Area moment of inertia and other physical quantities for special cross-sectional areas of bend specimens

Hans Jelitto

Institute of Advanced Ceramics, Hamburg University of Technology (TUHH), Germany,
May 31st, 2022, CC BY-NC-SA 4.0, E-mail: h.jelitto@tuhh.de, doi: 10.15480/882.4355

Abstract

With regard to materials in nature having extraordinary properties, one focus of our research is nacre-inspired materials. On the basis of 3D-printing, it was possible to prepare tiny bending bars in order to be tested in three-point bending (3PB). However, their cross sections are not rectangular but have a special geometric shape. Therefore, analytical calculations concerning the area moment of inertia and other physical quantities were performed for the correct analysis of the measured data.

1. Introduction

The original motivation for this report was a rough estimate of forces that are expected in the strength measurements of tiny specimens by three-point bending (3PB). The required stress for failure of the material in 3PB assuming a rectangular cross section is:

$$\sigma = \frac{3 F_{max} L}{2 h^2 b} \tag{1}$$

with σ being the maximum stress on the tensile surface before inelastic deformation or fracture and F_{max} being the corresponding force. L is the support distance, h the (average) height, and b the (average) width of the sample. It follows that the expected force F_{max} is:

$$F_{max} = \frac{2 \sigma h^2 b}{3 L} \tag{2}$$

In Table 1, we assume a support distance of $L = 5$ mm and an average width of the sample of $b = 1$ mm. If other values are required, they can easily be estimated with the second equation.

Table 1: *Maximum force calculated for different strengths and average heights of the specimen (L = 5 mm, b = 1 mm).*

	h = 100 µm	250 µm	500 µm
σ = 10 MPa	0.013 N	0.083 N	0.33 N
30 MPa	0.04 N	0.25 N	1 N
100 MPa	0.13 N	0.83 N	3.3 N

For a relatively precise measurement with our equipment, the force should be 1 N or more. However, lower forces can also be measured.

2. Area moment of inertia (semi-elliptic cross section)

The area moment of inertia, also named the second (planar) moment of area and which is necessary for the evaluation of bending tests, can be calculated analytically if the shape is not too complicated. An example is the semi-elliptic shape (Fig. 1). The area moment of inertia with respect to the x-axis is given by:

$$I_x = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) a h^3 = 0.10976 \cdot a h^3 \quad (3)$$

This formula is the modified version of a semi-circular cross section. Nevertheless, it can be verified by taking the half area moment of inertia of a full ellipse and applying Steiner's theorem with the center of area at a height of $4h/(3\pi) = 0.424413\ h$. A more realistic shape is provided below.

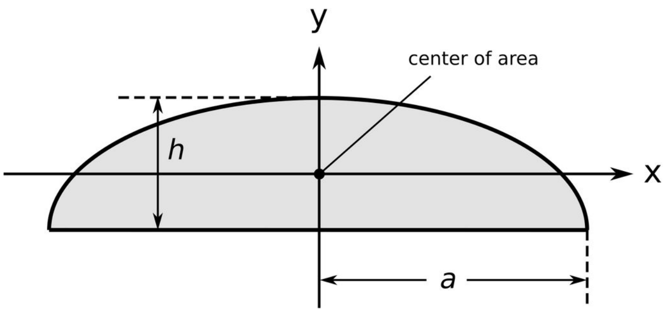


Figure 1: *Semi-elliptic cross section.*

3. Area moment of inertia (special cross section)

In the given experiment, the cross section of the samples is neither semi-elliptic nor triangular. Therefore, we analyze a special form which is between both given shapes (see Fig. 2). This cross section consists of one 90° sector and two right-angled triangles. (Ultimately, the shape will be altered to a flatter cross section with an elliptical sector.)

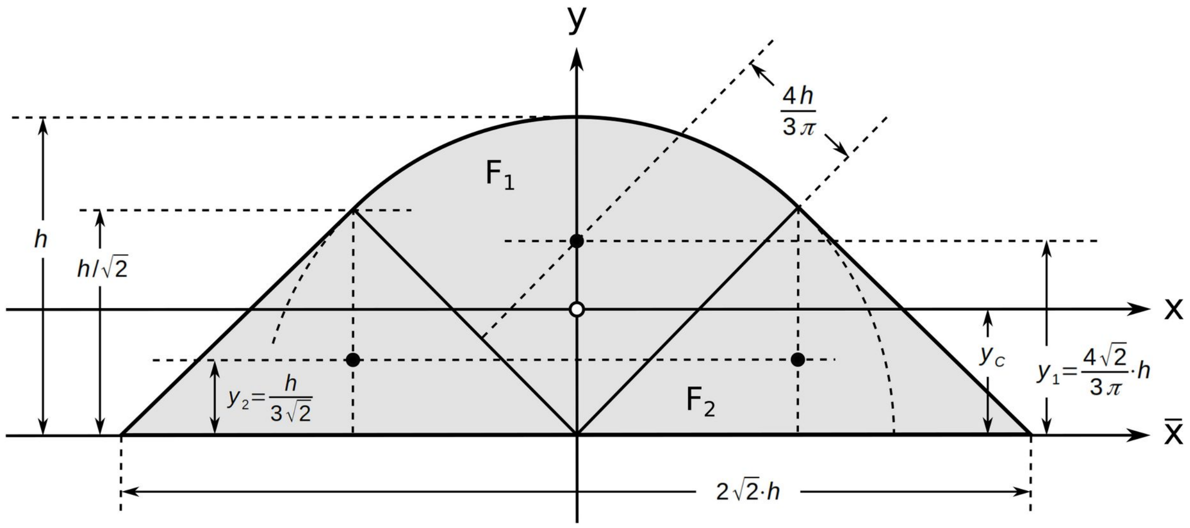


Figure 2: *Special cross section consisting of three areas: F_1 and twice F_2.*

We begin with the determination of the overall center of area, given by the white dot. The distance of the center of area of the 90° sector from the 45° line is the same as in Fig. 1. It is $4h/(3\pi)$, meaning that its distance from the $\bar{x}$-axis is $4h\sqrt{2}/(3\pi)$. The centers of area of the two triangles have the distance $y_2 = h/(3\sqrt{2})$ from the $\bar{x}$-axis. The x-coordinate of the overall center of area is $x_C = 0$ because of mirror symmetry. The corresponding y-coordinate, y_C, is given by taking into account the three centers of area (black points). With the areas $F_1 = \pi h^2/4$ and $F_2 = h^2/2$, we obtain:

$$y_C = \frac{\sum_{i=1}^{3} y_i \cdot F_i}{\sum_{i=1}^{3} F_i} = \frac{\frac{4\sqrt{2}h}{3\pi} \cdot \frac{\pi}{4} h^2 + 2 \cdot \frac{h}{3\sqrt{2}} \cdot \frac{h^2}{2}}{\frac{\pi}{4} h^2 + 2 \cdot \frac{h^2}{2}} = \frac{h}{\sqrt{2} \cdot \left(1 + \frac{\pi}{4}\right)} = 0.396050 \cdot h \qquad (4)$$

The task now is to calculate the area moments of inertia (also referred to as area moments) of the three areas in Fig. 2 with respect to the x-axis (running through y_C) and add these up.

Area moment of inertia of area F_1

The area moment of F_1 can be calculated in a simplified way. The area moment of the cross section in Fig. 3 is [1]:

$$I_x = (\theta - \sin\theta)\cdot\frac{r^4}{8} \tag{5}$$

With $\theta = \pi/2$, the right side of Eq. (5) becomes $(\pi/2-1)r^4/8$. Now, we obtain the area moment, $\bar{I}_1$, of F_1 with respect to the $\bar{x}$-axis in Fig. 2 by subtracting this term twice from the area moment of a full circle and dividing the result by 2.

With the area moment of a full circle of $\pi r^4/4$ and replacing r with h, we find:

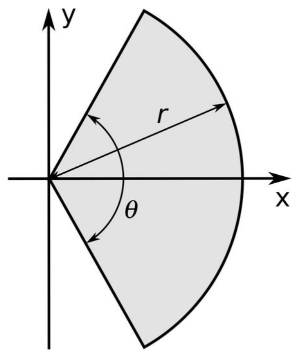


Figure 3: *Symmetric circle segment.*

$$\bar{I}_1 = \frac{1}{2}\cdot\left(\frac{\pi}{4}h^4 - 2\left(\frac{\pi}{2}-1\right)\frac{h^4}{8}\right) = \left(\frac{2+\pi}{16}\right)h^4 \tag{6}$$

Equation (6) represents the area moment of inertia of F_1 with respect to the $\bar{x}$-axis in Fig. 2. In order to obtain the value for the x-axis, which runs through the overall centroid (center of area), we must apply Steiner's theorem (parallel axis theorem) twice. First, we have to shift the axis to a parallel axis by a distance of y_1, and, second, this axis must be shifted back by the distance $y_1 - y_C$. Therefore, we obtain the area moment of inertia concerning the area F_1 with respect to the x-axis by:

$$I_1 = \bar{I}_1 - y_1^2 F_1 + (y_1 - y_C)^2 F_1 = \bar{I}_1 + y_C(y_C - 2y_1)F_1$$

$$= \frac{2+\pi}{16}\cdot h^4 + \frac{h}{\sqrt{2}\left(1+\frac{\pi}{4}\right)}\cdot\left(\frac{h}{\sqrt{2}\left(1+\frac{\pi}{4}\right)} - \frac{8\sqrt{2}h}{3\pi}\right)\cdot\frac{\pi}{4}h^2 = \frac{-416+64\pi+30\pi^2+3\pi^3}{48(4+\pi)^2}\cdot h^4 \tag{7}$$

This way (using Eq. (5)), the calculation of the area moment by integration is not required. However, since errors are possible, the result is validated by analytical integration, which is performed in the Appendix.

Area moment of inertia of area F_2

If b' and h' are the width and the height of the triangles, we have $b' = h\cdot\sqrt{2}$ and $h' = h/\sqrt{2}$. Thus, the area moment of one of the triangles with respect to the horizontal axis through the centroids of the triangles is $I_2' = b'h'^3/36 = h\cdot\sqrt{2}(h/\sqrt{2})^3/36 = h^4/72$. By shifting this axis by $y_C - y_2$ (see Fig. 2) and applying Steiner's theorem, for one triangle we obtain the following area moment of inertia, I_2, with respect to the overall center of area (x-axis):

$$I_2 = I_2' + (y_C - y_2)^2 F_2$$

$$= \frac{h^4}{72} + \left(\frac{h}{\sqrt{2}\left(1+\frac{\pi}{4}\right)} - \frac{h}{3\sqrt{2}}\right)^2\frac{h^2}{2} = \frac{48-8\pi+\pi^2}{24(4+\pi)^2}\cdot h^4 \tag{8}$$

Overall area moment of inertia I_x

The total area moment of inertia is the sum of the separate contributions. By using the results of Eqs. (7) and (8), this means that:

$$I_x = I_1 + 2 I_2 = \frac{-224 + 32\pi + 34\pi^2 + 3\pi^3}{48(4+\pi)^2} \cdot h^4 \tag{9}$$

This is the solution for the cross section in Fig. 2, which has a constant ratio $b/h = 2\sqrt{2}$. In the real experiment, this ratio could be different, e.g., as in Fig. 4b. Figure 4 shows a comparison between an experimental and the theoretical cross section. Fortunately, we can proportionally stretch or compress the cross section in the horizontal or vertical direction. The large prefactor in Eq. (9) would be unchanged if we replaced the factor h^4 by $bh^3/(2\sqrt{2})$. The final result is provided in Table 2 together with the formulas of some other cross sections.

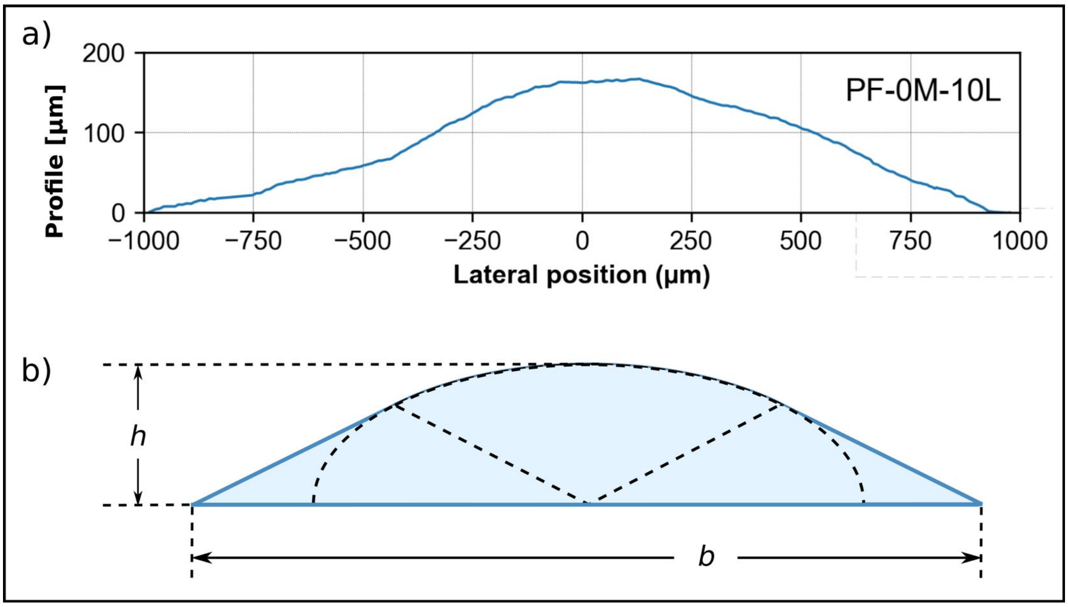
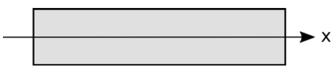
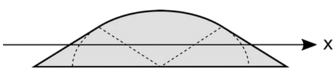
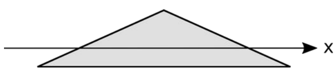


Figure 4:* *a) *Profile of experimental self-assembly specimen (prepared by Magnus Küpker at the TUHH);* ***b)*** *adapted theoretical cross section.*

Table 2: *Area moment of inertia for different typical cross sections (x-axis through the center of area, perpendicular to the symmetry axis).*

cross section	area moment of inertia
	$I_x = \frac{bh^3}{12} = 0.083333 \cdot bh^3$ (10)
	$I_x = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) bh^3 = 0.054878 \cdot bh^3$ (11)
	$I_x = \frac{-224 + 32\pi + 34\pi^2 + 3\pi^3}{96\sqrt{2}(4+\pi)^2} \cdot bh^3 = 0.044065 \cdot bh^3$ (12)
	$I_x = \frac{bh^3}{36} = 0.027778 \cdot bh^3$ (13)

4. Stress, strain, and Young's modulus for different cross sections

The relation between the stress σ and the force F in 3PB is:

$$\sigma_{3\mathrm{PB}} = \frac{F\,L\,z}{4\,I_x} = \frac{F\,L\,y_C}{4\,I_x} \tag{14}$$

where L is the support distance and z the distance between the center of area and the surface of the specimen with maximal tensile stress. In our case of Eq. (12), z is equal to y_C (Eq. (4) and Fig. 2). In the following, we add some useful physical quantities with respect to the geometric cross sections shown in Table 2. The maximum strain in 3PB at the outer fiber for a rectangular cross section is:

$$\epsilon_{3\mathrm{PB}} = \frac{6\,d\,h}{L^2} \tag{15}$$

with d being the displacement at the central load point. The general equation for the maximum strain of different shapes of the cross sections is:

$$\epsilon_{3\mathrm{PB}} = \frac{12\,d\,y_C}{L^2} \tag{16}$$

Consequently, the elastic modulus (Young's modulus) can be calculated using Eqs. (14) and (16):

$$E_{3\mathrm{PB}} = \frac{\sigma_{3\mathrm{PB}}}{\epsilon_{3\mathrm{PB}}} = \frac{F\,L\,y_C}{4\,I_x}\cdot\frac{L^2}{12\,d\,y_C} = \frac{L^3}{48\,I_x}\cdot\frac{F}{d} = \frac{L^3\,s}{48\,I_x} \tag{17}$$

with $s = F/d$ being the slope in the force-displacement diagram. Table 3 provides the distance between the center of area and the outer fiber as shown exemplarily in Fig. 2. The center of area defines the height position of the neutral fiber. Note that the lower edge of the cross section corresponds to maximum tensile stress. If the cross sections are turned upside down, all of the values of y_C, stress, and strain in Tables 3 and 4, except for the rectangular cross section, are changed. In this case the distance y_C has to be replaced by $h - y_C$.

Table 3: *Distances y_C between the neutral fiber (through the center of area) and the outer fiber under maximum tensile stress (compare Fig. 2).*

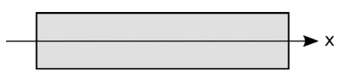
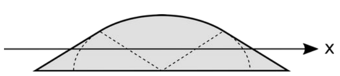
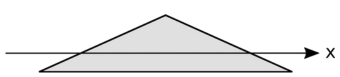
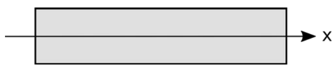
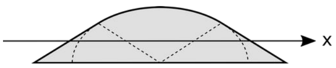
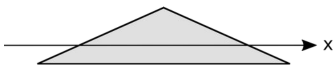
cross section	**center of area**
	$y_C = \frac{h}{2}$
	$y_C = 0.424413 \cdot h$
	$y_C = 0.396050 \cdot h$
	$y_C = \frac{h}{3}$

Table 4 summarizes the main quantities in a form allowing for an easy application. The equations for the rectangular cross section can be compared with the corresponding formulas in Table 1 of Ref. [2]. Note that the equations for the stress, σ, can also be used for four-point bending with the inner and outer support distances s_1 and s_2 if L is replaced by $s_2 - s_1$.

Table 4: *Stress, strain, and Young's modulus for 3PB and different shapes of the cross section (s = F/d).*

cross section	stress	strain	Young's modulus
	$\sigma = \frac{3}{2} \cdot \frac{F L}{b h^2}$	$\epsilon = 6 \cdot \frac{d h}{L^2}$	$E = \frac{1}{4} \cdot \frac{L^3 s}{b h^3}$
	$\sigma = 1.93342 \cdot \frac{F L}{b h^2}$	$\epsilon = 5.09296 \cdot \frac{d h}{L^2}$	$E = 0.37963 \cdot \frac{L^3 s}{b h^3}$
	$\sigma = 2.24699 \cdot \frac{F L}{b h^2}$	$\epsilon = 4.75260 \cdot \frac{d h}{L^2}$	$E = 0.47279 \cdot \frac{L^3 s}{b h^3}$
	$\sigma = 3 \cdot \frac{F L}{b h^2}$	$\epsilon = 4 \cdot \frac{d h}{L^2}$	$E = \frac{3}{4} \cdot \frac{L^3 s}{b h^3}$

5. Summary

The shape of the cross sections of tiny bending bars could be well approximated by a special geometric contour, and the corresponding physical quantities, being the area moment of inertia, the position of the neutral fiber, stress, strain, and Young's modulus, were determined analytically. The calculations and results are presented together with the known formulas of some common cross sections.

Acknowledgment

This work was supported by the German Research Foundation (DFG), project number 192346071, SFB 986.

Appendix

Area F_1 in Figure 2

The following integration of the area moment of inertia of area F_1 in Fig. 2 is a test of the correctness of Eq. (7). For the analytical integration, we need the two functions in Fig. 5 as integration bounds for the y-integration. These are:

$$f_1(x) = \sqrt{h^2 - x^2} - y_C \qquad \text{(A1)}$$

and

$$f_2(x) = x - y_C \qquad \text{(A2)}$$

with y_C given by Eq. (4).

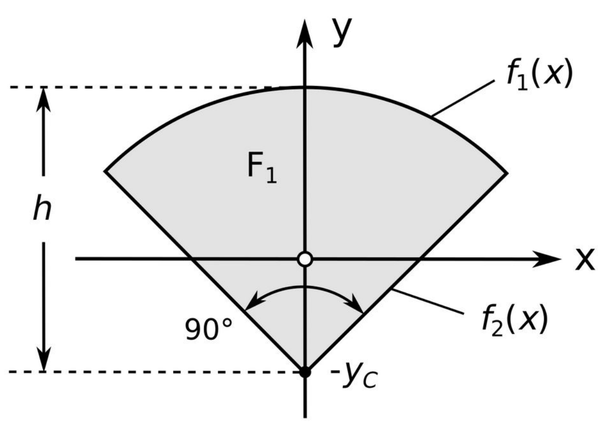


Figure 5*: Symmetric circle segment F_1 in Fig. 2.*

If we integrate the right half of the sector in Fig. 5 and multiply it by 2, we obtain the following area moment of inertia of the area F_1 with respect to the x-axis:

$$I_1 \;=\; \iint_{F_1} y^2 dydx \;=\; 2\cdot \int_0^{h/\sqrt{2}} \int_{x-y_C}^{\sqrt{h^2-x^2}-y_C} y^2 dydx$$

$$=\; 2\int_0^{h/\sqrt{2}} \left[\frac{y^3}{3}\right]_{x-y_C}^{\sqrt{h^2-x^2}-y_C} dx \;=\; \frac{2}{3}\int_0^{h/\sqrt{2}} \left(\left(\sqrt{h^2-x^2}-y_C\right)^3-(x-y_C)^3\right)dx$$

$$=\; \frac{2}{3}\int_0^{h/\sqrt{2}} \left((h^2-x^2)^{\frac{3}{2}}-3h^2 y_C+6x^2 y_C+3\sqrt{h^2-x^2}\,y_C^2-x^3-3x\,y_C^2\right)dx \qquad \text{(A3)}$$

The first summand in the integrand can be evaluated by the following indefinite integral taken from Ref. [3]:

$$\int (a^2-x^2)^{\frac{3}{2}} dx \;=\; \frac{1}{4}\left(x(a^2-x^2)^{\frac{3}{2}} \;+\; \frac{3a^2x}{2}\sqrt{a^2-x^2} \;+\; \frac{3a^4}{2}\sin^{-1}\frac{x}{a}\right) \qquad \text{(A4)}$$

Thus, we find:

$$\frac{2}{3}\int_0^{h/\sqrt{2}} (h^2-x^2)^{\frac{3}{2}} dx \;=\; \frac{2}{3}\left[\frac{1}{4}\left(x(h^2-x^2)^{\frac{3}{2}} \;+\; \frac{3h^2x}{2}\sqrt{h^2-x^2} \;+\; \frac{3h^4}{2}\sin^{-1}\frac{x}{h}\right)\right]_0^{h/\sqrt{2}}$$

$$=\; \frac{1}{6}\left(\frac{h}{\sqrt{2}}\left(h^2-\frac{h^2}{2}\right)^{\frac{3}{2}} \;+\; \frac{3h^3}{2\sqrt{2}}\sqrt{h^2-\frac{h^2}{2}} \;+\; \frac{3h^4}{2}\sin^{-1}\frac{1}{\sqrt{2}}\right)$$

$$=\; \frac{h^4}{6}\left(1+\frac{3\pi}{8}\right) \qquad \text{(A5)}$$

The fourth summand in the integrand of Eq. (A3) can be handled using Ref. [3]:

$$\int \sqrt{a^2-x^2}dx \;=\; \frac{1}{2}\left(x\sqrt{a^2-x^2} \;+\; a^2\sin^{-1}\frac{x}{a}\right) \qquad \text{(A6)}$$

Similarly, this yields:

$$\frac{2}{3}\int_0^{h/\sqrt{2}} \sqrt{h^2-x^2}dx \;=\; \frac{2}{3}\cdot 3y_C^2\left[\frac{1}{2}\left(x\sqrt{h^2-x^2} \;+\; h^2\sin^{-1}\frac{x}{h}\right)\right]_0^{h/\sqrt{2}} \;=\; y_C^2h^2\left(\frac{1}{2}+\frac{\pi}{4}\right) \qquad \text{(A7)}$$

Integrating the four remaining summands in Eq. (A3) is a standard procedure and performed as follows:

$$\frac{2}{3}\int_0^{h/\sqrt{2}} \left(-3h^2 y_C+6x^2 y_C-x^3-3x\,y_C^2\right)dx \;=\; \frac{2}{3}\left[-3h^2 y_C x \;+\; 2x^3 y_C-\frac{x^4}{4}-\frac{3}{2}x^2 y_C^2\right]_0^{h/\sqrt{2}}$$

$$= \frac{2}{3}\left(\frac{-3h^3 y_C}{\sqrt{2}} + \frac{2h^3 y_C}{2\sqrt{2}} - \frac{h^4}{16} - \frac{3}{2}\cdot\frac{h^2}{2}y_C^2\right)$$

$$= \frac{2}{3}\left(h^3 y_C\left(\frac{-3}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right) - \frac{h^4}{16} - \frac{3}{4}h^2 y_C^2\right) = -\frac{2\sqrt{2}}{3}h^3 y_C - \frac{h^4}{24} - \frac{1}{2}h^2 y_C^2 \qquad \text{(A8)}$$

By adding the results of Eqs. (A5), (A7), and (A8), for the area moment of inertia of the area F_1 we obtain:

$$I_1 = \frac{h^4}{6}\left(1+\frac{3\pi}{8}\right) + y_C^2 h^2\left(\frac{1}{2}+\frac{\pi}{4}\right) - \frac{2\sqrt{2}}{3}h^3 y_C - \frac{h^4}{24} - \frac{1}{2}h^2 y_C^2 \qquad \text{(A9)}$$

The last summand cancels with another term and $h^4/24$ is subtracted from the first bracket. Inserting y_C by means of Eq. (4) yields:

$$I_1 = h^4\left(\frac{6+3\pi}{48}\right) + \frac{h^4\pi}{8\left(1+\frac{\pi}{4}\right)^2} - \frac{2h^4}{3\left(1+\frac{\pi}{4}\right)} = \frac{(6+3\pi)\left(1+\frac{\pi}{4}\right)^2+6\pi-32\left(1+\frac{\pi}{4}\right)}{48\left(1+\frac{\pi}{4}\right)^2}\cdot h^4 \qquad \text{(A10)}$$

Further reduction leads to:

$$I_1 = \frac{-416+64\pi+30\pi^2+3\pi^3}{48(4+\pi)^2}\cdot h^4 \qquad \text{(A11)}$$

being identical to the result of Eq. (7).

Area F_2 in Figure 2

We also briefly calculate the area moment of inertia for the area F_2, at which F_2 is one of the triangles in Fig. 2. Thus, we validate Eq. (8). The two functions, being the integration limits in the following integration (see Fig. 6), are:

$$f_3(x) = \frac{h}{\sqrt{2}} - y_C - x \qquad \text{(A12)}$$

$$f_4(x) = -y_C \qquad \text{(A13)}$$

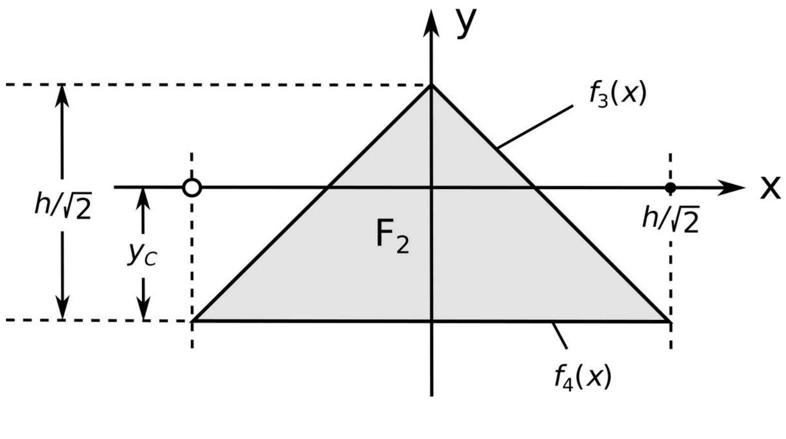


Figure 6: *One of the two triangular parts F_2 of the cross section in Fig. 2. The white dot indicates the centroid of the whole cross-sectional area in Fig. 2.*

Next, we calculate the area moment of inertia for the right half of the triangle in Fig. 6 and multiply it by 2:

$$I_2 = \iint_{F_2} y^2\,dydx = 2\cdot\int_0^{h/\sqrt{2}}\int_{-y_C}^{h/\sqrt{2}-y_C-x} y^2\,dydx \qquad \text{(A14)}$$

It follows:

$$I_2 \;=\; 2\int_0^{h/\sqrt{2}} \left[\frac{y^3}{3}\right]_{-y_C}^{h/\sqrt{2}-y_C-x} dx \;=\; \frac{2}{3}\int_0^{h/\sqrt{2}}\left(\left(\frac{h}{\sqrt{2}}-y_C-x\right)^3-(-y_C)^3\right)dx$$

$$= \frac{2}{3}\left[\frac{\left(h/\sqrt{2}-y_C-x\right)^4}{-4} + y_C^3\,x\right]_0^{h/\sqrt{2}} \;=\; \frac{2}{3}\left(\frac{y_C^4}{-4} + y_C^3\frac{h}{\sqrt{2}} - \frac{\left(h/\sqrt{2}-y_C\right)^4}{-4}\right) \qquad \text{(A15)}$$

For the sake of simplicity, y_C is replaced by: $\quad y_C \;=\; \frac{4h}{\sqrt{2}(4+\pi)}$

and the term in the last numerator of Eq. (A15) by: $\quad \frac{h}{\sqrt{2}}-y_C \;=\; \frac{h}{\sqrt{2}}-\frac{4h}{\sqrt{2}(4+\pi)} \;=\; \frac{\pi h}{\sqrt{2}(4+\pi)}$

Inserting these terms into Eq. (A15) yields:

$$I_2 \;=\; \frac{2}{3}\left(-\frac{1}{4}\cdot\left(\frac{4h}{\sqrt{2}(4+\pi)}\right)^4 + \left(\frac{4h}{\sqrt{2}(4+\pi)}\right)^3\frac{h}{\sqrt{2}} + \frac{1}{4}\left(\frac{\pi h}{\sqrt{2}(4+\pi)}\right)^4\right)$$

$$= \frac{2h^4}{3}\;\frac{-4^3+4^3(4+\pi)+\pi^4/4}{4(4+\pi)^4} \;=\; h^4\cdot\frac{768+256\pi+\pi^4}{24(4+\pi)^4} \;=\; \frac{48-8\pi+\pi^2}{24(4+\pi)^2}\cdot h^4 \qquad \text{(A16)}$$

This result is identical to that of Eq. (8) and shows that the previous calculation using Steiner's theorem is correct. The last equality in Eq. (A16) is not very obvious. However, it is valid because the polynomial $\pi^4+256\pi+768$ can be factorized as follows:

$$\pi^4+256\pi+768 \;=\; (\pi+4)^2(\pi^2-8\pi+48) \qquad \text{(A17)}$$

Remark: Although the calculation might appear easy, the integration of I_2 took hours of trial and error. Ultimately, a relatively simple way of performing the calculation was found. Apart from the fact that this validates the previous result, of course, it was much easier to apply Steiner's theorem.

References

[1] Wikipedia (English): List of second moments of area. (Unfortunately, the scientific source of the corresponding equation, Eq. (5), is not provided. However, the result in Eq. (7), based on this formula, is validated in an alternative way in the Appendix.)

[2] H. Jelitto: Area moment of inertia of a "flower-shaped" cross section and generalization for *N*-fold rotational symmetry and multipole moments. Technical report, Institute of Advanced Ceramics, Hamburg University of Technology (TUHH), published on the document server TORE (2020), **DOI**: 10.15480/882.3066

[3] I. N. Bronstein, K. A. Semendjajew, G. Musiol, H. Mühlig: Taschenbuch der Mathematik. 10. Auflage, Edition Harri Deutsch, Verlag Europa-Lehrmittel, Haan-Gruiten (2016) 1093–1094