DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 88-019
ITP-UH 1/88
February 1988

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ISSN 0418-9833

NOTKESTRASSE 85 · 2 HAMBURG 52

DESY 88-019 ISSN 0418-9833

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Gauge Fixing and the Cosmological Constant

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Abstract

We discuss a new class of gauges for the gravitational field, in which the volume element is fixed. The corresponding ghost lagrangian is constructed and the BRS-invariance of the full lagrangian is demonstrated. The cosmological term appears as part of the gauge fixing lagrangian. If the action is required to be invariant under general coordinate transformations, flat space-time is the uniquely determined ground state.

The Einstein action for the metric tensor $g_{\mu\nu}$,

$$S[g] = \frac{1}{2}\int d^4x \sqrt{g}\, R \quad , \tag{1}$$

is invariant under infinitesimal gauge transformations

$$\delta g_{\mu\nu} = \varepsilon^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu \varepsilon^\lambda g_{\lambda\nu} + \partial_\nu \varepsilon^\lambda g_{\mu\lambda} \quad , \tag{2}$$

where the fields ε^μ parametrize infinitesimal local coordinate transformations. Quantization of the gravitational field requires a choice of gauge [1]. Usually the harmonic gauge condition

$$C_\mu(g) = -\frac{1}{\sqrt{g}}\, g_{\mu\nu}\, \partial_\lambda(\sqrt{g}\, g^{\nu\lambda}) = 0 \tag{3}$$

is employed. In the linear approximation one obtains from eqs. (2) and (3) ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$)

$$\delta C_\mu(h) = \delta(\partial^\lambda h_{\lambda\mu} - \frac{1}{2}\partial_\mu h^\lambda{}_\lambda) = \Box\, \varepsilon_\mu \quad , \tag{4}$$

which shows that the conditions (3) lead to a well-defined propagator. In the harmonic gauge one has eight real Fadeev-Popov ghosts u^μ and $\bar{u}^\mu$, and the BRS invariance [2] as well as the unitarity of the physical S-matrix have been explicitely demonstrated [3].

In the following we will discuss an alternative set of gauge conditions. One of them fixes the volume element by requiring that some function C of $\sqrt{g} = (-\det(g_{\mu\nu}))^{1/2}$ vanishes, for instance,

$$C(g) = \sqrt{g} - \gamma = 0 \quad , \qquad \gamma = \text{const} . \tag{5}$$

As additional conditions*), which are required for a complete gauge fixing, one may choose

$$C_{\mu\nu}(g) = \partial_\mu C_\nu(g) - \partial_\nu C_\mu(g) = 0 \quad , \tag{6}$$

where $C_\mu(g)$ is the harmonic gauge condition (3). Bianchi identities imply that only three of the six equations (6) are independent, so that eqs. (5) and (6) are together four gauge fixing conditions as in the case of the harmonic gauge (3).

In the linear approximation one obtains from eqs. (2), (5) and (6):

$$\delta C(h) = \delta(\tfrac{1}{2} h^\lambda{}_\lambda) = \partial^\lambda \varepsilon_\lambda \quad , \tag{7a}$$

$$\delta C_{\mu\nu}(h) = \delta(\partial_\mu \partial^\lambda h_{\lambda\nu} - \partial_\nu \partial^\lambda h_{\lambda\mu})$$

$$= \Box(\partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu) \quad , \tag{7b}$$

i.e., the first condition determines the divergence of the gauge parameter ε_μ whereas the other three conditions fix the curl of ε_μ. Eqs. (7) show that the conditions (5) and (6) are independent, and that the corresponding propagator is well-defined.

*) These gauge conditions arise in the quantization of theories where the general coordinate invariance is restricted to transformations which leave the volume element invariant [4].

One easily verifies that it is not possible to supplement eq. (5) in a Lorentz-covariant way by three independent conditions which involve only first derivatives of the metric tensor $g_{\mu\nu}$.

Solutions of Einstein's equations which satisfy the gauge conditions $C_{\mu\nu} = C = 0$ are equally well-behaved as solutions in the harmonic gauge $C_\mu = 0$. This is obvious for gravitational waves in flat space, and can easily be verified for the Schwarzschild solution, maximally symmetric spaces and the Robertson-Walker metric.

Supplementing the generally covariant Einstein action by the gauge fixing lagrangian*) (cf. (5), (6))

$$\mathcal{L}_{GF} = -\frac{\alpha}{4} C^{\mu\nu} C_{\mu\nu} - \frac{\beta}{2} C^2 \tag{8}$$

yields a well-defined propagator for the gravitational field. Of course, one cannot just add a gauge fixing term to the lagrangian. One has to separate the gauge modes, which are now propagating, from the physical subspace and one has to introduce Fadeev-Popov ghost fields which compensate the contribution of the gauge modes in loops. Furthermore the gauge invariance of physical amplitudes has to be shown. All these required properties of the quantized theory can be derived from the invariance of the full lagrangian under BRS transformations [2],

$$s\mathcal{L} = 0 \quad , \tag{9}$$

*) Unless explicitly stated indices are raised and lowered with the flat metric $\eta_{\mu\nu}$.

where s is a real, nilpotent antiderivation,

$$s^2 = 0 \quad , \tag{10a}$$

$$s(AB) = (sA)B + (-)^{|A|} A(sB) \quad , \tag{10b}$$

$$(sA)^* = (-)^{|A|} (sA^*) \quad , \tag{10c}$$

with $|A| = 1(0)$ for anticommuting (commuting) fields.

The BRS invariant extension of the gauge fixing lagrangian (8) requires 10 ghosts: u^μ, v, $\bar{u}^\mu$ and $\bar{v}$, i.e., two additional ghosts compared to the harmonic gauge (3). The ghost lagrangian reads (cf. eqs. (7))

$$L_{GH} = i\alpha\, \bar{u}^\mu s(C_\mu + \kappa\, \partial_\mu C) + i\alpha\, \bar{u}^\mu \partial_\mu v + i\beta\, \bar{v}\, s\, C \quad , \tag{11}$$

where κ is an arbitrary real parameter. The BRS transformation on the field $g_{\mu\nu}$ is obtained in the usual way from the Lie derivative (2) by replacing the gauge parameter ε^μ by the ghost u^μ. The complete BRS transformation for all fields reads:

$$s\, g_{\mu\nu} = u^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu u^\lambda g_{\lambda\nu} + \partial_\nu u^\lambda g_{\mu\lambda} \quad , \tag{12a}$$

$$s\, u^\mu = u^\nu \partial_\nu u^\mu \quad , \tag{12b}$$

$$s\, v = 0 \quad , \tag{12c}$$

$$s\, \bar{u}^\mu = i\, \partial_\nu C^{\nu\mu} \quad , \tag{12d}$$

$$s\, \bar{v} = -i\, C \quad . \tag{12e}$$

Using $s^2 g_{\mu\nu} = s^2 u_\mu = sv = 0$ one easily proves the invariance under BRS transformations (cf. (8), (11)):

$$s \int d^4x \, (L_{GH} + L_{GF}) = 0 \quad . \tag{13}$$

On the antighosts $\bar{u}_\mu$ and $\bar{v}$ s is nilpotent only "on-shell", i.e., for

$$s\, \partial^\nu C_{\nu\mu} = s\, C = 0 \quad , \tag{14}$$

which follows from the ghost equations of motion.

In the linear approximation (cf. (4), (7)) the ghost lagrangian (11) becomes

$$L_{GH} = i\alpha\, \bar{u}^\mu (\Box \eta_{\mu\nu} + \kappa\, \partial_\mu \partial_\nu) u^\nu + i\alpha\, \bar{u}^\mu \partial_\mu v + i\beta\, \bar{v}\, \partial_\mu u^\mu \quad . \tag{15}$$

The corresponding path integral is nothing but the determinant of the d'Alembert operator for vector fields with vanishing divergence:

$$\begin{aligned} &\int Du\, D\bar{u}\, Dv\, D\bar{v}\, \exp\Big(i \int d^4x\, [\bar{u}^\mu (\Box \eta_{\mu\nu} + \kappa\, \partial_\mu \partial_\nu) u^\nu \\ &\qquad\qquad + \bar{u}^\mu \partial_\mu v + \bar{v}\, \partial_\mu u^\mu]\Big) \\ &= \int Du\, D\bar{u}\, \delta(\partial^\mu u_\mu)\, \delta(\partial^\mu \bar{u}_\mu)\, \exp\Big(i \int d^4x\, \bar{u}^\mu \Box u_\mu\Big) \end{aligned} \tag{16}$$

In the Fadeev-Popov procedure [5] this determinant results from the change of variables $(C, C_{\mu\nu}) \to (\partial^\lambda \varepsilon_\lambda, \varepsilon_{\mu\nu})$, $\varepsilon_{\mu\nu} = \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu$ (cf. eqs. (7)). This explains why the gauge condition $C_{\mu\nu} = 0$ leads to two additional ghosts compared to the harmonic gauge.

The ghosts v and $\bar{v}$ are necessary to obtain a well-defined ghost propagator for arbitrary values of κ. With $\omega = (u_\mu, v)$ one easily finds

$$\langle \omega(x)\bar{\omega}(y)\rangle_0 = \frac{1}{\Box}\begin{pmatrix} \eta_\mu{}^\nu - \frac{\partial_\mu \partial^\nu}{\Box} & \partial_\mu \\ \partial^\nu & -(1+\kappa) \end{pmatrix} \delta^4(x-y) \quad . \tag{17}$$

The complete ghost lagrangian (11) reads explicitly

$$\begin{aligned} L_{GH} = {} & i\alpha\,\bar{u}^\mu\big(g_{\mu\nu}(\partial_\lambda + \Gamma^\tau_{\tau\lambda})(g^{\nu\rho}D_\rho u^\lambda + g^{\lambda\rho}D_\rho u^\nu) \\ & - \partial_\mu((1-\kappa\sqrt{g})\,D_\nu u^\nu) + g^{\lambda\tau}\Gamma^\nu_{\lambda\tau}(g_{\nu\rho}D_\mu + g_{\mu\rho}D_\nu)u^\rho\big) \\ & + i\alpha\,\bar{u}^\mu\partial_\mu v \; + \; i\beta\,\bar{v}\sqrt{g}\,D_\mu u^\mu \quad . \end{aligned} \tag{18}$$

A straightforward calculation yields for the graviton propagator:

$$\begin{aligned} \langle h_{\mu\nu}(x)h_{\lambda\tau}(y)\rangle_0 = - \Big\{ & \tfrac{1}{2}\Big[\Big(\eta_{\mu\lambda} - \frac{\partial_\mu\partial_\lambda}{\Box}\Big)\Big(\eta_{\nu\tau} - \frac{\partial_\nu\partial_\tau}{\Box}\Big) + \Big(\eta_{\mu\tau} - \frac{\partial_\mu\partial_\tau}{\Box}\Big)\Big(\eta_{\nu\lambda} - \frac{\partial_\nu\partial_\lambda}{\Box}\Big) \\ & - \Big(\eta_{\mu\nu} - 2\frac{\partial_\mu\partial_\nu}{\Box}\Big)\Big(\eta_{\lambda\tau} - 2\frac{\partial_\lambda\partial_\tau}{\Box}\Big) - 2\,\frac{\partial_\mu\partial_\nu\partial_\lambda\partial_\tau}{\Box^2}\Big] \\ & + \frac{4}{\beta}\,\frac{\partial_\mu\partial_\nu\partial_\lambda\partial_\tau}{\Box} \\ & + \frac{1}{\alpha}\Big(\eta_{\mu\lambda}\partial_\nu\partial_\tau + \eta_{\nu\lambda}\partial_\mu\partial_\tau + \eta_{\mu\tau}\partial_\nu\partial_\lambda \\ & \quad + \eta_{\nu\tau}\partial_\mu\partial_\lambda - 4\,\frac{\partial_\mu\partial_\nu\partial_\lambda\partial_\tau}{\Box}\Big)\frac{1}{\Box^2}\Big\} \frac{1}{\Box}\,\delta^4(x-y) \quad . \end{aligned} \tag{19}$$

This result differs from the familiar propagator in the harmonic gauge by projection operators on longitudinal gauge degrees of freedom and by two terms which depend on the parameters α and β of the gauge fixing lagrangian. Since the term $\alpha(C_{\mu\nu})^2$ contains four derivatives the propagator (19) contains an infrared singular piece proportional to $1/\alpha$. Since the limit $\alpha \to \infty$ exists we do not expect that this infrared singular term causes any problems with respect to gauge invariant quantities.

Gauge conditions which fix the volume element are of interest in connection with the problem of the cosmological constant. The "volume gauge" (5) has the surprising property that, up to an irrelevant constant, a cosmological term can be absorbed into the gauge fixing part of the lagrangian:

$$-\sqrt{g}\,\Lambda + \frac{\beta}{2}(\sqrt{g} - \gamma)^2 = \frac{\beta}{2}(\sqrt{g} - \gamma')^2 - \frac{\Lambda^2}{2\beta} \quad , \tag{20}$$

where $\gamma' = \gamma + \Lambda/\beta$. This is a puzzling result since the cosmological constant determines the curvature of the ground state which is generally believed to be a gauge invariant observable quantity.

One might expect that the effect of the choice of gauge on the curvature is a pathological feature of "volume gauges", and that the familiar relation between the "physical" vacuum energy density Λ and the ground state curvature is restored if boundary conditions are properly taken into account. As we will see, however, this is not the case and the cosmological constant is indeed a gauge breaking term.

The variation of an arbitrary scalar lagrangian $L(\Phi)$ under infinitesimal gauge transformations is given by

$$\delta(\sqrt{g}\,L) = \partial_\mu(\varepsilon^\mu\sqrt{g}\,L) \quad . \tag{21}$$

Hence the action is invariant only if the boundary term

$$\delta S[\phi] = \int_{\partial M} d\sigma_\mu \, \varepsilon^\mu \sqrt{g}\, L(\phi) \tag{22}$$

vanishes for arbitrary parameters ε^μ, i.e., if the state Φ satisfies the boundary condition

$$L(\phi)\Big|_{\partial M} = 0 \quad . \tag{23}$$

In the case of the Einstein action with cosmological term,

$$L = \frac{1}{2} R - \Lambda \quad , \tag{24}$$

the equations of motions yield

$$R = 4\Lambda \quad . \tag{25}$$

For Lorentz signature of the metric there always exists a boundary ∂M either at $t=\pm\infty$ or $r = \infty$. Hence eqs. (23) and (25) imply:

$$\Lambda = 0 \quad . \tag{26}$$

We conclude that a gauge invariant ground state exists only for vanishing cosmological constant.

A cosmological term yields a covariantly constant contribution to the energy momentum tensor, which is the source of the gravitational field. The breaking of gauge invariance through a cosmological term is analogous to the breaking of gauge invariance in electrodynamics through an external current. The variation of the action

$$S[A] = \int_M d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \right) \tag{27}$$

under a gauge transformation $\delta A_\mu = \partial_\mu \lambda$ is given by

$$\delta S[A] = -\int_{\partial M} d\sigma_\mu \, \lambda \, j^\mu \quad , \tag{28}$$

which, for arbitrary $\lambda(x)$, is zero only if the current vanishes at the boundary. This is known [6] to be a consistency requirement for the quantization of the electromagnetic field. In particular a constant external current is excluded.

We have shown that the gravitational field can be consistently quantized in a new class of gauges in which the volume element is fixed. These gauges give rise to the question whether the cosmological constant breaks gauge invariance. An investigation of boundary terms shows that this is indeed the case and that the existence of a gauge invariant ground state requires vanishing cosmological constant. This suggests that the observed flatness of space-time may be understood as a consequence of gauge invariance.

Acknowledgements

We are indebted to M. Kreuzer for helpful discussions on gauge fixing, and thank M. Lüscher and G. Mack for drawing our attention to boundary terms.

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