Stability Indices of Non-Hyperbolic Equilibria in Two-Dimensional Systems of ODEs

Alexander Lohse
alexander.lohse@uni-hamburg.de
Fachbereich Mathematik, Universität Hamburg
Bundesstraße 55, 20146 Hamburg, Germany

Abstract

We consider families of systems of two-dimensional ordinary differential equations with the origin 0 as a non-hyperbolic equilibrium. For any number $s \in (-\infty, +\infty)$ we show that it is possible to choose a parameter in these equations such that the stability index $\sigma(0)$ is precisely $\sigma(0) = s$. In contrast to that, for a hyperbolic equilibrium x it is known that either $\sigma(x) = -\infty$ or $\sigma(x) = +\infty$. Furthermore, we discuss a system with an equilibrium that is locally unstable but globally attracting, highlighting some subtle differences between the local and non-local stability indices.

Keywords: stability, attraction, non-hyperbolic equilibrium

AMS classification: 34D20, 37C25, 37C75

1 Introduction

Attraction and stability of invariant sets are crucial concepts in the qualitative theory of dynamical systems: the degree to which a set possesses these properties is directly linked to the way it influences the overall (longterm) dynamics of a system. Beyond the classic notion of asymptotic (Lyapunov) stability several levels of so-called *non-asymptotic stability* have been identified. These include *fragmentary asymptotic stability (f.a.s.)* [12] and *essential asymptotic stability (e.a.s.)* [11] to mention probably the two most frequent ones. Loosely speaking, an f.a.s. set attracts something of positive measure while an e.a.s. set attracts "almost everything" in a small neighbourhood.

In 2011 Podvigina and Ashwin [13] introduced a *(local) stability index* as a means of quantifying stability and attraction of invariant sets in discrete and continuous dynamical systems. It is linked to the stability properties mentioned above: roughly speaking, positive indices correspond to essential asymptotic stability, while fragmentary asymptotic stability is associated with indices that are greater than $-\infty$, see [10] for a detailed discussion of this. In the last decade, this concept has been used to characterize various types of attractors, e.g. heteroclinic cycles/networks [3, 5, 6], invariant graphs in skew product systems [7] or attractors with riddled basins [14].

For the simple case of a hyperbolic equilibrium the stability index does not reveal significant information, since it turns out to be either $+\infty$ (for a sink) or $-\infty$ (for a saddle or source). In this paper we discuss two families of ordinary differential equations on $\mathbb{R}^2$ that possess the origin 0 as a non-hyperbolic equilibrium. We show that

(i) for any given real number $s > 0$ we can choose a parameter in the first family such that we obtain $\sigma(0) = s$, and

(ii) the same is possible for any $s < 0$ in the second family.

This confirms that non-hyperbolic equilibria can indeed be f.a.s. or e.a.s without being asymptotically stable.

We also present an example of a smooth system with a non-hyperbolic equilibrium that is strongly attracting (stability index equal to $+\infty$) but at the same time locally repels most initial conditions (local stability index equal to $-\infty$). Systems with similar properties in previous work [9] lacked smoothness.

In higher-dimensional systems our results may be useful for understanding the dynamics along the centre manifold of an equilibrium, thus helping to better describe stability and attraction properties of non-hyperbolic steady states. Moreover, the way we design these systems might serve as a prototype for controlling stability indices in more involved settings, e.g. along heteroclinic connections.

The paper is organized as follows: in section 2 we briefly discuss non-asymptotic stability and the (local) stability index. In section 3 we present our examples and prove that the equilibria possess the desired stability indices. We conclude with some comments in section 4.

2 Preliminaries

In this section we reproduce the definitions of fragmentary and essential asymptotic stability of a compact, invariant set $X \subset \mathbb{R}^n$ for a dynamical system on $\mathbb{R}^n$ given by $\dot{x} = f(x)$. Moreover, we recall the stability index that was introduced to quantify stability and attraction of such a set.

In line with standard notation we write $B_\varepsilon(x)$ for an ε-neighbourhood of a point $x \in \mathbb{R}^n$ and use $\ell(.)$ for Lebesgue measure. The basin of attraction of X, i.e. the set of points in $\mathbb{R}^n$ with ω-limit set in X, is denoted by $\mathcal{B}(X)$. For

$\delta > 0$ the δ-local basin of attraction $\mathcal{B}_\delta(X)$ is the subset of points in $\mathcal{B}(X)$ for which the trajectory never leaves $B_\delta(X)$ in positive time.

With this terminology we revisit the following definitions.

Definition 2.1 ([12], definition 2)**.** X is called *fragmentarily asymptotically stable (f.a.s.)* if $\ell(\mathcal{B}_\delta(X)) > 0$ for any $\delta > 0$.

As discussed in [8] being f.a.s. is equivalent to having a basin of attraction of positive measure.

Definition 2.2 ([4], definition 1.2)**.** X is called *essentially asymptotically stable (e.a.s.)* if it is asymptotically stable relative to a set $N \subset \mathbb{R}^n$ which satisfies

$$\lim_{\varepsilon \to 0} \frac{\ell(B_\varepsilon(X) \cap N)}{\ell(B_\varepsilon(X))} = 1.$$

Here *asymptotic stability relative to* N means that the usual conditions for asymptotic stability must be fulfilled for the intersection of a neighbourhood of X with N, but not necessarily in an entire neighbourhood.

Note that in [11] e.a.s. is used in the same sense as above, even though a slightly different definition is given.

Definition 2.3 ([13], definition 5)**.** For $x \in X$ and $\varepsilon, \delta > 0$ set

$$\Sigma_\varepsilon(x) := \frac{\ell(B_\varepsilon(x) \cap \mathcal{B}(X))}{\ell(B_\varepsilon(x))}, \qquad \Sigma_{\varepsilon,\delta}(x) := \frac{\ell(B_\varepsilon(x) \cap \mathcal{B}_\delta(X))}{\ell(B_\varepsilon(x))}.$$

Then the *stability index* at x with respect to X is defined as

$$\sigma(x) := \sigma_+(x) - \sigma_-(x),$$

with

$$\sigma_-(x) := \lim_{\varepsilon \to 0} \frac{\ln(\Sigma_\varepsilon(x))}{\ln(\varepsilon)}, \qquad \sigma_+(x) := \lim_{\varepsilon \to 0} \frac{\ln(1 - \Sigma_\varepsilon(x))}{\ln(\varepsilon)}.$$

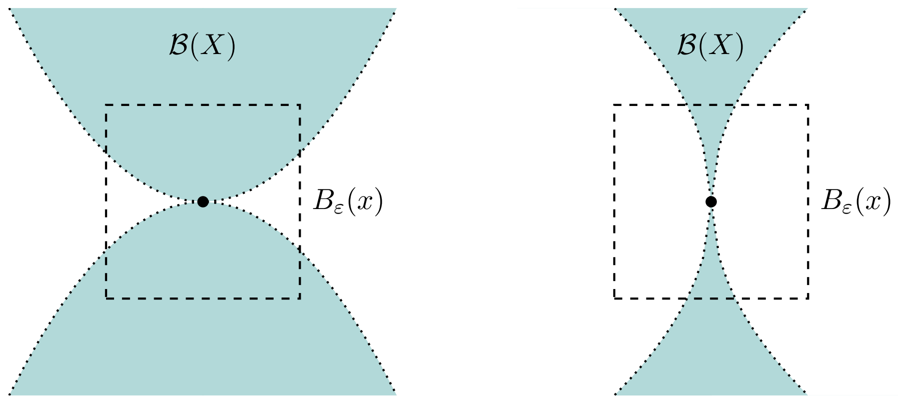
The convention that $\sigma_-(x) = \infty$ if $\Sigma_\varepsilon(x) = 0$ for some $\varepsilon > 0$, and $\sigma_+(x) = \infty$ if $\Sigma_\varepsilon(x) = 1$ for some $\varepsilon > 0$, implies $\sigma(x) \in [-\infty, \infty]$.

Analogously, the *local stability index* at $x \in X$ is defined to be

$$\sigma_{\mathrm{loc}}(x) := \sigma_{\mathrm{loc},+}(x) - \sigma_{\mathrm{loc},-}(x),$$

with

$$\sigma_{\mathrm{loc},-}(x) := \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \frac{\ln(\Sigma_{\varepsilon,\delta}(x))}{\ln(\varepsilon)}, \ \sigma_{\mathrm{loc},+}(x) := \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \frac{\ln(1 - \Sigma_{\varepsilon,\delta}(x))}{\ln(\varepsilon)}.$$



(a) (b)

Figure 1: (a) an e.a.s. equilibrium with a positive stability index; (b) an f.a.s. equilibrium with a negative stability index.

For an invariant set $X \subset \mathbb{R}^n$ and a point $x \in X$ the index $\sigma(x)$ quantifies attraction to X near x in the system. In the same way the local index $\sigma_{\mathrm{loc}}(x)$ characterizes (Lyapunov) stability of X near x. While these two properties often go hand in hand (and the local and non-local indices may coincide), it is well-known that they are independent of each other (so local and non-local indices may differ), see examples in [9].

For a geometric intuition consider Figure 1: if $\sigma(x) > 0$, then in a small neighbourhood of x an increasingly large portion of points is contained in the basin of attraction $\mathcal{B}(X)$ and therefore attracted to X. If on the other hand $\sigma(x) < 0$, then the portion of such points goes to zero as the neighbourhood $B_\varepsilon(x)$ shrinks. The meaning of signs for the local stability index may be illustrated analogously.

Since here we are interested in the stability of equilibria, we typically have $X = \{0\}$ in the following, which prompts us to conveniently shorten our notation to $\mathcal{B}(0) = \mathcal{B}(\{0\})$ etc.

3 Stability Indices

In this section we discuss several families of systems in $\mathbb{R}^2$, each with a non-hyperbolic equilibrium which, depending on a parameter in the equations, may possess any given real number as its stability index. Note that we define the systems only for $x, y \geq 0$, but they can easily be symmetrically extended to the whole plane. Most of the time local and non-local stability indices coincide – we therefore only distinguish between the two when this is not the case.

3.1 Positive Stability Indices

We first present a class of systems in $\mathbb{R}^2$ with the origin 0 as an equilibrium that can have any stability index in $(0, +\infty)$. With a parameter $a > 1$, for $x, y \geq 0$ our system reads:

$$\begin{cases} \dot{x} &= x(x^a - y) \\ \dot{y} &= y\left(\frac{1}{2}x^a - y\right) \end{cases} \tag{1}$$

We remark that the right-hand side is at least C^1, but not C^∞ if $a \notin \mathbb{N}$.

It is easy to see that 0 is a non-hyperbolic equilibrium of the system since the Jacobian is just the zero matrix. Both coordinate axes are invariant: for $y = 0$ we have $\dot{x} = x^{a+1} > 0$, so the x-axis belongs to the unstable set of 0. Similarly, for $x = 0$ we have $\dot{y} = -y^2 < 0$, so the y-axis belongs to the stable set of 0.

The x- and y-nullclines off the coordinate axes are given by:

$$\dot{x} = 0 \quad \Leftrightarrow \quad y = x^a \qquad \text{and} \qquad \dot{y} = 0 \quad \Leftrightarrow \quad y = \frac{1}{2}x^a$$

This enables us to sketch the dynamics of system (1) as in Figure 2. We now proceed to state and prove our result about the stability index.

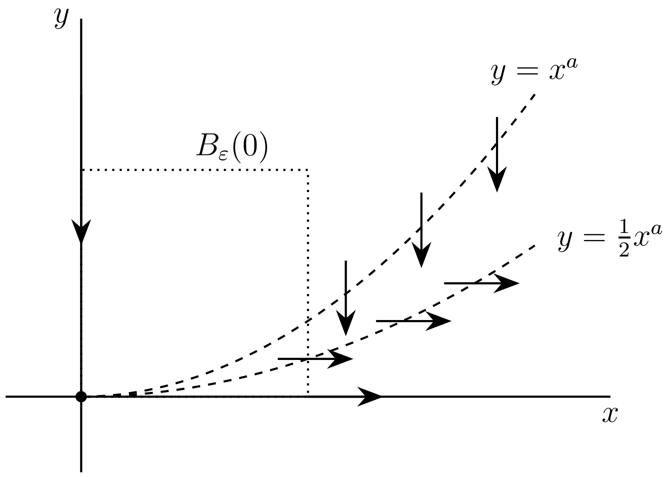


Figure 2: Nullclines for system (1) with $a > 1$.

Proposition 3.1. *In system (1), for $a > 1$ the stability index of the origin is $\sigma(0) = a - 1 > 0$.*

Proof. From Figure 2 it is clear that all points (x, y) with $y < x^a$ do not belong to the basin of attraction $\mathcal{B}(0)$. This enables our first estimate:

$$\ell(B_\varepsilon(0) \cap \mathcal{B}(0)) \leq \varepsilon^2 - \int_0^\varepsilon x^a \mathrm{d}x = \varepsilon^2 - \frac{1}{1+a}\varepsilon^{1+a}$$

and therefore

$$\Sigma_\varepsilon(0) = \frac{\ell(B_\varepsilon(0) \cap \mathcal{B}(0))}{\ell(B_\varepsilon(0))} \leq \frac{1}{\varepsilon^2}\left(\varepsilon^2 - \frac{1}{1+a}\varepsilon^{1+a}\right) = 1 - \frac{1}{1+a}\varepsilon^{a-1},$$

or equivalently

$$1 - \Sigma_\varepsilon(0) \geq \frac{1}{1+a}\varepsilon^{a-1}.$$

Hence

$$\sigma_+(0) = \lim_{\varepsilon \to 0} \frac{\ln(1 - \Sigma_\varepsilon(0))}{\ln(\varepsilon)} \leq \lim_{\varepsilon \to 0} \frac{\ln(\varepsilon^{a-1})}{\ln(\varepsilon)} = a - 1,$$

which finally implies $\sigma(0) = \sigma_+(0) - \sigma_-(0) \leq a - 1$.

For the other inequality we show that there is a constant $k > 1$ such that all (x, y) with $y > kx^a$ belong to $\mathcal{B}(0)$, in fact, even to all $\mathcal{B}_\delta(0)$ with suitable $\delta > 0$. In other words: we show that this region is forward invariant under the dynamics of system (1) and all trajectories in it converge to the origin.

We claim that for a given $a > 1$ a choice of $k > \frac{a - \frac{1}{2}}{a-1} > 1$ suffices. This we prove by showing that the vector $(\dot{x}, \dot{y})$ in this region always points downwards and "to the left" of the curve (x, kx^a), which means the corresponding solution is for all positive times confined between (x, kx^a) and the y-axis, and thus must limit to 0. To see this, first note that clearly $\dot{y} < -\frac{1}{2}y^2 < 0$ in this region. Furthermore, we calculate that the angle α between $(\dot{x}, \dot{y})$ and the normal vector $(-akx^{a-1}, 1)$ is always in $(-\frac{\pi}{2}, \frac{\pi}{2})$ along (x, kx^a), see Figure 3. To that end, consider the scalar product:

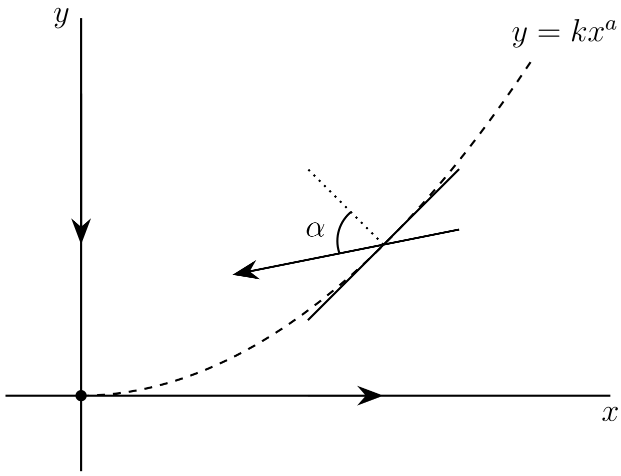


Figure 3: The angle α between the (dotted) normal vector to (x, kx^a) and the flow of system (1).

$$
\begin{aligned}
\langle (\dot{x}, \dot{y}), (-akx^{a-1}, 1) \rangle &= -akx^a(x^a - y) + y\left(\frac{1}{2}x^a - y\right) \\
&= -akx^a(x^a - kx^a) + kx^a\left(\frac{1}{2}x^a - kx^a\right) \\
&= kx^{2a}\left(a(k-1) + \frac{1}{2} - k\right) \\
&= kx^{2a}\left(k(a-1) - a + \frac{1}{2}\right),
\end{aligned}
$$

which is positive for all $x > 0$ if and only if $k > 1$ is chosen as above. Such a choice is obviously possible for any $a > 1$. An analogous calculation to that at the beginning of this proof now yields $\sigma_+(0) \geq a - 1$ and therefore $\sigma(0) \geq a - 1$. Therefore, $\sigma(0) = a - 1$ as claimed. □

Corollary 3.2. *Given any $s > 0$, set $a := s + 1 > 1$ to obtain $\sigma(0) = s$ in system (1).*

Corollary 3.3. *For $a > 1$ the origin in system (1) is e.a.s.*

3.2 Negative Stability Indices

We now strive for a similar result with negative stability indices. An analogous calculation for system (1) with $a < 1$ does not yield the desired flow, since no suitable k can be found to obtain a positive scalar product as above: we would need $k > 1$ as before, but with $a < 1$ obtaining a positive scalar product requires $k < \frac{a-\frac{1}{2}}{a-1} < 1$.

However, with $a \in (0,1)$ the following modification of system (1) does the job:

$$\begin{cases} \dot{x} &= x(\frac{1}{2}x^a - y) \\ \dot{y} &= y^2\,(x^a - y) \end{cases} \tag{2}$$

Note that the smoothness of system (2) is most severely limited by the x-term in the y-equation: since $a \in (0,1)$, the derivative of the second equation with respect to x is undefined at the origin. It is also worth pointing out that a stronger contraction in the y-direction than in system (1) is required to achieve the desired result, as becomes apparent in the calculations below.

As before the coordinate axes are invariant, and for $y = 0$ we have $\dot{x} = \frac{1}{2}x^{a+1} > 0$, so expanding dynamics on the x-axis; while for $x = 0$ we have $\dot{y} = -y^3 < 0$, so contracting dynamics on the y-axis. Note that the position of the x- and y-nullclines has been reversed compared to system (1), and we may sketch the phase portrait as in Figure 4.

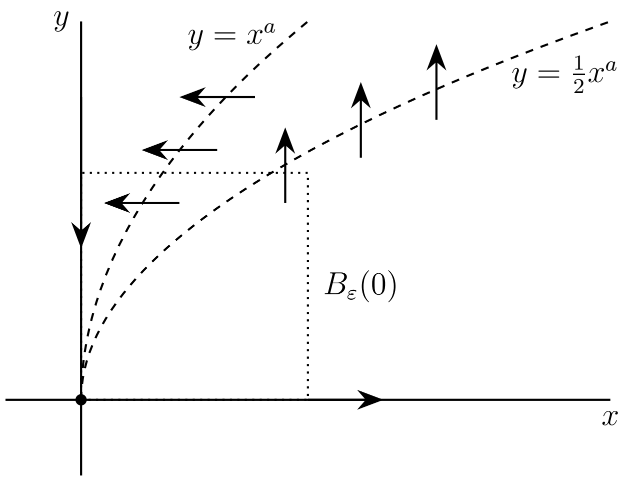


Figure 4: Nullclines for system (2) with $a < 1$.

Proposition 3.4. *In system (2), for $a < 1$ the stability index of the origin is $\sigma(0) = 1 - \frac{1}{a} < 0$.*

Proof. We argue in the same way as in the proof of Proposition 3.1 but with reversed justifications for the two inequalities: first observe from Figure 4 that for $y > x^a$ we have $\dot{x}, \dot{y} < 0$ and therefore all such points belong to the (local) basin of attraction of the origin. Thus, we obtain:

$$\ell(B_\varepsilon(0) \cap \mathcal{B}(0)) \geq \int_0^\varepsilon x^{\frac{1}{a}} \mathrm{d}x = \frac{a}{1+a} \varepsilon^{1+\frac{1}{a}}$$

and therefore

$$\Sigma_\varepsilon(0) = \frac{\ell(B_\varepsilon(0) \cap \mathcal{B}(0))}{\ell(B_\varepsilon(0))} \geq \frac{1}{\varepsilon^2} \frac{a}{1+a} \varepsilon^{1+\frac{1}{a}} = \frac{a}{1+a} \varepsilon^{\frac{1}{a}-1},$$

hence

$$\sigma_-(0) = \lim_{\varepsilon \to 0} \frac{\ln(\Sigma_\varepsilon(0))}{\ln(\varepsilon)} \leq \lim_{\varepsilon \to 0} \frac{\ln(\varepsilon^{\frac{1}{a}-1})}{\ln(\varepsilon)} = \frac{1}{a} - 1,$$

which finally implies $\sigma(0) = \sigma_+(0) - \sigma_-(0) \geq 1 - \frac{1}{a}$.

For the other inequality, we also proceed in a similar way as before, showing that along (x, kx^a) the angle between $(\dot{x}, \dot{y})$ and the normal vector $(akx^{a-1}, -1)$ is in $(-\frac{\pi}{2}, \frac{\pi}{2})$ for suitable $0 < k < \frac{1}{2}$. This implies that the region with $y < kx^a$ is forward invariant under the dynamics of system (2). Moreover, solutions with initial conditions in it do not limit to the origin in forward time and thus do not belong to $\mathcal{B}(0)$, which enables our second estimate for the stability index. Again we consider the scalar product:

$$\begin{aligned}
\langle (\dot{x}, \dot{y}), (akx^{a-1}, -1) \rangle &= akx^a \left(\frac{1}{2} x^a - y \right) - y^2 (x^a - y) \\
&= akx^a \left(\frac{1}{2} x^a - kx^a \right) - (kx^a)^2 (x^a - kx^a) \\
&= kx^{2a} \left(a \left(\frac{1}{2} - k \right) - kx^a (1-k) \right).
\end{aligned}$$

The second term in parentheses goes to zero when $x \to 0$, while the first one is constant in x and positive for $0 < k < \frac{1}{2}$. Thus, with $k \in \left(0, \frac{1}{2}\right)$ the scalar product is positive for sufficiently small $x > 0$. Again, similar calculations as above now yield $\sigma_-(0) \geq \frac{1}{a} - 1$ and thus finally $\sigma(0) = 1 - \frac{1}{a} < 0$. □

Corollary 3.5. *Given any $s < 0$, set $a := \frac{1}{1-s} \in (0,1)$ to obtain $\sigma(0) = s$ in system (2).*

Corollary 3.6. *For $a < 1$ the origin in system (2) is f.a.s., but not e.a.s.*

With Propositions 3.1 and 3.4 we have established that in these systems of equations we can obtain any positive or negative number as the stability index of the origin.

3.3 Infinite Stability Indices

More generally, instead of $x \mapsto x^a$ let us now take any function $x \mapsto \phi(x)$ and consider the following system for $x, y \geq 0$:

$$\begin{cases} \dot{x} &= x\left(y - \frac{1}{2}\phi(x)\right) \\ \dot{y} &= y(y - \phi(x)) \end{cases} \tag{3}$$

The smoothness of system (3) is determined by the smoothness of ϕ. If ϕ is non-negative and vanishes only at 0, we can draw similar initial conclusions as above: the coordinate axes are invariant with contraction along the x-axis, where $\dot{x} = -\frac{1}{2}x\phi(x) > 0$; and expansion along the y-axis, where $\dot{y} = y^2$. Looking at the nullclines we obtain the sketch of the dynamics in Figure 5.

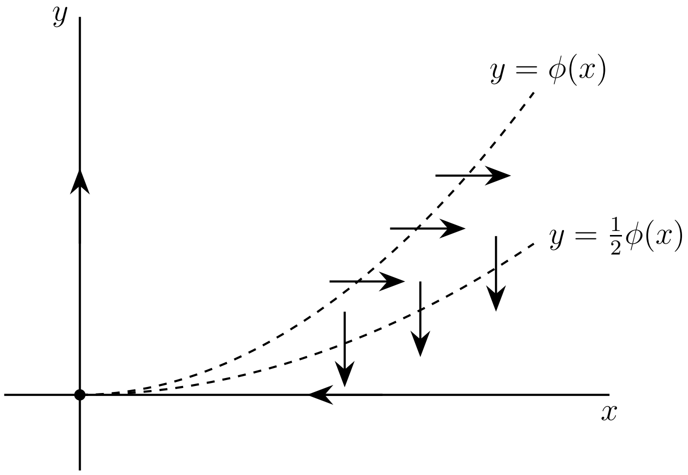


Figure 5: Dynamics for system (3).

We now pick a specific function for ϕ which is used in [9] to show that it is possible to have an equilibrium with a stability index equal to $+\infty$, but a local stability index equal to $-\infty$. This is achieved by making the equilibrium globally attracting, but confining the local basin of attraction within the region where $y < \phi(x)$. With this choice of ϕ for system (3), we obtain the same extreme discrepancy between the local and non-local stability index of the origin, but achieve a higher degree of smoothness of the system than in [9].

Proposition 3.7. *In system (3) define ϕ as $\phi(x) = (2x+1)\exp(-\frac{1}{x})$ for $x > 0$ and $\phi(0) = 0$. Then we have $\sigma_{\text{loc}}(0) = -\infty$ and $\sigma(0) = +\infty$.*

Proof. We start with the claim about the local stability index. It is clear from Figure 5 that all (x, y) with $y < \phi(x)$ belong to $\mathcal{B}(0)$, even to $\mathcal{B}_\delta(0)$ for suitable $\delta > 0$. For (x, y) with $y > \phi(x)$ we have $\dot{x}, \dot{y} > 0$, so these trajectories first move away from the origin in both coordinates and do not belong to $\mathcal{B}_\delta(0)$ for sufficiently small $\delta > 0$. Thus, by the same arguments as in [9], we have $\sigma_{\mathrm{loc}}(0) = -\infty$.

To prove the second claim we show that in system (3) all trajectories off the coordinate axes limit to 0 in forward time and are thus homoclinic to the origin. In fact, because of the above it suffices to ensure that all trajectories starting with $y > \phi(x)$ eventually cross the graph of ϕ.

To this end we show that $V(x, y) := \frac{x}{y}$ is a Lyapunov function for system (3):

$$\begin{aligned}
\frac{\partial}{\partial t} V(x, y) &= \frac{\dot{x}y - x\dot{y}}{y^2} \\
&= \frac{x(y - \frac{1}{2}\phi(x))y - xy(y - \phi(x))}{y^2} \\
&= \frac{x}{y}\left(y - \frac{1}{2}\phi(x) - (y - \phi(x))\right) \\
&= \frac{x\phi(x)}{2y} \\
&> 0
\end{aligned}$$

Thus, V increases along solutions to system (3). The level sets of V are straight lines through the origin, with the values of V increasing as the slope of these lines decreases. Since the derivative above is bounded away from zero off the coordinate axes, solutions cross level sets of V with non-vanishing speed and thus every solution eventually crosses the graph of ϕ, therefore converging to the origin. Thus, $\sigma(0) = +\infty$ as claimed. □

Note that the corresponding example in [9] has a right-hand side that is only continuous, not differentiable. Our choice of ϕ in Proposition 3.7 makes system (3) C^∞, so we provide a smooth example of this kind.

4 Concluding Remarks

We have discussed two families of systems of ordinary differential equations on $\mathbb{R}^2$ that possess a non-hyperbolic equilibrium with an arbitrary real number $s \in \mathbb{R} \setminus \{0\}$ as its stability index. While we give an explicit construction for any such s, it is worth pointing out that similar results can be obtained by taking system (1) or (2) with a fixed parameter a and transforming it through $(x, y) = (u^p, v)$. This coordinate change maps a curve given by $y = kx^a$ to that given by $v = kx^{pa}$ and thus yields a different stability index. For example, if $a > 1$ is fixed and system (1) is transformed with $p \in \mathbb{R}$ such that $pa > 1$, then it follows directly from Proposition 3.1 that $\sigma(0) = pa - 1$ in the transformed system.

Generalizing our construction and employing results from [9], in Proposition 3.7 we have designed a system with a strongly attracting equilibrium ($\sigma(0) = +\infty$) that is far from being asymptotically stable ($\sigma_{\text{loc}}(0) = -\infty$). In contrast to earlier such examples ours has a C^∞ right-hand side, answering an open question posed in [9].

In section 3 we have not considered the case $\sigma(0) = 0$. However, it is straightforward to write down such a system: one simply needs to make sure that $\Sigma_\varepsilon(0)$ is constant, i.e. independent of $\varepsilon > 0$. This is the case if the basin of attraction is linearly bounded, see e.g. the piecewise linear vector field on $\mathbb{R}^2$ displayed in Figure 6, where we have $\Sigma_\varepsilon(0) = \frac{1}{4}$ for all $\varepsilon > 0$.

Our work establishes explicit examples for non-asymptotically stable equilibria that are fragmentarily or essentially asymptotically stable. This may prove useful in future endeavors to develop more complicated systems with heteroclinic connections that possess a prescribed level of stability, thus extending previous efforts towards the design of systems with a desired connection structure between equilibria, see e.g. [1, 2].

Acknowledgements: The author is grateful for insightful comments by two anonymous reviewers of an earlier version of this work.

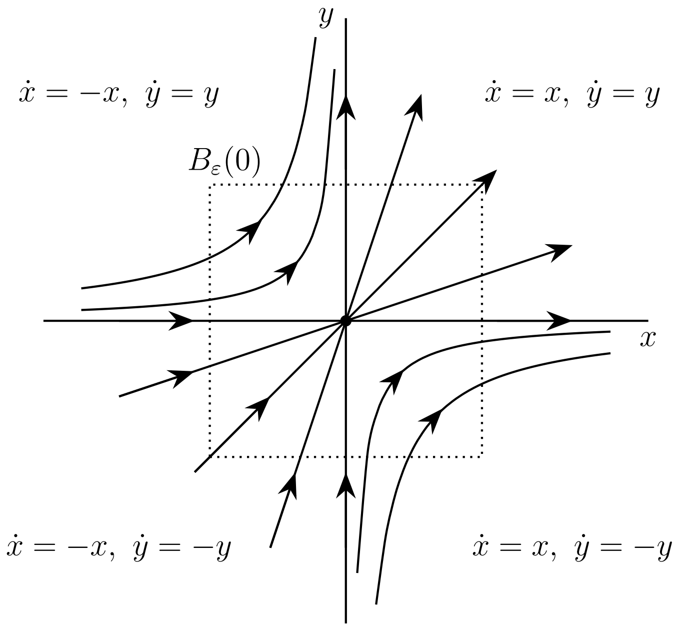


Figure 6: A system with $\sigma(0) = 0$.

References

[1] P. Ashwin and S. Castro and A. Lohse (2020) Almost complete and equable heteroclinic networks, *Journal of Nonlinear Science* 30 1–22.

[2] P. Ashwin and C. Postlethwaite (2013) On designing heteroclinic networks from graphs, *Physica D* 265, 26–39.

[3] C. Bick and A. Lohse (2019) Heteroclinic dynamics of localized frequency synchrony: Stability of heteroclinic cycles and networks, *Journal of Nonlinear Science* 29 (6), 2571–2600.

[4] W. Brannath (1994) Heteroclinic networks on the tetrahedron, *Nonlinearity* 7, 1367–1384.

[5] S.B.S.D. Castro and A. Lohse (2014) Stability in simple heteroclinic networks in $\mathbb{R}^4$, *Dynamical Systems* 29 (4), 451–481.

[6] L. Garrido-da-Silva and S.B.S.D. Castro (2019) Stability of quasi-simple heteroclinic cycles, *Dynamical Systems* 34 (1), 14–39.

[7] G. Keller (2014) Stability index for chaotically driven concave maps *J. London Math. Soc.* (2) 89, 603–622.

[8] A. Lohse (2014) Attraction properties and non-asymptotic stability of simple heteroclinic cycles and networks in $\mathbb{R}^4$, PhD Thesis, Universität Hamburg, Germany, https://ediss.sub.uni-hamburg.de/handle/ediss/5451.

[9] A. Lohse (2015) Unstable attractors: existence and stability indices, *Dynamical Systems* 30 (3), 324–332.

[10] A. Lohse (2015) Stability of heteroclinic cycles in transverse bifurcations, *Physica D* 310, 95–103.

[11] I. Melbourne (1991) An example of a non-asymptotically stable attractor, *Nonlinearity* 4, 835–844.

[12] O. Podvigina (2012) Stability and bifurcations of heteroclinic cycles of type Z, *Nonlinearity* 25, 1887–1917.

[13] O. Podvigina and P. Ashwin (2011) On local attraction properties and a stability index for heteroclinic connections, *Nonlinearity* 24, 887–929.

[14] U.A. Mohd Roslan and P. Ashwin (2016) Local and global stability indices for a riddled basin attractor of a piecewise linear map, *Dynamical Systems* 31 (3), 375-392.