

# String-net construction of RCFT correlators

Jürgen Fuchs <sup>a</sup>, Christoph Schweigert <sup>b</sup> and Yang Yang <sup>b</sup>

<sup>a</sup> *Teoretisk fysik, Karlstads Universitet  
Universitetsgatan 21, S-651 88 Karlstad*

<sup>b</sup> *Fachbereich Mathematik, Universität Hamburg  
Bereich Algebra und Zahlentheorie  
Bundesstraße 55, D-20 146 Hamburg*

## ABSTRACT

We use string-net models to accomplish a direct, purely two-dimensional, approach to correlators of two-dimensional rational conformal field theories. We obtain concise geometric expressions for the objects describing bulk and boundary fields in terms of idempotents in the cylinder category of the underlying modular fusion category, comprising more general classes of fields than is standard in the literature. Combining these idempotents with Frobenius graphs on the world sheet yields string nets that form a consistent system of correlators, i.e. a system of invariants under appropriate mapping class groups that are compatible with factorization. Using markings, we extract operator products of field objects from specific correlators; the resulting operator products are natural algebraic expressions that make sense beyond semisimplicity. We also derive an Eckmann-Hilton relation internal to a braided category, thereby demonstrating the utility of string nets for understanding algebra in braided tensor categories. Finally we introduce the notion of a universal correlator. This systematizes the treatment of situations in which different world sheets have the same correlator and allows for the definition of a more comprehensive mapping class group.

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# 1 Introduction

“Solving a quantum field theory” – for any type of quantum field theory, in any number of dimensions – is an important goal, with deep ramifications in mathematics and physics. This goal has been pursued in a variety of approaches. In the specific case of two-dimensional rational conformal field theories (RCFTs, in short), the task can be reduced to a tractable mathematical problem: correlators can be described as specific elements in spaces of conformal blocks, and these vector spaces are under control in terms of modular functors. The consistency conditions for these correlators have been discussed extensively in the literature, starting with [FriS, So], see e.g. [KoLR, FuS3]. They amount to requiring that the correlator for a world sheet is invariant under the action of the mapping class group of the world sheet, that upon sewing of world sheets correlators are mapped to correlators, and that the basic two-point correlators are non-vanishing.

Making use of three-dimensional topological field theories of Reshetikhin-Turaev type, it has been shown two decades ago that solutions to these constraints are in bijection to (Morita classes of) special symmetric Frobenius algebras [FRS1, FFRS2] in the modular fusion category  $\mathcal{C}$  that governs the monodromies of the conformal blocks. Let us briefly recapitulate this “TFT approach to RCFT correlators”, highlighting both successes and drawbacks. In this approach the correlator for an oriented world sheet  $\mathcal{S}$  is realized as an element of the vector space of conformal blocks for a double cover  $\widehat{\Sigma}_{\mathcal{S}}$  of the surface  $\Sigma_{\mathcal{S}}$  that underlies  $\mathcal{S}$ . The crucial idea is to use the Reshetikhin-Turaev topological field theory that is based on the modular fusion category  $\mathcal{C}$  together with the fact that any three-manifold with embedded Wilson lines whose boundary is  $\widehat{\Sigma}_{\mathcal{S}}$  determines a vector in the space of conformal blocks for  $\widehat{\Sigma}_{\mathcal{S}}$ . Accordingly, a vital step in the TFT construction is to determine, for given  $\mathcal{S}$ , a three-manifold  $M_{\mathcal{S}}$  (called the connecting manifold) with boundary  $\partial M_{\mathcal{S}} = \widehat{\Sigma}_{\mathcal{S}}$ . In this framework extensive existence and uniqueness results for correlators have been established. Three-dimensional geometry is a central ingredient in the construction, and occasionally it requires a certain amount of mental gymnastics, such as in the proof of factorization [FFRS1].

Avoiding such an effort is a motivation to develop an approach to CFT correlators that is inherently two-dimensional. But it is not the most relevant motivation. A more significant one is the desire to transcend the restriction to semisimplicity of the modular tensor category. This restriction is inherent to the TFT approach: an extended three-dimensional topological field theory with target the bicategory of finite tensor categories assigns to the circle a modular category which is necessarily semisimple [BDSV]. In contrast, many conformal field theories of direct physical relevance are based on chiral data that are captured by a non-semisimple modular category. To understand such theories is a long term goal of our research. While the present paper does restrict to the case of semisimple modular categories, i.e. modular fusion categories, the novel construction we present promises to admit an extension to non-semisimple situations, in particular to modular finite tensor categories. This is e.g. supported by the existence of a state-sum construction for a modular functor [FSS2] that does not require semisimplicity. (The construction in [FSS2] is, however, inconvenient for the construction of correlators.)

The new construction produces the same correlators as the one in [FRS1] in all cases in which a direct comparison is possible, in particular in the absence of defect lines. In that case, by the uniqueness result of [FFRS2], the fact that the two-point correlators on the sphere and on the disk coincide implies coincidence of *all* correlators. Crucially, besides being considerably

simpler than the TFT construction, our new approach is entirely two-dimensional and thereby avoids the main obstacle to an extension beyond semisimplicity. In fact, basic ingredients of the construction are formulated in such a manner that they still make sense in the non-semisimple case.

Any construction of correlators requires a thorough understanding of the field content. Accordingly, one starting point of our construction is the systematic mathematical formalization of defect fields and of their operator products in (not necessarily semisimple) conformal field theories that was developed in [FuS3]. As an essential tool this formalization uses the theory of internal natural transformations, as introduced in [FuS2] within the framework of modular finite tensor categories and pivotal module categories over them. This framework is convenient even in the semisimple case that applies to rational CFT. In the present paper we actually transcend the setting of [FuS3] in that we allow for more general defect fields, at which more than two defect lines can meet, and similarly for more general boundary fields. (See the illustrations in Examples 2.11 and 2.12 for a first impression, and subsections 2.3.1 and 2.3.2 for details.)

To get control over correlators for such general defect and boundary fields in the purely algebraic framework of internal natural transformations [FuS3] is hard, if not impossible, in particular when it comes to their invariance under the relevant mapping class group. More generally, beyond semisimplicity full control over correlators has been achieved so far only in limited cases, e.g. – by combining the results of [FuSS] and [FuS1] – for bulk fields on oriented world sheets in the case that the modular finite tensor category admits a description as a category of modules over a factorizable ribbon Hopf algebra.

The method we develop in the present paper is based on the *string-net construction* of modular functors. It is known [Ki, Go, BG] that this construction, which takes as an input a spherical fusion category, realizes a modular functor of Turaev-Viro type. The string-net approach exhibits several crucial advantages. First of all, in contrast to Reshetikhin-Turaev type constructions (see [Ly] for the non-semisimple case), the representations of mapping class groups are geometric. Also, string nets provide a geometric realization of specific vectors in the spaces of conformal blocks. Moreover, in contrast to approaches based on a Lego-Teichmüller game (see e.g. [FuS1]), the string-net construction does not introduce any auxiliary structure like a pair-of-pants decomposition whose combinatorics quickly becomes unwieldy, in particular in the presence of boundaries and of additional stratifications of the surface that result from defects. Accordingly we expect that the string-net construction can also be generalized beyond oriented surfaces, e.g. to unoriented surfaces or to surfaces with spin structure.

The advantages of string nets have already been exploited in [ScY] to construct correlators of bulk fields in the special case – called the *Cardy case* in the CFT literature – that the boundary conditions of the full conformal field theory are just the objects of the modular fusion category  $\mathcal{C}$  of chiral data. The present paper extends the treatment of [ScY] in two directions: beyond the Cardy case, allowing for general local field theories with chiral data given by  $\mathcal{C}$  (in which case the boundary conditions are objects of a module category over  $\mathcal{C}$ ), and to include (generalized) boundary and defect fields. It is worth stressing that we *construct* these fields, i.e. do not use (as has been done in [Tr]) the existence of a consistent system of bulk and boundary fields as an input. We construct the fields as objects in the Drinfeld center  $\mathcal{Z}(\mathcal{C})$  of  $\mathcal{C}$ . More specifically, we realize them in terms of idempotents in the category  $\mathcal{Cyl}^{\circ}(\mathcal{C}, S^1)$  of boundary values on the circle, whose Karoubification – the *cylinder category*  $\mathcal{Cyl}(\mathcal{C}, S^1)$ , see Definition 3.4 – is known, for  $\mathcal{C}$  a pivotal fusion category, to be equivalent to the Drinfeld center. (This constitutes in

fact a more categorical variant of the tube algebra, whose irreducible representations give the simple objects of the Drinfeld center.)

For bulk fields, this is suggested by the braided equivalence  $\mathcal{C}^{\text{rev}} \boxtimes \mathcal{C} \simeq \mathcal{Z}(\mathcal{C})$  which expresses the non-degeneracy of the braiding in a modular fusion category. However, we treat field insertions on the boundary as objects in the Drinfeld center as well. To this end we use the adjoint of the forgetful functor  $\mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ . (As explained in Remark 2.2, this can be motivated from ideas in factorization homology, which suggest that the embedding of an interval into a circle gives rise to this functor.) At a technical level, this is possible because of the cyclicity properties expressed in Lemma A.4. Owing to this novel treatment of boundary insertions, we can treat the open and closed sectors of the CFT uniformly in terms of string nets, even though we do not define string nets for surfaces with physical boundaries on which boundary conditions need to be prescribed.

The rest of this paper is organized as follows. In Section 2 we first explain how the problem of finding correlation functions is translated to the problem of finding specific elements in spaces of conformal blocks. These spaces of conformal blocks are then described in terms of an open-closed modular functor, a notion that is introduced in Section 2.1. Afterwards we present, in Section 2.2, the class of world sheets to which we want to associate correlators. Concretely, we wish to work on two-dimensional oriented manifolds with stratifications, so as to be able to account for topological line defects as well as boundaries. The significance of studying quantum field theory on stratified spaces is by now well established, not only in view of their role in the study [FröFRS2] of symmetries and dualities. Given a modular fusion category  $\mathcal{C}$ , the cells of the world sheet need representation-theoretic labels: two-cells are labeled by simple special symmetric Frobenius algebras in  $\mathcal{C}$ , labels for topological defect lines and for boundary conditions are bimodules and modules, respectively, internal to the category  $\mathcal{C}$ . This allows us to discuss, in the subsequent Section 2.3, the field content, that is, the different types of field insertions in the bulk and on the boundary. Finally, in Section 2.4 we discuss correlators as vectors in spaces of conformal blocks and give a precise mathematical meaning to factorization. This culminates in Definition 2.19 of a *consistent system of correlators*. At this point we have set up the problem that we are going to solve.

Section 3 is devoted to the solution of this problem. Our main tool are string nets; they are introduced in Section 3.1. To account for the gluing boundaries (as opposed to physical boundaries) of world sheets, we need to deal with string nets on surfaces with boundaries. In Section 3.2 we discuss categories of boundary values and string-net spaces for given boundary values. This allows us to set up, in Section 3.3, a modular functor in terms of string nets. In Section 3.4 we then explain how a world sheet (including the decorations of its two-, one- and zero-cells) determines a string net and thereby a vector in the space of conformal blocks. Readers who want to get a quick impression of this process are invited to see how the local situation described in the picture (3.56), which shows a world sheet containing two defect lines and a boundary circle that supports two boundary field insertions as well as two physical boundaries, is translated into the idempotent (3.59) of the cylinder category. This determines the form of the string net close to the boundary of a world sheet. To also take care of the two-cells of a world sheet we equip them with what we call a full Frobenius graph – a graph whose edges are labeled by the Frobenius algebra assigned to the two-cell and whose vertices are labeled by (multiple) products and coproducts (see Definition 3.20). The particular choice of full Frobenius graph is irrelevant, which is yet another incarnation of the well-known compatibility

of two-dimensional Pachner moves and Frobenius algebras. This feature is also at the basis of the invariance of the correlators under the mapping class group. The resulting prescription for the string-net correlators is summarized in Definition 3.26. Section 3 culminates in Theorem 3.28, which asserts that our purely two-dimensional construction indeed provides a consistent system of correlators.

Section 4 is devoted to the aspects of a few particular correlators and in particular to their relation with operator products. The goal is to extract from specific string-net correlators composition morphisms or, in other words, products, on the field objects. This translations from string nets to elements of morphism spaces of the relevant category requires additional choices, concretely a marking (akin to a pair-of-pants decomposition) of the world sheet. In Section 4.1 we show how the composition of bulk fields on the same line defect (as portrayed in picture (4.3)) leads to the vertical product of relative natural transformations as introduced in [FuS2]. (The proposal, put forward in [FuS3], that the vertical product describes this operator product, applies to all finite – not necessarily semisimple – conformal field theories; our string-net techniques at present only settle the case of semisimple categories, though.) In Section 4.2 we consider horizontal operator products, which are related to the fusion of two parallel topological line defects (compare the picture (4.18)). Here the situation is considerably more involved than for the vertical product: as we work with braided categories, there are two different horizontal products, corresponding to two non-isotopic string nets (with equal markings – or, equivalently, isotopic string nets with different markings). An immediate insight is that the two different horizontal products are related by a braiding, see Remark 4.2. In the two subsequent subsections we compute the bulk algebra and the torus partition function, finding full agreement with results in the literature. We finally show that the operator product of two boundary fields is given by the natural composition of internal Hom objects (Section 4.5) and that the bulk boundary operator products are the structure morphisms of the end that defines a bulk or, more generally, a disorder field (Section 4.6).

As is already apparent from the discussion of the two different horizontal products, string nets are a convincing tool for understanding algebraic facts in a braided monoidal setting. A further illustration of this point is given in Section 5, where we establish a braided version of an Eckmann-Hilton theorem.

It is known (see e.g. [FröFRS2]) that different world sheets can have correlators that are closely related. In particular there are identities of correlators which pinpoint symmetries of the conformal field theory that can be extracted from the presence of invertible topological defects. In Section 6 we introduce the new notion of a *universal correlator* which captures such relations. As it turns out, the most natural notion of mapping class group under which correlators are invariant is the stabilizer group of the universal correlator.

In nine appendices we collect mathematical background and fix notation. The background information concerns algebra and representation theory in finite tensor categories and features of their Drinfeld centers. The expert reader might wish to consult these appendices only when needed.

Much work remains to be done. In particular it is a challenge to extend the string-net construction beyond spherical fusion categories to finite tensor categories that are not semisimple. In this respect it is gratifying that the methods of [FuS2, FuS3] work for general finite conformal field theories. Furthermore, in the end even rigidity should be relaxed to the more general structure of a Grothendieck-Verdier duality, which is the natural duality structure for large

classes of vertex operator algebras [ALSW]. It is encouraging that for pivotal Grothendieck-Verdier categories one can still construct [MW] blocks that carry a representation of a handle body group.

## 2 Correlators in rational conformal field theory

### 2.1 Open-closed modular functor

A fundamental task in any quantum field theory is to determine, for each member of a class of manifolds with suitable additional structure – such as boundary conditions and field insertions – a correlation function. In this sense, to specify a quantum field theory one needs to provide first its field content and then an infinite set of correlation functions. Specifically, in two-dimensional conformal field theory (CFT) one wants to associate such a correlation function to a surface with additional data, which we call a *world sheet*.

In the approach to CFT correlation functions we take in this paper, the starting point is not a Lagrangian or a partial differential equation describing classical field equations, and we do not rely on any type of perturbation theory. Instead, we postulate that the correlation functions of the theory exhibit certain chiral symmetries, which means concretely that they are solutions to a collection of linear differential equations, also known as chiral Ward identities. For the two-dimensional conformal field theories of our interest, these symmetries can be encoded in the structure of a conformal vertex algebra, and the chiral Ward identities on any surface can be derived from that algebra. Locally, the solutions to these equations – which contain the correlation functions we are looking for as particular elements – form vector spaces. For the models of our interest these vector spaces are finite-dimensional; when regarded as elements of these spaces, we refer to the correlation functions for brevity also as *correlators*. Typically the solutions are multivalued, so the fundamental groups of the relevant parameter spaces (conformal structure of the surfaces and positions of insertion points), i.e. mapping class groups of surfaces, act on the solution spaces. The particular elements in these spaces that are correlation functions of *bulk fields* must be single-valued, and thus transform trivially under the action of the mapping class group, while correlation functions involving general *defect fields* are invariant under a subgroup of the mapping class group. (For the latter, see Definition 2.16 for an initial description and Eq. (6.10) for an extended definition based on the notion of “universal correlators” which we introduce in Section 6.) To appreciate these statements in our context, several ingredients need to be explained. First and foremost, we need a tractable notion of *conformal blocks* and a precise notion of world sheet. These will be provided in the present and the next subsection, respectively.

The spaces of solutions to the chiral Ward identities form vector bundles with projectively flat connection over moduli spaces of insertion points and conformal structures (see e.g. [DaGT]). One legitimate use of the term conformal blocks is to denominate these vector bundles, or also holomorphic sections in them. On the other hand, the conformal blocks depend functorially on the representations of the chiral symmetry structure, the conformal vertex algebra, at the insertion points. This leads to the notion of a *modular functor*, which furnishes a more algebraic formalization of the idea of conformal blocks: finite-dimensional vector spaces endowed with a representation of the mapping class group and depending functorially on the representation-theoretic data at the insertions. Indeed it is widely believed that one can as-

sociate to any rational chiral CFT a modular functor such that these mapping class group representations are related to the projectively flat connection on the vector bundles of the CFT by a Riemann-Hilbert correspondence. Accordingly, another use of the term conformal blocks refers to the vector spaces furnished by the modular functor; this is the use of the term that we adopt here.

A modular functor is then a certain 2-functor from a geometric bicategory of two-dimensional bordisms to an algebraic bicategory. For being relevant to CFT with the type of world sheets considered in this paper, which are allowed to have boundary field insertions, the bordism category has to be of *open-closed* type, meaning that its objects are finite disjoint unions of both circles and intervals, while on the algebraic side it turns out to be convenient to take the 1-morphisms as profunctors. This leads us to the following definition, where  $\mathbb{k}$  is an algebraically closed field of characteristic zero, chosen once and for all.

**Definition 2.1.** An *open-closed modular functor* is a symmetric monoidal 2-functor

$$\text{Bl} : \mathcal{B}ord_{2,o/c}^{\text{or}} \longrightarrow \mathcal{P}rof_{\mathbb{k}} \quad (2.1)$$

from the bicategory  $\mathcal{B}ord_{2,o/c}^{\text{or}}$  of two-dimensional open-closed bordisms to the bicategory of  $\mathbb{k}$ -linear profunctors.

Let us unwrap this definition. First, the bicategory  $\mathcal{B}ord_{2,o/c}^{\text{or}}$  of two-dimensional *open-closed bordisms* is the following symmetric monoidal bicategory: An object  $\alpha \in \mathcal{B}ord_{2,o/c}^{\text{or}}$  is a finite disjoint union of copies of the standard closed interval  $I = [0, 1] \subset \mathbb{R}$ , oriented from 1 to 0, and of the standard circle  $S^1 = \{z \in \mathbb{C} \mid |z| = 1\} \subset \mathbb{C}$ , with counter-clockwise orientation. A 1-morphism  $\alpha \rightarrow \alpha'$  is a compact oriented smooth surface  $\Sigma$  with boundary (and with induced orientation of  $\partial\Sigma$ ), together with a *parametrization*  $\phi \equiv \{\phi_-, \phi_+\}$  of a subset of  $\partial\Sigma$ ; the latter consists of an orientation reversing embedding  $\phi_-(\Sigma) : \alpha \hookrightarrow \partial\Sigma$  and an orientation preserving embedding  $\phi_+(\Sigma) : \alpha' \hookrightarrow \partial\Sigma$  whose images are disjoint and whose restriction to any connected component of  $\alpha$  and  $\alpha'$  is a diffeomorphism. A 2-morphism  $\gamma : \Sigma \rightarrow \Sigma'$  is an isotopy class of diffeomorphisms from  $\Sigma$  to  $\Sigma'$  such that  $\phi_{\pm}(\Sigma') = \gamma \circ \phi_{\pm}(\Sigma)$ . The composition of two 1-morphisms  $\Sigma : \alpha \rightarrow \alpha'$  and  $\Sigma' : \alpha' \rightarrow \alpha''$  is gluing along  $\alpha'$  (or rather, as usual, along a collar of  $\alpha'$ , such that the glued surface is again smooth), the vertical composition of 2-morphisms is induced by the composition of diffeomorphisms, and the horizontal composition of 2-morphisms is gluing. The tensor product is given by disjoint union, with the monoidal unit the empty set.

The bicategory  $\mathcal{P}rof_{\mathbb{k}}$  of  *$\mathbb{k}$ -linear profunctors* is a finite  $\mathbb{k}$ -linear version of the standard [Bo, Prop. 7.8.2] bicategory of profunctors: An object of  $\mathcal{P}rof_{\mathbb{k}}$  is a  $\mathbb{k}$ -linear finite abelian category  $\mathcal{A}$ . A 1-morphism  $\mathcal{A} \rightarrow \mathcal{A}'$  is a left exact functor  $F : \mathcal{A}^{\text{opp}} \boxtimes \mathcal{A}' \rightarrow \mathcal{V}ect_{\mathbb{k}}$  from the Deligne product of the opposite of  $\mathcal{A}$  and of  $\mathcal{A}'$  to the category of finite-dimensional  $\mathbb{k}$ -vector spaces. A 2-morphism  $F \rightarrow F'$  in  $\mathcal{P}rof_{\mathbb{k}}$  is a natural transformation from  $F$  to  $F'$ . The composition of 1-morphisms  $F : \mathcal{A} \rightarrow \mathcal{A}'$  and  $F' : \mathcal{A}' \rightarrow \mathcal{A}''$  is given by the coend

$$(F \circ F')(\bar{?} \boxtimes ?) = \int^{X \in \mathcal{A}'} F(\bar{?} \boxtimes X) \otimes_{\mathbb{k}} F'(\bar{X} \boxtimes ?), \quad (2.2)$$

where  $\bar{X}$  stands for  $X \in \mathcal{A}'$  regarded as an object of the opposite category.

In the sequel we tacitly strictify each of the bicategories  $\mathcal{B}ord_{2,o/c}^{\text{or}}$  and  $\mathcal{P}rof_{\mathbb{k}}$ , i.e. consider them as strict 2-categories. We then also take the 2-functor Bl as strict, and thus require strict preservation of composition.



That  $\text{Bl}$  is a 2-functor means in particular that any gluing from  $\Sigma: \alpha \rightarrow \alpha'$  and  $\Sigma': \alpha' \rightarrow \alpha''$  along the one-manifold  $\alpha'$  to the surface  $\Sigma \cup_{\alpha'} \Sigma'$  gives rise to a natural transformation

$$\text{Bl}(\Sigma)(\bar{?} \boxtimes Y) \otimes_{\mathbb{k}} \text{Bl}(\Sigma')(\bar{Y} \boxtimes ?) \longrightarrow \text{Bl}(\Sigma \cup_{\alpha'} \Sigma')(\bar{?} \boxtimes ?) \quad (2.3)$$

for  $Y \in \text{Bl}(\alpha')$ , given by the dinatural structure morphisms of the (left exact) coend

$$\text{Bl}(\Sigma \cup_{\alpha} \Sigma')(\bar{?} \boxtimes ?) = \int^{X \in \text{Bl}(\alpha')} \text{Bl}(\Sigma)(\bar{?} \boxtimes X) \otimes_{\mathbb{k}} \text{Bl}(\Sigma')(\bar{X} \boxtimes ?). \quad (2.4)$$

Moreover, for any 1-morphism  $\Sigma$  in  $\mathcal{B}ord_{2,o/c}^{\text{or}}$ , the functor  $\text{Bl}(\Sigma)$  carries an action of the mapping class group  $\text{Map}(\Sigma)$ , i.e. the group of 2-endomorphisms of  $\Sigma$ , by natural endotransformations.

To be relevant for applications to rational conformal field theory, such a modular functor has to satisfy constraints. One first formalizes the symmetries of the CFT in the structure of a vertex operator algebra  $\mathfrak{V}$ . Then one selects a good category  $\mathcal{C}$  of  $\mathfrak{V}$ -representations, e.g. by requiring that the HLZ theory of logarithmic tensor products [HLZ] should apply, which endows  $\mathcal{C}$  with the structure of a  $\mathbb{k}$ -linear braided monoidal category with a Grothendieck-Verdier duality. For a *rational* CFT – the case of interest in the present paper – the vertex operator algebra  $\mathfrak{V}$  is required to be rational, meaning that  $\mathcal{C}$  is even a modular fusion category (i.e., a semisimple modular tensor category); we fix this category  $\mathcal{C}$  throughout the paper. We thus assume that there is an open-closed modular functor that models the conformal blocks of a rational conformal field theory with given modular fusion category  $\mathcal{C}$ ; we denote this modular functor by  $\text{Bl}_{\mathcal{C}}$ .

Bulk fields of the CFT should be objects in the Deligne product  $\mathcal{C}^{\text{rev}} \boxtimes \mathcal{C}$  of the modular fusion category  $\mathcal{C}$  with its reverse  $\mathcal{C}^{\text{rev}}$ , i.e. the same spherical fusion category but endowed with the opposite braiding. This captures the informal statement that bulk fields are obtained by ‘combining left movers and right movers’. Remarkably, the nondegeneracy of the braiding of  $\mathcal{C}$  implies that the Deligne product  $\mathcal{C}^{\text{rev}} \boxtimes \mathcal{C}$  is equivalent, as a modular category, to the Drinfeld center  $\mathcal{Z}(\mathcal{C})$  of the spherical fusion category underlying the modular category  $\mathcal{C}$ . Accordingly we treat bulk fields as objects in  $\mathcal{Z}(\mathcal{C})$ . (For the precise notion of bulk fields, see Section 2.3.1 below.)

We are interested in this paper in a modular functor which when restricted to the *closed sector* reproduces the known description of the conformal blocks of rational conformal field theories. This includes in particular the requirement that the modular functor  $\text{Bl}_{\mathcal{C}}$  assigns to the circle  $S^1$  a category braided equivalent to the Drinfeld center  $\mathcal{Z}(\mathcal{C})$ , for  $\mathcal{C}$  a  $\mathbb{k}$ -linear spherical fusion category; in this paper,  $\mathcal{C}$  is the modular fusion category that we have already fixed above. More generally, we have to demand that the restriction of  $\text{Bl}_{\mathcal{C}}$  to the sub-bicategory of  $\mathcal{B}ord_{2,o/c}^{\text{or}}$  whose objects are disjoint unions of circles is isomorphic to the modular functor obtained by the Turaev-Viro state-sum construction for  $\mathcal{C}$  or, equivalently [TV], by the Reshetikhin-Turaev surgery construction for  $\mathcal{Z}(\mathcal{C})$ . For pertinent information on fusion categories and their Drinfeld center we refer to [EGNO, TV] and to Appendices A.1 and A.5. Without loss of generality we take the pivotal structure on  $\mathcal{C}$  to be strictified.

For ease of notation, in the sequel we write

$$\text{Bl}_{\mathcal{C}}(\Sigma)(\bar{?} \boxtimes ?) =: \text{Bl}_{\mathcal{C}}(\Sigma; ?; ?). \quad (2.5)$$

We also follow the common terminology to call, for a 1-morphism  $\Sigma: \alpha \rightarrow \alpha'$ , the circles and intervals in  $\partial\Sigma$  which are images under  $\phi_{-}(\Sigma)$  of the connected components of  $\alpha$  *incoming*,

and those which are images under  $\phi_+(\Sigma)$  of the connected components of  $\alpha'$  *outgoing*; in case that  $\alpha'$  is the empty set (regarded as a one-manifold), we briefly write  $\text{Bl}_{\mathcal{C}}(\Sigma; ?)$  instead of  $\text{Bl}_{\mathcal{C}}(\Sigma; ?; \emptyset)$ . For the closed sector,  $\text{Bl}_{\mathcal{C}}$  is the modular functor based on the spherical fusion category  $\mathcal{C}$  that is defined in terms of *string nets* colored by  $\mathcal{C}$ . This functor is implicit in the constructions in [Ki]; for the pertinent details about the string-net construction see Section 3.1.

Note that the string-net construction gives rise to a 3-2-1 topological field theory (TFT) that is isomorphic to the 3-2-1-extended Turaev-Viro state sum TFT for  $\mathcal{C}$  [Ki, Go]. This isomorphism of topological field theories restricts to an isomorphism between the respective modular functors. (In Section 3.3 we will demonstrate explicitly that this is indeed true for the string-net modular functor  $\text{SN}_{\mathcal{C}}$ .) Note that it follows e.g. that when the 1-morphism  $\Sigma$  is a connected surface of genus  $g$  with  $\partial_-\Sigma \cong \sqcup_{i=1}^p S^1$  and  $\partial_+\Sigma \cong \sqcup_{j=1}^q S^1$ , then there is an isomorphism [Ly, BakK]

$$\text{Bl}_{\mathcal{C}}\left(\Sigma; \boxtimes_{i=1}^p X_i; \boxtimes_{j=1}^q Y_j\right) \cong \text{Hom}_{\mathcal{Z}(\mathcal{C})}\left(\bigotimes_{i=1}^p \tilde{X}_i, \bigotimes_{j=1}^q \tilde{Y}_j \otimes K^{\otimes g}\right) \quad (2.6)$$

for  $X_i, Y_j \in \text{Bl}_{\mathcal{C}}(S^1)$ , where  $\tilde{X}_i$  and  $\tilde{Y}_j$  are their images under  $\text{Bl}_{\mathcal{C}}(S^1) \xrightarrow{\cong} \mathcal{Z}(\mathcal{C})$  and  $K \in \mathcal{Z}(\mathcal{C})$  is the distinguished Hopf algebra object

$$K := Z_{\mathcal{Z}(\mathcal{C})}(\mathbf{1}_{\mathcal{Z}(\mathcal{C})}) = \int^{X \in \mathcal{Z}(\mathcal{C})} X^{\vee} \otimes X \quad (2.7)$$

in  $\mathcal{Z}(\mathcal{C})$ , i.e. the image of the monoidal unit under the central monad  $Z_{\mathcal{Z}(\mathcal{C})}$  of  $\mathcal{Z}(\mathcal{C})$ . The isomorphism (2.6) is far from canonical, though. For a fixed choice of equivalence  $\text{Bl}_{\mathcal{C}}(S^1) \xrightarrow{\cong} \mathcal{Z}(\mathcal{C})$  it can, however, be specified uniquely up to unique isomorphism once the surface  $\Sigma$  is endowed with the auxiliary structure of a *marking*, which keeps track of the way that  $\Sigma$  can be assembled by sewing pairs of pants [Ly, BakK, FuS1].

Next we extend  $\text{Bl}_{\mathcal{C}}$  from the closed sector to all of  $\mathcal{B}ord_{2,o/c}^{\text{or}}$ . We implement this extension with the help of the functor

$$L := U^{\text{1.a.}} : \mathcal{C} \longrightarrow \mathcal{Z}(\mathcal{C}) \quad (2.8)$$

that is left adjoint to the forgetful functor from the Drinfeld center  $\mathcal{Z}(\mathcal{C})$  to  $\mathcal{C}$ . In particular we require that the category assigned to the interval  $I \in \mathcal{B}ord_{2,o/c}^{\text{or}}$  is, up to canonical equivalence,  $\text{Bl}_{\mathcal{C}}(I) = \mathcal{C}$  (to be precise,  $\mathcal{C}$  is assigned, as a bimodule category over itself, to the interior of  $I$ ). However, unlike in other approaches in the literature, we complement this assignment by a prescription that associates, via the functor (2.8), a single category with a geometric boundary circle of a 1-morphism that contains any finite number of embedded intervals, i.e. to a circle of the type



The string-net open-closed modular functor  $\text{SN}_{\mathcal{C}}$  that we will obtain in Section 3.3 indeed realizes the extension from the closed sector to the all of  $\mathcal{B}ord_{2,o/c}^{\text{or}}$  in this way.

More explicitly, for a boundary circle containing the image of the union  $I \sqcup I \sqcup \dots \sqcup I$  of  $n$  (ordered) copies of the interval we deal with  $n$  copies of  $\mathcal{C}$ , and we combine any ordered collection  $(X_1, X_2, \dots, X_n) \in \mathcal{C}^{\times n}$  of objects in  $\mathcal{C}$  to the object

$$L(X_1 \otimes X_2 \otimes \dots \otimes X_n) \in \mathcal{Z}(\mathcal{C}) \quad (2.10)$$

in the Drinfeld center. An advantage this prescription is that it allows us to treat the open and closed sectors uniformly: we assign to *any* connected component of  $\partial\Sigma$  one and the same category, namely (up to isomorphism) the Drinfeld center  $\mathcal{Z}(\mathcal{C})$ . This has various technical benefits. It will in particular make it possible to define CFT correlators with the help of the string-net construction even for world sheets with boundary field insertions.

**Remark 2.2.** The composite functor  $U \circ L$  is the underlying functor of the *central monad*  $Z$  of  $\mathcal{C}$ , as described in Appendix A.6. Accordingly, for any  $X \in \mathcal{C}$  the object underlying the object  $L(X)$  in  $\mathcal{Z}(\mathcal{C})$  is the coend given in (A.33). In particular, if  $\mathcal{C}$  is semisimple, as considered here, we have

$$U \circ L(X) = \bigoplus_{i \in \mathcal{I}(\mathcal{C})} i^\vee \otimes X \otimes i \quad (2.11)$$

with  $\mathcal{I}(\mathcal{C})$  the chosen set of representatives for the isomorphism classes of simple objects of  $\mathcal{C}$ .

The rationale behind the prescription (2.10) is as follows. When the assignment of categories to one-manifolds comes from factorization homology – which is indeed the case for the modular functor that is furnished by the string-net construction – then the embedding  $I \hookrightarrow S^1$  induces a functor, calculated via excision, from  $\text{Bl}_{\mathcal{C}}(I) \simeq \mathcal{C}$  to  $\text{Bl}_{\mathcal{C}}(S^1)$ , i.e. to the relative Deligne product  $\mathcal{C} \boxtimes_{\mathcal{C} \boxtimes_{\mathcal{C}^{\text{opp}}} \mathcal{C}} \mathcal{C}$ . This functor factorizes into the composite of the functor

$$\begin{aligned} \mathcal{C} &\longrightarrow \mathcal{C} \boxtimes \mathcal{C}, \\ c &\longmapsto c \boxtimes \mathbf{1} \end{aligned} \quad (2.12)$$

to the ordinary Deligne product and the structure functor  $\mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathcal{C} \boxtimes_{\mathcal{C} \boxtimes_{\mathcal{C}^{\text{opp}}} \mathcal{C}} \mathcal{C}$ . By Proposition 2.18(i) of [FSS1] one can represent the relative Deligne product as the twisted center  $\mathcal{Z}^{\text{D}}(\mathcal{C})$ , i.e. the Drinfeld center twisted by the double dual functor. Here the structure map is the left adjoint  $L$  of the forgetful functor  $U: \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$ . (Since a modular category is in particular unimodular,  $L$  is also a right adjoint of  $U$ .) Moreover, owing to the pivotality of  $\mathcal{C}$  the category  $\mathcal{Z}^{\text{D}}(\mathcal{C})$  is canonically equivalent to  $\mathcal{Z}(\mathcal{C})$ . This leads us to consider the left adjoint  $L$  in (2.8). In the same spirit, embedding two copies of  $I$  into  $S^1$  gives rise to the functor  $L \circ \otimes$ , and analogously for any number  $n > 2$  of copies of  $I$ .

Also note that it can happen that a boundary circle of a 1-morphism  $\Sigma$  does not contain the image of any interval or circle. In this case we can consider the sequence  $\emptyset \hookrightarrow I \hookrightarrow S^1$  of embeddings; from factorization homology the embedding  $\emptyset \hookrightarrow I$  gives us the object  $\mathbf{1} \in \mathcal{C}$ , and hence  $\emptyset \hookrightarrow S^1$  gives  $L(\mathbf{1}) \in \mathcal{Z}(\mathcal{C})$ . This is reproduced by setting  $n = 0$  in (2.10) and interpreting the empty tensor product as the monoidal unit.

## 2.2 World sheets

We now specify the world sheets to which a conformal field theory assigns correlators. A world sheet is an oriented surface equipped with quite a lot of additional structure. We therefore

proceed in two separate steps: First we describe purely geometric features of a world sheet; afterwards we complement these geometric structures with labels which come from the spherical fusion category  $\mathcal{C}$  (recall from Section 2.1 that  $\mathcal{C}$  has been selected once and for all). The former step will in particular allow us to extract from a world sheet specific 0-, 1- and 2-morphisms of the bicategory  $\mathcal{Bord}_{2,o/c}^{\text{or}}$ , while the latter makes it possible to take the spaces of conformal blocks in which the correlators of the CFT are elements as the vector spaces that are furnished by a modular functor  $\text{Bl}_{\mathcal{C}}$  of the type discussed in Section 2.1. (Later on, in Section 3.5 we will define correlators concretely with the help of the string-net modular functor  $\text{SN}_{\mathcal{C}}$ . But for the considerations from here on until the end of Section 2 we can just rely on the required properties of  $\text{Bl}_{\mathcal{C}}$  and do not need to invoke the concrete realization of  $\text{Bl}_{\mathcal{C}}$  as the particular modular functor  $\text{SN}_{\mathcal{C}}$ .)

**Definition 2.3.** An *unlabeled world sheet*  $\check{\mathfrak{S}}$  is a smooth oriented compact surface (possibly with boundary) equipped with a collection of submanifolds of dimension 0, 1 and 2 – to be referred to as zero-cells, one-cells and two-cells, respectively – satisfying the following conditions:

- The number of cells is finite.
- Each zero-cell is contained in the boundary of a one-cell, each one-cell is contained in the boundary of a two-cell, and  $\check{\mathfrak{S}}$  is the closure of the union of all two-cells.
- The interior of every one-cell is either contained in the interior of  $\check{\mathfrak{S}}$  or contained in the boundary  $\partial\check{\mathfrak{S}}$ .
- Every one-cell in the interior of  $\check{\mathfrak{S}}$  is oriented as a one-manifold.
- There are two types of *boundary one-cells*, i.e. one-cells on  $\partial\check{\mathfrak{S}}$ : oriented and unoriented. The 1-orientation of an oriented boundary one-cell is opposite to the one induced on it<sup>1</sup> from the 2-orientation of  $\check{\mathfrak{S}}$ .
- A zero-cell in  $\partial\check{\mathfrak{S}}$  at which two unoriented boundary one-cells meet is in addition met by precisely one oriented one-cell, whose interiors are contained in the interior of  $\check{\mathfrak{S}}$ . A zero-cell in  $\partial\check{\mathfrak{S}}$  at which an unoriented and an oriented boundary one-cell meet, is not met by any further one-cell.
- Every connected component of  $\partial\check{\mathfrak{S}}$  contains at least one zero-cell.

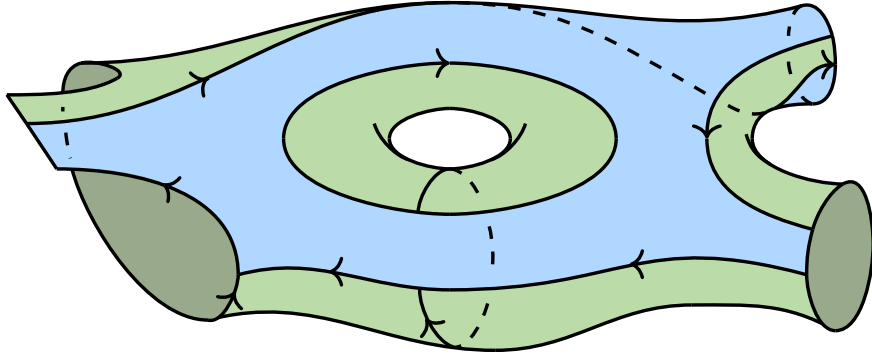
Note that according to this prescription it cannot happen that two one-cells whose interior is contained in the interior of  $\check{\mathfrak{S}}$  meet at a point on the boundary of  $\check{\mathfrak{S}}$ . The rationale for this prescription is that it suffices to describe all types of fields occurring in a conformal field theory, compare e.g. the field described in Example 2.12 below.

**Example 2.4.** The following picture shows an example of an unlabeled world sheet of genus 1

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<sup>1</sup>The 2-orientation of an oriented surface induces a 1-orientation on its boundary, see e.g. [BT, Ch. 3].

with three geometric boundary circles:



(2.13)

Here for clarity we have dyed the two two-cells of  $\mathcal{S}$  by different colors.

The following terminology is motivated by the role that world sheets play for conformal field theory correlators. The one-cells in the interior of an unlabeled world sheet  $\check{\mathcal{S}}$  are called *defect lines*. The oriented one-cells in  $\partial\check{\mathcal{S}}$  are called *physical boundaries*. Zero-cells in the interior and zero-cells on the boundary at which two physical boundaries meet are called *defect junctions*. The unoriented one-cells in  $\partial\check{\mathcal{S}}$  are called *gluing boundaries*, or also *field insertion boundaries*. We refer to the connected components of the boundary of  $\check{\mathcal{S}}$  as *geometric boundary circles* and denote the set of these by  $\pi_0(\partial\check{\mathcal{S}})$ . If all one-cells in a geometric boundary circle  $b$  are gluing boundaries, then  $b$  is called a *gluing circle*. If a geometric boundary circle  $b$  contains at least one physical boundary, then each the connected components of the complement of the physical boundaries in  $b$  is called a *gluing interval*. A geometric boundary circle can contain any finite number of gluing intervals.

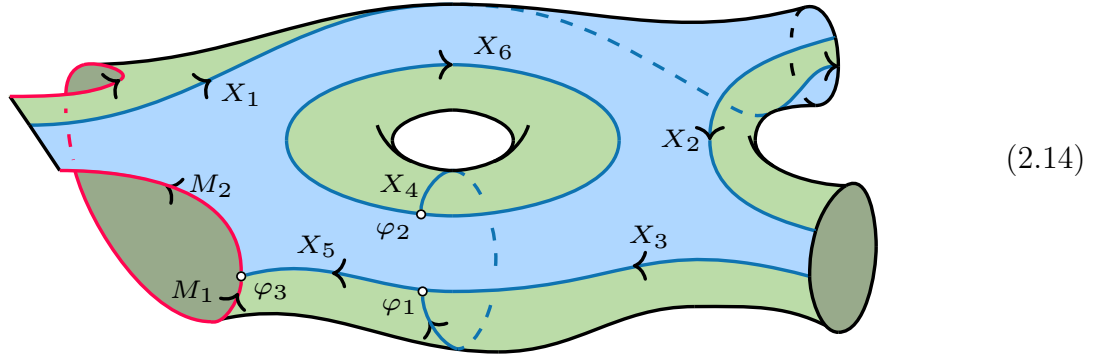
Having this terminology at hand, we can now formulate the prescription for adding labels to an unlabeled world sheet. These labels involve in particular Frobenius algebras in a fixed fusion category  $\mathcal{C}$  and (bi-)modules over them; for information about these structures and for pertinent conventions we refer to Appendices A.2 and A.3.

**Definition 2.5.** A *world sheet*  $\mathcal{S}$  is an unlabeled world sheet  $\check{\mathcal{S}}$  together with the following assignments of labels to the strata of  $\check{\mathcal{S}}$ :

- To any two-cell of  $\check{\mathcal{S}}$  there is assigned a simple special symmetric Frobenius algebra in  $\mathcal{C}$ .
- To any defect line there is assigned an  $A_l$ - $A_r$ -bimodule in  $\mathcal{C}$ , where  $A_l$  and  $A_r$  are the labels for the adjacent two-cells to the left and right of it, respectively, where the distinction between left and right is determined by the orientations of the one- and two-cells.
- To any physical boundary there is assigned a right  $A$ -module, where  $A$  is the label for the adjacent two-cell.
- To any defect junction  $v$  in the interior of  $\check{\mathcal{S}}$  there is assigned a bimodule morphism in a space  $H_v$  determined by the labels of the defect lines that meet at  $v$ , and to any defect junction  $w$  on the boundary of  $\check{\mathcal{S}}$  there is assigned a module morphism in a space  $H_w$  determined by the labels of the defect lines and the two physical boundaries that meet at  $w$ . (The spaces  $H_v$  and  $H_w$  will be specified in (2.17) and (2.19) below.)
- Gluing boundaries, as well as zero-cells that are not defect junctions (and thus are located at an end point of a gluing boundary), are not labeled.

To make this definition complete, we still must specify the morphism spaces in which the labels for labeled zero-cells take values. Before doing so, let us first illustrate the already somewhat lengthy description of a world sheet by an example. From here on, unless declared otherwise, by a “Frobenius algebra” we mean a simple special symmetric Frobenius algebra in  $\mathcal{C}$ .

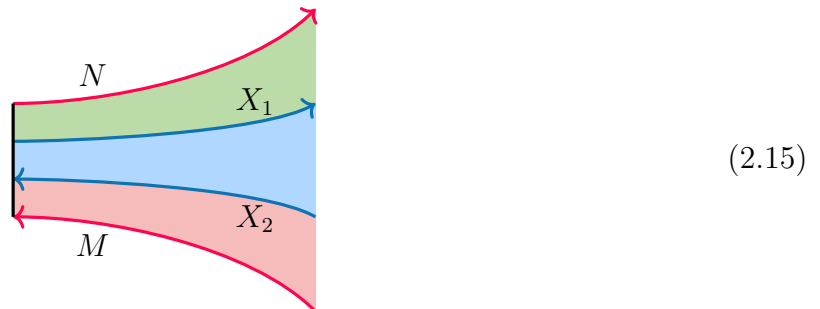
**Example 2.6.** The world sheet



with underlying unlabeled world sheet (2.13) has:

- two two-cells, shaded in blue and green, which are labeled by Frobenius algebras  $A$  and  $B$  in  $\mathcal{C}$ , respectively;
- six defect lines whose labels are an  $A$ - $B$ -bimodule  $X_6$ ,  $B$ - $A$ -bimodules  $X_1$ ,  $X_2$ ,  $X_3$  and  $X_5$ , and a  $B$ -bimodule  $X_4$ , respectively;
- two physical boundaries on the geometric boundary circle located to the left, labeled by an  $A$ -module  $M_2$  and a  $B$ -modules  $M_1$ , respectively;
- two gluing circles (the two geometric boundary circles on the right), and one gluing interval (drawn as a straight line) contained in the geometric boundary circle on the left;
- three defect junctions with morphism labels  $\varphi_1$ ,  $\varphi_2$  and  $\varphi_3$  (lying in morphism spaces that will be given in (2.21) below).

**Example 2.7.** Let us also display the situation near a slightly more general gluing interval in detail: the following picture shows part of a world sheet near a gluing interval that connects physical boundaries labeled by  $M$  and  $N$  and is in addition met by two defect lines labeled by bimodules  $X_1$  (outgoing) and  $X_2$  (incoming):



(The gluing interval, located on the left, is drawn in black, the physical boundaries in red, and the two defect lines in blue.)

Concerning the morphism space  $H_v$  assigned to a defect junction  $v$  in the interior of a world sheet, we first note that the orientation of the world sheet furnishes a cyclic order of the defect lines that meet at  $v$  in such a way that this cyclic ordering is clockwise if the orientation of the surface is counter-clockwise. Suppose that for some selected linear order compatible with this cyclic order, the defect lines meeting at  $v$  are labeled clockwise by bimodules  $X_i$  for  $i=1, 2, \dots, n$ . We write  $X_i^\vee$  for the bimodule dual to  $X_i$ , as defined at the end of Appendix A.3, and set

$$X^\epsilon := \begin{cases} X & \text{for } \epsilon = +, \\ X^\vee & \text{for } \epsilon = -. \end{cases} \quad (2.16)$$

Then  $X_i^{\epsilon_i}$  is an  $A_i$ - $A_{i+1}$ -bimodule for  $i < n$  and  $X_n^{\epsilon_n}$  is an  $A_n$ - $A_1$ -bimodule, for Frobenius algebras  $A_j$ , where  $\epsilon_i = +$  if the defect line labeled by  $X_i$  is oriented away from  $v$ , while  $\epsilon_i = -$  if that defect line is oriented towards  $v$ . For the chosen linear order, the space  $H_v$  is now defined to be

$$H_v := \text{Hom}_{A_1|A_1}(A_1, X_1^{\epsilon_1} \otimes_{A_2} X_2^{\epsilon_2} \otimes_{A_3} \cdots \otimes_{A_n} X_n^{\epsilon_n}). \quad (2.17)$$

If a different choice of linear order of the defect lines meeting at  $v$  is made, the same prescription gives instead the space

$$\text{Hom}_{A_j|A_j}(A_j, X_j^{\epsilon_j} \otimes_{A_{j+1}} X_{j+1}^{\epsilon_{j+1}} \otimes_{A_{j+2}} \cdots \otimes_{A_{j-1}} X_{j-1}^{\epsilon_{j-1}}) \quad (2.18)$$

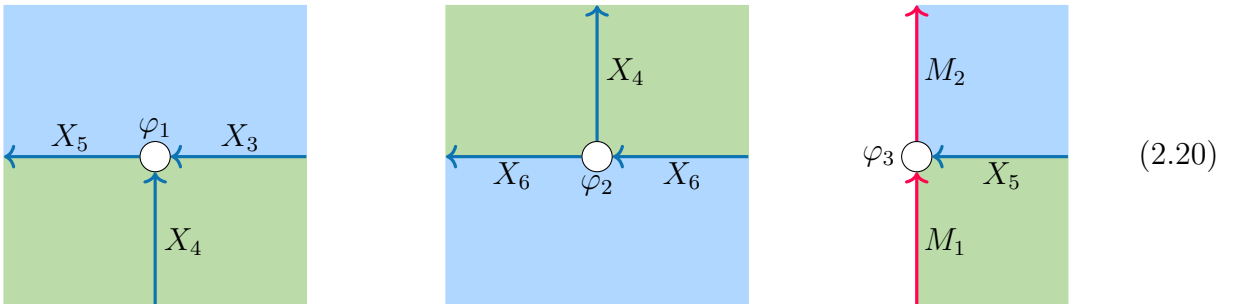
for some  $j \in \{2, 3, \dots, n\}$  (with labels counted modulo  $n$ ). Owing to pivotality, this space is isomorphic to  $H_v$  as defined in (2.17), and indeed pivotality provides a distinguished isomorphism of order  $n$ . Accordingly, the choice of linear order is immaterial.

For a defect junction  $w$  in the boundary  $\partial\mathcal{S}$ , there is directly a linear order on the set of physical boundaries and defect lines that meet at  $w$ . If the physical boundary that is oriented towards  $w$  is labeled by a right  $A_{n+1}$ -module  $M$  and the physical boundary oriented away from  $w$  is labeled by a right  $A_1$ -module  $N$ , then the morphism space  $H_w$  assigned to  $w$  is defined to be

$$H_w := \text{Hom}_{A_{n+1}}(M, N \otimes_{A_1} X_1^{\epsilon_1} \otimes_{A_2} X_2^{\epsilon_2} \otimes_{A_3} \cdots \otimes_{A_n} X_n^{\epsilon_n}), \quad (2.19)$$

with analogous conventions about the labels of the defect lines as in (2.17). In particular, in case  $n=0$ , so that only the two physical boundaries meet at  $w$ , we deal with the space  $\text{Hom}_{A_1}(M, N)$ .

**Example 2.8.** For the world sheet shown in Example 2.6 the situation near the three defect junctions looks as follows:



Then for a suitable choice of linear order the prescription (2.17) amounts to

$$\varphi_1 \in \text{Hom}_{B|B}(B, X_5 \otimes_A X_3^\vee \otimes_B X_4^\vee) \quad \text{and} \quad \varphi_2 \in \text{Hom}_{A|A}(A, X_6 \otimes_B X_4 \otimes_B X_6^\vee), \quad (2.21)$$

while (2.19) gives  $\varphi_3 \in \text{Hom}_B(M_1, M_2 \otimes_A X_5^\vee)$ .

**Remark 2.9.** Morita equivalent Frobenius algebras in  $\mathcal{C}$  describe the same full conformal field theory [FRS1]. To perform our string-net construction, for each two-cell of the world sheet we need to pick a specific Frobenius algebra representing its Morita class. The final result for the correlators will be independent of this choice. Our approach shares this feature with the TFT construction [FRS1, FRS2, FFRS1] of CFT correlators.

Now we would like to assign to any world sheet a correlator, and thus, as a first step, a space of conformal blocks in which the correlator is a specific element. To get the conformal blocks we associate to a world sheet  $\mathcal{S}$  a compact oriented smooth surface  $\Sigma_{\mathcal{S}}$  that carries the structure of a 1-morphism of the open-closed bordism category  $\mathcal{Bord}_{2,o/c}^{\text{or}}$ , from which we then obtain the conformal blocks by applying an open-closed modular functor  $\text{Bl}_{\mathcal{C}}$  that satisfies the requirements formulated in Section 2.1. As an oriented surface,  $\Sigma_{\mathcal{S}}$  is just the surface that underlies the world sheet  $\mathcal{S}$ , i.e. the surface obtained by forgetting all extra structure of the unlabeled world sheet  $\check{\mathcal{S}}$  except for the boundary one-cells. To make this surface into a 1-morphism we endow it with a *parametrization* – that is, we regard the union of all gluing circles and gluing intervals of  $\mathcal{S}$  as the image of an object of  $\mathcal{Bord}_{2,o/c}^{\text{or}}$  under a diffeomorphism. In particular, for every gluing interval  $r$  we specify a diffeomorphism from  $I \in \mathcal{Bord}_{2,o/c}^{\text{or}}$  to  $r$ , and for every gluing circle  $b \in \pi_0(\partial\check{\mathcal{S}})$  a diffeomorphism from  $S^1 \in \mathcal{Bord}_{2,o/c}^{\text{or}}$  to  $b$ , imposing in addition the condition that  $-1 \in S^1$  does not get mapped to any zero-cell of  $\mathcal{S}$ . These diffeomorphisms are required to preserve respectively reverse the orientation, depending on whether the intervals or circles are outgoing or incoming boundaries of  $\Sigma_{\mathcal{S}}$ . Moreover, for a geometric boundary circle containing several gluing intervals, the parametrization provides a linear order of those gluing intervals. From now on we will suppress the parametrization in the notation and just write  $\Sigma_{\mathcal{S}}$  for the 1-morphism of  $\mathcal{Bord}_{2,o/c}^{\text{or}}$  that is obtained from the world sheet  $\mathcal{S}$  in the manner just described. Also, for brevity we will say that  $\Sigma_{\mathcal{S}}$  is obtained from a *world sheet with parametrized boundary*, even though the subset of  $\partial\check{\mathcal{S}}$  that is given by the physical boundaries is not parametrized.

**Remark 2.10.** (i) Note that the parametrization is *not* part of the definition of a world sheet. The definition of the conformal block spaces  $\text{Bl}_{\mathcal{C}}(\mathcal{S})$  and of the string-net correlators  $\text{Cor}_{\text{SN}}(\mathcal{S})$  – see (2.41) and (3.68) below – makes use of the surface  $\Sigma_{\mathcal{S}}$ , but nevertheless  $\text{Bl}_{\mathcal{C}}(\mathcal{S})$  and  $\text{Cor}_{\text{SN}}(\mathcal{S})$  only depend on the world sheet  $\mathcal{S}$  but not on the parametrization of  $\Sigma_{\mathcal{S}}$ .

(ii) The parametrization also plays a role when discussing conformal blocks within the interpretation as vector bundles formed by the spaces of solutions to chiral Ward identities that we mentioned in the beginning of Section 2.1. In that context, the parametrization of a gluing circle corresponds to a local holomorphic coordinate on the disk that is bounded by the circle, and similarly for gluing intervals. A coordinate independent definition of conformal blocks can be obtained in this framework by using the action of the Virasoro algebra.

## 2.3 Field contents

A further ingredient needed for defining the concept of a correlator is a precise notion of *field insertions*. To achieve this, we now provide a prescription for associating an object in  $\mathcal{Z}(\mathcal{C})$ , to be called a field insertion, with each geometric boundary circle of a world sheet  $\mathcal{S}$  with parametrized boundary: suppressing again the dependence on the parametrization in the notation, we define a map

$$\mathbb{F} : \pi_0(\partial\check{\mathcal{S}}) \rightarrow \mathcal{Z}(\mathcal{C}) \quad (2.22)$$



from the set  $\pi_0(\partial\check{\mathfrak{S}})$  of geometric boundary circles of  $\check{\mathfrak{S}}$  to the Drinfeld center of  $\mathcal{C}$ . We refer to  $\mathbb{F}$  as the *field map*. In the literature, it is customary to associate field insertions, often called *boundary fields*, also to individual gluing intervals. In our approach we combine instead such boundary fields for all gluing intervals on a given geometric boundary circle  $b$  into a single compound field insertion  $\mathbb{F}(b)$  that is assigned to the entire circle  $b$ ; for the precise form of this ‘central lift’ of boundary fields see formula (2.38) below. This has the advantage that the so obtained compound fields are objects in  $\mathcal{Z}(\mathcal{C})$ , just like the bulk fields associated to gluing circles are, whereas the boundary fields for individual gluing intervals are objects in  $\mathcal{C}$ . Also, when defining the map  $\mathbb{F}$  below, in the concrete formulas we will assume that the circles or intervals are *incoming* boundaries of  $\Sigma_{\mathfrak{g}}$ ; in the case of outgoing boundaries, the objects given in those formulas must be replaced by their duals.

To define the field map  $\mathbb{F}$  we make use of further categorical tools. We first note that for  $\mathcal{M}$  and  $\mathcal{N}$  left module categories over  $\mathcal{C}$ , to any pair of module functors  $F$  and  $F'$  from  $\mathcal{M}$  to  $\mathcal{N}$  there is associated an object in the Drinfeld center of  $\mathcal{C}$ , namely the internal Hom

$$\underline{\text{Nat}}_{\mathcal{M},\mathcal{N}}(F, F') := \underline{\text{Hom}}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})}(F, F') \in \mathcal{Z}(\mathcal{C}) \quad (2.23)$$

for  $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$  as a left module category over the Drinfeld center  $\mathcal{Z}(\mathcal{C})$ . (See Appendix A.5 for the description of the category  $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$  of right exact module functors as a  $\mathcal{Z}(\mathcal{C})$ -module category.) The notation  $\underline{\text{Nat}}$  is chosen here in order to indicate that these objects arise as internal Homs for a functor category. For the same reason they are [FuS2] also called *internal natural transformations*.<sup>2</sup> This terminology is further justified by the fact that the standard expression of ordinary natural transformations as an end has an internalized analogue as an end over internal Homs: we have [FuS2, Thm. 9]

$$\underline{\text{Nat}}_{\mathcal{M},\mathcal{N}}(F, F') = \int_{M \in \mathcal{M}} \underline{\text{Hom}}_{\mathcal{N}}(F(M), F'(M)). \quad (2.24)$$

When the module categories  $\mathcal{M}$  and  $\mathcal{N}$  are obvious from the context, we omit them in the notation and just write  $\underline{\text{Nat}}(F, F')$ .

In the considerations below,  $\mathcal{M}$  and  $\mathcal{N}$  will often be represented in the form  $\mathcal{M} = \text{mod-}A$  and  $\mathcal{N} = \text{mod-}A'$  for  $A$  and  $A'$  (simple special symmetric) Frobenius algebras in  $\mathcal{C}$ . The module functors of our interest are then of the form

$$G^Y = - \otimes_A Y : \text{mod-}A \rightarrow \text{mod-}A' \quad (2.25)$$

for some  $A$ - $A'$ -bimodule  $Y$  (compare Eq. (A.27)).

### 2.3.1 Fields in the bulk

Internal natural transformations for functors of the form (2.25) provide us with the field insertions for those geometric boundary circles of  $\check{\mathfrak{S}}$  which are gluing circles, i.e. which do not contain any physical boundary. For giving the precise prescription, consider such a boundary circle  $b \in \pi_0(\partial\check{\mathfrak{S}})$ , and denote by  $O_b$  the set of zero-cells contained in  $b$  (which are all unlabeled, as none of them lies a physical boundary). The orientation of  $\check{\mathfrak{S}}$  induces a cyclic order on

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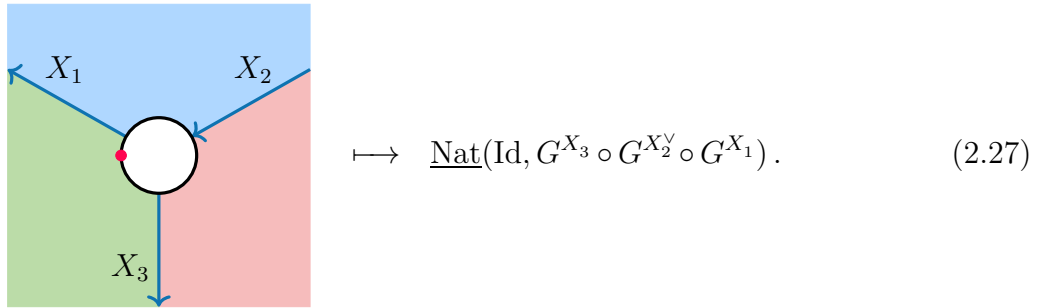
<sup>2</sup> The  $\mathcal{Z}(\mathcal{C})$ -valued internal Homs (2.23) also play a role in the context of the so-called full center functor from  $\mathcal{C}$  to  $\mathcal{Z}(\mathcal{C})$  [DavKR].

$O_b$ , such that if the induced orientation of the circle is clockwise, then the points are ordered clockwise as well. The parametrization chosen for the world sheet  $\mathcal{S}$  provides a linear order on  $O_b$  compatible with that cyclic order.

We can thus proceed analogously as in the case of the morphism spaces  $H_v$  in (2.17): Suppose that for the chosen linear order the defect lines that meet the zero-cells in  $O_b$  are labeled clockwise by bimodules  $X_i$  for  $i = 1, 2, \dots, n$ . Set  $\epsilon_i = +$  if the defect line labeled by  $X_i$  is oriented away from the zero-cell at which it meets  $b$  and  $\epsilon_i = -$  if it is oriented towards the zero-cell, and count labels modulo  $n$ . Then  $X_i^{\epsilon_i}$ , with  $X^\epsilon$  as defined in (2.16), is an  $A_i$ - $A_{i+1}$ -bimodule for Frobenius algebras  $A_j$ , for  $1 \leq i \leq n$ . Hereby we are ready to define the value of the field map on the gluing circle  $b$ : we set

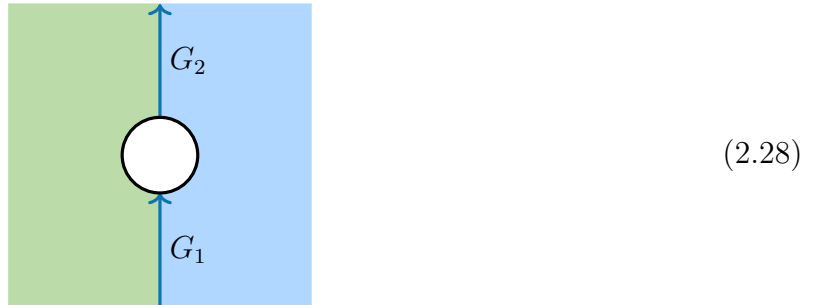
$$\mathbb{F}(b) := \underline{\text{Nat}}(\text{Id}_{A_1\text{-mod-}A_1}, G^{X_n^{\epsilon_n}} \circ G^{X_{n-1}^{\epsilon_{n-1}}} \circ \dots \circ G^{X_1^{\epsilon_1}}) \in \mathcal{Z}(\mathcal{C}). \quad (2.26)$$

**Example 2.11.** To illustrate this prescription, in the following picture we show the world sheet near a gluing circle with three zero-cells; the position of the image of  $-1 \in S^1 \subset \mathbb{C}$  is marked by a red dot:



(Note that the ordering of the labels  $X_i$  gets reversed when the functors are realized explicitly as tensor products over the relevant algebras, as e.g. done in the expression (2.17).)

It is worth indicating how the prescription (2.26) relates to more traditional approaches to fields in conformal field theory (see [FuS3] and references given there). Consider first the special case of a gluing circle  $b$  with two zero-cells, to which there are attached an incoming and an outgoing defect line, respectively:



In this case we refer to the resulting field insertion  $\mathbb{F}(b)$  as a (two-pronged) *defect field*. If the two two-cells adjacent to a defect line are labeled by the same Frobenius algebra and the label of a defect line is that Frobenius algebra, regarded as a bimodule over itself, then the defect line is called *transparent*. A *bulk field* is a special defect field  $\mathbb{F}(b)$  for which the two defect lines that meet the gluing circle  $b$  are both transparent. If one of the defect lines is transparent, but the other is not, then  $\mathbb{F}(b)$  is called a *disorder field*.

Now recall that, for  $\mathcal{C}$  a modular fusion category, any full local conformal field theory based on chiral data that are encoded in  $\mathcal{C}$  is characterized by an indecomposable pivotal left module category  $\mathcal{M}$  over  $\mathcal{C}$ . According to the proposal in [FuS3], the types of defect lines separating two full CFTs based on the same chiral data and on  $\mathcal{C}$ -module categories  $\mathcal{M}$  and  $\mathcal{M}'$ , respectively, are given by  $\mathcal{C}$ -module functors from  $\mathcal{M}$  to  $\mathcal{M}'$ , and the defect field for a gluing circle with one incoming defect line of type  $G_1$  and one outgoing defect line of type  $G_2$  – regarded as a *field insertion on the defect line* that changes the type of defect from  $G_1$  to  $G_2$  – is the object

$$\mathbb{D}^{G_1, G_2} := \underline{\text{Nat}}(G_1, G_2) \in \mathcal{Z}(\mathcal{C}) \quad (2.29)$$

of internal natural transformations. This is indeed in full accordance with the prescription (2.26) for the field map  $\mathbb{F}$ : there is a canonical isomorphism

$$\mathbb{D}^{G_1, G_2} = \underline{\text{Nat}}(G_1, G_2) \cong \underline{\text{Nat}}(\text{Id}, G_1^{\text{r.a.}} \circ G_2) = \mathbb{F}(b) \quad (2.30)$$

of objects in  $\mathcal{Z}(\mathcal{C})$ , with the gluing circle  $b$  as in (2.28), namely the one obtained by combining the description (2.24) of internal natural transformations with the functorial isomorphism [FuS2, Lemma 2]  $\underline{\text{Hom}}_{\mathcal{M}'}(G_1(M), G_2(M)) \cong \underline{\text{Hom}}_{\mathcal{M}'}(\text{Id}, G_1^{\text{r.a.}}(M) \circ G_2(M))$  for  $M \in \mathcal{M}$ . It is also worth pointing out that taking the Poincaré dual of the picture on the right hand side of (2.28) gives

$$\begin{array}{ccc} & G_2 & \\ & \curvearrowright & \\ \mathcal{M} & \xrightarrow{\mathbb{D}^{G_1, G_2}} & \mathcal{M}' \\ & \curvearrowleft & \\ & G_1 & \end{array} \quad (2.31)$$

The reminiscence of this picture with the standard graphical description of natural transformations fits well with our choice of terminology for  $\underline{\text{Nat}}$ .

In the case of a general gluing circle  $b$  with any number of zero-cells, in analogy with the previous special case we refer to the object  $\mathbb{F}(b)$  a *generalized defect field*. In the same vein as in (2.28) we can think of the generalized defect field for a gluing circle with  $n$  zero-cells as describing an  $n$ -pronged defect junction.<sup>3</sup> If all defect lines that meet a gluing circle  $b$  are transparent, then the field  $\mathbb{F}(b)$  is of the form  $\underline{\text{Nat}}(\text{Id}, \text{Id})$  and is again called a bulk field. Since we may trade gluing circles for defect junctions, we refer to all fields  $\mathbb{F}(b)$ , for  $b$  any type of gluing circle, also collectively as *fields in the bulk*.

To make contact to the expressions for defect fields, and in particular for bulk fields, that are familiar from the conformal field theory literature, we just note that using the semisimplicity of  $\mathcal{C}$  it can be shown [FuS3, Eq. (3.18)] that for  $A$ - $A'$ -bimodules  $X_1$  and  $X_2$  one has

$$\underline{\text{Nat}}(G^{X_1}, G^{X_2}) \cong \bigoplus_{i, j \in \mathcal{I}(\mathcal{C})} \text{Hom}_{A|A'}(i \otimes^- X_1 \otimes^+ j, X_2) \otimes_{\mathbb{k}} \Xi_{\mathcal{C}}(i \boxtimes j) \quad (2.32)$$

as objects in  $\mathcal{Z}(\mathcal{C})$ . Here  $\mathcal{I}(\mathcal{C})$  is a set of representatives for the isomorphism classes of simple objects of  $\mathcal{C}$ ; we fix the finite set  $\mathcal{I}(\mathcal{C})$  once and for all, in such a way that it contains the monoidal unit  $\mathbf{1}$  of  $\mathcal{C}$ .  $\Xi_{\mathcal{C}}$  denotes the equivalence (A.32) between  $\mathcal{C}^{\text{rev}} \boxtimes \mathcal{C}$  and  $\mathcal{Z}(\mathcal{C})$ , and  $i \otimes^- X_1 \otimes^+ j$  is an  $A$ - $A'$ -bimodule with underlying object  $i \otimes X_1 \otimes j$  and specific  $A$ - and  $A'$ -actions that are

<sup>3</sup> This is in fact the way defect junctions are treated in e.g. [FSS2, CR].

defined with the help of the braiding of  $\mathcal{C}$  (for details see [FuS3, p.21]). In the case of bulk fields for the full conformal field theory that corresponds to a  $\mathcal{C}$ -module category  $\mathcal{M} \simeq \text{mod-}A$ , (2.32) specializes to

$$\underline{\text{Nat}}(\text{Id}_{A\text{-mod-}A}, \text{Id}_{A\text{-mod-}A}) \cong \bigoplus_{i,j \in \mathcal{I}(\mathcal{C})} \text{Hom}_{A|A}(i \otimes^- A \otimes^+ j, A) \otimes_{\mathbb{k}} \Xi_{\mathcal{C}}(i \boxtimes j). \quad (2.33)$$

The dimensions of the spaces  $\text{Hom}_{A|A'}(i \otimes^- X_1 \otimes^+ j, X_2)$ , respectively  $\text{Hom}_{A|A}(i \otimes^- A \otimes^+ j, A)$ , are known as the coefficients of the *torus partition function* for defect fields and bulk fields, respectively (see e.g. Table 1 of [FRS1]).

When dealing with the objects (2.26) various results about internal natural transformations that were obtained in [FuS2] (in the setting of finite tensor categories) will be instrumental.

### 2.3.2 Fields on physical boundaries

Next we define the field map on boundary components which do contain a physical boundary. Given such a geometric boundary circle  $b \in \pi_0(\partial \check{\mathcal{S}})$ , we proceed in two separate steps. First consider an individual gluing interval  $r$  that is contained in  $b$ . We associate to  $r$  an object  $\mathbb{B}_r \in \mathcal{C}$  as follows. Denote the set of zero-cells at which a defect line meets  $r$  by  $O_r$ . On this set there is a linear order inherited from the prescription for the orientation of physical boundaries. Assume that the physical boundary that is oriented towards one of the end points of  $r$  is labeled by a right  $A_{n+1}$ -module  $M$  and the physical boundary oriented away from the other end point of  $r$  is labeled by a right  $A_1$ -module  $N$ , and that the defect lines that meet the zero-cells in  $O_r$  are labeled clockwise by bimodules  $X_i$  for  $i = 1, 2, \dots, n$ . Define  $\epsilon_i \in \{\pm 1\}$  and  $X_i^{\epsilon_i}$  in the same way as in the case of gluing circles, so that  $X_i^{\epsilon_i}$  is an  $A_i$ - $A_{i+1}$ -bimodule for  $1 \leq i \leq n$ . Then we define the object  $\mathbb{B}_r$  as the internal Hom

$$\mathbb{B}_r := \underline{\text{Hom}}_{\text{mod-}A_{n+1}}(M, G^{X_n^{\epsilon_n}} \circ G^{X_{n-1}^{\epsilon_{n-1}}} \circ \dots \circ G^{X_1^{\epsilon_1}}(N)) \in \mathcal{C}. \quad (2.34)$$

In particular, in case  $n=0$  we have

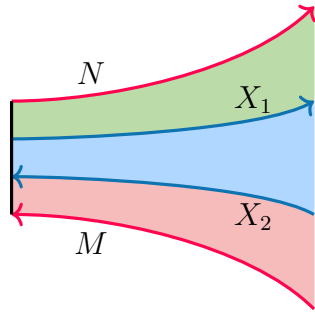
$$\mathbb{B}_r = \underline{\text{Hom}}_{\text{mod-}A_1}(M, N) =: \mathbb{B}^{M,N}. \quad (2.35)$$

Note that [FuS3, Eqs. (2.10),(2.11)]

$$\underline{\text{Hom}}_{\text{mod-}A}(M, N) \cong \int^{C \in \mathcal{C}} \text{Hom}_A(C \triangleright M, N) \otimes_{\mathbb{k}} C \cong \bigoplus_{i \in \mathcal{I}(\mathcal{C})} \text{Hom}_A(i \triangleright M, N) \otimes_{\mathbb{k}} i, \quad (2.36)$$

where the second isomorphism uses semisimplicity of  $\mathcal{C}$ . This reproduces the description of boundary fields familiar from earlier approaches to full conformal field theory (see again Table 1 of [FRS1]). Accordingly we call the object  $\mathbb{B}_r$  in  $\mathcal{C}$  that is assigned to a gluing interval  $r$  *boundary field* if  $r$  is not met by any defect line, so that  $\mathbb{B}_r$  is of the form (2.35), and a *generalized boundary field* otherwise.

**Example 2.12.** For the gluing interval described in Example 2.7, our prescription gives



$$\mapsto \underline{\text{Hom}}(M, G^{X_2^\vee} \circ G^{X_1}(N)). \quad (2.37)$$

Thus the field  $\underline{\text{Hom}}(M, G^{X_2^\vee} \circ G^{X_1}(N))$  changes the type of boundary condition from  $M$  to  $N$  in a way such that also defect lines of type  $X_1$  and  $X_2$  intervene.

It should be appreciated that  $\mathbb{B}_r$  as defined by (2.34) is an object in  $\mathcal{C}$ , not in  $\mathcal{Z}(\mathcal{C})$ , unlike the field insertions for gluing circles. Still, we can attain a uniform treatment of all geometric boundary circles – both gluing circles and circles containing gluing interval: In a second step we construct from the objects  $\mathbb{B}_r \in \mathcal{C}$  for all intervals  $r$  on a geometric boundary circle  $b$  an object in the Drinfeld center, by treating boundary intervals analogously as we did for open-closed modular functors  $\text{Bl}_{\mathcal{C}}$  in Section 2.1. Thus we take the tensor product in  $\mathcal{C}$  of the objects  $\mathbb{B}_r$  for all gluing intervals on  $b$  and then apply the left adjoint (2.8) of the forgetful functor from  $\mathcal{Z}(\mathcal{C})$  to  $\mathcal{C}$ . Denoting by  $Q_b$  the linearly ordered set of gluing intervals in  $b$ , this gives

$$\mathbb{F}(b) := L\left(\bigotimes_{r \in Q_b} \mathbb{B}_r\right) \in \mathcal{Z}(\mathcal{C}), \quad (2.38)$$

where the ordering of factors in the tensor product is according to the linear order of the boundary intervals  $r \in Q_b$  that results from the parametrization of the world sheet. (If the parametrization is changed in such a way that the linear order on  $Q_b$  is changed, while still keeping the cyclic order that results from the orientation of the world sheet, then the prescription (2.38) yields an isomorphic object, since according to Lemma A.4 we have  $L(C \otimes C') \cong L(C' \otimes C)$  for  $C, C' \in \mathcal{C}$ .) Analogously as for (2.10), in the extremal case that the set  $Q_b$  is empty, i.e. that the boundary circle  $b$  does not contain any gluing boundary, the expression (2.38) is interpreted as the empty tensor product, i.e.  $\mathbb{F}(b) = L(\mathbf{1}) \in \mathcal{Z}(\mathcal{C})$ .

We summarize our prescriptions in

**Definition 2.13.** The field map  $\mathbb{F}$  from the set of geometric boundary circles of  $\check{\mathcal{S}}$  to the Drinfeld center of  $\mathcal{C}$  is given by

$$\mathbb{F}(b) := \begin{cases} (2.26) & \text{if } b \text{ does not contain a physical boundary,} \\ (2.38) & \text{if } b \text{ contains a physical boundary.} \end{cases} \quad (2.39)$$

We call the field insertion  $\mathbb{F}(b)$  in the Drinfeld center  $\mathcal{Z}(\mathcal{C})$  that according to (2.38) is assigned to an entire boundary circle  $b$  with physical boundaries the *central lift* of the collection of (generalized) boundary fields that are associated to the individual gluing intervals on the circle.

## 2.4 Correlators as vectors in spaces of conformal blocks

As seen in Section 2.2, our prescriptions associate to every unlabeled world sheet  $\check{\mathcal{S}}$  endowed with a boundary parametrization a 1-morphism  $\Sigma_{\mathcal{S}}$  of the category  $\mathcal{B}ord_{2,o/c}^{\text{or}}$ . And as seen in Section 2.3, for every world sheet  $\mathcal{S}$  with boundary parametrization, with labels given in terms of the input datum of a spherical fusion category  $\mathcal{C}$ , they associate to every gluing circle of  $\mathcal{S}$  an object in  $\mathcal{Z}(\mathcal{C}) \simeq \text{Bl}_{\mathcal{C}}(S^1)$  and to every gluing interval of  $\mathcal{S}$  an object in  $\mathcal{C} \simeq \text{Bl}_{\mathcal{C}}(I)$ . (Also recall that in the formulas that were given for those objects in Section 2.3, it is assumed that the relevant boundary circle or interval is incoming; for an outgoing boundary one must replace the object by its dual.) We may then rephrase the statements above as follows: For any world sheet  $\mathcal{S}$  with boundary parametrization we are given a functor

$$\text{Bl}_{\mathcal{C}}(\Sigma_{\mathcal{S}}; -) : \text{Bl}_{\mathcal{C}}(S^1)^{\boxtimes p} \boxtimes \text{Bl}_{\mathcal{C}}(I)^{\boxtimes q} \longrightarrow \text{Vect}_{\mathbb{k}} \quad (2.40)$$

with  $p$  and  $q$  the number of gluing circles and gluing intervals of  $\mathcal{S}$ , respectively, as well as an object  $\mathbb{F}_{\mathcal{S}}$  in the domain of this functor which comprises the (generalized) defect and boundary fields of the world sheet  $\mathcal{S}$ . Combining this information we get a finite-dimensional vector space

$$\text{Bl}_{\mathcal{C}}(\mathcal{S}) := \text{Bl}_{\mathcal{C}}(\Sigma_{\mathcal{S}}; \mathbb{F}_{\mathcal{S}}); \quad (2.41)$$

we call this vector space the *space of conformal blocks* for the world sheet  $\mathcal{S}$ . By definition of  $\text{Bl}_{\mathcal{C}}$ , the functor  $\text{Bl}_{\mathcal{C}}(\Sigma_{\mathcal{S}}; -)$ , and thus also the space  $\text{Bl}_{\mathcal{C}}(\mathcal{S})$  of conformal blocks, carries an action of the mapping class group  $\text{Map}(\Sigma_{\mathcal{S}})$ . Note that up to isomorphism the space  $\text{Bl}_{\mathcal{C}}(\mathcal{S})$  only depends on the topology of the unlabeled world sheet  $\check{\mathcal{S}}$  and on the objects that the field map  $\mathbb{F}$  assigns to the geometric boundary circles of  $\mathcal{S}$ . In particular, world sheets which differ in the configuration of defect lines in their interior but agree on their boundary have the same conformal block spaces and the same representations of the mapping class group.

**Remark 2.14.** By suitably defining a category of world sheets, the prescription (2.41) can be turned into a functor from that category to vector spaces. When restricting to world sheets for which all field insertions are bulk fields, this gives the *pinned block functor* considered in [FuS1, Sect. 3.3].

Recall further that the vector spaces  $\text{Bl}_{\mathcal{C}}(\mathcal{S})$  are isomorphic to morphism spaces in  $\mathcal{Z}(\mathcal{C})$ .<sup>4</sup> In particular, if the world sheet  $\mathcal{S}$  is connected, then according to (2.6) we have isomorphisms

$$\text{Bl}_{\mathcal{C}}(\mathcal{S}) \cong \text{Hom}_{\mathcal{Z}(\mathcal{C})} \left( \bigotimes_{b \in \pi_0(\partial\check{\mathcal{S}})} \mathbb{F}(b), K^{\otimes g} \right), \quad (2.42)$$

where  $\mathbb{F}(b)$  is the generalized defect field, respectively central lift of boundary fields, for the geometric boundary circle  $b$ , as given by (2.39), while  $K = \int^{X \in \mathcal{Z}(\mathcal{C})} X^{\vee} \otimes X = \bigoplus_{j \in \mathcal{I}(\mathcal{Z}(\mathcal{C}))} j^{\vee} \otimes j$  is the distinguished object (2.7). If  $\mathcal{S}$  is the disjoint union of connected world sheets  $\mathcal{S}^{(\ell)}$  with  $\ell \in \{1, 2, \dots, N\}$ , then, by monoidality of the modular functor,  $\text{Bl}_{\mathcal{C}}(\mathcal{S}) \cong \bigotimes_{\ell=1}^N \text{Bl}_{\mathcal{C}}(\mathcal{S}^{(\ell)})$ .

<sup>4</sup>Also recall that these isomorphisms, like (2.6) and hence (2.42), are not canonical, but can be specified uniquely by the extra datum of a marking of  $\Sigma_{\mathcal{S}}$ . The conformal block space  $\text{Bl}_{\mathcal{C}}(\mathcal{S})$  can thus be understood as a colimit of marked blocks over a category of markings, compare e.g. [ScW, Sect. 2.3].

In short, for each world sheet the modular functor provides the conformal block space  $\text{Bl}(\mathcal{S})$ . These spaces come with an action of the mapping class group of the underlying surface as well as with sewing maps that relate the conformal block spaces for different world sheets. To explain the mapping class group actions and the behavior of conformal blocks under sewing, it is convenient to introduce a few further concepts. Recall first that the data of a world sheet  $\mathcal{S}$  do not include any labels for those zero-cells on the boundary  $\partial\check{\mathcal{S}}$  which are not defect junctions. There is, however, a natural way to label these zero-cells:

- To an unlabeled zero-cell at which a defect line starts, assign the object  $X$  of  $\mathcal{C}$  that underlies the bimodule labeling the defect line, and to one at which a defect line ends, assign the dual object  $X^\vee$ .
- To a zero-cell at which a physical boundary starts or ends, respectively, assign the object  $M$  of  $\mathcal{C}$  underlying the module label of the physical boundary, respectively the object  $M^\vee$ . (In the special case that a geometric boundary circle consists of only a single physical boundary and a single zero-cell at which the physical boundary both starts and ends, we take  $M$  as a label.)

While these additional labels are entirely redundant, it will be convenient to introduce them anyway, as they turn out to facilitate the formulation of various statements below. For concreteness, we refer to them as the *natural labels* for the zero-cells on the boundary.

**Definition 2.15.** (i) The *boundary datum*  $\mathfrak{B}(b)$  of a connected component  $b$  of the boundary of a world sheet with boundary parametrization is the set consisting of the labels for the defect junctions on  $b$  and of the natural labels of the other zero-cells on  $b$ , ordered according to the linear ordering of the zero-cells that is furnished by the boundary parametrization.

(ii) Given a boundary datum  $\mathfrak{B} = (\beta_1, \beta_2, \dots, \beta_n)$ , the associated *dual boundary datum*  $\mathfrak{B}^\vee$  is the cyclically ordered set

$$\mathfrak{B}^\vee = (\beta_1, \beta_2, \dots, \beta_n)^\vee := (\beta_n^\vee, \beta_{n-1}^\vee, \dots, \beta_1^\vee) \quad (2.43)$$

that is obtained from  $\mathfrak{B}$  by dualizing all elements as objects and morphisms in  $\mathcal{C}$  and  $\mathcal{Z}(\mathcal{C})$ , respectively, and inverting the order.

The version of mapping class group that is relevant for the definition of correlators is now as follows.

**Definition 2.16.** The *mapping class group*  $\text{Map}(\mathcal{S})$  of a world sheet  $\mathcal{S}$  is the group of homotopy classes of orientation preserving diffeomorphisms from  $\check{\mathcal{S}}$  to  $\check{\mathcal{S}}$  that satisfy the following conditions:

- Each non-transparent defect line and each defect junction that meets at least one non-transparent defect line is mapped to itself.
- Each geometric boundary circle is mapped to a geometric boundary circle with compatible boundary parametrization and with the same boundary datum.

**Example 2.17.** In the case of the world sheet (2.14), the boundary data for its three boundary circles are

$$(\varphi_3, M_2^\vee, X_1, M_1), \quad (X_2^\vee, X_3) \quad \text{and} \quad (X_1^\vee, X_2). \quad (2.44)$$

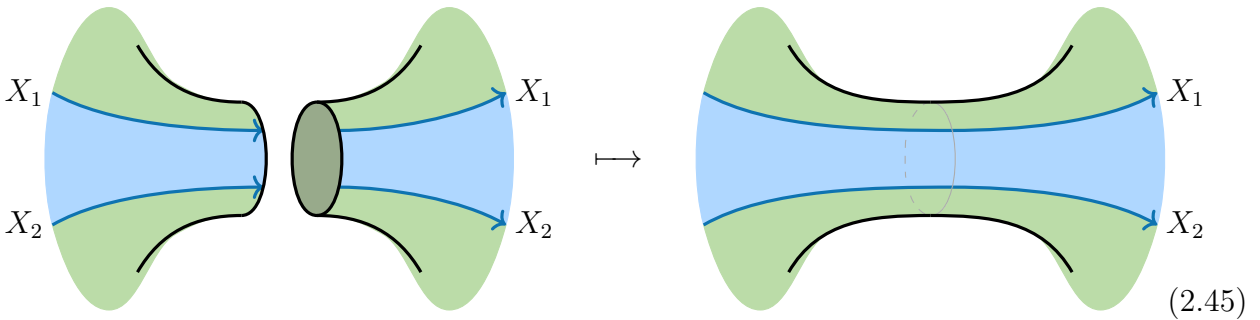
Thus in particular any mapping class has to map each of the three boundary circles to itself.

**Remark 2.18.** (i) Certain boundary circles, subject to the second condition, can be interchanged by acting with an element of  $\text{Map}(\mathcal{S})$ ; thus  $\text{Map}(\mathcal{S})$  can in particular contain braid groups. Also,  $\text{Map}(\mathcal{S})$  contains Dehn twists around gluing circles for bulk field insertions; these are rigidified, by the choice of a distinguished point (the image of  $-1 \in S^1$  as given by the parametrization) on each such circle.

(ii) Recall that by its construction through a modular functor, the conformal block space  $\text{Bl}(\mathcal{S})$  is actually equipped with an action of the entire mapping class group  $\text{Map}(\Sigma_{\mathcal{S}})$  of the geometric surface  $\Sigma_{\mathcal{S}}$ , of which  $\text{Map}(\mathcal{S})$  generically is a proper subgroup. In Section 6 we will, however, explain that besides the group  $\text{Map}(\mathcal{S})$  indeed also a larger subgroup of  $\text{Map}(\Sigma_{\mathcal{S}})$  plays a role, for which the requirements on the diffeomorphisms are relaxed.

(iii) The isomorphisms (2.42) are isomorphisms of vector spaces with  $\text{Map}(\Sigma_{\mathcal{S}})$ -action. In Section 3, conformal blocks will be expressed in terms of string nets. When doing so, the mapping class group action is induced by the action of diffeomorphisms on graphs embedded in  $\Sigma_{\mathcal{S}}$  (see Section 3.1) and is thus purely geometric. In contrast, on the morphism spaces  $\text{Hom}_{\mathcal{Z}(\mathcal{C})}(\otimes_b \mathbb{F}(b), K^{\otimes g})$  furnished by the isomorphisms (2.42), the action is in the first place given algebraically by composition with braiding and twist isomorphisms and with automorphisms of the object  $K^{\otimes g}$  that involve the structural morphisms of  $K$  as a Hopf algebra in  $\mathcal{Z}(\mathcal{C})$  [Ly]. There is, however, also a more geometric skein theoretical interpretation of this action [DeGGPR].

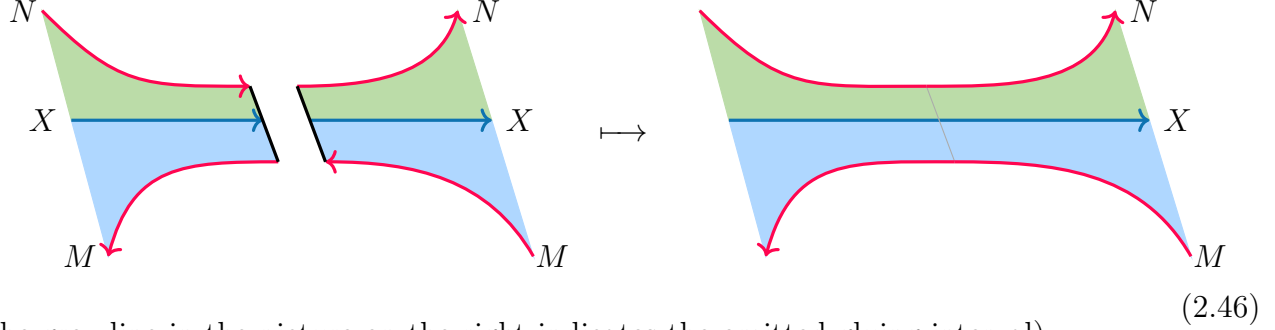
Next we describe how world sheets are sewn. We say that two distinct gluing circles  $b'$  and  $b''$  of a world sheet  $\mathcal{S}$  *match* iff there exists a diffeomorphism from  $b'$  to  $b''$  that is compatible with the boundary parametrization and for which the boundary data satisfy  $\mathfrak{B}(b'') = (\mathfrak{B}(b'))^\vee$  as ordered sets (which implies in particular that  $\mathbb{F}(b'') = \mathbb{F}(b')^\vee$ ). Matching gluing intervals are defined analogously. The *sewing*, or gluing, of a world sheet  $\mathcal{S}$  is performed either along matching gluing circles or along matching gluing intervals of  $\mathcal{S}$ . In the case of circles, the sewn world sheet  $\cup_{b', b''} \mathcal{S}$  is obtained from  $\mathcal{S}$  by a diffeomorphism, compatible with the boundary parametrization, between two matching gluing circles  $b'$  and  $b''$  of  $\mathcal{S}$ , whereby every zero-cell of  $b'$  is mapped to a zero-cell of  $b''$  with dual label, and afterwards omitting the gluing circle. (More precisely, the one-cells of the gluing circle are deleted, while each zero-cell together with the two defect lines meeting at it are replaced by a single defect line. Further aspects, like the fact that the gluing is actually along collars, in order for  $\cup_{b', b''} \mathcal{S}$  to be smooth, are inessential for our purposes. For a few pertinent details see e.g. Section 2.1 of [FuS1].) Locally the sewing along a circle looks schematically like



(the grey circle in the picture on the right hand side indicates the omitted gluing circle, whose intersections with the defect lines are the omitted zero-cells).



The sewing of a world sheet along matching gluing intervals  $r' \in O_{b'}$  and  $r'' \in O_{b''}$  proceeds analogously, with the additional requirement that the labels of the physical boundaries that meet the gluing interval must match. The local picture is now schematically



(the grey line in the picture on the right indicates the omitted gluing interval).

Any such sewing gives rise to a gluing transformation between the corresponding 1-morphisms  $\Sigma_{\mathcal{S}}$  and  $\Sigma_{\cup_{b',b''}\mathcal{S}}$  in  $\mathcal{B}ord_{2,o/c}^{or}$  which generalizes the natural transformation (2.3). As a consequence it induces a *sewing map*

$$s_{b',b''} : \text{Bl}_{\mathcal{C}}(\mathcal{S}) \rightarrow \text{Bl}_{\mathcal{C}}(\cup_{b',b''}\mathcal{S}) \quad (2.47)$$

between the respective spaces of conformal blocks. The sewing map can be defined with the help of the dinatural structure morphism  $\iota$  of a coend that describes the sewing procedure, similarly as for the coend (2.4). This will be explained in detail in Section 3.3. For now, let us just mention that upon applying the isomorphisms (2.42) we obtain corresponding coends over morphism spaces in the Drinfeld center  $\mathcal{Z}(\mathcal{C})$ , and present the so obtained coends for the simplest case of sewing along a gluing circle. The precise form of the coend depends on whether  $b'$  and  $b''$  lie on the same connected component of  $\check{\mathcal{S}}$  or not. We present only the former case, whose description involves less technicalities. We can then without loss of generality assume that  $\check{\mathcal{S}}$  is connected. Then  $s_{b',b''}$  is the morphism that makes the diagram

$$\begin{array}{ccc} \text{Bl}_{\mathcal{C}}(\mathcal{S}) & \xrightarrow{\cong} & \text{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathbb{F}(b')^{\vee} \otimes \mathbb{F}(b') \otimes \bigotimes_{\substack{b \in \pi_0(\partial\check{\mathcal{S}}) \\ b \neq b', b''}} \mathbb{F}(b), K^{\otimes g}) \\ \downarrow s_{b',b''} & & \downarrow \iota_{\mathbb{F}(b')} \\ \text{Bl}_{\mathcal{C}}(\cup_{b',b''}\mathcal{S}) & \xrightarrow{\cong} & \int^{Y \in \mathcal{Z}(\mathcal{C})} \text{Hom}_{\mathcal{Z}(\mathcal{C})}(Y^{\vee} \otimes Y \otimes \bigotimes_{\substack{b \in \pi_0(\partial\check{\mathcal{S}}) \\ b \neq b', b''}} \mathbb{F}(b), K^{\otimes g}) \end{array} \quad (2.48)$$

commute, where  $\iota_{\mathbb{F}(b')}$  is the  $\mathbb{F}(b')$ -component of the structure morphism of the coend. Since the two horizontal isomorphisms in this diagram depend on the extra datum of a marking of  $\Sigma_{\mathcal{S}}$  and of  $\Sigma_{\cup_{b',b''}\mathcal{S}}$ , respectively, actually some further effort is needed to make (2.48) into a full definition of the sewing map  $s_{b',b''}$ . We refrain from giving any details since, as already stated, below we will give a definition of  $s_{b',b''}$  directly in terms of string nets. (Also, in the generalization to non-semisimple  $\mathcal{C}$  the coend has to be taken in the sense of coends of left exact functors, see Appendix B of [Ly] and Section 2.3 of [FuS1] for details.)

It is worth noting that when realizing a sewing as a 1-morphisms in  $\mathcal{B}ord_{2,o/c}^{or}$ , all outgoing boundary circles of  $\mathcal{S}'$  and all incoming boundary circles of  $\mathcal{S}''$  participate in the sewing. To

realize also ‘partial sewings’ as 1-morphisms one has to invoke in addition dualities to turn incoming to outgoing circles or vice versa. This indicates that the structure of a bicategory might not be the ideal paradigm for formalizing bordisms.

We are now finally ready to introduce the objects of our primary interest, the correlators that are assigned to world sheets:

**Definition 2.19.** A *consistent system of correlators*, for a given spherical fusion category  $\mathcal{C}$ , is a collection of vectors  $\text{Cor}(\mathcal{S}) \in \text{Bl}_{\mathcal{C}}(\mathcal{S})$ , one for each world sheet  $\mathcal{S}$ , that satisfy:

(i) Invariance under the mapping class group of  $\mathcal{S}$ :

$$\gamma(\text{Cor}(\mathcal{S})) = \text{Cor}(\mathcal{S}) \tag{2.49}$$

for every mapping class  $\gamma \in \text{Map}(\mathcal{S})$ .

(ii) Compatibility with sewing:

$$\text{Cor}(\cup_{b,b'}\mathcal{S}) = s_{b,b'}(\text{Cor}(\mathcal{S})) \tag{2.50}$$

for every sewing of a world sheet along gluing circles, and

$$\text{Cor}(\cup_{r,r'}\mathcal{S}) = s_{r,r'}(\text{Cor}(\mathcal{S})) \tag{2.51}$$

for every sewing of a world sheet along gluing intervals.

**Remark 2.20.** (i) We may think of a correlator also as the linear map from  $\mathbb{k}$  to  $\text{Bl}_{\mathcal{C}}(\mathcal{S})$  that takes the value  $\text{Cor}(\mathcal{S})$  at  $1 \in \mathbb{k}$ .

(ii) World sheets can be taken to constitute the objects of a symmetric monoidal category, with tensor product given by disjoint union and with morphisms given by combinations of mapping classes and sewings. The collection of conformal blocks then furnishes a symmetric monoidal functor from the category of world sheets to the category of finite-dimensional  $\mathbb{k}$ -vector spaces. In this setting, a consistent system of correlators is the same as a monoidal natural transformation from a trivial functor that assigns to every world sheet the ground field  $\mathbb{k}$  to the functor of conformal blocks. For the case of surfaces without physical boundaries or defect lines, this approach to correlators is described in detail in [FuS1, Sect. 3.4].

## 3 Correlators from string nets

### 3.1 String nets

As we will see in Section 3.4, the data of a world sheet  $\mathcal{S}$  provide us in particular with sufficient information to apply the string-net construction as formulated in [Ki] to the surface  $\Sigma_{\mathcal{S}}$  that is obtained from  $\mathcal{S}$ , thereby yielding finite-dimensional vector spaces carrying an action of the mapping class group  $\text{Map}(\mathcal{S})$ . As a preparation, we here provide some pertinent information about this construction (for further details see [Ki, ScY]). We start with the notions of a coloring by  $\mathcal{C}$  of an oriented graph and of embedding a graph into a surface. By a *surface*  $\Sigma$  we mean a compact oriented smooth surface which may have a non-empty boundary. We consider embeddings  $\Gamma \rightarrow \Sigma$  of finite graphs  $\Gamma$  in a surface  $\Sigma$  for which the intersection of the image of

$\Gamma$  with the boundary  $\partial\Sigma$  consists of isolated points each of which is the image of a univalent vertex of  $\Gamma$ . In this case we simply say that the graph  $\Gamma$  is *embedded in the surface*  $\Sigma$ , and we denote by  $V_{\partial}(\Gamma, \Sigma)$  the set of the latter univalent vertices of  $\Gamma$ , and by  $V(\Gamma, \Sigma)$  the set of all other vertices of  $\Gamma$ .

Together with each edge  $e$  of an oriented graph  $\Gamma$  we consider also the same underlying unoriented edge endowed with the opposite orientation, which we denote by  $\bar{e}$ . We write  $E(\Gamma)$  for the set of edges of  $\Gamma$  and  $\widehat{E}(\Gamma) := E(\Gamma) \cup \{\bar{e} \mid e \in E(\Gamma)\}$ . For  $v$  a vertex of  $\Gamma$  we denote by  $E_v$  the set of edges that are incident to  $v$ .

**Definition 3.1.** Let  $\Gamma$  be a finite graph embedded in a surface  $\Sigma$ .

(i) A *coloring* of  $\Gamma$  by the category  $\mathcal{C}$  consists of a map

$$\text{col}_E : \widehat{E}(\Gamma) \longrightarrow \mathcal{C} \quad (3.1)$$

satisfying  $\text{col}_E(\bar{e}) = (\text{col}_E(e))^\vee$  for all  $e \in \widehat{E}(\Gamma)$ , and of an assignment  $\text{col}_V$  of a morphism

$$\text{col}_V(v) \in \text{Hom}_{\mathcal{C}}(\mathbf{1}, \text{col}_E(e_1) \otimes \text{col}_E(e_2) \otimes \cdots \otimes \text{col}_E(e_n)) \quad (3.2)$$

to any vertex  $v$  of  $\Gamma$ , with  $e_1, e_2, \dots, e_n$  the edges in  $E_v$ , taken in clockwise order<sup>5</sup> and with orientation away from  $v$ .

(ii) An *isomorphism* of colorings  $(\text{col}_E, \text{col}_V)$  and  $(\text{col}'_E, \text{col}'_V)$  of  $\Gamma$  is a collection  $\{f_e \mid e \in \widehat{E}(\Gamma)\}$  of isomorphisms  $f_e : \text{col}_E(e) \xrightarrow{\cong} \text{col}'_E(e)$  satisfying  $f_e^\vee \circ f_{\bar{e}} = \text{id}_{\text{col}_E(\bar{e})}$  and

$$\text{col}'_V(v) = \left( \bigotimes_{e \in E_v} f_e \right) \circ \text{col}_V(v). \quad (3.3)$$

(iii) The *boundary value* for a coloring of  $\Gamma$  is the map

$$\text{col}_{\partial}^{\Gamma} : V_{\partial}(\Gamma, \Sigma) \rightarrow \mathcal{C} \quad (3.4)$$

that is given by  $\text{col}_{\partial}^{\Gamma}(v) := \text{col}_E(e_v)$  with  $e_v$  the edge incident to  $v$  in case  $e_v$  is oriented towards  $v$ , and by  $\text{col}_{\partial}^{\Gamma}(v) := (\text{col}_E(e_v))^\vee$  otherwise.

**Definition 3.2.** Let  $\Sigma$  be a surface.

(i) A *boundary value* for  $\Sigma$  is a finite collection  $B$  of mutually disjoint points on  $\partial\Sigma$  together with a map  $\mathbf{B}^\circ : B \rightarrow \mathcal{C}$  to the objects of  $\mathcal{C}$ .

(ii) The set  $\mathbf{G}(\Sigma, \mathbf{B}^\circ)$  is the set of all  $\mathcal{C}$ -colored finite graphs  $\Gamma$  embedded in  $\Sigma$  with boundary value  $\text{col}_{\partial}^{\Gamma} = \mathbf{B}^\circ$ .

(iii)  $\mathbb{k}\mathbf{G}(\Sigma, \mathbf{B}^\circ)$  is the  $\mathbb{k}$ -vector space freely generated by the set  $\mathbf{G}(\Sigma, \mathbf{B}^\circ)$ .

If  $\Sigma = \mathbb{D}$  is a disk with counter-clockwise oriented boundary and  $X_i \in \mathcal{C}$  for  $i = 1, 2, \dots, n$ , denote by  $\mathbf{B}_X^\circ$  the map that associates the objects  $X_i$  in clockwise cyclic order to a collection of  $n$  points on  $\partial\mathbb{D}$ . Via the graphical description of morphisms in a spherical fusion category as

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<sup>5</sup> Strictly speaking,  $\text{col}_V(v)$  is not an element of the morphism space shown in (3.2), but rather in the space that is obtained as a limit over all linear orders compatible with the cyclic order, analogously as in (A.42). By abuse of notation, we suppress this issue.

given in (A.43), a graph  $\Gamma \in G(D, \mathbf{B}_X^\circ)$  can then in an obvious way be regarded as representing a morphism in  $\text{Hom}_{\mathcal{C}}(\mathbf{1}, X_1 \otimes \cdots \otimes X_n)$ , i.e. we have a canonical surjection

$$\langle - \rangle_D : \mathbb{k}G(D, \mathbf{B}_X^\circ) \twoheadrightarrow \text{Hom}_{\mathcal{C}}(\mathbf{1}, X_1 \otimes \cdots \otimes X_n) \quad (3.5)$$

of vector spaces. Thus the morphism space  $\text{Hom}_{\mathcal{C}}(\mathbf{1}, X_1 \otimes \cdots \otimes X_n)$  is isomorphic to the quotient of  $\mathbb{k}G(D, \mathbf{B}_X^\circ)/\ker(\langle - \rangle_D)$  in which any two elements of  $\mathbb{k}G(D, \mathbf{B}_X^\circ)$  that represent the same morphism are identified. By mapping graphs via (3.5) to morphism spaces we recover the graphical calculus for the monoidal category  $\mathcal{C}$  as a calculus for graphs on the disk  $D$ .

It is natural to extend this procedure to arbitrary surfaces  $\Sigma$ , such that we can use the graphical calculus for  $\mathcal{C}$  locally for any disk  $D \subset \Sigma$  on  $\Sigma$ . To this end we must provide a suitable analogue of the quotient  $\mathbb{k}G(D, \mathbf{B}_X^\circ)/\ker(\langle - \rangle_D)$  for general surfaces.

**Definition 3.3.** Let  $\Sigma$  be a surface and  $\mathbf{B}^\circ$  a boundary value for  $\Sigma$ .

- (i) Let  $D \subseteq \Sigma$  be an embedded disk with  $\partial D \cap \partial \Sigma = \emptyset$ . A *null graph with respect to  $D$*  is an element  $\sum_i c_i \Gamma_i$  of  $\mathbb{k}G(\Sigma, \mathbf{B}^\circ)$  such that, for each  $j$ , no vertex of  $\Gamma_j$  lies on  $\partial D$  and every edge of  $\Gamma_j$  intersects  $\partial D$  transversally, that all  $\Gamma_i$  coincide on  $\Sigma \setminus D$ , and such that  $\sum_i c_i \langle \Gamma_i \cap D \rangle_D = 0$ .
- (ii) The *null space*  $N(\Sigma, \mathbf{B}^\circ)$  for  $\Sigma$  and  $\mathbf{B}^\circ$  is the subspace of  $\mathbb{k}G(\Sigma, \mathbf{B}^\circ)$  that is spanned by all null graphs for all disks embedded in  $\Sigma$ .
- (iii) The *string-net space* for  $\Sigma$  and  $\mathbf{B}^\circ$  is the quotient

$$\text{SN}_{\mathcal{C}}^\circ(\Sigma, \mathbf{B}^\circ) := \mathbb{k}G(\Sigma, \mathbf{B}^\circ)/N(\Sigma, \mathbf{B}^\circ) \quad (3.6)$$

of  $\mathbb{k}G(\Sigma, \mathbf{B}^\circ)$ .

We call a vector in the quotient space  $\text{SN}_{\mathcal{C}}^\circ(\Sigma, \mathbf{B}^\circ)$  that is the image of an element of the generating set  $G(\Sigma, \mathbf{B}^\circ)$  of  $\mathbb{k}G(\Sigma, \mathbf{B}^\circ)$  a *bare string net*, or also just a *string net*. (The reason for the qualification “bare” and for the choice of notation  $\mathbf{B}^\circ$  will become clear in the next subsection.) A string net that has a graph  $\Gamma$  as representative is denoted by  $[\Gamma]$ ; by abuse of language, the term string net is also used for individual graphs that represent an element  $[\Gamma] \in \text{SN}_{\mathcal{C}}^\circ(\Sigma, \mathbf{B}^\circ)$ . The string-net space is linear in the color of each vertex of a graph and additive with respect to taking direct sums of objects labeling the edges. Isotopic graphs and graphs with isomorphic colorings give the same string net. Furthermore, all identities valid in the graphical calculus for  $\mathcal{C}$  also hold inside any disk embedded in  $\Sigma$ . Thus for string nets the graphical calculus for  $\mathcal{C}$  applies locally on  $\Sigma$ .

Diffeomorphisms of the surface  $\Sigma$  act naturally on embedded graphs. Since in the string-net space isotopic graphs are identified, this action descends to an action of the mapping class group of  $\Sigma$  on  $\text{SN}_{\mathcal{C}}^\circ(\Sigma, \mathbf{B}^\circ)$ .

String nets can be *concatenated*, in a manner similar to the sewing of world sheets. Consider a string net  $[\Gamma] \in \text{SN}_{\mathcal{C}}^\circ(\Sigma, \mathbf{B}^\circ)$  represented by an embedded graph  $\Gamma$ . Given the set  $B$  of points on  $\partial \Sigma$  that underlies the boundary datum  $\mathbf{B}^\circ$ , denote by  $B|_b$  and  $B|_{b'}$  the subsets of  $B$  consisting of those points which lie on two distinct boundary circles  $b$  and  $b'$  of  $\Sigma$ . Assume that  $B|_b$  and  $B|_{b'}$  have the same cardinality, and let  $\varphi: b \rightarrow b'$  be a diffeomorphism of one-manifolds that reverses orientation (with respect to the orientations on the circles induced by the 2-orientation of  $\Sigma$ ) and maps each point in  $B|_b$  to a point in  $B|_{b'}$ . Assume further that  $\mathbf{B}^\circ(\varphi(\beta)) = (\mathbf{B}^\circ(\beta))^\vee$  for each  $\beta \in B|_b$ . In this situation we can sew the surface  $\Sigma$  by identifying the circles  $b$  and  $b'$  in the same

way as is done when sewing world sheets (see Section 2.4). The sewing identifies in particular the (univalent) vertices of the graph  $\Gamma$  that lie on the boundary circle  $b$  with those that lie on  $b'$ . This results in a finite embedded graph  $\Gamma'$  that intersects the identified circles  $b$  and  $b'$  at points  $v$  which are two-valent vertices of  $\Gamma'$  with an incoming and an outgoing edge carrying the same label  $X(v) \in \mathcal{C}$ . We denote by  $\cup_{b,b'}\Gamma$  the graph obtained from  $\Gamma'$  by replacing each such two-valent vertex  $v$  and the two edges that meet at  $v$  by a single edge with label  $X(v)$ . The *concatenated string net*  $[\cup_{b,b'}\Gamma]$  is the string net on the sewn surface  $\cup_{b,b'}\Sigma$  that is represented by the graph  $\cup_{b,b'}\Gamma$ . Note that by construction we have  $[\cup_{b,b'}\Gamma] \cong [\cup_{b',b}\Gamma]$ . Similarly, if  $r, r' \subset \partial\Sigma$  are intervals such that  $r \cap r' = \emptyset$  and  $\Gamma \cap \partial r = \emptyset = \Gamma \cap \partial r'$ , and if  $\mathbf{B}^\circ(\varphi(\beta)) = (\mathbf{B}^\circ(\beta))^\vee$  for each  $\beta \in B|_r$  for an orientation reversing diffeomorphism  $\varphi: r \rightarrow r'$ , then we can concatenate the string net along the intervals, resulting in a string net  $[\cup_{r,r'}\Gamma]$  on the sewn surface  $\cup_{r,r'}\Sigma$ .

### 3.2 Idempotent completion

A further ingredient needed for a string-net construction based on world sheets is a suitable abelian category constructed from boundary values. This is obtained by first considering a bare variant, which is not abelian, but for which taking its Karoubian envelope, i.e. completing it such that any idempotent has an image, yields an abelian category (using that the category  $\mathcal{C}$  of input data is semisimple). Recall that an object of the Karoubian envelope  $\text{Kar}(\mathcal{D})$  of a category  $\mathcal{D}$  is a pair  $(D, p)$  consisting of an object  $D \in \mathcal{D}$  and an idempotent  $p \in \text{Hom}_{\mathcal{D}}(D, D)$ , such that  $\text{id}_{(D,p)} = p$ . Following [Ki, Sect.6] we give

**Definition 3.4.** Let  $S$  be an oriented one-dimensional manifold, possibly with non-empty boundary.

(i) Let  $B$  be a finite collection of points on  $S$ . The *category  $\mathcal{Cyl}^\circ(\mathcal{C}, S)$  of boundary values* for  $S$  is the following category: An object  $\mathbf{B}^\circ$  of  $\mathcal{Cyl}^\circ(\mathcal{C}, S)$  is a finite collection  $B$  of points on  $S$  together with a map  $\mathbf{B}^\circ: B \rightarrow \mathcal{C}$ . The space of morphisms between two objects is the string-net space on a cylinder  $S \times I$  over  $S$  with boundary value induced by the two inclusions of  $S$  as the boundary of the cylinder,

$$\text{Hom}_{\mathcal{Cyl}^\circ(\mathcal{C}, S)}(\mathbf{B}^\circ, \mathbf{B}'^\circ) := \text{SN}_{\mathcal{C}}^\circ(S \times I, \mathbf{B}^{\circ\vee} \cup \mathbf{B}'^\circ). \quad (3.7)$$

Composition of morphisms is given by concatenating string nets.

(ii) The *cylinder category* for  $S$  associated with  $\mathcal{C}$  is the Karoubian envelope

$$\mathcal{Cyl}(\mathcal{C}, S) := \text{Kar}(\mathcal{Cyl}^\circ(\mathcal{C}, S)) \quad (3.8)$$

of the category of boundary values for  $S$ .

For any two one-manifolds  $S$  and  $S'$  we have

$$\mathcal{Cyl}(\mathcal{C}, S \sqcup S') = \mathcal{Cyl}(\mathcal{C}, S) \times \mathcal{Cyl}(\mathcal{C}, S'); \quad (3.9)$$

since all functors with cylinder categories as domains that we consider in this paper are left exact in each factor, we may, and will, replace  $\mathcal{Cyl}(\mathcal{C}, S) \times \mathcal{Cyl}(\mathcal{C}, S')$  with  $\mathcal{Cyl}(\mathcal{C}, S) \boxtimes \mathcal{Cyl}(\mathcal{C}, S')$ . As a consequence, the cylinder categories for the interval and for the circle are the most interesting

ones. Consider thus first the case that  $S = I = [0, 1]$  is a single interval; there is then a canonical equivalence

$$\mathcal{Cyl}^\circ(\mathcal{C}, I) \xrightarrow{\simeq} \mathcal{C} \quad (3.10)$$

which sends the object in  $\mathcal{Cyl}^\circ(\mathcal{C}, I)$  that is given by a collection  $\{\xi_j\}$  of points on  $I$  satisfying  $0 < \xi_1 < \xi_2 < \dots < \xi_n < 1$  and labeled by  $\mathbf{B}^\circ(\xi_j) = X_j \in \mathcal{C}$  to the tensor product  $X_1 \otimes X_2 \otimes \dots \otimes X_n$ . Moreover, since  $\mathcal{C}$  is idempotent complete, after choosing a splitting for each idempotent in  $\mathcal{C}$  there is a unique extension of the functor (3.10) to an equivalence

$$\mathcal{Cyl}(\mathcal{C}, I) = \text{Kar}(\mathcal{Cyl}^\circ(\mathcal{C}, I)) \xrightarrow{\simeq} \mathcal{C}. \quad (3.11)$$

Similarly, in the case that  $S = S^1$  is a single circle with distinguished point  $-1 \in S^1$  (and analogously for any pointed connected compact one-manifold), there is a canonical functor

$$\Phi^\circ : \mathcal{Cyl}^\circ(\mathcal{C}, S^1) \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C}). \quad (3.12)$$

The functor (3.12) is fully faithful. It sends the object in  $\mathcal{Cyl}^\circ(\mathcal{C}, S^1)$  that is given by a collection  $\{\xi_j\}$  of points on  $S^1$  satisfying  $\pi > \arg(\xi_1) > \arg(\xi_2) > \dots > \arg(\xi_n) > -\pi$  and labeled by  $\mathbf{B}^\circ(\xi_j) = X_j \in \mathcal{C}$  to  $L(X_1 \otimes X_2 \otimes \dots \otimes X_n)$ . To describe the action of  $\Phi^\circ$  on morphisms, consider  $\varphi \in \text{Hom}_{\mathcal{C}}(X, Z^\vee \otimes Y \otimes Z)$  and set

$$\tilde{\varphi} := \begin{array}{c} \text{Y} \\ \uparrow \\ \text{---} \circlearrowleft \text{---} \\ \uparrow \\ \text{X} \\ \text{---} \circlearrowright \text{---} \\ \downarrow \\ \text{Z} \end{array} \in \text{Hom}_{\mathcal{Cyl}^\circ(\mathcal{C}, S^1)}(S_X, S_Y). \quad (3.13)$$

The morphism  $\Phi^\circ(\tilde{\varphi}) : L(X) \rightarrow L(Y)$  in  $\mathcal{Z}(\mathcal{C})$  is then defined by the dinatural family

$$U^\vee \otimes X \otimes U \xrightarrow{\text{id}_{U^\vee} \otimes \varphi \otimes \text{id}_U} U^\vee \otimes Z^\vee \otimes Y \otimes Z \otimes U \xrightarrow{i_{Y, Z \otimes U}^Z} L(Y), \quad (3.14)$$

where  $i_{Y, -}^Z$  is the dinatural structure morphism of the coend  $L(Y)$  (compare (A.34)). Using that  $\mathcal{C}$  is semisimple, and adopting the summation convention (A.46), we have

$$\Phi^\circ(\tilde{\varphi}) = \sum_{i, j \in \mathcal{I}(\mathcal{C})} d_j \begin{array}{c} j \\ \downarrow \\ \alpha \\ \downarrow \\ i \end{array} \begin{array}{c} Z \\ \rightarrow \\ \varphi \\ \leftarrow \\ Z \end{array} \begin{array}{c} Y \\ \uparrow \\ \alpha \\ \uparrow \\ i \end{array} \quad (3.15)$$

Since  $\mathcal{Z}(\mathcal{C})$  is idempotent complete, after choosing a splitting for each idempotent in  $\mathcal{Z}(\mathcal{C})$  there is a unique extension of  $\Phi^\circ$  to a functor

$$\Phi : \mathcal{Cyl}(\mathcal{C}, S^1) = \text{Kar}(\mathcal{Cyl}^\circ(\mathcal{C}, S^1)) \xrightarrow{\simeq} \mathcal{Z}(\mathcal{C}) \quad (3.16)$$

which sends  $(\mathbf{B}^\circ, p) \in \mathcal{Cyl}^\circ(\mathcal{C}, S^1)$  to  $\text{Im}(\Phi^\circ(p))$  with  $\Phi^\circ(p) \in \text{Hom}_{\mathcal{Z}(\mathcal{C})}(\Phi^\circ(\mathbf{B}^\circ), \Phi^\circ(\mathbf{B}^\circ))$ . The so obtained functor  $\Phi$  is an equivalence [Ki, Thm. 6.4]. Below we will largely work directly with the cylinder category  $\mathcal{Cyl}(\mathcal{C}, S^1)$  rather than with  $\mathcal{Z}(\mathcal{C})$ , so that the specific form of the equivalence (3.16) will actually not be important.

**Example 3.5.** For  $Y = (U(Y), \gamma) \in \mathcal{Z}(\mathcal{C})$  consider the string net

$$p_Y^{\text{can}} := \text{Diagram} \tag{3.17}$$

in  $\text{Hom}_{\mathcal{Cyl}^\circ(\mathcal{C}, S^1)}(S_{U(Y)}, S_{U(Y)})$ , where the unlabeled edge, which runs along the non-contractible cycle of the cylinder, stands for the *canonical color* (A.5) and the half-braiding  $\gamma$  is indicated by an over-crossing, as explained in (A.48). Using the identity (A.47) it is readily seen that  $p_Y^{\text{can}}$  is an idempotent (see e.g. [ScY, Rem. 2.6]). We have

$$\Phi^\circ(p_Y^{\text{can}}) = \sum_{i,j,k \in \mathcal{I}(\mathcal{C})} \frac{d_j d_k}{D_{\mathcal{C}}^2} \text{Diagram} = \sum_{i,j \in \mathcal{I}(\mathcal{C})} \frac{d_j}{D_{\mathcal{C}}^2} \text{Diagram} \tag{3.18}$$

with  $D_{\mathcal{C}}^2$  the global dimension (A.4) of  $\mathcal{C}$ . It is then easily seen [Ki, Lemma 8.3] that the image of the morphism  $\Phi^\circ(p_Y^{\text{can}}) \in \text{Hom}_{\mathcal{Z}(\mathcal{C})}(Y, Y)$  is the object  $Y \in \mathcal{Z}(\mathcal{C})$ . We write

$$\mathbf{B}_Y^{\text{can}} := (S_{U(Y)}, p_Y^{\text{can}}) \tag{3.19}$$

for the object of  $\mathcal{Cyl}(\mathcal{C}, S^1)$  that is given by the boundary value  $S_{U(Y)} \in \mathcal{Cyl}^\circ(\mathcal{C}, S^1)$  and the idempotent (3.17). We thus have

$$\Phi(\mathbf{B}_Y^{\text{can}}) = \text{Im}(\Phi^\circ(p_Y^{\text{can}})) = Y \in \mathcal{Z}(\mathcal{C}) \tag{3.20}$$

for every  $Y \in \mathcal{Z}(\mathcal{C})$ .

**Example 3.6.** For any two Frobenius algebras  $A$  and  $B$  in  $\mathcal{C}$  and any two  $A$ - $B$ -bimodules  $X$  and  $Y$  consider the string net

$$p_{X,Y} := \text{Diagram} \tag{3.21}$$

in  $\text{Hom}_{\mathcal{C} \text{yl}^\circ(\mathcal{C}, S^1)}(S_{Y, X^\vee}, S_{Y, X^\vee})$ . Using that  $A$  and  $B$  are special Frobenius algebras, one sees again directly that  $p_{X, Y}$  is an idempotent. Furthermore we find

$$\text{Im}(\Phi^\circ(p_{X, Y})) \cong \underline{\text{Nat}}(G^X, G^Y) \stackrel{(2.29)}{=} \mathbb{D}^{X, Y} \in \mathcal{Z}(\mathcal{C}). \quad (3.22)$$

Let us explain this isomorphism in detail, using manifestly that  $\mathcal{C}$  is semisimple. (The statement is, however, in fact valid beyond semisimplicity.) Inserting the explicit form (2.25) of the functors  $G^X$  and  $G^Y$ , in terms of the bimodules  $X$  and  $Y$  we have

$$\mathbb{D}^{X, Y} = \bigoplus_{m \in \mathcal{I}(\text{mod-}A)} m \otimes_A Y \otimes_B X^\vee \otimes_A m^\vee. \quad (3.23)$$

Now consider the two string nets

$$e_{X, Y} := \sum_{m \in \mathcal{I}(\text{mod-}A)} \frac{d_m}{D_{\mathcal{C}}^2} \quad \text{and} \quad r_{X, Y} := \sum_{m \in \mathcal{I}(\text{mod-}A)} \quad (3.24)$$

where  $d_m = d_{\hat{m}}$  is the dimension of the object in  $\mathcal{C}$  that underlies  $m \in \text{mod-}A$ . Using again that  $A$  and  $B$  are special Frobenius and invoking Corollary A.7 we see that

$$e_{X, Y} \circ r_{X, Y} = p_{X, Y}. \quad (3.25)$$

For the composition in the opposite order we get

$$r_{X, Y} \circ e_{X, Y} = p_{\mathbb{D}^{X, Y}}^{\text{can}} \quad (3.26)$$



with  $p_Z^{\text{can}}$  as in (3.17); this is shown by the following identities between of string nets:

$$\begin{aligned}
r_{X,Y} \circ e_{X,Y} &= \sum_{m,n \in \mathcal{I}(\text{mod-}A)} \frac{d_m}{D_{\mathcal{C}}^4} \\
&= \sum_{\substack{i \in \mathcal{I}(\mathcal{C}) \\ m,n \in \mathcal{I}(\text{mod-}A)}} \frac{d_i d_m}{D_{\mathcal{C}}^2} \\
&= p_{\mathbb{D}^{X,Y}}^{\text{can}}.
\end{aligned}
\tag{3.27}$$

(Here the cylinder is drawn in a slightly different manner – deformed by a diffeomorphism – than in (3.24). The  $\alpha$ -summation is over module morphisms, and the middle equality uses the identity (A.66).) We write

$$\mathbf{B}_{X,Y} := (S_{X,Y}, p_{X,Y}) \tag{3.28}$$

for the object of the category  $\mathcal{Cyl}(\mathcal{C}, S^1)$  that is given by the boundary value consisting of two points labeled by  $U(X^\vee)$  and by  $U(Y)$ , respectively, and by the idempotent (3.21). We have  $\Phi(\mathbf{B}_{X,Y}) = \text{Nat}(G^X, G^Y) = \mathbb{D}^{X,Y} \in \mathcal{Z}(\mathcal{C})$ . Also, in the Karoubian envelope the morphisms  $p_{X,Y}$  and  $p_{\mathbb{D}^{X,Y}}^{\text{can}}$  are nothing but the identity morphisms on the objects  $\mathbf{B}_{X,Y}$  and  $\mathbf{B}_{\mathbb{D}^{X,Y}}^{\text{can}}$ , respectively. It follows that the string nets (3.24) provide inverse isomorphisms

$$e_{X,Y} : \mathbf{B}_{\mathbb{D}^{X,Y}}^{\text{can}} \xrightarrow{\cong} \mathbf{B}_{X,Y} : r_{X,Y} \tag{3.29}$$

for any pair of  $A$ - $B$ -bimodules  $X$  and  $Y$  over (simple special symmetric) Frobenius algebras  $A$  and  $B$ . Since according to Example 3.5 we have  $\Phi(\mathbf{B}_{\mathbb{D}^{X,Y}}^{\text{can}}) = \mathbb{D}^{X,Y} \in \mathcal{Z}(\mathcal{C})$ , we then get

$$\text{Im}(\Phi^\circ(p_{X,Y})) \cong \Phi(\mathbf{B}_{X,Y}) \cong \Phi(\mathbf{B}_{\mathbb{D}^{X,Y}}^{\text{can}}) = \mathbb{D}^{X,Y}. \tag{3.30}$$

**Definition 3.7.** Let  $\Sigma$  be a surface,  $\mathbf{B}^\circ$  be a boundary value for  $\partial\Sigma$ , and  $p \in \text{Hom}_{\mathcal{Cyl}^\circ(\mathcal{C}, \partial\Sigma)}(X, X)$  be an idempotent. The map

$$p_* : \text{SN}_{\mathcal{C}}^\circ(\Sigma, \mathbf{B}^\circ) \rightarrow \text{SN}_{\mathcal{C}}^\circ(\Sigma, \mathbf{B}^\circ) \tag{3.31}$$

is the linear endomorphism of the string-net space  $\text{SN}_{\mathcal{C}}^\circ(\Sigma, \mathbf{B}^\circ)$  that maps  $[I] \in \text{SN}_{\mathcal{C}}^\circ(\Sigma, \mathbf{B}^\circ)$  to the string net that is obtained by concatenating  $[I]$  with the string net  $p$  close to the boundary.

The following observation is immediate:

**Lemma 3.8.** For any idempotent  $p \in \text{Hom}_{\text{Cyl}^\circ(\mathcal{C}, \partial\Sigma)}(X, X)$  the map  $p_*$  defined by (3.31) is an idempotent.

With this result at hand, we can adapt the definition of string-net spaces to the Karoubification of the boundary category.

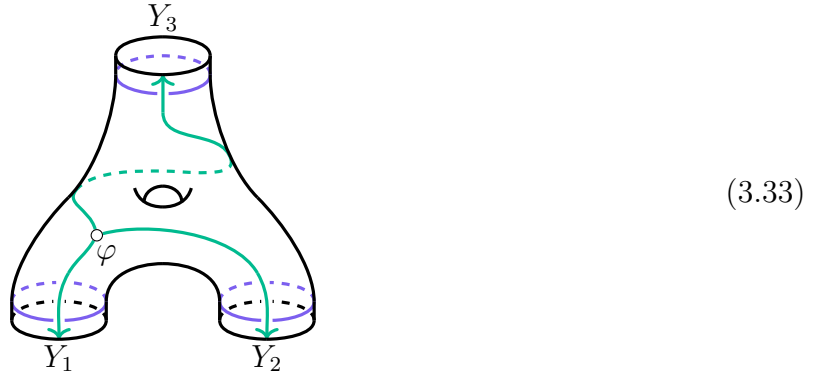
**Definition 3.9.** Let  $\Sigma$  be a surface and let  $\mathbf{B} = (\mathbf{B}^\circ, p_{\mathbf{B}})$  be an object in the cylinder category  $\text{Cyl}(\mathcal{C}, \partial\Sigma)$ . The *string-net space* for  $\Sigma$  and  $\mathbf{B}$  is the subspace

$$\text{SN}_{\mathcal{C}}(\Sigma, \mathbf{B}) := \text{Im}\left((p_{\mathbf{B}})_*\right) \quad (3.32)$$

of  $\text{SN}_{\mathcal{C}}^\circ(\Sigma, \mathbf{B}^\circ)$ .

We refer to an element of  $\text{SN}_{\mathcal{C}}(\Sigma, \mathbf{B})$  again as a string net, or also, if we want to stress that it is not a bare string net, as a *Karoubified string net*. The subspace  $\text{SN}_{\mathcal{C}}(\Sigma, \mathbf{B})$  is invariant under the action of the mapping class group of  $\Sigma$  on  $\text{SN}_{\mathcal{C}}^\circ(\Sigma, \mathbf{B}^\circ)$ , and thus carries an action of that group as well.

**Example 3.10.** The picture



shows a (Karoubified) string net on a torus  $T$  with three boundary circles which is an element of the space  $\text{SN}_{\mathcal{C}}(T, \mathbf{B}_{Y_1}^{\text{can}} \boxtimes \mathbf{B}_{Y_2}^{\text{can}} \boxtimes \mathbf{B}_{Y_3}^{\text{can}})$ .

### 3.3 The string-net modular functor

Next we note that as a consequence of the equivalence (3.16) we have a non-canonical equivalence

$$\text{Cyl}(\mathcal{C}, \partial\Sigma) \simeq \mathcal{Z}(\mathcal{C})^{\boxtimes \ell} \quad \text{if} \quad \partial\Sigma \cong (S^1)^{\sqcup \ell}. \quad (3.34)$$

**Proposition 3.11.** Let  $\Sigma$  be a connected surface with boundary  $\partial\Sigma$  diffeomorphic to  $(S^1)^{\sqcup \ell}$ . Let  $\Phi_{\partial\Sigma}: \text{Cyl}(\mathcal{C}, \partial\Sigma) \rightarrow \mathcal{Z}(\mathcal{C})^{\boxtimes \ell}$  be an equivalence, and let  $\mathbf{B} = (\mathbf{B}^\circ, p_{\mathbf{B}}) \in \text{Cyl}(\mathcal{C}, \partial\Sigma)$  be such that  $\Phi_{\partial\Sigma}(\mathbf{B}) \cong \boxtimes_i Y_i \in \mathcal{Z}(\mathcal{C})^{\boxtimes \ell}$ . Then there is an isomorphism

$$\text{SN}_{\mathcal{C}}(\Sigma, \mathbf{B}) \cong \text{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathbf{1}, \Phi_{\partial\Sigma}^{\boxtimes}(\mathbf{B}) \otimes K^{\otimes g}) \quad (3.35)$$

of vector spaces, where  $g$  is the genus of  $\Sigma$  and  $\Phi_{\partial\Sigma}^{\boxtimes}(\mathbf{B}) = \boxtimes_i Y_i \in \mathcal{Z}(\mathcal{C})$ . Moreover, the isomorphism (3.35) intertwines the action of the mapping class group of  $\Sigma$  on these vector spaces.

*Proof.* As has been shown in [Ki, Go], the (Karoubified) string-net construction based on a spherical fusion category  $\mathcal{C}$  gives rise to a 3-2-1 topological field theory (TFT) that is isomorphic to the (3-2-1-extended) Turaev-Viro state sum TFT based on  $\mathcal{C}$ . The latter Turaev-Viro TFT is, in turn, isomorphic as a 3-2-1 TFT to the Reshetikhin-Turaev surgery TFT based on the Drinfeld center  $\mathcal{Z}(\mathcal{C})$  [BalK, TV]. These isomorphisms of TFTs restrict to isomorphisms between the respective induced modular functors.  $\square$

**Remark 3.12.** By the conventions entering the notion of boundary value (see Definition 3.2), the unparametrized string-net functor  $\text{SN}_{\mathcal{C}}(\Sigma, -)$  is naturally a covariant functor with domain  $\mathcal{Cyl}(\mathcal{C}, \partial\Sigma)$ . That field insertions appear in the contravariant entry of the Hom functor (see e.g. (2.42)) is due to our choice of convention for the field map  $\mathbb{F}$ .

It is worth recalling that the mapping class group action on the string-net space on the left hand side of (3.35) is directly induced by the action of diffeomorphisms on graphs embedded in  $\Sigma_g$  and is thus geometric, while the action on the morphism spaces on the right hand side of (3.35) is more intricate (see Remark 2.18(iii)).

The result of Proposition 3.11 extends in an obvious manner to non-connected surfaces: If  $\Sigma$  is the disjoint union of connected surfaces  $\Sigma^{(q)}$  with  $q \in \{1, 2, \dots, N\}$ , then

$$\text{SN}_{\mathcal{C}}(\Sigma, \mathbf{B}) = \bigotimes_{q=1}^N \text{SN}_{\mathcal{C}}(\Sigma^{(q)}, \mathbf{B}|_{\Sigma^{(q)}}), \quad (3.36)$$

and for each  $\text{SN}_{\mathcal{C}}(\Sigma^{(q)}, \mathbf{B}|_{\Sigma^{(q)}})$  there is an isomorphism of the form (3.35).

It follows in particular that for any surface  $\Sigma$ , string nets furnish an exact functor  $\text{SN}_{\mathcal{C}}(\Sigma, -) : \mathcal{Cyl}(\mathcal{C}, \partial\Sigma) \rightarrow \mathcal{Vect}$ . It is worth pointing out that the string-net spaces considered so far are constructed for surfaces without a boundary parametrization, while the surfaces obtained from world sheets are equipped with this extra datum. To account for the latter, we observe that the assignment

$$b \longmapsto \mathcal{Cyl}(\mathcal{C}, b) \quad (3.37)$$

extends to a symmetric monoidal 2-functor from the bicategory of one-manifolds, with orientation preserving embeddings as 1-morphisms and isotopies as 2-morphisms, to the 2-category of categories. Moreover, denoting by  $\bar{b}$  the one-manifold  $b$  with opposite orientation, there is a canonical equivalence

$$\mathcal{Cyl}(\mathcal{C}, \bar{b}) \xrightarrow{\simeq} \mathcal{Cyl}(\mathcal{C}, b)^{\text{opp}} \quad (3.38)$$

which maps a boundary value  $\mathfrak{B}$  to its dual  $\mathfrak{B}^{\vee}$ , i.e. to the boundary value consisting of the same points on the one-manifold  $b$  as  $\mathfrak{B}$ , but each labeled with the dual of the original object of  $\mathcal{C}$ . This implies that from any boundary parametrization

$$\phi : (S^1)^{\sqcup(p+q)} \sqcup I^{\sqcup(r+s)} \longrightarrow \partial\Sigma \quad (3.39)$$

of a surface  $\Sigma$  with  $p$  incoming and  $q$  outgoing boundary circles and with  $r$  incoming and  $s$  outgoing boundary intervals we naturally obtain a functor

$$\phi_* : \mathcal{Cyl}(\mathcal{C}, S^1)^{\boxtimes(p+q)} \boxtimes \mathcal{Cyl}(\mathcal{C}, I)^{\boxtimes(r+s)} \longrightarrow \mathcal{Cyl}(\mathcal{C}, \partial\Sigma). \quad (3.40)$$

**Remark 3.13.** To treat all surfaces that can be obtained as  $\Sigma = \Sigma_{\mathfrak{S}}$  for any arbitrary world sheet  $\mathfrak{S}$ , we must also account for the case that a geometric boundary circle of  $\Sigma$  is not parametrized at all, which happens if the corresponding boundary circle of  $\mathfrak{S}$  entirely consists of a single physical boundary (together with a single zero-cell). Accordingly, if  $\mathfrak{S}$  has  $\ell$  such boundary circles, then one deals with a functor  $\phi_*$  with domain  $\mathcal{Cyl}(\mathcal{C}, S^1)^{\boxtimes(p+q)} \boxtimes \mathcal{Cyl}(\mathcal{C}, I)^{\boxtimes(r+s)} \boxtimes \mathcal{Vect}^{\boxtimes\ell}$  instead of (3.40). (Here we use that  $\mathcal{Cyl}(\mathcal{C}, \emptyset) = \mathcal{Vect}$ ). Since  $\mathcal{Vect}$  acts as a monoidal unit under the Deligne product, we can safely suppress this complication.

We can now define a functor  $\text{SN}_{\mathcal{C}}$  for surfaces with boundary parametrization:

**Definition 3.14.** Let  $\Sigma$  be a surface and  $\phi$  a boundary parametrization of  $\Sigma$ . The functor  $\text{SN}_{\mathcal{C}}(\Sigma_{\phi}, -)$  is the composite

$$\text{SN}_{\mathcal{C}}(\Sigma_{\phi}, -) := \text{SN}_{\mathcal{C}}(\Sigma, -) \circ \phi_* : \quad \mathcal{Cyl}(\mathcal{C}, S^1)^{\boxtimes(p+q)} \boxtimes \mathcal{Cyl}(\mathcal{C}, I)^{\boxtimes(r+s)} \longrightarrow \mathcal{Vect}. \quad (3.41)$$

When the boundary parametrization is clear from the context, we often abuse notation and simply write  $\text{SN}_{\mathcal{C}}(\Sigma, -)$  in place of  $\text{SN}_{\mathcal{C}}(\Sigma_{\phi}, -)$ .

We are now ready to state

**Theorem 3.15.** *Let  $\Sigma$  be a (compact oriented smooth) surface with a boundary parametrization  $\phi(\Sigma) = \{\phi_-(\Sigma), \phi_+(\Sigma)\}$  of the form*

$$\phi_-(\Sigma) : \alpha \sqcup \beta \longrightarrow \partial\Sigma \quad \text{and} \quad \phi_+(\Sigma) : \beta \sqcup \alpha' \longrightarrow \partial\Sigma, \quad (3.42)$$

where  $\alpha$ ,  $\alpha'$  and  $\beta$  are finite disjoint unions of the circle  $S^1$  and the interval  $I$ . Denote by  $\cup_{\beta}\Sigma$  the surface obtained by gluing  $\Sigma$  along the images of  $\beta$  under  $\phi_{\pm}(\Sigma)$  via the orientation reversing diffeomorphism  $\phi_+(\Sigma)|_{\beta} \circ (\phi_-(\Sigma)|_{\beta})^{-1}$ . Then the corresponding gluing transformation

$$\text{SN}_{\mathcal{C}}(\Sigma, \bar{?} \boxtimes \mathbf{B} \boxtimes \bar{\mathbf{B}} \boxtimes ?) \longrightarrow \text{SN}_{\mathcal{C}}(\cup_{\beta}\Sigma, \bar{?} \boxtimes ?) \quad (3.43)$$

exhibits the functor  $\text{SN}_{\mathcal{C}}(\cup_{\beta}\Sigma, -)$  as the coend

$$\text{SN}_{\mathcal{C}}(\cup_{\beta}\Sigma, \bar{?} \boxtimes ?) = \int^{\mathbf{B}' \in \mathcal{Cyl}(\mathcal{C}, \beta)} \text{SN}_{\mathcal{C}}(\Sigma, \bar{?} \boxtimes \mathbf{B}' \boxtimes \bar{\mathbf{B}}' \boxtimes ?) \quad (3.44)$$

(taken in the category of left exact functors), i.e. (3.44) has (3.43) as its structure morphisms.

*Proof.* Fix for the moment boundary values  $\mathbf{B}_{\alpha} \in \mathcal{Cyl}(\mathcal{C}, \alpha)$  and  $\mathbf{B}_{\alpha'} \in \mathcal{Cyl}(\mathcal{C}, \alpha')$ . We are going to show that the family of gluing morphisms

$$\iota_{\mathbf{B}^{\circ}} : \quad \text{SN}_{\mathcal{C}}(\Sigma, \bar{\mathbf{B}}_{\alpha} \boxtimes \mathbf{B}^{\circ} \boxtimes \bar{\mathbf{B}}^{\circ} \boxtimes \mathbf{B}_{\alpha'}) \longrightarrow \text{SN}_{\mathcal{C}}(\cup_{\beta}\Sigma, \bar{\mathbf{B}}_{\alpha} \boxtimes \mathbf{B}_{\alpha'}), \quad (3.45)$$

for boundary values  $\mathbf{B}^{\circ} \in \mathcal{Cyl}^{\circ}(\mathcal{C}, \beta)$ , is dinatural and satisfies the universal property of the structure morphism of the coend (3.44). Note that here we restrict the family to the subcategory  $\mathcal{Cyl}^{\circ}(\mathcal{C}, \beta) \hookrightarrow \text{Kar}(\mathcal{Cyl}^{\circ}(\mathcal{C}, \beta)) = \mathcal{Cyl}(\mathcal{C}, \beta)$ ; this is sufficient, owing to Lemmas A.2 and A.3. Dinaturality of the family (3.45) is immediate: The two expressions that are required to be equal can both be described as an additional gluing with a cylinder  $\Sigma' : \beta \rightarrow \beta$ , namely either gluing the images of  $\beta$  under  $\phi_+(\Sigma)$  and  $\phi_-(\Sigma')$  via  $\phi_+(\Sigma)|_{\beta} \circ (\phi_-(\Sigma')|_{\beta})^{-1}$  first, or gluing the images under  $\phi_+(\Sigma')$  and  $\phi_-(\Sigma)$  via  $\phi_+(\Sigma')|_{\beta} \circ (\phi_-(\Sigma)|_{\beta})^{-1}$  first; dinaturality then directly follows

from the associativity of composition.

To address universality of the family  $\{g_{\mathbf{B}^\circ}\}$  we construct, given an arbitrary dinatural family

$$\{g_{\mathbf{B}^\circ} : \text{SN}_{\mathcal{C}}(\Sigma, \overline{\mathbf{B}}_\alpha \boxtimes \mathbf{B}^\circ \boxtimes \overline{\mathbf{B}}^\circ \boxtimes \mathbf{B}_{\alpha'}) \rightarrow V\}_{\mathbf{B}^\circ \in \mathcal{Cyl}^\circ(\mathcal{C}, \beta)} \quad (3.46)$$

to some vector space  $V$ , a linear map

$$g : \text{SN}_{\mathcal{C}}(\cup_\beta \Sigma, \overline{\mathbf{B}}_\alpha \boxtimes \mathbf{B}_{\alpha'}) \rightarrow V \quad (3.47)$$

as follows: Any string net  $[I] \in \text{SN}_{\mathcal{C}}(\cup_\beta \Sigma, \overline{\mathbf{B}}_\alpha \boxtimes \mathbf{B}_{\alpha'})$  contains a representative graph  $I$  that intersects the one-manifold  $\beta \hookrightarrow \cup_\beta \Sigma$  transversally. Cutting open  $\cup_\beta \Sigma$  along  $\beta$  thus gives rise to an object  $\mathbf{B}_I \in \mathcal{Cyl}^\circ(\mathcal{C}, \beta)$  and a string net on the cut surface. We denote the latter by

$$[\text{cut}(I)] \in \text{SN}_{\mathcal{C}}(\Sigma, \overline{\mathbf{B}}_\alpha \boxtimes \mathbf{B}_I \boxtimes \overline{\mathbf{B}}_I \boxtimes \mathbf{B}_{\alpha'}) \quad (3.48)$$

and set  $g([I]) := g_{\mathbf{B}_I}([\text{cut}(I)])$ . By the dinaturality of the family (3.46) and the defining invariance of string nets under local relations, the so obtained linear map (3.47) is well defined. Moreover, by construction it is the unique map through which all members  $g_{\mathbf{B}^\circ}$  of the family (3.46) factor via the gluing map. This proves universality.

In summary, we have shown that

$$\begin{aligned} \text{SN}_{\mathcal{C}}(\cup_\beta \Sigma, \overline{\mathbf{B}}_\alpha \boxtimes \mathbf{B}_{\alpha'}) &\cong \int^{\mathbf{B}^\circ \in \mathcal{Cyl}^\circ(\mathcal{C}, \beta)} \text{SN}_{\mathcal{C}}(\Sigma, \overline{\mathbf{B}}_\alpha \boxtimes \mathbf{B}^\circ \boxtimes \overline{\mathbf{B}}^\circ \boxtimes \mathbf{B}_{\alpha'}) \\ &\cong \int^{\mathbf{B}' \in \mathcal{Cyl}(\mathcal{C}, \beta)} \text{SN}_{\mathcal{C}}(\Sigma, \overline{\mathbf{B}}_\alpha \boxtimes \mathbf{B}' \boxtimes \overline{\mathbf{B}}' \boxtimes \mathbf{B}_{\alpha'}) \end{aligned} \quad (3.49)$$

for all  $\mathbf{B}_\alpha \in \mathcal{Cyl}(\mathcal{C}, \alpha)$  and  $\mathbf{B}_{\alpha'} \in \mathcal{Cyl}(\mathcal{C}, \alpha')$ , where the second isomorphism holds by Lemmas A.2 and A.3. This directly upgrades to a coend of functors as in (3.51) (where we write an equality instead of an isomorphism because the coend is determined up to unique isomorphism), and indeed, by exactness of  $\text{SN}_{\mathcal{C}}(\Sigma, -)$ , to a left exact coend.  $\square$

In particular, when  $\Sigma$  is a disjoint union of two surfaces that are glued together, we have

**Corollary 3.16.** *Let  $\Sigma = \Sigma_1 \sqcup \Sigma_2$  be a (compact oriented smooth) surface with a boundary parametrization of the form (3.42) such that  $\Sigma_1: \alpha \rightarrow \beta$  and  $\Sigma_2: \beta \rightarrow \alpha'$  are composable 1-morphisms in  $\mathcal{Bord}_{2, \circ/c}^{\text{or}}$ . Then the gluing transformation*

$$\text{SN}_{\mathcal{C}}(\Sigma_1, \overline{\mathbf{B}} \boxtimes \mathbf{B}) \boxtimes \text{SN}_{\mathcal{C}}(\Sigma_2, \overline{\mathbf{B}} \boxtimes \mathbf{B}') \longrightarrow \text{SN}_{\mathcal{C}}(\cup_\beta \Sigma, \overline{\mathbf{B}} \boxtimes \mathbf{B}') \equiv \text{SN}_{\mathcal{C}}(\Sigma_1 \cup_\beta \Sigma_2, \overline{\mathbf{B}} \boxtimes \mathbf{B}') \quad (3.50)$$

exhibits the functor  $\text{SN}_{\mathcal{C}}(\Sigma_1 \cup_\beta \Sigma_2, \overline{\mathbf{B}} \boxtimes \mathbf{B}')$  as the (left exact) coend

$$\begin{aligned} \text{SN}_{\mathcal{C}}(\Sigma_1 \cup_\beta \Sigma_2, \overline{\mathbf{B}} \boxtimes \mathbf{B}') &= \int^{\mathbf{B}' \in \mathcal{Cyl}(\mathcal{C}, \beta)} \text{SN}_{\mathcal{C}}(\Sigma_1 \sqcup \Sigma_2, \overline{\mathbf{B}} \boxtimes \mathbf{B}' \boxtimes \overline{\mathbf{B}}' \boxtimes \mathbf{B}') \\ &= \int^{\mathbf{B}' \in \mathcal{Cyl}(\mathcal{C}, \beta)} \text{SN}_{\mathcal{C}}(\Sigma_1, \overline{\mathbf{B}} \boxtimes \mathbf{B}') \otimes_{\mathbb{k}} \text{SN}_{\mathcal{C}}(\Sigma_2, \overline{\mathbf{B}}' \boxtimes \mathbf{B}'). \end{aligned} \quad (3.51)$$

As a consequence, the two-functor  $\text{SN}_{\mathcal{C}}$  obtained by the string-net construction is an open-closed modular functor  $\text{SN}_{\mathcal{C}}: \mathcal{Bord}_{2, \circ/c}^{\text{or}} \rightarrow \text{Prof}_{\mathbb{k}}$  in the sense of Definition 2.1.

*Proof.* The first statement is just the specialization of Theorem 3.15 to the particular type of surface considered here. The second statement holds because in this case the gluing can be seen as the composition in  $\mathcal{B}ord_{2,o/c}^{\text{ot}}$  of the two 1-morphisms  $\Sigma_1: \alpha \rightarrow \beta$  and  $\Sigma_2: \beta \rightarrow \alpha'$ .  $\square$

In summary, we have shown that  $\text{SN}_{\mathcal{C}}$  is an open-closed modular functor and that it satisfies the two requirements that we demand for  $\text{Bl}_{\mathcal{C}}$ : When restricted to the closed sector it is isomorphic to the Turaev-Viro modular functor for  $\mathcal{C}$ , and thus to the Reshetikhin-Turaev modular functor for  $\mathcal{Z}(\mathcal{C})$ , which (conjecturally) models the conformal blocks for a conformal field theory with chiral data encoded in  $\mathcal{C}$ ; and the extension to the open-closed case is implemented by the functor  $L$ . Accordingly, from now on we take, as already announced,  $\text{Bl}_{\mathcal{C}}$  to be realized as the string-net modular functor  $\text{SN}_{\mathcal{C}}$ .

### 3.4 String nets from world sheets

To any world sheet  $\mathcal{S}$  without physical boundaries we can naturally associate a string net in the space  $\text{SN}_{\mathcal{C}}^{\circ}(\Sigma_{\mathcal{S}}, \mathbf{B}_{\mathcal{S}}^{\circ})$ , where  $\Sigma_{\mathcal{S}}$  is the surface that underlies  $\mathcal{S}$  and where the boundary value  $\mathbf{B}_{\mathcal{S}}^{\circ}$  is obtained from the collection of boundary data  $\mathfrak{B}$ , in the sense of Definition 2.15, for all geometric boundary circles of  $\mathcal{S}$ , seen as boundary circles of  $\Sigma_{\mathcal{S}}$ . (That is, for each boundary circle  $b \in \pi_0(\partial\check{\mathcal{S}})$ , take  $B|_b$  to be the set of zero-cells of  $b$  and define  $\mathbf{B}_{\mathcal{S}}^{\circ}(v) := \mathfrak{B}(v)$  for  $v \in B|_b$ .) We specify the string net associated to  $\mathcal{S}$  by the following prescription that yields a representative graph  $\Gamma_{\mathcal{S}}$ : As an embedded graph,  $\Gamma_{\mathcal{S}}$  consists of all one-cells in the interior of  $\check{\mathcal{S}}$  as oriented edges and of all zero-cells of  $\check{\mathcal{S}}$  as vertices. The coloring map  $\text{col}_E$  from Definition 3.1 assigns to each oriented edge  $e$  of  $\Gamma_{\mathcal{S}}$  the object in  $\mathcal{C}$  that underlies the bimodule label of  $e$  when seen as a defect line, and the map  $\text{col}_V$  assigns to each vertex  $v$  in the interior of  $\Sigma_{\mathcal{S}}$  the morphism label of  $v$  when seen as a defect junction on  $\mathcal{S}$ .

We refer to the graph  $\Gamma_{\mathcal{S}}$ , respectively the string net

$$[\Gamma_{\mathcal{S}}] \in \text{SN}_{\mathcal{C}}^{\circ}(\Sigma_{\mathcal{S}}, \mathbf{B}_{\mathcal{S}}^{\circ}) \quad (3.52)$$

represented by it, as the *partial defect network* on  $\Sigma_{\mathcal{S}}$  that is associated with the world sheet  $\mathcal{S}$ . Here we include the qualification ‘partial’ in order to remind us of the fact that the prescription ignores all information related to the Frobenius algebra labels of the two-cells of the world sheet  $\mathcal{S}$ . The partial defect network constitutes an important intermediate step towards the construction of correlators in Section 3.5.

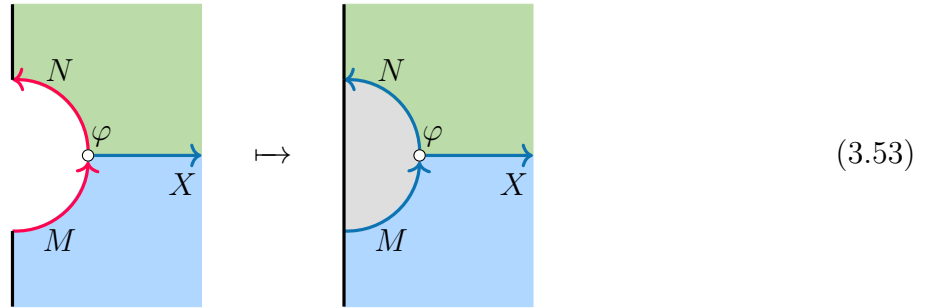
**Remark 3.17.** Recall that a defect junction is labeled by a bimodule morphism between objects that are of the form  $Y \otimes_B Y'$ , with  $Y$  an  $A$ - $B$ -bimodule,  $Y'$  a  $B$ - $A'$ -bimodule and  $\otimes_B$  the tensor product over the Frobenius algebra  $B$ . In the present context, such a morphism is to be regarded as a morphism involving the tensor product  $\otimes$  in  $\mathcal{C}$  via the identification  ${}_A\text{Hom}_{A'}(X, Y \otimes_B Y') = {}_A\text{Hom}_{A'}^{(B)}(X, Y \otimes Y')$ , with  ${}_A\text{Hom}_{A'}^{(B)}(X, Y \otimes Y')$  the subspace of  ${}_A\text{Hom}_{A'}(X, Y \otimes Y')$  that is invariant under pre-composition with the idempotent  $P_{Y \otimes_B Y'}$  as explained in Appendix A.3.

In the case of a general world sheet, which may possess physical boundaries, the prescription given above is no longer applicable. Indeed, if  $\mathcal{S}$  has at least one physical boundary, then the corresponding geometric boundary circle of  $\mathcal{S}$  contains a labeled and unparameterized one-cell, and for such one-cells the string-net construction does not provide a suitable counterpart.

However, it is not hard to reformulate the general situation in such a manner that it effectively reduces to the one for world sheets without physical boundaries. To this end we convert a world sheet  $\mathcal{S}$  with non-empty physical boundary to a new world sheet  $\tilde{\mathcal{S}}$ , to be referred to as the *complemented world sheet*, that resembles  $\mathcal{S}$  but whose physical boundaries are all labeled by the monoidal unit of  $\mathcal{C}$ . We then also consider an associated *complemented surface*  $\tilde{\Sigma}_{\mathcal{S}} \equiv \Sigma_{\tilde{\mathcal{S}}}$ .

The complemented world sheet  $\tilde{\mathcal{S}}$  is constructed as follows. Given a boundary circle  $b$  of  $\mathcal{S}$  containing physical boundaries, but no defect junctions, we partially fill  $b$  at each physical boundary by a two-cell that is labeled by the monoidal unit  $\mathbf{1}$  of  $\mathcal{C}$ . Hereby the circle  $b$  is replaced by a new geometric boundary circle  $\tilde{b}$ , and the former physical boundary, which is labeled by a right  $A$ -module for some Frobenius algebra  $A$ , is turned into a defect line labeled by the same object, but regarded as a  $\mathbf{1}$ - $A$ -bimodule. The boundary of the new  $\mathbf{1}$ -labeled two-cell is the union of this defect line and a new boundary segment  $b_+$ .

In case the boundary circle  $b$  contains two physical boundaries that meet, together with some defect line, at a defect junction on  $b$ , we partially fill  $b$  by a single new two-cell labeled by  $\mathbf{1}$  that is bounded by the two former physical boundaries, which have both turned into defect lines, and a single new boundary segment  $b_+$ . Analogously we proceed if more than two physical boundaries consecutively meet at defect junctions. We refrain from writing out the resulting prescription for the complemented world sheet  $\tilde{\mathcal{S}}$  in a formal definition, but content ourselves to illustrate what  $\tilde{\mathcal{S}}$  looks like locally in the case of two physical boundaries labeled by  $M$  and  $N$  that meet at a defect junction:



In order that the prescription for the complemented world sheet complies with the requirements on a world sheet in Definitions 2.3 and 2.5, the point at which a new defect line of  $\tilde{\mathcal{S}}$  (resulting from a physical boundary of  $\mathcal{S}$ ) meets the circle  $\tilde{b}$  must lie in the interior of a gluing interval. Accordingly we slightly enlarge the gluing intervals that are adjacent to the new boundary segment  $b_+ \subset \tilde{b}$ . To this end we regard  $b_+$  as the union of three intervals and declare the two outer ones to be new parts of the enlarged adjacent gluing intervals. The middle part of  $b_+$  must then be regarded as a new physical boundary; as such it must be labeled by a right  $\mathbf{1}$ -module, i.e. by an object of  $\mathcal{C}$ , and we take the distinguished object  $\mathbf{1} \in \mathcal{C}$  as that label. Let us illustrate this part of the prescription by redrawing the new boundary segment on the right

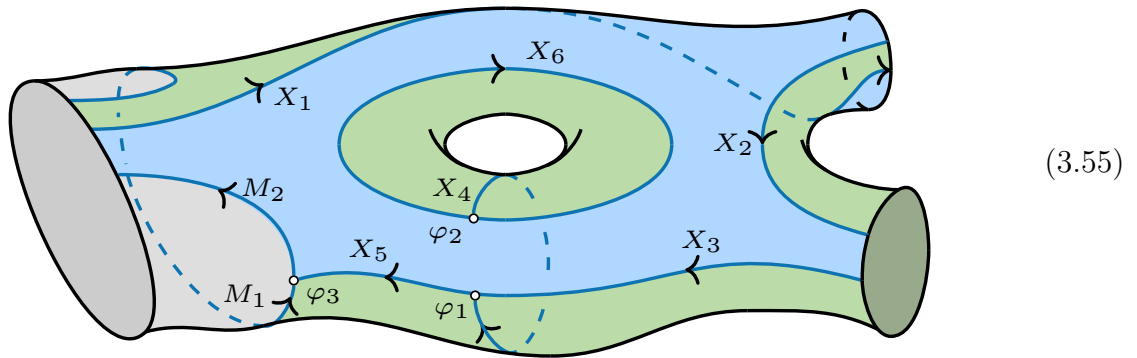
hand side of (3.53) in full detail:



Since the new physical boundaries of  $\tilde{\mathcal{S}}$  that arise this way all carry the same distinguished label  $\mathbf{1}$ , we can, and will, safely suppress this label in all pictures below. Moreover, the presence of these new physical boundaries, as well as of the adjacent  $\mathbf{1}$ -labeled two-cells, is in fact entirely irrelevant for the rest of the construction, for the same reason as we can ignore transparent defect lines. Specifically, we may treat the geometric boundary circles  $\tilde{b}$  of  $\tilde{\mathcal{S}}$  just as if they were gluing circles. As a consequence, for all practical purposes the complemented world sheet can be thought of as a world sheet that does not have any physical boundaries; this implies in particular that after converting  $\mathcal{S}$  to  $\tilde{\mathcal{S}}$  there is no longer any conflict with the string-net construction. In the pictures below, we account for this fact by suppressing the detailed structure of the new boundary segments of  $\tilde{\mathcal{S}}$  that is shown in (3.54), and instead just draw the geometric boundary circles  $\tilde{b}$  in the same way as we do for ordinary gluing circles.

Note that a right  $A$ -module  $N_A$  is the same as a  $\mathbf{1}$ - $A$ -bimodule  $\mathbf{1}N_A$ , and hence indeed the one-cells in  $\tilde{\mathcal{S}}$  coming from physical boundaries in  $\mathcal{S}$  are properly labeled defect lines. Moreover, a module morphism in  $\text{Hom}_B(M_B, N_A \otimes_A X_B)$  is the same as a bimodule morphism in  $\mathbf{1}\text{Hom}_B(\mathbf{1}M_B, \mathbf{1}N_A \otimes_A X_B)$ . Thus  $\tilde{\mathcal{S}}$  as obtained from  $\mathcal{S}$  by our construction is indeed again a proper world sheet in the sense of Definition 2.5.

**Example 3.18.** As an illustration, for the world sheet  $\mathcal{S}$  displayed in (2.14) the complemented world sheet  $\tilde{\mathcal{S}}$  is given by

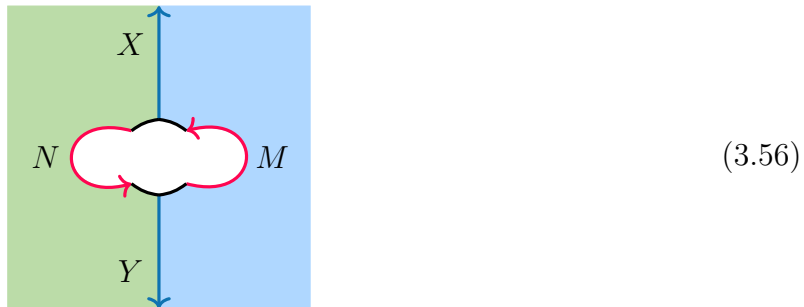


Replacing  $\mathcal{S}$  by the complemented world sheet  $\tilde{\mathcal{S}}$  does not change the field insertions, in the following sense:

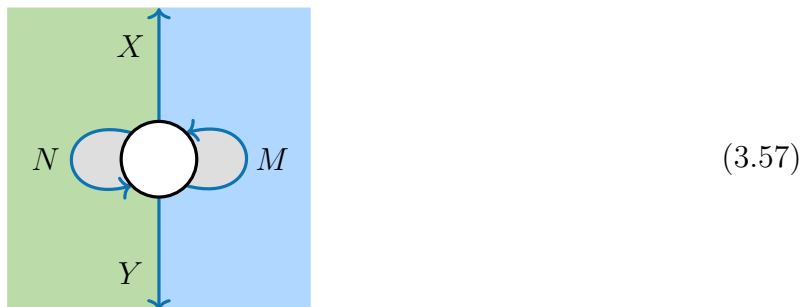
**Lemma 3.19.** *Let  $\mathcal{S}$  be a world sheet and  $b$  be a geometric boundary circle of  $\mathcal{S}$  containing a physical boundary. Let the boundary circle  $\tilde{b}$  of  $\tilde{\mathcal{S}}$  be obtained from  $b$  by the construction described above, and regard  $\tilde{b}$  as a gluing circle. Then  $\mathbb{F}(\tilde{b}) \cong \mathbb{F}(b)$  as objects in  $\mathcal{Z}(\mathcal{C})$ .*



*Proof.* We prove the statement for the case that  $b$  is a boundary circle with two (generalized) boundary field insertions that has the specific form



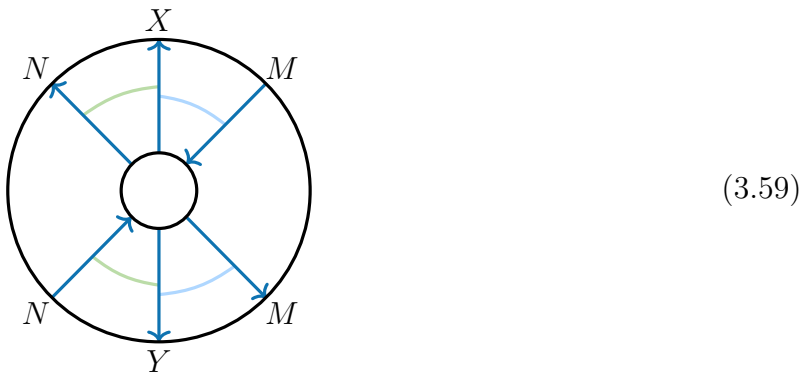
with  $X$  an  $A$ - $B$ -bimodule,  $Y$  a  $B$ - $A$ -bimodule,  $M$  a right  $B$ -module and  $N$  a right  $A$ -module for Frobenius algebras  $A$  and  $B$ . The general case can be treated analogously. In the case at hand the circle  $\tilde{b}$  is given by



Consider now the following two elements  $\mathbf{B} = (\mathbf{B}^\circ, p)$  and  $\tilde{\mathbf{B}} = (\tilde{\mathbf{B}}^\circ, \tilde{p})$  of the cylinder category  $\mathcal{Cyl}(\mathcal{C}, S^1)$ :  $\mathbf{B}^\circ$  consists of two points on  $S^1$  that are mapped to

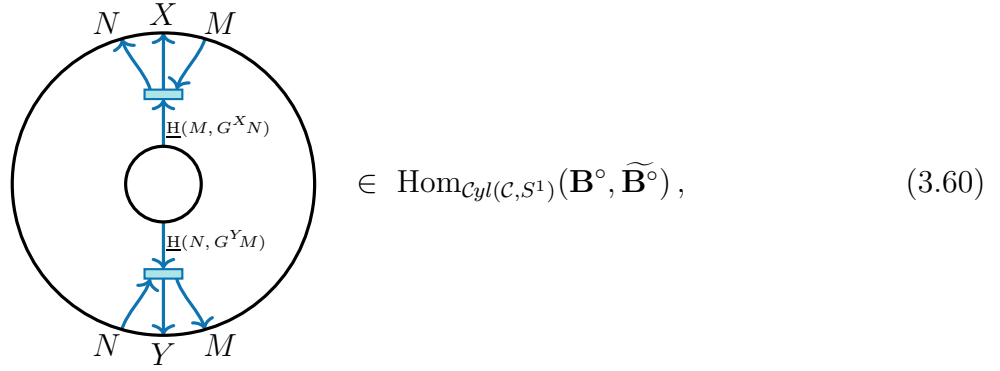
$$\underline{\text{Hom}}(M, G^X(N)) = \underline{\text{Hom}}(M, N \otimes_A X) \quad \text{and} \quad \underline{\text{Hom}}(N, G^Y(M)) = \underline{\text{Hom}}(N, M \otimes_B Y), \quad (3.58)$$

respectively, and  $p = \text{id}_{\mathbf{B}^\circ}$ , while  $\tilde{\mathbf{B}}^\circ$  consists of six points on  $S^1$  that are mapped to  $N$ ,  $X$ ,  $M^\vee$ ,  $M$ ,  $Y$  and  $N^\vee$ , respectively, and  $\tilde{p}$  is the string net represented by



(Note that here the two-cells whose boundary contains two segments labeled by  $M$  and  $N$ , respectively, are the  $\mathbf{1}$ -labeled two-cells that arose from complementing the world sheet.) By a slight variation of Example 3.6 one sees that  $\text{Im}(\Phi(\tilde{p})) = \mathbb{F}(\tilde{b})$ . Moreover,  $\mathbf{B}^\circ$  is isomorphic to

$\widetilde{\mathbf{B}}^\circ$  via the string net



$$\in \text{Hom}_{\text{Cyl}(\mathcal{C}, S^1)}(\mathbf{B}^\circ, \widetilde{\mathbf{B}}^\circ), \quad (3.60)$$

where we abbreviate  $\underline{\text{Hom}}$  by  $\underline{\text{H}}$ , and where the unlabeled boxes represent the canonical morphisms from the internal Hom's to the corresponding (non-relative) tensor products.

Choosing any equivalence  $\Phi_{S^1}: \text{Cyl}(\mathcal{C}, S^1) \xrightarrow{\cong} \mathcal{Z}(\mathcal{C})$  we thus arrive at

$$\begin{aligned} \mathbb{F}(b) &= L(\underline{\text{Hom}}(M, G^X(N)) \otimes \underline{\text{Hom}}(N, G^Y(M))) = \text{Im}(\Phi(\text{id}_{\mathbf{B}^\circ})) \\ &\cong \Phi_{S^1}(\mathbf{B}) \cong \Phi_{S^1}(\widetilde{\mathbf{B}}) \cong \text{Im}(\Phi(\tilde{p})) = \mathbb{F}(\tilde{b}). \end{aligned} \quad (3.61)$$

Thus  $\mathbb{F}(b) \cong \mathbb{F}(\tilde{b})$ , as claimed.  $\square$

Analogously as we did for the world sheet  $\mathcal{S}$  at the end of Section 2.2, we associate also to the complemented world sheet  $\tilde{\mathcal{S}}$  a surface, the complemented surface  $\tilde{\Sigma}_{\tilde{\mathcal{S}}} \equiv \Sigma_{\tilde{\mathcal{S}}}$ . To this end we must specify a parametrization for  $\tilde{\mathcal{S}}$ . Since when complementing the world sheet we extend some gluing intervals  $r$  to slightly larger intervals  $\tilde{r} \supset r$ , we cannot just take over a parametrization for  $\mathcal{S}$ , but rather specify a diffeomorphism from  $I \in \mathcal{Bord}_{2,o/c}^{\text{or}}$  to  $\tilde{r}$  instead of to  $r$ . This modification also guarantees that any sewing of the surface  $\tilde{\Sigma}_{\tilde{\mathcal{S}}}$  that is associated with  $\tilde{\mathcal{S}}$  and its parametrization agrees with the surface that arises when first sewing the surface associated with the original world sheet  $\mathcal{S}$  and its parametrization and then complementing, i.e. that (with a slight abuse of notation)  $\bigcup_{\tilde{\beta}} \tilde{\Sigma}_{\tilde{\mathcal{S}}} = \widetilde{\bigcup_{\beta} \Sigma_{\mathcal{S}}}$ .

As a consequence of Lemma 3.19 the spaces of conformal blocks for the world sheets  $\mathcal{S}$  and  $\tilde{\mathcal{S}}$  are isomorphic, albeit not canonically isomorphic.

### 3.5 String-net correlators

We are now ready to assign to a world sheet  $\mathcal{S}$  a string net that realizes a correlator for  $\mathcal{S}$ . We obtain this string net from the partial defect network  $[I_{\mathcal{S}}]$ , as constructed in the previous subsection, by embedding into each two-cell of the world sheet a suitable graph whose edges are labeled by the Frobenius algebra label of the two-cell. (If  $\mathcal{S}$  has physical boundaries, then we use instead the corresponding complemented world sheet  $\tilde{\mathcal{S}}$ .)

We say that two graphs of the latter type are related by a *Frobenius move* if the graphs can be transformed to each other by using, locally on the world sheet, a finite sequence of moves each of which implements a defining relation for the structural morphisms of the special symmetric Frobenius algebra  $A$  that labels the two-cell and of the representation morphisms for the left or right  $A$ -module structures carried by the defect lines that are contained in the boundary of the two-cell.

**Definition 3.20.** Let  $\vartheta$  be a two-cell of a world sheet and  $A(\vartheta) = (A, \mu, \eta, \Delta, \varepsilon)$  its Frobenius algebra label. A *Frobenius graph on  $\vartheta$*  is a  $\mathcal{C}$ -colored oriented graph  $\Gamma_\vartheta$  embedded in  $\vartheta$  having the following features:

- The vertices of  $\Gamma_\vartheta$  are univalent or trivalent. Each of the vertices of  $\Gamma_\vartheta$  that lies on the boundary  $\partial\vartheta$  is univalent and lies in the interior of a labeled one-cell (i.e. of a defect line).
- Each (oriented) edge of  $\Gamma_\vartheta$  is labeled by the object  $A$ .
- Each trivalent vertex is labeled either by the product  $\mu$  or by the coproduct  $\Delta$ , and each univalent vertex in the interior of  $\vartheta$  either by the unit  $\eta$  or by the counit  $\varepsilon$ . Each univalent vertex on a defect line in  $\partial\vartheta$  is labeled by the representation morphism  $\rho_X$  or  $\varrho_X$  for the left or right  $A$ -module structure that is carried by the defect line.

A *full Frobenius graph on  $\vartheta$*  is a Frobenius graph  $\Gamma_\vartheta$  on  $\vartheta$  such that in addition the following condition is satisfied:

- $\Gamma_\vartheta$  is *full* in the sense (compare [KopMRS]) that adding any further edges and vertices in a way compatible with the previous requirements results in a graph that can be reduced to  $\Gamma_\vartheta$  by a Frobenius move.

When drawing a Frobenius graph, it is convenient to use the simplified graphical notation for morphisms involving symmetric Frobenius algebras in which, as explained at the end of Appendix A.8, orientations as well as univalent vertices are omitted. We refer to the thus obtained simplified version of a (full) Frobenius graph as a *simplified* (full) Frobenius graph.

**Example 3.21.** The following pictures show an example of a full Frobenius graph on a two-cell (left) and its simplified version (right).

$X_1$   $X_2$   $\equiv$   $X_1$   $X_2$  (3.62)

Note that this Frobenius graph can be reduced to the one that appears in (A.53) by a suitable Frobenius move.

**Lemma 3.22.** *Any two full Frobenius graphs on a two-cell  $\vartheta \subseteq \mathcal{S}$  are related by a Frobenius move up to isotopy.*

*Proof.* The assertion follows by combining the following two statements, each of which is easy to verify: First, any two realizations of a *simplified* (not necessarily full) Frobenius graph as an ordinary Frobenius graph are related by a Frobenius move up to isotopy. Second, any two simplified full Frobenius graphs on  $\vartheta$  can be related by a finite sequence of the elementary moves that are shown in (A.54), (A.55) and (A.56) in Appendix A.8. □

**Remark 3.23.** (i) A convenient way to construct a (simplified) full Frobenius graph  $\Gamma_\vartheta$  on a given two-cell  $\vartheta$  is to remove any number of disks  $D_i$  from  $\vartheta$  and take  $\Gamma_\vartheta$  to be the graph that is obtained from  $\vartheta \setminus \cup_i D_i$  as a deformation retract. (This procedure of ‘punching holes’ is similar to the way in which the presence of a triangulation of the world sheet in the TFT construction of correlators [FRS1] is explained in [KaS] and Section 6 of [FuSV].)

(ii) The definition of full Frobenius graph applies also to the situation that the boundary  $\partial\vartheta$  contains a physical boundary labeled by some right  $A$ -module  $M$ , by regarding  $M$  as a defect line labeled by  $M$  taken as a  $\mathbf{1}$ - $A$ -bimodule.

(iii) If the two-cell  $\vartheta$  is labeled by the monoidal unit  $\mathbf{1} \in \mathcal{C}$ , like e.g. any of the additional two-cells that result from turning physical boundaries into defect lines, then each edge of a full  $\mathbf{1}$ -graph on  $\vartheta$  is labeled by  $\mathbf{1}$  and each vertex by an identity morphism. As a consequence the graph is *transparent* and may be replaced by the empty graph, whereby one deals with a situation that has already been discussed, from a somewhat different point of view, in [ScY].

Next we note that to any boundary circle  $b$  of a world sheet  $\mathcal{S}$  without physical boundaries we can naturally associate two string nets on the cylinder over the circle  $b$ : First, the string net  $[\Gamma_{\mathfrak{B}}^\circ]$  that directly results from taking the cylinder over the circle  $b$ , with boundary value  $\mathbf{B}^\circ \in \mathcal{Cyl}^\circ(\mathcal{C}, b)$  given by the boundary datum  $\mathfrak{B}$  on  $b$ , like in the construction of the partial defect network  $\Gamma_{\mathfrak{B}}$ . And second, the string net

$$p_b := [\Gamma_{\mathfrak{B}}^\circ \cup \cup_\vartheta P(\vartheta)] \quad (3.63)$$

on the so obtained cylinder that is obtained by inserting into each of its two-cells  $\vartheta$  the graph  $P(\vartheta)$  that represents the idempotent for the tensor product over the Frobenius algebra  $A(\vartheta)$  which labels the one-cell on  $b$  adjacent to  $\vartheta$ . (In the simplified graphical calculus,  $P(\vartheta)$  consists of a single  $A(\vartheta)$ -line, see the picture (A.53).) The string net  $p_b$  is an idempotent in  $\mathcal{Cyl}^\circ(\mathcal{C}, b)$  and thus defines an object

$$\mathbf{B} = (\mathbf{B}^\circ, p_b) \quad (3.64)$$

in the cylinder category  $\mathcal{Cyl}(\mathcal{C}, b)$ . The image of  $\mathbf{B}$  under the equivalence (3.16) between  $\mathcal{Cyl}(\mathcal{C}, b)$  and the Drinfeld center is isomorphic to the field insertion for the circle  $b$ ,

$$\text{Im}(\Phi(p_b)) \cong \mathbb{F}(b). \quad (3.65)$$

**Example 3.24.** An example already encountered before is the idempotent  $p_{X,Y}$  in (3.21), with image (3.22). Another example for an idempotent  $p_b$  obtained this way from a boundary circle  $b$  is shown in the following picture:

$$b = \begin{array}{c} X \\ \bullet \\ \text{---} \\ \bullet \\ Z \end{array} \quad \Longrightarrow \quad p_b = \begin{array}{c} X \\ \uparrow \\ \text{---} \\ \downarrow \\ Z \end{array} \quad (3.66)$$

Here the Frobenius algebra labels of the one-cells of  $b$  are indicated by different colors.

If  $\mathcal{S}$  has physical boundaries,  $p_b$  can be defined analogously, after first replacing  $\mathcal{S}$  by the complemented world sheet  $\tilde{\mathcal{S}}$ .

Combining this observation with Proposition 3.11 (and recalling the isomorphism (2.42)) directly gives

**Corollary 3.25.** *Let  $\mathcal{S}$  be a world sheet without physical boundaries. Then there is an isomorphism*

$$\text{Bl}_c(\mathcal{S}) \cong \text{SN}_c(\Sigma_{\mathcal{S}}, \mathbf{B}) \quad (3.67)$$

of vector spaces intertwining the action of the mapping class group, where  $\Sigma_{\mathcal{S}}$  is the surface underlying  $\mathcal{S}$  and the boundary value  $\mathbf{B}$  is provided by the collection of boundary data of  $\mathcal{S}$ .

Below we will regard a correlator  $\text{Cor}(\mathcal{S})$  for the world sheet  $\mathcal{S}$  as a correlator  $\text{Cor}(\tilde{\mathcal{S}}) \in \text{Bl}(\tilde{\mathcal{S}})$  for the complemented world sheet  $\tilde{\mathcal{S}}$ . In view of Corollary 3.25 we can regard  $\text{Cor}(\mathcal{S})$  as an element of the string-net space  $\text{SN}_c(\Sigma, \mathbf{B})$  with  $\Sigma = \Sigma_{\mathcal{S}}$  the surface underlying  $\mathcal{S}$ , respectively, in the case of world sheets with physical boundaries, with  $\Sigma = \Sigma_{\tilde{\mathcal{S}}}$  the surface underlying the complemented world sheet  $\tilde{\mathcal{S}}$ .

**Definition 3.26.** (i) Let  $\mathcal{S}$  be a world sheet without physical boundaries. Let  $\Sigma_{\mathcal{S}}$  be the surface underlying  $\mathcal{S}$  and  $\mathbf{B}$  be the boundary value for  $\Sigma_{\mathcal{S}}$  that is provided by the collection of boundary data of  $\mathcal{S}$ . The *string-net correlator* for  $\mathcal{S}$  is the string net

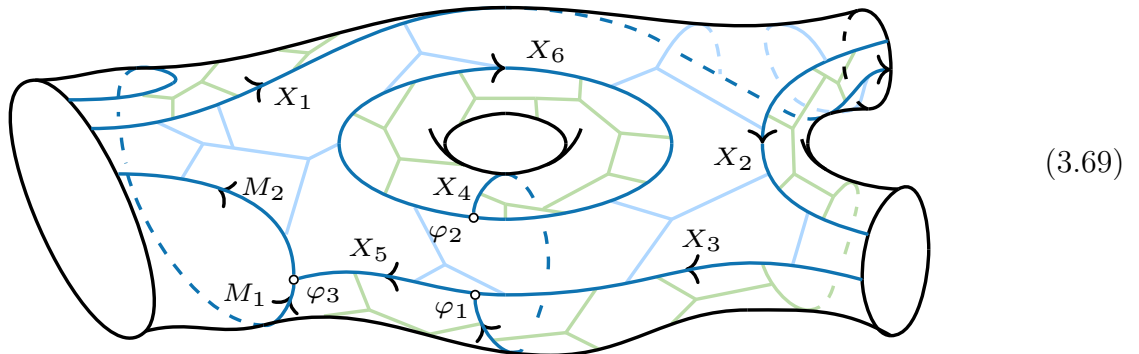
$$\text{Cor}_{\text{SN}}(\mathcal{S}) := \left[ \Gamma_{\mathcal{S}} \cup \bigcup_{\vartheta \subset \mathcal{S}} \Gamma_{\vartheta} \right] \in \text{SN}_c(\Sigma_{\mathcal{S}}, \mathbf{B}) \quad (3.68)$$

that is obtained as the union of the partial defect network (3.52) with full Frobenius graphs on all two-cells  $\vartheta$  of  $\mathcal{S}$ .

(ii) Let  $\mathcal{S}$  be a world sheet with physical boundaries and  $\tilde{\mathcal{S}}$  the complemented world sheet obtained from  $\mathcal{S}$  by turning physical boundaries into defect lines. Then the string-net correlator is obtained analogously as in (i), but with  $\tilde{\mathcal{S}}$  taking over the role of  $\mathcal{S}$ , i.e.  $\text{Cor}_{\text{SN}}(\mathcal{S}) := [\tilde{\Gamma}_{\tilde{\mathcal{S}}} \cup \bigcup_{\vartheta \subset \tilde{\mathcal{S}}} \tilde{\Gamma}_{\vartheta}]$  with  $\tilde{\Gamma}_{\tilde{\mathcal{S}}}$  the graph embedded in  $\tilde{\mathcal{S}}$  that results from the partial defect network  $\tilde{\Gamma}_{\tilde{\mathcal{S}}}$ .

Note that the definition of  $\text{Cor}_{\text{SN}}(\mathcal{S})$  involves a choice of full Frobenius graph on each two-cell of  $\mathcal{S}$ . However, by Lemma 3.22 any two such choices are related by a Frobenius move. Now each Frobenius move corresponds to an equality of morphisms in  $\mathcal{C}$ , and thus also to an equality of string nets. The correlator  $\text{Cor}_{\text{SN}}(\mathcal{S})$  therefore does not depend on these choices.

**Example 3.27.** The string-net correlator obtained for the world sheet  $\mathcal{S}$  displayed in (2.14), for which  $\tilde{\mathcal{S}}$  is given by (3.55), is represented by the string net



One of the main results of this paper is that the string nets (3.68) indeed deserve to be referred to as correlators:

**Theorem 3.28.** *Assigning to any world sheet  $\mathcal{S}$  the string-net correlator  $\text{Cor}_{\text{SN}}(\mathcal{S})$  provides a consistent system of correlators in the sense of Definition 2.19.*

*Proof.* (i) Invariance under the mapping class group  $\text{Map}(\mathcal{S})$ : By definition, any  $\gamma \in \text{Map}(\mathcal{S})$  maps the partial defect network  $I_{\mathcal{S}}$  on  $\Sigma_{\mathcal{S}}$  to itself. Thus when studying the correlator (3.68) we only have to deal with the full Frobenius graphs  $I_{\vartheta}$  on all two-cells  $\vartheta \subseteq \mathcal{S}$ . Let thus  $I_{\vartheta}$  be such a graph and  $\gamma \in \text{Map}(\mathcal{S})$ . Then the graph  $\gamma(I_{\vartheta})$  is clearly again a Frobenius graph; the following consideration shows that it is even a full Frobenius graph: Let  $(\gamma(I_{\vartheta}))^+$  be a Frobenius graph obtained by adding an edge to  $\gamma(I_{\vartheta})$ . Then the graph  $\gamma^{-1}((\gamma(I_{\vartheta}))^+)$  can be obtained from  $I_{\vartheta}$  by adding an edge. Since  $I_{\vartheta}$  is full, it is related to  $\gamma^{-1}((\gamma(I_{\vartheta}))^+)$  by some Frobenius move. Upon applying  $\gamma$ , that move transports to a Frobenius move that relates  $\gamma(I_{\vartheta})$  to  $(\gamma(I_{\vartheta}))^+$ . Hence both  $I_{\vartheta}$  and  $\gamma(I_{\vartheta})$  are full Frobenius graphs; by Lemma 3.22 they are thus related by a Frobenius move. Since, as already pointed out, a Frobenius move corresponds to an equality of string nets, this implies that the two graphs on  $\Sigma_{\mathcal{S}}$  that are related by replacing  $I_{\vartheta}$  by  $\gamma(I_{\vartheta})$  represent one and the same string net. Altogether we then have

$$\gamma(\text{Cor}_{\text{SN}}(\mathcal{S})) = \left[ \gamma(I_{\mathcal{S}} \cup \bigcup_{\vartheta \subseteq \mathcal{S}} I_{\vartheta}) \right] = \text{Cor}_{\text{SN}}(\mathcal{S}) \quad (3.70)$$

for every  $\gamma \in \text{Map}(\mathcal{S})$ .

(ii) Compatibility with sewing: Without loss of generality we can restrict our attention to what happens when two two-cells  $\vartheta'$  and  $\vartheta''$ , both labeled by the same Frobenius algebra, are sewn to a single new two-cell  $\vartheta = \vartheta' \cup_{\alpha} \vartheta''$ , whereby the full Frobenius graphs  $I_{\vartheta'}$  and  $I_{\vartheta''}$  combine to a graph  $I_{\vartheta'} \cup_{\alpha} I_{\vartheta''} := \Gamma$ . The graph  $\Gamma$  is clearly a Frobenius graph; we must show that it is even a full Frobenius graph. To this end, we add an edge to  $\Gamma$ , resulting in a new graph  $\Gamma^+$ , and show that  $\Gamma^+$  is related to  $\Gamma$  by a Frobenius move. If the new edge of  $\Gamma^+$  lies entirely in either  $\vartheta'$  or  $\vartheta''$  (regarded as embedded in  $\vartheta$ ), then with obvious notation we have either  $\Gamma^+ = I_{\vartheta'}^+ \cup_{\alpha} I_{\vartheta''}$  or  $\Gamma^+ = I_{\vartheta'} \cup_{\alpha} I_{\vartheta''}^+$ , so that the statement follows immediately, just because  $I_{\vartheta'}$  and  $I_{\vartheta''}$  are both full. Otherwise, i.e. if the new edge lies partly in  $\vartheta'$  and partly in  $\vartheta''$ , we can perform a suitable Frobenius move together with isotopy to transform the graph  $\Gamma$  in such a way that we deal again with the previous situation.

(iii) In part (ii) above it is actually implicitly assumed that neither of the two two-cells  $\vartheta'$  and  $\vartheta''$  has arisen from complementing a world sheet. However, the reasoning still applies to the case of two-cells that result from complementation: any full Frobenius graph in such a two-cell is transparent, so that compatibility with sewing reduces to the fact that the processes of complementation and of sewing commute.  $\square$

## 4 Correlators of particular interest

### 4.1 Vertical operator product

Consider a gluing circle  $b$  with two zero-cells, to which there are attached an incoming defect line of type  $G_1$  and an outgoing defect line of type  $G_2$ . Recall from Section 2.3.1 that the field

$\mathbb{F}(b) \in \mathcal{Z}(\mathcal{C})$  assigned to  $b$  is called a *defect field* and is denoted by  $\mathbb{D}^{G_1, G_2}$ , and that  $G_i$  are module functors between  $\mathcal{C}$ -modules  $\mathcal{M}$  and  $\mathcal{M}'$ .

Let us restrict our attention to the case that the module categories  $\mathcal{M}$  and  $\mathcal{M}'$  are indecomposable. Then there are connected special symmetric Frobenius algebras  $A$  and  $A'$  in  $\mathcal{C}$  such that  $\mathcal{M} \simeq \text{mod-}A$  and  $\mathcal{M}' \simeq \text{mod-}A'$ , and  $A$ - $A'$ -bimodules  $X_1$  and  $X_2$  such that  $G_i$  is isomorphic to the functor  $G^{X_i} = - \otimes_A X_i$  for  $i \in \{1, 2\}$  (see Appendix A.3). We abbreviate  $\mathbb{D}^{G^{X_1}, G^{X_2}}$  as  $\mathbb{D}^{X_1, X_2}$ . Using (2.24) and (A.28), the defect field  $\mathbb{D}^{X_1, X_2}$  can thus be written as

$$\begin{aligned} \mathbb{D}^{X_1, X_2} &= \underline{\text{Nat}}(G^{X_1}, G^{X_2}) = \int_{M \in \text{mod-}A} \underline{\text{Hom}}(M \otimes_A X_1, M \otimes_A X_2) \\ &\cong \left( \bigoplus_{m \in \mathcal{I}(\text{mod-}A)} m \otimes_A X_2 \otimes_{A'} X_1^\vee \otimes_A m^\vee, \gamma_{\mathbb{D}^{X_1, X_2}} \right) \in \mathcal{Z}(\mathcal{C}). \end{aligned} \quad (4.1)$$

The half-braiding  $\gamma_{\mathbb{D}^{X_1, X_2}}$  is obtained from the universal coaction of the central monad of  $\mathcal{M}$  as described in (A.38); graphically,

$$\gamma_{\mathbb{D}^{X_1, X_2}; Y} := \bigoplus_{m, n \in \mathcal{I}(\text{mod-}A)} d_m \quad \begin{array}{c} Y \\ \uparrow \\ \alpha \\ \uparrow \\ m \\ \downarrow \\ X_2 \end{array} \quad \begin{array}{c} n \\ \uparrow \\ X_1 \\ \downarrow \\ n \\ \downarrow \\ m \\ \downarrow \\ Y \end{array} \quad \text{for } Y \in \mathcal{C}. \quad (4.2)$$

Here a summation convention analogous to (A.46) is used, but with the implicit  $\alpha$ -summation being only over module morphisms,

A crucial feature of a full conformal field theory are the *operator products* among the various types of fields. In the case of defect fields, there are two ways of forming a product, either along a given defect line, or such that a fusion of two defect lines is involved. In terms of the description (2.29) of defect fields, these correspond to the vertical and horizontal composition of internal natural transformations, respectively [FuS3]. The vertical and horizontal products coincide if and only if  $\mathcal{M} = \mathcal{M}' = \mathcal{C}$  and  $G_1 = G_2 = \text{Id}_{\mathcal{C}}$  is the identity functor for  $\mathcal{C}$  as a bimodule over itself; the corresponding special defect field  $\mathbb{D}^{\text{Id}, \text{Id}} \in \mathcal{Z}(\mathcal{C})$  is known as the object of *bulk fields*. The purpose of the present subsection is to exhibit how the vertical product of defect fields can be related to the string-net construction of correlators; in the next subsection we will do the same for the horizontal product.

Thinking of defect fields in terms of two-pronged defect junctions, as already visualized in the picture (2.28), the situation on the world sheet that is relevant for the vertical composition amounts to a sewing operation, according to

$$s \left( \begin{array}{c} \text{Green} \\ \uparrow X_3 \\ \bigcirc \\ \downarrow X_1 \\ \text{Blue} \end{array} \sqcup \begin{array}{c} \text{Green} \\ \uparrow X_3 \\ \bigcirc \\ \downarrow X_2 \\ \bigcirc \\ \downarrow X_1 \\ \text{Blue} \end{array} \right) = \begin{array}{c} \text{Green} \\ \uparrow X_3 \\ \bigcirc \\ \downarrow X_2 \\ \bigcirc \\ \downarrow X_1 \\ \text{Blue} \end{array} \quad (4.3)$$

It is worth stressing that a string-net correlator as constructed in Section 3.5 is directly assigned to a world sheet  $\mathcal{S}$ , via the underlying surface  $\Sigma_{\mathcal{S}}$  and the boundary value for  $\Sigma_{\mathcal{S}}$  that comes from the boundary data of  $\mathcal{S}$ . In particular it does not require the choice of any auxiliary data such as, say, a fine marking of the surface  $\mathcal{S}$  (the latter is, for example, needed in the Lego-Teichmüller based approach of [FuS1]). In contrast, relating algebraic data – like the compositions of internal natural transformations that formalize operator products – to correlators does require such auxiliary data. Via the string-net construction of correlators one thus achieves a more invariant description of operator products than what can be seen based on the underlying purely algebraic structures alone.

The vertical composition

$$\underline{\mu}_{\text{ver}} \equiv \underline{\mu}_{\text{ver}}(X_1, X_2, X_3) \in \text{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathbb{D}^{X_2, X_3} \otimes \mathbb{D}^{X_1, X_2}, \mathbb{D}^{X_1, X_3}) \quad (4.4)$$

of internal natural transformations is nothing but a particular instance of the canonical product of internal Homs [FuS2, Def. 23]. Just like for ordinary natural transformations it amounts to the composition of components, according to

$$\underline{\mu}_{\text{ver}} := \bigoplus_{m \in \mathcal{I}(\mathcal{M})} \begin{array}{c} \begin{array}{c} m \ X_3 \\ \uparrow \uparrow \\ \text{---} \text{---} \\ \uparrow \uparrow \\ X_2 \ m \quad X_1 \ m \end{array} \end{array} \in \text{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathbb{D}^{X_2, X_3} \otimes \mathbb{D}^{X_1, X_2}, \mathbb{D}^{X_1, X_3}). \quad (4.5)$$

To relate this morphism to a string-net correlator, take the surface to be a *pair of pants*  $\Sigma_{\text{p.o.p.}}$  – a sphere with three boundary circles, two of them regarded as incoming and one as outgoing – and fix a marking  $w$  without cuts (in the sense of [FuS1, Def. 2.3]) on  $\Sigma_{\text{p.o.p.}}$ . Given these data, we will specify a world sheet  $\mathcal{S}_w^{\text{ver}} \equiv \mathcal{S}_w^{\text{ver}}(X_1, X_2, X_3)$  with underlying surface  $\Sigma_{\text{p.o.p.}}$  and a boundary value  $\mathbf{B}_w$  on  $\Sigma_{\text{p.o.p.}}$  that comes from boundary data for  $\mathcal{S}_w^{\text{ver}}$ , as well as a linear isomorphism

$$\varphi_w : \text{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathbb{D}^{X_2, X_3} \otimes \mathbb{D}^{X_1, X_2}, \mathbb{D}^{X_1, X_3}) \xrightarrow{\cong} \text{SN}_{\mathcal{C}}(\Sigma_{\text{p.o.p.}}, \mathbf{B}_w), \quad (4.6)$$

in such a way that  $\varphi_w$  maps the vertical composition (4.5) to the correlator  $\text{Cor}_{\text{SN}}(\mathcal{S}_w^{\text{ver}})$  that the string-net construction yields for the world sheet  $\mathcal{S}_w^{\text{ver}}$ , i.e.

$$\varphi_w(\underline{\mu}_{\text{ver}}) = \text{Cor}_{\text{SN}}(\mathcal{S}_w^{\text{ver}}). \quad (4.7)$$

We first present the isomorphism (4.6) for a particular marking  $w$ ; the prescription for any other marking without cuts is then obtained via an action of the mapping class group, as will be described in more detail below. Moreover, for describing the isomorphism (4.6) it is convenient to split it up into the composition of two simpler isomorphisms: In a first step, we consider an isomorphism

$$\varphi_w^{\text{can}} : \text{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathbb{D}^{X_2, X_3} \otimes \mathbb{D}^{X_1, X_2}, \mathbb{D}^{X_1, X_3}) \xrightarrow{\cong} \text{SN}_{\mathcal{C}}(\Sigma_{\text{p.o.p.}}, \mathbf{B}_w^{\text{can}}) \quad (4.8)$$



to the string-net space for a pair of pants with a different boundary value  $\mathbf{B}_w^{\text{can}}$ , namely one that involves the boundary value  $\mathbf{B}_Y^{\text{can}}$ , as defined in (3.19), for  $Y \in \mathcal{Z}(\mathcal{C})$  being the object  $\mathbb{D}^{X_1, X_2}$ ,  $\mathbb{D}^{X_2, X_3}$  and  $\mathbb{D}^{X_1, X_3}$ , respectively. The second step then consists of implementing the isomorphism (3.29) between  $\mathbf{B}_{\mathbb{D}^{X, Y}}^{\text{can}}$  and the boundary value  $\mathbf{B}_{X, Y}$  defined in (3.28).

For a standard pair of pants  $\Sigma_{\text{p.o.p.}}$ , which we draw as a disk with two holes, the marking on  $\Sigma_{\text{p.o.p.}}$  of our choice is

$$w := \text{[Diagram: A large circle containing two smaller circles. A dashed red line starts from the top boundary, goes down to a junction point between the two inner circles, and then splits into two lines going to each inner circle.]}$$
(4.9)

The corresponding boundary value  $\mathbf{B}_w^{\text{can}}$  is

$$\mathbf{B}_w^{\text{can}} = (\mathbf{B}_{\mathbb{D}^{X_2, X_3}}^{\text{can}})^\vee \boxtimes (\mathbf{B}_{\mathbb{D}^{X_1, X_2}}^{\text{can}})^\vee \boxtimes \mathbf{B}_{\mathbb{D}^{X_1, X_3}}^{\text{can}} .$$
(4.10)

The isomorphism  $\varphi_w^{\text{can}}$  is then defined by

$$\text{[Diagram: On the left, a green junction point 'g' with three outgoing lines labeled } \mathbb{D}^{X_1, X_3}, \mathbb{D}^{X_2, X_3}, \text{ and } \mathbb{D}^{X_1, X_2}. \text{ An arrow labeled } \varphi_w^{\text{can}} \text{ points to the right. On the right, a large circle with two inner circles. The junction point 'g' is inside, with green lines connecting it to the inner circles and the outer boundary. The lines are labeled } \mathbb{D}^{X_1, X_3}, \mathbb{D}^{X_2, X_3}, \text{ and } \mathbb{D}^{X_1, X_2}. \text{ The inner circles and the boundary are outlined in purple.]}$$
(4.11)

Hereby the vertical composition  $\underline{\mu}_{\text{ver}}$  is mapped to the string net

$$\begin{aligned}
\varphi_w^{\text{can}}(\underline{\mu}_{\text{ver}}) &= \bigoplus_{\substack{i,j,k \in \mathcal{I}(\mathcal{C}) \\ m,n,p,q \in \mathcal{I}(-\text{mod}A)}} \frac{d_i d_j d_k d_m d_n d_p}{D_{\mathcal{C}}^6} \quad \text{(Diagram 1)} \\
&= \sum_{m,n,q \in \mathcal{I}(-\text{mod}A)} \frac{d_m d_n}{D_{\mathcal{C}}^4} \quad \text{(Diagram 2)}
\end{aligned} \tag{4.12}$$

in  $\text{SN}_{\mathcal{C}}(\Sigma_{\text{p.o.p.}}, \mathbf{B}_w^{\text{can}})$ . Here the second equality holds as a consequence of the identity (A.66) and Corollary A.7. (Also, in the first picture – and likewise in several other pictures later on – we omit, for lack of space, the summation labels of the module morphisms; the appropriate pairings are instead indicated by matching colors.) The second step in the construction of  $\varphi_w$  then amounts to setting

$$\varphi_w(-) := e_{X_1, X_3} \circ \varphi_w^{\text{can}}(-) \circ (r_{X_2, X_3} \boxtimes r_{X_1, X_2}), \tag{4.13}$$

with the morphisms  $e_{-, -}$  and  $r_{-, -}$  as defined in (3.24). Accordingly, the boundary value  $\mathbf{B}_w$  is given by

$$\mathbf{B}_w = (\mathbf{B}_{X_2, X_3})^\vee \boxtimes (\mathbf{B}_{X_1, X_2})^\vee \boxtimes \mathbf{B}_{X_1, X_3}. \tag{4.14}$$

Via (4.13), the string net (4.12) gets mapped to

$$\varphi_w(\underline{\mu}_{\text{ver}}) = \text{(Diagram 3)} \in \text{SN}_{\mathcal{C}}(\Sigma_{\text{p.o.p.}}, \mathbf{B}_w) \tag{4.15}$$

We can now read off that we have indeed achieved to express  $\varphi_w(\underline{\mu}_{\text{ver}})$  as a string-net correlator  $\text{Cor}_{\text{SN}}(\mathcal{S}_w^{\text{ver}})$ , namely as the one for the world sheet

$$\mathcal{S}_w^{\text{ver}} := \text{Diagram} \tag{4.16}$$

that is, a pair of pants with three defect lines labeled by  $\mathbb{D}^{X_2, X_3}$ ,  $\mathbb{D}^{X_1, X_2}$  and  $\mathbb{D}^{X_1, X_3}$ , respectively, which pairwise connect the boundary circles.

**Remark 4.1.** By a diffeomorphism, the world sheet (4.16) can be redrawn as follows:

$$\mathcal{S}_w^{\text{ver}} = \text{Diagram} \tag{4.17}$$

This way of presenting  $\mathcal{S}_w^{\text{ver}}$  slightly obscures its relevance for the vertical operator product. On the other hand, it clarifies the relation to other approaches: It makes it obvious that one deals with an ‘operator product along a defect line’, and it is precisely what commonly is called the *fundamental world sheet* for three defect fields on the sphere, see e.g. Section 4.5 of [FRS2].

We finally comment on the dependence of the construction on the marking of the surface. A different choice  $w'$  of a marking without cuts on the pair of pants leads to a string net which, in general, differs from (4.15). The two world sheets are related by the unique element  $\gamma_{w, w'}$  of the mapping class group of  $\Sigma$  that corresponds (see [FuS1, Sect. 3.1]) to a move of markings mapping  $w$  to  $w'$ . In particular, two markings  $w$  and  $w'$  without cut and with the same end points on the boundary circles give the same world sheet, and thus the same correlator, if and only if they are isotopic (in this case  $\gamma_{w, w'}$  belongs to both  $\text{Map}(\mathcal{S}_w^{\text{ver}})$  and  $\text{Map}(\mathcal{S}_{w'}^{\text{ver}})$ , and the two groups actually coincide).

As a particular case, any two markings without cuts give the same correlator when each of the three defect fields is actually a bulk field, i.e. when  $X_1 = X_2 = X_3 = A = A'$ .

## 4.2 Horizontal operator products

We now analyze the horizontal composition of defect fields, in analogy with the study of the vertical composition in Section 4.1. Horizontal composition happens in combination with the

*fusion of defect lines*, in which two parallel segments of defect lines get replaced by a single one. That such a fusion process is possible is an integrated ingredient of the description of defect lines in other approaches to CFT. In the present setting, fusion is algebraically realized as the tensor product over the relevant Frobenius algebra: the fusion of two defect lines that are labeled by an  $A$ - $A'$ -bimodule  $X$  and an  $A'$ - $A''$ -bimodule  $X'$ , respectively, yields a defect line labeled by the  $A$ - $A''$ -bimodule  $X \otimes_{A'} X'$ .

In the same way as done in (4.3) for the vertical operator product, the horizontal composition can be expressed as a specific sewing operation, according to

$$s \left( \begin{array}{c} \text{Large circle with 4 lines } X_1, X_2, X_3, X_4 \\ \square \\ \text{Small circle with 4 lines } X_1, X_2, X_3, X_4 \end{array} \right) = \begin{array}{c} \text{Large circle with 2 lines } X_1, X_2 \end{array} \quad (4.18)$$

When attempting to translate this picture into an expression for composition

$$\mathbb{D}^{X_2, X_4} \otimes \mathbb{D}^{X_1, X_3} \longrightarrow \mathbb{D}^{X_1 \otimes_{A'} X_2, X_3 \otimes_{A'} X_4} \quad (4.19)$$

of internal natural transformations, some care is needed. To explain the subtleties involved, it is convenient to recall first how the horizontal composition of *ordinary* natural transformations is described in components: The horizontal product of two natural transformations  $d^{G_1, G_3}$  between functors  $G_1, G_3: \mathcal{M} \rightarrow \mathcal{M}'$  and  $d^{G_2, G_4}$  between functors  $G_2, G_4: \mathcal{M}' \rightarrow \mathcal{M}''$  – a natural transformation from  $G_2 \circ G_1$  to  $G_4 \circ G_3$ , which are functors from  $\mathcal{M}$  to  $\mathcal{M}''$  – amounts to a suitable composition of their components, i.e. of morphisms  $d_M^{G_1, G_3} \in \text{Hom}_{\mathcal{M}'}(G_1(M), G_3(M))$  and  $d_{M'}^{G_2, G_4} \in \text{Hom}_{\mathcal{M}''}(G_2(M'), G_4(M'))$ , respectively. In more detail, the composition can be expressed both as

$$G_2 \circ G_1(M) \xrightarrow{G_2(d_M^{G_1, G_3})} G_2 \circ G_3(M) \xrightarrow{d_{G_3(M)}^{G_2, G_4}} G_4 \circ G_3(M), \quad (4.20)$$

and as

$$G_2 \circ G_1(M) \xrightarrow{d_{G_1(M)}^{G_2, G_4}} G_4 \circ G_1(M) \xrightarrow{G_4(d_M^{G_1, G_3})} G_4 \circ G_3(M). \quad (4.21)$$

Equality of the two composite morphisms (4.20) and (4.21) for all  $M \in \mathcal{M}$  holds by naturality of  $d^{G_2, G_4}$ . Now in the case of *internal* natural transformations  $\mathbb{D}^{X_1, X_3}$  and  $\mathbb{D}^{X_2, X_4}$ , the components are internal Homs, and accordingly it is appropriate to interpret the dinatural structure morphisms

$$J_M \equiv J_M^{F, F'} : \underline{\text{Nat}}(F, F') \longrightarrow \underline{\text{Hom}}_{\mathcal{N}}(F(M), F'(M)). \quad (4.22)$$

of the end (2.24) as the ‘projection to components’. The analogues of the morphisms (4.20) and (4.21) are then the composites

$$\begin{aligned} & \underline{\text{Nat}}(G_2, G_4) \otimes \underline{\text{Nat}}(G_1, G_3) \\ & \xrightarrow{J_{G_3(M)}^{G_2, G_4} \otimes J_M^{G_1, G_3}} \underline{\text{Hom}}(G_2 \circ G_3(M), G_4 \circ G_3(M)) \otimes \underline{\text{Hom}}(G_1(M), G_3(M)) \\ & \xrightarrow{\text{id} \otimes \underline{G}_2} \underline{\text{Hom}}(G_2 \circ G_3(M), G_4 \circ G_3(M)) \otimes \underline{\text{Hom}}(G_2 \circ G_1(M), G_2 \circ G_3(M)) \\ & \xrightarrow{\underline{\iota}} \underline{\text{Hom}}(G_2 \circ G_1(M), G_4 \circ G_3(M)) \end{aligned} \quad (4.23)$$

and

$$\begin{aligned}
& \underline{\text{Nat}}(G_1, G_3) \otimes \underline{\text{Nat}}(G_2, G_4) \\
& \xrightarrow{j_M^{G_1, G_1} \otimes j_{G_1(M)}^{G_2, G_4}} \underline{\text{Hom}}(G_1(M), G_3(M)) \otimes \underline{\text{Hom}}(G_2 \circ G_1(M), G_4 \circ G_1(M)) \\
& \xrightarrow{\underline{G}_4 \otimes \text{id}} \underline{\text{Hom}}(G_4 \circ G_1(M), G_4 \circ G_3(M)) \otimes \underline{\text{Hom}}(G_2 \circ G_1(M), G_4 \circ G_1(M)) \\
& \xrightarrow{\underline{\mu}} \underline{\text{Hom}}(G_2 \circ G_1(M), G_4 \circ G_3(M))
\end{aligned} \tag{4.24}$$

for  $M \in \mathcal{M}$ , respectively. Here  $\underline{\mu}$  is the standard multiplication (A.23) on internal Homs, while the morphisms  $\underline{G}_2$  and  $\underline{G}_4$  are defined as in (A.25).

It is straightforward to see that both of the families (4.23) and (4.24) are dinatural in  $m \in \mathcal{M}$ . Owing to the universal property of  $\underline{\text{Nat}}$  as an end, they thus factorize to morphisms

$$\underline{\mu}_{\text{hor}}^l : \underline{\text{Nat}}(G_2, G_4) \otimes \underline{\text{Nat}}(G_1, G_3) \longrightarrow \underline{\text{Nat}}(G_2 \circ G_1, G_4 \circ G_3) \tag{4.25}$$

and

$$\underline{\mu}_{\text{hor}}^r : \underline{\text{Nat}}(G_1, G_3) \otimes \underline{\text{Nat}}(G_2, G_4) \longrightarrow \underline{\text{Nat}}(G_2 \circ G_1, G_4 \circ G_3), \tag{4.26}$$

respectively. It can further be shown that (4.25) and (4.26), which are defined as morphisms in  $\mathcal{C}$ , are in fact even morphisms in  $\mathcal{Z}(\mathcal{C})$ . When expressing the module functors  $G_i$  through bimodules  $X_i$  as in (2.25), we can write the morphisms (4.25) and (4.26) in terms of defect fields as

$$\underline{\mu}_{\text{hor}}^l : \mathbb{D}^{X_2, X_4} \otimes \mathbb{D}^{X_1, X_3} \longrightarrow \mathbb{D}^{X_1 \otimes_{A'} X_2, X_3 \otimes_{A'} X_4} \tag{4.27}$$

and

$$\underline{\mu}_{\text{hor}}^r : \mathbb{D}^{X_1, X_3} \otimes \mathbb{D}^{X_2, X_4} \longrightarrow \mathbb{D}^{X_1 \otimes_{A'} X_2, X_3 \otimes_{A'} X_4} \tag{4.28}$$

respectively.

We call the morphisms  $\underline{\mu}_{\text{hor}}^l$  and  $\underline{\mu}_{\text{hor}}^r$  the left and right horizontal operator product. The reason for this choice of terminology is the description of (4.27) and (4.28) in terms of (ordinary, non-cyclic) string diagrams, in which the basis morphisms that are summed over are located in the left and right half of the graph, respectively: we have

$$\underline{\mu}_{\text{hor}}^l = \bigoplus_{\substack{m \in \mathcal{I}(-\text{mod } A') \\ n \in \mathcal{I}(-\text{mod } A)}} d_m \tag{4.29}$$

and

$$\underline{\mu}_{\text{hor}}^r = \bigoplus_{\substack{m \in \mathcal{I}(-\text{mod } A') \\ n \in \mathcal{I}(-\text{mod } A)}} d_m \quad (4.30)$$

Let us now express these two compositions through string nets. Recall from Section 4.1 that this requires the specification of auxiliary data beyond the structure of a world sheet, concretely the choice of a marking without cuts. Note that these auxiliary data do not appear in the purely algebraic treatment of horizontal composition that is given in [FuS2]. (Also, in our discussion of the horizontal composition we partly deviate from the exposition in [FuS2].)

In the same vein as done for the vertical composition  $\underline{\mu}_{\text{ver}}$ , we will now determine the image of the morphism  $\underline{\mu}_{\text{hor}}^1 \in \text{Hom}_{\mathcal{Z}(C)}(\mathbb{D}^{X_2, X_4} \otimes \mathbb{D}^{X_1, X_3}, \mathbb{D}^{X_1 \otimes_{A'} X_2, X_3 \otimes_{A'} X_4})$  under the composite map

$$\begin{aligned} \varphi_w : \quad \text{Hom}_{\mathcal{Z}(C)}(\mathbb{D}^{X_2, X_4} \otimes \mathbb{D}^{X_1, X_3}, \mathbb{D}^{X_1 \otimes_{A'} X_2, X_3 \otimes_{A'} X_4}) &\xrightarrow{\varphi_w^{\text{can}}} \text{SN}_C(\Sigma_{\text{p.o.p.}}, \mathbf{B}_w^{\text{can}}) \\ &\xrightarrow{e_{X_1 \otimes_{A'} X_2, X_3 \otimes_{A'} X_4} \circ (-) \circ (r_{X_2, X_4} \boxtimes r_{X_1, X_3})} \text{SN}_C(\Sigma_{\text{p.o.p.}}, \mathbf{B}_w), \end{aligned} \quad (4.31)$$

Here  $w$  is the marking (4.9) on the pair of pants  $\Sigma_{\text{p.o.p.}}$ , the isomorphism  $\varphi_w^{\text{can}}$  is defined analogously as in (4.11), and the boundary values are

$$\begin{aligned} \mathbf{B}_w^{\text{can}} &= (\mathbf{B}_{\mathbb{D}^{X_2, X_4}}^{\text{can}})^{\vee} \boxtimes (\mathbf{B}_{\mathbb{D}^{X_1, X_3}}^{\text{can}})^{\vee} \boxtimes \mathbf{B}_{\mathbb{D}^{X_1 \otimes_{A'} X_2, X_3 \otimes_{A'} X_4}}^{\text{can}} \\ \text{and } \mathbf{B}_w &= (\mathbf{B}_{X_2, X_4})^{\vee} \boxtimes (\mathbf{B}_{X_1, X_3})^{\vee} \boxtimes \mathbf{B}_{X_1 \otimes_{A'} X_2, X_3 \otimes_{A'} X_4} \end{aligned} \quad (4.32)$$

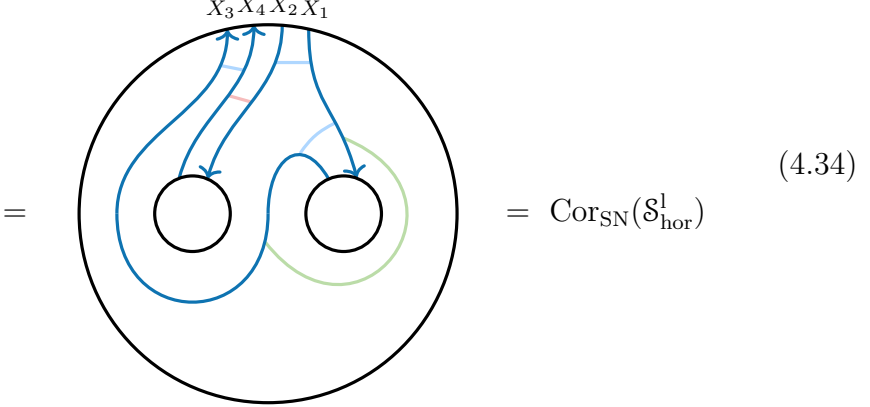
analogously as in (4.10) in (4.14).

By direct calculation we obtain

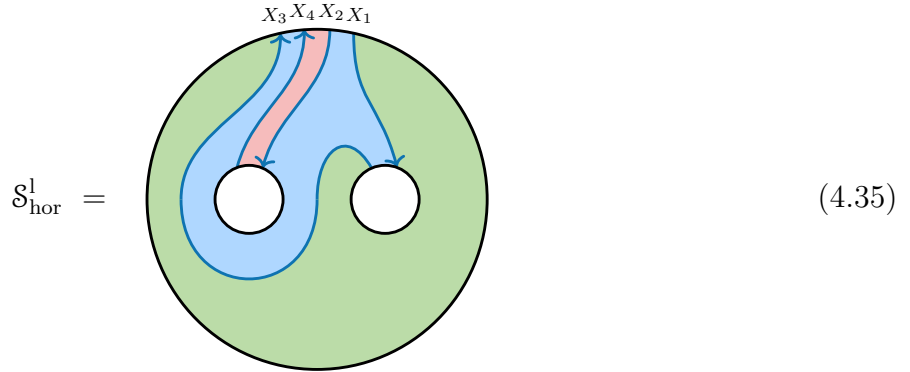
$$\begin{aligned}
 \varphi_w^{\text{can}}(\underline{\mu}_{\text{hor}}^1) &= \text{Diagram 1} \\
 &= \sum_{\substack{i,j,k \in \mathcal{I}(\mathcal{C}) \\ m,p \in \mathcal{I}(\text{mod-}A') \\ n,q,r \in \mathcal{I}(\text{mod-}A'')}} \frac{d_i d_j d_k d_m d_n d_p d_q}{D_{\mathcal{C}}^6} \text{Diagram 2} \\
 &= \sum_{\substack{m,p \in \mathcal{I}(\text{mod-}A') \\ n,q,r \in \mathcal{I}(\text{mod-}A'')}} \frac{d_m d_n d_p d_q}{D_{\mathcal{C}}^6} \text{Diagram 3} \\
 &= \sum_{\substack{m \in \mathcal{I}(\text{mod-}A') \\ n,r \in \mathcal{I}(\text{mod-}A'')}} \frac{d_m d_n}{D_{\mathcal{C}}^4} \text{Diagram 4}
 \end{aligned}
 \tag{4.33}$$

The diagrams are as follows:
   
 Diagram 1: A large circle containing two smaller circles. A central point is labeled  $\underline{\mu}_{\text{hor}}^1$ . Green arrows point from this point to the two smaller circles, labeled  $\mathbb{D}_{X_2, X_4}$  and  $\mathbb{D}_{X_1, X_3}$ . A larger green arrow points from the central point to the top boundary of the large circle, labeled  $\mathbb{D}^{X_1 \otimes_{A'} X_2, X_3 \otimes_{A'} X_4}$ .
   
 Diagram 2: The same setup as Diagram 1, but with blue paths. The paths are labeled  $r, X_3, X_4, X_2, X_1, r$  at the top. Points  $m, m$  and  $n, n$  are marked on the two smaller circles. Points  $i, j, k$  are marked on the large circle's boundary. Points  $p, q$  are marked on the paths.
   
 Diagram 3: Similar to Diagram 2, but with a different configuration of paths and points. Points  $m, n$  are on the smaller circles, and  $r$  is on the boundary. Points  $p, q$  are on the paths.
   
 Diagram 4: Similar to Diagram 3, but with a different configuration of paths and points. Points  $m, n$  are on the smaller circles, and  $r$  is on the boundary.

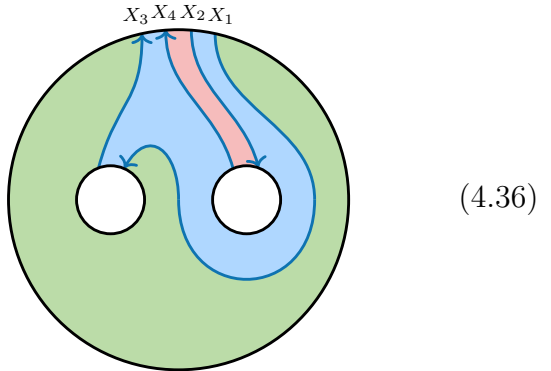
This implies that  $\varphi_w(\underline{\mu}_{\text{hor}}^1)$  gives the world sheet for the left horizontal composition as follows:

$$\varphi_w(\underline{\mu}_{\text{hor}}^1) = e_{X_1 \otimes_{A'} X_2, X_3 \otimes_{A'} X_4} \circ \varphi_w^{\text{can}}(\underline{\mu}_{\text{hor}}^1) \circ (r_{X_2, X_4} \boxtimes r_{X_1, X_3})$$


with



Likewise, the image of  $\underline{\mu}_{\text{hor}}^r$  under  $\varphi_w$  gives the world sheet for the right horizontal composition:

$$\varphi_w(\underline{\mu}_{\text{hor}}^r) = \text{Cor}_{\text{SN}}(\mathcal{S}_{\text{hor}}^r) \quad \text{with} \quad \mathcal{S}_{\text{hor}}^r =$$


**Remark 4.2.** (i) Up to isotopy we have

$$\mathcal{S}_{\text{hor}}^1 = \beta(\mathcal{S}_{\text{hor}}^r), \tag{4.37}$$

where  $\beta$  is the action of the *braid move* [BakK], which (when expressed in terms of markings)



acts as

(4.38)

As we will observe in Remark 5.5 below, this implies that the left and right horizontal compositions  $\underline{\mu}_{\text{hor}}^l$  and  $\underline{\mu}_{\text{hor}}^r$  merely differ by a half-braiding.

(ii) By applying suitable diffeomorphisms, both world sheets  $\mathcal{S}_{\text{hor}}^l$  and  $\mathcal{S}_{\text{hor}}^r$  may be redrawn as

(4.39)

which is the form familiar from the sewing operation shown in picture (4.18).

### 4.3 Bulk algebras

The most basic type of field insertion in the bulk is the one obtained for a gluing circle  $b = b_{\text{bulk}}^A$  of the following form:  $b = b_{\text{bulk}}^A$  has two zero-cells, and the two adjacent two-cells are both labeled by one and the same Frobenius algebra  $A$ , while the two defect lines attached to the zero-cells are both transparent, i.e. labeled by the functor  $G^A = - \otimes_A A$ , which is canonically isomorphic to the identity functor  $\text{Id}_{\text{mod-}A}$ . Thus  $b = b_{\text{bulk}}^A$  is a special case of the gluing circle describing defect fields, which has already been displayed in the picture (2.28):

(4.40)

The resulting field insertion is the bulk field object

$$\mathbb{D}^{A,A} \equiv \mathbb{F}(b_{\text{bulk}}^A) = \underline{\text{Nat}}(G^A, G^A) \cong \underline{\text{Nat}}(\text{Id}_{\text{mod-}A}, \text{Id}_{\text{mod-}A}) = \int_{M \in \text{mod-}A} \underline{\text{Hom}}(M, M). \quad (4.41)$$

The end  $\int_M \underline{\text{Hom}}(M, M)$  is the *full center*  $Z(A) \in \mathcal{Z}(\mathcal{C})$  [FröFRS1, Dav] of the algebra  $A \in \mathcal{C}$ . In the semisimple case of our interest, we thus have

$$\mathbb{D}^{A,A} \cong Z(A) = \left( \bigoplus_{m \in \mathcal{I}(\text{mod-}A)} m \otimes_A m^\vee, \gamma_{Z(A)} \right) \in \mathcal{Z}(\mathcal{C}). \quad (4.42)$$

The half-braiding is the one already described in (4.2), which now specializes to

$$\gamma_{Z(A);Y} = \bigoplus_{m,n \in \mathcal{I}(\text{mod-}A)} d_m \quad \text{for } Y \in \mathcal{C}. \quad (4.43)$$

As already mentioned in Section 2.3.1, the special case (4.42) of the defect fields (4.1) is known as the *bulk fields* of a CFT. For bulk fields, the vertical product  $\underline{\mu}_{\text{ver}}(A, A, A)$  as defined in (4.5) and the left and right horizontal products  $\underline{\mu}_{\text{hor}}^l(A, A, A)$  and  $\underline{\mu}_{\text{hor}}^r(A, A, A)$  as defined in (4.25) and in (4.26), respectively, all coincide. The canonical isomorphism  $\mathbb{D}^{A,A} \cong Z(A)$  transports each of them to the morphism

$$\mu_{Z(A)} = \bigoplus_{m \in \mathcal{I}(\text{mod-}A)} \quad (4.44)$$

By the same strategy that we already pursued in the more general case of defect fields we can, after fixing a marking  $w$  without cuts on the pair of pants  $\Sigma_{\text{p.o.p.}}$ , map the product (4.44) to the string-net correlator on a world sheet  $\mathcal{S}$ : as a special case of (4.7) we get

$$\varphi_w(\mu_{Z(A)}) = \text{Cor}_{\text{SN}}(\mathcal{S}) \in \text{SN}_{\mathcal{C}}(\Sigma_{\text{p.o.p.}}, \mathbf{B}_{Z(A),w}) \quad (4.45)$$

with

$$\mathcal{S} \equiv \mathcal{S}_w := \quad (4.46)$$

Note that the so obtained world sheet  $\mathcal{S}$  is nothing but any of the world sheets  $\mathcal{S}_w^{\text{ver}}$ ,  $\mathcal{S}_{\text{hor}}^1$  and  $\mathcal{S}_{\text{hor}}^r$  as displayed in (4.16), (4.35) and (4.36), respectively, each specialized to the case that the two two-cells are both labeled by the same Frobenius algebra and the three defect lines are all transparently labeled. Owing to their transparency the defect lines can be omitted from the world sheet, and the boundary value  $\mathbf{B}_{Z(A),w}$  – a choice of points on  $S^1 \sqcup S^1 \sqcup S^1$  at which the transparent defect lines start or end – is immaterial. For the same reason, the choice of marking  $w$  is actually irrelevant as well. It follows that  $\mu_{Z(A)}$  endows the object  $Z(A) \in \mathcal{Z}(\mathcal{C})$  with the structure of a commutative special symmetric Frobenius algebra, and that the correlator  $\text{Cor}_{\text{SN}}(\mathcal{S})$  is invariant under the mapping class group  $\text{Map}(\Sigma_{\text{p.o.p.}})$  of the three-holed sphere. Further, invoking Theorem 3.4 of [KonR], it follows that  $Z(A)$  is even a *modular* Frobenius algebra in the sense of [FuS1, Def. 4.9].

**Remark 4.3.** The Frobenius algebra  $Z(\mathbf{1})$  that is obtained when  $A$  is the monoidal unit  $\mathbf{1}$  of  $\mathcal{C}$  is the *Cardy bulk algebra* considered in [ScY]. It should be noted that in [ScY] a less natural boundary value  $\mathbf{B}_{\tilde{F}}^{\text{can}} = (S_{\tilde{F}}, p_{\tilde{F}}^{\text{can}})$ , was considered than here, consisting of a circle with a point labeled by  $\tilde{F}$  together with the idempotent

$$p_{\tilde{F}}^{\text{can}} := \text{Diagram} \tag{4.47}$$

This choice results in more complicated graphs than the ones appearing above, having additional edges around the boundary circles. However, the boundary values  $\mathbf{B}_{Z(A),w}$  and  $\mathbf{B}_{\tilde{F}}^{\text{can}}$  are in fact isomorphic, and as a consequence the present description and the one used in [ScY] indeed yield the same correlator of three bulk fields on the sphere.

## 4.4 Torus partition function

Next we consider the correlator  $\text{Cor}(\text{T})$  for a torus  $\text{T}$  without field insertions and without non-transparent defect lines. This correlator, commonly called the *torus partition function*, is of much interest. For instance, in rational CFT it allows one to directly read off the decomposition of the bulk field object  $\mathbb{D}^{A,A} = \mathbb{F}(b_{\text{bulk}}^A)$  into simple objects of  $\mathcal{Z}(\mathcal{C})$ . Also, modular invariance of  $\text{Cor}(\text{T})$  is an important constraint on the consistency of a full conformal field theory, to the extent that often it has even been assumed, erroneously, to be even a sufficient condition for consistency.

The string-net form of the torus partition function follows immediately from Definition 3.26:

we have

$$\text{Cor}_{\text{SN}}(\mathbb{T}) = \text{Diagram} \quad (4.48)$$

Using the expression (4.42) for the bulk field object  $\mathbb{D}^{A,A} \cong Z(A)$  together with Corollary A.7 and the identity (A.66), this can be rewritten as

$$\text{Cor}_{\text{SN}}(\mathbb{T}) = \bigoplus_{\substack{k \in \mathcal{I}(\mathcal{C}) \\ m \in \mathcal{I}(-\text{mod} A)}} \frac{d_k d_m}{D_{\mathcal{C}}^2} \text{Diagram} = \text{Diagram} \quad (4.49)$$

Moreover, since  $\mathcal{C}$  is semisimple the object  $Z(A) \in \mathcal{Z}(\mathcal{C}) \simeq \mathcal{C}^{\text{rev}} \boxtimes \mathcal{C}$  can be decomposed into a direct sum of simple objects of  $\mathcal{Z}(\mathcal{C})$  as

$$Z(A) \cong \bigoplus_{i,j \in \mathcal{I}(\mathcal{C})} Z(A)_{i,j} \otimes_{\mathbb{k}} \Xi_{\mathcal{C}}(i \boxtimes j). \quad (4.50)$$

Here  $\Xi_{\mathcal{C}}$  the equivalence (A.32) between  $\mathcal{C}^{\text{rev}} \boxtimes \mathcal{C}$  and  $\mathcal{Z}(\mathcal{C})$ , while  $Z(A)_{i,j}$  are the vector spaces

$$\begin{aligned} Z(A)_{i,j} &:= \text{Hom}_{\mathcal{Z}(\mathcal{C})}(\Xi_{\mathcal{C}}(i \boxtimes j), \underline{\text{Nat}}(G^A, G^A)) \\ &\cong \text{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\text{mod-}A, \text{mod-}A)}(\Xi_{\mathcal{C}}(i \boxtimes j) \triangleright G^A, G^A) \\ &\cong \text{Hom}_{A|A}(i \otimes^- A \otimes^+ j, A). \end{aligned} \quad (4.51)$$

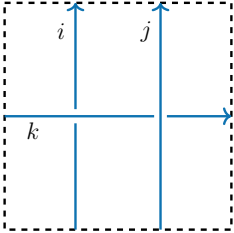
This in particular reproduces the decomposition rule (2.33) for bulk fields. For the torus partition function we thus get

$$\text{Cor}_{\text{SN}}(\mathbb{T}) = \sum_{i,j \in \mathcal{I}(\mathcal{C})} z(A)_{i,j} \text{Diagram} \quad (4.52)$$

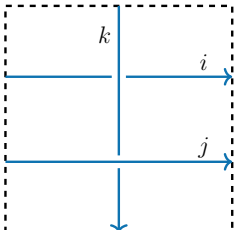
with  $z(A)_{i,j} := \dim_{\mathbb{C}}(Z(A)_{i,j})$ .

**Remark 4.4.** The full Frobenius graph in the string-net correlator (4.48) is the same as the one appearing in the TFT construction of  $\text{Cor}(\mathbb{T})$ . In that case, the torus in which the graph is embedded is the subset  $\{0\} \times \mathbb{T}$  of the three-manifold  $[-1, 1] \times \mathbb{T}$  (see [FRS1, Eq. (5.24)]). An analogous relationship between the two constructions holds for all other partition functions, i.e. for the correlator of any world sheet that neither has field insertions nor contains any physical boundaries or non-trivial defect lines.

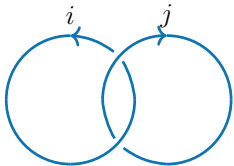
The expression (4.48) for  $\text{Cor}_{\text{SN}}(\mathbb{T})$  shows manifestly that the torus partition function is invariant under the geometric action of the modular group  $\text{Map}(\mathbb{T}) \cong \text{PSL}(2, \mathbb{Z})$  of the torus. The result (4.52) allows us to translate this geometric modular invariance to the algebraic modular invariance of the  $\mathcal{I}(\mathcal{C}) \times \mathcal{I}(\mathcal{C})$ -matrix  $z(A) = (z(A)_{i,j})$ . To see this, we first note that the set  $\{G_{i,j} \mid i, j \in \mathcal{I}(\mathcal{C})\}$  with

$$G_{i,j} := \sum_{k \in \mathcal{I}(\mathcal{C})} \frac{d_k}{D_{\mathcal{C}}^2} \quad (4.53)$$


is a basis of the string-net space  $\text{SN}_{\mathcal{C}}(\mathbb{T})$  [Ru, Prop. 4.8]. It is therefore sufficient to show that the image

$$S(G_{i,j}) = \sum_{k \in \mathcal{I}(\mathcal{C})} \frac{d_k}{D^2} \quad (4.54)$$


of  $G_{i,j}$  under the modular  $S$ -transformation satisfies

$$S(G_{i,j}) = \sum_{i',j' \in \mathcal{I}(\mathcal{C})} \frac{s_{i',i'} s_{j,j'}}{D_{\mathcal{C}}^2} G_{i',j'} \quad \text{with} \quad s_{i,j} := \quad (4.55)$$


The validity of (4.55) is established by the following chain of equalities:

$$\begin{aligned}
\sum_{i',j' \in \mathcal{I}(C)} \frac{s_{i',i'} s_{j,j'}}{D_C^2} G_{i',j'} &= \sum_{i',j',k \in \mathcal{I}(C)} \frac{s_{i',i'} s_{j,j'} d_k}{D_C^4} \begin{array}{|c|c|} \hline \uparrow i' & \uparrow j' \\ \hline \leftarrow k & \rightarrow \\ \hline \downarrow & \downarrow \\ \hline \end{array} \\
&= \sum_{i',j',k \in \mathcal{I}(C)} \frac{d_{i'} d_{j'} d_k}{D_C^4} \begin{array}{|c|c|} \hline \uparrow i' & \uparrow j' \\ \hline \leftarrow i & \rightarrow \\ \hline \downarrow & \downarrow \\ \hline \end{array} = \sum_{i',j',k,l \in \mathcal{I}(C)} \frac{d_{i'} d_{j'} d_k d_l}{D_C^4} \begin{array}{|c|c|} \hline \uparrow i' & \uparrow j' \\ \hline \leftarrow i & \rightarrow \\ \hline \downarrow & \downarrow \\ \hline \end{array} \\
&= \sum_{i',j',l \in \mathcal{I}(C)} \frac{d_{i'} d_{j'} d_l}{D_C^4} \begin{array}{|c|c|} \hline \uparrow i' & \uparrow j' \\ \hline \leftarrow i & \rightarrow \\ \hline \downarrow & \downarrow \\ \hline \end{array} = \sum_{i',j',l,m \in \mathcal{I}(C)} \frac{d_{i'} d_{j'} d_l d_m}{D_C^4} \begin{array}{|c|c|} \hline \uparrow m & \uparrow i \\ \hline \leftarrow i' & \rightarrow \\ \hline \downarrow & \downarrow \\ \hline \end{array} \\
&= \sum_{i',l,m \in \mathcal{I}(C)} \frac{d_{i'} d_l d_m}{D_C^4} \begin{array}{|c|c|} \hline \uparrow m & \uparrow i \\ \hline \leftarrow i' & \rightarrow \\ \hline \downarrow & \downarrow \\ \hline \end{array} = \sum_{l,m \in \mathcal{I}(C)} \frac{d_l d_m}{D_C^2} \delta_{l,0} \begin{array}{|c|c|} \hline \uparrow m & \uparrow i \\ \hline \leftarrow & \rightarrow \\ \hline \downarrow & \downarrow \\ \hline \end{array} \\
&= \sum_{m \in \mathcal{I}(C)} \frac{d_m}{D_C^2} \begin{array}{|c|c|} \hline \uparrow m & \uparrow i \\ \hline \leftarrow & \rightarrow \\ \hline \downarrow & \downarrow \\ \hline \end{array} = S(G_{i,j}). \tag{4.56}
\end{aligned}$$

Recalling the expansion (4.52), it follows that

$$\begin{aligned}
S(\text{SN}_C(\mathbb{T})) &= \sum_{i,j \in \mathcal{I}(C)} z(A)_{i,j} S(G_{i,j}) \\
&= \frac{1}{D_C^2} \sum_{i',i',j',j' \in \mathcal{I}(C)} s_{i',i'} z(A)_{i,j} s_{j,j'} G_{i',j'} = \sum_{i',j' \in \mathcal{I}(C)} (S^{-1} z(A) S)_{i',j'} G_{i',j'}
\end{aligned} \tag{4.57}$$

with  $S_{i,j} := s_{i,j}/D_C$ . Hence indeed we have

$$S(\text{SN}_C(\mathbb{T})) = \text{SN}_C(\mathbb{T}) \iff [S, z(A)] = 0. \tag{4.58}$$

Similarly, invariance of the correlator  $\text{SN}_C(\mathbb{T})$  under the modular  $T$ -transformation is equivalent to  $[T, z(A)] = 0$  which, in turn, is equivalent to the statement that the object  $Z(A) \in \mathcal{Z}(C)$  has trivial twist.

## 4.5 Boundary operator product

Among the correlators involving only field insertions on the boundary of the world sheet, the most basic one is the correlator for three boundary insertions on a disk, which describes the operator product of boundary insertions. The world sheet for this correlator is a disk  $D$  with a single two-cell, labeled by a Frobenius algebra  $A$ , and with three physical boundary segments labeled by  $A$ -modules  $M_1$ ,  $M_2$  and  $M_3$ . We denote this world sheet by  $D_{M_1, M_2, M_3}$ , and the corresponding complemented world sheet by  $\tilde{D}_{M_1, M_2, M_3}$ . In pictures,

$$D_{M_1, M_2, M_3} = \text{[Diagram: Green disk with three boundary segments labeled } M_1, M_2, M_3 \text{ and red arrows pointing outwards]} \quad \text{and} \quad \tilde{D}_{M_1, M_2, M_3} = \text{[Diagram: Green disk with three boundary segments labeled } M_1, M_2, M_3 \text{ and blue arrows pointing inwards]} \quad (4.59)$$

Analogously as we did in the case of defect fields, we want to obtain the world sheet  $\tilde{D}$  from a suitable string net on the disk  $D$  that encodes the algebraic information about the boundary operator product. We choose conventions such that two of the boundary fields, say  $\mathbb{B}^{M_1, M_2}$  and  $\mathbb{B}^{M_2, M_3}$  are incoming, while the third is outgoing and is thus given by  $\mathbb{B}^{M_1, M_3}$ . Recall from (2.35) that these fields are internal Homs,  $\mathbb{B}^{M_i, M_j} = \underline{\text{Hom}}_{\text{mod-}A}(M_i, M_j)$ . The natural candidate for their operator product is thus the canonical composition

$$\underline{\mu}(M_1, M_2, M_3) \in \text{Hom}_C(\underline{\text{Hom}}(M_2, M_3) \otimes \underline{\text{Hom}}(M_1, M_2), \underline{\text{Hom}}(M_1, M_3)) \quad (4.60)$$

of internal Homs (see (A.23)). Accordingly we consider the string net

$$\Gamma_{D; M_1, M_2, M_3} := \text{[Diagram: Circle with three points labeled } \mathbb{B}^{M_1, M_3}, \mathbb{B}^{M_2, M_3}, \mathbb{B}^{M_1, M_2} \text{ and a central vertex labeled } \underline{\mu} \text{ with arrows pointing towards it]} \in \text{SN}_C(D, \mathbf{B}_{M_1, M_2, M_3}), \quad (4.61)$$

where the boundary value  $\mathbf{B}_{M_1, M_2, M_3}$  is a circle with three points labeled by  $(\mathbb{B}^{M_1, M_2})^\vee$ ,  $(\mathbb{B}^{M_2, M_3})^\vee$  and  $\mathbb{B}^{M_1, M_3}$ . Note that

$$\begin{aligned} \text{SN}_C(D, \mathbf{B}_{M_1, M_2, M_3}) &\cong \text{Hom}_{\mathcal{Z}(C)}(\mathbf{1}, L(\mathbb{B}^{M_1, M_3} \otimes (\mathbb{B}^{M_1, M_2})^\vee \otimes (\mathbb{B}^{M_2, M_3})^\vee)) \\ &\cong \text{Hom}_C(\mathbf{1}, \mathbb{B}^{M_1, M_3} \otimes (\mathbb{B}^{M_1, M_2})^\vee \otimes (\mathbb{B}^{M_2, M_3})^\vee) \\ &\cong \text{Hom}_C(\mathbb{B}^{M_2, M_3} \otimes \mathbb{B}^{M_1, M_2}, \mathbb{B}^{M_1, M_3}). \end{aligned} \quad (4.62)$$

Now since  $\mathcal{C}$  is semisimple, we can invoke (A.28) to write  $\mathbb{B}^{M_i, M_j} = M_j \otimes_A M_i^\vee$ , and we then identify the boundary value  $\mathbf{B}_{M_1, M_2, M_3}$  with an isomorphic one so as to rewrite the string net (4.61) as

$$\Gamma_{D; M_1, M_2, M_3} = \text{Diagram} \quad (4.63)$$

Note that the surface obtained this way is the one underlying the complemented world sheet  $\tilde{D}_{M_1, M_2, M_3}$ . We can thus read off that we indeed have

$$\Gamma_{D; M_1, M_2, M_3} = \text{Cor}_{\text{SN}}(D_{M_1, M_2, M_3}). \quad (4.64)$$

## 4.6 Bulk-boundary operator product

Among the correlators involving field insertions both in the bulk and on the boundary, the most basic one is the correlator for one bulk and one boundary insertion on a disk. This correlator encodes a connection between bulk and boundary insertions, which is called the bulk-boundary operator product. Since in our approach defect fields can be treated in much the same way as bulk fields, we consider here the more general situation of one defect and one boundary insertion on a disk. The corresponding world sheet, which we denote by  $D_{X, Y; M}$ , is a disk  $D$  having two two-cells labeled by Frobenius algebra  $A$  and  $B$ , respectively, two defect lines labeled by  $A$ - $B$ -bimodules  $X$  and  $Y$ , and a physical boundary segment labeled by a right  $A$ -module  $M$ ; it looks as follows:

$$D_{X, Y; M} = \text{Diagram} \quad (4.65)$$

The crucial algebraic datum describing the connection between bulk and boundary in this situation is the component

$$J_M = \bigoplus_{m \in \mathcal{I}(\text{mod-}A)} d_m \text{Diagram} \quad (4.66)$$

$$\in \text{Hom}_{\mathcal{C}} \left( \int_{M' \in \text{mod-}A} M' \otimes_A Y \otimes_B X^\vee \otimes_A M'^\vee, M \otimes_A Y \otimes_B X^\vee \otimes_A M^\vee \right)$$



at the object  $M \otimes_A Y \otimes_B X^\vee \otimes_A M^\vee = \mathbb{B}^{M \otimes_A X, M \otimes_A Y}$  of the structure morphism  $j$  of the end

$$\int_{M' \in \text{mod-}A} M' \otimes_A Y \otimes_B X^\vee \otimes_A M'^\vee = U(\mathbb{D}^{X,Y}). \quad (4.67)$$

Accordingly we consider the string net

$$\Gamma_{\mathbb{D};X,Y;M} := \text{Diagram} \in \text{SN}_{\mathcal{C}}(S^1 \times I, (\mathbf{B}_{\mathbb{D}^{X,Y}}^{\text{can}})^\vee \boxtimes \mathbf{B}_{X,Y;M}). \quad (4.68)$$

Here the boundary value  $\mathbf{B}_{\mathbb{D}^{X,Y}}^{\text{can}}$  is the one defined in (3.19), while  $\mathbf{B}_{X,Y;M}$  in the first place consists of four points on  $S^1$  labeled by  $M, Y, X^\vee$  and  $M^\vee$ , respectively, together with an idempotent consisting of three relative tensor product idempotents combined, but is in an obvious manner isomorphic to the boundary value that consists of a single point on  $S^1$  labeled by  $\mathbb{B}^{M \otimes_A X, M \otimes_A Y}$  together with the identity cylinder. Note that

$$\begin{aligned} \text{SN}_{\mathcal{C}}(S^1 \times I, (\mathbf{B}_{\mathbb{D}^{X,Y}}^{\text{can}})^\vee \boxtimes \mathbf{B}_{X,Y;M}) &\cong \text{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathbb{D}^{X,Y}, L(\mathbb{B}^{M \otimes_A X, M \otimes_A Y})) \\ &\cong \text{Hom}_{\mathcal{C}}(U(\mathbb{D}^{X,Y}), \mathbb{B}^{M \otimes_A X, M \otimes_A Y}) \end{aligned} \quad (4.69)$$

(where we use that,  $\mathcal{C}$  being semisimple,  $L$  is also right adjoint to the forgetful functor  $U$ ). Inserting the explicit form (4.66) of  $j_M$  and the identities (A.65) and (A.66), we can write

$$\begin{aligned} \Gamma_{\mathbb{D};X,Y;M} &= \sum_{\substack{i \in \mathcal{I}(\mathcal{C}) \\ m, n \in \mathcal{I}(\text{mod-}A)}} \frac{d_i d_m d_n}{D_{\mathcal{C}}^2} \text{Diagram}_1 \\ &= \sum_{m \in \mathcal{I}(\text{mod-}A)} \frac{d_m}{D_{\mathcal{C}}^2} \text{Diagram}_2 \end{aligned} \quad (4.70)$$

Now recall from Example 3.6 the string net  $r_{X,Y}$  defined in (3.24), which furnishes an isomorphism  $\mathbf{B}_{X,Y} \xrightarrow{\cong} \mathbf{B}_{\mathbb{D}^{X,Y}}^{\text{can}}$ . Precomposing with  $r_{X,Y}$  yields the string net

$$\Gamma_{\mathbb{D};X,Y;M} \circ r_{X,Y} = \text{Diagram} \quad (4.71)$$

in  $\text{SN}_{\mathcal{C}}(S^1 \times I, (\mathbf{B}_{X,Y})^{\vee} \boxtimes \mathbf{B}_{X,Y;M})$ . By comparison with Definition 3.26, we thus conclude that

$$\Gamma_{\mathbb{D};X,Y;M} \circ r_{X,Y} = \text{Cor}_{\text{SN}}(\mathbb{D}_{X,Y;M}). \quad (4.72)$$

In short, the component (4.66) of the structure morphism of the end  $U(\mathbb{D}^{X,Y})$  indeed reproduces the string-net correlator for the world sheet  $\mathbb{D}_{X,Y;M}$ . In particular, specializing to the case that  $B = A$  and that  $X = A = Y$  as bimodules, we obtain the world sheet describing the bulk-boundary operator product.

## 5 Braided Eckmann-Hilton relation

Recall the diagram (2.31) which gives a Poincaré-dual view of (two-pronged) defect fields. In this section we consider multiple operator products of defect fields for which the description in terms of (2.31) looks as follows (for better readability we suppress the labels of the defect fields):

$$\text{Diagram} \quad (5.1)$$

with  $\mathcal{M} = \text{mod-}A$ ,  $\mathcal{M}' = \text{mod-}A'$  and  $\mathcal{M}'' = \text{mod-}A''$  for simple special symmetric Frobenius algebras  $A, A', A'' \in \mathcal{C}$ .

Now recall that for ordinary natural transformations (and, more generally, for 2-morphisms in a bicategory), the horizontal and vertical compositions satisfy the interchange law

$$\mu_{\text{ver}} \circ (\mu_{\text{hor}} \otimes \mu_{\text{hor}}) = \mu_{\text{hor}} \circ (\mu_{\text{ver}} \otimes \mu_{\text{ver}}). \quad (5.2)$$

It should be appreciated that this is a statement about elements of sets, and it implicitly relies on the fact that the category  $\text{Set}$  of sets is a *symmetric* monoidal category.

In contrast, internal natural transformations are objects of the category  $\mathcal{Z}(\mathcal{C})$  which is braided but (generically) not symmetric. However, we can show the following generalization to the braided setting:

**Theorem 5.1.** *Let  $\mathcal{C}$  be a modular fusion category and  $A, A'$  and  $A''$  be simple special symmetric Frobenius algebras in  $\mathcal{C}$ . Then the diagram*

$$\begin{array}{ccc}
& \mathbb{D}^{X_4, X_6} \otimes \mathbb{D}^{X_2, X_4} \otimes \mathbb{D}^{X_3, X_5} \otimes \mathbb{D}^{X_1, X_3} & \\
& \swarrow \text{id} \otimes \gamma_{\mathbb{D}^{X_2, X_4}; \mathbb{D}^{X_3, X_5}} \otimes \text{id} & \searrow \underline{\mu}_{\text{ver}} \otimes \underline{\mu}_{\text{ver}} \\
\mathbb{D}^{X_4, X_6} \otimes \mathbb{D}^{X_3, X_5} \otimes \mathbb{D}^{X_2, X_4} \otimes \mathbb{D}^{X_1, X_3} & & \mathbb{D}^{X_2, X_6} \otimes \mathbb{D}^{X_1, X_5} \quad (5.3) \\
\downarrow \underline{\mu}_{\text{hor}}^1 \otimes \underline{\mu}_{\text{hor}}^1 & & \downarrow \underline{\mu}_{\text{hor}}^1 \\
\mathbb{D}^{X_3 \otimes_{A'} X_4, X_5 \otimes_{A'} X_6} \otimes \mathbb{D}^{X_1 \otimes_{A'} X_2, X_3 \otimes_{A'} X_4} & \xrightarrow{\underline{\mu}_{\text{ver}}} & \mathbb{D}^{X_1 \otimes_{A'} X_2, X_5 \otimes_{A'} X_6}
\end{array}$$

involving the horizontal and vertical compositions of internal natural transformations and the half-braiding  $\gamma$  of  $\mathcal{Z}(\mathcal{C})$  commutes for all  $X_1, X_3, X_5 \in A\text{-mod-}A'$  and all  $X_2, X_4, X_6 \in A'\text{-mod-}A''$ .

Since the equality (5.2) provides the basis for the Eckmann-Hilton argument, we refer to the commutativity of (5.3) as the *braided Eckmann-Hilton relation*.

**Remark 5.2.** (i) Theorem 5.1 renders Proposition 14 in [FuS2] precise. In the latter, neither the relevant (left or right) horizontal composition nor the relevant (half-)braiding were specified. (ii) There is, of course, also a variant in which the right rather than left horizontal composition appears: the diagram

$$\begin{array}{ccc}
& \mathbb{D}^{X_3, X_5} \otimes \mathbb{D}^{X_1, X_3} \otimes \mathbb{D}^{X_4, X_6} \otimes \mathbb{D}^{X_2, X_4} & \\
& \swarrow \text{id} \otimes \gamma_{\mathbb{D}^{X_1, X_3}; \mathbb{D}^{X_4, X_6}} \otimes \text{id} & \searrow \underline{\mu}_{\text{ver}} \otimes \underline{\mu}_{\text{ver}} \\
\mathbb{D}^{X_3, X_5} \otimes \mathbb{D}^{X_4, X_6} \otimes \mathbb{D}^{X_1, X_3} \otimes \mathbb{D}^{X_2, X_4} & & \mathbb{D}^{X_1, X_5} \otimes \mathbb{D}^{X_2, X_6} \quad (5.4) \\
\downarrow \underline{\mu}_{\text{hor}}^r \otimes \underline{\mu}_{\text{hor}}^r & & \downarrow \underline{\mu}_{\text{hor}}^r \\
\mathbb{D}^{X_3 \otimes_{A'} X_4, X_5 \otimes_{A'} X_6} \otimes \mathbb{D}^{X_1 \otimes_{A'} X_2, X_3 \otimes_{A'} X_4} & \xrightarrow{\underline{\mu}_{\text{ver}}} & \mathbb{D}^{X_1 \otimes_{A'} X_2, X_5 \otimes_{A'} X_6}
\end{array}$$

commutes as well. This follows directly by combining Theorem 5.1 with the identity (5.22) which will be established below.

Preparing and giving the proof of Theorem 5.1 will occupy most of the rest of this section. A main ingredient are certain braided colored operads in  $\text{Set}$  [Ya]. With the help of string nets we will establish morphisms between these operads. The operads in question are:

- The braided colored operad  $\mathcal{OWS}_{\mathcal{C}}$  of genus-0 world sheets with labels from  $\mathcal{C}$ .  
The colors of  $\mathcal{OWS}_{\mathcal{C}}$  are all possible boundary data of world sheets. The set of operations of  $\mathcal{OWS}_{\mathcal{C}}$  consists of all genus-0 world sheets, with the boundary data on their boundary circles

regarded as in- and outputs, and taken up to isotopy; e.g. the binary products are given by

$$\mathcal{O}WS_c \left( \begin{array}{c} c \\ a \quad b \end{array} \right) = \left\{ \left( \text{Diagram} \right) \right\} / \text{isotopy}, \quad (5.5)$$

i.e. by the set of all genus-0 world sheets with boundary datum  $a^\vee \times b^\vee \times c$ , up to isotopy. The operadic composition on  $\mathcal{O}WS_c$  is the sewing of world sheets, and the braid group action on  $\mathcal{O}WS_c$  is obtained by identifying the braid group  $B_n$  as a subgroup of  $\text{Map}(\Sigma_{n+1}^0)$ , with  $\Sigma_{n+1}^0$  a standard sphere with  $n+1$  holes.

- The braided colored *operad*  $\mathcal{O}SN_c$  of string nets.

The colors of  $\mathcal{O}SN_c$  are the objects of the cylinder category  $\mathcal{Cyl}(\mathcal{C}, S^1)$ . The set of operations consists of (the sets underlying) the string-net spaces on genus-0 surfaces with appropriate boundary values, e.g.

$$\mathcal{O}SN_c \left( \begin{array}{c} \mathbf{C} \\ \mathbf{A} \quad \mathbf{B} \end{array} \right) = \text{SN}_c \left( \Sigma_3^0, \mathbf{A}^\vee \boxtimes \mathbf{B}^\vee \boxtimes \mathbf{C} \right). \quad (5.6)$$

The operadic composition on  $\mathcal{O}SN_c$  is the gluing of string nets, and the braid group action on  $\mathcal{O}WS_c$  is again obtained by identifying  $B_n$  as a subgroup of  $\text{Map}(\Sigma_{n+1}^0)$ .

- The braided colored *endomorphism operad*  $\mathcal{O}\text{Hom}_{\mathcal{Z}(\mathcal{C})}$ .

The colors of  $\mathcal{O}\text{Hom}_{\mathcal{Z}(\mathcal{C})}$  are the objects of the Drinfeld center  $\mathcal{Z}(\mathcal{C})$ . The set of operations consists of (the sets underlying) the morphisms in  $\mathcal{Z}(\mathcal{C})$ , e.g.

$$\mathcal{O}\text{Hom}_{\mathcal{Z}(\mathcal{C})} \left( \begin{array}{c} Z \\ X \quad Y \end{array} \right) = \text{Hom}_{\mathcal{Z}(\mathcal{C})}(X \otimes Y, Z). \quad (5.7)$$

The operadic composition on  $\mathcal{O}\text{Hom}_{\mathcal{Z}(\mathcal{C})}$  is the composition of morphisms in  $\mathcal{Z}(\mathcal{C})$ . We define the braid group action to be generated by

$$\left( \text{Diagram 1} \right) \xrightarrow{\beta} \left( \text{Diagram 2} \right) \longmapsto \left( \text{Diagram 3} \right) \xrightarrow{\beta} \left( \text{Diagram 4} \right) \quad (5.8)$$

for  $X_1, X_2, Y \in \mathcal{Z}(\mathcal{C})$ , with the overbraiding standing for the half-braiding  $\gamma_{X_2, X_1}$ .

**Remark 5.3.** For both  $\mathcal{O}SN_c$  and  $\mathcal{O}\text{Hom}_{\mathcal{Z}(\mathcal{C})}$  there is an obvious linear version, in which the operations are given by the vector spaces  $\text{SN}_c(-)$  and  $\text{Hom}_{\mathcal{Z}(\mathcal{C})}(-)$ , respectively, instead of by

their underlying sets. We could indeed formulate the present considerations in a linear setting, by linearizing also the world sheet operad  $\mathcal{O}WS_{\mathcal{C}}$ . The latter can be achieved by regarding a world sheet as colored by the linear bicategory  $\mathcal{Fr}(\mathcal{C})$  of Frobenius algebras (which we will introduce in Section 6) and treating those  $\mathcal{Fr}(\mathcal{C})$ -colored world sheets in the same way as  $\mathcal{C}$ -colored graphs are treated in the string-net construction.

Now we define morphisms

$$\text{Cor}_{\text{SN}} : \mathcal{O}WS_{\mathcal{C}} \longrightarrow \mathcal{O}SN_{\mathcal{C}} \quad (5.9)$$

and

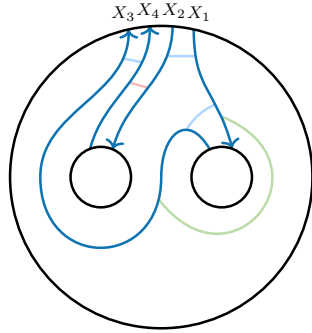
$$\varphi_{\text{Hom}} : \mathcal{O}SN_{\mathcal{C}} \longrightarrow \mathcal{O}\text{Hom}_{\mathcal{Z}(\mathcal{C})} \quad (5.10)$$

of braided colored operads in *Set* by the following prescriptions:

- $\text{Cor}_{\text{SN}}$  acts on colors by sending a boundary datum to the corresponding object in  $\mathcal{Cyl}(\mathcal{C}, S^1)$  (compare Example 3.24).  $\text{Cor}_{\text{SN}}$  acts on operations by sending a world sheet  $\mathcal{S}$  to its string-net correlator  $\text{Cor}_{\text{SN}}(\mathcal{S})$ . Compatibility with the operadic compositions and braid group equivariance of these prescriptions are evident.
- The definition of  $\varphi_{\text{Hom}}$  is slightly more involved. It depends on two types of auxiliary data: for each genus-0 surface  $\Sigma$  a marking  $w = w(\Sigma)$  without cuts, and for each object  $\mathbf{B} \in \mathcal{Cyl}(\mathcal{C}, S^1)$  an isomorphism  $\psi = \psi(\mathbf{B})$  from  $\mathbf{B}$  to its “canonical form”  $\mathbf{B}_{\Phi(\mathbf{B})}^{\text{can}}$ , i.e. the one that is analogous to (3.19), with  $\Phi : \mathcal{Cyl}(\mathcal{C}, S^1) \xrightarrow{\cong} \mathcal{Z}(\mathcal{C})$  the equivalence (3.16). With a fixed choice for these data,  $\varphi_{\text{Hom}}$  acts on colors as  $\mathbf{B} \mapsto \Phi(\mathbf{B})$ . Its action on operations is by the inverses of the isomorphisms  $\text{Hom}_{\mathcal{Z}(\mathcal{C})}(\dots, \dots) \xrightarrow{\cong} \text{SN}_{\mathcal{C}}(\Sigma, \mathbf{B}_w)$  that are analogous to (4.13) and (4.31). To give an example, when choosing again the marking (4.9) on the pair of pants we have

$$\begin{aligned}
 \varphi_{\text{Hom}} : & \quad \left( \text{Diagram 1} \right) \xrightarrow{\Psi} \sum_{m,n,q \in \mathcal{I}(\mathcal{M})} \frac{d_m d_n}{D_{\mathcal{C}}^4} \left( \text{Diagram 2} \right) \\
 & \quad = \left( \text{Diagram 3} \right) \\
 & \quad \xrightarrow{\varphi_w^{-1}} \underline{\mu}_{\text{ver}} \in \mathcal{O}\text{Hom}_{\mathcal{Z}(\mathcal{C})} \left( \begin{array}{c} \mathbb{D}^{X_1, X_3} \\ \mathbb{D}^{X_2, X_3} \quad \mathbb{D}^{X_1, X_2} \end{array} \right).
 \end{aligned} \quad (5.11)$$

where  $\Psi$  stands for pre- and post-composition with appropriate isomorphisms  $\psi$  and  $\varphi_w$  is given by (4.13). Similarly, with  $\varphi_w$  as in (4.31) we have

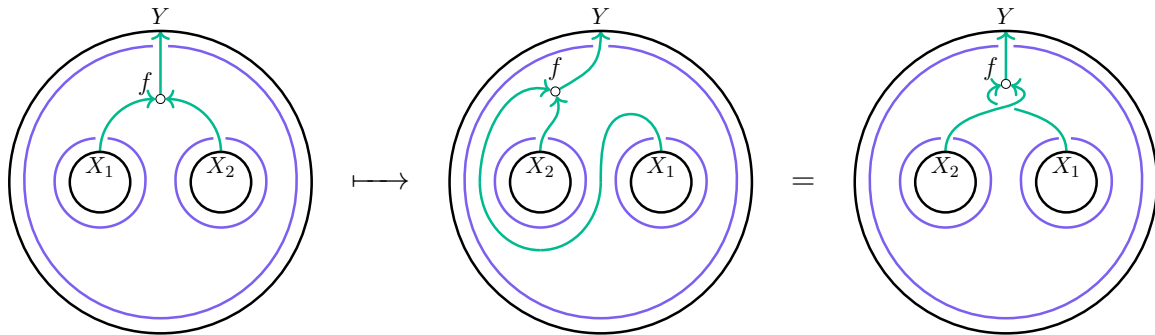


$$\xrightarrow{\varphi_{\text{Hom}}} \underline{\mu}_{\text{hor}}^1 \in \mathcal{O}\text{Hom}_{\mathcal{Z}(C)} \left( \begin{array}{c} \mathbb{D}^{X_1 \otimes_{A'} X_2, X_3 \otimes_{A'} X_4} \\ \mathbb{D}^{X_2, X_4} \quad \mathbb{D}^{X_1, X_3} \end{array} \right). \quad (5.12)$$

Operadic composition results in a marking with cuts with an internal edge  $e$  which connects the outgoing circle of the inserted surface to the new root of the marking. Compatibility with the operadic compositions is achieved by complementing the prescription for  $\varphi_{\text{Hom}}$  given above by the requirement to replace this marking by the marking without cuts that is obtained by contracting the internal edge  $e$ . (This is analogous to the F-move on markings, compare Figure 5.7 of [BakK].)

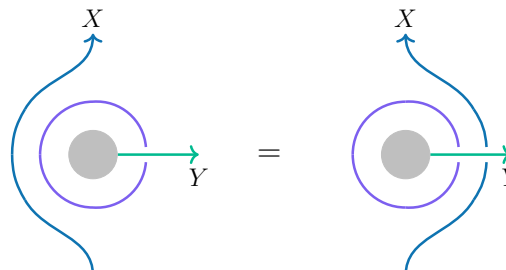
**Remark 5.4.** A different choice of the isomorphisms between the boundary values and their canonical forms results in composing every element in a given morphism space with one and the same isomorphism or its inverse. Similarly, a different choice of the markings results in composing them with braiding and twist isomorphisms. The choices we make are particularly convenient for revealing the braided Eckmann-Hilton relation, but other choices will lead to that same relation as well.

Let us verify that our prescription (5.8) leads to the correct braid group action on  $\mathcal{O}\text{Hom}_{\mathcal{Z}(C)}$ : Under the move  $\beta$  that is shown on the left hand side of (5.8) we have



$$\xrightarrow{\beta} = \quad (5.13)$$

where the equality holds by the cloaking relation [ScY, Lemma 3.7]



$$= \quad (5.14)$$

The overbraiding on the right hand side of (5.13) is the half-braiding  $\gamma_{X_2;X_1}$ . Thus we indeed obtain the braid group action on  $\mathcal{O}\text{Hom}_{\mathcal{Z}(C)}$  as defined in (5.8). (A priori, in (5.8) one might have considered instead the inverse half-braiding  $\gamma_{X_1;X_2}^{-1}$ ; the present calculation shows that the choice made in (5.8) is the correct one.)

*Proof of Theorem 5.1.* Consider the composition

$$\widetilde{\text{Cor}}_{\text{SN}} := \varphi_{\text{Hom}} \circ \text{Cor}_{\text{SN}} : \mathcal{O}\text{WS}_C \longrightarrow \mathcal{O}\text{Hom}_{\mathcal{Z}(C)} \quad (5.15)$$

of the morphisms (5.9) and (5.10). We have

$$\mathcal{S}_{\text{ver}}^w \stackrel{(4.16)}{=} \begin{array}{c} \text{Diagram: A green circle with two white circles inside. Blue regions with arrows labeled } X_1, X_2, X_3 \text{ and a dashed red line.} \end{array} \xrightarrow{\widetilde{\text{Cor}}_{\text{SN}}} \underline{\mu}_{\text{ver}}^w \quad (5.16)$$

and

$$\mathcal{S}_{\text{hor}}^1 \stackrel{(4.35)}{=} \begin{array}{c} \text{Diagram: A green circle with two white circles inside. Blue regions with arrows labeled } X_1, X_2, X_3, X_4 \text{ and a dashed red line.} \end{array} \xrightarrow{\widetilde{\text{Cor}}_{\text{SN}}} \underline{\mu}_{\text{hor}}^1 \quad (5.17)$$

Invoking the compatibility of  $\widetilde{\text{Cor}}_{\text{SN}}$  with operadic composition, we further get

$$\mathcal{S}_{\text{h};\text{v},\text{v}} := \begin{array}{c} \text{Diagram: A green circle with four white circles inside. Blue regions with arrows labeled } X_1, X_2, X_3, X_4, X_5, X_6 \text{ and a dashed red line.} \end{array} \xrightarrow{\widetilde{\text{Cor}}_{\text{SN}}} \begin{array}{c} \text{Diagram: A tree-like structure with nodes and arrows labeled } \mathbb{D}^{X_1,2,X_5,6}, \underline{\mu}_{\text{hor}}^1, \underline{\mu}_{\text{ver}}, \mathbb{D}^{X_2,X_6}, \mathbb{D}^{X_1,X_5}, \mathbb{D}^{X_4,X_6}, \mathbb{D}^{X_2,X_4}, \mathbb{D}^{X_3,X_5}, \mathbb{D}^{X_1,X_3} \end{array} \quad (5.18)$$

as well as

$$\mathcal{S}_{v;h,h} := \xrightarrow{\widetilde{\text{CorSN}}} \text{String Net Diagram} \quad (5.19)$$

where we use the short-hand  $X_{i,j} := X_i \otimes_{A'} X_j$ . Moreover, applying the braid group element  $\beta_{2,3}$  to (5.19) gives

$$\beta_{2,3}(\mathcal{S}_{v;h,h}) = \xrightarrow{\widetilde{\text{CorSN}}} \text{String Net Diagram} \quad (5.20)$$

Notice that

$$\beta_{2,3}(\mathcal{S}_{v;h,h}) = \mathcal{S}_{h;v,v}. \quad (5.21)$$

By comparison of (5.18) and (5.20) we arrive at the desired equality that states the commutativity of the diagram (5.3).  $\square$

**Remark 5.5.** By similar arguments one sees that the equality  $\mathcal{S}_{\text{hor}}^1 = \beta(\mathcal{S}_{\text{hor}}^r)$  obtained in (4.37) gets mapped under  $\widetilde{\text{CorSN}}$  to

$$\underline{\mu}_{\text{hor}}^1 = \beta(\underline{\mu}_{\text{hor}}^r) = \underline{\mu}_{\text{hor}}^r \circ \gamma_{\mathbb{D}^{X_2, X_4}; \mathbb{D}^{X_1, X_3}}, \quad (5.22)$$

showing that the left and right horizontal compositions are related by a half-braiding.

**Remark 5.6.** Consider the special case that  $A = A' = A''$  and that all defect fields involved are actually bulk fields  $\mathbb{D}^{A,A}$ , for which vertical and horizontal compositions coincide (see (4.44)). Then commutativity of the diagram (5.3) amounts to the statement that the bulk algebra  $\mathbb{D}^{A,A}$  in  $\mathcal{Z}(\mathcal{C})$  is braided commutative (compare Corollary 15 in [FuS2]).

## 6 Universal correlators

According to Definition 2.5 the labels involved in the specification of a world sheet  $\mathcal{S}$  are simple special symmetric Frobenius algebras in the monoidal category  $\mathcal{C}$  (for the two-cells of  $\mathcal{S}$ ),



bimodules between these (for the one-cells), and bimodule morphisms (for the zero-cells). We call this assignment a *defect pattern*. A pertinent feature of a defect pattern is that these labels form the objects, 1-morphisms, and 2-morphisms, respectively, of a linear pivotal bicategory (as defined e.g. in [CR, Eq.(2.12)] or [CMS, Def.3.6]). We denote this bicategory by  $\mathcal{Fr}(\mathcal{C})$ . So far we have ignored this feature of our construction; now we put it into use. (A further motivation for considering the bicategory  $\mathcal{Fr}(\mathcal{C})$  is the desire to linearize the operad  $\mathcal{OWS}_{\mathcal{C}}$  of world sheets, see Remark 5.3.)

As any pivotal bicategory,  $\mathcal{Fr}(\mathcal{C})$  comes with a graphical calculus on disks, which relates disks having the same boundary value but different defect patterns. As in the case of the ordinary string-net construction, we promote this feature to local relations, so that we can identify different defect patterns locally on disks embedded in a world sheet  $\mathcal{S}$ .

This observation motivates us to consider a variant of the string-net construction for which the coloring is not by a pivotal fusion category  $\mathcal{C}$ , but rather by some pivotal bicategory  $\mathcal{B}$ . Taking the quotient of the vector space spanned by all defect patterns with given boundary values by the local relations, we obtain a vector space  $\text{SN}_{\mathcal{B}}^{\circ}(\Sigma, \mathbf{B}_{\mathcal{B}})$ , which we call again the bare string-net space. A boundary value for this string-net construction is now an object  $\mathbf{B}_{\mathcal{B}}$  in the appropriate cylinder category  $\mathcal{Cyl}^{\circ}(\mathcal{B}, \partial\Sigma)$  instead of  $\mathcal{Cyl}^{\circ}(\mathcal{C}, \partial\Sigma)$ . Let us illustrate the structure of  $\mathcal{Cyl}^{\circ}(\mathcal{B}, \partial\Sigma)$ , restricting for brevity to the case that  $\partial\Sigma = S^1$ : As an example, consider

$$(6.1)$$

where  $a, b$  and  $c$  are objects in  $\mathcal{B}$  and  $X \in 1\text{-Hom}_{\mathcal{B}}(a, b)$ ,  $Y \in 1\text{-Hom}_{\mathcal{B}}(c, b)$  and  $Z \in 1\text{-Hom}_{\mathcal{B}}(c, a)$ . Here the symbol  $\vee$  indicates the pivotal duality, compare [CMS, Def.3.6]. Morphisms in  $\mathcal{Cyl}^{\circ}(\mathcal{B}, S^1)$  are equivalence classes of graphs on  $S^1 \times I$  modulo the local relations resulting from the graphical calculus for  $\mathcal{B}$ . The picture

$$(6.2)$$

shows an example of a morphism; here  $\alpha$  and  $\beta$  are 2-morphisms in  $\mathcal{B}$ . The mapping class group of  $\Sigma$  acts on the bicategorical string-net spaces.

In the present context – the construction of CFT correlators for given chiral data, captured by a pivotal fusion category  $\mathcal{C}$  – the pivotal bicategory of our interest is  $\mathcal{Fr}(\mathcal{C})$ . In this case a world sheet  $\mathcal{S}$  with underlying surface  $\Sigma = \Sigma_{\mathcal{S}}$  and boundary value  $\mathbf{B}_{\mathcal{Fr}(\mathcal{C})} \in \mathcal{Cyl}^{\circ}(\mathcal{Fr}(\mathcal{C}), \partial\Sigma)$  provides us with a defect pattern which determines a vector in the bare string-net space. We describe it as a linear map

$$\begin{aligned} \delta_{\mathcal{S}} : \quad \mathbb{C} &\longrightarrow \text{SN}_{\mathcal{Fr}(\mathcal{C})}^{\circ}(\Sigma, \mathbf{B}_{\mathcal{Fr}(\mathcal{C})}), \\ 1 &\longmapsto [\mathcal{S}], \end{aligned} \tag{6.3}$$

where  $[\mathcal{S}]$  is the equivalence class of the  $\mathcal{F}r(\mathcal{C})$ -colored world sheet  $\mathcal{S}$ . On the other hand, we can view the vector in the string-net space based on  $\mathcal{C}$  that describes the string-net correlator  $\text{Cor}_{\text{SN}}(\mathcal{S})$  as a linear map

$$\begin{aligned} \overline{\text{Cor}}_{\text{SN}}(\mathcal{S}) : \quad \mathbb{C} &\longrightarrow \text{SN}_{\mathcal{C}}(\Sigma, \widetilde{\mathbb{F}}(\mathbf{B}_{\mathcal{F}r(\mathcal{C})})), \\ 1 &\longmapsto \text{Cor}_{\text{SN}}(\mathcal{S}). \end{aligned} \quad (6.4)$$

Here

$$\widetilde{\mathbb{F}} : \quad \text{Cyl}^{\circ}(\mathcal{F}r(\mathcal{C}), \partial\Sigma) \longrightarrow \text{Cyl}(\mathcal{C}, \partial\Sigma) \quad (6.5)$$

is the obvious map between the two objects of the two categories; as an illustration, we have

$$(6.6)$$

With the help of  $\widetilde{\mathbb{F}}$  the field map, which was introduced in (2.22) just as a map from the set of geometric boundary circles of the world sheet to the class of objects in the Drinfeld center, can be promoted to a functor. For brevity, we describe this functor only for the case that  $\partial\Sigma \cong (S^1)^{\sqcup n}$  with all  $n$  boundary components incoming. Then the field map becomes a functor

$$\mathbb{F} : \quad \text{Cyl}^{\circ}(\mathcal{F}r(\mathcal{C}), \partial\Sigma)^{\text{opp}} \longrightarrow \mathcal{Z}(\mathcal{C})^{\boxtimes n}, \quad (6.7)$$

namely the composite

$$\mathbb{F} : \quad \text{Cyl}^{\circ}(\mathcal{F}r(\mathcal{C}), \partial\Sigma)^{\text{opp}} \xrightarrow{\widetilde{\mathbb{F}}^{\text{opp}}} \text{Cyl}(\mathcal{C}, \partial\Sigma)^{\text{opp}} \xrightarrow{(\phi^{-1})_*} \text{Cyl}(\mathcal{C}, S^1)^{\boxtimes n} \xrightarrow{\Phi^{\boxtimes n}} \mathcal{Z}(\mathcal{C})^{\boxtimes n}, \quad (6.8)$$

with  $\phi : (S^1)^{\sqcup n} \rightarrow \partial\Sigma$  an orientation reversing diffeomorphism describing the parametrization of the  $n$  ingoing boundaries and  $\Phi$  the functor (3.16).

Using the bare string nets based on the pivotal bicategory  $\mathcal{F}r(\mathcal{C})$ , we can enrich our description of correlators with further information and sharpen the concept of a mapping class group action. First, note that world sheets related by local relations from  $\mathcal{F}r(\mathcal{C})$  have the same boundary data and thus have the same space  $\text{Bl}_{\mathcal{C}}$  of conformal blocks. Their correlators thus take value in the same vector space. We claim that a much stronger statement holds: such world sheets that give the same vector in the bicategorical string-net spaces even have the same correlator  $\text{Cor}_{\text{SN}}$ . Concretely, we claim that, given a surface  $\Sigma$  with boundary value  $\mathbf{B}_{\mathcal{F}r(\mathcal{C})}$ , there exists a unique linear map  $\text{UCor}_{\text{SN}}(\Sigma, \mathbf{B}_{\mathcal{F}r(\mathcal{C})})$  – which we call the *universal correlator* for  $(\Sigma, \mathbf{B}_{\mathcal{F}r(\mathcal{C})})$  – such that the triangle

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\delta_{\mathcal{S}}} & \text{SN}_{\mathcal{F}r(\mathcal{C})}^{\circ}(\Sigma, \mathbf{B}_{\mathcal{F}r(\mathcal{C})}) \\ & \searrow \overline{\text{Cor}}_{\text{SN}}(\mathcal{S}) & \downarrow \text{UCor}_{\text{SN}}(\Sigma, \mathbf{B}_{\mathcal{F}r(\mathcal{C})}) \\ & & \text{SN}_{\mathcal{C}}(\Sigma, \mathbb{F}(\mathbf{B}_{\mathcal{F}r(\mathcal{C})})) \end{array} \quad (6.9)$$

commutes for every world sheet  $\mathcal{S}$  with underlying surface  $\Sigma$  and boundary data corresponding to  $\mathbf{B}_{\mathcal{F}_r(\mathcal{C})}$ . Moreover, this map  $\text{UCor}_{\text{SN}}(\Sigma, \mathbf{B}_{\mathcal{F}_r(\mathcal{C})})$  intertwines the action of the mapping class group  $\text{Map}(\Sigma)$  on  $\text{SN}_{\mathcal{F}_r(\mathcal{C})}^\circ(\Sigma, \mathbf{B}_{\mathcal{F}_r(\mathcal{C})})$  and  $\text{SN}_{\mathcal{C}}(\Sigma, \mathbb{F}(\mathbf{B}_{\mathcal{F}_r(\mathcal{C})}))$ .

The vector space  $\text{SN}_{\mathcal{F}_r(\mathcal{C})}^\circ(\Sigma, \mathbf{B}_{\mathcal{F}_r(\mathcal{C})})$  is by definition spanned by string nets. We therefore define  $\text{UCor}_{\text{SN}}(\Sigma, \mathbf{B}_{\mathcal{F}_r(\mathcal{C})})$  as the map that sends the vector in the bare string-net space that is determined by the defect pattern of a world sheet  $\mathcal{S}$  to the string-net correlator of  $\mathcal{S}$ . The crucial observation is now that the so obtained map is actually well defined and that it intertwines the  $\text{Map}(\Sigma)$ -action. This is not hard to see, but we leave a detailed proof to a forthcoming paper. Instead, we content ourselves to make a few immediate comments:

- In the context of universal correlators there is a more comprehensive notion of the mapping class group of a world sheet that supersedes the Definition 2.16 of  $\text{Map}(\mathcal{S})$ , namely as the stabilizer of the vector  $\delta_{\mathcal{S}}$  under the  $\text{Map}(\Sigma)$ -action on  $\text{SN}_{\mathcal{F}_r(\mathcal{C})}^\circ(\Sigma, \mathbf{B}_{\mathcal{F}_r(\mathcal{C})})$ :

$$\widehat{\text{Map}}(\mathcal{S}) := \text{Stab}_{\text{Map}(\Sigma)}(\delta_{\mathcal{S}}). \quad (6.10)$$

As an illustration, consider the following world sheets  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , where  $X$  is an invertible bimodule, which are identified by the local relations in  $\text{SN}_{\mathcal{F}_r(\mathcal{C})}^\circ$ :

$$\mathcal{S}_1 := \text{[Diagram]} = \text{[Diagram]} := \mathcal{S}_2. \quad (6.11)$$

In this case, the mapping class group  $\text{Map}(\mathcal{S}_1)$  does not include Dehn twists along the boundary circles, while  $\text{Map}(\mathcal{S}_2)$ , and hence also  $\widehat{\text{Map}}(\mathcal{S}_1) \equiv \widehat{\text{Map}}(\mathcal{S}_2)$  does.

That the two world sheets in (6.11) have the same correlator is a consequence of the invertibility of the bimodule  $X$ . This type of equality of correlators is also at the basis of the fact that invertible bimodules describe symmetries of conformal field theories [FröFRS2].

- More generally, the fact that the linear map  $\text{UCor}_{\text{SN}}(\Sigma, \mathbf{B}_{\mathcal{F}_r(\mathcal{C})})$  from  $\text{SN}_{\mathcal{F}_r(\mathcal{C})}^\circ(\Sigma, \mathbf{B}_{\mathcal{F}_r(\mathcal{C})})$  to  $\text{SN}_{\mathcal{C}}(\Sigma, \mathbb{F}(\mathbf{B}_{\mathcal{F}_r(\mathcal{C})}))$  is well defined and intertwines the action of  $\text{Map}(\Sigma)$  implies that the correlator  $\text{Cor}_{\text{SN}}(\mathcal{S})$  is invariant under the action of the group  $\widehat{\text{Map}}(\mathcal{S})$  as defined in (6.10).
- That the collection  $\{\text{UCor}_{\text{SN}}(\Sigma, \mathbf{B}_{\mathcal{F}_r(\mathcal{C})})\}_{(\Sigma, \mathbf{B}_{\mathcal{F}_r(\mathcal{C})})}$  also intertwines with the sewing of surfaces is equivalent to the statement that the string-net construction of correlators is compatible with all sewing constraints.

# A Appendix

## A.1 Spherical fusion categories

Let  $\mathbb{k}$  be an algebraically closed field of characteristic zero. A *fusion category* over  $\mathbb{k}$  is a finitely semisimple  $\mathbb{k}$ -linear rigid monoidal category with simple monoidal unit.

To fix notations, we briefly comment on the qualifications in this definition. A *monoidal* category is a category  $\mathcal{C}$  equipped with a tensor product functor, which we denote by  $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ , and with a monoidal unit object, which we write as  $\mathbf{1} \in \mathcal{C}$ , and with associativity and unit constraints satisfying pentagon and triangle identities. Without loss of generality we take the tensor product to be strict, i.e.  $\otimes$  is associative on the nose and  $\mathbf{1} \otimes C = C = C \otimes \mathbf{1}$  for all  $C \in \mathcal{C}$ .  $\mathbb{k}$ -*linearity* of a monoidal category means that the morphism sets  $\text{Hom}_{\mathcal{C}}(-, -)$  are  $\mathbb{k}$ -vector spaces and that composition as well as the tensor product of morphisms is bilinear.

Being *finitely semisimple* means that the isomorphism classes of simple objects of  $\mathcal{C}$  form a finite set and that every object is a finite direct sum of simple objects. We select a set  $\mathcal{I}(\mathcal{C})$  of representatives for the isomorphism classes of simple objects, in such a way that  $\mathbf{1} \in \mathcal{I}(\mathcal{C})$ . In a  $\mathbb{k}$ -linear finitely semisimple category the morphism spaces are finite-dimensional. These finiteness properties are crucial. Keeping them while removing the semisimplicity requirement generalizes fusion categories to the class of *finite tensor categories*, which in many respects behave much like fusion categories. A fusion category is the same as a semisimple finite tensor category. Since  $\mathbf{1}$  is simple, we have  $\text{Hom}_{\mathcal{C}}(\mathbf{1}, \mathbf{1}) = \mathbb{k} \text{id}_{\mathbf{1}}$ , and we can canonically identify  $\text{Hom}_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$  with  $\mathbb{k}$ .

Finally, in a *rigid* monoidal category every object  $C \in \mathcal{C}$  has a *right dual object*  $C^{\vee}$  accompanied by an *evaluation* morphism  $\text{ev}_C^r \in \text{Hom}_{\mathcal{C}}(C^{\vee} \otimes C, \mathbf{1})$  and by a *coevaluation* morphism  $\text{coev}_C^r \in \text{Hom}_{\mathcal{C}}(\mathbf{1}, C \otimes C^{\vee})$  that satisfy the snake identities

$$(\text{id}_C \otimes \text{ev}_C^r) \circ (\text{coev}_C^r \otimes \text{id}_C) = \text{id}_C \quad \text{and} \quad (\text{ev}_C^r \otimes \text{id}_{C^{\vee}}) \circ (\text{id}_{C^{\vee}} \otimes \text{coev}_C^r) = \text{id}_{C^{\vee}}, \quad (\text{A.1})$$

as well as a *left dual object*  ${}^{\vee}C$  accompanied by corresponding evaluation and coevaluation morphisms. The right and left dualities extend to functors  $(-)^{\vee}$  and  ${}^{\vee}(-)$  from  $\mathcal{C}$  to its opposite category with opposite tensor product.

A *pivotal structure* on a rigid monoidal category is a monoidal natural isomorphism  $\pi$  from the identity functor to the double (right, say) dual functor  $(-)^{\vee\vee}$ . Again without loss of generality we take a pivotal structure (if it exists) to be strict, i.e. take  $\pi$  to be the identity natural transformation. A category admitting a pivotal structure is called a *pivotal category*. In a pivotal category  $\mathcal{C}$  one can associate to any endomorphism  $f \in \text{Hom}_{\mathcal{C}}(C, C)$  its right and left *traces*, which in the strict case are given by

$$\text{tr}_r(f) = \text{ev}_{C^{\vee}}^r \circ (f \otimes \text{id}_{C^{\vee}}) \circ \text{coev}_C^r \quad \text{and} \quad \text{tr}_l(f) = \text{ev}_C^r \circ (\text{id}_{C^{\vee}} \otimes f) \circ \text{coev}_{C^{\vee}}^r, \quad (\text{A.2})$$

respectively. A pivotal structure is called *spherical* if the right and left traces of any endomorphism coincide.

A *spherical fusion category* over  $\mathbb{k}$  is a fusion category over  $\mathbb{k}$  admitting, and endowed with, a spherical pivotal structure. In a spherical category one can identify left and right duals, and we tacitly do so.

We denote the (right and left) *dimension* of an object  $C$  of a spherical fusion category by  $\dim(C) = \text{tr}(\text{id}_C)$ ; for simple objects  $i \in \mathcal{I}(\mathcal{C})$  we abbreviate

$$\dim(i) =: d_i \text{id}_i. \quad (\text{A.3})$$

The *global dimension* of a spherical fusion category  $\mathcal{C}$  is the number

$$D_{\mathcal{C}}^2 := \sum_{i \in \mathcal{I}(\mathcal{C})} d_i^2 \quad (\text{A.4})$$

(no choice of square root implied); this number is non-zero [ENO, Thm. 2.3].

One quantity in which the global dimension shows up is the *canonical color* (or *Kirby color*, or *surgery color*). This is by definition the morphism

$$\varphi_{\text{can}} := \sum_{i \in \mathcal{I}(\mathcal{C})} \frac{d_i}{D_{\mathcal{C}}^2} \text{id}_i \in \text{End}_{\mathcal{C}}\left(\bigoplus_{i \in \mathcal{I}(\mathcal{C})} i\right). \quad (\text{A.5})$$

## A.2 Frobenius algebras

A unital associative algebra object internal to a monoidal category  $\mathcal{C}$  – or an *algebra* in  $\mathcal{C}$ , for short – is a triple  $A \equiv (A, \mu, \eta)$  consisting of an object  $A \in \mathcal{C}$ , a multiplication morphism  $\mu \in \text{Hom}_{\mathcal{C}}(A \otimes A, A)$  that satisfies the associativity property

$$\mu \circ (\text{id}_A \otimes \mu) = \mu \circ (\mu \otimes \text{id}_A), \quad (\text{A.6})$$

and a unit morphism  $\eta \in \text{Hom}_{\mathcal{C}}(\mathbf{1}, A)$  obeying the unit properties

$$\mu \circ (\eta \otimes \text{id}_A) = \text{id}_A = \mu \circ (\text{id}_A \otimes \eta). \quad (\text{A.7})$$

An algebra  $A$  in  $\mathcal{C}$  is called *haploid*, or *connected*, iff  $\dim_{\mathbb{k}} \text{Hom}_{\mathcal{C}}(\mathbf{1}, A) = 1$ .

Dually, a *coalgebra* in  $\mathcal{C}$  is a triple  $(C, \Delta, \varepsilon)$  consisting of an object  $C \in \mathcal{C}$  and comultiplication and counit morphisms  $\Delta \in \text{Hom}_{\mathcal{C}}(C, C \otimes C)$  and  $\varepsilon \in \text{Hom}_{\mathcal{C}}(C, \mathbf{1})$  obeying the coassociativity property  $(\text{id}_C \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}_C) \circ \Delta$  and  $(\varepsilon \otimes \text{id}_C) \circ \Delta = \text{id}_C = (\text{id}_C \otimes \varepsilon) \circ \Delta$ .

A *Frobenius algebra* in  $\mathcal{C}$  is a quintuple  $F \equiv (F, \mu, \eta, \Delta, \varepsilon)$  such that  $(F, \mu, \eta)$  is an algebra in  $\mathcal{C}$ ,  $(F, \Delta, \varepsilon)$  is a coalgebra in  $\mathcal{C}$ , and such that in addition the compatibility conditions

$$(\mu \otimes \text{id}_F) \circ (\text{id}_F \otimes \Delta) = \Delta \circ \mu = (\text{id}_F \otimes \mu) \circ (\Delta \otimes \text{id}_F) \quad (\text{A.8})$$

between multiplication and comultiplication are satisfied. (The conditions (A.8) are not independent; when combined with the algebra and coalgebra structures, each one implies the other.)

A Frobenius algebra  $F$  in a spherical fusion category  $\mathcal{C}$  is called *special* iff  $\dim(F) \neq 0$  and the two equalities

$$\mu \circ \Delta = \text{id}_F \quad \text{and} \quad \varepsilon \circ \eta = \dim(F) \text{id}_{\mathbf{1}} \quad (\text{A.9})$$

are satisfied, and  $F$  is called *symmetric* iff the equality

$$\left( (\varepsilon \circ \mu) \otimes \text{id}_{F^{\vee}} \right) \circ (\text{id}_F \otimes \text{coev}_F^r) = \left( \text{id}_{F^{\vee}} \otimes (\varepsilon \circ \mu) \right) \circ (\text{coev}_F^l \otimes \text{id}_F) \quad (\text{A.10})$$

of morphisms from  $F$  to  $F^{\vee} = {}^{\vee}F$  holds.

Owing to the Frobenius relation (A.8), the morphisms (A.10) are actually *isomorphisms*. Indeed, any Frobenius algebra in a rigid monoidal category is self-dual, i.e. is isomorphic to its right and to its left dual, and moreover the Frobenius-Schur indicator of  $F$  is  $+1$ . Specifically, a symmetric Frobenius algebra  $F$  in a spherical fusion category is self-dual in a particularly strong form: we can (and do) *identify*  $F$  with its (right and left) dual object and can choose both the right and left evaluation and coevaluation morphisms of  $F$  in such a way that they are expressible in terms of the structure morphisms of the algebra-coalgebra  $F$ , according to

$$\text{ev}_F^r = \varepsilon \circ \mu = \text{ev}_F^l \quad \text{and} \quad \text{coev}_F^r = \Delta \circ \eta = \text{coev}_F^l. \quad (\text{A.11})$$

### A.3 Modules and bimodules

A *left module*  $M \equiv (M, \rho)$  over an algebra  $(A, \mu, \eta)$  in a monoidal category  $\mathcal{C}$  is a pair consisting of an object  $M \in \mathcal{C}$  and a *representation morphism*  $\rho \in \text{Hom}_{\mathcal{C}}(A \otimes M, M)$  satisfying

$$\rho \circ (\text{id}_A \otimes \rho) = \rho \circ (\mu \otimes \text{id}_M) \quad \text{and} \quad \rho \circ (\eta \otimes \text{id}_M) = \text{id}_M. \quad (\text{A.12})$$

Similarly, a *right module*  $N \equiv (N, \mathfrak{q})$  over  $A$  is a pair consisting of an object  $N$  and a morphism  $\mathfrak{q} \in \text{Hom}_{\mathcal{C}}(N \otimes A, N)$  such that  $\mathfrak{q} \circ (\mathfrak{q} \otimes \text{id}_A) = \mathfrak{q} \circ (\text{id}_N \otimes \mu)$  and  $\mathfrak{q} \circ (\text{id}_N \otimes \eta) = \text{id}_N$ . Further, given two algebras  $(A, \mu, \eta)$  and  $(A', \mu', \eta')$  in  $\mathcal{C}$ , an  $A$ - $A'$ -bimodule in  $\mathcal{C}$  is a triple  $B = (B, \rho, \mathfrak{q})$  such that  $(N, \rho)$  is a left  $A$ -module,  $(B, \mathfrak{q})$  is a right  $A'$ -module and such that the left  $A$ -action and right  $A'$ -action commute, i.e.

$$\rho \circ (\text{id}_A \otimes \mathfrak{q}) = \mathfrak{q} \circ (\rho \otimes \text{id}_{A'}). \quad (\text{A.13})$$

Every algebra  $A$  has a natural structure of an  $A$ -bimodule. An algebra is called *simple* iff it is simple as a bimodule over itself; every connected algebra is simple.

Dual to the notion of a left module over an algebra is the one of a left *comodule*  $W \equiv (W, \delta)$  over a coalgebra  $C \equiv (C, \Delta, \varepsilon)$ : an object  $W \in \mathcal{C}$  together with a morphism  $\delta \in \text{Hom}_{\mathcal{C}}(W, C \otimes W)$  such that

$$(\text{id}_C \otimes \delta) \circ \delta = (\Delta \otimes \text{id}_W) \circ \delta \quad \text{and} \quad (\varepsilon \otimes \text{id}_W) \circ \delta = \text{id}_W. \quad (\text{A.14})$$

Right comodules are defined analogously. If  $A$  is a special Frobenius algebra, then any right  $A$ -module  $M = (M, \mathfrak{q})$  has a canonical structure of a right  $A$ -comodule, with the right  $A$ -coaction given by

$$\mathfrak{b} := (\mathfrak{q} \otimes \text{id}_A) \circ (\text{id}_M \otimes (\Delta \circ \eta)), \quad (\text{A.15})$$

and analogously for left modules.

Given algebras  $A, A'$  and  $A''$  and an  $A$ - $A'$ -bimodule  $(B, \rho, \mathfrak{q})$  as well as an  $A'$ - $A''$ -bimodule  $(B', \rho', \mathfrak{q}')$ , the *tensor product over  $A'$*  of  $B$  and  $B'$  is the  $A$ - $A''$ -bimodule  $B \otimes_{A'} B'$  that is defined by the universal property of coequalizing the two morphisms

$$B \otimes A' \otimes B' \begin{array}{c} \xrightarrow{\mathfrak{q} \otimes \text{id}_{B'}} \\ \xrightarrow{\text{id}_B \otimes \rho'} \end{array} B \otimes B'. \quad (\text{A.16})$$

In case  $A'$  is a special Frobenius algebra, the tensor product bimodule  $B \otimes_{A'} B'$  is isomorphic to the image of the split idempotent

$$P_{B \otimes_{A'} B'} := (\mathfrak{q} \otimes \rho') \circ (\text{id}_B \otimes (\Delta' \circ \eta') \otimes \text{id}_{B'}). \quad (\text{A.17})$$

For any object  $X \in \mathcal{C}$  the morphism space  $\text{Hom}_{\mathcal{C}}(X, M \otimes_A N)$  is isomorphic to the subspace  $\text{Hom}_{\mathcal{C}}^{(A)}(X, M \otimes N) \subseteq \text{Hom}_{\mathcal{C}}(X, M \otimes N)$  that consists of those morphisms which are invariant under post-composition with the idempotent  $P_{M \otimes_A N}$  defined according to (A.17). We use this isomorphism to tacitly identify

$$\text{Hom}_{\mathcal{C}}(X, M \otimes_A N) = \text{Hom}_{\mathcal{C}}^{(A)}(X, M \otimes N). \quad (\text{A.18})$$

Analogously we identify the space  $\text{Hom}_{\mathcal{C}}(M \otimes_A N, X)$  with the subspace  $\text{Hom}_{\mathcal{C}}^{(A)}(M \otimes N, X)$  of  $\text{Hom}_{\mathcal{C}}(M \otimes N, X)$  of morphisms invariant under pre-composition with  $P_{M \otimes_A N}$ , and similarly

for morphism spaces involving objects of the form  $M \otimes_{A_1} B_1 \otimes_{A_2} \dots \otimes_{A_n} B_n \otimes_{A_{n+1}} N$  with  $M$  a right  $A_1$ -module,  $N$  a left  $A_{n+1}$ -module, and, for  $i \in \{1, 2, \dots, n\}$ ,  $B_i$  an  $A_i$ - $A_{i+1}$ -bimodule.

Given an algebra  $A$  in a monoidal category  $\mathcal{C}$ , the right  $A$ -modules constitute the objects of the *category of the right  $A$ -modules*, which we denote by  $\text{mod-}A$ . The morphisms in  $\text{mod-}A$  are those morphisms in  $\mathcal{C}$  which intertwine the  $A$ -action, i.e.

$$\text{Hom}_{\text{mod-}A}((M, \mathfrak{q}), (M', \mathfrak{q}')) = \{f \in \text{Hom}_{\mathcal{C}}(M, M') \mid \mathfrak{q}' \circ (f \otimes \text{id}_A) = f \circ \mathfrak{q}\}. \quad (\text{A.19})$$

The categories  $A\text{-mod}$  of left  $A$ -modules and  $A\text{-mod-}A'$  of  $A$ - $A'$ -bimodules are defined analogously. We abbreviate

$$\text{Hom}_{\text{mod-}A} \equiv \text{Hom}_A \quad \text{and} \quad \text{Hom}_{A\text{-mod-}A'} \equiv \text{Hom}_{A|A'}. \quad (\text{A.20})$$

If  $\mathcal{C}$  is rigid and  $X = (\dot{X}, \rho)$  is a left module over an algebra  $A$  in  $\mathcal{C}$ , then the object  $\dot{X}^\vee$  that is right dual to the underlying object  $\dot{X}$  admits a natural structure of a right  $A$ -module, while the left dual object  ${}^\vee\dot{X}$  admits a natural structure of a right  ${}^\vee\vee A$ -module. Similarly, for a right module  $Y = (\dot{Y}, \mathfrak{q})$  over an algebra  $B$  in  $\mathcal{C}$ ,  ${}^\vee\dot{Y}$  and  $\dot{Y}^\vee$  admit natural structures of a left  $B$ -module and a left  $B^{\vee\vee}$ -module, respectively. In particular, if  $\mathcal{C}$  is strictly pivotal and  $A$  and  $B$  are symmetric Frobenius algebras in  $\mathcal{C}$ , then for  $X$  an  $A$ - $B$ -bimodule, the dual  $\dot{X}^\vee = {}^\vee\dot{X}$  admits a natural structure of a  $B$ - $A$ -bimodule. We call this  $B$ - $A$ -bimodule the bimodule *dual* to  $X$  and denote it by  $X^\vee$ . The assignment  $X \mapsto X^\vee$  is functorial. Strict pivotality also allows us to identify

$$X^{\vee\vee} = X \quad (\text{A.21})$$

as  $A$ - $B$ -bimodules.

## A.4 Module categories

A (*left*) *module category*  $\mathcal{M}$  over a monoidal category  $\mathcal{C}$  is an abelian category equipped with an exact functor  $\triangleright : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$ , called the *action* of  $\mathcal{C}$  on  $\mathcal{M}$ , and with natural families of isomorphisms  $(C \otimes C') \triangleright M \rightarrow C \triangleright (C' \triangleright M)$  and  $\mathbf{1} \triangleright M \rightarrow M$  for  $C, C' \in \mathcal{C}$  and  $M \in \mathcal{M}$  that satisfy pentagon and triangle relations analogous to those for the associator and unitors of a non-strict monoidal category. By invoking the equivalence  $\mathcal{M} \simeq \text{mod-}A_{\mathcal{M}}$  (see below) and strictifying the underlying monoidal category, for our purposes we can (and do) take also these isomorphisms to be identities. Right module categories as well as  *$\mathcal{C}$ - $\mathcal{D}$ -bimodule categories* over a pair of monoidal categories  $\mathcal{C}$  and  $\mathcal{D}$  are defined analogously.

A *module functor* between two module categories  $\mathcal{M}$  and  $\mathcal{N}$  over  $\mathcal{C}$  is a functor  $F : \mathcal{M} \rightarrow \mathcal{N}$  together with a natural family of isomorphisms  $F(X \triangleright M) \rightarrow X \triangleright F(M)$  for  $X \in \mathcal{C}$  and  $M \in \mathcal{M}$  that satisfy a pentagon and a triangle identity that express compatibility with the module constraints of  $\mathcal{M}$  and  $\mathcal{N}$ . Since we take the latter to be identities, we can take the module structure morphism of a module functor to be the identity as well.

The *direct sum*  $\mathcal{M}_1 \oplus \mathcal{M}_2$  of two module categories  $\mathcal{M}_1$  and  $\mathcal{M}_2$  over  $\mathcal{C}$  is the direct sum of  $\mathcal{M}_1$  and  $\mathcal{M}_2$  as abelian categories with  $\mathcal{C}$ -action (and analogously mixed associator and unitor) given by the sum of the  $\mathcal{C}$ -actions for  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . A module category is called *indecomposable* iff it is not equivalent to a non-trivial direct sum of module categories.

For a (left) module category  $\mathcal{M}$  over a monoidal category  $\mathcal{C}$  and any pair of objects  $M, M' \in \mathcal{M}$ , the *internal Hom*  $\underline{\text{Hom}}_{\mathcal{M}}(M, M')$  is an object in  $\mathcal{C}$  together with a natural family of isomorphisms

$$\text{Hom}_{\mathcal{C}}(C, \underline{\text{Hom}}_{\mathcal{M}}(M, M')) \xrightarrow{\cong} \text{Hom}_{\mathcal{M}}(C \triangleright M, M') \quad (\text{A.22})$$

for  $C \in \mathcal{C}$ . In other words, for every  $M' \in \mathcal{M}$  the functor  $\underline{\text{Hom}}_{\mathcal{M}}(-, M')$  is right adjoint to the action functor  $\triangleright$ . For a generic module category internal Homs need not exist. But they do exist under suitable finiteness and exactness conditions, which are in particular met for finite module categories over finite tensor categories, and in particular for all semisimple module categories over fusion categories. For any triple  $M, M', M''$  of objects in  $\mathcal{C}$  there is an associative multiplication morphism

$$\underline{\mu} \equiv \underline{\mu}_{M, M', M''} : \underline{\text{Hom}}(M', M'') \otimes \underline{\text{Hom}}(M, M') \rightarrow \underline{\text{Hom}}(M, M'') \quad (\text{A.23})$$

in  $\mathcal{C}$ . In particular, for any  $M \in \mathcal{M}$  the internal End  $\underline{\text{Hom}}(M, M)$  comes with a natural structure of an algebra in  $\mathcal{C}$ .

Let further  $G$  be a right exact  $\mathcal{C}$ -module functor between  $\mathcal{C}$ -module categories  $\mathcal{M}$  and  $\mathcal{N}$ . Then for any  $M, M' \in \mathcal{M}$ , by taking the composite

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(C, \underline{\text{Hom}}_{\mathcal{M}}(M, M')) &\xrightarrow{\cong} \text{Hom}_{\mathcal{M}}(C \triangleright M, M') \\ &\xrightarrow{G} \text{Hom}_{\mathcal{M}}(G(C \triangleright M), G(M')) \xrightarrow{\cong} \text{Hom}_{\mathcal{M}}(C \triangleright G(M), G(M')) \\ &\xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(C, \underline{\text{Hom}}_{\mathcal{N}}(G(M), G(M'))) \end{aligned} \quad (\text{A.24})$$

for  $C \in \mathcal{C}$  one defines a natural transformation from  $\mathcal{C}^{\text{opp}}$  to the category of vector spaces and thus a morphism

$$\underline{G} : \underline{\text{Hom}}_{\mathcal{M}}(M, M') \rightarrow \underline{\text{Hom}}_{\mathcal{N}}(G(M), G(M')) \quad (\text{A.25})$$

of internal Hom objects in  $\mathcal{C}$ .

A *pivotal* module category over a pivotal finite tensor category  $\mathcal{C}$  is a module category  $\mathcal{M}$  over  $\mathcal{C}$  such that for any  $M, M' \in \mathcal{M}$  there are functorial isomorphisms

$$\underline{\text{Hom}}(M, M')^{\vee} \xrightarrow{\cong} \underline{\text{Hom}}(M', M) \quad (\text{A.26})$$

between internal Homs and dual internal Homs which are compatible with the pivotal structure of  $\mathcal{C}$ .

In case that the finite tensor category  $\mathcal{C}$  is semisimple and thus a fusion category – the situation studied in the main text – an indecomposable pivotal module category over  $\mathcal{C}$  is the same as a *semisimple* module category. We prefer to use the qualification ‘pivotal’ also in the semisimple case because various results for pivotal (and thus semisimple) module categories over fusion categories extend to pivotal module categories over finite tensor categories, see [Sc, Sh, FuS2]. For instance, if the module category  $\mathcal{M}$  is pivotal, then for any  $M \in \mathcal{M}$  the algebra  $\underline{\text{Hom}}(M, M)$  in  $\mathcal{C}$  has a natural structure of a *symmetric Frobenius* algebra. If  $\mathcal{C}$  is semisimple and thus a fusion category, then the Frobenius algebra  $\underline{\text{Hom}}(M, M)$  is in addition special [Sc, Thm. 6.6].

By setting  $C \triangleright (M, \mathfrak{q}) := (C \otimes M, \text{id}_C \otimes \mathfrak{q})$ , the category  $\text{mod-}A$  of *right* modules over an algebra  $A$  in  $\mathcal{C}$  becomes a *left* module category over  $\mathcal{C}$ . Conversely, under suitable finiteness



conditions which are in particular fulfilled in the case of fusion categories, for any left module category  $\mathcal{M}$  over  $\mathcal{C}$  there exists an algebra  $A_{\mathcal{M}}$  in  $\mathcal{C}$  such that  $\mathcal{M}$  and  $\text{mod-}A_{\mathcal{M}}$  are equivalent as module categories over  $\mathcal{C}$ . This algebra is not unique; algebras  $A$  and  $A'$  such that  $\text{mod-}A$  and  $\text{mod-}A'$  are equivalent as module categories are called *Morita equivalent*. In fact, any such algebra is an internal Hom  $\underline{\text{Hom}}(M, M)$  for some  $M \in \mathcal{M}$ .

If  $\mathcal{M}$  is indecomposable, then the Morita class contains a representative that is connected. Thus in particular for every indecomposable pivotal module category over a pivotal fusion category  $\mathcal{C}$  there is a connected special symmetric Frobenius algebra  $A$  in  $\mathcal{C}$  such that  $\text{mod-}A$  is equivalent to  $\mathcal{M}$  as a module category. Moreover, for any two such algebras  $A$  and  $A'$ , every right exact  $\mathcal{C}$ -module functor from  $\mathcal{M} \simeq \text{mod-}A$  to  $\mathcal{M}' \simeq \text{mod-}A'$  is isomorphic to the functor

$$G^X := - \otimes_A X : \mathcal{M} \rightarrow \mathcal{M}' \quad (\text{A.27})$$

of forming the tensor product, over  $A$ , with some  $A$ - $A'$ -bimodule  $X$ . Finally, in terms of the Frobenius algebra  $A$  the internal Hom is given by

$$\underline{\text{Hom}}_{\text{mod-}A}(M, M') = M' \otimes_A M^\vee. \quad (\text{A.28})$$

## A.5 Drinfeld center

A *braiding*  $\beta$  on a monoidal category  $\mathcal{C}$  is a natural isomorphism from the tensor product functor to the opposite tensor product, i.e. a natural family of isomorphisms  $\beta_{C,C'} : C \otimes C' \rightarrow C' \otimes C$  that satisfy two *hexagon identities*, which express compatibility with the tensor product and for strict  $\mathcal{C}$  reduce to

$$\begin{aligned} \beta_{C,C' \otimes C''} &= (\text{id}_{C'} \otimes \beta_{C,C''}) \circ (\beta_{C,C'} \otimes \text{id}_{C''}) \\ \text{and } \beta_{C \otimes C', C''} &= (\beta_{C,C''} \otimes \text{id}_{C'}) \circ (\text{id}_C \otimes \beta_{C',C''}) \end{aligned} \quad (\text{A.29})$$

for  $C, C', C'' \in \mathcal{C}$ . A *half-braiding*  $\gamma$  on an object  $C \in \mathcal{C}$  is a natural family of isomorphisms

$$\gamma_X : C \otimes X \rightarrow X \otimes C \quad (\text{A.30})$$

for  $X \in \mathcal{C}$  obeying the single hexagon identity  $\gamma_{X \otimes X'} = (\text{id}_X \otimes \gamma_{X'}) \circ (\gamma_X \otimes \text{id}_{X'})$  for  $X, X' \in \mathcal{C}$ .

The *Drinfeld center*  $\mathcal{Z}(\mathcal{C})$  of a monoidal category  $\mathcal{C}$  is the category whose objects are pairs  $(C, \gamma)$  consisting of an object  $C \in \mathcal{C}$  and a half-braiding on  $C$ , and whose morphisms are those morphisms in  $\mathcal{C}$  which intertwine the half-braiding, i.e.  $\text{Hom}_{\mathcal{Z}(\mathcal{C})}((C, \gamma), (C', \gamma'))$  is the subset of those morphisms  $f \in \text{Hom}_{\mathcal{C}}(C, C')$  for which  $(f \otimes \text{id}_X) \circ \gamma_X = \gamma'_{X'} \circ (\text{id}_X \otimes f)$  for all  $X \in \mathcal{C}$ .

The half-braidings of its objects endow the Drinfeld center of any monoidal category with a braiding. The Drinfeld center of a fusion category is again a fusion category, and the Drinfeld center of a finite tensor category is again a finite tensor category. The Drinfeld center of a spherical fusion category is naturally a modular fusion category. If  $\mathcal{M}$  and  $\mathcal{N}$  are finite module categories over a finite tensor category  $\mathcal{C}$ , then the finite category  $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$  of right exact module functors is a finite module category over the Drinfeld center  $\mathcal{Z}(\mathcal{C})$ . Indeed, given a functor  $F \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$  and an object  $X = (\dot{X}, \gamma) \in \mathcal{Z}(\mathcal{C})$ , the right exact functor  $X \triangleright F$  defined by  $(X \triangleright F)(M) := \dot{X} \triangleright (F(M))$  is endowed with the structure of a  $\mathcal{C}$ -module functor via the isomorphisms

$$\begin{aligned} (X \triangleright F)(C \triangleright M) &= \dot{X} \triangleright F(C \triangleright M) \xrightarrow{\cong} (\dot{X} \otimes C) \triangleright F(M) \\ &\xrightarrow[\cong]{\gamma_{C \triangleright F(M)}} (C \otimes \dot{X}) \triangleright F(M) = C \triangleright ((X \triangleright F)(M)) \end{aligned} \quad (\text{A.31})$$

for  $C \in \mathcal{C}$ .

Given a braiding  $\beta$  on a monoidal category  $\mathcal{C}$ , the family  $\beta^{\text{rev}}$  consisting of the isomorphisms  $(\beta^{\text{rev}})_{C,C'} = \beta_{C',C}^{-1}$  is a braiding on  $\mathcal{C}$  as well. We denote the braided category with underlying monoidal category  $\mathcal{C}$  and braiding  $\beta^{\text{rev}}$  by  $\mathcal{C}^{\text{rev}}$ . A pivotal braided fusion category (and, more generally, a pivotal braided finite tensor category)  $\mathcal{C}$  is called *modular* iff a (distinguished) braided monoidal functor

$$\Xi_{\mathcal{C}} : \mathcal{C}^{\text{rev}} \boxtimes \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C}) \quad (\text{A.32})$$

from the Deligne product the categories  $\mathcal{C}^{\text{rev}}$  and  $\mathcal{C}$  to the Drinfeld center of  $\mathcal{C}$  is an equivalence.

## A.6 The central monad and comonad

A *monad* on a category  $\mathcal{C}$  is an endofunctor  $T : \mathcal{C} \rightarrow \mathcal{C}$  endowed with the structure of an algebra in the monoidal category of endofunctors of  $\mathcal{C}$ . Analogously, a *comonad* on  $\mathcal{C}$  is an endofunctor endowed with a coalgebra structure. A (left) *module* over a monad  $T$  on  $\mathcal{C}$  is a pair  $(C, \rho)$  consisting of an object  $C \in \mathcal{C}$  and a representation morphism  $\rho : T(C) \rightarrow C$  that is compatible in the standard way with the algebra structure of  $T$ .

On any finite tensor category there are canonically a monad  $Z \equiv Z_{\mathcal{C}}$  and a comonad  $\Delta \equiv \Delta_{\mathcal{C}}$ , called the *central monad* and *central comonad*, respectively. On objects their underlying endofunctors are given by

$$Z : C \mapsto \int^{X \in \mathcal{C}} X^{\vee} \otimes C \otimes X \quad \text{and} \quad \Delta : C \mapsto \int_{X \in \mathcal{C}} X \otimes C \otimes X^{\vee}, \quad (\text{A.33})$$

respectively. Here  $\int^{X \in \mathcal{C}}$  is a *coend* and  $\int_{X \in \mathcal{C}}$  an *end*, i.e.  $Z$  and  $\Delta$  come equipped with families of structure morphisms

$$i_{C;Y}^Z : Y^{\vee} \otimes C \otimes Y \rightarrow Z(C) \quad \text{and} \quad i_{C;Y}^{\Delta} : \Delta(C) \rightarrow Y \otimes C \otimes Y^{\vee} \quad (\text{A.34})$$

for  $C, Y \in \mathcal{C}$  that are dinatural in  $Y$  and obey a universal property with respect to dinaturality.

Being defined through a universal property, the objects  $Z(C)$  and  $\Delta(C)$  are (if they exist) unique up to unique isomorphism. If  $\mathcal{C}$  is semisimple and thus a fusion category – the case considered in the main text – as well as pivotal, so that we can identify left and right duals, they can both be expressed as a finite direct sum

$$Z(C) = \Delta(C) = \bigoplus_{i \in \mathcal{I}(\mathcal{C})} i^{\vee} \otimes C \otimes i \quad (\text{A.35})$$

over the isomorphism classes of simple objects.

The forgetful functor  $U$  from the Drinfeld center  $\mathcal{Z}(\mathcal{C})$  to  $\mathcal{C}$  that omits the half-braiding is exact and thus has both a left adjoint  $U^{1.a.}$  and a right adjoint  $U^{r.a.}$ . The endofunctors  $Z$  and  $\Delta$  on  $\mathcal{C}$  are related to these by

$$Z = U \circ U^{1.a.} \quad \text{and} \quad \Delta = U \circ U^{r.a.} \quad (\text{A.36})$$

Moreover, the adjunctions  $U^{1.a.} \vdash U$  and  $U \vdash U^{r.a.}$  are monadic and comonadic respectively, implying that there are canonical equivalences between  $\mathcal{Z}(\mathcal{C})$  and the categories of  $Z$ -modules and of  $\Delta$ -comodules. Indeed, the family of morphisms

$$\partial_X^{\mathcal{C}} := (\text{id}_X \otimes i_{C;X}^Z) \circ (\text{coev}_X^r \otimes \text{id}_C \otimes \text{id}_X) : C \otimes X \rightarrow X \otimes Z(C) \quad (\text{A.37})$$

for  $C, X \in \mathcal{C}$  generalizes the natural right coaction  $\partial_X^1: X \rightarrow X \otimes Z(\mathbf{1})$  of the coalgebra  $Z(\mathbf{1})$  on  $X \in \mathcal{C}$ ; this family is called the *universal coaction* associated with  $Z$  [TV, Ch. 9.1.2]. Given any  $Z$ -module  $(C, \rho)$ , the family

$$\gamma_X^{(C, \rho)} := (\text{id}_X \otimes \rho) \circ \partial_X^C : C \otimes X \rightarrow X \otimes C \quad (\text{A.38})$$

of morphisms in  $\mathcal{C}$  defines a half-braiding on  $C$ . This allows one to define a functor  $Z\text{-mod} \rightarrow \mathcal{Z}(\mathcal{C})$  by setting  $(C, \rho) \mapsto (C, \gamma^{(C, \rho)})$ . One can show (see e.g. [TV, Thm. 9.3]) that this functor is a monoidal isomorphism. The case of  $\Delta$ -comodules can be treated dually.

## A.7 Coends for generating subcategories

**Definition A.1.** [KeL, Def. 5.1.6]

Let  $\mathcal{A}$  be a finite  $\mathbb{k}$ -linear abelian category. A subset  $U$  of objects of  $\mathcal{A}$  is said to *p-generate*  $\mathcal{A}$  iff for any object  $X \in \mathcal{A}$  there exists an epimorphism  $h_X: \bigoplus_{i \in I} U_i \twoheadrightarrow X$  with a finite set  $I$  and  $U_i \in U$  for  $i \in I$ , and for any  $U \in U$  and any morphism  $f: U \rightarrow X$  there exists a morphism  $g: U \rightarrow \bigoplus_{i \in I} U_i$  such that  $f = h_X \circ g$ . We denote by  $\mathcal{U}$  the full subcategory of  $\mathcal{A}$  having  $U$  as its set of objects.

**Lemma A.2.** [KeL, Thm. 5.1.7]

Let  $\mathcal{A}$  and  $\mathcal{B}$  be finite  $\mathbb{k}$ -linear abelian categories and  $\mathcal{U} \subset \mathcal{A}$  be a full subcategory that p-generates  $\mathcal{A}$ . Let  $G: \mathcal{A} \times \mathcal{A}^{\text{opp}} \rightarrow \mathcal{B}$  be an exact functor and  $G': \mathcal{U} \times \mathcal{U}^{\text{opp}} \rightarrow \mathcal{B}$  its restriction to  $\mathcal{U}$ . Then the coends of  $G$  and  $G'$  exist, and the canonical morphism

$$\int^{X \in \mathcal{U}} G'(X, \bar{X}) \longrightarrow \int^{Y \in \mathcal{A}} G(Y, \bar{Y}) \quad (\text{A.39})$$

is an isomorphism.

**Lemma A.3.** Let  $\mathcal{D}$  be a  $\mathbb{k}$ -linear additive category. Then  $\mathcal{D}$  p-generates its Karoubian envelope  $\text{Kar}(\mathcal{D})$ .

*Proof.* Given any object  $X = (\dot{X}, p_X) \in \text{Kar}(\mathcal{D})$ , with  $\dot{X} \in \mathcal{D}$ , we can set  $h_X := p_X: (\dot{X}, \text{id}_{\dot{X}}) \twoheadrightarrow X$ . Further, for any object  $Y = (\dot{Y}, \text{id}_{\dot{Y}}) \in \mathcal{D} \subseteq \text{Kar}(\mathcal{D})$  and any morphism  $f: Y \rightarrow X$ , we have  $h_X \circ f \equiv p_X \circ f = f$ . Thus the conditions in Definition A.1 are satisfied by just taking  $h_X = p_X$  and  $g = f$ .  $\square$

## A.8 Graphical calculus for spherical fusion categories

A convenient tool for manipulating the string nets of our interest is the graphical calculus for morphisms in monoidal categories (see e.g. [TV, Ch. 2]), including the treatment of algebras and their modules and bimodules (see e.g. [FFRS1, App. A]). This calculus is tailored to the case that the monoidal category is strict, albeit as long as one deals with equalities between morphisms it still makes full sense for the general case, since all associativity and unit constraints involved are easily reconstructed.

In the context of *spherical fusion* categories, the graphical calculus can be simplified in several ways: First, taking the pivotal structure to be strict we can unambiguously identify

$$\begin{array}{c} \uparrow \\ X \\ \downarrow \end{array} = \begin{array}{c} \downarrow \\ X^\vee \\ \uparrow \end{array} \quad (\text{A.40})$$

and can identify the left (co)evaluation morphism for  $X$  with the right (co)evaluation morphism for  $X^\vee = {}^\vee X$ .

It also follows that for any pair  $X$  and  $Y$  of objects we have a distinguished linear isomorphism  $\zeta_{X,Y} : \text{Hom}_{\mathcal{C}}(\mathbf{1}, X \otimes Y) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(\mathbf{1}, Y \otimes X)$  given by

$$\zeta_{X,Y} : \begin{array}{c} X \quad Y \\ \uparrow \quad \uparrow \\ \boxed{f} \end{array} \mapsto \begin{array}{c} Y \quad X \\ \uparrow \quad \uparrow \\ \boxed{f} \end{array} \quad (\text{A.41})$$

Pivotality implies that  $\zeta_{Y,X} \circ \zeta_{X,Y} = \text{id}$ . As an important consequence, we can think of a morphism  $\varphi \in \text{Hom}_{\mathcal{C}}(\mathbf{1}, X_1 \otimes \cdots \otimes X_n)$  as depending only on the cyclic order of the tensor factors  $X_1, X_2, \dots, X_n$ . This can be made precise by associating to the cyclically ordered set  $(X_1, X_2, \dots, X_n)$  not the individual vector space  $\text{Hom}_{\mathcal{C}}(\mathbf{1}, X_1 \otimes \cdots \otimes X_n)$ , but rather the limit of the diagram

$$\begin{array}{c} \dots \xrightarrow{\zeta_{X_{i+1} \otimes \cdots \otimes X_{i-1}, X_i}} \text{Hom}_{\mathcal{C}}(\mathbf{1}, X_i \otimes X_{i+1} \otimes \cdots \otimes X_{i-1}) \\ \xrightarrow{\zeta_{X_i \otimes \cdots \otimes X_{i-2}, X_{i-1}}} \text{Hom}_{\mathcal{C}}(\mathbf{1}, X_{i-1} \otimes X_i \otimes \cdots \otimes X_{i-2}) \longrightarrow \dots \end{array} \quad (\text{A.42})$$

but for ease of notation we represent this limit by any of the vector spaces appearing in the diagram. To stress this cyclic aspect of morphisms, it is convenient to introduce the graphical notation [Ki]

$$\varphi = \begin{array}{c} X_n \quad X_1 \\ \uparrow \quad \uparrow \\ \circlearrowleft \varphi \\ \downarrow \quad \downarrow \end{array} \quad (\text{A.43})$$

in which the edges attached to a round coupon are to be thought of as being only cyclically ordered. Further, for suitable pairs of morphisms of this type there are partial composition maps

$$\begin{aligned} \circ_X : \quad \text{Hom}_{\mathcal{C}}(\mathbf{1}, Y \otimes X^\vee) \otimes_{\mathbb{k}} \text{Hom}_{\mathcal{C}}(\mathbf{1}, X \otimes Y') &\longrightarrow \text{Hom}_{\mathcal{C}}(\mathbf{1}, Y \otimes Y'), \\ \varphi \otimes \varphi' &\longmapsto (\text{id}_Y \otimes \text{ev}_X^r \otimes \text{id}_{Y'}) \circ (\varphi \otimes \varphi'). \end{aligned} \quad (\text{A.44})$$

In particular for each  $X \in \mathcal{C}$  this provides a non-degenerate pairing

$$\text{Hom}_{\mathcal{C}}(\mathbf{1}, X^\vee) \otimes_{\mathbb{k}} \text{Hom}_{\mathcal{C}}(\mathbf{1}, X) \longrightarrow \mathbb{k}. \quad (\text{A.45})$$

Given a basis  $\{\varphi^\alpha\}_{\alpha \in A}$  of  $\text{Hom}_{\mathcal{C}}(\mathbf{1}, X)$  we denote the basis of  $\text{Hom}_{\mathcal{C}}(\mathbf{1}, X^\vee)$  that is dual to  $\{\varphi^\alpha\}$  with respect to this pairing by  $\{\varphi_\alpha\}_{\alpha \in A}$ . Following [Ki] we suppress the symbols for summing over a pair of such bases, i.e. use the short-hand notation

$$\sum_{\alpha \in A} \varphi^\alpha \otimes \varphi_\alpha =: \begin{array}{c} X_n \quad X_1 \\ \nearrow \quad \nearrow \\ \circlearrowleft \alpha \\ \searrow \quad \searrow \\ X_1 \quad X_n \end{array} \quad \begin{array}{c} X_1 \quad X_n \\ \nearrow \quad \nearrow \\ \circlearrowright \alpha \\ \searrow \quad \searrow \\ X_n \quad X_1 \end{array} \quad (\text{A.46})$$

In addition to the convention (A.43), it is frequently convenient to regard a morphism in  $\text{Hom}_{\mathcal{C}}(\mathbf{1}, X)$  as an element of an isomorphic morphism space that is related to  $\text{Hom}_{\mathcal{C}}(\mathbf{1}, X)$  by rigidity, and also (in particular when product, coproduct or representation morphisms are involved) to suppress the round coupon in the notation. As an illustration, the completeness relation for  $\text{Hom}_{\mathcal{C}}(\mathbf{1}, X)$  that follows from the semisimplicity of  $\mathcal{C}$ , which we draw as

$$\bigoplus_{i \in \mathcal{I}(\mathcal{C})} d_i \quad \begin{array}{c} X_1 \cdots X_n \\ \nearrow \quad \nearrow \\ \circlearrowleft \alpha \\ \uparrow i \\ \circlearrowright \alpha \\ \searrow \quad \searrow \\ X_1 \cdots X_n \end{array} = \begin{array}{c} \uparrow \\ \uparrow \\ X_1 \cdots X_n \end{array} \quad (\text{A.47})$$

is in fact a family of equalities in various different morphism spaces, rather than a single equality: we may e.g. regard it as an equality in  $\text{End}_{\mathcal{C}}(X_1 \otimes \cdots \otimes X_n)$ , in which case the right hand side stands for the identity morphism  $\text{id}_{X_1 \otimes \cdots \otimes X_n}$ , while when regarding it as an equality in  $\text{Hom}_{\mathcal{C}}(\mathbf{1}, X_1 \otimes \cdots \otimes X_n \otimes X_n^\vee \otimes \cdots \otimes X_1^\vee)$ , the right hand side stands instead for the coevaluation  $\text{coev}_{X_1 \otimes \cdots \otimes X_n}^r$ .

As an example in which a coupon is omitted, we often think of the components of the half-braiding of an object  $Y = (U(Y), \gamma) \in \mathcal{Z}(\mathcal{C})$  as morphisms in  $\text{Hom}_{\mathcal{C}}(U(Y) \otimes X, X \otimes U(Y))$  and accordingly draw them as

$$\begin{array}{c} X \quad U(Y) \\ \nearrow \quad \nearrow \\ \circlearrowleft \gamma_X \\ \searrow \quad \searrow \\ U(Y) \quad X \end{array} = \begin{array}{c} X \quad U(Y) \\ \nearrow \quad \nearrow \\ \gamma_X \\ \searrow \quad \searrow \\ U(Y) \quad X \end{array} = \begin{array}{c} X \quad Y \\ \nearrow \quad \nearrow \\ \searrow \quad \searrow \\ Y \quad X \end{array} \quad (\text{A.48})$$

Here in the last picture we slightly abuse notation by using the label  $Y \in \mathcal{Z}(\mathcal{C})$  in the description of a morphism in  $\mathcal{C}$ , which has the advantage that the label at the over-crossing cannot be mixed up with a braiding in  $\mathcal{C}$  and can thus be omitted. (For clarity, in addition we draw here strands labeled by objects of  $\mathcal{C}$  in blue, while strands labeled by objects of  $\mathcal{Z}(\mathcal{C})$  are drawn in green.)

Similarly, instead of thinking of the product  $\mu$  of an algebra  $A$  as an element of the morphism space  $\text{Hom}_{\mathcal{C}}(A \otimes A, A)$ , we may also regard it as an element of  $\text{Hom}_{\mathcal{C}}(\mathbf{1}, A \otimes A^\vee, A^\vee)$ , which

amounts to the identification

$$\begin{array}{c} A \\ \uparrow \\ \textcircled{\mu} \\ \downarrow \downarrow \\ A^\vee \quad A^\vee \end{array} = \begin{array}{c} A \\ \uparrow \\ \bullet \\ \downarrow \downarrow \\ A \quad A \end{array} \tag{A.49}$$

The defining relations for an algebra-coalgebra to be Frobenius, special, and symmetric, respectively, are then drawn as

$$\begin{array}{c} A \quad A \\ \uparrow \quad \uparrow \\ \bullet \quad \bullet \\ \downarrow \downarrow \\ A \quad A \end{array} = \begin{array}{c} A \quad A \\ \uparrow \quad \uparrow \\ \bullet \\ \downarrow \downarrow \\ A \quad A \end{array} = \begin{array}{c} A \quad A \\ \uparrow \quad \uparrow \\ \bullet \quad \bullet \\ \downarrow \downarrow \\ A \quad A \end{array} \tag{A.50}$$

and as

$$\begin{array}{c} A \\ \uparrow \\ \bullet \\ \downarrow \downarrow \\ \bullet \\ \downarrow \\ A \end{array} = \begin{array}{c} \uparrow \\ \bullet \\ \downarrow \\ A \end{array} \quad \text{and} \quad \begin{array}{c} A \\ \uparrow \\ \bullet \\ \downarrow \downarrow \\ A \end{array} = \begin{array}{c} A \\ \downarrow \downarrow \\ \bullet \\ \downarrow \\ A \end{array} \tag{A.51}$$

respectively.

Finally, when dealing with a symmetric Frobenius algebra in a spherical fusion category we may further simplify the graphical description of morphisms involving  $A$  by using that its evaluation and coevaluation morphisms can be taken to be expressed through its structural morphisms as in (A.11). Specifically, we can identify  $A$  with its (left and right) dual and accordingly omit the orientation of an  $A$ -line, and we can use (A.11) to remove all occurrences of the unit and counit of  $F$ . The resulting *simplified graphs* have unoriented  $A$ -lines and do not involve univalent coupons for the unit or counit. Let us give two examples of such simplified graphs: First, the  $A$ -coaction defined in (A.15) can be simplified according to

$$\mathfrak{z} = \begin{array}{c} A \\ \uparrow \\ \bullet \\ \downarrow \downarrow \\ M \end{array} =: \begin{array}{c} \uparrow \\ \downarrow \\ M \end{array} \tag{A.52}$$

Second, the idempotent  $P_{B \otimes_{A'} B'}$  defined in (A.17) simplifies as

$$P_{B \otimes_{A'} B'} = \begin{array}{c} \uparrow \quad \uparrow \\ \downarrow \downarrow \\ B \quad B' \end{array} =: \begin{array}{c} \uparrow \quad \uparrow \\ \text{---} \\ B \quad B' \end{array} \tag{A.53}$$

As one specific application of the simplified graphical calculus we note that any *Frobenius move* between two (simplified) full Frobenius graphs in the sense of Definition 3.20 can be presented as a sequence of elementary moves of the following form:

- The a-move

$$(A.54)$$

- The b-move

$$(A.55)$$

- The r-move

$$(A.56)$$

Note that each of these elementary moves between simplified graphs stands for a whole collection of moves between non-simplified graphs. For instance, the moves

$$(A.57)$$

constitute three possible realizations of the b-move as a move between non-simplified graphs.

## A.9 Three useful lemmas

In this appendix we provide the proofs of observations that are used in the main text. We freely use the graphical calculus.

The following result is an important ingredient in the definition of the field map in Section 2.3, which allows us to treat boundary fields on the same footing as bulk and defect fields:

**Lemma A.4.** *Let  $\mathcal{C}$  be a finite tensor category and  $C$  and  $D$  be any two objects in  $\mathcal{C}$ . Then the objects  $L(C \otimes D)$  and  $L(D \otimes C)$  in  $\mathcal{Z}(\mathcal{C})$  are isomorphic.*

*Proof.* Consider morphisms  $f: L(C \otimes D) \rightarrow L(D \otimes C)$  and  $g: L(D \otimes C) \rightarrow L(C \otimes D)$  in  $\mathcal{Z}(\mathcal{C})$  that are defined, by means of the dinatural structure morphisms (A.34), by the equalities

$$\begin{array}{ccc}
 \begin{array}{c} L(D \otimes C) \\ \uparrow \\ \boxed{f} \\ \uparrow L(C \otimes D) \\ \begin{array}{c} \text{cap} \\ \downarrow \\ X \quad C \quad D \quad X \end{array} \end{array} & = & \begin{array}{c} L(D \otimes C) \\ \uparrow \\ \begin{array}{c} \text{cap}' \\ \downarrow \\ X \quad C \quad D \quad X \end{array} \end{array} \\
 \text{(A.58)} & & 
 \end{array}$$

and

$$\begin{array}{ccc}
 \begin{array}{c} L(C \otimes D) \\ \uparrow \\ \boxed{g} \\ \uparrow L(D \otimes C) \\ \begin{array}{c} \text{cap}' \\ \downarrow \\ X \quad D \quad C \quad X \end{array} \end{array} & = & \begin{array}{c} L(C \otimes D) \\ \uparrow \\ \begin{array}{c} \text{cap} \\ \downarrow \\ X \quad D \quad C \quad X \end{array} \end{array} \\
 \text{(A.59)} & & 
 \end{array}$$

respectively. The composite  $g \circ f$  satisfies

$$\begin{array}{ccc}
 \begin{array}{c} L(C \otimes D) \\ \uparrow \\ \boxed{g \circ f} \\ \uparrow L(C \otimes D) \\ \begin{array}{c} \text{cap} \\ \downarrow \\ X \quad C \quad D \quad X \end{array} \end{array} & = & \begin{array}{c} L(C \otimes D) \\ \uparrow \\ \begin{array}{c} \text{cap}' \\ \downarrow \\ X \quad C \quad D \quad X \end{array} \end{array} & = & \begin{array}{c} L(C \otimes D) \\ \uparrow \\ \begin{array}{c} \text{cap} \\ \downarrow \\ X \quad C \quad D \quad X \end{array} \end{array} \\
 \text{(A.60)} & & & & 
 \end{array}$$

where the second equality follows by dinaturality. Together with the snake identity (A.1) this shows that  $g \circ f = \text{id}_{L(C \otimes D)}$ . Similarly one sees that  $f \circ g = \text{id}_{L(D \otimes C)}$ . Moreover, it is readily checked that all morphisms involved are compatible with half-braidings, so that they are actually morphisms in  $\mathcal{Z}(\mathcal{C})$ . It thus follows that  $f$  is a (non-canonical) isomorphism  $L(C \otimes D) \xrightarrow{\cong} L(D \otimes C)$  in  $\mathcal{Z}(\mathcal{C})$ .  $\square$



**Lemma A.5.** For  $A$  a simple special symmetric Frobenius algebra in a pivotal fusion category  $\mathcal{C}$  and  $M$  a right  $A$ -module the equality

$$\begin{array}{c} M \\ \circlearrowright \\ \uparrow A \end{array} = \frac{\dim(M)}{\dim(A)} \begin{array}{c} \circ \\ \uparrow A \end{array} \tag{A.61}$$

holds, where the morphism on the right hand side is the counit  $\varepsilon$  of  $A$ .

*Proof.* Consider the endomorphism

$$f_M := \begin{array}{c} M \\ \circlearrowright \\ \uparrow A \end{array} \tag{A.62}$$

of  $A$ , with  $\Delta$  the coproduct of  $A$ . Since  $A$  is symmetric Frobenius,  $f_M$  is not just a morphism in  $\mathcal{C}$ , but even a morphism of  $A$ -bimodules (with respect to the regular  $A$ -bimodule structure on  $A$ ). Since  $A$  is simple, this implies that  $f_M$  is a multiple  $\xi_M \text{id}_A$  of the identity morphism. Post-composing with the counit then shows that the morphism on the left hand side of (A.61) equals  $\xi_M \varepsilon$ . Further pre-composing with the unit gives  $\dim(M) = \xi_M \varepsilon \circ \eta$ . Since  $A$  is special, this implies (A.61).  $\square$

**Lemma A.6.** For  $A$  a symmetric Frobenius algebra in a pivotal fusion category  $\mathcal{C}$  we have

$$\sum_{m \in \mathcal{I}(\text{mod-}A)} d_m^2 = \dim(A) D_{\mathcal{C}}^2, \tag{A.63}$$

with  $D_{\mathcal{C}}^2$  the global dimension of  $\mathcal{C}$ .

*Proof.* We have

$$\begin{aligned}
 \dim(A) D_{\mathcal{C}}^2 &\stackrel{(A.4)}{=} \sum_{i \in \mathcal{I}(\mathcal{C})} d_i \begin{array}{c} \text{---} \circlearrowright \text{---} \\ \uparrow A \end{array} = \sum_{\substack{i \in \mathcal{I}(\mathcal{C}) \\ m \in \mathcal{I}(\text{mod-}A)}} d_i d_m \begin{array}{c} \text{---} \circlearrowright \text{---} \\ \uparrow \alpha \\ \uparrow m \\ \uparrow \alpha \\ \uparrow A \end{array} \\
 &= \sum_{m \in \mathcal{I}(\text{mod-}A)} d_m \begin{array}{c} \text{---} \circlearrowright \text{---} \\ \uparrow m \\ \uparrow A \end{array} = \sum_{m \in \mathcal{I}(\text{mod-}A)} d_m^2.
 \end{aligned} \tag{A.64}$$

Here in the second equality we use the identity

$$\bigoplus_{m \in \mathcal{I}(A\text{-mod-}B)} d_m \begin{array}{c} X \\ \uparrow \\ \alpha \\ \uparrow \\ m \\ \uparrow \\ \alpha \\ \uparrow \\ X \end{array} = \begin{array}{c} \uparrow \\ X \end{array} \quad (\text{A.65})$$

valid for any right  $A$ - $B$ -bimodule  $X$ , where the  $\alpha$ -summation is over a basis of  $\text{Hom}_{A\text{-mod-}B}(m, X)$ ; this is a variant of the completeness relation (A.47). The third equality of (A.64) follows from the identity [FFRS1, Lemma 4.3]

$$\bigoplus_{i \in \mathcal{I}(C)} d_i \begin{array}{c} M \quad N \\ \uparrow \quad \uparrow \\ \alpha \\ \uparrow \\ i \\ \uparrow \\ \alpha \\ \uparrow \\ M \quad N \end{array} = \begin{array}{c} \uparrow \quad \uparrow \\ \text{---} \\ M \quad N \end{array} \quad (\text{A.66})$$

which holds for any right  $A$ -module  $M$  and left  $A$ -module  $N$  (with the  $\alpha$ -summation being over a basis of  $\text{Hom}_C^{(A)}(i, M \otimes N) \cong \text{Hom}_C(X, M \otimes_A N)$ ),  $\square$

Combining Lemma A.5 and Lemma A.6 we obtain

**Corollary A.7.** *For  $A$  a simple special symmetric Frobenius algebra in a pivotal fusion category the equality*

$$\sum_{m \in \mathcal{I}(\text{mod-}A)} \frac{d_m}{D_C^2} \begin{array}{c} m \\ \circlearrowright \\ \uparrow \\ A \end{array} = \begin{array}{c} \circ \\ \uparrow \\ A \end{array} \quad (\text{A.67})$$

holds.

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