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Davydov–Yetter cohomology and relative homological algebra

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Abstract

Davydov–Yetter (DY) cohomology classifies infinitesimal deformations of the monoidal structure of tensor functors and tensor categories. We consider such deformations of finite tensor categories and exact tensor functors between them. In [GHS19], DY cohomology with coefficients was introduced and related to the comonad cohomology for a certain adjunction; in the case of tensor categories it reduces to the adjunction for the forgetful functor of the Drinfeld center. We first prove that the DY cohomology groups are isomorphic to the relative Ext groups for this adjunction. From this, we derive the following main results: the vanishing of the first DY cohomology group, long exact sequences of DY cohomology groups which allow to express the groups in terms of Hom spaces, and the existence of a Yoneda-type product on DY cocycles. We apply these results to the category of finite-dimensional modules over a finite-dimensional Hopf algebra and provide a method to compute explicit DY cocycles. We study in detail the examples of the bosonization of exterior algebras $\Lambda\mathbb{C}^k \rtimes \mathbb{C}[\mathbb{Z}_2]$, the Taft algebras and the restricted quantum group of $\mathfrak{sl}_2$ at a fourth root of unity $\bar{U}_{\mathbf{i}}(\mathfrak{sl}_2)$.

1 Introduction

A deformation theory of monoidal structures has been introduced and studied by Davydov, Crane and Yetter [Dav97, CY98, Yet98, Yet03]; it describes deformations of the monoidal structure of a k-linear monoidal functor or the associator of a k-linear monoidal category, without changing the underlying categories and functors except that the scalars are extended from k to $k[[h]]$, where k is a field. This theory is the first step to the classification problem of monoidal structures [Dav97] but is also related to quantum algebra and low-dimensional topology. Within this deformation theory, one recovers the category of modules over the quantum group $U_q(\mathfrak{g})$ as a deformation of the category of modules over the envelopping algebra $U(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ [DE19]. Also, this theory allows to deform the braiding of a tensor category and this can be used to produce link invariants; see [Yet98] where a relation with Vassiliev invariants was established.

We first recall a bit more precisely this deformation theory. Let $\mathcal{C}, \mathcal{D}$ be k-linear monoidal categories, assumed strict for simplicity, and let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor, *i.e.* a k-linear monoidal functor. By definition, there exists a natural isomorphism $\theta_{X,Y} : F(X \otimes Y) \xrightarrow{\sim} F(X) \otimes F(Y)$ such that the diagram

$$\begin{array}{ccc} F(X \otimes Y \otimes Z) & \xrightarrow{\theta_{X \otimes Y, Z}} & F(X \otimes Y) \otimes F(Z) \\ \downarrow{\scriptstyle \theta_{X, Y \otimes Z}} & & \downarrow{\scriptstyle \theta_{X,Y} \otimes \mathrm{id}_{F(Z)}} \\ F(X) \otimes F(Y \otimes Z) & \xrightarrow[\mathrm{id}_{F(X)} \otimes \theta_{Y,Z}]{} & F(X) \otimes F(Y) \otimes F(Z) \end{array} \tag{1}$$

is commutative. In Davydov–Yetter theory we are looking for deformations of θ, which are expansions over $k[[h]]$ of the form $\theta_h = \theta + \sum_{n \geq 1} h^n f_n$, where the f_n's are natural transformations $(f_n)_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y)$, such that the diagram (1) remains commutative with θ_h instead of θ. Without loss of generality, we assume from now on that F is strict: $\theta = \mathrm{id}$ [JS93, Th. 1.7].

In order to get the cochain complex $C^{\bullet}_{\mathrm{DY}}(F)$ associated to this deformation problem, we restrict to infinitesimal deformations $\theta_h = \mathrm{id} + hf$, with $h^2 = 0$. Then the condition (1) on θ_h implies that the natural transformation $f : F \otimes F \to F \otimes F$ satisfies

$$\mathrm{id}_{F(X_1)} \otimes f_{X_2,X_3} - f_{X_1 \otimes X_2, X_3} + f_{X_1, X_2 \otimes X_3} - f_{X_1,X_2} \otimes \mathrm{id}_{F(X_3)} = 0. \tag{2}$$

The space of 2-cochains $C^2_{\mathrm{DY}}(F)$ is defined as the space of all natural transformations $F \otimes F \to F \otimes F$ while the subspace of 2-cocycles contains the solutions of (2). In other words, the differential $\delta^2(f)_{X_1,X_2,X_3}$ is defined as the expression at the left hand-side of (2). The other cochain spaces $C^n_{\mathrm{DY}}(F)$ and differentials δ^n are defined as a generalization of the case $n = 2$. The resulting cohomology, denoted by $H^{\bullet}_{\mathrm{DY}}(F)$, is called the Davydov–Yetter (DY) cohomology of F. The infinitesimal deformations of the monoidal structure of F are classified by $H^2_{\mathrm{DY}}(F)$, and it was shown in [Yet98] that the obstructions are contained in $H^3_{\mathrm{DY}}(F)$.

We note that the identity functor $F = \mathrm{Id}_{\mathcal{C}}$ deserves a special attention because $H^3_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}})$ classifies the infinitesimal deformations of the (trivial) associator of $\mathcal{C}$. Such a deformation is an expansion $a_h = \mathrm{id} + hg$ over $k[h]/(h^2)$ which satisfies the pentagon equation, where g is a natural transformation $g_{X,Y,Z} : X \otimes Y \otimes Z \to X \otimes Y \otimes Z$. The obstructions are contained in $H^4_{\mathrm{DY}}(\mathrm{Id}_{\mathcal{C}})$, at least for the extension of an infinitesimal deformation to the order 2 [BD20, Prop. 3.21].

In this paper we study DY cohomology for finite tensor categories and exact tensor functors between them. Such categories are equivalent to A-mod for a finite-dimensional k-algebra A and are equipped with a rigid monoidal structure (see §3). They are ubiquitous in quantum algebra and mathematical physics, for instance from logarithmic CFTs [GR17, FGR17, CGR19], small quantum groups [GLO18, N18] or more generally finite-dimensional Hopf algebras.

In general, cohomology theories admit coefficients; for instance in Hochschild cohomology the coefficients are bimodules. Coefficients for DY cohomology have been introduced in [GHS19] and are objects in $\mathcal{Z}(F)$, namely the centralizer of the tensor functor $F : \mathcal{C} \to \mathcal{D}$. The category $\mathcal{Z}(F)$ is rigid and monoidal and plays a crucial role in this paper; its objects are pairs $\mathsf{V} = (V, \rho^V)$ where $V \in \mathcal{D}$ and $\rho^V : V \otimes F(?) \to F(?) \otimes V$ is like a half-braiding for V (see more details in §4.1). In particular $\mathcal{Z}(\mathrm{Id}_{\mathcal{C}})$ is just the Drinfeld center $\mathcal{Z}(\mathcal{C})$. The DY cohomology with coefficients is denoted by $H^{\bullet}_{\mathrm{DY}}(F; \mathsf{V}, \mathsf{W})$ and the cohomology without coefficients $H^{\bullet}_{\mathrm{DY}}(F)$ is recovered as $H^{\bullet}_{\mathrm{DY}}(F; \mathbf{1}, \mathbf{1})$, where $\mathbf{1}$ is the tensor unit of $\mathcal{Z}(F)$ (trivial coefficients).

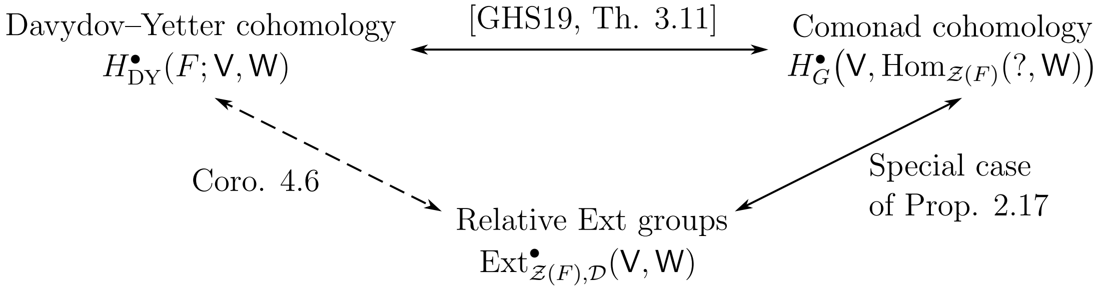
An obvious question is: how to compute Davydov–Yetter cohomology? This can be divided in two problems:

(P1) Compute the dimension of the Davydov–Yetter cohomology groups.

(P2) Determine explicit cocycles. This question is especially relevant for 2-cocycles (or 3-cocycles for the identity functor) since they give rise to infinitesimal deformations.

In general it is difficult to solve these problems using only the definition of the DY complex $C^\bullet_{\mathrm{DY}}(F)$ because the cochain spaces grow fast and natural transformations are in general not suitable for explicit computations. A more succesful strategy is to use methods from homological algebra. For instance in [DE19] the DY cohomology of $U(\mathfrak{g})$-mod for a semisimple Lie algebra $\mathfrak{g}$ has been computed thanks to a natural filtration on this DY complex and using the associated spectral sequence. But in general the DY complex of a tensor category does not admit a natural filtration.

In the case of finite tensor categories, DY cohomology was related to comonad cohomology in [GHS19] for a comonad on $\mathcal{Z}(F)$. In this paper we further extend this result and find a link to relative homological algebra. The relations of DY cohomology with these more classical cohomology theories can be schematized as:



(3)

Comonad cohomology [BB96] and relative Ext groups [Hoc56, ML75] are general theories which are defined in the context of an adjunction $\mathcal{A} \underset{\mathcal{F}}{\overset{\mathcal{U}}{\rightleftarrows}} \mathcal{B}$ between abelian categories. For comonad cohomology, the relevant comonad is $G = \mathcal{F} \circ \mathcal{U}$ and the cohomology is computed thanks to G-resolutions, see §2.4. For relative homological algebra the adjunction must be a resolvent pair, which means that $\mathcal{U}$ is k-linear, exact and faithful, and the relative Ext groups are computed thanks to relatively projective resolutions, see §2.1. We show in Proposition 2.17 that these two types of resolutions are equivalent.

In [GHS19] it was shown that Davydov–Yetter cohomology is isomorphic to the comonad cohomology for the adjunction $\mathcal{Z}(F) \underset{\mathcal{F}}{\overset{\mathcal{U}}{\rightleftarrows}} \mathcal{D}$, where $\mathcal{U}$ is the forgetful functor. This adjunction is a resolvent pair. In Proposition 2.17 we observe that the relative Ext groups of any resolvent pair are a special case of comonad cohomology. Hence, as indicated in diagram (3) above, we obtain the key point of this paper:

Theorem. *The Davydov–Yetter cohomology of a tensor functor $F : \mathcal{C} \to \mathcal{D}$ with coefficients $\mathsf{U}, \mathsf{V} \in \mathcal{Z}(F)$ is isomorphic to the relative Ext groups of the adjunction $\mathcal{Z}(F) \leftrightarrows \mathcal{D}$:*

$$H^n_{\mathrm{DY}}(F; \mathsf{U}, \mathsf{V}) \cong \mathrm{Ext}^n_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{U}, \mathsf{V})$$

for all $n \geq 0$.

We also have the diagram (3) at the level of complexes, with all relations made explicit in §4.2.

It has been explained in [GHS19] that the comonad cohomology reformulation of DY theory helps to determine the dimensions of the DY cohomology groups. The point is that one is free to use any G-projective resolution. The computation is feasible if the G-resolution is simple enough, as was demonstrated in [GHS19, §5] for the identity functor of the category of modules over the Hopf algebra $\Lambda(\mathbb{C}^k) \rtimes \mathbb{C}[\mathbb{Z}_2]$ (bosonization of the exterior algebra). Hence the computation of Davydov–Yetter cohomology is reduced to the construction of a sufficiently simple G-projective resolution. However, G-projective resolutions can be very difficult to find and it may happen that even the minimal one is too complicated, which for instance is the case with the category of modules over the restricted quantum group $\bar{U}_{\mathfrak{i}}(\mathfrak{sl}_2)$ studied in §5.4: even the low degrees cochain spaces of the resolution are difficult to construct.

Working with relative Ext groups instead of comonad cohomology allows us to use powerful results from relative homological algebra, which is a well-established subject [ML75]: existence of a long exact sequence of Ext groups associated to a short exact sequence, Yoneda product, Yoneda description of relative Ext^n groups in terms of certain n-fold exact sequences. These results are reviewed in §2.2. Our goal here is to use these properties in the case of the resolvent pair $\mathcal{Z}(F) \leftrightarrows \mathcal{D}$ and to obtain the corresponding properties for Davydov–Yetter cohomology through the isomorphism in Theorem 1. In that way we obtain the following results:

1. Proposition 4.7: $H^1_{\mathrm{DY}}(F) = 0$ provided that the tensor functor F is exact.
2. Corollary 4.16: long exact sequence for Davydov–Yetter cohomology.
3. Corollary 4.9: dimension formulas for the Davydov–Yetter cohomology groups.
4. Theorem 4.11: explicit formula of the Yoneda product on Davydov–Yetter cohomology.
5. Proposition 4.14: graded module structure of $H^\bullet_{\mathrm{DY}}(F; \mathsf{U}, \mathsf{V})$ over the Yoneda product algebra $H^\bullet_{\mathrm{DY}}(F)$.
6. The method in §5.1.4 for constructing explicit DY cocycles for $\mathcal{C} = H$-mod and $F = \mathrm{Id}_{\mathcal{C}}$.

These properties are helpful to compute the DY cohomology on examples, as demonstrated in §5.2, §5.3, §5.4 for certain finite-dimensional Hopf algebras. Let us now give more details on how they are obtained.

The first item follows from the fact that $\mathcal{Z}(F)$ is a finite tensor category when F is exact; therefore $\mathrm{Ext}^1_{\mathcal{Z}(F)}(\mathbf{1}, \mathbf{1}) = 0$ from which we derive $\mathrm{Ext}^1_{\mathcal{Z}(F),\mathcal{D}}(\mathbf{1}, \mathbf{1}) = 0$. The second item is immediate from the corresponding statement on relative Ext groups recalled in §2.2.3.

A simple but important property is that the forgetful functor $\mathcal{U} : \mathcal{Z}(F) \to \mathcal{D}$ is a tensor functor; we say that $\mathcal{Z}(F) \leftrightarrows \mathcal{D}$ is a monoidal resolvent pair. We show in §3.1 that the relative Ext groups of a monoidal resolvent pair have properties similar to those of usual Ext groups for tensor categories: the relatively projective objects form a tensor ideal and relative Ext groups are compatible with the duality.

We express the relative Ext groups for monoidal resolvent pairs in terms of Hom spaces in Proposition 3.4. This is based on the long exact sequence for relative Ext groups and on their compatibility with duality. The associated dimension formula in Corollary 3.5 requires only to find a relatively projective cover (see definition in §2.3) and to determine the dimension of certain Hom spaces; hence it replaces the construction of a relatively projective resolution by a problem of representation theory. Applying this result to the monoidal resolvent pair $\mathcal{Z}(F) \leftrightarrows \mathcal{D}$ we get the third item; the formula for $\dim\big(H^2_{\mathrm{DY}}(F)\big)$ is especially efficient in practice and gives the dimension of the infinitesimal deformations of F. In §5.4.1 we apply the dimension formula to the example of $\mathcal{C} = \bar{U}_{\mathbf{i}}(\mathfrak{sl}_2)$-mod with $F = \mathrm{Id}_{\mathcal{C}}$ and we compute $\dim\big(H^n_{\mathrm{DY}}(F)\big)$ for $n \leq 4$; this calculation was not accessible with previous methods. Therefore we have a new method to address the problem (P1).

To address the problem (P2), we recall in §2.2.1 the Yoneda description $\mathrm{YExt}^n_{\mathcal{A},\mathcal{B}}$ of the relative Ext groups $\mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}$, based on exact sequences in $\mathcal{A}$ of length n that are split in $\mathcal{B}$. The point is that these n-fold exact sequences are not so difficult to construct (in particular they are easier than relatively projective resolutions) and that we have the isomorphism

$$\mathrm{YExt}^n_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{V}, \mathsf{W}) \xrightarrow{\sim} \mathrm{Ext}^n_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{V}, \mathsf{W}) \xrightarrow{\sim} H^n_{\mathrm{DY}}(F; \mathsf{V}, \mathsf{W}) \tag{4}$$

which associates DY cocycles to exact sequences.

Long exact sequences have a natural gluing operation $\circ : \mathrm{YExt}^n_{\mathcal{A},\mathcal{B}}(V, W) \times \mathrm{YExt}^m_{\mathcal{A},\mathcal{B}}(U, V) \to \mathrm{YExt}^{m+n}_{\mathcal{A},\mathcal{B}}(U, W)$ called Yoneda product and recalled in §2.2.2. We transport the Yoneda product through the isomorphism (4), in order to get an associative product on DY cohomology:

$$\circ : H^n_{\mathrm{DY}}(F; \mathsf{V}, \mathsf{W}) \times H^m_{\mathrm{DY}}(F; \mathsf{U}, \mathsf{V}) \to H^{m+n}_{\mathrm{DY}}(F; \mathsf{U}, \mathsf{W}).$$

The explicit expression of this product is determined in Theorem 4.11 and turns out to be very simple; it is an important tool in practice because every cocycle can be factored as a product of 1-cocycles. The fifth item follows from the fact that $\circ$ endows $H^\bullet(F)$ with a structure of graded associative algebra and that $H^\bullet_{\mathrm{DY}}(F; \mathsf{U}, \mathsf{V})$ can be naturally endowed with a structure of graded $H^\bullet(F)$-module; we also determine the formula of the action.

In section 5 we turn to the case $\mathcal{C} = H$-mod where H is a finite-dimensional Hopf algebra over an algebraically closed field k and we obtain fully explicit results for $F = \mathrm{Id}_{\mathcal{C}}$. Note that $\mathcal{Z}(\mathrm{Id}_{H\text{-mod}}) \cong D(H)$-mod, where $D(H)$ is the Drinfeld double of H and $\mathcal{U} : D(H)\text{-mod} \to H\text{-mod}$ is the restriction functor for the standard embedding $H \hookrightarrow D(H) = (H^*)^{\mathrm{op}} \otimes H$. The isomorphism of Theorem 1 then becomes

$$H^n_{\mathrm{DY}}(\mathrm{Id}_{H\text{-mod}}; V, W) \cong \mathrm{Ext}^n_{D(H),H}(V, W)$$

for all $n \geq 0$. The Yoneda Ext group $\mathrm{YExt}^n_{D(H),H}(V, W)$ consists of exact sequences of $D(H)$-modules of length n from W to V which split as sequences in H-mod. The method of §5.1.4, based on the isomorphism (4) and on its compatibility with the Yoneda product, associates to such an exact sequence an explicit DY n-cocycle. A key feature in this case is that the Davydov–Yetter cochains can be encoded in a very explicit manner in terms of H-linear maps, suitable for computations. We finally apply these results and methods to examples:

- In §5.2 we consider the bosonization $H = \Lambda(\mathbb{C}^k) \rtimes \mathbb{C}[\mathbb{Z}_2]$ of the exterior algebra of $\mathbb{C}^k$. The dimensions of the Davydov-Yetter cohomology groups of this example have been already computed in [GHS19] thanks to comonad cohomology; here we give a shorter computation based on relative Ext groups for algebras in tensor categories (Proposition 3.8) and we moreover determine explicit 2-cocycles which generate $H^\bullet_{\mathrm{DY}}(\mathrm{Id}_H)$ thanks to the Yoneda product.

- In §5.3 we consider the Taft algebras $H = T_q$ for every root of unity q. We show that $H^\bullet_{\mathrm{DY}}(T_q\text{-mod}) \cong \mathbb{C}[X^2]$ as a graded algebra and we compute the explicit Davydov–Yetter cocycle in each even degree.

- In §5.4 we consider $H = \bar{U}_{\mathbf{i}}(\mathfrak{sl}_2)$, namely the restricted quantum group of $\mathfrak{sl}_2$ at $\mathbf{i} = \sqrt{-1}$. This example is much more involved: it is too difficult to construct a relatively projective resolution of the trivial $D(\bar{U}_{\mathbf{i}}(\mathfrak{sl}_2))$-module $\mathbb{C}$. Instead we determine only the relatively projective cover of $\mathbb{C}$ and we apply the dimension formulas of Corollary 4.9. This allows to easily find that $\dim\big(H^2_{\mathrm{DY}}(\bar{U}_{\mathbf{i}})\big) = 0$ while it is computationnaly harder to get $\dim\big(H^3_{\mathrm{DY}}(\bar{U}_{\mathbf{i}})\big) = 3$, $\dim\big(H^4_{\mathrm{DY}}(\bar{U}_{\mathbf{i}})\big) = 0$. We construct two explicit DY 3-cocycles with the method of §5.1.4; then using a GAP program we prove that they are linearly independent and we find a third basis element of $H^3_{\mathrm{DY}}(\bar{U}_{\mathbf{i}})$. This gives a complete picture of the infinitesimal associators on $\bar{U}_{\mathbf{i}}$-mod.

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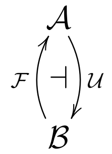
2 Relative homological algebra

This section collect results on resolvent pairs, relatively projective objects, relative Ext groups, Yoneda product, *etc* following [ML75]. It also contains results that we have not been able to find in the literature: for instance the existence and properties of relatively projective covers for general resolvent pairs in §2.3 or the relation between the notions from relative homological algebra and those from comonad cohomology in §2.4.

2.1 Resolvent pairs and relative Ext groups

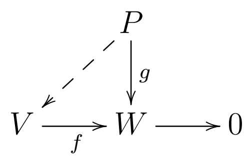
Let $\mathcal{A}, \mathcal{B}$ be abelian categories and



(5)

be a pair of adjoint functors ($\mathcal{F}$ left adjoint to $\mathcal{U}$). We first recall some notions of relative homological algebra from [ML75, Chap. IX].

- If the functor $\mathcal{U} : \mathcal{A} \to \mathcal{B}$ is additive, exact and faithful then we say that the adjunction in (5) is a *resolvent pair* of abelian categories.
- A morphism $f \in \mathrm{Hom}_{\mathcal{A}}(V, W)$ is called *allowable* if there exists $s \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{U}(W), \mathcal{U}(V))$ such that $\mathcal{U}(f)s\mathcal{U}(f) = \mathcal{U}(f)$. This condition is equivalent to the requirement that $\mathcal{U}(\ker(f))$ is a direct summand[1] of $\mathcal{U}(V)$ and $\mathcal{U}(\mathrm{im}(f))$ is a direct summand of $\mathcal{U}(W)$.
- An object P of $\mathcal{A}$ is called *relatively projective* if the diagram (with exact row)

  can always be filled, whenever f is allowable and g is any morphism in $\mathcal{A}$.
- A *relatively projective resolution* of $V \in \mathcal{A}$ is an exact sequence

  $$0 \longleftarrow V \xleftarrow{d_0} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} \ldots \tag{6}$$

  such that each $P_i \in \mathcal{A}$ is relatively projective and each d_i is allowable. The latter is equivalent to the statement that the image by $\mathcal{U}$ of the exact sequence is split, *i.e.* $\mathcal{U}(P_i) = \mathrm{im}\big(\mathcal{U}(d_{i+1})\big) \oplus \ker\big(\mathcal{U}(d_i)\big)$ for all i.

Note that the definitions are analogous to the ones in usual homological algebra, the difference being the notion of allowable morphism.

Let $\varepsilon : \mathcal{F}\mathcal{U} \to \mathrm{Id}_{\mathcal{A}}$ be the counit of the adjunction (5). Then for all $V \in \mathcal{A}$, $\varepsilon_V : \mathcal{F}\mathcal{U}(V) \to V$ is an epimorphism and is allowable since we have $\mathcal{U}(\varepsilon_V)\eta_{\mathcal{U}(V)} = \mathrm{id}_{\mathcal{U}(V)}$, where $\eta : \mathrm{Id}_{\mathcal{B}} \to \mathcal{U}\mathcal{F}$ is the unit of the adjunction. Thanks to ε_V one can construct an important relatively projective resolution of V, called the bar resolution [ML75, Th. 6.3]:

$$\mathrm{Bar}^{\bullet}_{\mathcal{A},\mathcal{B}}(V) = \Big(0 \longleftarrow V \xleftarrow{\varepsilon_V} \mathcal{F}\mathcal{U}(V) \xleftarrow{d_1} (\mathcal{F}\mathcal{U})^2(V) \xleftarrow{d_2} \ldots\Big) \tag{7}$$

where

$$d_n = \sum_{i=0}^{n} (-1)^i (\mathcal{F}\mathcal{U})^{n-i}\big(\varepsilon_{(\mathcal{F}\mathcal{U})^i(V)}\big). \tag{8}$$

This shows in particular that any $V \in \mathcal{A}$ admits at least one relatively projective resolution. However, bar resolutions are not suitable for concrete computations because the objects $(\mathcal{F}\mathcal{U})^n(V)$ grow too fast. One can obtain smaller relatively projective objects thanks to the following basic properties:

[1] Recall that in an additive category an object X is a direct summand of Y if there exist morphisms $\iota : X \to Y$ and $\pi : Y \to X$ such that $\pi\iota = \mathrm{id}_X$.

Lemma 2.1. *1. Let $M = V \oplus W$ be a direct sum in $\mathcal{A}$. Then M is relatively projective if and only if V and W are relatively projective.*
2. An object $P \in \mathcal{A}$ is relatively projective if and only if it is a direct summand of $\mathcal{F}(X)$ for some $X \in \mathcal{B}$.

Proof. 1. This is the same statement as in usual homological algebra. The easy proof is left to the reader.
2. We know by [ML75, Th. 6.1] that $\mathcal{F}(X)$ is relatively projective. Hence, direct summands of $\mathcal{F}(X)$ are relatively projective as well thanks to the previous item. Conversely, assume that P is relatively projective. Then since $\varepsilon_P : \mathcal{F}\mathcal{U}(P) \to P$ is an allowable epimorphism we can fill the diagram

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{\iota} & \downarrow{\scriptstyle \mathrm{id}_P} & & \\ \mathcal{F}\mathcal{U}(P) & \xrightarrow[\varepsilon_P]{} & P & \longrightarrow & 0 \end{array}$$

Hence $\varepsilon_P \iota = \mathrm{id}_P$ and P is a direct summand of $\mathcal{F}(\mathcal{U}(P))$. □

Definition 2.2. *Given a resolvent pair as in* (5) *and a relatively projective resolution as in* (6), *the n-th cohomology group of the complex of abelian groups*

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P_0, W) \xrightarrow{d_1^*} \mathrm{Hom}_{\mathcal{A}}(P_1, W) \xrightarrow{d_2^*} \mathrm{Hom}_{\mathcal{A}}(P_2, W) \xrightarrow{d_3^*} \dots$$

is denoted by $\mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(V, W)$ and is called the n-th relative Ext *group of V and W. We denote by $[\alpha] \in \mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(V, W)$ the equivalence class of a cocycle $\alpha \in \mathrm{Hom}_{\mathcal{A},\mathcal{B}}(P_n, W)$.*

Thanks to the fundamental lemma of relative homological algebra [ML75, Chap. IX, Th. 4.3] (which is the generalization of the well-known statement in usual homological algebra), $\mathrm{Ext}^\bullet_{\mathcal{A},\mathcal{B}}(V, W)$ does not depend on the choice of a relatively projective resolution. For the particular choice of the bar resolution (7), we denote the complex in Definition 2.2 by

$$\mathrm{Bar}^\bullet_{\mathcal{A},\mathcal{B}}(V, W) = \Big(0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}\big(\mathcal{F}\mathcal{U}(V), W\big) \xrightarrow{d_1^*} \mathrm{Hom}_{\mathcal{A}}\big((\mathcal{F}\mathcal{U})^2(V), W\big) \xrightarrow{d_2^*} \dots\Big).$$

Note that the adjunction is not apparent in the notation $\mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(V, W)$ but different adjunctions yield different cohomologies since they yield different allowable morphisms and relatively projective objects.

Example 2.3. If the category $\mathcal{B}$ is semisimple, then $\mathrm{Ext}^\bullet_{\mathcal{A},\mathcal{B}}(V, W) = \mathrm{Ext}^\bullet_{\mathcal{A}}(V, W)$. Indeed, any subobject in $\mathcal{B}$ is a direct summand. In particular for any $f \in \mathrm{Hom}_{\mathcal{A}}(V, W)$, the objects $\mathcal{U}(\ker(f))$ and $\mathcal{U}(\mathrm{im}(f))$ are direct summands of $\mathcal{U}(V)$ and $\mathcal{U}(W)$ respectively and thus f is allowable. It follows that a relatively projective object is projective in $\mathcal{A}$ and that any projective resolution in $\mathcal{A}$ is relatively projective.

Let A be a finite-dimensional algebra over a field k and $R \subset A$ be a subalgebra. The obvious forgetful functor $\mathcal{U} : A\text{-mod} \to R\text{-mod}$ has induction as left adjoint: $\mathcal{F}(X) = A \otimes_R X$. This adjunction is an example of a resolvent pair. We write $\mathrm{Ext}_{A,R}$ instead of $\mathrm{Ext}_{A\text{-mod},R\text{-mod}}$. The next example is a special case of this type of resolvent pair:

Example 2.4. Let A, B be two finite-dimensional k-algebras. Take $\mathcal{A} = (A \otimes B)\text{-mod}, \mathcal{B} = B\text{-mod}$, where B is identified with the subalgebra $1 \otimes B \subset A \otimes B$. If V, V' are (finite-dimensional) A-modules and S, S' are (finite-dimensional) B-modules such that $\mathrm{Hom}_B(S, S') \cong k$, then

$$\mathrm{Ext}^n_{A\otimes B,B}(V \boxtimes S, V' \boxtimes S') \cong \mathrm{Ext}^n_A(V, V') \tag{9}$$

where $V \boxtimes W$ is $V \otimes_k W$ with the action $(a \otimes b) \cdot (v \otimes w) = (a \cdot v) \otimes (b \cdot w)$. To see this isomorphism, note first that the induction functor is $\mathcal{F}(W) = A \boxtimes W$. If P is a projective A-module, then the

$(A\otimes B)$-module $P\boxtimes W$ is relatively projective for any $W \in B$-mod. Indeed, as a projective A-module, P is a direct summand of a free module $A^{\oplus n}$ for some $n \geq 0$ so that $P \boxtimes W$ is a direct summand of

$$A^{\oplus n} \boxtimes W \cong (A \boxtimes W)^{\oplus n} \cong A \boxtimes W^{\oplus n} = \mathcal{F}(W^{\oplus n}).$$

Now, take a projective resolution $0 \leftarrow V \xleftarrow{d_0} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} \ldots$ in A-mod and consider

$$0 \longleftarrow V \boxtimes S \xleftarrow{d_0\otimes \mathrm{id}} P_0 \boxtimes S \xleftarrow{d_1\otimes \mathrm{id}} P_1 \boxtimes S \xleftarrow{d_2\otimes \mathrm{id}} \ldots. \tag{10}$$

It is an exact sequence of relatively projective modules. Moreover each d_i is allowable: indeed, since the B-action on $\mathcal{U}(P_i \boxtimes S)$ is simply $b\cdot(x\otimes s) = x\otimes(b\cdot s)$, $\mathcal{U}\big(\ker(d_i\otimes \mathrm{id})\big) = \mathcal{U}\big(\ker(d_i)\boxtimes S\big)$ is a direct summand of $\mathcal{U}(P_i\boxtimes S)$ and similarly $\mathcal{U}\big(\mathrm{im}(d_i\otimes\mathrm{id})\big) = \mathcal{U}\big((\mathrm{im}(d_i)\boxtimes S\big)$ is a direct summand of $\mathcal{U}(P_{i-1}\boxtimes S))$. Hence (10) is a relatively projective resolution. The isomorphism (9) follows from the fact that $\mathrm{Hom}_{A\otimes B}(P_i\boxtimes S, V'\boxtimes S') \cong \mathrm{Hom}_A(P_i,V')\otimes \mathrm{Hom}_B(S,S') \cong \mathrm{Hom}_A(P_i,V')$. This example will be generalized in section 3.2 to algebras in braided tensor categories.

The next lemma is an immediate generalization of a well-known fact in homological algebra and will be useful later; it can be found in [ML75, Chap. IX, Prop. 4.2] but we recall the proof for convenience.

Lemma 2.5. *An object $P \in \mathcal{A}$ is relatively projective if and only if the functor* $\mathrm{Hom}_{\mathcal{A}}(P,-)$ *sends allowable short exact sequences in $\mathcal{A}$ to short exact sequences (of abelian groups).*

Proof. Let $0 \to X \xrightarrow{j} Y \xrightarrow{\sigma} Z \to 0$ be an allowable short exact sequence in $\mathcal{A}$. Recall that the functor $\mathrm{Hom}_{\mathcal{A}}(P,-)$ is left exact for all $P\in\mathcal{A}$ which means that

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P,X) \xrightarrow{j_*} \mathrm{Hom}_{\mathcal{A}}(P,Y) \xrightarrow{\sigma_*} \mathrm{Hom}_{\mathcal{A}}(P,Z)$$

is automatically exact. Thus the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P,X) \xrightarrow{j_*} \mathrm{Hom}_{\mathcal{A}}(P,Y) \xrightarrow{\sigma_*} \mathrm{Hom}_{\mathcal{A}}(P,Z) \longrightarrow 0$$

is exact if and only if $\sigma_* = \mathrm{Hom}_{\mathcal{A}}(P,\sigma)$ is surjective. But note that the definition of a relatively projective object is equivalent to the statement that if σ is an allowable epimorphism then $\mathrm{Hom}_{\mathcal{A}}(P,\sigma)$ is surjective. Hence the equivalence. □

2.2 Properties of relative Ext groups

In this section we consider a resolvent pair of abelian categories as in (5) and we discuss properties of the associated relative Ext groups: the Yoneda description of these groups, the Yoneda product and the long exact sequence of Ext groups associated to a short exact sequence of coefficients.

2.2.1 Yoneda description of relative Ext groups

A n-fold exact sequence from W to V is an exact sequence in $\mathcal{A}$ of the form

$$0 \longrightarrow W \xrightarrow{f_n} X_n \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_1} X_1 \xrightarrow{f_0} V \longrightarrow 0. \tag{11}$$

It is called allowable if each f_i is allowable. This condition is equivalent to the property that the exact sequence

$$0 \longrightarrow \mathcal{U}(W) \xrightarrow{\mathcal{U}(f_n)} \mathcal{U}(X_n) \xrightarrow{\mathcal{U}(f_{n-1})} \ldots \xrightarrow{\mathcal{U}(f_1)} \mathcal{U}(X_1) \xrightarrow{\mathcal{U}(f_0)} \mathcal{U}(V) \longrightarrow 0$$

splits in $\mathcal{B}$.

Definition 2.6. *The n-th relative Yoneda Ext group of V and W, denoted by $\mathrm{YExt}^n_{\mathcal{A},\mathcal{B}}(V,W)$ is the set of all allowable n-fold exact sequences from W to V, modulo congruence relations. We denote by $[S]$ the congruence class of a sequence S.*

We do not explain what are the congruence relations in $\mathrm{YExt}^n_{\mathcal{A},\mathcal{B}}(V,W)$ and refer to [ML75], Chap. III (§1 and §5) and Chap. XII (§4).

For further use, let us recall from [ML75, Chap. III, Th. 6.4][2] the construction of the isomorphism

$$\overline{\eta} : \mathrm{YExt}^n_{\mathcal{A},\mathcal{B}}(V,W) \xrightarrow{\sim} \mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(V,W). \tag{12}$$

We first define a map η at the level of cocycles. Let S be a n-fold allowable exact sequence as in (11) and let $\ldots \xrightarrow{d_1} P_0 \xrightarrow{d_0} V \longrightarrow 0$ be a relatively projective resolution of V. Fill the dashed arrows in the diagram (which bottom row is S)

$$\begin{array}{ccccccccccc}
P_n & \xrightarrow{d_n} & P_{n-1} & \xrightarrow{d_{n-1}} & \ldots & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & V & \longrightarrow & 0 \\
\downarrow{\scriptstyle \eta_n=\eta(S)} & & \downarrow{\scriptstyle \eta_{n-1}} & & & & \downarrow{\scriptstyle \eta_0} & & \| & & \\
W & \xrightarrow[f_n]{} & X_n & \xrightarrow[f_{n-1}]{} & \ldots & \xrightarrow[f_1]{} & X_1 & \xrightarrow[f_0]{} & V & \longrightarrow & 0
\end{array} \tag{13}$$

such that it becomes commutative. This is always possible by the fundamental lemma of relative homological algebra [ML75, Chap. IX, Th. 4.3], since the top row is a relatively projective resolution. Then the morphism $\eta_n \in \mathrm{Hom}_{\mathcal{A}}(P_n, W)$ is a cocycle of the complex from Definition 2.2 which we denote by $\eta(S)$. Indeed, $f_n\eta_n d_{n+1} = \eta_{n-1}d_n d_{n+1} = 0$ and since f_n is a monomorphism we get $\eta_n d_{n+1} = 0$. This cocycle depends on the choice of the relatively projective resolution of V but one shows that η sends congruent sequences to cohomologous cocycles, yielding a morphism $\overline{\eta} : \mathrm{YExt}^n_{\mathcal{A},\mathcal{B}}(V,W) \to \mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(V,W)$ defined by $\overline{\eta}([S]) = [\eta(S)]$; in particular $\overline{\eta}([S])$ does not longer depend on the choice of the relatively projective resolution. Finally, $\overline{\eta}$ turns out to be an isomorphism.

With the Yoneda description of Ext groups, it is easy to see that there is an injective morphism

$$\mathrm{Ext}^1_{\mathcal{A},\mathcal{B}}(V,W) \hookrightarrow \mathrm{Ext}^1_{\mathcal{A}}(V,W). \tag{14}$$

Indeed, $\mathrm{YExt}^1_{\mathcal{A},\mathcal{B}}(V,W)$ (resp. $\mathrm{YExt}^1_{\mathcal{A}}(V,W)$) consists of congruence classes of allowable short exact sequences (resp. congruence classes of short exact sequences) from W to V. In both cases, the zero element is the split exact sequence $0 \to W \to V \oplus W \to V \to 0$ and two sequences $0 \to W \to Z \to V \to 0$, $0 \to W \to Z' \to V \to 0$ are congruent if there exists a morphism in $\mathrm{Hom}_{\mathcal{A}}(Z, Z')$ such that the obvious diagram is commutative (see pages 64 and 368 in [ML75]). Hence, if a sequence is equal to 0 in $\mathrm{YExt}^1_{\mathcal{A}}(V,W)$, then it is equal to 0 in $\mathrm{YExt}^1_{\mathcal{A},\mathcal{B}}(V,W)$. This property is not true for higher Ext groups; indeed, two n-fold exact sequences (resp. allowable exact sequences) S, T are equal in $\mathrm{YExt}^n_{\mathcal{A}}(V,W)$ (resp. in $\mathrm{YExt}^n_{\mathcal{A},\mathcal{B}}(V,W)$) if there exists a chain of morphisms like for instance $S \to C_1 \leftarrow C_2 \to C_3 \leftarrow T$, where the C_i are n-fold exact sequences (resp. allowable exact sequences). Hence it might happen that two allowable exact sequences are not equal $\mathrm{YExt}^n_{\mathcal{A},\mathcal{B}}(V,W)$ but they are equal in $\mathrm{YExt}^n_{\mathcal{A}}(V,W)$ because they can be related by a chain of morphisms using non-allowable exact sequences. For an example, see item 2 in Remark 5.10 below.

In general, it is not easy to prove that a n-fold allowable exact sequence is not congruent to 0 (*i.e.* that it is not equal to 0 in the corresponding Yoneda Ext group), except for $n = 1$ where congruence is just isomorphism of short exact sequences. For $n = 2$ we have this useful criterion [ML75, Chap. XII, Lemma 5.3]:

Lemma 2.7. *Let $S = \big(0 \to W \to X_2 \xrightarrow{\pi} K \to 0\big)$ and $T = \big(0 \to K \xrightarrow{j} X_1 \to V \to 0\big)$ be allowable short exact sequences in $\mathcal{A}$ and let $S \circ T = \big(0 \to W \to X_2 \xrightarrow{j\pi} X_1 \to V \to 0\big)$ be their Yoneda product (see §2.2.2 below). The following are equivalent:*

[2] In [ML75] both Ext groups are denote by Ext^n. In Theorem 6.4 of Chapter III, YExt^n is denoted Ext^n while Ext^n is denoted H^n. Moreover this theorem is stated for usual Ext groups, but in §5, Chap. XII it is said that it holds for relative Ext groups.

1. $S \circ T \equiv 0$.

2. *There exists an allowable short exact sequence* $P = \big(0 \to W \to M \to X_1 \to 0\big)$ *such that* $S = Pj$*, where* Pj *denotes the pullback of* P *by* j*.*

3. *There exists an allowable short exact sequence* $Q = \big(0 \to X_2 \to N \to V \to 0\big)$ *such that* $T = \pi Q$*, where* πQ *denotes the push-forward of* Q *by* π*.*

By definition of the pullback and of the push-forward of a short exact sequence, items 2 and 3 mean respectively that there exist commutative diagrams (with exact rows):

$$\begin{array}{ccccccccccc} S = \big(0 & \longrightarrow & W & \longrightarrow & X_2 & \longrightarrow & K & \longrightarrow & 0\big) \\ & & \Vert & & \downarrow & & \downarrow{\scriptstyle j} & & \\ P = \big(0 & \longrightarrow & W & \longrightarrow & M & \longrightarrow & X_1 & \longrightarrow & 0\big) \end{array} \qquad \text{and} \qquad \begin{array}{ccccccccccc} Q = \big(0 & \longrightarrow & X_2 & \longrightarrow & N & \longrightarrow & V & \longrightarrow & 0\big) \\ & & \downarrow{\scriptstyle \pi} & & \downarrow & & \Vert & & \\ T = \big(0 & \longrightarrow & K & \longrightarrow & X_1 & \longrightarrow & V & \longrightarrow & 0\big) \end{array}$$

See §5.2.2 for a use of this criterion on an example. This lemma can be extended to n-fold exact sequences (see [ML75, Chap. XII, Lemma 5.5]) but it becomes much more difficult to apply in practice.

Corollary 2.8. *Let* S, T *be allowable short exact sequences as above.*

1. *If* $S \not\equiv 0$ *and* $\mathrm{Ext}^1_{\mathcal{A},\mathcal{B}}(X_1, W) = 0$*, then* $S \circ T \not\equiv 0$*.*

2. *If* $T \not\equiv 0$ *and* $\mathrm{Ext}^1_{\mathcal{A},\mathcal{B}}(V, X_2) = 0$*, then* $S \circ T \not\equiv 0$

Proof. We show the first item, the proof of the second being analogous. Any allowable short exact sequence P as in Lemma 2.7 belongs to $\mathrm{YExt}^1_{\mathcal{A},\mathcal{B}}(X_1, W) \cong \mathrm{Ext}^1_{\mathcal{A},\mathcal{B}}(X_1, W) = 0$ and thus is congruent to 0. It follows that $Pj \equiv 0$ as well. But then $S = Pj$ is impossible since $S \not\equiv 0$. Hence, by the equivalence in Lemma 2.7, $S \circ T \not\equiv 0$. □

See §5.3.2 for a use of this corollary on an example.

2.2.2 The Yoneda product

For two allowable exact sequences from W to V and from V to U, their Yoneda product $\circ$ is an allowable exact sequence from W to U defined by

$$\begin{aligned} &\big(0 \longrightarrow W \xrightarrow{f_n} Y_n \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_1} Y_1 \xrightarrow{f_0} V \longrightarrow 0\big) \circ \big(0 \longrightarrow V \xrightarrow{g_m} X_m \xrightarrow{g_{m-1}} \ldots \xrightarrow{g_1} X_1 \xrightarrow{g_0} U \longrightarrow 0\big) \\ &= 0 \longrightarrow W \xrightarrow{f_n} Y_n \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_1} Y_1 \xrightarrow{g_m f_0} X_m \xrightarrow{g_{m-1}} \ldots \xrightarrow{g_1} X_1 \xrightarrow{g_0} U \longrightarrow 0 \end{aligned} \tag{15}$$

Any n-fold allowable exact sequence can be written as a product of n allowable short exact sequences [ML75, Chap. III, §5].

The Yoneda product is compatible with the congruence relations and yields a bilinear map

$$\begin{array}{rrcl} \circ : & \mathrm{YExt}^n_{\mathcal{A},\mathcal{B}}(V, W) \times \mathrm{YExt}^m_{\mathcal{A},\mathcal{B}}(U, V) & \to & \mathrm{YExt}^{m+n}_{\mathcal{A},\mathcal{B}}(U, W) \\ & ([S], [S']) & \mapsto & [S \circ S'] \end{array}$$

We recall how to compute the corresponding product (also called Yoneda product) on cocycles:

$$\circ : \mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(V, W) \times \mathrm{Ext}^m_{\mathcal{A},\mathcal{B}}(U, V) \to \mathrm{Ext}^{m+n}_{\mathcal{A},\mathcal{B}}(U, W) \tag{16}$$

along the isomorphism $\overline{\eta}$ from (12). Let $\ldots \xrightarrow{d_1^U} P_0 \xrightarrow{d_0^U} U \longrightarrow 0$ and $\ldots \xrightarrow{d_1^V} Q_0 \xrightarrow{d_0^V} V \longrightarrow 0$ be relatively projective resolutions of U and V respectively and let $\alpha \in \mathrm{Hom}_{\mathcal{A}}(Q_n, W)$ and $\beta \in \mathrm{Hom}_{\mathcal{A}}(P_m, V)$ be cocycles with respect to these resolutions. Fill the dashed arrows in the diagram

$$\begin{array}{ccccccccccc} P_{m+n} & \xrightarrow{d^U_{m+n}} & \ldots & \xrightarrow{d^U_{m+2}} & P_{m+1} & \xrightarrow{d^U_{m+1}} & P_m & & & & \\ \downarrow{\scriptstyle \widetilde{\beta}_n} & & & & \downarrow{\scriptstyle \widetilde{\beta}_1} & & \downarrow{\scriptstyle \widetilde{\beta}_0} & \searrow{\scriptstyle \beta} & & & \\ Q_n & \xrightarrow{d^V_n} & \ldots & \xrightarrow{d^V_2} & Q_1 & \xrightarrow{d^V_1} & Q_0 & \xrightarrow{d^V_0} & V & \longrightarrow & 0 \end{array} \tag{17}$$

such that it becomes commutative (this is always possible thanks to the fundamental lemma of relative homological algebra [ML75, Chap. IX, Th. 4.3]). Then $\alpha \circ \beta$ is defined to be the composition $\alpha\widetilde{\beta}_n \in \mathrm{Hom}_{\mathcal{A}}(P_{m+n}, W)$ and $[\alpha] \circ [\beta]$ is defined to be $[\alpha \circ \beta] \in \mathrm{Ext}^{m+n}_{\mathcal{A},\mathcal{B}}(V, W)$. The definition is such that the map η from (13) satisfies $\eta(S \circ S') = \eta(S) \circ \eta(S')$, and thus in particular $\overline{\eta}([S] \circ [S']) = \overline{\eta}([S]) \circ \overline{\eta}([S'])$.

2.2.3 The long exact sequence for relative Ext groups

Note that $\mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(-,-)$ is a bifunctor $\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Ab}$ for all n, because it is an instance of a derived functor [ML75, Chap. XII, §9]. Let U, V, W be objects in $\mathcal{A}$ and let $f : U \to V$ be a morphism in $\mathcal{A}$; here we only need to recall how to define the morphism of abelian groups

$$f^* = \mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(f, W) : \mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(V, W) \to \mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(U, W).$$

We first define f^* on cochains. Take relative projective resolutions $\ldots \to P_1 \to P_0 \to U \to 0$, $\ldots \to Q_1 \to Q_0 \to V \to 0$ of U, V respectively, and let $\alpha \in \mathrm{Hom}_{\mathcal{A}}(Q_n, W)$ be a cocycle representing a class $[\alpha] \in \mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(V, W)$ (see Definition 2.2). Fill the dashed arrows in the diagram

$$\begin{array}{ccccccccccc} P_n & \longrightarrow & P_{n-1} & \longrightarrow & \ldots & \longrightarrow & P_0 & \longrightarrow & U & \longrightarrow & 0 \\ \downarrow{\scriptstyle \widetilde{f}_n} & & \downarrow{\scriptstyle \widetilde{f}_{n-1}} & & & & \downarrow{\scriptstyle \widetilde{f}_0} & & \downarrow{\scriptstyle f} & & \\ Q_n & \longrightarrow & Q_{n-1} & \longrightarrow & \ldots & \longrightarrow & Q_0 & \longrightarrow & V & \longrightarrow & 0 \end{array}$$

such that it becomes commutative (again this is always possible thanks to the fundamental lemma of relative homological algebra). Then $\alpha\widetilde{f}_n \in \mathrm{Hom}_{\mathcal{A}}(P_n, W)$ is a cocycle which we denote by $f^*(\alpha)$ and we put $f^*([\alpha]) = [f^*(\alpha)] \in \mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(U, W)$.

Theorem 2.9. [ML75, Chap. XII, Th. 5.1][3] *Let $S = \big(0 \longrightarrow U \xrightarrow{j} V \xrightarrow{\pi} W \longrightarrow 0\big)$ be an allowable short exact sequence in $\mathcal{A}$ and let N be any object in $\mathcal{A}$. Then we have the long exact sequence of abelian groups*

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(W, N) \xrightarrow{\pi^*} \mathrm{Hom}_{\mathcal{A}}(V, N) \xrightarrow{j^*} \mathrm{Hom}_{\mathcal{A}}(U, N) \\ \xrightarrow{c^0} \mathrm{Ext}^1_{\mathcal{A},\mathcal{B}}(W, N) \xrightarrow{\pi^*} \mathrm{Ext}^1_{\mathcal{A},\mathcal{B}}(V, N) \xrightarrow{j^*} \mathrm{Ext}^1_{\mathcal{A},\mathcal{B}}(U, N) \\ \xrightarrow{c^1} \mathrm{Ext}^2_{\mathcal{A},\mathcal{B}}(W, N) \xrightarrow{\pi^*} \mathrm{Ext}^2_{\mathcal{A},\mathcal{B}}(V, N) \xrightarrow{j^*} \mathrm{Ext}^2_{\mathcal{A},\mathcal{B}}(U, N) \ldots \end{array}$$

For each n, π^* and j^* are the pullbacks $\mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(\pi, N)$ and $\mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(j, N)$. The maps c^n are called connecting morphisms and are defined by

$$c^n(\alpha) = (-1)^n \alpha \circ \overline{\eta}(S)$$

[3] Note that in [ML75] this theorem is stated in a more general setting based on the notion of proper class of short exact sequences. Here we take the class of all allowable short exact sequences in $\mathcal{A}$ (*i.e.* the short exact sequences which arrows are allowable morphisms), which is a proper class as remarked in [ML75, p. 368].

where $\circ$ is the Yoneda product defined after (16) and $\overline{\eta}([S]) \in \mathrm{Ext}^1_{\mathcal{A},\mathcal{B}}(W,U)$ is the element associated to $[S] \in \mathrm{YExt}^1_{\mathcal{A},\mathcal{B}}(W,U)$ by the isomorphism from (12).

The long exact sequence from Theorem 2.9 gives in particular an inductive formula for the relative Ext groups:

Corollary 2.10. *Let* $0 \longrightarrow L \xrightarrow{i} Q \xrightarrow{p} V \longrightarrow 0$ *be an allowable short exact sequence in* $\mathcal{A}$*, with* Q *a relatively projective object. We have*

$$\mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(V,W) \cong \mathrm{Ext}^{n-1}_{\mathcal{A},\mathcal{B}}(L,W) \quad \textit{for } n>1,$$
$$\mathrm{Ext}^1_{\mathcal{A},\mathcal{B}}(V,W) \cong \mathrm{Hom}_{\mathcal{A}}(L,W)/\mathrm{im}(i^*)$$

where i^* *is the pullback* $\mathrm{Hom}_{\mathcal{A}}(Q,W) \to \mathrm{Hom}_{\mathcal{A}}(L,W)$.

Proof. Since Q is relatively projective we have $\mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(Q,-) = 0$ for all $n \geq 1$. Thanks to Theorem 2.9 we get for all $n > 1$ an exact sequence

$$\mathrm{Ext}^{n-1}_{\mathcal{A},\mathcal{B}}(Q,W) = 0 \xrightarrow{i^*} \mathrm{Ext}^{n-1}_{\mathcal{A},\mathcal{B}}(L,W) \xrightarrow{c^{n-1}} \mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(V,W) \xrightarrow{p^*} \mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(Q,W) = 0$$

which implies that c^{n-1} is an isomorphism of vector spaces. For the case $n = 1$, again thanks to Theorem 2.9 we have the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(V,W) \xrightarrow{i^*} \mathrm{Hom}_{\mathcal{A}}(Q,W) \xrightarrow{p^*} \mathrm{Hom}_{\mathcal{A}}(L,W) \xrightarrow{c^0} \mathrm{Ext}^1_{\mathcal{A},\mathcal{B}}(V,W) \xrightarrow{i^*} \mathrm{Ext}^1_{\mathcal{A},\mathcal{B}}(Q,W) = 0$$

which gives the desired isomorphism. □

2.3 Relatively projective cover

Relatively projective covers have been defined and studied in [Thé85] in the special case where $\mathcal{A}$ is the category of modules over a ring, $\mathcal{B}$ is the category of modules over a subring and the adjunction is given by the usual restriction and induction functors.

Here we only assume that $\mathcal{A}$ is the category A-mod of finite-dimensional modules over a k-algebra A (not necessarily finite-dimensional), where k is a field; the category $\mathcal{B}$ and the functors $\mathcal{U}, \mathcal{F}$ remain arbitrary, except of course that they form a resolvent pair as in (5). We prove existence, uniqueness and some properties of the relatively projective cover in this general setting. The assumption on $\mathcal{A}$ is important because our proofs are based on the fact that the objects are in particular k-vector spaces.

As explained in [Thé85, §1], there are two possible definitions of a relatively projective cover (as for usual projective covers) based respectively on the notions of minimal epimorphism and relatively essential epimorphism. Recall that an allowable epimorphism $\pi : M \to V$ in $\mathcal{A}$ is called

- *minimal* if for all $f \in \mathrm{End}_{\mathcal{A}}(M)$,

$$\pi f = \pi \quad\Longrightarrow\quad f \text{ is an isomorphism.}$$

- *relatively essential* if for every submodule $S \subset M$,

$$\pi|_S : S \to V \text{ is an allowable epimorphism} \quad\Longrightarrow\quad S = M$$

  where $\pi|_S$ is the restriction of π to S.

With our assumption that $\mathcal{A} \cong A$-mod these two notions are equivalent:

Lemma 2.11. *Let* P *be a relatively projective object and* $\pi : P \to V$ *be an allowable epimorphism. Then* π *is minimal if and only if it is relatively essential.*

Proof. Assume that π is minimal and that some restriction $\pi|_S$ is an allowable epimorphism. We can fill the diagram

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{f} & \downarrow{\scriptstyle \pi} & & \\ S & \xrightarrow[\pi|_S]{} & V & \longrightarrow & 0 \end{array}$$

Let $\iota : S \to P$ be the canonical embedding. Then $\pi\iota f = \pi|_S f = \pi$. Hence ιf is an isomorphism due to the minimality of π, which implies that ι is surjective and thus $S = P$. Conversely, assume that π is relatively essential and let $f : P \to P$ be such that $\pi f = \pi$. Then $\pi|_{\mathrm{im}(f)}$ is an epimorphism since so is π. Moreover $\pi|_{\mathrm{im}(f)}$ is allowable: indeed, since π is allowable and is an epimorphism there exists $s \in \mathrm{Hom}_{\mathcal{B}}\big(\mathcal{U}(V), \mathcal{U}(P)\big)$ such that $\mathcal{U}(\pi)s = \mathrm{id}_{\mathcal{U}(V)}$, hence we have

$$\mathcal{U}(\pi|_{\mathrm{im}(f)})\mathcal{U}(f)s = \mathcal{U}(\pi f)s = \mathcal{U}(\pi)s = \mathrm{id}_{\mathcal{U}(V)}.$$

It follows that $\mathrm{im}(f) = P$, so that f is a surjection from a finite-dimensional vector space to itself, which implies that f is an isomorphism. □

Definition 2.12. *A relatively projective cover of $V \in \mathcal{A}$ is a relatively projective object R_V together with a relatively essential allowable epimorphism $p_V : R_V \to V$.*

Proposition 2.13. *1. Any object $V \in \mathcal{A}$ has a relatively projective cover, which is unique up to isomorphism.*
2. Let P be any relatively projective object with an allowable epimorphism $\pi : P \to V$. Then R_V is (isomorphic to) a direct summand of P and $\pi|_{R_V} = p_V$.
3. If k is algebraically closed and $\mathcal{U}(V) \in \mathcal{B}$ is a simple object then $\mathrm{Hom}_{\mathcal{A}}(R_V, V) \cong k$, with basis element p_V.

Proof. 1. For existence, let P be a relatively projective object together with an allowable epimorphism $\pi : P \to V$. There always exists such a pair (P, π); for instance one can take $P = G(V)$ and $\pi = \varepsilon_V$ where $G = \mathcal{F}\mathcal{U}$ is the comonad on $\mathcal{A}$ and ε is the counit of G. Consider

$$\mathbb{S} = \big\{ S \subset P \,\big|\, \pi|_S \text{ is an allowable epimorphism} \big\}$$

and let R be an element of $\mathbb{S}$ with minimal dimension. Since P is relatively projective there exists $f : P \to R$ such that $\pi|_R f = \pi$. By the same arguments as in the proof of Lemma 2.11 we get that $\pi|_{\mathrm{im}(f)}$ is an allowable epimorphism; hence $\mathrm{im}(f) \in \mathbb{S}$. On one hand $\mathrm{im}(f) \subset R$ but on the other hand $\dim(\mathrm{im}(f)) \geq \dim(R)$ by the minimality assumption on R. As a result, $\mathrm{im}(f) = R$ and f is surjective. Now let $\iota : R \hookrightarrow P$ be the canonical embedding. Then $f\iota : R \to R$ is a surjective morphism from a finite-dimensional vector space to itself, so it is an isomorphism. If we define $\mathrm{pr} = (f\iota)^{-1} f : P \to R$ we have $\mathrm{pr}\,\iota = \mathrm{id}_R$, which shows that R is a direct summand of P. In particular, R is relatively projective as a direct summand of P by Lemma 2.1. We thus take $R_V = R$ and $p_V = \pi|_R$. The morphism p_V is relatively essential because R_V is an element in $\mathbb{S}$ with minimal dimension.
For uniqueness, let (R_V, p_V), (R'_V, p'_V) be two relatively projective covers of V. We can fill the diagrams

$$\begin{array}{ccccc} & & R_V & & \\ & \swarrow^{g} & \downarrow{\scriptstyle p_V} & & \\ R'_V & \xrightarrow[p'_V]{} & V & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccc} & & R'_V & & \\ & \swarrow^{h} & \downarrow{\scriptstyle p'_V} & & \\ R_V & \xrightarrow[p_V]{} & V & \longrightarrow & 0 \end{array}$$

Note that $p_V hg = p'_V g = p_V$ and $p'_V gh = p_V h = p'_V$. Since p_V and p'_V are minimal by Lemma 2.11, hg and gh are isomorphisms. It follows that g and h are isomorphisms.
2. By the proof of item 1 we know that P contains a direct summand R such that $(R, \pi|_R)$ is a

relatively projective cover. By uniqueness of the projective cover, (R_V, p_V) is isomorphic to $(R, \pi|_R)$.
3. By the previous item R_V is a direct summand of $G(V)$, so we have

$$\mathrm{Hom}_{\mathcal{A}}(R_V, V) \hookrightarrow \mathrm{Hom}_{\mathcal{A}}(G(V), V) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{FU}(V), V) \cong \mathrm{Hom}_{\mathcal{B}}(\mathcal{U}(V), \mathcal{U}(V)) \cong k. \tag{18}$$

The last equality is due to Schur's lemma, which applies since we assume that k is algebraically closed. Hence $\mathrm{Hom}_{\mathcal{A}}(R_V, V)$ is at most one-dimensional; but it contains p_V, so it is one-dimensional. □

Remark 2.14. It follows from the proof of Proposition 2.13 that the relatively projective cover R_V of V is isomorphic to any one of the direct summands of $G(V)$ which cover V by the restriction of ε_V and which have minimal dimension; in particular all such direct summands are isomorphic. If $\mathcal{U}(V)$ is simple then V is simple (because $\mathcal{U}$ is faithful) and R_V is the minimal direct summand of $G(V)$ on which ε_V does not vanish.

2.4 Comonad cohomology and relative Ext groups

Let $G = \mathcal{FU}$ be the comonad on $\mathcal{A}$ associated to an adjunction $\mathcal{F} \dashv \mathcal{U}$ as in (5). In this section we review the theory of comonad cohomology introduced in [BB96] and in the case of a resolvent pair we relate it to relative Ext groups. Our notations and conventions are those of [GHS19].

- An object P of $\mathcal{A}$ is called *G-projective* if there exists a morphism $s : P \to G(P)$ such that $\varepsilon_P\, s = \mathrm{id}_P$, where ε is the counit of G. By [GHS19, Lemma 2.5], this is equivalent to the requirement that P is a direct summand of some $G(V)$.

- A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$ is called *G-exact* if $gf = 0$ and the sequence of abelian groups

$$\mathrm{Hom}_{\mathcal{A}}(G(V), X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(G(V), Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(G(V), Z)$$

  is exact for all $V \in \mathcal{A}$. Note that thanks to the adjunction property, this last condition is equivalent to the exactness of

$$\mathrm{Hom}_{\mathcal{B}}\big(\mathcal{U}(V), \mathcal{U}(X)\big) \xrightarrow{\mathcal{U}(f)_*} \mathrm{Hom}_{\mathcal{B}}\big(\mathcal{U}(V), \mathcal{U}(Y)\big) \xrightarrow{\mathcal{U}(g)_*} \mathrm{Hom}_{\mathcal{B}}\big(\mathcal{U}(V), \mathcal{U}(Z)\big). \tag{19}$$

  for all $V \in \mathcal{A}$.

- A sequence $0 \longleftarrow V \xleftarrow{d_0} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} \ldots$ in $\mathcal{A}$ is called a *G-resolution* if each P_i is G-projective and the sequence is G-exact.

There always exists at least one G-projective resolution of any object V in $\mathcal{A}$, called the bar resolution:

$$\mathrm{Bar}_G^\bullet(V) = \Big(0 \longleftarrow V \xleftarrow{\varepsilon_V} G(V) \xleftarrow{d_1} G^2(V) \xleftarrow{d_2} \ldots\Big) \tag{20}$$

where $\varepsilon_V : G(V) \to V$ is the counit of the comonad G and

$$d_n = \sum_{i=0}^{n} (-1)^i G^{n-i}\big(\varepsilon_{G^i(V)}\big). \tag{21}$$

Definition 2.15. *Let $\mathcal{E}$ be an abelian category and $E : \mathcal{A} \to \mathcal{E}$ be a contravariant additive functor. Given a G-projective resolution as above, the cohomology of the complex in $\mathcal{E}$*

$$0 \longrightarrow E(P_0) \xrightarrow{E(d_1)} E(P_1) \xrightarrow{E(d_2)} E(P_2) \xrightarrow{E(d_3)} \ldots$$

is called the cohomology of V associated to G with coefficients in E, and is denoted $H_G^\bullet(V, E)$.

There is an analogue of the fundamental lemma of homological algebra for comonad cohomology which implies that $H^\bullet_G(V, E)$ does not depend on the choice of a G-projective resolution [BB96, §4.2]. For the particular choice of the bar resolution (20), we denote the complex in Definition 2.15 by

$$\mathrm{Bar}^\bullet_G(V, E) = \Big(0 \longrightarrow E(G(V)) \xrightarrow{E(d_1)} E(G^2(V)) \xrightarrow{E(d_2)} \ldots\Big) \tag{22}$$

We now relate the notions from comonad cohomology to those from relative homological algebra. Suppose we are given a resolvent pair of categories as in (5) and let $G = \mathcal{F}\mathcal{U}$ be the associated comonad on $\mathcal{A}$.

Lemma 2.16. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence in $\mathcal{A}$.*

1. *Let $\ker(g) = (k : K \to Y)$. The sequence is G-exact if and only if $gf = 0$ and there exists $t \in \mathrm{Hom}_{\mathcal{B}}\big(\mathcal{U}(K), \mathcal{U}(X)\big)$ such that $\mathcal{U}(f)t = \mathcal{U}(k)$.*
2. *If the sequence is G-exact and g is allowable, then f is allowable.*
3. *If the sequence is exact and f is allowable, then it is G-exact.*
4. *If the sequence is G-exact, then it is exact.*

Proof. Note that $\ker(\mathcal{U}(g)) = \mathcal{U}(\ker(g))$ and $\mathrm{im}(\mathcal{U}(f)) = \mathcal{U}(\mathrm{im}(f))$ since $\mathcal{U}$ is exact.
1. Since $gk = 0$, the exactness of (19) with $V = K$ implies that there exists $t : \mathcal{U}(K) \to \mathcal{U}(X)$ such that $\mathcal{U}(f)t = \mathcal{U}(k)$. Conversely, let $V \in \mathcal{A}$ and $h : \mathcal{U}(V) \to \mathcal{U}(Y)$ such that $\mathcal{U}(g)h = 0$. Thus by the universal property of the kernel, there exists $u : \mathcal{U}(V) \to \mathcal{U}(K)$ such that $h = \mathcal{U}(k)u = \mathcal{U}(f)tu$, which shows that $h \in \mathrm{im}(\mathcal{U}(f)_*)$. Hence the sequence (19) is exact.
2. By definition, there exists $s : \mathcal{U}(Z) \to \mathcal{U}(Y)$ such that $\mathcal{U}(g)s\mathcal{U}(g) = \mathcal{U}(g)$. Let $\pi = \mathrm{id}_{\mathcal{U}(Y)} - s\mathcal{U}(g) : \mathcal{U}(Y) \to \mathcal{U}(Y)$. We have $\mathcal{U}(g)\pi = 0$, so by the universal property of the kernel we get $u : \mathcal{U}(Y) \to \mathcal{U}(K)$ such that $\mathcal{U}(k)u = \pi$. Now since the sequence is G-exact, we have $t : \mathcal{U}(K) \to \mathcal{U}(X)$ from item 1. Then $tu : \mathcal{U}(Y) \to \mathcal{U}(X)$ satisfies

$$\mathcal{U}(f)tu\mathcal{U}(f) = \mathcal{U}(k)u\mathcal{U}(f) = \pi\mathcal{U}(f) = \mathcal{U}(f) - s\mathcal{U}(g)\mathcal{U}(f) = \mathcal{U}(f)$$

where for the last equality we used that $gf = 0$. Hence f is allowable.
3. Since f is allowable, we have some $s : \mathcal{U}(Y) \to \mathcal{U}(X)$ such that $\mathcal{U}(f)s\mathcal{U}(f) = \mathcal{U}(f)$. Since the sequence is exact, $\ker(g) = (k : K \to Y) = \mathrm{im}(f)$, and by definition of the image there exists an epimorphism $e : X \to K$ such that $f = ke$. We have

$$\mathcal{U}(f)\big(s\mathcal{U}(k)\big)\mathcal{U}(e) = \mathcal{U}(f)s\mathcal{U}(f) = \mathcal{U}(f) = \mathcal{U}(k)\mathcal{U}(e).$$

Since e is an epimorphism, we deduce $\mathcal{U}(f)\big(s\mathcal{U}(k)\big) = \mathcal{U}(k)$ and by item 1 the sequence is G-exact.
4. We show that $\ker(\mathcal{U}(g)) = \mathrm{im}(\mathcal{U}(f))$, then it follows that $\ker(g) = \mathrm{im}(f)$ since $\mathcal{U}$ is faithful. First, $\mathcal{U}(k)$ is a monomorphism and by definition of the kernel, since $\mathcal{U}(f)\mathcal{U}(g) = 0$, there exists $u : \mathcal{U}(X) \to \mathcal{U}(K)$ such that $\mathcal{U}(f) = \mathcal{U}(k)\mathcal{U}(u)$. It remains to prove that $\mathcal{U}(k)$ is universal for these properties. So let $m : I \to \mathcal{U}(Y)$ be a monomorphism such that there exists $e : \mathcal{U}(X) \to I$ with $\mathcal{U}(f) = me$. We want to show that there exists $v : \mathcal{U}(K) \to I$ such that $\mathcal{U}(k) = mv$. Using item 1, we have $\mathcal{U}(k) = \mathcal{U}(f)t = met$, hence $v = et$. □

Proposition 2.17. *Let $\mathcal{A} \underset{\mathcal{U}}{\overset{\mathcal{F}}{\leftrightarrows}} \mathcal{B}$ be a resolvent pair of abelian categories and $G = \mathcal{F}\mathcal{U}$ be the associated comonad on $\mathcal{A}$.*

1. *An object $P \in \mathcal{A}$ is G-projective if and only if it is relatively projective.*
2. *A sequence is a G-resolution if and only if it is a relatively projective resolution.*

3. We have an equality of complexes $\mathrm{Bar}^\bullet_G\big(V, \mathrm{Hom}_\mathcal{A}(?, W)\big) = \mathrm{Bar}^\bullet_{\mathcal{A},\mathcal{B}}(V, W)$*, for any* $V, W \in \mathcal{A}$*.*

4. This implies $\mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(V, W) \cong H^n_G\big(V, \mathrm{Hom}_\mathcal{A}(?, W)\big)$ *for all* $n \geq 0$*.*

Proof. 1. If P is G-projective then it is a direct summand of some $G(V) = \mathcal{F}(\mathcal{U}(V))$ and hence is relatively projective by Lemma 2.1. Conversely, assume that P is a direct summand of some $\mathcal{F}(X)$, with morphisms $\iota : P \to \mathcal{F}(X)$ and $\pi : \mathcal{F}(X) \to P$ such that $\pi\iota = \mathrm{id}_P$. Let $\eta : \mathrm{Id}_\mathcal{B} \to \mathcal{U}\mathcal{F}$ and $\varepsilon : \mathcal{F}\mathcal{U} \to \mathrm{Id}_\mathcal{A}$ be respectively the unit and the counit of the adjunction. Define

$$\iota' : P \xrightarrow{\iota} \mathcal{F}(X) \xrightarrow{\mathcal{F}(\eta_X)} \mathcal{F}\mathcal{U}\mathcal{F}(X) = G(\mathcal{F}(X)),$$
$$\pi' : G(\mathcal{F}(X)) = \mathcal{F}\mathcal{U}\mathcal{F}(X) \xrightarrow{\varepsilon_{\mathcal{F}(X)}} \mathcal{F}(X) \xrightarrow{\pi} P.$$

Then $\pi'\iota' = \pi\varepsilon_{\mathcal{F}(X)}\mathcal{F}(\eta_X)\iota = \pi\iota = \mathrm{id}_P$, thanks to one of the defining properties of ε and η. Hence P is G-projective, as a direct summand of $G(\mathcal{F}(X))$.
2. Let $\ldots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} V \longrightarrow 0$ be a G-projective resolution. Since $P_0 \xrightarrow{d_0} V \longrightarrow 0$ is G-exact and $V \xrightarrow{0} 0$ is allowable, so is d_0 by the previous lemma. Assume now that some d_i is allowable. Since $P_{i+1} \xrightarrow{d_{i+1}} P_i \xrightarrow{d_i} P_{i-1}$ is G-exact, d_{i+1} is allowable by the previous lemma. By induction, all the d_i's are allowable. Since moreover we have seen that a G-exact sequence is exact, we have a relatively projective resolution.
The converse implication is obvious due to the third item in the previous lemma.
3. Trivial, the definitions are the same.
4. Follows immediately from any of the two previous items. □

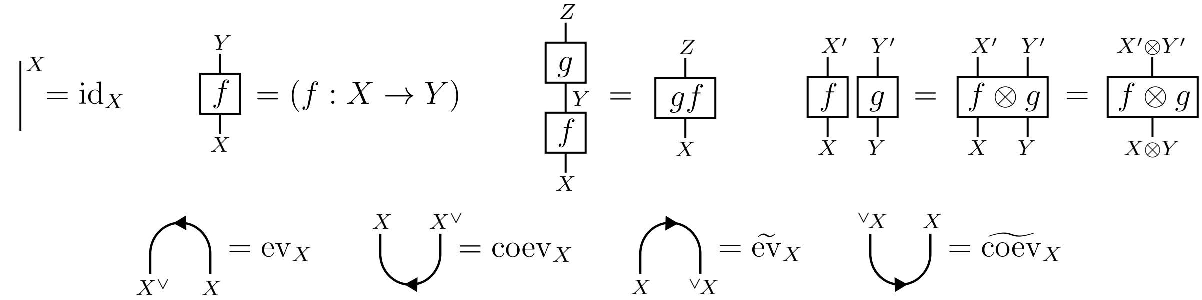
3 Relative Ext groups for tensor categories

Let k be a field, which we assume algebraically closed. A tensor category $\mathcal{C}$ is a k-linear, rigid and monoidal category with k-bilinear tensor product and with simple tensor unit object $\mathbf{1}$ (see e.g. [EGNO15, Chap. 4]). In a tensor category the tensor product is exact, because of rigidity [EGNO15, Prop. 4.2.1].

Recall that $\mathcal{C}$ is rigid if every object $X \in \mathcal{C}$ has a right dual $X^\vee$ and a left dual ${}^\vee X$ together with left and right (co)evaluation morphisms

$$\mathrm{ev}_X : X^\vee \otimes X \to k, \quad \mathrm{coev}_X : k \to X \otimes X^\vee,$$
$$\widetilde{\mathrm{ev}}_X : X \otimes {}^\vee X \to k, \quad \widetilde{\mathrm{coev}}_X : k \to {}^\vee X \otimes X,$$

satisfying the standard axioms. For certain computations it will be convenient to represent morphisms with diagrams, following the usual rules:



Note that we read diagrams from bottom to top.

3.1 Monoidal resolvent pairs

In this subsection, we consider a resolvent pair of categories as in (5) but moreover we assume that $\mathcal{A}, \mathcal{B}$ are tensor categories and that the exact, faithful and linear functor $\mathcal{U} : \mathcal{A} \to \mathcal{B}$ is monoidal.

Lemma 3.1. *If $f \in \mathrm{Hom}_{\mathcal{A}}(X, X')$ and $g \in \mathrm{Hom}_{\mathcal{A}}(Y, Y')$ are allowable morphisms then $f \otimes g$ is an allowable morphism.*

Proof. Let $s \in \mathrm{Hom}_{\mathcal{B}}(X', X), t \in \mathrm{Hom}_{\mathcal{B}}(Y', Y)$ be such that $\mathcal{U}(f)s\mathcal{U}(f) = \mathcal{U}(f)$ and $\mathcal{U}(g)\,t\,\mathcal{U}(g) = \mathcal{U}(g)$ and let $\mathcal{U}^{(2)}_{V,W} : \mathcal{U}(V) \otimes \mathcal{U}(W) \xrightarrow{\sim} \mathcal{U}(V \otimes W)$ be the monoidal structure of $\mathcal{U}$. Then

$$\begin{aligned}\mathcal{U}(f \otimes g)\left(\mathcal{U}^{(2)}_{X,Y}(s \otimes t)\,(\mathcal{U}^{(2)}_{X',Y'})^{-1}\right)\mathcal{U}(f \otimes g) &= \mathcal{U}^{(2)}_{X',Y'}\big(\mathcal{U}(f) \otimes \mathcal{U}(g)\big)(s \otimes t)\big(\mathcal{U}(f) \otimes \mathcal{U}(g)\big)(\mathcal{U}^{(2)}_{X,Y})^{-1} \\ &= \mathcal{U}^{(2)}_{X',Y'}\big(\mathcal{U}(f) \otimes \mathcal{U}(g)\big)(\mathcal{U}^{(2)}_{X,Y})^{-1} = \mathcal{U}(f \otimes g).\end{aligned}$$

□

Proposition 3.2. *The full subcategory $\mathrm{Proj}_{\mathcal{A},\mathcal{B}}$ of relatively projective objects is a tensor ideal in $\mathcal{A}$.*

Proof. Let P be a relatively projective object and V any object in $\mathcal{A}$, we want to show that $V \otimes P$ and $P \otimes V$ are relatively projective. So take $0 \to X \xrightarrow{j} Y \xrightarrow{\sigma} Z \to 0$ an allowable short exact sequence in $\mathcal{A}$. By exactness of $\otimes$ [EGNO15, Prop. 4.2.1], the sequence

$$0 \longrightarrow X \otimes V^\vee \xrightarrow{j \otimes \mathrm{id}} Y \otimes V^\vee \xrightarrow{\sigma \otimes \mathrm{id}} Z \otimes V^\vee \longrightarrow 0$$

is exact and it is allowable due to Lemma (3.1). Hence by Lemma 2.5 the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P, X \otimes V^\vee) \xrightarrow{(j \otimes \mathrm{id})_*} \mathrm{Hom}_{\mathcal{A}}(P, Y \otimes V^\vee) \xrightarrow{(\sigma \otimes \mathrm{id})_*} \mathrm{Hom}_{\mathcal{A}}(P, Z \otimes V^\vee) \longrightarrow 0$$

is exact. Due to the adjunction $(- \otimes V) \dashv (- \otimes V^\vee)$ [EGNO15, Prop. 2.10.8] it follows that

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P \otimes V, X) \xrightarrow{j_*} \mathrm{Hom}_{\mathcal{A}}(P \otimes V, Y) \xrightarrow{\sigma_*} \mathrm{Hom}_{\mathcal{A}}(P \otimes V, Z) \longrightarrow 0$$

is exact. Thus by Lemma 2.5, $P \otimes V$ is relatively projective. The proof for $V \otimes P$ is similar. □

Corollary 3.3. *It holds*

$$\mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(X, Y) \cong \mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(X \otimes {}^\vee Y, \mathbf{1}), \quad \mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(X, Y) \cong \mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(\mathbf{1}, Y \otimes X^\vee)$$

where $\mathbf{1}$ is the tensor unit of $\mathcal{A}$.

Proof. We show the first isomorphism, the proof of the second being similar. Let $0 \longleftarrow X \xleftarrow{d_0} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} \ldots$ be a relatively projective resolution of X. By exactness of $\otimes$, the sequence

$$0 \longleftarrow X \otimes {}^\vee Y \xleftarrow{d_0 \otimes \mathrm{id}} P_0 \otimes {}^\vee Y \xleftarrow{d_1 \otimes \mathrm{id}} P_1 \otimes {}^\vee Y \xleftarrow{d_2 \otimes \mathrm{id}} \ldots$$

is exact. By the previous proposition, each $P_i \otimes {}^\vee Y$ is relatively projective and by Lemma 3.1, $d_i \otimes \mathrm{id}$ is allowable. Hence it is a relatively projective resolution of $X \otimes {}^\vee Y$. The result follows from the natural isomorphism $\mathrm{Hom}_{\mathcal{A}}(X, -) \cong \mathrm{Hom}_{\mathcal{A}}(X \otimes {}^\vee(-), \mathbf{1})$. □

Note that thanks to the second isomorphism, it is enough to know a relatively projective resolution of the unit object $\mathbf{1}$ to compute any Ext group.

In general it is difficult to determine a relatively projective resolution which is simple enough to make the computation of (the dimension of) the Ext groups feasible. The following proposition, which is a refinement of Corollary 2.10, replaces this problem by the computation of the first step of a relatively projective resolution and of certain Hom spaces:

Proposition 3.4. *Let $V \in \mathcal{A}$ and let*

$$0 \longrightarrow K \xrightarrow{j} P \xrightarrow{\pi} \mathbf{1} \longrightarrow 0, \qquad 0 \longrightarrow L \xrightarrow{i} Q \xrightarrow{p} V \longrightarrow 0$$

be allowable short exact sequences in $\mathcal{A}$ where P, Q are relatively projective objects. Then for all $n \geq 2$,

$$\mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(V, W) \cong \mathrm{Hom}_{\mathcal{A}}\big(K, W \otimes L^\vee \otimes (K^\vee)^{\otimes(n-2)}\big)/\mathrm{im}(j^*)$$

where j^* is the pullback $\mathrm{Hom}_{\mathcal{A}}\big(P, W \otimes L^\vee \otimes (K^\vee)^{\otimes(n-2)}\big) \to \mathrm{Hom}_{\mathcal{A}}\big(K, W \otimes L^\vee \otimes (K^\vee)^{\otimes(n-2)}\big)$.

Proof. We use Corollary 2.10 and Corollary 3.3 several times:

$$\begin{aligned}\mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(V, W) &\cong \mathrm{Ext}^{n-1}_{\mathcal{A},\mathcal{B}}(L, W) \cong \mathrm{Ext}^{n-1}_{\mathcal{A},\mathcal{B}}(\mathbf{1}, W \otimes L^\vee)\\ &\cong \mathrm{Ext}^{n-2}_{\mathcal{A},\mathcal{B}}(K, W \otimes L^\vee) \cong \mathrm{Ext}^{n-2}_{\mathcal{A},\mathcal{B}}(\mathbf{1}, W \otimes L^\vee \otimes K^\vee)\\ &\cong \ldots\\ &\cong \mathrm{Ext}^1_{\mathcal{A},\mathcal{B}}\big(K, W \otimes L^\vee \otimes (K^\vee)^{\otimes(n-3)}\big) \cong \mathrm{Ext}^1_{\mathcal{A},\mathcal{B}}\big(\mathbf{1}, W \otimes L^\vee \otimes (K^\vee)^{\otimes(n-2)}\big)\\ &\cong \mathrm{Hom}_{\mathcal{A}}\big(K, W \otimes L^\vee \otimes (K^\vee)^{\otimes(n-2)}\big)/\mathrm{im}(j^*).\end{aligned}$$

□

Corollary 3.5. *With the notations of the previous proposition and $M = W \otimes L^\vee \otimes (K^\vee)^{\otimes(n-2)}$, we have for $n \geq 2$*

$$\dim \mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(V, W) = \dim \mathrm{Hom}_{\mathcal{A}}(K, M) - \dim \mathrm{Hom}_{\mathcal{A}}(P, M) + \dim \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, M).$$

In particular

$$\dim \mathrm{Ext}^n_{\mathcal{A},\mathcal{B}}(\mathbf{1}, \mathbf{1}) = \dim \mathrm{Hom}_{\mathcal{A}}\big(K, (K^\vee)^{\otimes(n-1)}\big) - \dim \mathrm{Hom}_{\mathcal{A}}\big(P, (K^\vee)^{\otimes(n-1)}\big) + \dim \mathrm{Hom}_{\mathcal{A}}\big(\mathbf{1}, (K^\vee)^{\otimes(n-1)}\big).$$

Proof. Thanks to the exactness of

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, M) \xrightarrow{\pi^*} \mathrm{Hom}_{\mathcal{A}}(P, M) \xrightarrow{j^*} \mathrm{Hom}_{\mathcal{A}}(K, M)$$

we have $\dim \mathrm{im}(j^*) = \dim \mathrm{Hom}_{\mathcal{A}}(P, M) - \dim \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, M)$ and then the isomorphism in the previous proposition gives the result. □

In order to use relatively projective covers (Definition 2.12) we assume that $\mathcal{A} = A$-mod as a k-linear category, for some k-algebra A; this is in particular the case if $\mathcal{A}$ is a finite category. If P is the relatively projective cover $R_{\mathbf{1}}$ of $\mathbf{1}$ (which exists by Proposition 2.13), the second formula of Corollary 3.5 admits a special case:

Corollary 3.6. *Let $K_{\mathbf{1}} = \ker(p_{\mathbf{1}} : R_{\mathbf{1}} \twoheadrightarrow \mathbf{1})$. We have*

$$\dim \mathrm{Ext}^2_{\mathcal{A},\mathcal{B}}(\mathbf{1}, \mathbf{1}) = \dim \mathrm{Hom}_{\mathcal{A}}(K_{\mathbf{1}}, K_{\mathbf{1}}^\vee) - \dim \mathrm{Hom}_{\mathcal{A}}(R_{\mathbf{1}}, K_{\mathbf{1}}^\vee).$$

Proof. By definition there is the short exact sequence $0 \longrightarrow K_{\mathbf{1}} \longrightarrow R_{\mathbf{1}} \longrightarrow \mathbf{1} \longrightarrow 0$. Hence Corollary (3.5) gives

$$\dim \mathrm{Ext}^2_{\mathcal{A},\mathcal{B}}(\mathbf{1}, \mathbf{1}) = \dim \mathrm{Hom}_{\mathcal{A}}(K_{\mathbf{1}}, K_{\mathbf{1}}^\vee) - \dim \mathrm{Hom}_{\mathcal{A}}(R_{\mathbf{1}}, K_{\mathbf{1}}^\vee) + \dim \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, K_{\mathbf{1}}^\vee).$$

We want to show that the last term is equal to 0. Using Theorem 2.9 we obtain an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, \mathbf{1}) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(R_{\mathbf{1}}, \mathbf{1}) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(K_{\mathbf{1}}, \mathbf{1}) \longrightarrow \mathrm{Ext}^1_{\mathcal{A},\mathcal{B}}(\mathbf{1}, \mathbf{1}).$$

Since $\mathbf{1}$ is a simple object and k is algebraically closed, Schur's lemma applies and we have $\mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, \mathbf{1}) \cong k$. Since the functor $\mathcal{U}$ is monoidal we have that $\mathcal{U}(\mathbf{1}) \cong \mathbf{1}$ is simple and thus $\mathrm{Hom}_{\mathcal{A}}(R_{\mathbf{1}}, \mathbf{1}) \cong k$ by Proposition 2.13. Finally $\mathrm{Ext}^1_{\mathcal{A},\mathcal{B}}(\mathbf{1}, \mathbf{1}) \subset \mathrm{Ext}^1_{\mathcal{A}}(\mathbf{1}, \mathbf{1}) = 0$, where we used (14) and the fact that in a finite tensor category the unit object does not have non-trivial self-extensions [EGNO15, Theorem 4.4.1]. Hence the exact sequence becomes

$$0 \longrightarrow k \longrightarrow k \longrightarrow \mathrm{Hom}_{\mathcal{A}}(K_{\mathbf{1}}, \mathbf{1}) \longrightarrow 0$$

which forces $\mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, K_{\mathbf{1}}^\vee) \cong \mathrm{Hom}_{\mathcal{A}}(K_{\mathbf{1}}, \mathbf{1}) = 0$. □

3.2 Tensor product of algebras in braided tensor categories

In this section we generalize Example 2.4 to algebras in a braided tensor category $\mathcal{C}$. We assume that $\mathcal{C}$ is strict and we denote its braiding by c:

$$c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X.$$

Recall that a (unital) algebra (A, m_A, η_A) in $\mathcal{C}$ is an object $A \in \mathcal{C}$ together with morphisms $m_A \in \mathrm{Hom}_{\mathcal{C}}(A \otimes A, A)$ and $\eta_A \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A)$ satisfying the usual axioms. Similarly, an A-module (V, ρ_V) in $\mathcal{C}$ is an object $V \in \mathcal{C}$ together with a morphism $\rho_V \in \mathrm{Hom}_{\mathcal{C}}(A \otimes V, V)$ satisfying the usual axioms. A morphism of A-modules is defined in the obvious way and we get the category $A\text{-mod}_{\mathcal{C}}$ of A-modules in $\mathcal{C}$. Note that $(A, m_A) \in A\text{-mod}_{\mathcal{C}}$. If (B, m_B, η_B) is another algebra in $\mathcal{C}$, then $A \otimes B$ is again an algebra in $\mathcal{C}$ thanks to the braiding, with

$$m_{A\otimes B} = (m_A \otimes m_B)(\mathrm{id}_A \otimes c_{A,B} \otimes \mathrm{id}_B), \quad \eta_{A\otimes B} = \eta_A \otimes \eta_B.$$

We have the forgetful functor $\mathcal{U} : (A \otimes B)\text{-mod}_{\mathcal{C}} \to B\text{-mod}_{\mathcal{C}}$ induced by the pullback along the morphism $\eta_A \otimes \mathrm{id}_B \in \mathrm{Hom}_{\mathcal{C}}(B, A \otimes B)$, namely $\mathcal{U}(V, \rho_V) = \big(V, \rho_V(\eta_A \otimes \mathrm{id}_{B\otimes V})\big)$ on objects and $\mathcal{U}(f) = f$ on morphisms. It is clear that $\mathcal{U}$ is exact and faithful; moreover, it has a left adjoint $\mathcal{F}$ given by

$$\mathcal{F}(W) = A \boxtimes W, \qquad \mathcal{F}(f) = \mathrm{id}_A \boxtimes f$$

where the functor $\boxtimes : A\text{-mod}_{\mathcal{C}} \times B\text{-mod}_{\mathcal{C}} \to (A\otimes B)\text{-mod}_{\mathcal{C}}$ is defined on objects by $(V, \rho_V) \boxtimes (W, \rho_W) = (V \otimes W, \rho_{V\boxtimes W})$ with

$$\rho_{V\boxtimes W} : A \otimes B \otimes V \otimes W \xrightarrow{\mathrm{id}\otimes c_{B,V}\otimes \mathrm{id}} A \otimes V \otimes B \otimes W \xrightarrow{\rho_V \otimes \rho_W} V \otimes W$$

and on morphisms by $f \boxtimes g = f \otimes g$. Indeed, one checks easily that the adjunction property holds:

$$\begin{array}{rcl} \mathrm{Hom}_{(A\otimes B)\text{-mod}_{\mathcal{C}}}(A \boxtimes W, M) & \xrightarrow{\sim} & \mathrm{Hom}_{B\text{-mod}_{\mathcal{C}}}(W, \mathcal{U}(M)) \\ f & \mapsto & f(\eta_A \otimes \mathrm{id}_V) \\ \rho_M(\mathrm{id}_A \otimes \eta_B \otimes g) & \leftarrow\!\shortmid & g \end{array}$$

We will see that the following resolvent pairs of categories are related:

$$\begin{array}{ccc} A\text{-mod}_{\mathcal{C}} & \qquad & (A \otimes B)\text{-mod}_{\mathcal{C}} \\ \mathfrak{F} \uparrow \dashv \downarrow \mathfrak{U} & & \mathcal{F} \uparrow \dashv \downarrow \mathcal{U} \\ \mathcal{C} & & B\text{-mod}_{\mathcal{C}} \end{array}$$

In the former $\mathfrak{U}$ is the obvious forgetful functor and $\mathfrak{F}(X) = (A \otimes X, m_A \otimes \mathrm{id}_X)$.

Lemma 3.7. *Let $W \in B\text{-mod}_{\mathcal{C}}$. If P is relatively projective in $A\text{-mod}_{\mathcal{C}}$ then $P \boxtimes W$ is relatively projective in $(A \otimes B)\text{-mod}_{\mathcal{C}}$.*

Proof. By assumption P is a direct summand of $\mathfrak{F}(X)$ for some $X \in \mathcal{C}$. Let

$$X_{\mathrm{triv}} \otimes W \underset{\mathrm{def}}{=} \big(X \otimes W, (\mathrm{id}_X \otimes \rho_W)(c_{B,X} \otimes \mathrm{id}_W)\big). \tag{23}$$

It is easy to check that $X_{\mathrm{triv}} \otimes W \in B\text{-mod}_{\mathcal{C}}$. Observe that $\mathcal{F}(X_{\mathrm{triv}} \otimes W) = \mathfrak{F}(X) \boxtimes W$. Indeed:

$$\mathcal{F}(X_{\mathrm{triv}} \otimes W) = \big(A \otimes (X \otimes W), \big(m_A \otimes ((\mathrm{id}_X \otimes \rho_W)(c_{B,X} \otimes \mathrm{id}_W))\big)(\mathrm{id}_A \otimes c_{B,A} \otimes \mathrm{id}_{X\otimes W})\big),$$
$$\mathfrak{F}(X) \boxtimes W = \big((A \otimes X) \otimes W, (m_A \otimes \mathrm{id}_X \otimes \rho_W)(\mathrm{id}_A \otimes c_{B,A\otimes X} \otimes \mathrm{id}_W)\big)$$

and the two actions coincide thanks to the property of the braiding. It follows that $P \boxtimes W$ is a direct summand of $\mathcal{F}(X_{\mathrm{triv}} \otimes W)$ and hence is relatively projective by Lemma 2.1. $\square$

Proposition 3.8. *Let $V, V' \in A\text{-mod}_{\mathcal{C}}$ and $S, S' \in B\text{-mod}_{\mathcal{C}}$ be such that $\mathrm{Hom}_{B\text{-mod}_{\mathcal{C}}}(S, S') \cong k$. It holds*

$$\mathrm{Ext}^n_{(A\otimes B)\text{-mod}_{\mathcal{C}},\, B\text{-mod}_{\mathcal{C}}}(V \boxtimes S, V' \boxtimes S') \cong \mathrm{Ext}^n_{A\text{-mod}_{\mathcal{C}}, \mathcal{C}}(V, V').$$

Proof. Let $0 \longleftarrow V \xleftarrow{d_0} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} \ldots$ be a relatively projective resolution in $A\text{-mod}_{\mathcal{C}}$ and since d_i is allowable let $t_i \in \mathrm{Hom}_{\mathcal{C}}(P_{i-1}, P_i)$ be such that $d_i\, t_i\, d_i = d_i$ (we write d_i instead of $\mathfrak{U}(d_i)$ because $\mathfrak{U}$ is the identity on morphisms). By the previous lemma

$$0 \longleftarrow V \boxtimes S \xleftarrow{d_0 \otimes \mathrm{id}_S} P_0 \boxtimes S \xleftarrow{d_1 \otimes \mathrm{id}_S} P_1 \boxtimes S \xleftarrow{d_2 \otimes \mathrm{id}_S} \ldots$$

is an exact sequence of relatively projective modules in $(A\otimes B)\text{-mod}_{\mathcal{C}}$. Let us show that it is allowable, and thus a relatively projective resolution. Note that $\mathcal{U}(P_i \boxtimes S) = (P_i)_{\mathrm{triv}} \otimes S$ with the notation introduced in (23). For $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ it is easy to see that $f \otimes \mathrm{id}_S \in \mathrm{Hom}_{B\text{-mod}_{\mathcal{C}}}(X_{\mathrm{triv}} \otimes S, Y_{\mathrm{triv}} \otimes S)$. Hence $t_i \otimes \mathrm{id}_S \in \mathrm{Hom}_{B\text{-mod}_{\mathcal{C}}}(\mathcal{U}(P_{i-1} \boxtimes S), \mathcal{U}(P_i \boxtimes S))$ and it holds $(d_i \otimes \mathrm{id}_S)(t_i \otimes \mathrm{id}_S)(d_i \otimes \mathrm{id}_S) = (d_i \otimes \mathrm{id}_S)$, which shows that $d_i \otimes \mathrm{id}_S$ is allowable (again, we do not write $\mathcal{U}(d_i \otimes \mathrm{id}_S)$ because $\mathcal{U}$ is the identity on morphisms). The claims follows from

$$\mathrm{Hom}_{(A\otimes B)\text{-mod}_{\mathcal{C}}}(P_i \boxtimes S, V' \boxtimes S') \cong \mathrm{Hom}_{A\text{-mod}_{\mathcal{C}}}(P_i, V') \otimes \mathrm{Hom}_{B\text{-mod}_{\mathcal{C}}}(S, S') \cong \mathrm{Hom}_{A\text{-mod}_{\mathcal{C}}}(P_i, V')$$

and the fact that these isomorphisms commutes with the differentials. □

Remark 3.9. Let H be a finite-dimensional quasi-triangular Hopf algebra (in Vect_k) and take $\mathcal{C} = H\text{-mod}$. Let A be a finite-dimensional H-module-algebra or equivalently an algebra in $\mathcal{C}$. Then it holds $A\text{-mod}_{\mathcal{C}} \cong (A\#H)\text{-mod}$, where $A\#H$ is the smash product of A and H. Let B be another finite-dimensional H-module algebra and let $A \widetilde{\otimes} B$ be their braided tensor product (in other words, their tensor product in $\mathcal{C}$). Then the proposition is rewritten as

$$\mathrm{Ext}^n_{(A\widetilde{\otimes}B)\#H, B\#H}(V \boxtimes S, V' \boxtimes S') \cong \mathrm{Ext}^n_{A\#H, H}(V, V').$$

Corollary 3.10. *Under the assumptions of Proposition of 3.8 and if the category $\mathcal{C}$ is semisimple, it holds*

$$\mathrm{Ext}^n_{(A\otimes B)\text{-mod}_{\mathcal{C}}, B\text{-mod}_{\mathcal{C}}}(V \boxtimes S, V' \boxtimes S') \cong \mathrm{Ext}^n_{A\text{-mod}_{\mathcal{C}}}(V, V').$$

Proof. Since $\mathcal{C}$ is semisimple, we know from Example 2.3 that $\mathrm{Ext}^n_{A\text{-mod}_{\mathcal{C}}, \mathcal{C}}(V, V') = \mathrm{Ext}^n_{A\text{-mod}_{\mathcal{C}}}(V, V')$ and the claim follows from Proposition 3.8. □

4 Davydov–Yetter cohomology and relative Ext groups

Davydov–Yetter cohomology classifies infinitesimal deformations of tensor structures, as reviewed in the Introduction. Here we will work in the setting of finite tensor categories and with the version of this cohomology theory with coefficients introduced in [GHS19].

A k-linear category is finite if it is equivalent as a k-linear category to the category of finite-dimensional modules over some finite-dimensional k-algebra. Here we assume that the field k is algebraically closed.

Recall from [ML71, Chap. IX, §6] the notion of a coend. Let $\mathcal{X}, \mathcal{Y}$ be categories and let $T : \mathcal{X}^{\mathrm{op}} \times \mathcal{X} \to \mathcal{Y}$ be a bifunctor. The coend of T, if it exists, is a universal pair (C, i), where C is an object in $\mathcal{Y}$ and $i = (i_X : T(X, X) \to C)_{X\in\mathcal{X}}$ is a dinatural transformation. This means that for any dinatural transformation $\psi_X : T(X, X) \to V$, there exists a unique morphism $\Psi \in \mathrm{Hom}_{\mathcal{Y}}(C, V)$ such that $\psi_X = \Psi\, i_X$ for all $X \in \mathcal{X}$. The object C is denoted by $\int^{X\in\mathcal{X}} T(X, X)$. Note in particular that morphisms $f, g \in \mathrm{Hom}_{\mathcal{Y}}(C, W)$ (where W is any object of $\mathcal{Y}$) are equal if, and only if, $f\, i_X = g\, i_X$ for all $X \in \mathcal{X}$.

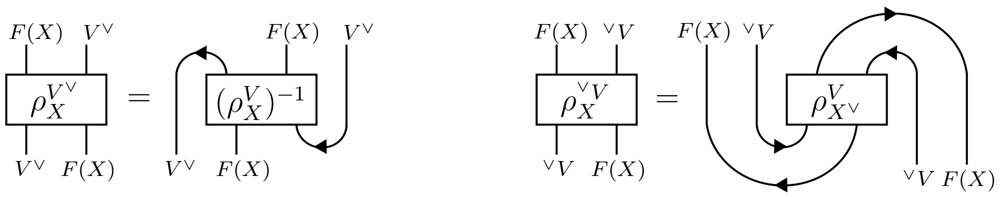
In this section, $\mathcal{C}$ and $\mathcal{D}$ denote finite tensor categories, assumed to be strict for simplicity.

4.1 The centralizer of a functor and the associated adjunction

Let $F : \mathcal{C} \to \mathcal{D}$ be a k-linear monoidal functor, assumed to be strict for simplicity. Recall that there exists a monoidal natural isomorphism $d_X : F(X^\vee) \xrightarrow{\sim} F(X)^\vee$ (see *e.g.* [NS07, Lemma 1.1]); in the sequel for simplicity we identify $F(X)^\vee$ with $F(X^\vee)$.

- A half-braiding relative to F is a natural isomorphism $\rho^V : V \otimes F(?) \to F(?) \otimes V$ such that
$$\rho^V_{X\otimes Y} = (\mathrm{id}_{F(X)} \otimes \rho^V_Y)(\rho^V_X \otimes \mathrm{id}_{F(Y)}) \tag{24}$$
for all $X, Y \in \mathcal{C}$.
- The centralizer $\mathcal{Z}(F)$ of the functor F is the category whose objects are pairs $\mathsf{V} = (V, \rho^V)$ where $V \in \mathcal{D}$ and ρ^V is a half-braiding relative to F and whose morphisms $f : (V, \rho^V) \to (W, \rho^W)$ are morphisms $f \in \mathrm{Hom}_{\mathcal{D}}(V, W)$ such that the diagram
$$\begin{array}{ccc} V \otimes F(X) & \xrightarrow{\rho^V_X} & F(X) \otimes V \\ {\scriptstyle f\otimes \mathrm{id}_{F(X)}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{F(X)}\otimes f} \\ W \otimes F(X) & \xrightarrow[\rho^W_X]{} & F(X) \otimes W \end{array}$$
commutes for all $X \in \mathcal{C}$.

$\mathcal{Z}(F)$ is a (strict) tensor category [Maj91], [Shi17b, §3.2] with tensor product
$$(V, \rho^V) \otimes (W, \rho^W) = \big(V \otimes W,\ \rho^{V\otimes W} = (\rho^V \otimes \mathrm{id}_W)(\mathrm{id}_V \otimes \rho^W)\big)$$
and with left and right duals given by $(V, \rho^V)^\vee = (V^\vee, \rho^{V^\vee})$, ${}^\vee(V, \rho^V) = ({}^\vee V, \rho^{{}^\vee V})$, where



(25)

In particular, if $F = \mathrm{Id}_{\mathcal{C}}$ is the identity functor, then $\mathcal{Z}(\mathrm{Id}_{\mathcal{C}})$ is $\mathcal{Z}(\mathcal{C})$, the usual Drinfeld center of $\mathcal{C}$.

Assumption: We assume from now on that F is an exact functor.

Under this assumption, $\mathcal{Z}(F)$ is a *finite* tensor category, see [Shi17b, §3.3].

Let $\mathcal{U} : \mathcal{Z}(F) \to \mathcal{D}$ be the forgetful functor defined by $\mathcal{U}(V, \rho^V) = V$, $\mathcal{U}(f) = f$. Let us recall the explicit construction of its left adjoint $\mathcal{F} : \mathcal{D} \to \mathcal{Z}(F)$ ([DS07, BV12, Shi17b], here we use the conventions of [GHS19, §3.3]). First, consider
$$Z_F(V) = \int^{X\in\mathcal{C}} F(X)^\vee \otimes V \otimes F(X) \tag{26}$$
and denote by
$$i_X(V) : F(X)^\vee \otimes V \otimes F(X) \to Z_F(V)$$
the associated universal dinatural transformation. The coend $Z_F(V)$ exists for all V by exactness of F, thanks to [KL11, Cor. 5.1.8.] or [Shi17, Th. 3.6]. Moreover, Z_F gives a functor $\mathcal{D} \to \mathcal{D}$; for a morphism $f : V \to W$, $Z_F(f)$ is defined to be the unique morphism such that
$$Z_F(f)\, i_X(V) = i_X(W)\big(\mathrm{id}_{F(X)^\vee} \otimes f \otimes \mathrm{id}_{F(X)}\big) \tag{27}$$

for all $X \in \mathcal{C}$. Here we use the universality of the coend, the right hand side of (27) being dinatural in X. In the sequel we often implicitly use universality of the coend to define morphisms. For any $Y \in \mathcal{C}$, we define $\rho_Y^{Z_F(V)} : Z_F(V) \otimes F(Y) \to F(Y) \otimes Z_F(V)$ as the unique morphism in $\mathcal{D}$ such that

$$\rho_Y^{Z_F(V)}\big(i_X(V) \otimes \mathrm{id}_{F(Y)}\big) = \big(\mathrm{id}_{F(Y)} \otimes i_{X\otimes Y}(V)\big)\big(\mathrm{coev}_{F(Y)} \otimes \mathrm{id}_{F(X)^\vee \otimes V \otimes F(X) \otimes F(Y)}\big) \tag{28}$$

for all $X \in \mathcal{C}$ (again we use the universality of the coend). Then one can show that $\rho^{Z_F(V)}$ is a half-braiding relative to F and that the left adjoint to $\mathcal{U}$ is given by

$$\mathcal{F}(V) = \big(Z_F(V), \rho^{Z_F(V)}\big).$$

The forgetful functor $\mathcal{U}$ is clearly additive, exact and faithful, and it has the left adjoint $\mathcal{F}$, so we have a resolvent pair of categories:

$$\mathcal{Z}(F) \underset{\mathcal{U}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{D} \quad (\mathcal{F} \dashv \mathcal{U}) \tag{29}$$

Since $\mathcal{U} : \mathcal{Z}(F) \to \mathcal{D}$ is obviously monoidal, Proposition 3.2 applies:

Corollary 4.1. *The relatively projective objects in $\mathcal{Z}(F)$ form a tensor ideal.*

In the sequel we will use intensively the bar resolution (20) (or equivalently (7)) for the adjunction (29), so in the rest of this section we describe it in detail. First, let G be the comonad on $\mathcal{Z}(F)$ for this adjunction:

$$G = \mathcal{F}\mathcal{U} : \mathcal{Z}(F) \to \mathcal{Z}(F), \qquad G(\mathsf{V}) = \big(Z_F(V), \rho^{Z_F(V)}\big), \qquad G(f) = Z_F(f) \tag{30}$$

where $\mathsf{V} = (V, \rho^V) \in \mathcal{Z}(F)$ and f is a morphism (recall that a morphism $f \in \mathrm{Hom}_{\mathcal{Z}(F)}(\mathsf{V}, \mathsf{W})$ is just a morphism $f \in \mathrm{Hom}_{\mathcal{D}}(V, W)$ which commutes with the half-braidings ρ^V, ρ^W). For any object V, let $\varepsilon_{\mathsf{V}} \in \mathrm{Hom}_{\mathcal{D}}\big(Z_F(V), V\big)$ be the unique morphism such that

$$\varepsilon_{\mathsf{V}}\, i_X(V) = \big(\mathrm{ev}_{F(X)} \otimes \mathrm{id}_V\big)\big(\mathrm{id}_{F(X)^\vee} \otimes \rho^V_X\big). \tag{31}$$

One can show that $\varepsilon_{\mathsf{V}} \in \mathrm{Hom}_{\mathcal{Z}(F)}\big(G(\mathsf{V}), \mathsf{V}\big)$ and that $\varepsilon : G \to \mathrm{Id}$ is the counit of G.

The chain objects of the bar resolution of V are

$$\mathrm{Bar}^n_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{V}) = \mathrm{Bar}^n_G(\mathsf{V}) = G^{n+1}(\mathsf{V}) = \left(Z_F^{n+1}(V), \rho^{Z_F^{n+1}(V)}\right)$$

for $\mathsf{V} = (V, \rho^V) \in \mathcal{Z}(F)$, and where the first equality simply reminds the two possible notations. Using the Fubini theorem for coends [ML71, Prop. IX.8], we have

$$Z_F^n(V) = \int^{(X_1,\ldots,X_n)\in\mathcal{C}^n} F(X_n)^\vee \otimes \ldots \otimes F(X_1)^\vee \otimes V \otimes F(X_1) \otimes \ldots \otimes F(X_n)$$

with the universal dinatural transformation $i^{(n)}_{X_1,\ldots,X_n}(V)$ defined inductively by

$$\begin{aligned} i^{(n)}_{X_1,\ldots,X_n}(V) &= i_{X_n}\big(Z_F^{n-1}(V)\big)\big(\mathrm{id}_{F(X_n)^\vee} \otimes i^{(n-1)}_{X_1,\ldots,X_{n-1}}(V) \otimes \mathrm{id}_{F(X_n)}\big) \\ &= i^{(n-1)}_{X_2,\ldots,X_n}\big(Z_F(V)\big)\big(\mathrm{id}_{F(X_n)^\vee \otimes \ldots \otimes F(X_2)^\vee} \otimes i_{X_1}(V) \otimes \mathrm{id}_{F(X_2)\otimes\ldots\otimes F(X_n)}\big). \end{aligned}$$

We can characterize the bar differential $d_n^{\mathsf{V}} : \mathrm{Bar}^n_G(\mathsf{V}) \to \mathrm{Bar}^{n-1}_G(\mathsf{V})$ thanks to the universal dinatural transformations:

Lemma 4.2. *For the comonad G on $\mathcal{Z}(F)$, the differential of the bar resolution for an object $\mathsf{V} = (V, \rho^V)$ satisfies*

$$d_n^{\mathsf{V}}\, i^{(n+1)}_{X_1,\ldots,X_{n+1}}(V) = i^{(n)}_{X_2,\ldots,X_{n+1}}(V)\big(\mathrm{id}_{F(X_{n+1})^\vee\otimes\ldots\otimes F(X_2)^\vee} \otimes \varepsilon_{\mathsf{V}} i_{X_1}(V) \otimes \mathrm{id}_{F(X_2)\otimes\ldots\otimes F(X_{n+1})}\big) \\ + \sum_{j=1}^{n} (-1)^j\, i^{(n)}_{X_1,\ldots,X_j\otimes X_{j+1},\ldots,X_{n+1}}(V)$$

where ε_{V} is the counit of G, defined in (31).

Proof. Recall the general definition of the bar differential in (21). The result is obtained by a computation based on the definitions of the counit and of the value of G on a morphism (see (30) and (27)); we skip the details. □

Let $\mathsf{V} = (V, \rho^V), \mathsf{W} = (W, \rho^W) \in \mathcal{Z}(F)$ and for $n \in \mathbb{N}$ let

$$\Psi_n^{\mathsf{V},\mathsf{W}} \in \mathrm{Hom}_{\mathcal{D}}\big(Z_F^n(V\otimes W), Z_F^n(V)\otimes W\big)$$

be the unique morphism such that $\Psi_n^{\mathsf{V},\mathsf{W}}\, i^{(n)}_{X_1,\ldots,X_n}(V\otimes W)$ is equal to the following dinatural transformation

$$F(X_n)^\vee \otimes \ldots \otimes F(X_1)^\vee \otimes V \otimes W \otimes F(X_1) \otimes \ldots \otimes F(X_n) \\ \xrightarrow{\mathrm{id}_{F(X_n)^\vee\otimes\ldots\otimes F(X_1)^\vee\otimes V}\otimes\rho^W_{X_1\otimes\ldots\otimes X_n}} F(X_n)^\vee \otimes \ldots \otimes F(X_1)^\vee \otimes V \otimes F(X_1) \otimes \ldots \otimes F(X_n) \otimes W \\ \xrightarrow{i^{(n)}_{X_1,\ldots,X_n}(V)\otimes \mathrm{id}_W} Z_F(V) \otimes W.$$

Proposition 4.3. *For each n, $\Psi_n^{\mathsf{V},\mathsf{W}}$ is an isomorphism in $\mathcal{Z}(F)$:*

$$\Psi_n^{\mathsf{V},\mathsf{W}} : G^n(\mathsf{V}\otimes\mathsf{W}) \xrightarrow{\sim} G^n(\mathsf{V})\otimes \mathsf{W}.$$

Moreover, the family $\big(\Psi_n^{\mathsf{V},\mathsf{W}}\big)_{n\in\mathbb{N}}$ is an isomorphism of complexes from $\mathrm{Bar}^\bullet_G(\mathsf{V}\otimes\mathsf{W})$ to $\mathrm{Bar}^\bullet_G(\mathsf{V})\otimes\mathsf{W}$.

Proof. Let us first show that $\Psi_1^{\mathsf{V},\mathsf{W}}$ is an isomorphism in $\mathcal{Z}(F)$. The inverse of $\Psi_1^{\mathsf{V},\mathsf{W}}$ is simply the unique morphism in $\mathrm{Hom}_{\mathcal{D}}\big(Z_F(V)\otimes W, Z_F(V\otimes W)\big)$ determined by the following dinatural transformation:

$$F(X)^\vee \otimes V \otimes F(X) \otimes W \xrightarrow{\mathrm{id}_{F(X)^\vee}\otimes(\rho^W_X)^{-1}} F(X)^\vee \otimes V \otimes W \otimes F(X) \xrightarrow{i_X(V\otimes W)} Z_F(V\otimes W).$$

A morphism in $\mathcal{Z}(F)$ is a morphism in $\mathcal{D}$ which commutes with the half-braidings, so we have to check that the following diagram is commuative:

$$\begin{array}{ccc} Z_F(V\otimes W)\otimes F(Y) & \xrightarrow{\rho_Y^{Z_F(V\otimes W)}} & F(Y)\otimes Z_F(V\otimes W) \\ \downarrow{\scriptstyle \Psi_1^{\mathsf{V},\mathsf{W}}\otimes\mathrm{id}_{F(Y)}} & & \downarrow{\scriptstyle \mathrm{id}_{F(Y)}\otimes\Psi_1^{\mathsf{V},\mathsf{W}}} \\ Z_F(V)\otimes W\otimes F(Y) & \xrightarrow[\rho_Y^{Z_F(V)\otimes W}]{} & F(Y)\otimes Z_F(V)\otimes W \end{array} \tag{32}$$

for all $Y \in \mathcal{C}$. The computation is displayed in Figure 1. The first equality uses the definition of $\Psi_1^{\mathsf{V},\mathsf{W}}$ and $\rho_Y^{Z_F(V)\otimes W} = \big(\rho_Y^{Z_F(V)}\otimes\mathrm{id}_W\big)\big(\mathrm{id}_{Z_F(V)}\otimes\rho^W_Y\big)$, the second equality uses the definition of $\rho^{Z_F(V)}$ (see (28)) and $(\mathrm{id}_{F(X)}\otimes\rho^W_Y)(\rho^W_X\otimes\mathrm{id}_{F(Y)}) = \rho^W_{X\otimes Y}$, and the third and fourth equalities are the definitions of $\Psi_1^{\mathsf{V},\mathsf{W}}$ and $\rho^{Z_F(V\otimes W)}$ respectively. Since this holds for any $X \in \mathcal{C}$, the diagram is commutative. Now for general n, it is not difficult to show that $\Psi_{n+1}^{\mathsf{V},\mathsf{W}}$ can be constructed as follows:

$$Z_F^{n+1}(V\otimes W) \xrightarrow{Z_F(\Psi_n^{\mathsf{V},\mathsf{W}})} Z_F\big(Z_F^n(V)\otimes W\big) \xrightarrow{\Psi_1^{Z_F^n(V),W}} Z_F^{n+1}(V)\otimes W.$$

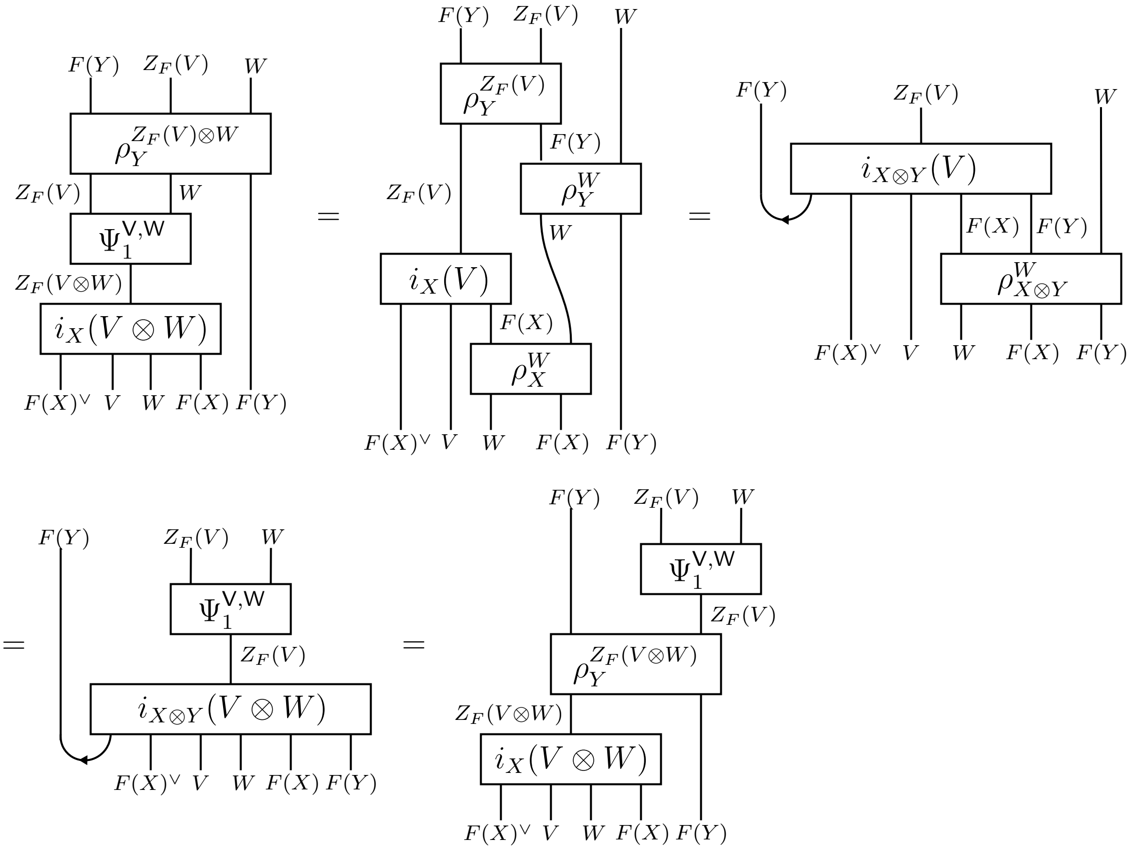


Figure 1: Proof of the commutation of the diagram (32).

Hence by induction we obtain that $\Psi_n^{\mathsf{V},\mathsf{W}}$ is an isomorphism in $\mathcal{Z}(F)$ for all n. To prove the last claim, we must check that

$$\begin{array}{ccc} G^{n+1}(\mathsf{V}\otimes\mathsf{W}) & \xrightarrow{d_n^{\mathsf{V}\otimes\mathsf{W}}} & G^n(\mathsf{V}\otimes\mathsf{W}) \\ \downarrow{\scriptstyle \Psi_{n+1}^{\mathsf{V},\mathsf{W}}} & & \downarrow{\scriptstyle \Psi_n^{\mathsf{V},\mathsf{W}}} \\ G^{n+1}(\mathsf{V})\otimes\mathsf{W} & \xrightarrow{d_n^{\mathsf{V}}\otimes\mathrm{id}_{\mathsf{W}}} & G^n(\mathsf{V})\otimes\mathsf{W} \end{array}$$

is commutative, where $d^{\mathsf{V}\otimes\mathsf{W}}, d^{\mathsf{V}}$ are the bar differentials for $\mathsf{V}\otimes\mathsf{W}$ and V respectively. This follows from a straightforward computation using Lemma 4.2 and is left to the reader. □

Remark 4.4. One can similarly construct isomorphisms $G^n(\mathsf{V}\otimes\mathsf{W}) \xrightarrow{\sim} \mathsf{V}\otimes G^n(\mathsf{W})$ which yield an isomorphism of complexes from $\mathrm{Bar}_G^\bullet(\mathsf{V}\otimes\mathsf{W})$ to $\mathsf{V}\otimes\mathrm{Bar}_G^\bullet(\mathsf{W})$.

Recall that for coefficients V,W, the complex associated to the bar resolution of V is

$$\mathrm{Bar}^n_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{V},\mathsf{W}) = \mathrm{Hom}_{\mathcal{Z}(F)}\big(\mathrm{Bar}^n_G(\mathsf{V}),\mathsf{W}\big) = \mathrm{Hom}_{\mathcal{Z}(F)}\big(G^{n+1}(\mathsf{V}),\mathsf{W}\big).$$

The isomorphism in Proposition 4.3 allows us to write down explicitly the first isomorphism from Corollary 3.3 (and a similar description applies for the second). Indeed,

$$\mathrm{Hom}_{\mathcal{Z}(F)}\big(G^{n+1}(\mathsf{V}),\mathsf{W}\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{Z}(F)}\big(G^{n+1}(\mathsf{V})\otimes{}^\vee\mathsf{W},\mathbf{1}\big) \xrightarrow{(\Psi_{n+1}^{\mathsf{V},{}^\vee\mathsf{W}})^*} \mathrm{Hom}_{\mathcal{Z}(F)}\big(G^{n+1}(\mathsf{V}\otimes{}^\vee\mathsf{W}),\mathbf{1}\big)$$

is an isomorphism and due to the naturality of the first arrow and to Proposition 4.3 it yields an isomorphism of complexes

$$\begin{array}{rcl} \mathrm{Bar}^n_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{V},\mathsf{W}) & \xrightarrow{\sim} & \mathrm{Bar}^n_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{V}\otimes{}^\vee\mathsf{W},\mathbf{1}) \\ \alpha & \mapsto & \widetilde{\mathrm{ev}}_W(\alpha\otimes\mathrm{id}_{{}^\vee W})\Psi_{n+1}^{\mathsf{V},{}^\vee\mathsf{W}} \end{array} \tag{33}$$

which in turn gives an isomorphism $\mathrm{Ext}^\bullet_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{V},\mathsf{W}) \xrightarrow{\sim} \mathrm{Ext}^\bullet_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{V}\otimes{}^\vee\mathsf{W},\mathbf{1})$.

4.2 Relation to Davydov-Yetter cohomology

Recall first the following notions:

- Let $\mathsf{U} = (U, \rho^U), \mathsf{V} = (V, \rho^V) \in \mathcal{Z}(F)$. The n-th Davydov–Yetter cochain space of F with coefficients U, V, denoted by $C^n_{\mathrm{DY}}(F, \mathsf{U}, \mathsf{V})$, is the space of all natural transformations f of the form
$$f_{X_1,\ldots,X_n} : U \otimes F(X_1) \otimes \ldots \otimes F(X_n) \to F(X_1) \otimes \ldots \otimes F(X_n) \otimes V.$$
The case $n = 0$ is $C^0_{\mathrm{DY}}(F, \mathsf{U}, \mathsf{V}) = \mathrm{Hom}_{\mathcal{D}}(U, V)$.

- The Davydov–Yetter differential $\delta^n : C^n_{\mathrm{DY}}(F, \mathsf{U}, \mathsf{V}) \to C^{n+1}_{\mathrm{DY}}(F, \mathsf{U}, \mathsf{V})$ is defined by
$$\begin{aligned}\delta^n(f)_{X_0,\ldots,X_n} =& (\mathrm{id}_{F(X_0)} \otimes f_{X_1,\ldots,X_n})(\rho^U_{X_0} \otimes \mathrm{id}_{F(X_1)\otimes\ldots\otimes F(X_n)}) \\ &+ \sum_{i=1}^{n} (-1)^i f_{X_0,\ldots,X_{i-1}\otimes X_i,\ldots,X_n} \\ &+ (-1)^{n+1}(\mathrm{id}_{F(X_0)\otimes\ldots\otimes F(X_{n-1})} \otimes \rho^V_{X_n})(f_{X_0,\ldots,X_{n-1}} \otimes \mathrm{id}_{F(X_n)}).\end{aligned}$$
The case $n = 0$ is $\delta^0(f)_{X_0} = (\mathrm{id}_{F(X_0)} \otimes f)\rho^U_{X_0} - \rho^V_{X_0}(f \otimes \mathrm{id}_{F(X_0)})$.

- The n-th Davydov–Yetter cohomology[4] group of F with coefficients U, V, denoted by $H^n_{\mathrm{DY}}(F, \mathsf{U}, \mathsf{V})$, is $\ker(\delta^n)/\mathrm{im}(\delta^{n-1})$. The case $n = 0$ is $H^0_{\mathrm{DY}}(F, \mathsf{U}, \mathsf{V}) = \ker(\delta^0)$.

- If coefficients are trivial ($\mathsf{U} = \mathsf{V} = \mathbf{1}$), then we write simply $C^n_{\mathrm{DY}}(F)$ and $H^n_{\mathrm{DY}}(F)$. If the functor F is the identity functor $\mathrm{Id}_{\mathcal{C}}$, then we write $C^n_{\mathrm{DY}}(\mathcal{C}, \mathsf{U}, \mathsf{V})$ and $H^n_{\mathrm{DY}}(\mathcal{C}, \mathsf{U}, \mathsf{V})$. Finally, $C^n_{\mathrm{DY}}(\mathcal{C})$ and $H^n_{\mathrm{DY}}(\mathcal{C})$ mean respectively the DY cochains and DY cohomology of the identity functor with trivial coefficients.

Let $G = \mathcal{F}\mathcal{U}$ be the comonad on $\mathcal{Z}(F)$ associated to the adjunction (29) discussed in the previous subsection and recall that the bar complex of a comonad is defined in (22). We have the key result:

Theorem 4.5. [GHS19, Th. 3.11] *Let $\mathsf{U} = (U, \rho^U), \mathsf{V} = (V, \rho^V) \in \mathcal{Z}(F)$. Under the previous assumptions, there is an isomorphism of cochain complexes*
$$C^\bullet_{\mathrm{DY}}(F, \mathsf{U}, \mathsf{V}) \cong \mathrm{Bar}^\bullet_G\big(\mathsf{U}, \mathrm{Hom}_{\mathcal{Z}(F)}(?, \mathsf{V})\big).$$
It follows that
$$H^n_{\mathrm{DY}}(F, \mathsf{U}, \mathsf{V}) \cong H^n_G\big(\mathsf{U}, \mathrm{Hom}_{\mathcal{Z}(F)}(?, \mathsf{V})\big).$$

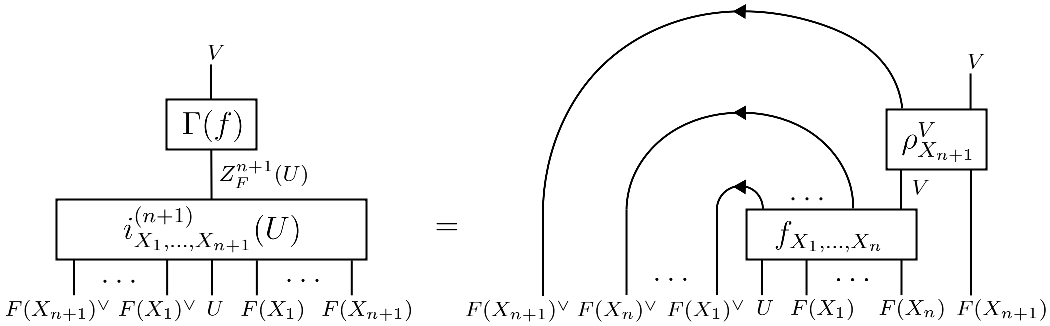
Now, since the adjunction (29) is a resolvent pair of categories, this theorem can be restated as follows, thanks to Proposition 2.17:

Corollary 4.6. *With the same notations, there is an isomorphism of cochain complexes*
$$C^\bullet_{\mathrm{DY}}(F, \mathsf{U}, \mathsf{V}) \cong \mathrm{Bar}^\bullet_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{U}, \mathsf{V}).$$
It follows that
$$H^n_{\mathrm{DY}}(F, \mathsf{U}, \mathsf{V}) \cong \mathrm{Ext}^n_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{U}, \mathsf{V}).$$

Since we will need it in the sequel, we recall from [GHS19] the explicit form of the isomorphism of cochain complexes in Corollary 4.6, which we denote by Γ:
$$\forall\, n,\ \ \Gamma : C^n_{\mathrm{DY}}(F, \mathsf{U}, \mathsf{V}) \xrightarrow{\sim} \mathrm{Bar}^n_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{U}, \mathsf{V}) = \mathrm{Hom}_{\mathcal{Z}(F)}\big(G^{n+1}(\mathsf{U}), \mathsf{V}\big). \tag{34}$$

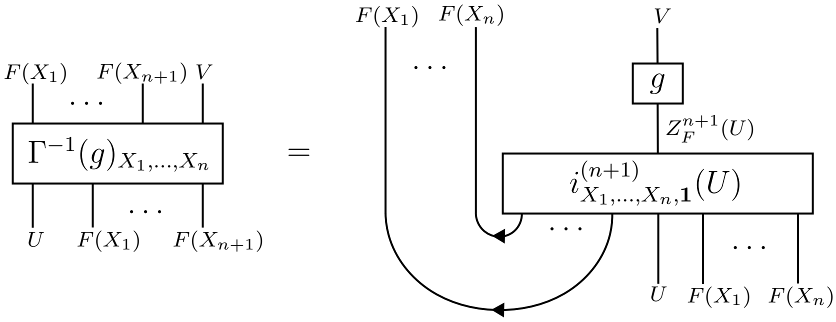
[4] We will also use the shortand "DY cohomology".

For $f \in C^n_{\mathrm{DY}}(F, \mathsf{U}, \mathsf{V})$, $\Gamma(f) : G^{n+1}(\mathsf{U}) \to \mathsf{V}$ is the unique morphism such that



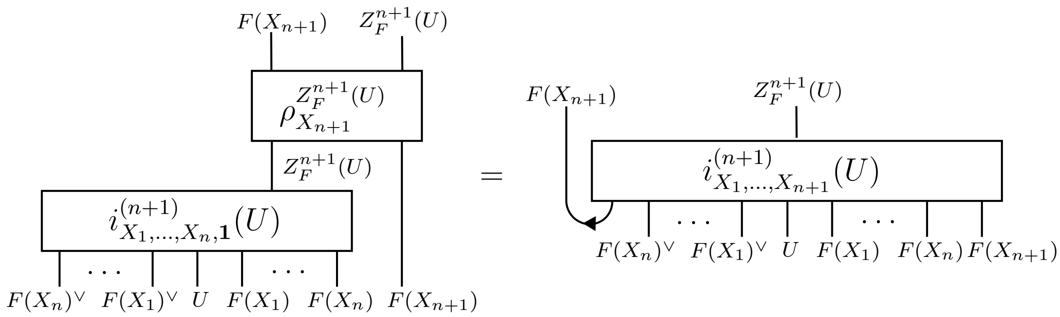
(35)

where as before $\mathsf{U} = (U, \rho^U)$ and $\mathsf{V} = (V, \rho^V)$. Conversely, for $g \in \mathrm{Hom}_{\mathcal{Z}(F)}\big(G^{n+1}(\mathsf{U}), \mathsf{V}\big)$, the components of the natural transformation $\Gamma^{-1}(g)$ are given by



(36)

where $\mathbf{1}$ is the tensor unit of $\mathcal{C}$. To show that Γ and Γ^{-1} are inverse to each other, one uses the following equality, which is an easy consequence of the definitions and which will also be used in the next section:



(37)

4.3 First consequences

A nice consequence of Corollary 4.6 is:

Proposition 4.7. *For every exact tensor functor $F : \mathcal{C} \to \mathcal{D}$, it holds $H^1_{\mathrm{DY}}(F) = 0$.*

Proof. This is due to (14):

$$H^1_{\mathrm{DY}}(F) \cong \mathrm{Ext}^1_{\mathcal{Z}(F),\mathcal{D}}(\mathbf{1}, \mathbf{1}) \subset \mathrm{Ext}^1_{\mathcal{Z}(F)}(\mathbf{1}, \mathbf{1}) = 0$$

where the last equality is the fact that in a finite tensor category the unit object does not have non-trivial self-extensions [EGNO15, Theorem 4.4.1], and we already recalled in the previous section that $\mathcal{Z}(F)$ is a finite tensor category when F is exact. □

Remark 4.8. A cocycle in $C^1_{\mathrm{DY}}(F)$ is a F-derivation, that is an element $f \in \mathrm{Nat}(F, F)$ such that $f_{X \otimes Y} = f_X \otimes \mathrm{id}_{F(Y)} + \mathrm{id}_{F(X)} \otimes f_Y$. For trivial coefficients we have $\delta_0 = 0$, and thus $H^1_{\mathrm{DY}}(F) = \ker(\delta_1)$. Hence the proposition means that non-zero F-derivations do not exist. Note that if $\mathcal{C} = H$-mod and if $F = U : \mathcal{C} \to \mathrm{Vect}_k$ is the forgetful functor, this means that a finite-dimensional Hopf algebra H does not have non-zero primitive elements, thanks to the description of the DY complex for U given in (54) in section 5 below. This known fact is actually true for finite-dimensional bialgebras [EGNO15, Corollary 5.9.1].

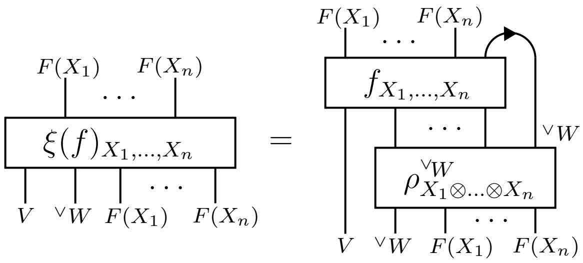
Next, combining Corollary 4.6 and Corollary 3.3, we obtain

$$H^n_{\mathrm{DY}}(F; \mathsf{V}, \mathsf{W}) \cong H^n_{\mathrm{DY}}(F; \mathsf{V} \otimes {}^\vee\mathsf{W}, \mathbf{1}), \quad H^n_{\mathrm{DY}}(F; \mathsf{V}, \mathsf{W}) \cong H^n_{\mathrm{DY}}(F; \mathbf{1}, \mathsf{W} \otimes \mathsf{V}^\vee) \tag{38}$$

but even better we can transport the explicit isomorphism from (33) through Γ, which defines

$$\forall n,\ \xi : C^n_{\mathrm{DY}}(F; \mathsf{V}, \mathsf{W}) \xrightarrow{\Gamma} \mathrm{Bar}^n_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{V}, \mathsf{W}) \xrightarrow{(33)} \mathrm{Bar}^n_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{V} \otimes {}^\vee\mathsf{W}, \mathbf{1}) \xrightarrow{\Gamma^{-1}} C^n_{\mathrm{DY}}(F; \mathsf{V} \otimes {}^\vee\mathsf{W}, \mathbf{1}). \tag{39}$$

Then for all n, ξ descends to an isomorphism $\overline{\xi} : H^n_{\mathrm{DY}}(F; \mathsf{V}, \mathsf{W}) \cong H^n_{\mathrm{DY}}(F; \mathsf{V} \otimes {}^\vee\mathsf{W}, \mathbf{1})$. A similar description applies for the second isomorphism in (38). A simple diagrammatic computation reveals that for $f \in C^n_{\mathrm{DY}}(F; \mathsf{V}, \mathsf{W})$, the components of the natural transformation $\xi(f)$ are



(40)

Finally, the dimension formulas from Corollaries 3.5 and 3.6 translate into dimension formulas for the DY cohomology groups:

Corollary 4.9. *Let*

$$0 \longrightarrow \mathsf{K} \xrightarrow{j} \mathsf{P} \xrightarrow{\pi} \mathbf{1} \longrightarrow 0, \qquad 0 \longrightarrow \mathsf{L} \xrightarrow{i} \mathsf{Q} \xrightarrow{p} \mathsf{V} \longrightarrow 0$$

be allowable short exact sequences in $\mathcal{Z}(F)$ where P, Q are relatively projective objects. Then for $n \geq 2$,

$$\dim H^n_{\mathrm{DY}}(F; \mathsf{V}, \mathsf{W}) = \dim \mathrm{Hom}_{\mathcal{A}}(\mathsf{K}, \mathsf{M}) - \dim \mathrm{Hom}_{\mathcal{A}}(\mathsf{P}, \mathsf{M}) + \dim \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, \mathsf{M})$$

where $\mathsf{M} = \mathsf{W} \otimes \mathsf{L}^\vee \otimes (\mathsf{K}^\vee)^{\otimes(n-2)}$. In particular,

$$\dim H^n_{\mathrm{DY}}(F) = \dim \mathrm{Hom}_{\mathcal{A}}\big(\mathsf{K}, (\mathsf{K}^\vee)^{\otimes(n-1)}\big) - \dim \mathrm{Hom}_{\mathcal{A}}\big(\mathsf{P}, (\mathsf{K}^\vee)^{\otimes(n-1)}\big) + \dim \mathrm{Hom}_{\mathcal{A}}\big(\mathbf{1}, (\mathsf{K}^\vee)^{\otimes(n-1)}\big).$$

If moreover P is the relatively projective cover (Definition 2.12) of $\mathbf{1}$,

$$\dim H^2_{\mathrm{DY}}(F) = \dim \mathrm{Hom}_{\mathcal{A}}\big(\mathsf{K}, \mathsf{K}^\vee\big) - \dim \mathrm{Hom}_{\mathcal{A}}\big(\mathsf{P}, \mathsf{K}^\vee\big).$$

4.4 The Yoneda product on DY cohomology

As recalled in section 2.1, there is the Yoneda product $\circ$ on relative Ext cohomology. Thanks to the isomorphism of cochain complexes Γ in (34) we get a product on the DY side, which we still denote by $\circ$:

$$f \circ g = \Gamma^{-1}\big(\Gamma(f) \circ \Gamma(g)\big). \tag{41}$$

This is an associative product on DY cocycles which descends to the cohomology groups.

From (13) and (15), we know that any n-cocycle can be written as a Yoneda product of n 1-cocycles. Hence the same is true for DY cocycles, through the isomorphism Γ:

Lemma 4.10. *Let $f \in C^n_{\mathrm{DY}}(F, \mathsf{U}, \mathsf{W})$ be a cocycle. Then there exists cocycles*

$$g_1 \in C^1_{\mathrm{DY}}(F, \mathsf{V}_1, \mathsf{W}),\ g_2 \in C^1_{\mathrm{DY}}(F, \mathsf{V}_2, \mathsf{V}_1), \ldots,\ g_n \in C^1_{\mathrm{DY}}(F, \mathsf{U}, \mathsf{V}_{n-1})$$

such that $f = g_1 \circ \ldots \circ g_n$.

The product (41) has the following simple expression:

Theorem 4.11. *Let $f \in C^n_{\mathrm{DY}}(F, \mathsf{V}, \mathsf{W})$ and $g \in C^m_{\mathrm{DY}}(F, \mathsf{U}, \mathsf{V})$ be cocycles. Then the components of the cocycle $f \circ g \in C^{m+n}_{\mathrm{DY}}(F, \mathsf{U}, \mathsf{W})$ are*

$$(f \circ g)_{X_1,\ldots,X_m,Y_1,\ldots,Y_n} = (-1)^{nm}\big(\mathrm{id}_{F(X_1)\otimes\ldots\otimes F(X_m)} \otimes f_{Y_1,\ldots,Y_n}\big)\big(g_{X_1,\ldots,X_m} \otimes \mathrm{id}_{F(Y_1)\otimes\ldots\otimes F(Y_n)}\big).$$

The proof of this formula is divided in three steps:

- We first determine the Yoneda product of a n-cocycle $\alpha \in \mathrm{Bar}^n_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{V}, \mathsf{W}) = \mathrm{Hom}_{\mathcal{Z}(F)}\big(G^{n+1}(\mathsf{V}), \mathsf{W}\big)$ with a 1-cocycle $\beta \in \mathrm{Bar}^1_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{U}, \mathsf{V}) = \mathrm{Hom}_{\mathcal{Z}(F)}\big(G^2(\mathsf{U}), \mathsf{V}\big)$. See Lemma 4.12.
- From this we compute the product (41) for a DY n-cocycle $f \in C^n_{\mathrm{DY}}(F, \mathsf{V}, \mathsf{W})$ and a DY 1-cocycle $g \in C^1_{\mathrm{DY}}(F, \mathsf{U}, \mathsf{V})$. See Lemma 4.13.
- Then we deduce the product (41) for general DY cocycles f, g by induction on m thanks to Lemma 4.10.

Restricting g to be a 1-cocycle in the two first steps makes the proofs of the corresponding lemmas less cumbersome. We write as usual $\mathsf{U} = (U, \rho^U)$, $\mathsf{V} = (V, \rho^V)$, $\mathsf{W} = (W, \rho^W)$.

Lemma 4.12. *Let $\alpha \in \mathrm{Bar}^n_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{V}, \mathsf{W})$, $\beta \in \mathrm{Bar}^1_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{U}, \mathsf{V})$ be cocycles. The Yoneda product $\alpha \circ \beta \in \mathrm{Bar}^{n+1}_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{U}, \mathsf{W})$ is the unique morphism such that*

$$(\alpha \circ \beta)\, i^{(n+2)}_{X,Y_1,\ldots,Y_{n+1}}(U) = (-1)^n\, \alpha\, i^{(n+1)}_{Y_1,\ldots,Y_{n+1}}(V)\big(\mathrm{id}_{F(Y_{n+1})^\vee\otimes\ldots\otimes F(Y_1)^\vee} \otimes \beta\, i^{(2)}_{X,\mathbf{1}}(U) \otimes \mathrm{id}_{F(Y_1)\otimes\ldots\otimes F(Y_{n+1})}\big)$$

where $\mathbf{1}$ is the tensor unit of $\mathcal{C}$.

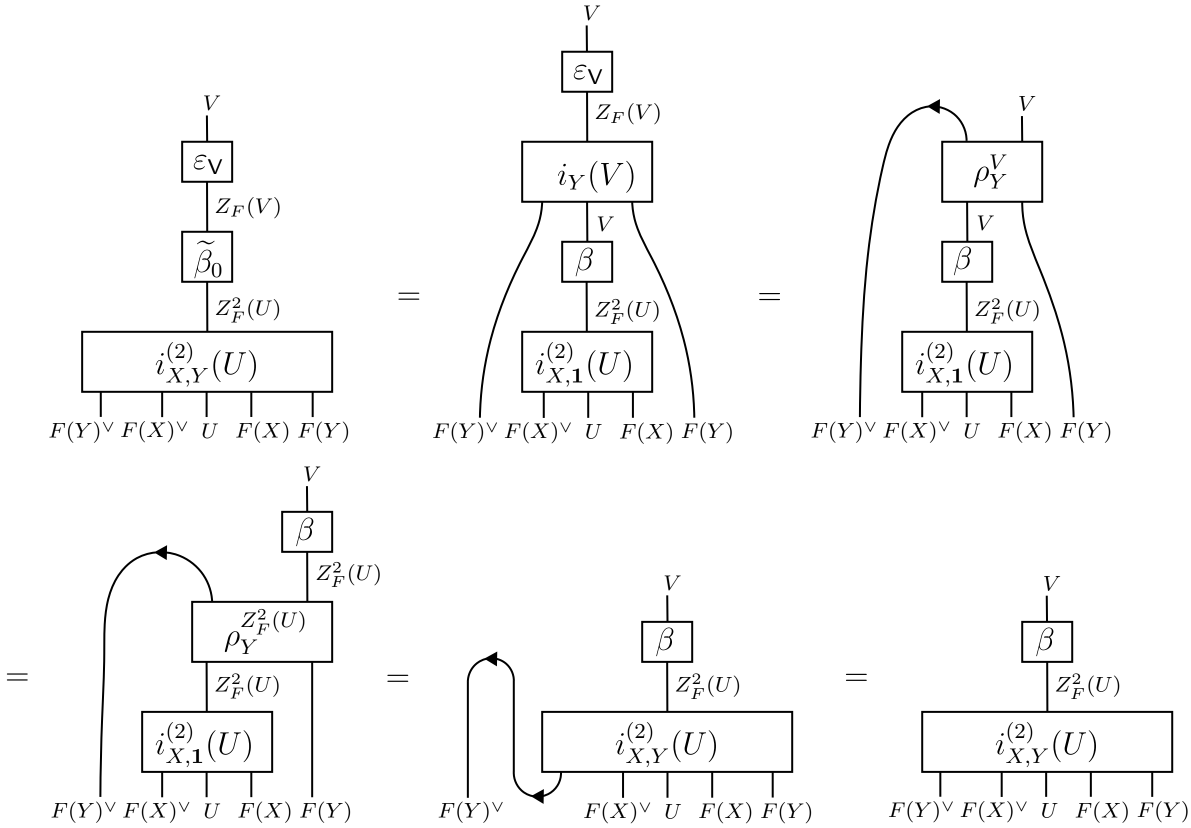
Proof. Since we use the bar resolution, the diagram in (17) becomes

$$\begin{array}{ccccccccccccc}
G^{n+2}(\mathsf{U}) & \xrightarrow{d^{\mathsf{U}}_{n+1}} & \ldots & \xrightarrow{d^{\mathsf{U}}_{l+3}} & G^{l+3}(\mathsf{U}) & \xrightarrow{d^{\mathsf{U}}_{l+2}} & G^{l+2}(\mathsf{U}) & \xrightarrow{d^{\mathsf{U}}_{l+1}} & \ldots & \xrightarrow{d^{\mathsf{U}}_{2}} & G^{2}(\mathsf{U}) & & \\
\downarrow \widetilde{\beta}_n & & & & \downarrow \widetilde{\beta}_{l+1} & & \downarrow \widetilde{\beta}_{l} & & & & \downarrow \widetilde{\beta}_0 & \searrow \beta & \\
G^{n+1}(\mathsf{V}) & \xrightarrow{d^{\mathsf{V}}_{n}} & \ldots & \xrightarrow{d^{\mathsf{U}}_{l+2}} & G^{l+2}(\mathsf{V}) & \xrightarrow{d^{\mathsf{V}}_{l+1}} & G^{l+1}(\mathsf{V}) & \xrightarrow{d^{\mathsf{V}}_{l}} & \ldots & \xrightarrow{d^{\mathsf{V}}_{1}} & G(\mathsf{V}) & \xrightarrow{\varepsilon_{\mathsf{V}}} & \mathsf{V} \longrightarrow 0
\end{array} \tag{42}$$

where G is the comonad on $\mathcal{Z}(F)$ defined in (30), $d^{\mathsf{U}}, d^{\mathsf{V}}$ are the differential of the bar resolutions of U, V respectively and ε_{V} is the counit of G. We have to find the dashed arrows and the Yoneda product $\alpha \circ \beta$ is then $\alpha\widetilde{\beta}_n$. Define $\widetilde{\beta}_l$ as the unique morphism such that

$$\widetilde{\beta}_l\, i^{(l+2)}_{X,Y_1,\ldots,Y_{l+1}}(U) = (-1)^l\, i^{(l+1)}_{Y_1,\ldots,Y_{l+1}}(V)\big(\mathrm{id}_{F(Y_{l+1})^\vee\otimes\ldots\otimes F(Y_1)^\vee} \otimes \beta\, i^{(2)}_{X,\mathbf{1}}(U) \otimes \mathrm{id}_{F(Y_1)\otimes\ldots\otimes F(Y_{l+1})}\big).$$

We left to the reader to check that $\widetilde{\beta}_l$ is a morphism in $\mathcal{Z}(F)$, *i.e.* that it commutes with the half-braidings relative to F. We first show that the triangle in (42) commutes:



and the claim follows since this holds for any $X, Y \in \mathcal{C}$. The first and second equalities are just the definitions of $\widetilde{\beta}_0$ and ε_{V} (recall (31)), for the third equality we used that β is a morphism in $\mathcal{Z}(F)$ and for the fourth equality we used (37).

Now we show that the squares in (42) commutes. Recall the expression of the bar differential in Lemma 4.2. On the one hand we have

$$\begin{aligned}
\widetilde{\beta}_l\, d^{\mathsf{U}}_{l+2}\, i^{(l+3)}_{X,Y_1,\dots,Y_{l+2}} &= \widetilde{\beta}_l\, i^{(l+2)}_{Y_1,\dots,Y_{l+2}}(U)\big(\mathrm{id}_{F(Y_{l+2})^\vee\otimes\dots\otimes F(Y_1)^\vee} \otimes \varepsilon_{\mathsf{U}}\, i_X(U) \otimes \mathrm{id}_{F(Y_1)\otimes\dots\otimes F(Y_{l+2})}\big)\\
&\quad - \widetilde{\beta}_l\, i^{(l+2)}_{X\otimes Y_1,Y_2,\dots,Y_{l+2}}(V) + \sum_{j=2}^{l+2}(-1)^j\, \widetilde{\beta}_l\, i^{(l+2)}_{X,Y_1,\dots,Y_{j-1}\otimes Y_j,\dots,Y_{l+2}}(V)\\
&= (-1)^l\, i^{(l+1)}_{Y_2,\dots,Y_{l+2}}(V)\, \Big(\mathrm{id}_{F(Y_{l+2})^\vee\otimes\dots\otimes F(Y_2)^\vee} \otimes \beta\, i^{(2)}_{Y_1,\mathbf{1}}(U)\big(\mathrm{id}_{F(Y_1)^\vee} \otimes \varepsilon_{\mathsf{U}}\, i_X(U) \otimes \mathrm{id}_{F(Y_1)}\big) \otimes \mathrm{id}_{F(Y_2)\otimes\dots\otimes F(Y_{l+2})}\Big)\\
&\quad - (-1)^l\, i^{(l+1)}_{Y_2,\dots,Y_{l+2}}(V)\big(\mathrm{id}_{F(Y_{l+2})^\vee\otimes\dots\otimes F(Y_2)^\vee} \otimes \beta\, i^{(2)}_{X\otimes Y_1,\mathbf{1}}(U) \otimes \mathrm{id}_{F(Y_2)\otimes\dots\otimes F(Y_{l+2})}\big)\\
&\quad + (-1)^l \sum_{j=2}^{l+2}(-1)^j\, i^{(l+1)}_{Y_1,\dots,Y_{j-1}\otimes Y_j,\dots,Y_{l+2}}(V)\big(\mathrm{id}_{F(Y_{l+2})^\vee\otimes\dots\otimes F(Y_1)^\vee} \otimes \beta\, i^{(2)}_{X,\mathbf{1}}(U) \otimes \mathrm{id}_{F(Y_1)\otimes\dots\otimes F(Y_{l+2})}\big)\\
&= (-1)^{l+1}\, i^{(l+1)}_{Y_2,\dots,Y_{l+2}}(V)\big(\mathrm{id}_{F(Y_{l+2})^\vee\otimes\dots\otimes F(Y_2)^\vee} \otimes \beta\, i^{(2)}_{X,Y_1}(U) \otimes \mathrm{id}_{F(Y_2)\otimes\dots\otimes F(Y_{l+2})}\big)\\
&\quad + (-1)^l \sum_{j=2}^{l+2}(-1)^j\, i^{(l+1)}_{Y_1,\dots,Y_{j-1}\otimes Y_j,\dots,Y_{l+2}}(V)\big(\mathrm{id}_{F(Y_{l+2})^\vee\otimes\dots\otimes F(Y_1)^\vee} \otimes \beta\, i^{(2)}_{X,\mathbf{1}}(U) \otimes \mathrm{id}_{F(Y_1)\otimes\dots\otimes F(Y_{l+2})}\big).
\end{aligned}$$

For the last equality we used $\beta i^{(2)}_{Y_1,\mathbf{1}}\big(\mathrm{id}_{F(Y_1)^\vee} \otimes \varepsilon_{\mathsf{U}} i_X(U) \otimes \mathrm{id}_{F(Y_1)}\big) - \beta i^{(2)}_{X\otimes Y_1,\mathbf{1}}(U) = -\beta i^{(2)}_{X,Y_1}(U)$, due to the assumption that β is a cocycle ($\beta d^{\mathsf{U}}_2 = 0$). On the other hand and still using Lemma 4.2 we get

$$
\begin{aligned}
&d^{\vee}_{l+1}\,\widetilde{\beta}_{l+1}\, i^{(l+3)}_{X,Y_1,\dots,Y_{l+2}} = (-1)^{l+1}\, d^{\vee}_{l+1}\, i^{(l+2)}_{Y_1,\dots,Y_{l+2}}(V)\big(\mathrm{id}_{F(Y_{l+2})^\vee\otimes\dots\otimes F(Y_1)^\vee}\otimes \beta\, i^{(2)}_{X,\mathbf{1}}(U)\otimes \mathrm{id}_{F(Y_1)\otimes\dots\otimes F(Y_{l+2})}\big)\\
&= (-1)^{l+1}\, i^{(n)}_{Y_2,\dots,Y_{l+2}}(V)\,\Big(\mathrm{id}_{F(Y_{l+2})^\vee\otimes\dots\otimes F(Y_2)^\vee}\otimes \varepsilon_{\vee} i_{Y_1}(V)\big(\mathrm{id}_{F(Y_1)^\vee}\otimes \beta\, i^{(2)}_{X,\mathbf{1}}(U)\otimes \mathrm{id}_{F(Y_1)}\big)\otimes \mathrm{id}_{F(Y_2)\otimes\dots\otimes F(Y_{l+2})}\Big)\\
&\quad + (-1)^{l+1}\sum_{j=1}^{l+1}(-1)^j\, i^{(l+1)}_{Y_1,\dots,Y_j\otimes Y_{j+1},\dots,Y_{n+1}}(V)\big(\mathrm{id}_{F(Y_{l+2})^\vee\otimes\dots\otimes F(Y_1)^\vee}\otimes \beta\, i^{(2)}_{X,\mathbf{1}}(U)\otimes \mathrm{id}_{F(Y_1)\otimes\dots\otimes F(Y_{l+2})}\big)\\
&= (-1)^{l+1}\, i^{(l+1)}_{Y_2,\dots,Y_{l+2}}(V)\big(\mathrm{id}_{F(Y_{l+2})^\vee\otimes\dots\otimes F(Y_2)^\vee}\otimes \beta\, i^{(2)}_{X,Y_1}(U)\otimes \mathrm{id}_{F(Y_2)\otimes\dots\otimes F(Y_{l+2})}\big)\\
&\quad + (-1)^{l}\sum_{j=2}^{l+2}(-1)^j\, i^{(l+1)}_{Y_1,\dots,Y_{j-1}\otimes Y_j,\dots,Y_{n+1}}(V)\big(\mathrm{id}_{F(Y_{l+2})^\vee\otimes\dots\otimes F(Y_1)^\vee}\otimes \beta\, i^{(2)}_{X,\mathbf{1}}(U)\otimes \mathrm{id}_{F(Y_1)\otimes\dots\otimes F(Y_{l+2})}\big)
\end{aligned}
$$

which agrees with the result of the previous computation; since this holds for any $X, Y_1, \dots, Y_{l+2} \in \mathcal{C}$, the desired equality follows . Note that for the last equality we used that

$$
\varepsilon_{\vee}\, i_{Y_1}(V)\,\big(\mathrm{id}_{F(Y_1)^\vee}\otimes \beta\, i^{(2)}_{X,\mathbf{1}}(U)\otimes \mathrm{id}_{F(Y_1)}\big) = \varepsilon_{\vee}\,\widetilde{\beta}_0\, i^{(2)}_{X,Y_1}(U) = \beta\, i^{(2)}_{X,Y_1}(U)
$$

due to the commutation of the triangle in (42). □

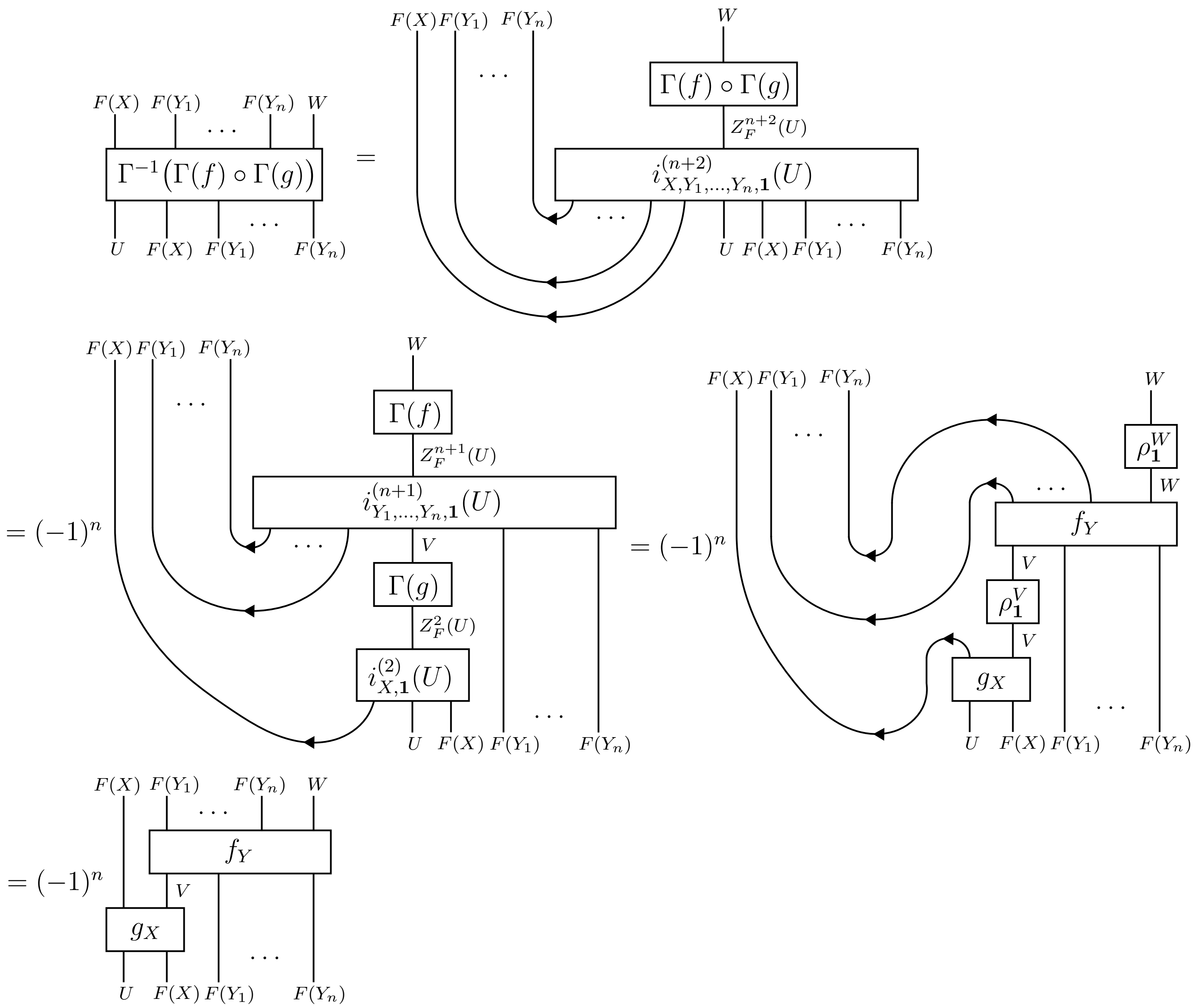


Figure 2: Proof of Lemma 4.13

Lemma 4.13. *Let $f \in C^n_{\mathrm{DY}}(F,\mathsf{V},\mathsf{W})$ and $g \in C^1_{\mathrm{DY}}(F,\mathsf{U},\mathsf{V})$ be cocycles. Then it holds*

$$(f \circ g)_{X,Y_1,\dots,Y_n} = (-1)^n \left(\mathrm{id}_{F(X)} \otimes f_{Y_1,\dots,Y_n}\right)\left(g_X \otimes \mathrm{id}_{F(Y_1)\otimes\dots\otimes F(Y_n)}\right).$$

Proof. The proof is the diagrammatic computation displayed in Figure 2. We use the previous lemma, the formulas for Γ in (35) and its inverse in (36), and the fact that $\rho^V_{\mathbf{1}} = \mathrm{id}_V$. □

Proof of Theorem 4.11. We will conclude by induction on m thanks to the previous lemma. Due to Lemma 4.10, we can write g as $g' \circ h$ with $g' \in C^{m-1}_{\mathrm{DY}}(F,\mathsf{U}',\mathsf{V})$ and $h \in C^1_{\mathrm{DY}}(F,\mathsf{U},\mathsf{U}')$. Then by associativity of $\circ$ we get

$$\begin{aligned}
&(f\circ(g'\circ h))_{X_1,\dots,X_m,Y_1,\dots,Y_n} = \big((f\circ g')\circ h\big)_{X_1,X_2,\dots,X_m,Y_1,\dots,Y_n}\\
&= (-1)^{n+m-1}\big(\mathrm{id}_{F(X_1)}\otimes (f\circ g')_{X_2,\dots,X_m,Y_1,\dots,Y_n}\big)\big(h_{X_1}\otimes \mathrm{id}_{F(X_2)\otimes\dots\otimes F(X_m)\otimes F(Y_1)\otimes\dots\otimes F(Y_n)}\big)\\
&= (-1)^{nm+m-1}\big(\mathrm{id}_{F(X_1)\otimes\dots\otimes F(X_m)}\otimes f_{Y_1,\dots,Y_n}\big)\big(\mathrm{id}_{F(X_1)}\otimes g'_{X_2,\dots,X_m}\otimes \mathrm{id}_{F(Y_1)\otimes\dots\otimes F(Y_n)}\big)\\
&\qquad\qquad \big(h_{X_1}\otimes \mathrm{id}_{F(X_2)\otimes\dots\otimes F(X_m)\otimes F(Y_1)\otimes\dots\otimes F(Y_n)}\big)\\
&= (-1)^{nm}\big(\mathrm{id}_{F(X_1)\otimes\dots\otimes F(X_m)}\otimes f_{Y_1,\dots,Y_n}\big)\big((g'\circ h)_{X_1,\dots,X_m}\otimes \mathrm{id}_{F(Y_1)\otimes\dots\otimes F(Y_n)}\big).
\end{aligned}$$ □

For $\mathsf{U} = (U,\rho^U), \mathsf{V} = (V,\rho^V) \in \mathcal{Z}(F)$, we write as usual

$$C^\bullet_{\mathrm{DY}}(F;\mathsf{U},\mathsf{V}) = \bigoplus_{n=0}^{\infty} C^n_{\mathrm{DY}}(F;\mathsf{U},\mathsf{V}), \quad H^\bullet_{\mathrm{DY}}(F;\mathsf{U},\mathsf{V}) = \bigoplus_{n=0}^{\infty} H^n_{\mathrm{DY}}(F;\mathsf{U},\mathsf{V}).$$

Then $C^\bullet_{\mathrm{DY}}(F) = C^\bullet_{\mathrm{DY}}(F;\mathbf{1},\mathbf{1})$ and $H^\bullet_{\mathrm{DY}}(F) = H^\bullet_{\mathrm{DY}}(F;\mathbf{1},\mathbf{1})$ are associative graded k-algebras with respect to the product $\circ$. In [DE19, eq. (3)] and [BD20, §3.1], a product $\cup$ is defined by

$$(f\cup g)_{X_1,\dots,X_n,Y_1,\dots,Y_m} = (-1)^{(n-1)(m-1)} f_{X_1,\dots,X_n}\otimes g_{Y_1,\dots,Y_m}$$

for $f \in C^n_{\mathrm{DY}}(F)$, $g \in C^m_{\mathrm{DY}}(F)$ and is shown to be commutative (up to sign) on $H^\bullet_{\mathrm{DY}}(F)$: $f\cup g = (-1)^{mn} g\cup f$. Thanks to Theorem 4.11, we see that $f\cup g = (-1)^{m+n+1} g\circ f$, and thus $\circ$ is commutative (up to sign) on $H^\bullet_{\mathrm{DY}}(F)$ as well.

Recall the isomorphism ξ in (39); for any $\mathsf{U},\mathsf{V} \in \mathcal{Z}(F)$ we define the map

$$\begin{aligned}
\triangleright : C^n_{\mathrm{DY}}(F)\otimes C^m_{\mathrm{DY}}(F;\mathsf{U},\mathsf{V}) &\xrightarrow{\mathrm{id}\otimes\xi} C^n_{\mathrm{DY}}(F;\mathbf{1},\mathbf{1})\otimes C^m_{\mathrm{DY}}(F;\mathsf{U}\otimes{}^\vee\mathsf{V},\mathbf{1})\\
&\xrightarrow{\circ} C^{n+m}_{\mathrm{DY}}(F;\mathsf{U}\otimes{}^\vee\mathsf{V},\mathbf{1}) \xrightarrow{\xi^{-1}} C^{n+m}_{\mathrm{DY}}(F;\mathsf{U},\mathsf{V})
\end{aligned} \tag{43}$$

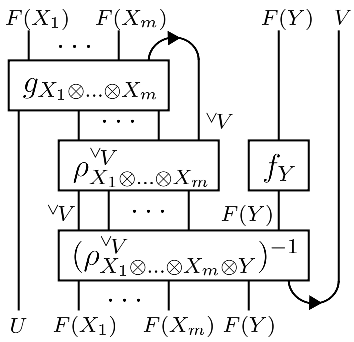
Proposition 4.14. *For $f \in C^n_{\mathrm{DY}}(F)$ and $g \in C^m_{\mathrm{DY}}(F;\mathsf{U},\mathsf{V})$, the components of $f\triangleright g \in C^{n+m}_{\mathrm{DY}}(F;\mathsf{U},\mathsf{V})$ are*

$$(f\triangleright g)_{X_1,\dots,X_m,Y_1,\dots,Y_n} = (-1)^{mn}\big(\mathrm{id}_{F(X_1)\otimes\dots\otimes F(X_m)}\otimes \rho^V_{Y_1\otimes\dots\otimes Y_n}\big)\big(g_{X_1,\dots,X_m}\otimes f_{Y_1,\dots,Y_n}\big)$$

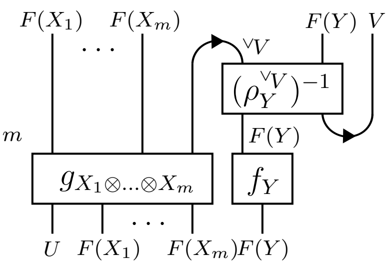
This endows $C^\bullet_{\mathrm{DY}}(F;\mathsf{U},\mathsf{V})$ with a structure of graded $C^\bullet_{\mathrm{DY}}(F)$-module which descends to a structure of graded $H^\bullet_{\mathrm{DY}}(F)$-module on $H^\bullet_{\mathrm{DY}}(F;\mathsf{U},\mathsf{V})$.

Proof. For simplicity of notation, take for instance f to be a 1-cochain (for the general case simply replace Y by a tensor product $Y_1\otimes\ldots\otimes Y_n$ in the pictures below). We have:

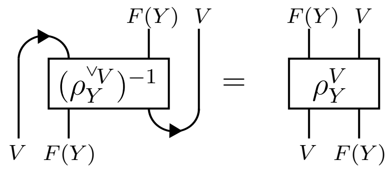
$\xi^{-1}\big(f\circ\xi(g)\big)_{X_1,\dots,X_m,Y} = (-1)^m$



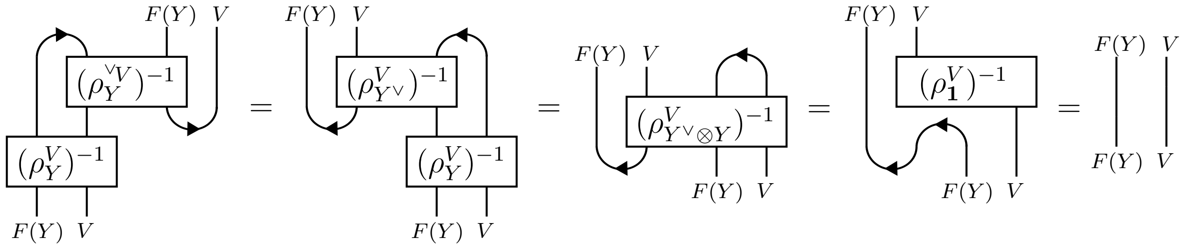
$= (-1)^m$



where the first equality is the output of a straightforward diagrammatic computation using (40) and the formula in Theorem 4.11, while for the second we used (24) and the naturality of the half-braiding. To obtain the desired expression, observe that



Indeed:



where the first equality is by definition of $\rho^{\vee V}$ (25), for the second equality we used the half-braiding property (24) and the third equality is by naturality of the half-braiding. With the explicit expression of $\triangleright$ just obtained, it is easy to check that $(f \circ f') \triangleright g = f \triangleright (f' \triangleright g)$. Moreover, ξ and $\circ$ descend to cohomology, so that the representation $\triangleright$ descends to cohomology as well. □

4.5 The long exact sequence for DY cohomology

We can transport the long exact sequence of relative Ext groups from Theorem 2.9 through the isomorphism Γ from (35) in order to obtain a long exact sequence for the Davydov–Yetter cohomology groups, which we now describe.

Let $f : \mathsf{U} \to \mathsf{V}$ be a morphism in $\mathcal{Z}(F)$. It induces linear maps

$$\forall n, \;\; f^* : \mathrm{Bar}^n_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{V},\mathsf{W}) \to \mathrm{Bar}^n_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{U},\mathsf{W})$$

as defined at the beginning of §2.2.3. Here we choose the bar resolution $\ldots \to G^2(\mathsf{X}) \to G(\mathsf{X}) \to \mathsf{X} \to 0$, which yields the bar complex $\mathrm{Bar}^n_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{X},\mathsf{W}) = \mathrm{Hom}_{\mathcal{Z}(F)}\big(G^{n+1}(\mathsf{X}),\mathsf{W}\big)$, where X is U or V and G is the comonad on $\mathcal{Z}(F)$ associated to the adjunction (29). Thanks to the isomorphism $\Gamma : C^n_{\mathrm{DY}}(F;\mathsf{X},\mathsf{W}) \to \mathrm{Bar}^n_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{X},\mathsf{W})$, we have the corresponding linear maps $\Gamma^{-1} f^* \Gamma$ on DY cochains, which we also denote by f^* for brevity:

$$\forall n, \;\; f^* : C^n_{\mathrm{DY}}(F;\mathsf{V},\mathsf{W}) \to C^n_{\mathrm{DY}}(F;\mathsf{U},\mathsf{W}).$$

Lemma 4.15. *For $g \in C^n_{\mathrm{DY}}(F;\mathsf{V},\mathsf{W})$, the components of $f^*(g) \in C^n_{\mathrm{DY}}(F;\mathsf{U},\mathsf{W})$ are*

$$f^*(g)_{X_1,\ldots,X_n} = g_{X_1,\ldots,X_n}\left(f \otimes \mathrm{id}_{F(X_1)\otimes\ldots\otimes F(X_n)}\right).$$

Proof. First note a general fact. Take a resolvent pair of abelian categories as in (5) and let G be the associated comonad on $\mathcal{A}$. Then for any objects $U, V \in \mathcal{A}$, the diagram

$$\begin{array}{ccccccccccc}
G^{n+1}(U) & \xrightarrow{d_n^U} & G^n(U) & \xrightarrow{d_{n-1}^U} & \ldots & \xrightarrow{d_1^U} & G(U) & \xrightarrow{d_0^U} & U & \longrightarrow & 0 \\
\downarrow {\scriptstyle G^{n+1}(f)} & & \downarrow {\scriptstyle G^n(f)} & & & & \downarrow {\scriptstyle G(f)} & & \downarrow {\scriptstyle f} & & \\
G^{n+1}(V) & \xrightarrow[d_n^V]{} & G^n(V) & \xrightarrow[d_{n-1}^V]{} & \ldots & \xrightarrow[d_1^V]{} & G(V) & \xrightarrow[d_0^V]{} & V & \longrightarrow & 0
\end{array}$$

commutes. The first (resp. second) row is the bar resolution of U (resp. V), which differential is defined in (21). This is based on an easy computation using the naturality of the counit $\varepsilon : G \to \mathrm{Id}$. Then by definition (see §2.2.3), $f^* : \mathrm{Hom}_{\mathcal{A}}\big(G^{n+1}(V),W\big) \to \mathrm{Hom}_{\mathcal{A}}\big(G^{n+1}(U),W\big)$ is simply given by

$f^*(\alpha) = \alpha G^{n+1}(f)$.
In our case, $G^{n+1}(f) = Z_F^{n+1}(f)$ and the result is the outcome of an easy diagrammatic computation using the definition of Γ and Γ^{-1} from (35)–(36) and the definition of $Z_F(f)$ from (27). □

Corollary 4.16. *Let* $S = \big(0 \longrightarrow \mathsf{U} \xrightarrow{j} \mathsf{V} \xrightarrow{\pi} \mathsf{W} \longrightarrow 0\big)$ *be an allowable short exact sequence in* $\mathcal{Z}(F)$ *and let* N *be any object in* $\mathcal{Z}(F)$*. Then we have a long exact sequence of* k*-vector spaces*

$$0 \longrightarrow H^0_{\mathrm{DY}}(F;\mathsf{W},\mathsf{N}) \xrightarrow{\pi^*} H^0_{\mathrm{DY}}(F;\mathsf{V},\mathsf{N}) \xrightarrow{j^*} H^0_{\mathrm{DY}}(F;\mathsf{U},\mathsf{N}) \xrightarrow{c^0}$$
$$H^1_{\mathrm{DY}}(F;\mathsf{W},\mathsf{N}) \xrightarrow{\pi^*} H^1_{\mathrm{DY}}(F;\mathsf{V},\mathsf{N}) \xrightarrow{j^*} H^1_{\mathrm{DY}}(F;\mathsf{U},\mathsf{N}) \xrightarrow{c^1}$$
$$H^2_{\mathrm{DY}}(F;\mathsf{W},\mathsf{N}) \xrightarrow{\pi^*} H^2_{\mathrm{DY}}(F;\mathsf{V},\mathsf{N}) \xrightarrow{j^*} H^2_{\mathrm{DY}}(F;\mathsf{U},\mathsf{N}) \ldots$$

A nice feature of this long exact sequence is that its arrows are easy to describe. Indeed, the pullback morphisms π^*, j^* have a very simple expression (Lemma 4.15). The connecting morphisms c^n are obtained by transporting the connecting morphisms from Theorem 2.9 through the isomorphism Γ from (35). Let
$$S_{\mathrm{DY}} = \Gamma^{-1}\big(\eta(S)\big) \in H^1_{\mathrm{DY}}(F;\mathsf{W},\mathsf{U})$$
be the DY cocycle associated to $S \in \mathrm{YExt}^1_{\mathcal{Z}(F),\mathcal{D}}(\mathsf{W},\mathsf{U})$ (recall (12)). Then $c^n(g) = (-1)^n g \circ S_{\mathrm{DY}}$, where the product $\circ$ is given by the simple formula in Theorem 4.11. The only non-trivial point might be to determine the DY cocycle S_{DY} associated to S.

To conclude, we reformulate Corollary 2.10 for DY cohomology. Let $S = \big(0 \to \mathsf{K} \to \mathsf{P} \to \mathsf{W} \to 0\big)$ be an allowable short exact sequence in $\mathcal{Z}(F)$, with P a relatively projective object. Then for $n \geq 1$ the connecting morphism is an isomorphism:

$$\begin{array}{rrcl} c^n: & H^n_{\mathrm{DY}}(F;\mathsf{K},\mathsf{N}) & \xrightarrow{\sim} & H^{n+1}_{\mathrm{DY}}(F;\mathsf{W},\mathsf{N}) \\ & g & \mapsto & (-1)^n g \circ S_{\mathrm{DY}} \end{array}. \tag{44}$$

5 Finite-dimensional Hopf algebras

Let H be a finite-dimensional Hopf algebra over an algebraically closed field k, with coproduct $\Delta : H \to H \otimes H$, counit $\varepsilon : H \to k$ and antipode $S : H \to H$. We will specialize the general results obtained above to the case $\mathcal{C} = H$-mod (which is a finite k-linear tensor category) and study examples.

In the sequel we denote the iterated coproduct by $\Delta^{(n)}$:

$$\Delta^{(-1)} = \varepsilon, \quad \Delta^{(0)} = \mathrm{id}, \quad \Delta^{(1)} = \Delta, \quad \Delta^{(2)} = (\Delta \otimes \mathrm{id})\Delta, \ \ldots \tag{45}$$

and we use Sweedler's notation with implicit summation:

$$\Delta(h) = h' \otimes h'', \quad \Delta^{(2)}(h) = h' \otimes h'' \otimes h''', \quad \ldots, \quad \Delta^{(n)}(h) = h^{(1)} \otimes \ldots \otimes h^{(n+1)}.$$

5.1 Specializing to H-mod

We are mainly interested in DY cohomology for the identity functor on H-mod. However, another interesting functor is the forgetful functor $U : H\text{-mod} \to \mathrm{Vect}_k$ which we study first.

5.1.1 DY cohomology of the forgetful functor

Let $(H^*)^{\mathrm{op}}$ be H^* with the product $\varphi\psi = (\psi \otimes \varphi)\Delta$ and the coproduct defined by $\Delta(\varphi)(x \otimes y) = \varphi(xy)$. It has been shown in [Dav97, Prop. 7] that $C^\bullet_{\mathrm{DY}}(U)$ is isomorphic to the Hochschild cochains of $(H^*)^{\mathrm{op}}$ with the trivial coefficient. As noted in [GHS19, Remark 5.11] this implies that $H^\bullet_{\mathrm{DY}}(U) \cong \mathrm{Ext}_{(H^*)^{\mathrm{op}}}(\mathbb{C}, \mathbb{C})$. This isomorphism can be generalized with coefficients and is easily proved thanks to Corollary 4.6:

Lemma 5.1. *1. We have an equivalence of tensor categories $\mathcal{Z}(U) \cong (H^*)^{\mathrm{op}}$-mod.*
2. For all $V, W \in (H^*)^{\mathrm{op}}$-mod, $H^n_{\mathrm{DY}}(U; V, W) \cong \mathrm{Ext}^n_{(H^*)^{\mathrm{op}}}(V, W)$.

Proof. 1. An object in $\mathcal{Z}(U)$ is a pair (V, ρ^V), where V is simply a vector space. For such a pair, the formula

$$\varphi \cdot v = (\varphi \otimes \mathrm{id}_V)\big(\rho^V_H(v \otimes 1)\big)$$

with $\varphi \in H^*$, $v \in V$ endows V with a structure of $(H^*)^{\mathrm{op}}$-module. Conversely if $(V, \cdot)$ is a $(H^*)^{\mathrm{op}}$-module, the formula

$$\rho^V_H(v \otimes x) = h_i x \otimes h^i \cdot v$$

(where (h_i) is a basis of H and (h^i) is its dual basis) can be extended by naturality to a half-braiding relative to U. It is straightforward to check that this correspondence defines a strict tensor equivalence.
2. We have

$$H^n_{\mathrm{DY}}(U; V, W) \cong \mathrm{Ext}^n_{\mathcal{Z}(U),\mathrm{Vect}_k}(V, W) \cong \mathrm{Ext}^n_{(H^*)^{\mathrm{op}}\text{-mod},\mathrm{Vect}_k}(V, W) = \mathrm{Ext}^n_{(H^*)^{\mathrm{op}}}(V, W).$$

The first isomorphism is due to Corollary 4.6, the second isomorphism is due to the previous item (actually we only use the equivalence of abelian categories) and the equality is due to the semisimplicity of Vect_k (see Example 2.3). □

To compute $\mathrm{Ext}^n_{(H^*)^{\mathrm{op}}}(V, W)$ one just needs to find a usual projective resolution of the $(H^*)^{\mathrm{op}}$-module V.

5.1.2 The resolvent pair $\mathcal{Z}(\mathrm{Id}_{H\text{-mod}}) \leftrightarrows H$-mod

Let $(H^*)^{\mathrm{op}}$ be H^* with the product $\varphi * \psi = (\psi \otimes \varphi)\Delta)$. The Drinfeld double of H, denoted $D(H)$, is $(H^*)^{\mathrm{op}} \otimes H$ as a coalgebra and has the product

$$(\varphi \otimes h)(\psi \otimes g) = \varphi\psi\big(S(h')?h'''\big) \otimes h''g.$$

As usual, we identify $\varphi \in (H^*)^{\mathrm{op}}$ with $\varphi \otimes 1 \in D(H)$ and $h \in H$ with $1 \otimes h \in D(H)$, so that $(H^*)^{\mathrm{op}}$ and H become subalgebras of $D(H)$. Then $\varphi \otimes h$ can be written as φh and $h\psi = \psi\big(S(h')?h'''\big)h''$.

Recall that $\mathcal{Z}(\mathrm{Id}_{H\text{-mod}}) = \mathcal{Z}(H\text{-mod}) \cong {}^H_H\mathcal{YD}$. Indeed, if $(U, \rho^U) \in \mathcal{Z}(H\text{-mod})$ then

$$\lambda_U = \rho^U_H(? \otimes 1) : U \to H \otimes U$$

is a left comodule structure and (U, λ_U) is a left-left Yetter–Drinfeld module. Moreover, it is well-known that ${}^H_H\mathcal{YD} \cong D(H)$-mod, since one can define a $(H^*)^{\mathrm{op}}$-action on $(V, \lambda_V) \in {}^H_H\mathcal{YD}$ by

$$\varphi \cdot v = (\varphi \otimes \mathrm{id}_V)(\lambda_V(v))$$

and a left coaction on $W \in D(H)$-mod by

$$\lambda_W(w) = h_i \otimes h^i \cdot w \tag{46}$$

where (h_i) is a basis of H with dual basis (h^i). In the sequel we identify all these categories with $D(H)$-mod.

Let us rephrase the resolvent pair $\mathcal{Z}(\mathrm{Id}_{H\text{-mod}}) \leftrightarrows H\text{-mod}$ (recall (29)) under the identification $\mathcal{Z}(\mathrm{Id}_{H\text{-mod}}) = D(H)\text{-mod}$, following [GHS19, §4]. The forgetful functor $\mathcal{U} : D(H)\text{-mod} \to H\text{-mod}$ is induced by the obvious injective morphism $H \to D(H)$ and forgets the $(H^*)^{\mathrm{op}}$-action. The associated induction functor $\mathcal{F} : H\text{-mod} \to D(H)\text{-mod}$ is given by

$$\mathcal{F}(X) = D(H) \otimes_H X \cong (H^* \otimes_k X)_{\mathrm{coad}}, \qquad \mathcal{F}(f) = \mathrm{id}_{(H^*)^{\mathrm{op}}} \otimes f \tag{47}$$

where the $D(H)$-module structure on $(H^* \otimes X)_{\mathrm{coad}}$ is

$$h \cdot (\psi \otimes x) = \psi\big(S(h')?h'''\big) \otimes h''x, \qquad \varphi \cdot (\psi \otimes x) = (\varphi * \psi) \otimes x \tag{48}$$

with $h \in H, \varphi \in H^*$ and $* = (\Delta^{\mathrm{op}})^*$ is the product in $(H^*)^{\mathrm{op}}$. Then we get the resolvent pair $D(H)\text{-mod} \leftrightarrows H\text{-mod}$ and Corollary (4.6) gives

$$H^n_{\mathrm{DY}}(H\text{-mod}; V, W) \cong \mathrm{Ext}^n_{D(H),H}(V, W) \tag{49}$$

where V, W are $D(H)$-modules.

5.1.3 DY cohomology of the identity functor for H-mod

Here we explain that the DY complex of H-mod with coefficients $V, W \in D(H)$-mod can be encoded with more tractable data than in the general definition with natural transformations. The point is that a natural transformation is entirely determined by its value on the regular representation.

Let $(H^{\otimes n} \otimes W)_{\mathrm{ad}}$ be the H-module structure defined by

$$h \cdot (x_1 \otimes \ldots \otimes x_n \otimes w) = h^{(1)}x_1S(h^{(2n+1)}) \otimes \ldots \otimes h^{(n)}x_nS(h^{(n+2)}) \otimes h^{(n+1)} \cdot w.$$

We have a linear map

$$\begin{array}{rrcl} \forall\, n,\ \Psi : & C^n_{\mathrm{DY}}(H\text{-mod}; V, W) & \to & \mathrm{Hom}_H\big(V, (H^{\otimes n} \otimes W)_{\mathrm{ad}}\big) \\ & f & \mapsto & \big(v \mapsto f_{H,\ldots,H}(v \otimes 1 \otimes \ldots \otimes 1)\big) \end{array} \tag{50}$$

where on the right-hand side the natural transformation f is evaluated on the regular representation H on each slot. Let us show that $\Psi(f)$ is indeed H-linear. For $a \in H$, let $r_a \in \mathrm{End}_H(H)$ be the right multiplication by a: $r_a(x) = xa$. Write $\Psi(f) = \Psi(f)^1_H \otimes \ldots \otimes \Psi(f)^n_H \otimes \Psi(f)_W$ (with implicit summation); then by H-linearity and naturality of f we get

$$\begin{aligned} \Psi(f)(h \cdot v) &= f_{H,\ldots,H}(h \cdot v \otimes 1 \otimes \ldots \otimes 1) = f_{H,\ldots,H}\big(h^{(1)} \cdot v \otimes h^{(2)}S(h^{(2n+1)}) \otimes \ldots \otimes h^{(n+1)}S(h^{(n+2)})\big) \\ &= \big(h^{(1)} \otimes \ldots \otimes h^{(n+1)}\big) \cdot f_{H,\ldots,H}\big(\mathrm{id}_V \otimes r_{S(h^{(2n+1)})} \otimes \ldots \otimes r_{S(h^{(n+2)})}\big)(v \otimes 1 \otimes \ldots \otimes 1) \\ &= \big(h^{(1)} \otimes \ldots \otimes h^{(n+1)}\big) \cdot \big(r_{S(h^{(2n+1)})} \otimes \ldots \otimes r_{S(h^{(n+2)})} \otimes \mathrm{id}_W\big) f_{H,\ldots,H}(v \otimes 1 \otimes \ldots \otimes 1) \\ &= h^{(1)}\Psi(f)^1_H(v)S(h^{(2n+1)}) \otimes \ldots \otimes h^{(n)}\Psi(f)^n_H(v)S(h^{(n+2)}) \otimes h^{(n+1)} \cdot \Psi(f)_W(v) \end{aligned}$$

which shows that $\Psi(f) \in \mathrm{Hom}_H\big(V, (H^{\otimes n} \otimes W)_{\mathrm{ad}}\big)$.

For each n, Ψ is an isomorphism of vector spaces, which inverse is given by

$$\forall\, n,\ \Psi^{-1}(\varphi)_{X_1,\ldots,X_n}(v \otimes x_1 \otimes \ldots \otimes x_n) = \varphi^1_H(v) \cdot x_1 \otimes \ldots \otimes \varphi^n_H(v) \cdot x_n \otimes \varphi_W(v)$$

where again we write $\varphi = \varphi^1_H \otimes \ldots \otimes \varphi^n_H \otimes \varphi_W$ with implicit summation. We can thus transport the Davydov–Yetter differential through the isomorphism Ψ and an easy computation gives:

Lemma 5.2. *The Davydov–Yetter complex of H-mod with coefficients $V, W \in D(H)$ is isomorphic to the cochain complex of vector spaces*

$$0 \longrightarrow \mathrm{Hom}_H\big(V, W\big) \xrightarrow{\delta^0} \mathrm{Hom}_H\big(V, (H \otimes W)_{\mathrm{ad}}\big) \xrightarrow{\delta^1} \mathrm{Hom}_H\big(V, (H^{\otimes 2} \otimes W)_{\mathrm{ad}}\big) \xrightarrow{\delta^2} \ldots$$

with differential

$$\delta^n(\varphi) = \big(\mathrm{id}_H \otimes \varphi\big)\lambda_V + \sum_{i=1}^{n}(-1)^i\big(\mathrm{id}^{\otimes(i-1)} \otimes \Delta \otimes \mathrm{id}^{\otimes(n-i)} \otimes \mathrm{id}_W\big)\varphi + (-1)^{n+1}\big(\mathrm{id}_H^{\otimes n} \otimes \lambda_W\big)\varphi$$

where λ_V, λ_W are the left H-coactions defined in (46).

For trivial coefficients ($V = W = k$), this can be made even more explicit. Since in this case the H-action on $(H^{\otimes n})_{\mathrm{ad}}$ is $h \cdot (x_1 \otimes \ldots \otimes x_n) = \Delta^{(n-1)}\big(h'\big)(x_1 \otimes \ldots \otimes x_n)\Delta^{(n-1)}\big(S(h'')\big)$, we see that

$$C^n_{\mathrm{DY}}(H\text{-mod}) \underset{\Psi}{\xrightarrow{\sim}} \mathrm{Hom}_H\big(k, (H^{\otimes n})_{\mathrm{ad}}\big) = \mathcal{Z}\big(\Delta^{(n-1)}(H)\big) \tag{51}$$

where $\mathcal{Z}\big(\Delta^{(n-1)}(H)\big) \subset H^{\otimes n}$ is the centralizer of the image of the iterated coproduct (45). The isomorphism of Lemma 5.2 is then

$$C^\bullet_{\mathrm{DY}}(H\text{-mod}) \cong \Big(0 \longrightarrow k \xrightarrow{\delta^0} \mathcal{Z}(H) \xrightarrow{\delta^1} \mathcal{Z}(\Delta(H)) \xrightarrow{\delta^2} \ldots\Big) \tag{52}$$

where the complex on the right has the differential

$$\delta^n(\mathbf{x}) = 1 \otimes \mathbf{x} + \sum_{i=1}^{n}(-1)^i(\mathrm{id}^{\otimes(i-1)} \otimes \Delta \otimes \mathrm{id}^{\otimes(n-i)})(\mathbf{x}) + (-1)^{n+1}\mathbf{x} \otimes 1. \tag{53}$$

Hence $C^\bullet_{\mathrm{DY}}(H\text{-mod})$ is a subcomplex of the Cartier complex for the coalgebra H. This was already noted in [Dav97, Prop. 8] and taken as a definition in [ENO05, §6]. Note that $\delta_0 = 0$.

Remark 5.3. As noted in [Dav97, Prop. 7], for the forgetful functor $U : H\text{-mod} \to \mathrm{Vect}_k$ the centralizer condition in (52) disappears and $C^\bullet_{\mathrm{DY}}(U)$ is isomorphic to the complex

$$0 \longrightarrow k \xrightarrow{\delta^0} H \xrightarrow{\delta^1} H^{\otimes 2} \xrightarrow{\delta^2} \ldots \tag{54}$$

with the differential (53). This gives an injective morphism of complexes $\iota_n : C^n_{\mathrm{DY}}(H\text{-mod}) \hookrightarrow C^n_{\mathrm{DY}}(U)$; the induced morphisms $\bar{\iota}_n : H^n_{\mathrm{DY}}(H\text{-mod}) \hookrightarrow H^n_{\mathrm{DY}}(U)$ are not injective in general (see e.g. Remark 5.10 below), except for $n = 1$ because $\delta_0 = 0$.

We use the isomorphism (50) to transport the product of Theorem 4.11: if $\varphi \in \mathrm{Hom}_H\big(V, (H^{\otimes n} \otimes W)_{\mathrm{ad}}\big)$ and $\psi \in \mathrm{Hom}_H\big(U, (H^{\otimes m} \otimes V)_{\mathrm{ad}}\big)$ are cocycles then $\varphi \circ \psi = \Psi\big(\Psi^{-1}(\varphi) \circ \Psi^{-1}(\psi)\big)$ is given by

$$\varphi \circ \psi = (-1)^{nm}(\mathrm{id}_{H^{\otimes m}} \otimes \varphi)\psi \in \mathrm{Hom}_H\big(U, (H^{\otimes(n+m)} \otimes W)_{\mathrm{ad}}\big). \tag{55}$$

In view of the following section, let us restate the isomorphism $\Gamma : C^n_{\mathrm{DY}}(H\text{-mod}; V, W) \xrightarrow{\sim} \mathrm{Bar}^n_{D(H),H}(V, W)$ defined in (35). We use the desciption of the bar resolution $\mathrm{Bar}^\bullet_{D(H),H}(V)$ for the adjunction $D(H)\text{-mod} \leftrightarrows H\text{-mod}$ given in [GHS19, Corollary 4.6]. Let $\big((H^*)^{\otimes n} \otimes V\big)_{\mathrm{coad}}$ be the $D(H)$-module defined by

$$h \cdot (\varphi_1 \otimes \ldots \otimes \varphi_n \otimes v) = \varphi_1\big(S(h^{(1)})?h^{(2n+1)}\big) \otimes \ldots \otimes \varphi_n\big(S(h^{(n)})?h^{(n+2)}\big) \otimes h^{(n+1)}v$$
$$\psi \cdot (\varphi_1 \otimes \ldots \otimes \varphi_n \otimes v) = (\psi * \varphi_1) \otimes \ldots \otimes \varphi_n \otimes v$$

where $*$ is the product in $(H^*)^{\mathrm{op}}$. Then

$$\mathrm{Bar}^n_{D(H),H}(V) = \big((H^*)^{\otimes(n+1)} \otimes V\big)_{\mathrm{coad}} \tag{56}$$

with the differential

$$d_n(\varphi_1 \otimes \ldots \otimes \varphi_{n+1} \otimes v) = \varphi_1 \otimes \ldots \otimes \varphi_n \otimes \varphi_{n+1}v + \sum_{i=1}^{n}(-1)^{n-i+1}\varphi_1 \otimes \ldots \otimes (\varphi_i * \varphi_{i+1}) \otimes \ldots \otimes \varphi_{n+1} \otimes v.$$

We have the isomorphism of complexes

$$\forall n, \quad \widetilde{\Gamma} : \mathrm{Hom}_H\big(V, (H^{\otimes n} \otimes W)_{\mathrm{ad}}\big) \underset{\Psi^{-1}}{\xrightarrow{\sim}} C^n_{\mathrm{DY}}(H\text{-mod}; V, W) \underset{\Gamma}{\xrightarrow{\sim}} \mathrm{Bar}^n_{D(H),H}(V, W) \tag{57}$$

between the complex of Lemma 5.2 and $\mathrm{Bar}^\bullet_{D(H),H}(V, W) = \mathrm{Hom}_{D(H)}\Big(\big((H^*)^{\otimes(\bullet+1)} \otimes V\big)_{\mathrm{coad}}, W\Big)$. In the sequel we will need its inverse, which is simply given by

$$\forall n, \quad \widetilde{\Gamma}^{-1}(\alpha) : v \mapsto h_{i_1} \otimes \ldots \otimes h_{i_n} \otimes \alpha\big(\varepsilon \otimes h^{i_n} \otimes \ldots \otimes h^{i_1} \otimes v\big) \tag{58}$$

where (h_i) is a basis of H with dual basis (h^i) and ε is the counit of H. Finally, note that by definition $\widetilde{\Gamma}$ is compatible with the products $\circ$: $\widetilde{\Gamma}(f \circ g) = \widetilde{\Gamma}(f) \circ \widetilde{\Gamma}(g)$.

5.1.4 Finding DY cocycles thanks to allowable exact sequences

The isomorphism between the groups $\mathrm{Ext}^{\bullet}_{D(H),H}$ and the DY cohomology groups (49) allows us to determine the dimension of the latter thanks to a relatively projective resolution. If some dimension is not 0, one would like to find explicit DY cocycles. A possible way to achieve this without too much computations is to find allowable n-fold exact sequences of $D(H)$-modules. Indeed, there is a DY cocycle associated to such a sequence, thanks to the maps

$$\begin{aligned}&\big\{\text{allowable } n\text{-fold exact sequences from } W \text{ to } V\big\}\\ &\xrightarrow{\eta} \big\{\text{cocycles in } \mathrm{Bar}^n_{D(H),H}(V,W)\big\} \xrightarrow{\widetilde{\Gamma}^{-1}} \big\{\text{cocycles in } \mathrm{Hom}_H\big(V,(H^{\otimes n}\otimes W)_{\mathrm{ad}}\big)\big\}\end{aligned}$$

defined in (13) and (58) respectively. So assume that we have found an allowable n-fold exact sequence $S = \big(0 \to W \to X_n \to \ldots \to X_1 \to V \to 0\big)$. Since we are using the bar resolution descibed explicitly in (56), computing $\eta(S)$ amounts to fill the following diagram:

$$\begin{array}{ccccccccccc}
\big((H^*)^{\otimes(n+1)}\otimes V\big)_{\mathrm{coad}} & \xrightarrow{d_{n+1}} & \big((H^*)^{\otimes n}\otimes V\big)_{\mathrm{coad}} & \xrightarrow{d_n} & \ldots & \xrightarrow{d_1} & \big(H^*\otimes V\big)_{\mathrm{coad}} & \xrightarrow{d_0} & V & \longrightarrow & 0\\
\Big\downarrow{\scriptstyle \eta(S)} & & \Big\downarrow & & & & \Big\downarrow & & \Big\| & & \\
W & \longrightarrow & X_n & \longrightarrow & \ldots & \longrightarrow & X_1 & \longrightarrow & V & \longrightarrow & 0
\end{array}$$

The longer S is, the harder it becomes to fill the diagram. So we decompose S as the Yoneda product of n short exact sequences $S = S_1 \circ \ldots \circ S_n$ and we compute $\eta(S_i)$ for each i. Then we note that

$$\widetilde{\Gamma}^{-1}\eta(S) = \widetilde{\Gamma}^{-1}\eta(S_1\circ\ldots\circ S_n) = \widetilde{\Gamma}^{-1}\big(\eta(S_1)\circ\ldots\circ\eta(S_n)\big) = \widetilde{\Gamma}^{-1}\eta(S_1)\circ\ldots\circ\widetilde{\Gamma}^{-1}\eta(S_1)$$

where on the right hand side $\circ$ is the product (55). Since this product has a very simple expression, the computation becomes feasible and we get an explicit DY n-cocycle. Note moreover that due to the formula (58) for $\widetilde{\Gamma}^{-1}$, it is sufficient to determine the values of the form $\eta(S_i)(\varepsilon\otimes?\otimes?)$. This method will be demonstrated on the examples below.

After this, one would like to show that the so obtained cocycle is not cohomologous to 0. It is equivalent to show that the exact sequence S is not congruent to 0. For 2-fold exact sequences one can use Lemma 2.7 or Corollary 2.8.

5.2 Example: bosonization of exterior algebras

Let $B_k = \Lambda\mathbb{C}^k \rtimes \mathbb{C}[\mathbb{Z}_2]$ be the $\mathbb{C}$-algebra with the presentation

$$B_k = \mathbb{C}\langle x_1,\ldots,x_k,g \,|\, \forall\, i,j,\ \ x_ix_j = -x_jx_i,\ \ gx_i = -x_ig,\ \ g^2 = 1\rangle. \tag{59}$$

It has dimension 2^{k+1}, with basis elements $x_1^{\alpha_1}\ldots x_k^{\alpha_k}g^{\alpha_{k+1}}$ ($\alpha_i \in \{0,1\}$). Its Hopf structure is

$$\begin{array}{lll}\Delta(x_i) = 1\otimes x_i + x_i\otimes g, & \varepsilon(x_i) = 0, & S(x_i) = gx_i,\\ \Delta(g) = g\otimes g, & \varepsilon(g) = 1, & S(g) = g^{-1}.\end{array}$$

5.2.1 Dimension of DY cohomology groups

In [GHS19, §5], the DY cohomologies with trivial coefficients of the identity and forgetful functors have been computed for the category B_k-mod thanks to Theorem 4.5 by finding an explicit G-projective resolution of $\mathbb{C}$. Here we present a faster way to get these results, based on Corollary 4.6 and Corollary 3.10, which moreover explains why the cohomologies for the identity and forgetful functors are isomorphic in the case of trivial coefficients (Proposition 5.4).

Recall that sVect is the category of super-vector spaces, which objects are $\mathbb{Z}_2$-graded $\mathbb{C}$-vector spaces $V = V_0 \oplus V_1$ and which morphisms are $\mathbb{C}$-linear maps respecting the gradings. The tensor product in sVect is defined by

$$(V\otimes W)_0 = (V_0\otimes W_0)\oplus(V_1\otimes W_1), \qquad (V\otimes W)_1 = (V_1\otimes W_0)\oplus(V_0\otimes W_1).$$

The symmetry $c_{V,W} : V \otimes W \to W \otimes V$ is defined by

$$c_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v$$

where v and w are homogeneous elements and $|v| = j$ if $v \in V_j$ ($j \in \{0, 1\}$).

sVect is equivalent as a tensor category to $\mathbb{C}[\mathbb{Z}_2]$-mod. Let H be a Hopf algebra in sVect and write $\mathbb{C}[\mathbb{Z}_2] = \mathbb{C}\langle g | g^2 = 1\rangle$. Then g acts on H by $g \triangleright h = (-1)^{|h|}h$ and by assumption H is a $\mathbb{C}[\mathbb{Z}_2]$-module-algebra. Hence the smash product $H\#\mathbb{C}[\mathbb{Z}_2]$ is an algebra in Vect. With $\Delta(g) = g \otimes g$ then it actually is a Hopf algebra (in Vect) which we denote by $H \rtimes \mathbb{C}[\mathbb{Z}_2]$, and called the bosonization of H. We have an equivalence of tensor categories

$$H\text{-sVect} \cong (H \rtimes \mathbb{C}[\mathbb{Z}_2])\text{-mod} \tag{60}$$

defined by $V_0 = \ker(g - \mathrm{id}), V_1 = \ker(g + \mathrm{id})$, and where we denote H-sVect instead of $H\text{-mod}_{\mathrm{sVect}}$ (which is the category of H-modules internal to sVect).

Now take

$$H = \Lambda\mathbb{C}^k = \langle x_1, \ldots, x_k \,|\, \forall\, i, j,\; x_i x_j = -x_j x_i\rangle$$

with coproduct $\Delta(x_i) = 1 \otimes x_i + x_i \otimes 1$. It is a Hopf algebra in sVect. Its bosonization $\Lambda\mathbb{C}^k \rtimes \mathbb{C}[\mathbb{Z}_2]$ is the Hopf algebra B_k defined in (59).

We want to compute $H^n_{\mathrm{DY}}(B_k\text{-mod})$. According to (49), this is isomorphic to $\mathrm{Ext}^n_{D(B_k),B_k}(\mathbb{C}, \mathbb{C})$. Defining relations for $D(B_k)$ were obtained in [FGR17, App. C], however here we use its presentation from [GHS19, §5]:

$$D(B_k) = \left\langle x_1, \ldots, x_k, y_1, \ldots, y_k, g, h \;\middle|\; \forall\, i, j, \begin{array}{lll} x_i x_j = -x_j x_i, & g x_i = -x_i g, & g^2 = 1, \\ y_i y_j = -y_j y_i, & h y_i = -y_i h, & h^2 = 1, \\ h x_i = -x_i h, & g y_i = -y_i g, & gh = hg, \\ x_i y_j + y_j x_i = \delta_{i,j}(1 - gh) & & \end{array} \right\rangle$$

where $\delta_{i,j}$ is the Kronecker symbol. We see that

$$\pi_\pm = \frac{1 \pm gh}{2}$$

are central orthogonal idempotents and that we have an isomorphism of algebras $D(B_k)\pi_+ \xrightarrow{\sim} B_{2k}$ given by

$$x_i\pi_+ \mapsto x_i, \quad y_i\pi_+ \mapsto x_{k+i}, \quad g\pi_+ = h\pi_+ \mapsto g.$$

Hence the algebra map

$$\iota_+ : B_k \to D(B_k)\pi_+, \quad x \mapsto \iota(x)\pi_+$$

(where $\iota : B_k \to D(B_k)$ is the canonical injection) can be viewed as a homomorphism $\iota_+ : B_k \to B_{2k}$; it is simply given by $\iota_+(x_i) = x_i$, $\iota_+(g) = g$. In particular it is injective and yields the resolvent pair (B_{2k}, B_k). Moreover

$$\mathrm{Ext}^n_{D(B_k),B_k}(\mathbb{C}, \mathbb{C}) \cong \mathrm{Ext}^n_{B_{2k},B_k}(\mathbb{C}, \mathbb{C}). \tag{61}$$

Indeed, if $0 \leftarrow \mathbb{C} \leftarrow P_0 \leftarrow P_1 \leftarrow \ldots$ is a $\big(D(B_k), B_k\big)$-relatively projective resolution, then $0 \leftarrow \pi_+\mathbb{C} = \mathbb{C} \leftarrow \pi_+P_0 \leftarrow \pi_+P_1 \leftarrow \ldots$ is both a $\big(D(B_k), B_k\big)$-relatively projective resolution and a (B_{2k}, B_k)-relatively projective resolution.

Let $U : B_k\text{-mod} \to \mathrm{Vect}$ be the forgetful functor.

Proposition 5.4. *$H^n_{\mathrm{DY}}(B_k\text{-mod}) \cong H^n_{\mathrm{DY}}(U)$.*

Proof. Note that $\Lambda\mathbb{C}^{k+l} = \Lambda\mathbb{C}^k \otimes \Lambda\mathbb{C}^l$ in sVect and that sVect is semisimple. Hence, using (61), (60), Corollary 4.6, Corollary 3.10, the fact that $B_k \cong (B_k^*)^{\mathrm{op}}$ and Lemma 5.1, we get

$$\begin{aligned} H^n_{\mathrm{DY}}(B_k\text{-mod}) &\cong \mathrm{Ext}^n_{D(B_k),B_k}(\mathbb{C},\mathbb{C}) \cong \mathrm{Ext}^n_{B_{2k},B_k}(\mathbb{C},\mathbb{C}) \cong \mathrm{Ext}^n_{\Lambda\mathbb{C}^{2k}\text{-sVect},\Lambda\mathbb{C}^k\text{-sVect}}(\mathbb{C}^{1|0},\mathbb{C}^{1|0}) \\ &\cong \mathrm{Ext}^n_{(\Lambda\mathbb{C}^k\otimes\Lambda\mathbb{C}^k)\text{-sVect},\Lambda\mathbb{C}^k\text{-sVect}}(\mathbb{C}^{1|0}\boxtimes\mathbb{C}^{1|0},\mathbb{C}^{1|0}\boxtimes\mathbb{C}^{1|0}) \cong \mathrm{Ext}^n_{\Lambda\mathbb{C}^k\text{-sVect}}(\mathbb{C}^{1|0},\mathbb{C}^{1|0}) \\ &\cong \mathrm{Ext}^n_{B_k}(\mathbb{C},\mathbb{C}) \cong \mathrm{Ext}^n_{(B_k^*)^{\mathrm{op}}}(\mathbb{C},\mathbb{C}) \cong H^n_{\mathrm{DY}}(U) \end{aligned}$$

where $\mathbb{C}^{1|0} = \mathbb{C} \oplus 0$ (no non-zero elements in odd degree). □

Using the isomorphism $H^n_{\mathrm{DY}}(B_k\text{-mod}) \cong \mathrm{Ext}^n_{(B_k^*)^{\mathrm{op}}}(\mathbb{C},\mathbb{C})$, it is not difficult to compute that (see [GHS19, Rem. 5.11])

$$\dim\big(H^n_{\mathrm{DY}}(B_k\text{-mod})\big) \cong \begin{cases} \binom{k+n-1}{n} & \text{if } n \text{ is even.} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

5.2.2 Explicit cocycles from allowable exact sequences

Here we apply the method described in §5.1.4. For $\epsilon \in \{\pm\}$, let $\mathbb{C}_\epsilon$ be the $D(B_k)$-module defined by $x_i \cdot 1 = y_i \cdot 1 = 0$, $g \cdot 1 = h \cdot 1 = \epsilon$. For $1 \le i \le n$, let Y_i^ϵ be the 2-dimensional $D(B_k)$-module with basis $(u_\epsilon^{(i)}, v_{-\epsilon}^{(i)})$ and defined by

$$\begin{array}{llll} x_j u_\epsilon^{(i)} = 0, & y_j u_\epsilon^{(i)} = \delta_{i,j} v_{-\epsilon}^{(i)}, & g u_\epsilon^{(i)} = \epsilon u_\epsilon^{(i)}, & h u_\epsilon^{(i)} = \epsilon u_\epsilon^{(i)}, \\ x_j v_{-\epsilon}^{(i)} = 0, & y_j v_{-\epsilon}^{(i)} = 0, & g v_{-\epsilon}^{(i)} = -\epsilon v_{-\epsilon}^{(i)}, & h v_{-\epsilon}^{(i)} = -\epsilon v_{-\epsilon}^{(i)}. \end{array}$$

Its subquotient structure can be depicted by

$$Y_i^\epsilon = \begin{array}{c} \mathbb{C}_\epsilon \\ \big\downarrow {\scriptstyle y_i} \\ \mathbb{C}_{-\epsilon} \end{array}$$

so that it is a non-trivial extension of $\mathbb{C}_\epsilon$ by $\mathbb{C}_{-\epsilon}$, thus yielding the non-trivial exact sequence

$$\mathbf{y}_i^\epsilon = \big(0 \longrightarrow \mathbb{C}_{-\epsilon} \longrightarrow Y_i^\epsilon \longrightarrow \mathbb{C}_\epsilon \longrightarrow 0\big).$$

Moreover this short exact sequence splits when we look it in B_k-mod, so it is allowable.

Proposition 5.5. *For any $1 \le i, j \le n$, it holds*

$$\mathbf{y}_i^- \circ \mathbf{y}_j^+ \not\equiv 0 \qquad \textit{and} \qquad \mathbf{y}_i^- \circ \mathbf{y}_j^+ \equiv \mathbf{y}_j^- \circ \mathbf{y}_i^+.$$

The explicit DY cocycle associated to $\mathbf{y}_i^- \circ \mathbf{y}_j^+$ is

$$\widetilde{\Gamma}^{-1}\eta\big(\mathbf{y}_i^- \circ \mathbf{y}_j^+\big) = x_j \otimes x_i g$$

which is an element in $\mathcal{Z}\big(\Delta(B_k)\big) \cong C^2_{\mathrm{DY}}(B_k\text{-mod})$.

Proof. For the first congruence, we use the second item of Lemma 2.7. Take a diagram with exact rows

$$\begin{array}{ccccccccccc} \mathbf{y}_i^- = \big(0 & \longrightarrow & \mathbb{C}_+ & \xrightarrow{\iota_1} & Y_i^- & \longrightarrow & \mathbb{C}_- & \longrightarrow & 0\big) \\ & & \| & & \downarrow {\scriptstyle f} & & \downarrow & & \\ P = \big(0 & \longrightarrow & \mathbb{C}_+ & \xrightarrow{\iota_2} & M & \longrightarrow & Y_j^+ & \longrightarrow & 0\big) \end{array}$$

where the vertical arrow at the right is the obvious injection and where we denote by ι_1 and ι_2 the monomorphisms of $\mathbf{y}_i^-$ and P respectively. As a vector space, M is $Y_j^+ \oplus \mathbb{C}_+$, and hence has the basis

$(u_+^{(j)}, v_-^{(j)}, 1_+)$. Since the trivial representation $\mathbb{C}_+$ is a submodule of M, the representation matrix of M in this basis has the block form

$$\rho_M = \begin{pmatrix} \rho_{Y_j^+} & 0 \\ \sigma & \varepsilon \end{pmatrix}$$

where $\sigma = (\sigma_+, \sigma_-)$ with $\sigma_\pm : D(B_k) \to \mathbb{C}$. Since ρ_M is a morphism of algebras, it holds $\sigma(st) = \sigma(s)\rho_{Y_j^+}(t) + \varepsilon(s)\sigma(t)$ for all $s, t \in D(B_k)$. In particular $\sigma(y_i y_j) = (\sigma_-(y_i), 0)$ while $\sigma(y_j y_i) = (0, 0)$ which implies $\sigma_-(y_i) = 0$ since $y_i y_j = -y_j y_i$. Now, due to the weights, the only possibility for f is $f(u_-^{(i)}) = \lambda v_-^{(j)}$ for some $\lambda \in \mathbb{C}$ and thus $f(v_+^{(i)}) = y_i f(u_-^{(i)}) = \lambda\big(\rho_{Y_j^+}(y_i)(v_-^{(j)}) + \sigma_-(y_i)1_+\big) = 0$. It follows that $f\iota_1(1_+) = f(v_+^{(i)}) = 0$ while $\iota_2(1_+) = 1_+$. Hence it is impossible to fill the diagram.
Now we prove the second congruence. Consider the following modules:

$$\mathrm{V}_+^{i,j} = \begin{array}{ccc} \mathbb{C}_- & & \mathbb{C}_- \\ & {\scriptstyle y_i}\searrow \quad \swarrow{\scriptstyle y_j} & \\ & \mathbb{C}_+ & \end{array} \qquad\qquad \Lambda_{i,j}^+ = \begin{array}{ccc} & \mathbb{C}_+ & \\ & {\scriptstyle y_i}\swarrow \quad \searrow{\scriptstyle y_j} & \\ \mathbb{C}_- & & \mathbb{C}_- \end{array}$$

These diagrams mean that $\mathrm{V}_+^{i,j}$ has the basis $(a_-^{(i)}, a_-^{(j)}, a_+)$ where the subscripts indicate the weight for the action of h and g (they are equal) and $y_i a_-^{(i)} = a_+$, $y_j a_-^{(j)} = a_+$ while all the other actions are equal to 0. Similarly, $\Lambda_{i,j}^+$ has the basis $(b_+, b_-^{(i)}, b_-^{(j)})$ with the obvious actions. Now consider the following diagram, with allowable exact rows:

$$\begin{array}{rccccccccccc}
\mathbf{y}_i^- \circ \mathbf{y}_j^+ = & 0 & \longrightarrow & \mathbb{C}_+ & \longrightarrow & Y_i^- & \longrightarrow & Y_j^+ & \longrightarrow & \mathbb{C}_+ & \longrightarrow & 0 \\
& & & \| & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \| & & \\
& 0 & \longrightarrow & \mathbb{C}_+ & \longrightarrow & \mathrm{V}_+^{i,j} & \xrightarrow{\tau} & \Lambda_{i,j}^+ & \longrightarrow & \mathbb{C}_+ & \longrightarrow & 0 \\
& & & \| & & {\scriptstyle g_1}\uparrow & & {\scriptstyle g_2}\uparrow & & \| & & \\
\mathbf{y}_j^- \circ \mathbf{y}_i^+ = & 0 & \longrightarrow & \mathbb{C}_+ & \longrightarrow & Y_j^- & \longrightarrow & Y_i^+ & \longrightarrow & \mathbb{C}_+ & \longrightarrow & 0
\end{array}$$

where $f_1(u_-^{(i)}) = a_-^{(i)}$, $f_2(u_+^{(j)}) = b_+$, $\tau(a_-^{(i)}) = b_-^{(j)}$, $\tau(a_-^{(j)}) = b_-^{(i)}$, $g_1(u_-^{(j)}) = a_-^{(j)}$, $g_2(u_+^{(i)}) = b_+$. It is straightforward to check that the diagram is commutative, so that the vertical arrows are morphisms of exact sequences. The existence of such a chain of morphisms of allowable exact sequences between $\mathbf{y}_i^- \circ \mathbf{y}_j^+$ and $\mathbf{y}_j^- \circ \mathbf{y}_i^+$ is equivalent to the desired congruence, see Prop 5.2 of Chap. III and p. 370 in [ML75].
Let us now compute $\eta(\mathbf{y}_i^-)$ and $\eta(\mathbf{y}_j^+)$. For $\mathbf{y}_i^-$, we have to fill the commutative diagram of $D(B_k)$-modules

$$\begin{array}{ccccccc}
\big((B_k^*)^{\otimes 2} \otimes \mathbb{C}_-\big)_{\mathrm{coad}} & \xrightarrow{d_1} & \big(B_k^* \otimes \mathbb{C}_-\big)_{\mathrm{coad}} & \xrightarrow{\mathrm{act}} & \mathbb{C}_- & \longrightarrow & 0 \\
\downarrow{\scriptstyle \eta_1 = \eta(\mathbf{y}_i^-)} & & \downarrow{\scriptstyle \eta_0} & & \| & & \\
\mathbb{C}_+ & \longrightarrow & Y_i^- & \longrightarrow & \mathbb{C}_- & \longrightarrow & 0
\end{array}$$

Let $\varphi_\pm = \frac{\varepsilon \pm h}{2}$, then the elements $y_{i_1} \ldots y_{i_l}\varphi_\pm$ (with $i_1 < \ldots < i_l$) are a basis of $(B_k^*)^{\mathrm{op}}$. A solution is easily found:

$\eta_0(\varphi_- \otimes 1_-) = v_-$, $\eta_0(y_i\varphi_- \otimes 1_-) = v_+$, and 0 on the other basis elements,
$\eta\big(\mathbf{y}_i^-\big)(\varepsilon \otimes y_i\varphi_- \otimes 1_-) = -1_+$, and 0 on the other basis elements of the form $\varepsilon \otimes (\ldots) \otimes 1_-$.

$\eta\big(\mathbf{y}_i^-\big)$ can be extended to all the elements by $(B_k^*)^{\mathrm{op}}$-linearity but we do not need this. One similarly finds

$\eta\big(\mathbf{y}_j^+\big)(\varepsilon \otimes y_j\varphi_+ \otimes 1_+) = -1_-$, and 0 on the other basis elements of the form $\varepsilon \otimes (\ldots) \otimes 1_+$.

Let $(x_{i_1}\dots x_{i_l})^*, (x_{i_1}\dots x_{i_l}g)^*$ be the elements of the dual basis of the monomial basis of B_k. Recall from [GHS19, §5] that by definition $y_i = x_i^* - (x_ig)^*$, $h = 1^* - g^*$. It follows that $x_i^* = y_i\varphi_+$ and $(x_ig)^* = -y_i\varphi_-$, and thus

$$\widetilde{\Gamma}^{-1}\eta\big(\mathbf{y}_i^-\big) : \mathbb{C}_- \to B_k \otimes \mathbb{C}_+, \quad 1_- \mapsto x_ig \otimes 1_+,$$
$$\widetilde{\Gamma}^{-1}\eta\big(\mathbf{y}_j^+\big) : \mathbb{C}_+ \to B_k \otimes \mathbb{C}_-, \quad 1_+ \mapsto -x_j \otimes 1_-.$$

Finally, thanks to the product (55), we get $\widetilde{\Gamma}^{-1}\eta\big(\mathbf{y}_i^- \circ \mathbf{y}_j^+\big)(1_+) = \widetilde{\Gamma}^{-1}\eta\big(\mathbf{y}_i^-\big) \circ \widetilde{\Gamma}^{-1}\eta\big(\mathbf{y}_j^+\big)(1_+) = x_j \otimes x_ig \otimes 1_+$. □

Note that for n even, the Yoneda products $\mathbf{y}_{i_1}^- \circ \mathbf{y}_{i_2}^+ \circ \dots \circ \mathbf{y}_{i_{n-1}}^- \circ \mathbf{y}_{i_n}^+$ with $1 \leq i_1, \dots, i_n \leq k$ provide several examples of n-fold allowable exact sequences from $\mathbb{C}$ to $\mathbb{C}$. Thanks to the second congruence in the previous proposition any such product is congruent to an ordered product, in which $1 \leq i_1 \leq \dots \leq i_n \leq k$. There are $\binom{k+n-1}{n}$ such ordered products, which is exactly the dimension of $\mathrm{YExt}^n_{D(B_k),B_k}(\mathbb{C},\mathbb{C})$ as we have seen previously. Hence this family of allowable exact sequences is an obvious candidate for a basis of $\mathrm{YExt}^n_{D(B_k),B_k}(\mathbb{C},\mathbb{C})$. The corresponding explicit DY n-cocycles are simply the tensor products of the DY 2-cocycles found previously:

$$\widetilde{\Gamma}^{-1}\eta\big(\mathbf{y}_{i_1}^- \circ \mathbf{y}_{i_2}^+ \circ \dots \circ \mathbf{y}_{i_{n-1}}^- \circ \mathbf{y}_{i_n}^+\big) = \widetilde{\Gamma}^{-1}\eta\big(\mathbf{y}_{i_{n-1}}^- \circ \mathbf{y}_{i_n}^+\big) \otimes \dots \otimes \widetilde{\Gamma}^{-1}\eta\big(\mathbf{y}_{i_1}^- \circ \mathbf{y}_{i_2}^+\big).$$

We conjecture that these elements form a basis of $H^n_{\mathrm{DY}}(B_k\text{-mod})$.

5.3 Example: Taft algebras

Let $q \in \mathbb{C}$ be a primitive n-th root of unity ($n \geq 2$). The Taft algebra T_q is

$$T_q = \big\langle x, g \,|\, gx = qxg,\ x^n = 0,\ g^n = 1\big\rangle.$$

It is a n^2-dimensional Hopf $\mathbb{C}$-algebra with basis $(x^ig^j)_{0\leq i,j\leq n-1}$ and with Hopf structure given by

$$\begin{array}{lll} \Delta(x) = 1 \otimes x + x \otimes g, & S(x) = -xg^{-1}, & \varepsilon(x) = 0, \\ \Delta(g) = g \otimes g, & S(g) = g^{-1}, & \varepsilon(g) = 1. \end{array}$$

Let $X_d^{(s)} = \mathrm{vect}(v_0^{(s)}, \dots, v_{d-1}^{(s)})$ ($1 \leq d \leq n$, $0 \leq s \leq n-1$) be the T_q-module defined by

$$gv_i^{(s)} = q^{s+i}v_i^{(s)}, \quad xv_i^{(s)} = v_{i+1}^{(s)}, \quad xv_{d-1}^{(s)} = 0.$$

Any indecomposable T_q-module is isomorphic to some $X_d^{(s)}$.

5.3.1 Dimension of DY cohomology groups

We will compute the DY cohomology of T_q-mod thanks to (49). The first task is to describe the Drinfeld double $D(T_q) = (T_q^*)^{\mathrm{op}} \otimes T_q$. In order to determine $(T_q^*)^{\mathrm{op}}$, we use the matrix coefficients of $X_2^{(0)}$ in its basis $(v_0^{(0)}, v_1^{(0)})$. The representation matrix has the form

$$\rho_{X_2^{(0)}} = \begin{pmatrix} \varepsilon & 0 \\ y & h \end{pmatrix}$$

where ε is the counit. It is not difficult to show that

$$\langle y^kh^l, x^ig^j\rangle = q^{lj}\,(k)_q!\,\delta_{i,k} \tag{62}$$

(where $(k)_q = \frac{1-q^k}{1-q}$ and $(k)_q! = (1)_q(2)_q\dots(k)_q$) and it easily follows that the monomials y^kh^l form a basis of $(T_q^*)^{\mathrm{op}}$. Moreover, we have the presentation

$$(T_q^*)^{\mathrm{op}} = \big\langle y, h \,|\, hy = qyh,\ y^n = 0,\ h^n = 1\big\rangle.$$

The Hopf structure is given by

$$\begin{array}{lll}\Delta(y) = y \otimes 1 + h \otimes y, & S(y) = -h^{-1}y, & \varepsilon(y) = 0,\\ \Delta(h) = h \otimes h, & S(h) = h^{-1}, & \varepsilon(h) = 1.\end{array}$$

The exchange relations in $D(T_q)$ are easily computed and we find

$$D(T_q) = \left\langle x, y, g, h \;\middle|\; \begin{array}{llll} gx = qxg, & x^n = 0, & g^n = 1, & \\ hy = qyh, & y^n = 0, & h^n = 1, & \\ hx = q^{-1}xh, & gy = q^{-1}yg, & gh = hg, & [x,y] = h - g \end{array} \right\rangle.$$

Let $\mathcal{V}^{(s,t)}$ (with $0 \leq s, t \leq n-1$) be the $D(T_q)$-module generated by a vector $v^{(s,t)}$ such that

$$xv^{(s,t)} = 0, \quad gv^{(s,t)} = q^s v^{(s,t)}, \quad hv^{(s,t)} = q^t v^{(s,t)}$$

(formally it is an induced representation). It has the basis $v_i^{(s,t)} = y^i v^{(s,t)}$ ($0 \leq i \leq n-1$) with actions

$$xv_i^{(s,t)} = (i)_q(q^t - q^{s+1-i})v_{i-1}^{(s,t)}, \quad yv_i^{(s,t)} = v_{i+1}^{(s,t)}, \quad gv_i^{(s,t)} = q^{s-i}v_i^{(s,t)}, \quad hv_i^{(s,t)} = q^{t+i}v_i^{(s,t)}$$

where we used that $xy^i = y^i x + (i)_q y^{i-1}\big(h - q^{1-i}g\big)$. This gives the familiar ladder structure:

$$\begin{array}{ll} v_0^{(s,t)} \underset{x}{\overset{y}{\rightleftarrows}} \;\ldots\; \underset{x}{\overset{y}{\rightleftarrows}} v_{s-t}^{(s,t)} \xrightarrow{\;y\;} v_{s-t+1}^{(s,t)} \underset{x}{\overset{y}{\rightleftarrows}} \;\ldots\; \underset{x}{\overset{y}{\rightleftarrows}} v_{n-1}^{(s,t)} & \text{if } t \leq s+1, \\ v_0^{(s,t)} \underset{x}{\overset{y}{\rightleftarrows}} \;\ldots\; \underset{x}{\overset{y}{\rightleftarrows}} v_{n+s-t}^{(s,t)} \xrightarrow{\;y\;} v_{n+s-t+1}^{(s,t)} \underset{x}{\overset{y}{\rightleftarrows}} \;\ldots\; \underset{x}{\overset{y}{\rightleftarrows}} v_{n-1}^{(s,t)} & \text{if } t > s+1. \end{array} \tag{63}$$

Lemma 5.6. *We have $\mathcal{F}\big(X_1^{(s)}\big) = \bigoplus_{t=0}^{n-1} \mathcal{V}^{(s,t)}$, where $\mathcal{F}$ is the induction functor described in* (47)*; hence the modules $\mathcal{V}^{(s,t)}$ are relatively projective.*

Proof. We have $\mathcal{F}\big(X_1^{(s)}\big) = (T_q^*)^{\mathrm{op}} \otimes X_1^{(s)}$ with the $D(T_q)$-action from (48). For $0 \leq t \leq n-1$, let $\varphi_t = \frac{1}{n}\sum_{j=0}^{n-1} q^{-jt}h^j$. Then $h\varphi_t = q^t\varphi_t$ and $(y^i\varphi_t)_{0\leq i,t\leq n-1}$ is a basis of $(T_q^*)^{\mathrm{op}}$. Let $v^{(s)}$ be the basis vector of $X_1^{(s)}$. Since

$$\begin{aligned} x(\varphi_t \otimes v^{(s)}) &= \varphi_{t-1} \otimes xv^{(s)} = 0,\\ g(\varphi_t \otimes v^{(s)}) &= \varphi_t \otimes gv^{(s)} = q^s(\varphi_t \otimes v^{(s)}),\\ h(\varphi_t \otimes v^{(s)}) &= h\varphi_t \otimes v^{(s)} = q^t(\varphi_t \otimes v^{(s)}), \end{aligned}$$

the subspace $\mathrm{vect}\big(y^i\varphi_t \otimes v^{(s)}\big)_{0\leq i\leq n-1}$ is isomorphic to $\mathcal{V}^{(s,t)}$. The last claim is due to Lemma 2.1. $\square$

Note in particular that $\mathcal{V}^{(0,0)}$ is the relatively projective cover of $\mathbb{C}$ (Definition 2.12).

Consider the sequence of $D(T_q)$-modules

$$0 \longleftarrow \mathbb{C} \xleftarrow{d_0} \mathcal{V}^{(0,0)} \xleftarrow{d_1} \mathcal{V}^{(n-1,1)} \xleftarrow{d_2} \mathcal{V}^{(0,0)} \xleftarrow{d_1} \mathcal{V}^{(n-1,1)} \xleftarrow{d_2} \ldots \tag{64}$$

where

$$d_0(v_i^{(0,0)}) = \begin{cases} 1 & \text{if } i = 0\\ 0 & \text{else}\end{cases}, \quad d_1(v_i^{(n-1,1)}) = \begin{cases} v_{i+1}^{(0,0)} & \text{if } i < n-1\\ 0 & \text{else}\end{cases}, \quad d_2(v_i^{(0,0)}) = \begin{cases} v_{n-1}^{(n-1,1)} & \text{if } i = 0\\ 0 & \text{else}\end{cases}$$

This is clearly an exact sequence. Moreover, if we apply the forgetful functor $\mathcal{U} : D(T_q)\text{-mod} \to T_q\text{-mod}$ to it, we obtain the split sequence

$$0 \leftarrow \mathbb{C} = X_1^{(0)} \leftarrow X_1^{(0)} \oplus X_{n-1}^{(n-1)} \leftarrow X_{n-1}^{(n-1)} \oplus X_1^{(0)} \leftarrow X_1^{(0)} \oplus X_{n-1}^{(n-1)} \leftarrow \ldots$$

This implies that (64) is allowable and hence is a relatively projective resolution. For trivial coefficients, the resulting sequence is then

$$\begin{aligned}&\mathrm{Hom}_{D(T_q)}(\mathcal{V}^{(0,0)},\mathbb{C}) \longrightarrow \mathrm{Hom}_{D(T_q)}(\mathcal{V}^{(n-1,1)},\mathbb{C}) \longrightarrow \mathrm{Hom}_{D(T_q)}(\mathcal{V}^{(0,0)},\mathbb{C}) \longrightarrow \ldots\\ &= \mathbb{C} \longrightarrow 0 \longrightarrow \mathbb{C} \longrightarrow \ldots\end{aligned}$$

and we get

$$H^k_{\mathrm{DY}}(T_q\text{-mod}) \cong \mathrm{Ext}^k_{D(T_q),T_q}(\mathbb{C},\mathbb{C}) = \begin{cases}\mathbb{C} & \text{if } k \text{ is even}\\ 0 & \text{if } k \text{ is odd}\end{cases}$$

Remark 5.7. The complex (64) is in particular a projective resolution of $\mathbb{C}$ in $(T_q^*)^{\mathrm{op}}$-mod. Since

$$\mathrm{Hom}_{D(T_q)}(\mathcal{V}^{(0,0)},\mathbb{C}) = \mathrm{Hom}_{(T_q^*)^{\mathrm{op}}}(\mathcal{V}^{(0,0)},\mathbb{C}), \quad \mathrm{Hom}_{D(T_q)}(\mathcal{V}^{(n-1,1)},\mathbb{C}) = \mathrm{Hom}_{(T_q^*)^{\mathrm{op}}}(\mathcal{V}^{(n-1,1)},\mathbb{C}),$$

it holds by Lemma 5.1 that $H^k_{\mathrm{DY}}(T_q\text{-mod}) \cong \mathrm{Ext}^k_{(T_q^*)^{\mathrm{op}}}(\mathbb{C},\mathbb{C}) \cong H^k_{\mathrm{DY}}(U)$, where $U : T_q\text{-mod} \to \mathrm{Vect}_{\mathbb{C}}$ is the forgetful functor.

5.3.2 Explicit cocycles from allowable exact sequences

Here again we apply the method described in §5.1.4. Let $\mathcal{K}$ be the simple module which is the quotient of $\mathcal{V}^{(n-1,1)}$ by $\mathbb{C}v^{(n-1,1)}_{n-1}$ (see the picture in (63)); we denote by $(s_i)_{0\le i\le n-2}$ the basis of $\mathcal{K}$. It is also a submodule of $\mathcal{V}^{(0,0)}$. Hence we obtain two allowable short exact sequences:

$$S_1 = \big(0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{V}^{(n-1,1)} \longrightarrow \mathcal{K} \longrightarrow 0\big), \qquad S_2 = \big(0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{V}^{(0,0)} \longrightarrow \mathbb{C} \longrightarrow 0\big).$$

We denote by S their Yoneda product:

$$S = S_1 \circ S_2 = \big(0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{V}^{(n-1,1)} \longrightarrow \mathcal{V}^{(0,0)} \longrightarrow \mathbb{C} \longrightarrow 0\big). \tag{65}$$

For the next proposition, recall from §4.4 that $H^\bullet_{\mathrm{DY}}(F) = \bigoplus_{k\ge 0} H^k_{\mathrm{DY}}(F)$ is an algebra for the product $f \circ g = (-1)^{|f||g|} g \otimes f$ (where f, g are homogeneous elements of degree $|f|, |g|$ respectively).

Proposition 5.8. *1. The* 2*-fold exact sequence* S *in* (65) *is not congruent to* 0 *and thus is a basis of* $\mathrm{YExt}^2_{D(T_q),T_q}(\mathbb{C},\mathbb{C})$.
2. The explicit DY cocycle associated to S *is*

$$\widetilde{\Gamma}^{-1}\eta(S) = -\sum_{i=1}^{n-1} \frac{1}{(i)_q!(n-i)_q!} x^i \otimes x^{n-i} g^i$$

which is an element in $\mathcal{Z}\big(\Delta(T_q)\big) \cong C^2_{\mathrm{DY}}(T_q\text{-mod})$*. It is a basis of* $H^2_{\mathrm{DY}}(T_q\text{-mod})$*.*
3. The DY cocycle $\widetilde{\Gamma}^{-1}\eta(S)^{\circ k} = \widetilde{\Gamma}^{-1}\eta(S)^{\otimes k}$ *is a basis of* $H^{2k}_{\mathrm{DY}}(T_q\text{-mod})$ *for all* k *and it follows that* $H^\bullet_{\mathrm{DY}}(T_q\text{-mod}) \cong \mathbb{C}[X^2]$ *as a graded* $\mathbb{C}$*-algebra.*

Proof. 1. We show that $\mathrm{Ext}^1_{D(T_q),T_q}\big(\mathbb{C},\mathcal{V}^{(n-1,1)}\big) = 0$; then according to Corollary 2.8 it follows that $S = S_1 \circ S_2 \not\equiv 0$. Recall the relatively projective resolution of $\mathbb{C}$ in (64) and consider

$$\mathrm{Hom}_{D(T_q)}\big(\mathcal{V}^{(0,0)},\mathcal{V}^{(n-1,1)}\big) \xrightarrow{d_1^*} \mathrm{Hom}_{D(T_q)}\big(\mathcal{V}^{(n-1,1)},\mathcal{V}^{(n-1,1)}\big) \xrightarrow{d_2^*} \mathrm{Hom}_{D(T_q)}\big(\mathcal{V}^{(0,0)},\mathcal{V}^{(n-1,1)}\big)$$

It is easy to see that $\mathrm{Hom}_{D(T_q)}\big(\mathcal{V}^{(n-1,1)},\mathcal{V}^{(n-1,1)}\big) = \mathbb{C}\,\mathrm{id}$ and thus $\ker(d_2^*) = 0$, whence the claim.
2. Let us determine $\eta(S_1), \eta(S_2)$. For S_1 we must fill

$$\begin{array}{ccccccc}\big((T_q^*)^{\otimes 2}\otimes\mathcal{K}\big)_{\mathrm{coad}} & \xrightarrow{d_1} & \big(T_q^*\otimes\mathcal{K}\big)_{\mathrm{coad}} & \xrightarrow{\mathrm{act}} & \mathcal{K} & \longrightarrow & 0\\ \downarrow{\scriptstyle \eta_1=\eta(S_1)} & & \downarrow{\scriptstyle \eta_0} & & \| & & \\ \mathbb{C} & \longrightarrow & \mathcal{V}^{(n-1,1)} & \longrightarrow & \mathcal{K} & \longrightarrow & 0\end{array}$$

It is straightforward to find a solution:

$$\eta_0(y^i\varphi_j \otimes s_k) = \delta_{j,k+1}\delta_{i+k\le n-1}v_{i+k}^{(n-1,1)}, \qquad \eta(S_1)(\varepsilon \otimes y^i\varphi_j \otimes s_k) = -\delta_{j,k+1}\delta_{n-i,k+1}.$$

where $\varphi_j = \frac{1}{n}\sum_{l=0}^{n-1} q^{-lj}h^l$. For S_2 one finds similarly:

$$\eta_0(y^i\varphi_j \otimes 1) = \delta_{j,0}v_i^{(0,0)}, \qquad \eta(S_2)(\varepsilon \otimes y^i\varphi_j \otimes 1) = -\delta_{i>0}\delta_{j,0}x_{i-1}$$

The morphisms $\eta(S_1), \eta(S_2)$ can be easily extended from these values by the $(T_q^*)^{\mathrm{op}}$-linearity, but we do not need this. From (62), the dual basis $\big((x^ig^j)^*\big)_{0\le i,j\le n-1}$ of the monomial basis $\big(x^ig^j\big)_{0\le k,l\le n-1}$ is given by $(x^ig^j)^* = \frac{1}{(i)_q!}y^i\varphi_j$. With this we can compute the DY cocycles associated to $\eta(S_1), \eta(S_2)$ through the isomorphism (58):

$$\widetilde{\Gamma}^{-1}(\eta(S_1)) : s_k \mapsto \sum_{i,j=0}^{n-1} x^ig^j \otimes \frac{1}{(i)_q!}\eta(S_1)(\varepsilon \otimes y^i\varphi_k \otimes s_k) = \frac{-1}{(n-k-1)_q!}x^{n-k-1}g^{k+1} \otimes 1,$$

$$\widetilde{\Gamma}^{-1}(\eta(S_2)) : 1 \mapsto \sum_{i,j=0}^{n-1} x^ig^j \otimes \frac{1}{(i)_q!}\eta(S_2)(\varepsilon \otimes y^i\varphi_k \otimes 1) = -\sum_{i=1}^{n-1} x^i \otimes \frac{1}{(i)_q!}s_{i-1}.$$

Then $\widetilde{\Gamma}^{-1}(\eta(S)) = \widetilde{\Gamma}^{-1}(\eta(S_1)) \circ \widetilde{\Gamma}^{-1}(\eta(S_2))$ and the formula (55) for the product $\circ$ gives the result.
3. Since $\mathcal{V}^{(0,0)}$ and $\mathcal{V}^{(n-1,1)}$ are relatively projective objects (Lemma 5.6), the sequences S_1, S_2 yield for $k \ge 1$ the connecting isomorphisms described in (44):

$$\begin{array}{ccccc} H^k_{\mathrm{DY}}(T_q\text{-mod}, \mathbb{C}, \mathbb{C}) & \to & H^{k+1}_{\mathrm{DY}}(T_q\text{-mod}, \mathcal{K}, \mathbb{C}) & \to & H^{k+2}_{\mathrm{DY}}(T_q\text{-mod}, \mathbb{C}, \mathbb{C}) \\ g & \mapsto & (-1)^l g \circ \Gamma^{-1}(\eta(S_1)) & \mapsto & -g \circ \Gamma^{-1}(\eta(S_1)) \circ \Gamma^{-1}(\eta(S_2)) \end{array}$$

Hence $H^{2k}_{\mathrm{DY}}(T_q\text{-mod}, \mathbb{C}, \mathbb{C})$ is generated by $\big(\Gamma^{-1}(\eta(S_1)) \circ \Gamma^{-1}(\eta(S_2))\big)^{\circ k} = \Gamma^{-1}(\eta(S))^{\otimes k}$. Applying the isomorphism Ψ to this element, we get the claim; indeed, recall from (57) that $\widetilde{\Gamma}^{-1} = \Psi\Gamma^{-1}$, where Ψ simply encodes an element in $C^k_{\mathrm{DY}}(H\text{-mod})$ by a more concrete element in $\mathcal{Z}\big(\Delta^{(k-1)}(H)\big)$. □

5.4 Example: restricted quantum group $\bar{U}_{\mathbf{i}}(\mathfrak{sl}_2)$

Let $\mathbf{i} = \sqrt{-1}$ and $\bar{U}_{\mathbf{i}} = \bar{U}_{\mathbf{i}}(\mathfrak{sl}_2)$ be the $\mathbb{C}$-algebra generated by E, F, K modulo

$$KE = -EK, \quad KF = -FK, \quad EF - FE = -\frac{\mathbf{i}}{2}(K - K^3), \quad E^2 = F^2 = 0, \quad K^4 = 1. \tag{66}$$

The Hopf structure is given by

$$\begin{array}{lll} \Delta(E) = 1 \otimes E + E \otimes K, & \Delta(F) = F \otimes 1 + K^{-1} \otimes F, & \Delta(K) = K \otimes K, \\ \varepsilon(E) = 0, & \varepsilon(F) = 0, & \varepsilon(K) = 1, \\ S(E) = -EK^{-1}, & S(F) = -KF, & S(K) = K^{-1}. \end{array}$$

Let $\rho : \bar{U}_i \to \mathrm{End}(\mathbb{C}^2)$ be the fundamental representation defined by $\rho(E) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\rho(F) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $\rho(K) = \left(\begin{smallmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{smallmatrix}\right)$. Write $\rho = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, so that $a, b, c, d \in \bar{U}_{\mathbf{i}}^*$. Then $(\bar{U}_{\mathbf{i}}^*)^{\mathrm{op}}$ is generated by b, c, d with relations

$$bc = cb, \quad db = -\mathbf{i}bd, \quad dc = -\mathbf{i}cd, \quad b^2 = c^2 = 0, \quad d^4 = 1. \tag{67}$$

The element a is not required as a generator due to the q-determinant relation $da + \mathbf{i}cb = 1$, which gives $a = d^3 + \mathbf{i}bcd^3$. The exchange relations of the Drinfeld double are easily computed: $D(\bar{U}_{\mathbf{i}})$ is the $\mathbb{C}$-algebra generated by E, F, K, b, c, d modulo the relations (66)–(67) above and

$$\begin{array}{llll} Ea = \mathbf{i}aE + cK, & Eb = -\mathbf{i}bE + a - dK, & Ec = \mathbf{i}cE, & Ed = -\mathbf{i}dE + c, \\ Fa = \mathbf{i}aF + \mathbf{i}bK^{-1}, & Fb = \mathbf{i}bF, & Fc = -\mathbf{i}cF + \mathbf{i}a - \mathbf{i}dK^{-1}, & Fd = -\mathbf{i}dF + ib, \\ Ka = aK, & Kb = -bK, & Kc = -cK, & Kd = dK. \end{array}$$

It is useful to introduce the orthogonal idempotents

$$\varphi_l = \frac{1}{4}\sum_{j=0}^{3} \mathbf{i}^{-jl} d^j \tag{68}$$

where $l \in \mathbb{Z}$ is taken modulo 4. They satisfy $\varepsilon(\varphi_l) = \delta_{l,0}$ and

$$d\varphi_l = \mathbf{i}^l \varphi_l, \qquad E\varphi_l = \varphi_{l+1}E + \frac{\mathbf{i}^{-l}}{2}c(\varphi_l + \varphi_{l+2}), \qquad F\varphi_l = \varphi_{l+1}F + \frac{\mathbf{i}^{1-l}}{2}b(\varphi_l + \varphi_{l+2}).$$

Moreover $\big(b^i c^j \varphi_l\big)_{0\leq i,j\leq 1,\, 0\leq l\leq 3}$ is a basis of $(\bar{U}_{\mathbf{i}}^*)^{\mathrm{op}}$.

5.4.1 Dimension of DY cohomology groups

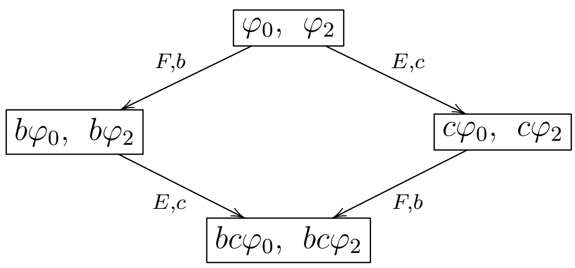
It is too difficult to find a relatively projective resolution in this example. Instead, we will use the dimension formulas from Corollary 4.9. So let us determine the relatively projective cover $R_{\mathbb{C}}$ of the trivial $D(\bar{U}_{\mathbf{i}})$-module $\mathbb{C}$; recall from Remark 2.14 that $R_{\mathbb{C}}$ is the minimal direct summand of $G(\mathbb{C})$ on which the counit $\varepsilon : G(\mathbb{C}) = (\bar{U}_{\mathbf{i}}^*)_{\mathrm{coad}} \to \mathbb{C}$ does not vanish. We find that

$$(\bar{U}_{\mathbf{i}}^*)_{\mathrm{coad}} = \big\langle b^i c^j \varphi_1 \big\rangle_{0\leq i,j\leq 1} \oplus \big\langle b^i c^j \varphi_3 \big\rangle_{0\leq i,j\leq 1} \oplus \big\langle b^i c^j \varphi_0,\, b^i c^j \varphi_2 \big\rangle_{0\leq i,j\leq 1}$$

as a $D(\bar{U}_{\mathbf{i}})$-module, where $\langle \ldots \rangle$ means linear span. The direct summands $\big\langle b^i c^j \varphi_1 \big\rangle_{0\leq i,j\leq 1}$ and $\big\langle b^i c^j \varphi_3 \big\rangle_{0\leq i,j\leq 1}$ both are simple; the counit vanishes on them because $\varepsilon(\varphi_l) = \delta_{l,0}$. So the relevant direct summand is $R_{\mathbb{C}} = \big\langle b^i c^j \varphi_0,\, b^i c^j \varphi_2 \big\rangle_{0\leq i,j\leq 1}$. It is indecomposable and its generating elements are a basis of eigenvectors for d on which the $(\bar{U}_{\mathbf{i}}^*)^{\mathrm{op}}$-action is obvious (given by multiplication) and where the $\bar{U}_{\mathbf{i}}$-action is entirely determined by

$$\begin{aligned}
&E \cdot \varphi_0 = \tfrac{1}{2}(c\varphi_0 + c\varphi_2), \quad E \cdot \varphi_2 = -\tfrac{1}{2}(c\varphi_0 + c\varphi_2),\\
&F \cdot \varphi_0 = \tfrac{\mathbf{i}}{2}(b\varphi_0 + b\varphi_2), \quad F \cdot \varphi_2 = -\tfrac{\mathbf{i}}{2}(b\varphi_0 + b\varphi_2),\\
&\forall x,\ K \cdot x = d^2 \cdot x.
\end{aligned}$$

The action can be schematized by



Let $K_{\mathbb{C}} = \ker(\varepsilon_{|R_{\mathbb{C}}})$, which is generated by φ_2. By definition we have the allowable short exact sequence $0 \longrightarrow K_{\mathbb{C}} \longrightarrow R_{\mathbb{C}} \longrightarrow \mathbb{C} \longrightarrow 0$. It is not difficult to show that

$$R_{\mathbb{C}}^{\vee} \cong R_{\mathbb{C}} \quad \text{and} \quad K_{\mathbb{C}}^{\vee} \cong R_{\mathbb{C}}/\langle bc\varphi_2 \rangle.$$

Note that $H^0_{\mathrm{DY}}(\bar{U}_{\mathbf{i}}\text{-mod}) \cong \mathbb{C}$ (as always for identity functor with trivial coefficients) and $H^1_{\mathrm{DY}}(\bar{U}_{\mathbf{i}}\text{-mod}) = 0$ by Proposition 4.7. For $H^2_{\mathrm{DY}}(\bar{U}_{\mathbf{i}}\text{-mod})$, Corollary 3.6 and easy computations gives

$$\dim H^2_{\mathrm{DY}}(\bar{U}_{\mathbf{i}}\text{-mod}) = \dim \mathrm{Hom}_{D(\bar{U}_{\mathbf{i}})}(K_{\mathbb{C}}, K_{\mathbb{C}}^{\vee}) - \dim \mathrm{Hom}_{D(\bar{U}_{\mathbf{i}})}(R_{\mathbb{C}}, K_{\mathbb{C}}^{\vee}) = 2 - 2 = 0.$$

For higher cohomology groups, the computation of the Hom spaces appearing in Corollary 3.5 becomes much more difficult because it requires to determine the indecomposable decomposition of

$(K_{\mathbb{C}}^{\vee})^{\otimes(n-1)}$. However, using that $\mathrm{Hom}_{\mathcal{C}}(V,W) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, W \otimes V^{\vee})$ in any tensor category $\mathcal{C}$ and the fact that $R_{\mathbb{C}}$ is self-dual we get

$$\dim H^n_{\mathrm{DY}}(\bar{U}_{\mathbf{i}}\text{-mod}) = \dim \mathrm{Inv}\big((K_{\mathbb{C}}^{\vee})^{\otimes n}\big) - \dim \mathrm{Inv}\big((K_{\mathbb{C}}^{\vee})^{\otimes(n-1)} \otimes R_{\mathbb{C}}\big) + \dim \mathrm{Inv}\big((K_{\mathbb{C}}^{\vee})^{\otimes(n-1)}\big) \quad (69)$$

where $\mathrm{Inv}(M) = \mathrm{Hom}_{D(\bar{U}_{\mathbf{i}})}(\mathbb{C}, M)$ is the subspace of invariant elements in a $D(\bar{U}_{\mathbf{i}})$-module M. Computing the dimension of these invariant subspaces simply amounts to find the number of solutions of a homogeneous linear system, which we did with a program written in GAP4[5]. For $n = 3$ and $n = 4$ we find:

$$\dim H^3_{\mathrm{DY}}(\bar{U}_{\mathbf{i}}\text{-mod}) = 10 - 9 + 2 = 3, \qquad \dim H^4_{\mathrm{DY}}(\bar{U}_{\mathbf{i}}\text{-mod}) = 44 - 54 + 10 = 0. \quad (70)$$

5.4.2 Explicit cocycles

Here we address the problem of finding explicit cocycles in $H^3_{\mathrm{DY}}(\bar{U}_{\mathbf{i}}\text{-mod})$. Thanks to two allowable 3-fold exact sequences we constructed two explicit DY 3-cocycles with the method of §5.1.4, as we now explain. For $\lambda, \mu \in \{\pm 1, \pm \mathbf{i}\}$, let $\mathcal{W}(\lambda,\mu)$ be the $D(\bar{U}_{\mathbf{i}})$-module with basis (w, bw, Fw, Fbw) where w satisfies

$$Ew = 0, \;\; cw = 0, \;\; Kw = \lambda w, \;\; dw = \mu w$$

(formally it is the induced representation $\mathbb{C}\langle F, b\rangle \otimes_{\mathbb{C}\langle E,c,K,d\rangle} \mathbb{C}_{\lambda,\mu}$). Define

$$w_{\lambda,\mu} = w, \quad w_{-\lambda,\mathbf{i}\mu} = Fw - \frac{\mathbf{i}}{2\mu} bw, \quad w_{-\lambda,-\mathbf{i}\mu} = bw, \quad w'_{\lambda,\mu} = Fbw. \quad (71)$$

These elements form a basis of weight vectors for the actions of K and d and the subscripts indicate the weights for K and d respectively. The actions of the other generators are:

$$Ew_{\lambda,\mu} = 0, \quad Fw_{\lambda,\mu} = w_{-\lambda,\mathbf{i}\mu} + \frac{\mathbf{i}}{2\mu} w_{-\lambda,-\mathbf{i}\mu}, \quad bw_{\lambda,\mu} = w_{-\lambda,-\mathbf{i}\mu}, \quad cw_{\lambda,\mu} = 0,$$
$$Ew_{-\lambda,\mathbf{i}\mu} = \frac{\mathbf{i}}{2}(\lambda^{-1} - \mu^2) w_{\lambda,\mu}, \quad Fw_{-\lambda,\mathbf{i}\mu} = \frac{-\mathbf{i}}{2\mu} w'_{\lambda,\mu}, \quad bw_{-\lambda,\mathbf{i}\mu} = -\mathbf{i} w'_{\lambda,\mu}, \quad cw_{-\lambda,\mathbf{i}\mu} = \mu(\mu^2 - \lambda^{-1}) w_{\lambda,\mu},$$
$$Ew_{-\lambda,-\mathbf{i}\mu} = \mu(\mu^2 - \lambda) w_{\lambda,\mu}, \quad Fw_{-\lambda,-\mathbf{i}\mu} = w'_{\lambda,\mu}, \quad bw_{-\lambda,-\mathbf{i}\mu} = 0, \quad cw_{-\lambda,-\mathbf{i}\mu} = 0,$$
$$Ew'_{\lambda,\mu} = \mathbf{i}\frac{\mu^2 - \lambda^{-1}}{2} w_{-\lambda,-\mathbf{i}\mu} - \mu(\lambda - \mu^2) w_{-\lambda,\mathbf{i}\mu}, \quad Fw'_{\lambda,\mu} = 0, \quad bw'_{\lambda,\mu} = 0, \quad cw'_{\lambda,\mu} = \mathbf{i}\mu(\mu^2 - \lambda^{-1}) w_{-\lambda,-\mathbf{i}\mu}$$

From these formulas it is not difficult to show that $\mathcal{W}(\lambda,\mu)$ is simple if, and only if, $\lambda \neq \mu^2$.[6] For $\lambda = \mu^2$ we write $\mathcal{W}(\mu)$ instead of $\mathcal{W}(\mu^2,\mu)$ and we denote the basis of weight vectors by $w_\mu, w_{\mathbf{i}\mu}, w_{\mathbf{i}\mu}, w'_\mu$ since K acts as d^2. We also notice that E and c act by 0 on $\mathcal{W}(\mu)$, so that the action is depicted by:

$$\begin{array}{c} \mathbb{C}_\mu = \langle w_\mu \rangle \\ \downarrow{\scriptstyle F,b} \\ \mathbb{C}_{\mathbf{i}\mu} \oplus \mathbb{C}_{-\mathbf{i}\mu} = \langle w_{\mathbf{i}\mu}, w_{-\mathbf{i}\mu} \rangle \\ \downarrow{\scriptstyle F,b} \\ \mathbb{C}_\mu = \langle w'_\mu \rangle \end{array}$$

where $\langle \ldots \rangle$ means linear span. Consider the following modules:

$$M = \mathcal{W}(\mathbf{i})/\langle w'_{\mathbf{i}} \rangle = \langle \overline{w}_{\mathbf{i}}, \overline{w}_{-1}, \overline{w}_1 \rangle, \quad N = \langle w_1, w_{-1}, w'_{-\mathbf{i}} \rangle \subset \mathcal{W}(-\mathbf{i}),$$
$$V_{\mathbf{i}} = \langle w_{\mathbf{i}}, w'_{-1} \rangle \subset \mathcal{W}(-1), \qquad V_{-1} = \mathcal{W}(-1)/V_{\mathbf{i}} = \langle \overline{w}_{-1}, \overline{w}_{-\mathbf{i}} \rangle.$$

[5] https://www.gap-system.org

[6] The complete list of simple $D(\bar{U}_{\mathbf{i}})$-modules consists actually of the $\mathcal{W}(\lambda,\mu)$ with $\lambda \neq \mu^2$ together with the $\mathbb{C}_\mu$, which is the 1-dimensional module defined by $K \cdot 1 = \mu^2$, $c \cdot 1 = \mu$ and the other generators act by 0.

They give three allowable short exact sequences:

$$S_1 = \big(0 \to \mathbb{C} \to M \to V_{\mathbf{i}} \to 0\big), \quad S_2 = \big(0 \to V_{\mathbf{i}} \to \mathcal{W}(-1) \to V_{-1} \to 0\big),$$
$$S_3 = \big(0 \to V_{-1} \to N \to \mathbb{C} \to 0\big),$$

such that $S_1 \circ S_2 \circ S_3$ is a 3-fold sequence from $\mathbb{C}$ to $\mathbb{C}$. Let us compute $\widetilde{\Gamma}^{-1}\eta(S_i)$. For instance for S_3 we must first fill the following diagram:

$$\begin{array}{ccccccc}
\big((\bar{U}_{\mathbf{i}}^*)^{\otimes 2} \otimes \mathbb{C}\big)_{\mathrm{coad}} & \xrightarrow{d_1} & \big(\bar{U}_{\mathbf{i}}^* \otimes \mathbb{C}\big)_{\mathrm{coad}} & \xrightarrow{\mathrm{act}} & \mathbb{C} & \longrightarrow & 0 \\
\big\downarrow {\scriptstyle \eta_1 = \eta(S_1)} & & \big\downarrow {\scriptstyle \eta_0} & & \big\| & & \\
V_{-1} & \longrightarrow & N & \longrightarrow & \mathbb{C} & \longrightarrow & 0
\end{array}$$

(act is equal to ε in this case: $\mathrm{act}(\psi \otimes 1) = \varepsilon(\psi)$). Let us give a few details on how to find a solution. Recall the elements φ_l defined in (68). Since $\mathrm{act}(\varphi_1 \otimes 1) = \mathrm{act}(\varphi_3 \otimes 1) = 0$, we set $\eta_0(\varphi_1 \otimes 1) = \eta_0(\varphi_3 \otimes 1) = 0$. On the other hand, due to the weights and to the commutativity of the right square we necessarily have $\eta_0(\varphi_0 \otimes 1) = w_1$ and $\eta_0(\varphi_2 \otimes 1) = \lambda w_{-1}$ for some scalar λ. Then we note that

$$\eta_0(F \cdot (\varphi_2 \otimes 1)) = \eta_0(\varphi_3 \otimes F \cdot 1) - \frac{\mathbf{i}}{2}\eta_0(b(\varphi_0 + \varphi_2) \otimes 1) = -\frac{1}{2}w'_{-\mathbf{i}}, \quad F \cdot \eta_0(\varphi_2 \otimes 1) = \lambda w'_{-\mathbf{i}}$$

which reveals that $\lambda = -\frac{1}{2}$. Hence

$$\eta_0\big(b^j c^k \varphi_l \otimes 1\big) = b^j c^k \eta_0\big(\varphi_l \otimes 1\big) = \delta_{j,0}\delta_{k,0}\delta_{l,0}w_1 - \mathbf{i}\delta_{j,1}\delta_{k,0}\delta_{l,0}w'_{-\mathbf{i}} - \frac{1}{2}\delta_{j,0}\delta_{k,0}\delta_{l,2}w_{-1}.$$

It follows that

$$\eta_0 d_1\big(\varepsilon \otimes b^j c^k \varphi_l \otimes 1\big) = \eta_0\big(\delta_{j,0}\delta_{k,0}\delta_{l,0}\varepsilon \otimes 1 - b^j c^k \varphi_l \otimes 1\big) = \frac{1}{2}\delta_{j,0}\delta_{k,0}(\delta_{l,2} - \delta_{l,0})w_{-1} + \mathbf{i}\delta_{j,1}\delta_{k,0}\delta_{l,0}w_{-\mathbf{i}}$$

and a solution is simply given by

$$\eta(S_1)\big(\varepsilon \otimes b^j c^k \varphi_l \otimes 1\big) = \frac{1}{2}\delta_{j,0}\delta_{k,0}(\delta_{l,2} - \delta_{l,0})\overline{w}_{-1} + \mathbf{i}\delta_{j,1}\delta_{k,0}\delta_{l,0}\overline{w}_{-\mathbf{i}}.$$

Recall from (58) that to compute $\widetilde{\Gamma}^{-1}$ we need a pair of dual bases. It is not difficult to find the dual basis of the monomial basis of $\bar{U}_{\mathbf{i}}$:

$$(K^l)^* = \varphi_{-l}, \quad (EK^l)^* = \mathbf{i}^l b\varphi_{-l}, \quad (FK^l)^* = \mathbf{i}^{-l} c\varphi_{1-l}, \quad (EFK^l)^* = bc\varphi_{1-l}.$$

(recall that the products on the right-hand sides are in $(\bar{U}_{\mathbf{i}}^*)^{\mathrm{op}}$). It follows that

$$\widetilde{\Gamma}^{-1}\eta(S_3) : \mathbb{C} \to \bar{U}_{\mathbf{i}} \otimes V_{-1}, \quad 1 \mapsto -\boldsymbol{e} \otimes \overline{w}_{-1} + \mathbf{i}E \otimes \overline{w}_{-\mathbf{i}}$$

where $\boldsymbol{e} = \frac{1-K^2}{2}$. One similarly finds

$$\widetilde{\Gamma}^{-1}\eta(S_1) : V_{\mathbf{i}} \to \bar{U}_{\mathbf{i}} \otimes \mathbb{C}, \quad w_{\mathbf{i}} \mapsto \mathbf{i}EK^3 \otimes 1, \quad w'_{-1} \mapsto -\boldsymbol{e} \otimes 1,$$
$$\widetilde{\Gamma}^{-1}\eta(S_2) : V_{-1} \to \bar{U}_{\mathbf{i}} \otimes V_{\mathbf{i}}, \quad \overline{w}_{-1} \mapsto EK^2 \otimes w_{\mathbf{i}}, \quad \overline{w}_{-\mathbf{i}} \mapsto -\frac{\mathbf{i}}{2}K\boldsymbol{e} \otimes w_{\mathbf{i}} - EK \otimes w_{-1}.$$

Thanks to the formula (55) for the product $\circ$ we find

$$\widetilde{\Gamma}^{-1}\eta(S_1 \circ S_2 \circ S_3) = \widetilde{\Gamma}^{-1}\eta(S_1) \circ \widetilde{\Gamma}^{-1}\eta(S_2) \circ \widetilde{\Gamma}^{-1}\eta(S_3) = \mathbf{i}c_1$$

where c_1 is given in Proposition 5.9 below.

In a completely analogous fashion, we can consider the $D(\bar{U}_{\mathbf{i}})$-module $\overline{\mathcal{W}}(\lambda,\mu)$ with basis (w, cw, Ew, Ecw) where w satisfies

$$Fw = 0,\ \ bw = 0,\ \ Kw = \lambda w,\ \ dw = \mu w$$

and if we reproduce the above reasoning with $\overline{\mathcal{W}}(\lambda,\mu)$ instead of $\mathcal{W}(\lambda,\mu)$, we will discover the cocycle c_2 in Proposition 5.9 below (up to a scalar).

Unfortunately, we have not been able to show that the cocycles c_1, c_2 so obtained are are a free family in $H^3_{\mathrm{DY}}(\bar{U}_{\mathbf{i}}\text{-mod})$, or to find another explicit 3-cocycle. Recall from (70) that $\dim\big(H^3_{\mathrm{DY}}(\bar{U}_{\mathbf{i}}\text{-mod})\big) = 3$; with the help of a program written in GAP4, we get the following:

Proposition 5.9. *A basis of explicit cocycles for* $H^3_{\mathrm{DY}}(\bar{U}_{\mathbf{i}}\text{-mod})$ *is*

$$\begin{aligned}
c_1 &= \boldsymbol{e} \otimes EK^2 \otimes EK^3 - E \otimes K\boldsymbol{e} \otimes EK^3 - E \otimes EK \otimes \boldsymbol{e},\\
c_2 &= \boldsymbol{e} \otimes FK^3 \otimes E - FK \otimes K\boldsymbol{e} \otimes F - FK \otimes FK^2 \otimes \boldsymbol{e},\\
c_3 &= \boldsymbol{e} \otimes EK^2 \otimes FK^2 - \boldsymbol{e} \otimes FK^3 \otimes EK^3 + EK^2 \otimes K\boldsymbol{e} \otimes FK^2\\
&\quad + FK \otimes K\boldsymbol{e} \otimes EK^3 - EK^2 \otimes FK^2 \otimes \boldsymbol{e} + FK \otimes EK \otimes \boldsymbol{e}
\end{aligned}$$

where $\boldsymbol{e} = \frac{1-K^2}{2}$.

Proof. We use the description of the DY complex given in (52)–(53). It is a purely linear problem. In GAP4, we represent $\bar{U}_{\mathbf{i}}$ as a 16-dimensional vector space over $\mathsf{Qi} := \mathsf{GaussianRationals}$, together with a unit tensor $\mathsf{un} : \mathsf{Qi} \to \bar{U}_{\mathbf{i}}$ and matrices $\mathsf{m}_E, \mathsf{m}_F, \mathsf{m}_K$ for the left multiplication by E, F, K respectively. From these matrices, we easily construct

- the multiplication tensor $\mathsf{m} : \bar{U}_{\mathbf{i}}^{\otimes 2} \to \bar{U}_{\mathbf{i}}$ (a matrix of size 16×256),
- the comultiplication tensor $\mathsf{com} : \bar{U}_{\mathbf{i}} \to \bar{U}_{\mathbf{i}}^{\otimes 2}$ (a matrix of size 256×16),
- the matrices $\mathsf{r}_E, \mathsf{r}_F, \mathsf{r}_K$ for the right multiplication by E, F, K respectively.

From this data it is easy to determine the centralizers of the iterated coproduct and to define the matrices of the differentials δ^2, δ^3, recall (53). Finally, one looks for 3 basis elements of $\ker(\delta^3)$ which generate a supplementary to $\mathrm{im}(\delta^2) \subset \ker(\delta^3)$. □

Remark 5.10. 1. We have seen two methods to compute the DY cohomology using a computer program: at the end of §5.4.1 and in the previous proof. If one is just interested in the dimensions of the DY cohomology groups, the method explained at the end of §5.4.1 is much faster, although it requires to do a bit a of representation theory in order to determine the relatively projective cover of $\mathbb{C}$.
2. By (49) and (70), we get $\mathrm{Ext}^3_{D(\bar{U}_{\mathbf{i}}),\bar{U}_{\mathbf{i}}}(\mathbb{C},\mathbb{C}) \cong \mathbb{C}^3$. On the other hand by finding a usual projective resolution of $\mathbb{C}$ one can compute that $\mathrm{Ext}^3_{D(\bar{U}_{\mathbf{i}})}(\mathbb{C},\mathbb{C}) = 0$. Hence the two allowable 3-fold exact sequences constructed above form a free family in $\mathrm{YExt}^3_{D(\bar{U}_{\mathbf{i}}),\bar{U}_{\mathbf{i}}}(\mathbb{C},\mathbb{C}) \cong \mathbb{C}^3$ but are equal to 0 in $\mathrm{YExt}^3_{D(\bar{U}_{\mathbf{i}})}(\mathbb{C},\mathbb{C}) = 0$.
3. It is straightforward to compute a usual projective resolution of $\mathbb{C}$ over $(U_{\mathbf{i}}^*)^{\mathrm{op}}$ by using the defining relations (67). Thanks to Lemma 5.1 we get $H^3_{\mathrm{DY}}(U) = 0$, where $U : \bar{U}_{\mathbf{i}}\text{-mod} \to \mathrm{Vect}_{\mathbb{C}}$ is the forgetful functor. This means that the map $\bar{\iota}_3 : H^3_{\mathrm{DY}}(\bar{U}_{\mathbf{i}}\text{-mod}) \to H^3_{\mathrm{DY}}(U)$ defined in Remark 5.3 is equal to 0. In other words, there exist elements $b_i \in \bar{U}_{\mathbf{i}}^{\otimes 2}$ such that $c_i = \delta^2(b_i)$ for $i = 1, 2, 3$.

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