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Generalizing the DGLAP Evolution of Fragmentation Functions to the Smallest x Values

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Abstract

An approach which unifies the Double Logarithmic Approximation at small x and the leading order DGLAP evolution of fragmentation functions at large x is presented. This approach reproduces exactly the Modified Leading Logarithm Approximation, but is more complete due to the degrees of freedom given to the quark sector and the inclusion of the fixed order terms. We find that data from the largest x values to the peak region can be better fitted than with other approaches.

The perturbative approach to Quantum Chromodynamics (QCD) is believed to solve all problems within its own limitations provided the correct choices of the expansion variable and the variable(s) to be fixed are used. However, perturbative QCD (pQCD) currently has the status of being a large collection of seemingly independent approaches, since a single unified approach valid for all processes is not known. This is problematic when one wants to use a range, qualitatively speaking, of different processes to constrain the same parameters, for example in global fits. What is needed is a single formalism valid over the union of all ranges that the various pQCD approaches allow. This unification must be consistent, i.e. it must agree with each approach in the set, when the expansion of that approach is used, up to the order being considered.

The evolution in the factorization scale Q^2 of fragmentation functions (FFs) $D(x,Q^2)$ (D is a vector containing all quark FFs D_q, all antiquark FFs $D_{\overline{q}}$ and the gluon FF D_g) at large and intermediate momentum fraction x is well described [1] by the leading order (LO) Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation [2]

$$\frac{d}{d\ln Q^2}D(x,Q^2)=\int_x^1\frac{dz}{z}a_s(Q^2)P^{(0)}(z)D\left(\frac{x}{z},Q^2\right), \tag{1}$$

where $P^{(0)}(z)$ are the LO splitting functions calculated from fixed order (FO) pQCD. We define $a_s=\alpha_s/(2\pi)$, which at LO obeys $a_s(Q^2)=1/(\beta_0\ln(Q^2/\Lambda^2_{\rm QCD}))$, where $\beta_0=(11/6)C_A-(2/3)T_Rn_f$ is the first coefficient of the beta function and $\Lambda_{\rm QCD}$ is the asymptotic scale parameter of QCD. For the color gauge group SU(3), the color factors appearing in this Letter are $C_F=3/4$, $C_A=3$, and $T_R=1/2$; n_f is the number of active quark flavors. On the other hand, at small x the Double Logarithmic Approximation (DLA) [3, 4]

$$\frac{d}{d\ln Q^2}D(x,Q^2)=\int_x^1\frac{dz}{z}\frac{2C_A}{z}Az^{2\frac{d}{d\ln Q^2}}\left[a_s(Q^2)D\left(\frac{x}{z},Q^2\right)\right]. \tag{2}$$

is required, where $A=0$ when D is a valence quark or non-singlet FF, while

$$A=\begin{pmatrix}0 & \frac{2C_F}{C_A}\\ 0 & 1\end{pmatrix} \tag{3}$$

when $D=(D_\Sigma,D_g)$, where $D_\Sigma=\frac{1}{n_f}\sum_{q=1}^{n_f}(D_q+D_{\overline{q}})$ is the singlet FF. The Modified Leading Logarithm Approximation (MLLA) [4, 5, 6] improves the description here by including a part of the FO contribution that is known to be important at small x. With certain qualifications [7], the MLLA leads to a good description of all data down to the smallest x values. However, what is still lacking is a single approach which can describe data from the largest to smallest values of x. We now construct such an approach, but leave the more detailed arguments to a future publication.

As $z \to 0$, the LO splitting function $a_s P^{(0)}(z)$ diverges due to terms of the form a_s/z. These double logarithms (DLs) occur at all orders in the FO splitting function, being generally of the form $(1/z)(a_s \ln z)^2 (a_s \ln^2 z)^r$ for $r = -1, ..., \infty$. As x decreases, Eq. (1) will therefore become a poor approximation once $\ln(1/x) = O(a_s^{-1/2})$. The reason why Eq. (2) is valid at low x is that it accounts for all double logarithms (DLs), by essentially summing them up. What we want, rather, is an evolution of the form of Eq. (1), but with the modification to the splitting function

$$a_s P^{(0)}(z) \to P^{\rm DL}(z, a_s) + a_s \overline{P}^{(0)}(z), \tag{4}$$

where $P^{\rm DL}(z, a_s)$ contains the complete contribution to the splitting function to all orders from the DLs, while $a_s \overline{P}^{(0)}(z)$ is the remaining FO contribution at LO. It is obtained by subtracting the LO DLs, already accounted for in $P^{\rm DL}$, from $a_s P^{(0)}(z)$ to prevent double counting. We will now use Eq. (2) to gain some understanding of $P^{\rm DL}$. For this we need to work in Mellin space, where the Mellin transform is defined by

$$f(\omega) = \int_0^1 dx x^\omega f(x). \tag{5}$$

Upon Mellin transformation, Eq. (2) becomes

$$\left(\omega + 2\frac{d}{d\ln Q^2}\right) \frac{d}{d\ln Q^2} D(\omega, Q^2) = 2C_A a_s(Q^2) A D(\omega, Q^2). \tag{6}$$

Making the replacement in Eq. (4) in Eq. (1) and neglecting the FO term $a_s \overline{P}^{(0)}(z)$ for now, taking its Mellin transform

$$\frac{d}{d\ln Q^2} D(\omega, Q^2) = P^{\rm DL}(\omega, a_s(Q^2)) D(\omega, Q^2). \tag{7}$$

and then substituting this into Eq. (6) gives an equation for $P^{\rm DL}$, viz.

$$2(P^{\rm DL})^2 + \omega P^{\rm DL} - 2C_A a_s A = 0. \tag{8}$$

We choose the solution

$$P^{\rm DL}(\omega, a_s) = \frac{A}{4}\left(-\omega + \sqrt{\omega^2 + 16 C_A a_s}\right), \tag{9}$$

since its expansion in a_s yields at LO the result

$$a_s P^{\rm DL(0)}(\omega, a_s) = \begin{pmatrix} 0 & a_s \frac{4C_F}{\omega} \\ 0 & a_s \frac{2C_A}{\omega} \end{pmatrix}, \tag{10}$$

which agrees with the LO DLs from the literature [8]. Equation (9) contains all terms in the splitting function of the form $(a_s/\omega)(a_s/\omega^2)^{r+1}$, being the DLs in Mellin space, and agrees with the results of Refs. [4, 5]. We now return to x space, where Eq. (9) reads

$$P^{\rm DL}(z, a_s) = \frac{A\sqrt{C_A a_s}}{z \ln \frac{1}{z}} J_1\left(4\sqrt{C_A a_s} \ln \frac{1}{z}\right), \tag{11}$$

with J_1 being the Bessel function of the first kind.

To summarize our approach, we evolve the fragmentation functions according to Eq. (1), but with the replacement of Eq. (4), where $P^{\rm DL}(z, a_s)$ is given by Eq. (11), and $a_s\overline{P}^{(0)}(z)$ is given by $a_s P^{(0)}(z)$ after the terms proportional to a_s/z have been subtracted.

Before we outline the phenomenological investigation of our approach, we note that it is more complete than the MLLA, which can be shown as follows. With $a_s\overline{P}^{(0)}(z)$ accounted for, Eq. (6) is modified to

$$\begin{aligned}\left(\omega + 2\frac{d}{d\ln Q^2}\right)\frac{d}{d\ln Q^2} D(\omega, Q^2) &= 2C_A a_s(Q^2) A D(\omega, Q^2) \\ &+ \left(\omega + 2\frac{d}{d\ln Q^2}\right) a_s(Q^2)\overline{P}^{(0)}(\omega) D(\omega, Q^2),\end{aligned} \tag{12}$$

up to terms which are being neglected in this Letter and which are neglected in the MLLA. If we approximate $a_s\overline{P}^{(0)}(\omega)$ by its single logarithms (SLs), defined at LO to be the coefficients of ω^0, equal to those in $a_s P^{(0)}(\omega)$,

$$P^{\rm SL(0)}(\omega) = \begin{pmatrix} 0 & -3C_F \\ \frac{2}{3}T_R n_f & -\frac{11}{6}C_A - \frac{2}{3}T_R n_f \end{pmatrix}, \tag{13}$$

then if we apply the approximate result that follows from the DLA at large Q,

$$D_{q,\overline{q}} = \frac{C_F}{C_A} D_g \tag{14}$$

(e.g. this can be derived from Eq. (11)), the gluon component of Eq. (12) becomes precisely the MLLA differential equation. Therefore we conclude that, since we do not use these two approximations, our approach is more complete and accurate than the MLLA.

We now test our approach by comparing its effects on fits of quark and gluon FFs to data to the standard FO DGLAP evolution. We use normalized differential cross section data for light charged hadron production in the process $e^+e^- \to (\gamma, Z) \to h + X$, where h is the observed hadron and X is anything else, from TASSO at $\sqrt{s} = 14$, 35, 44 GeV [9] and 22 GeV [10], MARK II [11] and TPC [12] at 29 GeV, TOPAZ at 58 GeV [13], ALEPH [14], DELPHI [15], L3 [16], OPAL [17] and MARK II [18] at 91 GeV, ALEPH [19] and OPAL [20] at 133 GeV, DELPHI at 161 GeV [21] and OPAL at 172, 183, 189 GeV [22] and 202 GeV [23]. We place a small x cut [7] on our data of

$$\xi = \ln(1/x) < \ln \frac{\sqrt{s}}{2M}, \tag{15}$$

where M is a mass scale of $O(1)$ GeV. We fit the gluon $g(x, Q_0^2)$, as well as the quark FFs

$$\begin{aligned} f_{uc}(x, Q_0^2) &= \frac{1}{2}\left(u(x, Q_0^2) + c(x, Q_0^2)\right), \\ f_{dsb}(x, Q_0^2) &= \frac{1}{3}\left(d(x, Q_0^2) + s(x, Q_0^2) + b(x, Q_0^2)\right), \end{aligned} \tag{16}$$

where $Q_0 = 14$ GeV. Since the hadron charge is summed over, we set $D_{\overline{q}} = D_q$. For each of these three FFs, we choose the parameterization

$$f(x, Q_0^2) = N \exp(-c \ln^2 x) x^\alpha (1-x)^\beta, \tag{17}$$

which at small x is a Gaussian in ξ for $c > 0$ with centre positive in ξ for $\alpha < 0$ as is found to be the case, while it reproduces the standard parameterization (i.e. that without the $\exp(-c\ln^2 x)$ factor) used in global fits at intermediate and large x. We use Eq. (14) to motivate the simplification

$$\begin{aligned} c_{uc} &= c_{dsb} = c_g, \\ \alpha_{uc} &= \alpha_{dsb} = \alpha_g \end{aligned} \tag{18}$$

to our parameterization. We also fit $\Lambda_{\rm QCD}$, giving 9 free parameters. Since we only use data for which $\sqrt{s} > m_b$, where $m_b \approx 5$ GeV is the mass of the bottom quark, and since $Q_0 > m_b$, we will take $n_f = 5$ in all our calculations. While the precise choice for n_f does not matter in the DLA, calculations in the FO approach depend strongly on it.

If the $(1-x)^\beta$ factors were absent, Eq. (14) would dictate that

$$N_{uc} = N_{dsb} = \frac{C_F}{C_A} N_g. \tag{19}$$

TABLE I: Parameter values for the FFs at $Q_0 = 14$ GeV parameterized as in Eq. (17) from a fit to all data listed in the text using DGLAP evolution in the FO approach to LO. $\Lambda_{\rm QCD} = 388$ MeV.

FF \ Parameter	N	β	α	c
g	0.22	−0.43	−2.38	0.25
(u+c)/2	0.49	2.30	[−2.38]	[0.25]
(d+s+b)/3	0.37	1.49	[−2.38]	[0.25]

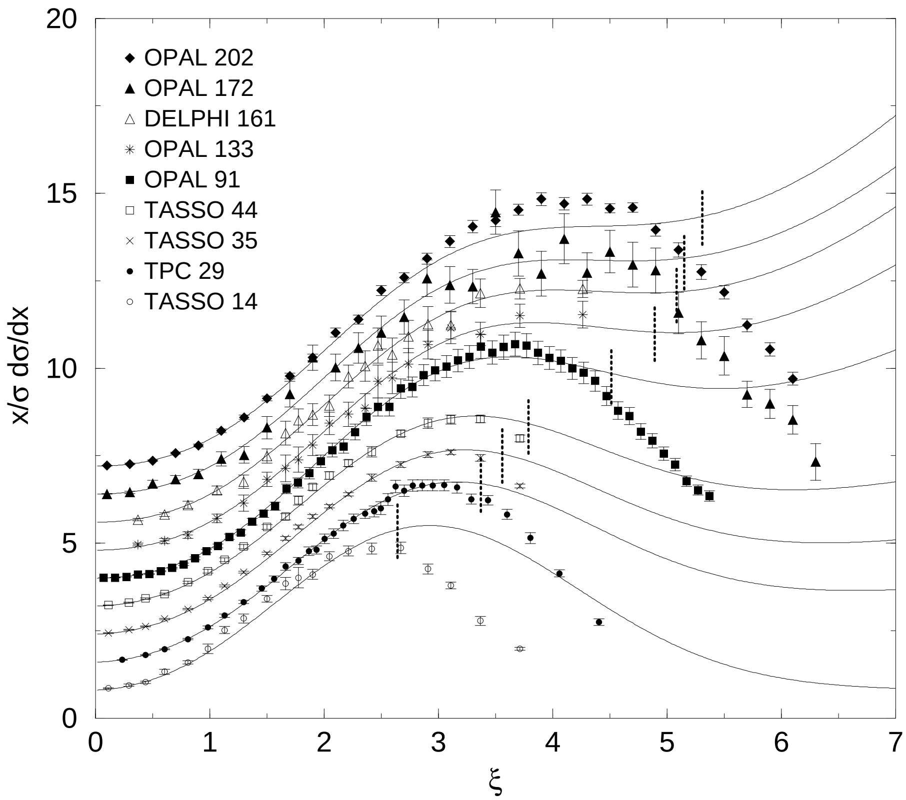

FIG. 1: Fit to data as described in Table I. Some of the data sets used for the fit are shown, together with their theoretical predictions from the results of the fit. Data to the right of the horizontal dotted lines were not used. Each curve is shifted up by 0.8 for clarity.

However, the $(1-x)^\beta$ factors are important at large x, and Eq. (14) is only an approximation at small x. Thus it will be interesting to see the deviations from Eq. (19) after fitting.

We first perform a fit to all data sets listed above using standard LO DGLAP evolution, i.e. Eq. (1) without the replacement in Eq. (4). We fit to those data for which Eq. (15) is obeyed with $M = 0.5$ GeV. This gives a total of 425 data points out of the available 492. We obtain $\chi^2_{\rm DF} = 3.0$, and the results are shown in Fig. 1 and Table I. The result for $\Lambda_{\rm QCD}$ is quite consistent with that of other analyses, at least within the theoretical error of a factor of $O(1)$. It is clear that FO DGLAP evolution fails in the description of the peak region and shows a different trend outside the fit range. The $\exp(-c\ln^2 x)$ factor does at least allow for the fit range to be extended to x values below that of $x = 0.1$, the lower limit of most global fits, to around $x = 0.05$ ($\xi = 3$) for data at the larger $\sqrt{s}$ values. Note that β_g is negative, while kinematics require it to be positive. However, this clearly does not make any noticeable difference to the cross section.

TABLE II: Parameter values for the FFs at $Q_0 = 14$ GeV parameterized as in Eq. (17) from a fit to all data listed in the text using DGLAP evolution in our approach. $\Lambda_{\rm QCD} = 801$ MeV.

FF \ Parameter	N	β	α	c
g	1.60	5.01	-2.63	0.35
(u+c)/2	0.39	1.46	[-2.63]	[0.35]
(d+s+b)/3	0.34	1.49	[-2.63]	[0.35]

Now we perform the same fit again, but using our approach, i.e. Eq. (1) with the replacement in Eq. (4), for the evolution. The results are shown in Table II and Fig. 2. We obtain $\chi^2_{\rm DF} = 2.1$, a significant improvement to the fit above with FO DGLAP evolution. This should also be compared to the fit to the same data in Ref. [24], where DL resummation was used within the MLLA but with neither FO terms nor quark freedom (i.e. Eq. (14) was imposed over the whole x range) and $\chi^2_{\rm DF} = 4.0$ was obtained. The data around the peak is now much better described. The energy dependence is well reproduced up to the largest $\sqrt{s}$ value, $\sqrt{s} = 202$ GeV. We conclude that, relative to the MLLA, the FO contributions in the evolution, together with freedom from the constraint of Eq. (14), make a significant improvement to the description of the data for ξ from zero to just beyond the peak. However, $\Lambda_{\rm QCD}$ is rather large, even within the theoretical errors. N_g is too large by a factor of about 2 relative to its prediction provided by Eq. (14). However, note that N_g is weakly constrained since the gluon FF couples to the data only through the evolution, requiring

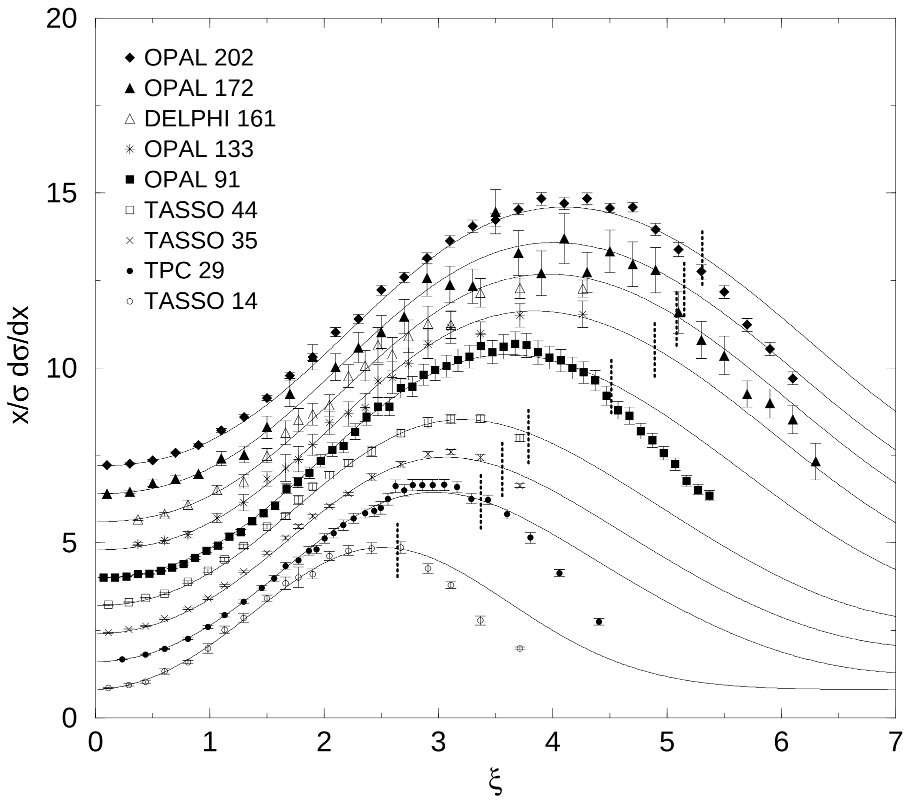

FIG. 2: Fit to data as described in Table II.

e.g. gluon data to be properly constrained. These problems are related to the worsening description of the data on moving beyond the peak, since fits in which the cuts were moved to larger ξ values gave an increase in $\Lambda_{\rm QCD}$ and N_g, as well as $\chi^2_{\rm DF}$. Figure 2 is repeated in Fig. 3, to show more clearly the good quality of the fit at intermediate and large x. A couple of points at $x = 0.9$ are not well described, although the data here are scarce and have larger errors.

In conclusion, we have proposed a single unified scheme which can describe a larger range in x than either FO DGLAP evolution or the DLA. Further improvement in the small x region can be expected from the inclusion of resummed SLs. Alternatively, improvement may be achieved by suppressing the higher moments's evolutions, since these are unstable yet formally of higher order [7], and the suppression of these effects provided by the FO contribution is unlikely to be sufficient. Our scheme allows a determination of quark and gluon FFs over a wider range of data than previously achieved, and should be incorporated into global fits of FFs such as that in Ref. [25] since the current range of $0.1 < x < 1$ is very limited.

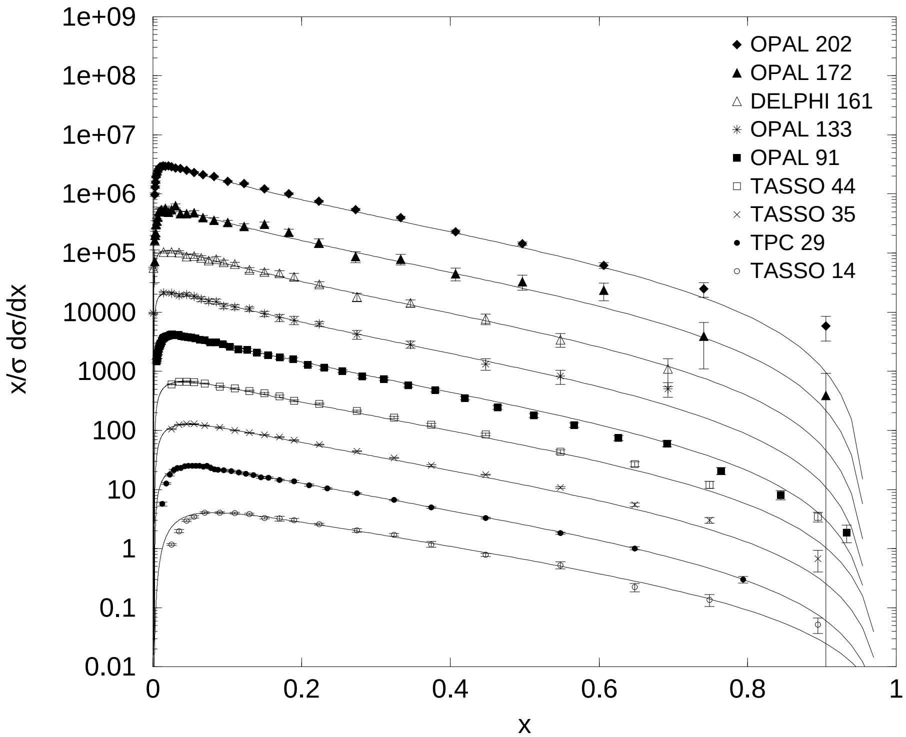

FIG. 3: As in Fig. 2, but with the cross section on a logarithmic scale versus x. Each curve, apart from the lowest one, has been rescaled relative to the one immediately below it by a factor of 5 for clarity.

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