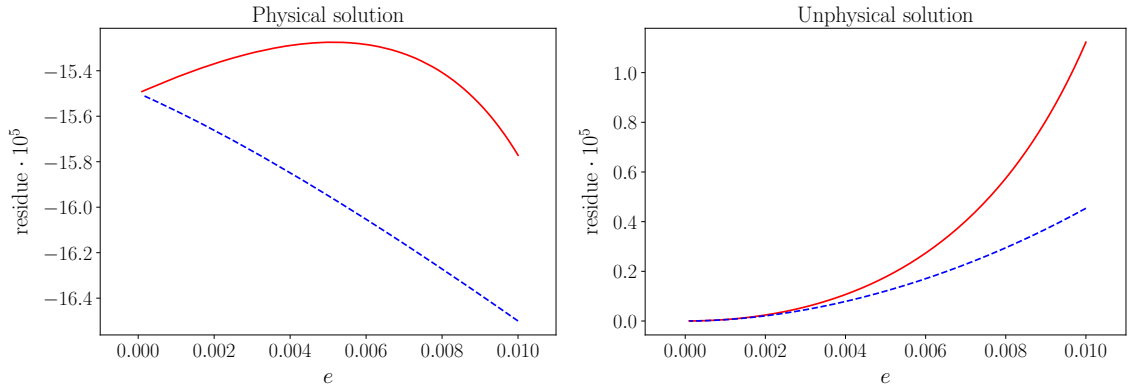

Figure 9. The e-dependence of the residua for the physical and unphysical levels. The blue and red curves correspond to the two different solutions of the quantization condition.

6 Conclusions

i) A novel method for the computation of the resonance form factors on the lattice has been proposed. Within this approach, one circumvents the calculation of the three-point function on the lattice, measuring instead the finite-volume energy spectrum in an external periodic field in space.

ii) It is known that the finite-volume three-point function, Fig. 1a, has an irregular dependence of the box size L that complicates the extraction of the infinite-volume

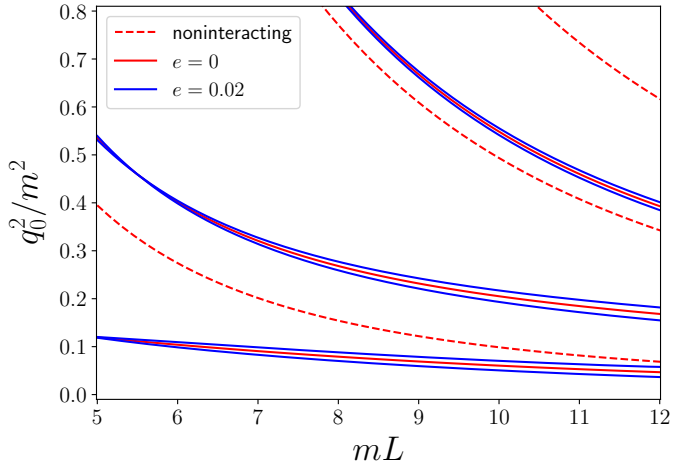

Figure 10. The L-dependence of the energy levels for $e = 0$, as well as for $e \neq 0$. For comparison, we plot the non-interacting energy levels as well, corresponding to the two free particles in a box. A single energy level at $e = 0$ splits into two nearby levels, when the external field is turned on.

form factor considerably. On the other hand, with this method, one merely needs to extract the parameters of the contact interaction with the external field from the fit to the energy levels at $e \neq 0$. These parameters, by definition, can only contain exponentially suppressed corrections in L.

iii) If the lattice simulations are performed at a non-zero external field, the formalism that is used to analyze the data should be also set in the presence of the external field. In order to match this objective, a generalization of the Lüscher equation in the presence of an external periodic field is obtained in this paper.

iv) Since in the vicinity of the free-particle poles the use of perturbation theory is questionable, an expression for the modified Lüscher function has been derived that avoids the expansion of the energy denominators. The limits of the use of perturbation theory in this context have been discussed in detail. On the other hand, for consistency, one is forced to use a strictly perturbative framework to to analyze data on the matrix elements in the "standard" approach [1, 16–18]. Of course, in the limit $e \to 0$, the matrix elements extracted within two different approaches, agree.

v) The Feynman-Hellmann theorem, which has been so far proven for stable particles only, is generalized to the case of resonances. It has been demonstrated that, finding the (complex) resonance pole position in the external field and in the Breit frame, and differentiating this quantity with respect to e, one arrives at the form factor. The theoretical arguments have been checked numerically, see Eq. (5.1).

vi) A numerical implementation of the framework is considered for a toy model. The qualitative structure of the energy levels is discussed.

vii) The present work provides a proof of principle only. Different improvements and generalizations will have to be considered. For example, it will be crucial to take into account relativistic corrections to all orders and write down a framework that is explicitly Lorentz-invariant. Furthermore, higher orders in the effective theory should be systematically included in order to write down the result in a form that does not explicitly rely on the effective-range expansion, and is also valid away from the elastic threshold. Partial-wave mixing should also be addressed appropriately. Finally, the numerical implementation should be considered for realistic values of the parameters that resemble the cases of existing low-lying resonances. All these technical issues will be addressed in the future.

Acknowledgments

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A Mathieu equation: essentials

The differential equation (3.4) admits variable separation by using the ansatz

$$\Phi(\mathbf{x},t) = e^{-iEt+i\mathbf{p}_\perp\mathbf{x}_\perp}\bar{\Phi}(z)\,, \qquad z=\frac{\omega x_\parallel}{2}\,, \tag{A.1}$$

where the function $\bar{\Phi}(z)$ obeys the differential equation

$$\left(\frac{d^2}{dz^2}+\frac{8m}{\omega^2}\left(E-m-\frac{\mathbf{p}_\perp^2}{2m}\right)+\frac{8me\mathit{\Gamma}A_0}{\omega^2}\cos 2z\right)\bar{\Phi}(z)=0\,. \tag{A.2}$$

This coincides with the Mathieu equation [32, 33]. Note that the potential in the above equation is periodic, and hence the solutions are given by Bloch's wave functions that have the property

$$\bar{\Phi}(z+\pi)=e^{i\nu\pi}\bar{\Phi}(z)\,, \qquad -1<\nu\leq 1\,. \tag{A.3}$$

The solutions corresponding to a particular ν (the so-called ν-periodic solutions) are denoted by $\mathrm{me}_{\nu+2n}(z,q)$ (with an integer n) and obey the equation

$$\left(\frac{d^2}{dz^2}+\lambda_{\nu+2n}(q)-2q\cos 2z\right)\mathrm{me}_{\nu+2n}(z,q)=0\,. \tag{A.4}$$

From the comparison of Eqs. (A.2) and (A.4) it follows that

$$\lambda_{\nu+2n}(q) = \frac{8m}{\omega^2}\left(E - m - \frac{\mathbf{p}_\perp^2}{2m}\right). \tag{A.5}$$

In case when ν becomes integer, one has

$$\lambda_n(q) = \begin{cases} a_n(q)\,, & n = 0, 1, \cdots \\ b_{-n}(q)\,, & n = -1, -2, \cdots \end{cases} \tag{A.6}$$

and

$$me_n(z,q) = \begin{cases} \sqrt{2}\, ce_n(z,q), & n = 0, 1, \cdots \\ -\sqrt{2}i\, se_{-n}(z,q), & n = -1, -2, \cdots \end{cases} \tag{A.7}$$

Due to the periodic boundary conditions, the parameter ν will be quantized. Indeed, from $\Phi(x_\parallel + L) = \Phi(x_\parallel)$ we get $\bar{\Phi}(z + \pi N) = e^{i\nu\pi N}\bar{\Phi}(z) = \bar{\Phi}(z)$, leading to the condition $e^{i\nu\pi N} = 1$. Together with the requirement $-1 < \nu \leq 1$ this leads to the conclusion that ν can take the following values

$$\begin{aligned} &N = 1: \ \ \nu = 0 \\ &N = 2: \ \ \nu = 0, 1 \\ &N = 3: \ \ \nu = -\tfrac{2}{3}, 0, \tfrac{2}{3} \\ &N = 4: \ \ \nu = -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1 \end{aligned} \tag{A.8}$$

and so on.

The Fourier expansion of the Mathieu functions takes the form

$$\mathrm{me}_\nu(z,q) = \sum_{a=-\infty}^{\infty} C_{2a}^\nu(q) e^{i(\nu+2a)z}\,. \tag{A.9}$$

The coefficients of this expansion, $C_{2a}^\nu(q)$, are known [33].

B Expansion of the propagator in powers of q

The expansion of Mathieu functions $me_\nu(z,q)$ in powers of q for the non-integer (ν) and integer ($k \geq 2$) values of the index is given by

$$\begin{aligned} \mathrm{me}_\nu(z,q) &= e^{i\nu z} - \frac{q}{4}\left(\frac{1}{\nu+1}\, e^{i(\nu+2)z} - \frac{1}{\nu-1}\, e^{i(\nu-2)z}\right) + O(q^2)\,, \\ \mathrm{me}_k(z,q) &= \sqrt{2}\left\{\cos kz - \frac{q}{4}\left(\frac{1}{k+1}\,\cos(k+2)z - \frac{1}{k-1}\,\cos(k-2)z\right) + O(q^2)\right\}, \\ \mathrm{me}_{-k}(z,q) &= -i\sqrt{2}\left\{\sin kz - \frac{q}{4}\left(\frac{1}{k+1}\,\sin(k+2)z - \frac{1}{k-1}\,\sin(k-2)z\right) + O(q^2)\right\}. \end{aligned} \tag{B.1}$$

If $k = 1, 0, -1$, the pertinent expressions take the form

$$\begin{aligned}
\mathrm{me}_1(z,q) &= \sqrt{2}\left\{\cos z - \frac{q}{8}\,\cos 3z + O(q^2)\right\}, \\
\mathrm{me}_0(z,q) &= 1 - \frac{q}{2}\,\cos 2z + O(q^2)\,, \\
\mathrm{me}_{-1}(z,q) &= -i\sqrt{2}\left\{\sin z - \frac{q}{8}\,\sin 3z + O(q^2)\right\}.
\end{aligned} \tag{B.2}$$

In order to perform the expansion of the two-point function $S(\mathbf{x},\mathbf{y};E)$, given by Eq. (3.9), one should consider the cases of odd and even N separately. In case of the odd N, the only integer value of the parameter ν in the interval $\nu \in]-1,1]$ is $\nu = 0$. In case of the even N, there are two integer values $\nu = 0, 1$. In the sum over all eigenvectors, one should separate the integer and non-integer values of ν, and carry out the expansion in q in each term.

Let us start from the more simple case of the odd N. Here, $i = 1$ corresponds to the value $\nu_i = 0$. The eigenvalues are given by $\lambda_{\nu_i+2n} = (\nu_i + 2m)^2 + O(q^2)$. The original expression of the propagator can be split into three terms $S = S_1 + S_2 + S_3$, where, at $O(q^2)$,

$$\begin{aligned}
S_1 &= \frac{1}{L^3}\sum_{\mathbf{p}_\perp}\sum_{i=2}^{N}\sum_{n=-\infty}^{\infty} \frac{e^{i\mathbf{p}_\perp(\mathbf{x}_\perp-\mathbf{y}_\perp)}}{m + \dfrac{\mathbf{p}_\perp^2}{2m} + \dfrac{\omega^2}{8m}(\nu_i+2n)^2 - E} \\
&\times \left[e^{i(\nu_i+2n)\frac{\omega x_\parallel}{2}} - \frac{q}{4}\left(\frac{e^{i(\nu_i+2n+2)\frac{\omega x_\parallel}{2}}}{\nu_i+2n+1} - \frac{e^{i(\nu_i+2n-2)\frac{\omega x_\parallel}{2}}}{\nu_i+2n-1}\right)\right] \\
&\times \left[e^{-i(\nu_i+2n)\frac{\omega y_\parallel}{2}} - \frac{q}{4}\left(\frac{e^{-i(\nu_i+2n+2)\frac{\omega y_\parallel}{2}}}{\nu_i+2n+1} - \frac{e^{-i(\nu_i+2n-2)\frac{\omega y_\parallel}{2}}}{\nu_i+2n-1}\right)\right], \\
&= \frac{1}{L^3}\sum_{\mathbf{p}_\perp}\sum_{i=2}^{N}\sum_{n=-\infty}^{\infty} \frac{e^{i\mathbf{p}_\perp(\mathbf{x}_\perp-\mathbf{y}_\perp)}}{m + \dfrac{\mathbf{p}_\perp^2}{2m} + \dfrac{\omega^2}{8m}(\nu_i+2n)^2 - E}\, e^{i(\nu_i+2n)\frac{\omega(x_\parallel-y_\parallel)}{2}} \\
&\times \left[1 - \frac{q}{4}\left(\frac{e^{i\omega x_\parallel}}{\nu_i+2n+1} - \frac{e^{-i\omega x_\parallel}}{\nu_i+2n-1} + \frac{e^{-i\omega y_\parallel}}{\nu_i+2n+1} - \frac{e^{i\omega y_\parallel}}{\nu_i+2n-1}\right)\right],
\end{aligned} \tag{B.3}$$

$$S_2 = \frac{1}{L^3}\sum_{\mathbf{p}_\perp}\sum_{n=0}^{\infty} \frac{e^{i\mathbf{p}_\perp(\mathbf{x}_\perp-\mathbf{y}_\perp)}}{m + \dfrac{\mathbf{p}_\perp^2}{2m} + \dfrac{\omega^2}{8m}(2n)^2 - E}\left[2\mathrm{ce}_{2n}\left(\frac{\omega x_\parallel}{2},q\right)\mathrm{ce}_{2n}\left(-\frac{\omega y_\parallel}{2},q\right)\right], \tag{B.4}$$

and

$$S_3 = \frac{1}{L^3}\sum_{\mathbf{p}_\perp}\sum_{n=-\infty}^{-1} \frac{e^{i\mathbf{p}_\perp(\mathbf{x}_\perp-\mathbf{y}_\perp)}}{m + \dfrac{p_\perp^2}{2m} + \dfrac{\omega^2}{8m}(2n)^2 - E}\left[-2\mathrm{se}_{2n}\left(\frac{\omega x_\parallel}{2},q\right)\mathrm{se}_{2n}\left(-\frac{\omega y_\parallel}{2},q\right)\right]. \tag{B.5}$$

Using Eq. (B.2), one could rewrite the last two terms at $O(q)$ in the following form:

$$S_2 + S_3 = = \sum_{n=-\infty}^{\infty} \sum_{\mathbf{p}_\perp} \frac{e^{i\mathbf{p}_\perp(\mathbf{x}_\perp - \mathbf{y}_\perp)}}{m + \dfrac{\mathbf{p}_\perp^2}{2m} + \dfrac{\omega^2 n^2}{2m} - E} e^{in\omega(x_\parallel - y_\parallel)}$$
$$\times \left\{1 - \frac{q}{4}\left(\frac{e^{-i\omega y_\parallel}}{2n+1} - \frac{e^{i\omega y_\parallel}}{2n-1} + \frac{e^{i\omega x_\parallel}}{2n+1} - \frac{e^{-i\omega x_\parallel}}{2n-1}\right)\right\}. \tag{B.6}$$

It is easy to see that Eq. (B.6) follows from Eq. (B.3) for $\nu_i = 0$. Hence, one could lump together these two expressions, extending the sum in Eq. (B.3) from $i = 1$ to $i = N$. Furthermore, defining $p_\parallel = \frac{\omega}{2}(\nu_i + 2n)$, it is easily seen that the sum over all i, n is equivalent to sum over all $p_\parallel = \frac{2\pi}{L} k$, where $k \in \mathbb{Z}$. Defining further $\mathbf{p} = (\mathbf{p}_\perp, p_\parallel)$ and $\mathbf{px} = \mathbf{p}_\perp \mathbf{x}_\perp - p_\parallel x_\parallel$, the two-point function can be rewritten in a more compact form:

$$S(\mathbf{x}, \mathbf{y}; E) = \frac{1}{L^3} \sum_{\mathbf{p}} \frac{e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}}{m + \dfrac{\mathbf{p}^2}{2m} - E}$$
$$\times \left\{1 - \frac{\omega q}{8}\left(\frac{e^{i\omega x_\parallel}}{p_\parallel + \frac{\omega}{2}} - \frac{e^{-i\omega x_\parallel}}{p_\parallel - \frac{\omega}{2}} + \frac{e^{-i\omega y_\parallel}}{p_\parallel + \frac{\omega}{2}} - \frac{e^{i\omega y_\parallel}}{p_\parallel - \frac{\omega}{2}}\right)\right\}. \tag{B.7}$$

One can now shift $p_\parallel \to p_\parallel - \omega$ and $p_\parallel \to p_\parallel + \omega$ in the third and fourth terms in the brackets, respectively. Then, we have

$$S(\mathbf{x}, \mathbf{y}, E) = \frac{1}{L^3} \sum_{\mathbf{p}} \frac{e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}}{m + \dfrac{\mathbf{p}^2}{2m} - E}$$
$$- \left\{\frac{\omega q}{8} \frac{1}{L^3} \sum_{\mathbf{p}} \frac{e^{i(\mathbf{p}+\boldsymbol{\omega})\mathbf{x} - i\mathbf{py}}}{p_\parallel + \frac{\omega}{2}} \left(\frac{1}{m + \dfrac{\mathbf{p}^2}{2m} - E} - \frac{1}{m + \dfrac{(\mathbf{p}+\boldsymbol{\omega})^2}{2m} - E}\right) + (\omega \to -\omega)\right\}$$
$$= \frac{1}{L^3} \sum_{\mathbf{p}} \frac{e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}}{m + \dfrac{\mathbf{p}^2}{2m} - E}$$
$$- \frac{\omega^2 q}{8} \frac{1}{L^3} \sum_{\mathbf{p}} \left\{\frac{e^{i(\mathbf{p}+\boldsymbol{\omega})\mathbf{x} - i\mathbf{py}}}{\left(m + \dfrac{\mathbf{p}^2}{2m} - E\right)\left(m + \dfrac{(\mathbf{p}+\boldsymbol{\omega})^2}{2m} - E\right)} + (\omega \to -\omega)\right\}. \tag{B.8}$$

Performing the Fourier transform and using Eq. (3.6), we finally arrive at Eq. (3.11).

The calculations in case of an even N are slightly more complicated. Now, the eigenvalue corresponding to $\nu_i = 1$ is also present, with $\lambda_{\pm 1}(q) = 1 \pm q + O(q^2)$. Hence, the denominators corresponding to this eigenvalue, should be expanded:

$$\frac{1}{m + \dfrac{\mathbf{p}_\perp^2}{2m} + \dfrac{\omega^2}{8m} \lambda_{\pm 1}(q) - E} = \frac{1}{m + \dfrac{\mathbf{p}_\perp^2}{2m} + \dfrac{\omega^2}{8m} - E}$$

$$\mp \frac{\omega^2 q}{8m} \frac{1}{\left(m + \frac{\mathbf{p}_\perp^2}{2m} + \frac{\omega^2}{8m} - E\right)^2} + O(q^2)\,. \tag{B.9}$$

Otherwise, the calculations follow exactly the same path. Adding all contributions carefully, one finally verifies that Eq. (3.11) holds in case of the even N as well.

C The Lüscher function at $e \neq 0$

Taking into account the fact that $\omega = 2\pi N/L$ and performing the variable transformation $x_\parallel = 2u/\omega$, $y_\parallel = 2v/\omega$ in Eq. (4.13), we get

$$\bar{\Pi}(P_\parallel, Q_\parallel; \mathbf{P}_\perp; E) = \frac{1}{\pi^2 L^2 N^2} \sum_{\mathbf{p}_\perp} \sum_{i,j=1}^{N} \sum_{n,m=-\infty}^{\infty} \int_0^{N\pi} du \int_0^{N\pi} dv\, D_{in,jm}(\mathbf{p}_\perp; \mathbf{P}_\perp; E)$$

$$\times\, e^{-iau+ibv}\, \mathrm{me}_{\nu_i+2n}(u,q)\mathrm{me}_{\nu_i+2n}(-v,q)\mathrm{me}_{\nu_j+2m}(u,q)\mathrm{me}_{\nu_j+2m}(-v,q)\,. \tag{C.1}$$

Here, $a = 2P_\parallel/\omega$ and $b = 2Q_\parallel/\omega$. Furthermore, using the periodicity property of the Mathieu functions, the integration over the variables u, v can be restricted to the interval from 0 to π:

$$\begin{aligned}
&\int_0^{N\pi} du \int_0^{N\pi} dv\, e^{-iau+ibv}\, \mathrm{me}_{\nu_i+2n}(u,q)\mathrm{me}_{\nu_i+2n}(-v,q)\mathrm{me}_{\nu_j+2m}(u,q)\mathrm{me}_{\nu_j+2m}(-v,q) \\
&= \sum_{k,l=1}^{N-1} e^{i\pi(\nu_i+2n+\nu_j+2m-a)(k-1)-i\pi(\nu_i+2n+\nu_j+2m-b)(l-1)} \\
&\times \int_0^{\pi} du \int_0^{\pi} dv\, e^{-iau+ibv}\, \mathrm{me}_{\nu_i+2n}(u,q)\mathrm{me}_{\nu_i+2n}(-v,q)\mathrm{me}_{\nu_j+2m}(u,q)\mathrm{me}_{\nu_j+2m}(-v,q) \\
&= N^2 \sum_{k,l=-\infty}^{\infty} \delta_{a-b,2k}\, \delta_{\nu_i+2n+\nu_j+2m-a,2l} \\
&\times \int_0^{\pi} du \int_0^{\pi} dv\, e^{-iau+ibv}\, \mathrm{me}_{\nu_i+2n}(u,q)\mathrm{me}_{\nu_i+2n}(-v,q)\mathrm{me}_{\nu_j+2m}(u,q)\mathrm{me}_{\nu_j+2m}(-v,q)\,.
\end{aligned} \tag{C.2}$$

Hence,

$$\bar{\Pi}(P_\parallel, Q_\parallel; \mathbf{P}_\perp; E) = \sum_{\ell=-\infty}^{\infty} L\delta_{P_\parallel - Q_\parallel, \ell\omega} \tilde{\Pi}(P_\parallel, Q_\parallel; \mathbf{P}_\perp; E)\,, \tag{C.3}$$

where

$$\tilde{\Pi}(P_\parallel, Q_\parallel; \mathbf{P}_\perp; E) = \frac{1}{\pi^2 L^3} \sum_{\mathbf{p}_\perp} \sum_{i,j=1}^{N} \sum_{n,m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta_{\nu_i+2m+\nu_j+2m-a,2k}$$

$$\times \int_0^{\pi} du \int_0^{\pi} dv\, D_{in,jm}(\mathbf{p}_\perp; \mathbf{P}_\perp; E) e^{-iau+ibv}$$

$$\times\ \mathrm{me}_{\nu_i+2n}(u,q)\mathrm{me}_{\nu_i+2n}(-v,q)\mathrm{me}_{\nu_j+2m}(u,q)\mathrm{me}_{\nu_j+2m}(-v,q)\,. \tag{C.4}$$

Since $\nu_i, \nu_j \in]-1,1]$, the sum over k in the above equation has a finite number of non-zero terms. Finally, one can carry out the summation over k, which yields

$$\tilde{\Pi}(P_\parallel, Q_\parallel; \mathbf{P}_\perp; E) = \frac{1}{\pi^2 L^3} \sum_{\mathbf{p}_\perp} \sum_{i,j=1}^{N} \sum_{n,m=-\infty}^{\infty} \int_0^\pi du \int_0^\pi dv\, D_{in,jm}(\mathbf{p}_\perp; \mathbf{P}_\perp; E) e^{-iau+ibv}$$
$$\times\ \mathrm{me}_{\nu_i+2n}(u,q)\mathrm{me}_{\nu_i+2n}(-v,q)\mathrm{me}_{\nu_j+2m}(u,q)\mathrm{me}_{\nu_j+2m}(-v,q)\,. \tag{C.5}$$

Note that the conservation of the "longitudinal momentum" takes the form

$$\frac{\omega}{2}(\nu_i+2n) + \frac{\omega}{2}(\nu_j+2m) - P_\parallel = k\omega\,. \tag{C.6}$$

Equation (C.5) is still too complicated for using it in the analysis of data. Here, we are interested in the shift of the energy levels that are linear in e. It would be therefore useful to get a simplified expression that allows one to extract the levels at this precision. To this end, one first expands the numerator, using the Eqs. (B.1) and (B.2). Furthermore, as we already know, the eigenvalues $\lambda_{\nu_i+2n}(q)$ up to the order q^2 correspond to those in the free theory, whereas the case $\nu_i + 2n = \pm 1$ is an exception, see Eq. (3.14). Expanding the denominator in $D_{in,jm}(\mathbf{p}_\perp; \mathbf{P}_\perp; E)$ up to the first order in q corresponds to the "perturbative" expression, whereas leaving the denominator intact leads to the "exact" one. The final result is displayed in Eqs. (4.15), (4.16), (4.17).

D Explicit expression for the form factor

An explicit expression for the form factor in the toy model considered here can be straightforwardly obtained by evaluating the expression given in Eq. (2.23). Below, we give the final result without derivation:

$$F(\mathbf{P},\mathbf{Q}) = F(\omega) = \frac{\sqrt{-q_R^2}}{4\pi\left(1+r\sqrt{-q_R^2}\right)} \left\{ -\kappa\omega^2 q_R^2 + 8\pi\Gamma\left(r + \frac{4}{\omega}\arcsin\frac{\omega}{\sqrt{\omega^2-16q_R^2}}\right)\right\}. \tag{D.1}$$

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