THE $K^{\aleph_0}$ GAME: VERTEX COLOURING

NATHAN BOWLER, MARIT EMDE & FLORIAN GUT

Abstract. We investigate games played between Maker and Breaker on an infinite complete graph whose vertices are coloured with colours from a given set, each colour appearing infinitely often. The players alternately claim edges, Maker's aim being to claim all edges of a sufficiently colourful infinite complete subgraph and Breaker's aim being to prevent this. We show that if there are only finitely many colours then Maker can obtain a complete subgraph in which all colours appear infinitely often, but that Breaker can prevent this if there are infinitely many colours. Even when there are infinitely many colours, we show that Maker can obtain a complete subgraph in which infinitely many of the colours each appear infinitely often.

1. Introduction

Games have been of interest to mathematicians for centuries. The field was sparked by the analysis of historic games such as tic-tac-toe. This is a finite game and naturally, this was the first class of games analysed. In recent decades infinite games have increasingly drawn the attention of researchers. An intuitive starting point is just moving the rules of a finite game to an infinite board. Consider the aforementioned Tic-tac-toe. It's counterpart on an infinite board became known as *unrestricted* 3*-in-a-row* and was further generalised to *n-in-a-row* [1]. As some results in the finite version simply rely on the fact that there are only finitely many possible plays, this often yields interesting insights. These types of games are called *semi-infinite*; a comprehensive overview of the known results about such games can be found in [1].

Games also give rise to an interesting field in set theory, which is e.g. described in [9, Chapter 6]: One may assume as an axiom that every game is determined, i.e. that for any two player game with complete information at least one of the players has a winning strategy. It is known that this is not consistent with the axiom of choice, but it nevertheless provides a good framework to study descriptive set theory, itself an essential field of research in present-day mathematical logic.

However, there has not yet been a systematic analysis of infinite combinatorial games, i.e. games played on an infinite board with perfect information where the players do moves sequentially. One classic variant is the following two player game, which we will call the *strong H-building game*. The game depends on a fixed graph H. During the course of the game the players alternately claim edges of a sufficiently large complete graph K^n. The game ends as soon as H is contained as a subgraph in the graph induced by some player's claimed edges. The word *strong* in the name of the game indicates that both players have the same aim (to build a copy of H), thus the two players' roles differ only in who plays first. This is to distinguish it from the *Maker-Breaker* variant of the game, in which only one of the players

Key words and phrases. Maker Breaker game, infinite game, vertex colouring, vertexcolouring.

(Maker) is trying to build a copy of H and the other (Breaker) is simply trying to prevent this. In particular, Breaker does not win simply by virtue of building his own copy of H first.

Finite strong games have been extensively studied, and in particular the strong H-building game is fairly well understood. Indeed, as long as the board is sufficiently large, any such game is a first player win: due to Ramsey's Theorem (see [4, 9.1.1]) after all edges of the K^n have been claimed, there will always be a copy of H contained in one of the players' graphs if $n \geq R(v(H))$. So one of the players must have a winning strategy, and using a technique called *strategy stealing* (see [6, Theorem 1.3.1]) it cannot be the second player. The argument can be roughly sketched as follows: suppose the second player has a winning strategy. Then the first player can make an arbitrary move in his first turn and from then on play according to the winning strategy as if she is the second player.

Note that even though we now know that there is a winning strategy for the first player if $n \geq R(v(H))$, we only have an abstract argument and can not deduce an actual strategy according to which the first player should play. In particular, this argument does not show the existence of an upper bound only depending on H for the number of moves the first player needs to win.

The problem of establishing such bounds is in general very hard, and for example if H is a complete graph this problem has only been resolved when H has at most 4 vertices (See [3]). By contrast, such bounds are well known for the Maker-Breaker H-building game (see [6, Chapter 2]). Indeed, Maker-Breaker variants are usually easier to analyse, and are often investigated as a preliminary step before analysing the strong game.

In contrast to the fertility of the study of finite combinatorial games, infinite combinatorial games have thus far proved barren ground, in that the strong games are too hard to analyse and the Maker-Breaker games are too easy. Let us consider first the strong H-building game, but now played on an infinite complete graph. Since there are now infinitely many edges on the board to choose from these edges need not be exhausted in the course of the game, and there is no guarantee that the players will ever even claim all edges of a finite complete subgraph K^n between them. Thus Ramsey's Theorem is no longer applicable. Play could continue forever without either player ever winning!

So even though we can still use a strategy-stealing argument to rule out the existence of a winning strategy for the second player, it is possible that he has a strategy to force a draw. Although it might seem implausible that this could really happen, in fact for the analogous games played on 5-regular hypergraphs rather than graphs cases are known in which the second player can force a draw. Such an example was constructed by Hefetz, Kusch, Narins, Pokrovskiy, Requilé and Sarid in [7].

In fact, it is folklore that the existence of a winning strategy for the first player in the infinite strong H-building game is equivalent to the existence of a finite upper bound on the number of moves the first player needs to win in the finite strong H-building games, a straightforward proof by a compactness argument can be found in [8, Proposition 4], and we saw above that such problems currently seem intractable.

Then again, the Maker-Breaker H-building game is trivial; the winning strategies for the finite variants also work in the infinite variant. However, Joshua Erde recently noticed that there are interesting Maker-Breaker H-building games on infinite graphs; one just has to take H itself to be infinite [5].

We believe that the investigation of such infinite Maker-Breaker games will prove very fruitful. In this paper we begin that investigation by considering a few simple variants on the $K^{\aleph_0}$-building game. We will present a winning strategy for Maker in the basic version of the game in Section 3.

A natural variation arises if we colour the vertices of the board beforehand and demand that Maker respects this colouring in such a way that her $K^{\aleph_0}$ again contains infinitely many vertices of every colour.

In Section 4 we will present a winning strategy for Maker if there are only finitely many colours. If there are infinitely many colours then, on the one hand, Maker can still incorporate infinitely many different colours infinitely often into their $K^{\aleph_0}$, which we will prove in Section 5.2. On the other hand, in case Makers' aim is to incorporate all colours that are present on the board, Breaker can stop Maker from doing so, which we will prove in Section 5.1 by giving a winning strategy for Breaker. In fact, we prove even more: Breaker can stop Maker from incorporating cofinitely many colours infinitely often into her $K^{\aleph_0}$.

Another possible variant of the game is to demand that Maker respect some predefined colouring of the edges of the board instead. We address this variation in a paper in preparation [2].

There are two new kinds of difficulty which arise for infinite games of this kind but which are not so relevant for finite games. First of all, infinite games need not have a winning strategy for either player. Indeed, assuming the continuum hypothesis there are undetermined infinite combinatorial Maker-Breaker games on an infinite board. We present such a game in Appendix A.

Another difficulty arising for the games studied in this paper is that it would be hopeless for Maker to attempt to build her copy of $K^{\aleph_0}$ up iteratively, by building larger and larger nested K^i. The difficulty is that for any fixed K^i with $i \geq 2$ it is straightforward for Breaker to claim at least one edge to it from each subsequent vertex, thus preventing it from being integrated into any $K^{\aleph_0}$ claimed by Maker. Thus Maker must proceed more speculatively, never knowing during the course of the game which vertices will be integrated into her eventual $K^{\aleph_0}$.

Let us now begin by introducing some necessary notation.

2. Preliminaries

Throughout the paper we draw on the standard definitions as established in [4], if not explicitly mentioned otherwise. There are two players, who alternately take *turns* throughout the game. In each individual turn they pick an edge from the *board* $G = (V, E) := (\mathbb{N}, \mathbb{N}^2) \cong K^{\aleph_0}$, and colour it in their respective colour. We will refer to Maker by *she* or *her* and likewise we refer to Breaker by *he* or *him*. We will assume that Maker colours her edges in magenta, which we will abbreviate with M and that Breaker uses the colour blue which we will abbreviate with B. Neither player may colour an edge that was already picked by either player in a previous turn. The goal for Maker will be to colour the edges in such a way that her subgraph contains a specific substructure, i.e. a $K^{\aleph_0}$. Breaker's goal is to stop Maker from doing so.

Definition 2.1 (G_γ)**.** *For a colour $\gamma \in \{M, B\}$ and for any point in the game we define $E(G_\gamma)$ to be the edges that have been coloured in the colour γ up to that point,*

$V(G_\gamma)$ to be all the vertices that are incident with at least one edge of $E(G_\gamma)$ and thus define the graph $G_\gamma = (V(G_\gamma), E(G_\gamma))$. $N_\gamma(v)$ are the neighbours *of a vertex $v \in V(G_\gamma)$ in G_γ and the* γ-degree $\deg_\gamma(v)$ *of a vertex $v \in G$ is $\deg_\gamma(v) = |N_\gamma(v)|$. Accordingly, we will say that two vertices v and w are γ-connected or γ-adjacent, if $vw \in G_\gamma$.*

When we say that a player $\gamma \in \{M, B\}$ γ-connects *a vertex v to a vertex w in a turn, then we mean that Maker or Breaker claims the edge vw in that turn respectively. We will mean the same when we say that Maker or Breaker* plays from *a vertex v* to *a vertex w.*

Definition 2.2 (fresh vertex)**.** *Whenever we talk about a* fresh *vertex, we mean a vertex $v \in V \setminus (V(G_M) \cup V(G_B))$.*

Since Maker adds vertices to G_M in distinct moves, the vertices become ordered in a natural way. We will make use of that and assign indices to the vertices accordingly, i.e. v_k is the k^{th} vertex that Maker adds to her subgraph G_M. When Maker claims an edge incident with two fresh vertices, she assigns the next two indices to these vertices arbitrarily, which will only happen on Maker's first turn in our construction.

Definition 2.3 ($[n]$)**.** *For natural numbers $n \in \mathbb{N}$ we set*

$$[n] := \{j \in \mathbb{N} \setminus \{0\} : j \leq n\}.$$

When we want to prove that a game is a win for Breaker, we shall always do this by means of a *pairing strategy.* That is, we will define a family of disjoint pairs of edges from $E(G)$ with the intention that whenever Maker claims one edge from a pair Breaker claims the other one in his following turn.

It will then suffice to verify, for the game in question, that any $K^{\aleph_0}$ of the kind that Maker is trying to build must include both edges of at least one such pair.

3. The basic version

We will begin by investigating the basic version of the $K^{\aleph_0}$-game. In this game the aim of Maker is that G_M contains a $K^{\aleph_0}$ at the end of the game. We will prove that Maker can win this game. We will achieve this by first describing a strategy according to which Maker should play and then verifying that, in fact, $K^{\aleph_0} \subseteq G_M$ holds true.

Our focus will be on two different kinds of activity by Maker. On the one hand, she will regularly want to add fresh vertices to her subgraph G_M. On the other hand, she must ensure that G_M is as interconnected as possible and thus contains large complete graphs. The same interplay between making G_M highly interconnected and regularly moving on to fresh vertices will also provide the basic rhythm for our strategies for Maker in later sections.

Definition 3.1 (structured greedy strategy)**.** *We will call the following strategy for Maker the* structured greedy strategy*.*
In her first turn, she picks some edge v_1v_2. In case Breaker was the first player, she picks one that only uses fresh vertices.
In a later turn, suppose v_n is the last vertex that was added to Maker's subgraph. Now, if there is some v_i, $1 \leq i < n$ such that

(□1) v_iv_n has not yet been claimed in either colour,

(□2) $N_M(v_n) \subseteq N_M(v_i)$, and
(□3) i is minimal subject to (□1) and (□2),

then Maker claims $v_i v_n$. If there is no such v_i, she picks a fresh vertex v_{n+1} and claims $v_1 v_{n+1}$.

Theorem 3.2. *The structured greedy strategy is a winning strategy for Maker in the basic version of the $K^{\aleph_0}$-game.*

Proof. We consider an arbitrary play of the game in which Maker follows the structured greedy strategy. We must show that at the end of the game G_M includes a $K^{\aleph_0}$. We will recursively construct a complete graph $K^n \subseteq G_M$ as well as a set of vertices $W_n \subseteq V(G_M)$ for every $n \in \mathbb{N} \setminus \{0\}$ such that:

(■1) $|K^n| = n$, $|W_n| = \aleph_0$,
(■2) $K^n \subseteq K^{n+1}$ for $n > 1$, and
(■3) for any $w \in W_n$ the first n vertices to which w was M-connected were $V(K^n)$.

If we successfully construct such a sequence $K^1 \subset K^2 \subset K^3 \subset \dots$, the claim follows immediately for

$$\bigcup_{i \in \mathbb{N}} K^i = K^{\aleph_0}.$$

The purpose of the sets W_n is to ensure that there will be a suitable candidate to enlarge the complete graph at each step.

Initial step: We can set $K^1 = (\{v_1\}, \emptyset)$ and $W_1 = V(G_M) \setminus \{v_1\}$. This immediately satisfies (■1) and (■2). (■3) holds true because every vertex in G_M other than v_1 got M-connected to v_1 right after it was chosen as a fresh vertex.

Recursion step: Now suppose K^n and W_n subject to (■1), (■2) and (■3) are given for some fixed $n \in \mathbb{N}$. Consider the first $n+1$ vertices that are completely M-adjacent to every $v \in V(K^n)$. Such vertices exist since every w in the infinite set W_n has this property by (■3). Let us call this set of vertices F.

Now consider any vertex $w \in W_n \setminus F$. At the point in the game n turns after w was chosen as a fresh vertex by Maker, it was already M-connected to $V(K^n)$, and at that point w was B-adjacent to at most n other vertices, so at least one vertex $v' \in F$ was still available. Let $\hat{v}$ be the vertex in F with this property and the smallest possible index. Then Maker claimed $w\hat{v}$ in her $(n+1)^{\text{st}}$ move of M-connecting w. As w was arbitrary, this is true for every one of the infinitely many vertices of $W_n \setminus F$ and so, as F is finite, at least one vertex of F gets chosen in this way for infinitely many vertices from W_n. We denote the smallest such vertex in F by v^* and set

$$K^{n+1} := \left(V(K^n) \cup \{v^*\}, E(K^n) \cup \{vv^* : v \in V(K^n)\}\right), \text{ as well as}$$
$$W_{n+1} := \{w \in W_n : v^* \text{ was } M\text{-connected in the } (n+1)^{\text{st}} \text{ turn after being picked as a fresh vertex}\}.$$

This takes care of (■2) and additionally, we have $\left|K^{n+1}\right| = n+1$ and $|W_{n+1}| = \aleph_0$, thus (■1) is satisfied. (■3) follows from the recursion assumption together with the choice of W_{n+1}. □

As we know now that the $K^{\aleph_0}$ game is a Maker win, we will go on to consider some variants in which we make life a little harder for her. The natural way to do so might be to allow Breaker to claim more than one edge for any edge that Maker claims. This kind of variant has also been studied in the finite case, see [6, Chapter 3] and is called a *biased* game. In our setting it doesn't make much difference, at least as long as Breaker is only allowed to claim the same finite number k of edges on each turn. Maker does not even need to adapt her strategy. In the verification we need to take F to be of size $kn+1$ rather than $n+1$, and the argument works just as before.

What if Breaker is allowed to claim a monotone increasing number of edges on his turns? It turns out that regardless of how slow the increment actually is, as long as the number of edges he claims tends to infinity he has a winning strategy:

At the beginning of the game, he picks an enumeration $e_1 = \{x_1, y_1\}, e_2 = \{x_2, y_2\}, e_3 = \{x_3, y_3\}, \ldots$ of the edges of E. For any $n \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that from the N^{th} turn on, Breaker is allowed to claim n edges in each of his turns, for any edge $e = \{x, y\}$ that Maker claims. Beginning at $i = 1$, for every $i \leq n$, whenever $G[\{x, y, x_i, y_i\}] \not\subseteq G_M \cup G_B$, Breaker claims one of the available edges from $G[\{x, y, x_i, y_i\}]$ in his i^{th} turn. This strategy ensures that e_n can be part of a complete graph in G_M of at most some finite size dependent on N. As this holds for any $n \in \mathbb{N}$, Maker cannot construct a $K^{\aleph_0} \subseteq G_M$.

In the following sections we will take a closer look at other, more challenging variations. One could also consider biased versions of the games considered later in this paper, but the theory of such biased games is always just the same as that outlined above and so we will not discuss it further.

4. Finitely many colours

A more interesting way to make Maker's objective more demanding is the following: Before the beginning of the game, every vertex of the board gets assigned one of $k \in \mathbb{N} \setminus \{0\}$ many colours.

Definition 4.1 (colouring)**.** *Let $k \in \mathbb{N}$. A map*

$$c \colon V(G) \longrightarrow [k]$$

or

$$c \colon V(G) \longrightarrow \mathbb{N}$$

is a colouring *of $V(G)$ if $\left|c^{-1}(i)\right| = \aleph_0$ for every colour $i \in [k]$. Moreover, for a set $W \subseteq V(G)$ we define $c[W] := \{c(v) \colon v \in W\}$.*

Definition 4.2 (colour class)**.** *Let c be a colouring and $j \in \operatorname{im}(c)$. We call*

$$c^{-1}(j) \subseteq V(G)$$

the colour class *of j.*

Maker's objective will be to build a $K^{\aleph_0} \subseteq G_M$ as before but with the additional property that it includes infinitely many vertices from every colour class. We will call this version of the game the finitely coloured $K^{\aleph_0}$-game. This is again a Makers' win with the following strategy.

Definition 4.3 (finite colour balanced greedy strategy)**.** *Let $k \in \mathbb{N} \setminus \{0\}$, suppose that the board is coloured by a colouring $c : V(G) \longrightarrow [k]$ and let v_n be the vertex added to G_M most recently. Now suppose $\deg_M(v_n) \equiv \ell \mod k$ and $v_n \in c^{-1}(h)$ for $h, \ell \in [k]$ not necessarily distinct. Then if Maker connects v_n to a vertex of colour ℓ in the following fashion, we say that she plays according to the* finite colour balanced greedy strategy*.*

Let $F \subseteq V(G_M)$ be the set of the first $k \cdot \deg_M(v_n) + 1$ many vertices such that for all $v_m \in F$:

- $N_M(v_n) \subseteq N_M(v_m)$,
- $v_m \in c^{-1}(\ell)$,
- $m < n$, *and*
- $a < m$ *for all* $v_a \in N_M(v_n)$.

If there are fewer than $k \cdot \deg_M(v_n) + 1$ vertices satisfying these conditions, Maker chooses a fresh vertex v_{n+1} of colour m, where $n + 1 \equiv m \mod k$, and claims $v_1 v_{n+1}$. Otherwise, she considers the set $K \subseteq V(G_M)$ of all vertices v_i satisfying:

- $j < i$ *for all* $v_j \in F$,
- $N_M(v_n) \supseteq N_M(v_i)$, *and*
- $v_i \in c^{-1}(h)$.

Maker assigns a tuple in $\mathbb{N} \times \mathbb{N}$ to every $v_i \in F$ via the injective map

$$\begin{aligned} f : F &\longrightarrow \mathbb{N} \times \mathbb{N}, \\ v_i &\longmapsto (|N_M(v_i) \cap K|, i) \end{aligned}$$

and then she orders $f(F)$ lexicographically, which results in an ordered set

$$(f(F), \leq). \tag{1}$$

Maker determines the smallest tuple $(|N_M(v_\delta) \cap K|, \delta) \in f(F)$ such that $v_\delta v_n \notin E(G_B)$ and claims this edge. By the size of F it is clear that there will be a vertex v_δ available, as Breaker had only $\deg_M(v_n) < k \cdot \deg_M(v_n) + 1$ many moves where he could have coloured edges that are incident with v_n.

Let us shed some light on two aspects of this strategy, namely the size of F and the purpose of the order on F induced by f.

Our verification that this strategy works will be similar to that in Section 3, in that we will again recursively build a nested sequence of complete graphs K^n for every n and in every step make sure that there is an infinite set $W_n \subseteq V(G_M)$ such that for every vertex $v \in W_n$ the entire K^n is contained in its neighbourhood, i.e. the induced subgraph on $V(K^n) \cup \{v\}$ is a potential candidate to continue the sequence. Then the crutial part is to carefully pick a vertex such that there still is an infinite set $W_{n+1} \subseteq W_n$ left. Note that the set K for some vertex v in the strategy will be contained in the corresponding set W_n in the proof if v is considered as a potential next vertex in the recursion. Because of the role the sets W_n and therefore the sets K play, we will informally refer to them as *reservoir*. In contrast to the proof in Section 3, we need to also make sure that the sets W_n also contain infinitely many vertices of every colour. This is precisely the motivation for the map f introduced in the strategy above: If Maker just chose to play to the vertex from F with the smallest possible index as she does in the basic version, Breaker could ensure that all elements of the reservoir of colour a are joined to some vertex v_a, but that all elements of the reservoir of colour b are joined to some other vertex v_b. Then there would be no vertex that has infinitely many neighbours of both colours. Thus, instead of designating one vertex that has infinitely many neighbours of every colour, Maker instead ensures that for any colour o, Breaker can bar at most $\deg_M(v_n)$ vertices of F from having infinitely many neighbours of colour o. This excludes at most $k \cdot m$ vertices of F (recall that k is the number of colours and m the current M-degree of v_n). Maker wants to utilise this fact in order to ensure that there is at least one suitable vertex, i.e. a vertex with infinitely many neighbours of every colour, in the recursion step of the proof. She can achieve this by ensuring that the connection from vertices in the reservoir are spread as evenly as possible across F. The tool to do this is the function f and the lexicographic ordering:

Picking v_δ minimally in 1 makes the choice of the vertex unique for Maker, this is ensured by the second entry of the ordering. More importantly, as we have argued above, the vertices of F must be M-connected in a balanced fashion and this is achieved by choosing v such that $|N_M(v) \cap K|$ is smallest possible. To illustrate what we mean by that, one may think of the vertices in K as being the set of vertices in G_M that are identical to v_n in the following sense: They were added to G_M later than all of the vertices in F, the vertices got M-connected to G_M during their first $\deg_M(v_n)$ many turns in the same manner as v_n, and they have the same colour as v_n, namely h. Via f, Maker finds the elements in F that have the fewest neighbours in K and out of these she chooses the one that has the smallest index.

Therefore, by ensuring that F has size $k \cdot m + 1$ playing to vertices of F as evenly as possible via f Maker ensures that there is a suitable vertex in the recursion step of the proof.

Theorem 4.4. *The finite colour balanced greedy strategy is a winning strategy for Maker in the finitely coloured version of the $K^{\aleph_0}$-game.*

Proof. We want to show that at the end of the game, if Maker plays according to the strategy given, there is a $K^{\aleph_0} \subseteq G_M$ that uses infinitely many vertices of each colour class.

Recursive construction: For every $n \in \mathbb{N}$ we will construct a complete graph $K^n \subseteq G_M$ together with a set of vertices $W_n \subseteq V(G_M)$ with the properties

- (▲1) $K^n \subset K^{n+1}$,
- (▲2) $\left|W_n \cap c^{-1}(i)\right| = \aleph_0$ for all $i \in [k]$,
- (▲3) for each $w \in W_n$ we have $N_M(w) \supseteq V(K^n)$ and the vertices of K^n were the first n to become M-connected to w, and
- (▲4) $|K^n| = n$ and there is an enumeration $\{v'_1, v'_2, \dots, v'_n\}$ of $V(K^n)$ such that v'_j is coloured in m and $j \equiv m \mod k$ for every $1 \leq j \leq n$.

Note that by (▲2) we have in particular $|W_n| = \aleph_0$. As in Section 3, we can get the desired $K^{\aleph_0}$ from properties (▲1) and (▲4) by considering

$$\bigcup_{n \in \mathbb{N}} K^n = K^{\aleph_0}.$$

Here (▲1) ensures that there is the sequence $K^1 \subset K^2 \subset K^3 \subset \dots$ of complete graphs. (▲4) ensures that there are infinitely many vertices of each colour. (▲2) and (▲3) are needed to ensure that there always is a next vertex that can be added to K^n to form K^{n+1}. It remains to show that the conditions above can be preserved in every step.

Initial step: Again, we can set $K^1 := (\{v_1\}, \emptyset)$ and $W_1 = V(G_M) \setminus \{v_1\}$. As this is the initial step, (▲1) holds true. Since Maker repeatedly added vertices of all colours to G_M, (▲2) is true as well. (▲3) holds, since v_1 was the first vertex to be joined to each v_i with $i \in \mathbb{N} \setminus \{1\}$. Finally, as $V(K^1) = \{v_1\}$ and v_1 is coloured with colour $1 \in [k]$, (▲4) is also true. This concludes the base case.

Recursion step: Let $n \in \mathbb{N}$, $1 \leq i \leq k$, let K^n and W_n subject to (▲1)—(▲4) be given and let $i \in [k]$ such that $n + 1 \equiv i \mod k$. We want to construct K^{n+1}

and W_{n+1} with the required properties. In order to do so, let F be the set of the first $kn+1$ vertices of colour i that have a common magenta edge with every vertex of K^n. Such a set exists, since all of the infinitely many vertices of colour i in W_n have this property.

Let j be the largest index of a vertex in F, fix an arbitrary colour $\ell \in [k]$ and let $w \in (W_n \setminus F) \cap c^{-1}(\ell)$ be a vertex with an index larger than j. After Maker claimed wv' for every $v' \in V(K^n)$ in her first n moves of connecting w to G_M, the statement

$$N_M(w) \subseteq N_M(v)$$

held for all $v \in F$. Since these are the first $kn+1$ such vertices, Maker chose the smallest available one of them with respect to the ordering derived from f as defined in (1). Breaker could block at most n edges vw for $v \in F$, thus there are at least $(k-1)n+1$ possible edges for Maker to choose from. Therefore, at most k vertices of F individually have only finitely many vertices in $W_n \cap c^{-1}(l)$ that chose them in the $(n+1)^{\text{st}}$ move of connecting to G_M.

As the colour l was arbitrary, the argument above holds true for each of the k different colours. Therefore there is at least one vertex u in F that has infinitely many neighbours in every colour class in W_n. We choose the vertex $u^* \in F$ of these with the smallest index and let K^{n+1} be the graph obtained from K^n by adding u^* and all edges from it to K^n. As W_{n+1} we take the set of vertices in W_n such that u^* was the $(n+1)^{\text{st}}$ vertex to which they were M-connected. This ensures (▲1). Moreover, it means that W_{n+1} contains infinitely many vertices of every colour class by the choice of u^*, therefore ensuring (▲2). The first $n+1$ vertices to be joined to any $w \in W_{n+1}$ were those of the K^{n+1} by the induction hypothesis and the construction of u^*, so we have (▲3). Lastly, the fact that we considered only vertices of colour $n+1 \equiv i \mod k$ for F, together with the assumption that (▲4) holds for n, ensures (▲4) for step $n+1$. □

5. Infinitely many colours

Next we consider what happens if there are infinitely many colours. As before, we will consider a surjective map $c\colon V(G) \longrightarrow \mathbb{N}$ and we will require $\left|c^{-1}(i)\right| = \aleph_0$ for every $i \in \mathbb{N}$.

If it is Maker's aim to construct a $K^{\aleph_0}$ that uses every colour class infinitely often, she is doomed to fail, as there is a strategy for Breaker with which he can keep Maker from doing so. However, if it is Maker's aim to construct a $K^{\aleph_0}$ that only uses infinitely many different colour classes, then there is a strategy with which she can secure this. We will first present Breaker's strategy for the first variant and after that we will give the strategy according to which Maker should play to win the second variant.

5.1. **Using all colours of the board.** Our aim is to define a pairing strategy such that for every edge e of the board there is some colour class i of which Maker may use no vertices together with e. First note that there are countably infinitely many edges in a $K^{\aleph_0}$ as $V(K^{\aleph_0})$ is countably infinite and the edges correspond to the two element subsets of V.

Before the beginning of the game, Breaker picks an enumeration $e_1, e_2, e_3, \dots$ of all the edges of the board G. He then recursively finds an enumeration $c_1, c_2, c_3, \dots$ of infinitely many colours that are present in the $K^{\aleph_0}$ such that for all $i \in \mathbb{N}$:

(★1) $c_i \neq c_j$ for all $j < i$ and
(★2) $c_i \notin c\left[\bigcup_{j\leq i} e_j\right]$.

Fix some $m \in \mathbb{N}$. We set $V_m := c^{-1}(c_m)$ and according to (★2) we have $V_m \cap e_m = \emptyset$. Suppose $e_m = xy$. Then for any $v \in c^{-1}(c_m)$ there are exactly two edges from v to e_m, namely vx and vy. Whenever Maker claims one of those edges, Breaker claims the other one in his following turn.

Lemma 5.1. *The defined strategy is a pairing strategy and furthermore a winning strategy for Breaker in the infinitely coloured version of the $K^{\aleph_0}$-game where Maker must have all colours contained in her $K^{\aleph_0}$.*

Proof. We first check whether the defined strategy actually is a pairing strategy, i.e. that any edge lies in at most one of the pairs of edges. Indeed, suppose for a contradiction, that there is an edge $e = uw$, that lies in two different pairs of edges.

Case 1: e lies in the pair of edges for two distinct edges e_i, e_j that are incident with the same vertex of e, u say: Then $w \in c^{-1}(c_i) \cap c^{-1}(c_j)$. Thus we have $c_i = c_j$, a contradiction to (★1), as either $i < j$ or $j < i$.

Case 2: e lies in the pair of edges for two non adjacent edges e_i, e_j: Without loss of generality we may assume $u \in e_i$ and $w \in e_j$. This can only happen, if $u \in V_j$ and $w \in V_i$, a contradiction to (★2), as either $i < j$ or $j < i$.

Thus the pairs used for the strategy are disjoint. Therefore the given strategy actually is a pairing strategy for Breaker.

Furthermore Maker cannot build a $K^{\aleph_0}$ that uses all of the colours present on the board, if Breaker plays according to the defined strategy: For any edge that she wants to incorporate into her $K^{\aleph_0}$, there is a colour class that corresponds to it according to the construction which she therefore cannot use in her $K^{\aleph_0}$. □

Note that this result can be strengthened in the following sense: As every edge in a $K^{\aleph_0}$ of Maker has a different colour assigned to it and the $K^{\aleph_0}$ contains infinitely many edges, there are infinitely many colours of which Maker cannot incorporate infinitely many into her $K^{\aleph_0}$, this means that Breaker can even stop Maker from using cofinitely many colours, each infinitely often, in a $K^{\aleph_0}$.

5.2. **Using infinitely many colours of the board.** Let us now investigate how Maker should play in order to ensure that G_M contains infinitely many vertices from infinitely many different colour classes. As before, she needs to add fresh vertices to G_M, making sure that she keeps track of the colours as well as taking care of the vertices that are already part of G_M, while also ensuring that they are as interconnected as possible in general and paying attention to the colours of these fresh vertices in particular.

We first introduce one additional difinition.

Definition 5.2 (φ_U). *For a finite subset $U \subseteq V(G_M)$ we let*

$$\varphi_U\colon\ [|U|] \longrightarrow \{i \in \mathbb{N}\colon v_i \in U\} \subseteq \mathbb{N}$$

be the unique order preserving bijection. For an infinite subset $W \subseteq V(G_M)$ we consider the unique order preserving bijection

$$\varphi_W\colon \mathbb{N} \longrightarrow \{i \in \mathbb{N}\colon\ v_i \in W\} \subseteq \mathbb{N}.$$

In this variant of the game, Maker cannot rotate through the colour classes in the same fashion as with finitely many colours and thus she has to work on her objective in a diagonal fashion. We further specify this in the strategy:

Definition 5.3 (infinite colour balanced greedy strategy)**.** *At the beginning of the game, Maker chooses a sequence $s_1, s_2, s_3, \dots$ of all the colours appearing on the board such that each individual colour appears infinitely often. Let us call this sequence S.*

We call the following strategy for Maker the infinite colour balanced greedy strategy*.*

In her first turn, she picks two fresh vertices of colours s_1 and s_2, calls them v_1 and v_2 respectively and claims the edge v_1v_2 for herself. When Maker adds vertices to her subgraph in later stages of the game, letting $|G_M| = n-1$, she adds a fresh vertex v_n of colour s_n to G_M.

After M-connecting a fresh vertex v_n of colour s_n by claiming v_1v_n, on the next few turns Maker determines which edge v_iv_n to claim by considering the set $U \subseteq V(G_M)$ of all vertices v_u that satisfy

- *$N_M(v_n) \subseteq N_M(v_u)$, moreover in the first $\deg_M(v_n)$ turns of M-connecting v_u Maker M-connected v_u to the same vertices as v_n, in the same order, and*
- *$i < u$ for all $v_i \in N_M(v_n)$.*

Then Maker considers the subset

$$U' := \{v \in U : c(v) = c(v_n)\} \tag{2}$$

and with φ_U from Definition 5.2 she determines that the vertex she plays to next should be of colour

$$j = c\left(v_{\varphi_U(|U'|)}\right). \tag{3}$$

Note that $v_n \in U$, thus $\varphi_U(|U'|)$ is well defined. Next, Maker lets $F \subseteq V(G_M)$ be the set of the first $(|c[N_M(v_n)]| + 2) \cdot \deg_M(v_n) + 1$ vertices, such that

- *$k < i$ for all $v_i \in F$ and all $v_k \in N_M(v_n)$,*
- *$i < n$ for all $v_i \in F$,*
- *$v_i \in c^{-1}(j)$ for all $v_i \in F$, and*
- *all $v_i \in F$ satisfy $N_M(v_n) \subseteq N_M(v_i)$.*

If there are fewer than $(|c[N_M(v_n)]| + 2) \cdot \deg_M(v_n) + 1$ vertices satisfying these conditions, Maker instead chooses a fresh vertex v_{n+1} as described above. If there is such a set, she chooses its subset of the first $(|c[N_M(v_n)]|+2) \cdot \deg_M(v_n)+1$ many and calls this F. Maker wants to M-connect a vertex from F to v_n analogously to the strategy in Definition 4.3: she considers the set $K \subseteq V(G_M)$ of all vertices v_k that satisfy

- *for all $v_i \in F$ we have $i < k$,*
- *for all $v_\ell \in N_M(v_n)$ we have $\ell < k$,*
- *$N_M(v_k) \supseteq N_M(v_n)$, and*
- *$c(v_k) = c(v_n)$.*

Maker assigns a tuple to every $v_i \in F$ as follows:

$$\begin{aligned} g\colon F &\longrightarrow \mathbb{N} \times \mathbb{N}, \\ v_i &\longmapsto \left(|N_M(v_i) \cap K|, i\right), \end{aligned}$$

and then orders $g(F)$ lexicographically, which results in an ordered set

$$(g(F), \leq) . \tag{4}$$

Maker determines the smallest $v_\delta \in F$ such that $v_\delta v_n \notin E(G_B)$ and claims this edge.

Note that as $|F| = (|c[N_M(v_n)]| + 2) \cdot \deg_M(v_n) + 1$, there will be a vertex v_δ available, as Breaker has only had $\deg_M(v_n)$ many moves where he could have B-connected vertices from F with v_n. As in Section 4 considering the ordering $(g(F), \leq)$ ensures that Maker plays from vertices similar to v_n to the vertices in F in a "balanced fashion", which is crucial in the verification step.

Let us investigate why the size of F should be $(|c[N_M(v_n)]| + 2) \cdot \deg_M(v_n) + 1$. Recall that the size of the corresponding set in Definition 4.3 was "(number of colours on the board $\cdot$ degree of the active vertex)+1". The sizes thus only differ by "number of colours on the board" vs "number of colours in the neighbourhood of the active vertex+2". It is clear that "number of colours on the board" cannot be used in the infinitely coloured game, as the size of F must be finite. It becomes clear, why the chosen number gives a good compromise for the following reason, when we suppose that the vertex v_n will be considered as an element of the set W_m of potential future vertices for some $m \in \mathbb{N}$. In the proof we must make sure that for any colour d already present on the K^m there are infinitely many vertices of colour d in W_m. This is ensured by the "number of colours in the neighbourhood of the active vertex"-part. On top of that, in order to eventually have infinitely many colours present on the $K^{\aleph_0}$, we need to (a) allow for one additional colour in case we want to add a new colour to the K^{m+1} and (b) ensure that there will still be infinitely many other potential colours to add in the future present in W_{m+1}. This gives rise to the "+2"-part.

Lastly, let us elucidate Makers' choice of the colour j given in Equation 3 before we move on. Breaker might render some colours unusable for Maker, but which these will be will not be clear until after the game. Thus, Maker needs a method to ensure that for each two such colours j and k she infinitely often tries to connect from a vertex of colour k down to one of colour j. The given function fulfils this purpose, which we will prove later (See Equation (5)).

Theorem 5.4. *The infinite colour balanced greedy strategy is a winning strategy for Maker in the infinitely coloured version of the $K^{\aleph_0}$-game where Maker must have infinitely many colours contained infinitely often in her $K^{\aleph_0}$*

Proof. We want to prove that, after infinitely many turns, there is a $K^{\aleph_0} \subseteq G_M$ that uses infinitely many vertices of infinitely many different colours, if Maker plays according to the strategy above. Before we begin with the recursion, we pick a sequence $\hat{C} = c_1, c_2, c_3, \dots$ of colours of $c[V]$ which contains every element of $c[V]$ infinitely often. We just require that $c_1 = s_1 = c(v_1)$.

Recursive construction: For every $n \in \mathbb{N} \setminus \{0\}$ we will construct a complete graph $K^n \subseteq G_M$ together with a set of vertices $W_n \subseteq V$, and a set of colours $C_n \subseteq c[V]$ with the properties

- (♦1) $K^n \subset K^{n+1}$,
- (♦2) $\left|W_n \cap c^{-1}(i)\right| = \aleph_0$ for every $i \in C_n$,
- (♦3) for each $w \in W_n$ the first n moves of connecting w to G_M by Maker were claiming the edges that join w to the K^n,

(♦4) $|K^n| = n$ and there is an enumeration $\{v'_1, v'_2, \dots, v'_n\}$ of $V(K^n)$ such that $c(v'_i) = c(v'_j)$ if and only if $c_i = c_j$ for $1 \le i \le j \le n$, and

(♦5) $|C_n| = \aleph_0$ and $c[V(K^n)] \subseteq C_n$.

Note that in (♦4) we do not require $c(v_i) = c_i$. This is indeed impossible to achieve. But it secures that any colour that appears really appears infinitely often and together with (♦5) it furthermore secures that infinitely many different colours appear in the inclusive chain

$$K^1 \subset K^2 \subset K^3 \subset K^4 \subset K^5 \subset \dots ,$$

thus

$$\bigcup_{n\in\mathbb{N}} K^n = K^{\aleph_0}$$

is the desired complete subgraph of G_M.

Initial step: Set $K^1 := (\{v_1\}, \emptyset)$, $W_1 := V(G_M) \setminus \{v_1\}$ and $C_1 = c[W_1]$. (♦1) holds true, since this is the initial step. As $|K_1| = 1$, (♦4) holds true as well. $|C_1| = \aleph_0$ is true and $c[V(K^1)] \subseteq C_1$ is satisfied because S contains every colour infinitely often, thus C_1 satisfies (♦5). Moreover, as every vertex of $G_M \setminus \{v_1\}$ was first M-connected to v_1, (♦3) is true. Finally, as S contains every colour infinitely often, this ensures (♦2) for C_1. This concludes the base case.

Recursion step: Let $n \ge 1$ and K^n, W_n and C_n subject to (♦1)—(♦5) be given and suppose c_m was the entry of $\hat{C}$ we worked with in the previous step. (This means in particular that $c(v_n) = c_m$.) We want to construct K^{n+1}, W_{n+1} and C_{n+1} with the required properties. As before, we will sometimes need to make sure that we add vertices of colours that are already present in the K^n but we will sometimes also need to add vertices of colours that are not. If there is $i \le n$ such that $c_i = c_{n+1}$, we set $c_p := c(v_i)$ and otherwise we choose $c_p \in C_n \setminus c[V(K^n)]$ arbitrarily. We want to add a vertex of colour c_p next and let F be the set of the first $(|c[V(K^n)]| + 2) \cdot n + 1$ vertices of colour c_p that have a common magenta edge with every vertex of the K^n. This set exists since there are infinitely many vertices of colour c_p in W_n by (♦2). Moreover, we need to restrict W_n to only contain vertices whose $(n + 1)^{\text{st}}$ turn of connecting it to G_M was a vertex of colour c_p and we want to ensure that (♦2) holds for this restriction of W_n as well. Note that since (♦1), (♦4) and (♦5) are independent of W_n, they still hold and (♦3) will hold for the restriction as it is a subset of W_n.

Fix an order preserving map $\psi\colon \mathbb{N} \setminus \{0\} \to I$ such that

$$\{v_{\psi(i)} : i \in \mathbb{N} \setminus \{0\}\} = W_n \cap c^{-1}(c_p). \tag{5}$$

Then, for every $m \in \mathbb{N} \setminus \{0\}$ and every $d \in C_n$ the $(\psi(m))^{\text{th}}$ vertex of colour d in W_n got M-connected to a vertex of colour c_p in the $(n+1)^{\text{st}}$ move of connecting it to G_M. Thus, there are infinitely many vertices of colour d in W_n whose $(n + 1)^{\text{st}}$ neighbour in G_M (according to the order in which they were connected to it) was a vertex of colour c_p. As d was arbitrary, this is true for every colour in C_n. We can thus restrict W_n to these vertices and work with this set W'_n from here on.

Let $\ell \in c[V(K^n)] \cup \{c_p\}$. Since Maker played to the vertices of F in a balanced fashion, there are at most n vertices $v \in F$ such that only finitely many vertices $w \in W'_n \cap c^{-1}(\ell)$ got M-connected to v in their $(n + 1)^{\text{st}}$ move of connecting them to G_M. As ℓ was arbitrary, this is true for every colour in $c[V(K^n)] \cup \{c_p\}$ and thus there are at least $n + 1$ vertices that have infinitely many such vertices in $W'_n \cap c^{-1}(p)$ for every $p \in c[V(K^n)] \cup \{c_p\}$. Conversely, regarding the infinitely many colours in $C_n \setminus (c[V(K^n)] \cup c_p)$, since Breaker can block at most n vertices for any of them, there are at most n vertices in F that are chosen by only finitely many vertices of cofinitely many colours not yet occurring in the K^n in the $(n+1)^{\text{st}}$ move of connecting them to G_M. Combining this means that there is at least one vertex $u' \in F$ that got chosen by infinitely many vertices of every colour in $c[V(K^n)] \cup \{c_p\}$ as well as infinitely many vertices of infinitely many distinct colours in C_n in their $(n+1)^{\text{st}}$ move of connecting them to G_M. We choose the smallest such vertex and call it v'_{n+1}. We set

- $K^{n+1} := G\left[V(K^n) \cup \{v'_{n+1}\}\right]$,
- C_{n+1} the set of colours $i \in C_n$ for which infinitely many vertices that lie in $c^{-1}(i) \cap W_n$ got M-connected to v'_{n+1} in their $(n+1)^{\text{st}}$ move of connecting them to G_M, and
- $W_{n+1} \subseteq W'_n$ as the vertices in W_n that got M-connected to v'_{n+1} in their $(n+1)^{\text{st}}$ move of connecting them to G_M and that are coloured with a colour in C_{n+1}.

This ensures (♦1), (♦2) and (♦5). (♦4) holds true by the choice of c_p and the definition of v'_{n+1}. All vertices of W_{n+1} are completely (M-)adjacent to the K^n by the induction hypothesis and to v'_{n+1} according to the construction, so all vertices of W_{n+1} are completely (M-)adjacent to the K^{n+1}. It follows from the induction hypothesis and the choice of v'_{n+1} that those were the first $n+1$ moves that Maker made for each element of W_{n+1}. This verifies (♦3).

This shows that all of the required properties are preserved throughout the induction and thus the claim is proved. □

This concludes our investigation into vertex colourings.

6. Outlook

In a paper in preparation [2] the present authors investigate a variant of the $K^{\aleph_0}$ building game in which the edges of the board are coloured instead of the vertices. While in the vertex case the colour classes are very symmetric, this is different for edge colourings: it could happen that a subgraph induced by a colour class is locally finite!

Appendix A. Determinacy of Maker-Breaker games

Although Maker-Breaker games are easier to handle than infinite games in general, we show here that they are still not necessarily determined. More precisely, assuming the continuum hypothesis we construct an infinite Maker-Breaker game which is not determined.

Let X be any countably infinite set. Let $(\sigma_i : i < \omega_1)$ be an enumeration of the possible strategies for Maker on this set and $(\tau_i : i < \omega_1)$ an enumeration of the possible strategies for Breaker. We will construct sequences $(S_i : i < \aleph_1)$ and $(T_i : i < \aleph_1)$ of infinite subsets of X recursively such that each S_i meets each T_j, and so that Maker may claim S_i if Breaker plays according to τ_i and Breaker may claim T_i if Maker plays according to σ_i.

Suppose that all S_j and T_j with $j < i$ have already been constructed. Let $(U_n : n < \omega)$ be a sequence of infinite subsets of X in which all T_j with $j < i$ appear. Then we take S_i to be the set of elements claimed by Maker in a play of the game in which Breaker plays according to τ_i and Maker, on her n^{th} move, claims some as yet unclaimed element of U_n. The construction of T_i is similar.

Now consider the Maker-Breaker game on X whose set of winning sets is $\{S_i : i < \omega_1\}$. No τ_i can be a winning strategy for Breaker, since Maker can still claim S_i and win if Breaker plays according to τ_i. Similarly, no σ_i can be a winning strategy for Maker, since Breaker can still claim T_i, which meets all S_j, and so prevent Maker from winning if Maker plays according to σ_i. So this game is not determined.

References

[1] J. Beck. *Combinatorial games: Tic-Tac-Toe theory.* 2008. ISBN 978-0-521-46100-9.

[2] N. Bowler, F. Gut. *Infinite Maker-Breaker-Games; The $K^{\aleph_0}$ game: Edge colouring. In preparation.*

[3] N. Bowler, F. Gut. *The strong K^4 game on an infinite board. In preparation.*

[4] R. Diestel. *Graph Theory.* 5. Edition, 2017. ISBN 978-3-662-53621-6.

[5] J. Erde. *personal communication.*

[6] D. Hefetz, M. Krivelevich, M. Stojaković, Tibor Szabó. *Positional games.* 2014. ISBN 978-3-0348-0824-8.

[7] D. Hefetz, C. Kusch, L. Narins, A. Pokrovskiy, C. Requilé, A. Sarid. *Strong Ramsey Games: Drawing on an infinite board.* Journal of Combinatorial Theory. Volume 150, pages 248-266, 2017.

[8] I. B. Leader, *Lecture on Hypergraph Games.* 2008.

[9] Y. N. Moschovakis. *Descriptive Set Theory.* 2. Edition, 2009. ISBN 978-0-8218-4813-5.

Universität Hamburg, Faculty of Mathematics, Informatics and Natural Sciences, Department of Mathematics, Bundesstrasse 55 (Geomatikum), 20146 Hamburg, Germany

Email address: nathan.bowler@uni-hamburg.de, florian.gut@uni-hamburg.de