THE LOVÁSZ-CHERKASSKY THEOREM FOR LOCALLY FINITE GRAPHS WITH ENDS

RAPHAEL W. JACOBS, ATTILA JOÓ, PAUL KNAPPE, JAN KURKOFKA, AND RUBEN MELCHER

Abstract. Lovász and Cherkassky discovered independently that, if G is a finite graph and $T \subseteq V(G)$ such that the degree $d_G(v)$ is even for every vertex $v \in V(G) \setminus T$, then the maximum number of edge-disjoint paths which are internally disjoint from T and connect distinct vertices of T is equal to $\frac{1}{2}\sum_{t\in T}\lambda_G(t, T \setminus \{t\})$ (where $\lambda_G(t, T \setminus \{t\})$ is the size of a smallest cut that separates t and $T \setminus \{t\}$). From another perspective, this means that for every vertex $t \in T$, in any optimal path-system there are $\lambda_G(t, T \setminus \{t\})$ many paths between t and $T \setminus \{t\}$. We extend the theorem of Lovász and Cherkassky based on this reformulation to all locally-finite infinite graphs and their ends. In our generalisation, T may contain not just vertices but ends as well, and paths are one-way (two-way) infinite when they establish a vertex-end (end-end) connection.

1. Introduction

A non-trivial path P is a T-*path* for a set T of vertices if P has its endvertices but no inner vertex in T. For disjoint vertex sets X and Y in a graph G, we write $\lambda_G(X, Y)$ for the size of a smallest cut in G that separates X and Y.

Now let T be any set of vertices in a finite graph G. In a set $\mathcal{P}$ of edge-disjoint T-paths, there are at most $\lambda_G(t, T \setminus \{t\})$ many paths that link a vertex $t \in T$ to $T \setminus \{t\}$. It follows that $|\mathcal{P}| \leqslant \frac{1}{2}\sum_{t\in T}\lambda_G(t, T \setminus \{t\})$. The question about the sharpness of this upper bound can be formulated in the following structural way. For every vertex $t \in T$, let $\mathcal{P}_t$ be a set of $\lambda_G(t, T \setminus \{t\})$ many edge-disjoint t–$(T \setminus \{t\})$ paths.

Question 1.1. *Can we choose the paths in the sets $\mathcal{P}_t$ for each vertex $t \in T$ in such a way that the union of all the sets $\mathcal{P}_t$ is an edge-disjoint path-system?*

Clearly, the answer is no: the three leaves of a star $K_{1,3}$ form a set T where each set $\mathcal{P}_t$ must consist of a single path, but the union $\bigcup_{t\in T}\mathcal{P}_t$ always contains two distinct paths that share an edge, no matter how we choose the paths in each set $\mathcal{P}_t$. Lovász and Cherkassky independently showed that, perhaps surprisingly, the answer is yes under the additional assumption that the graph G is *inner-Eulerian* for T in that every vertex of G which is not in T has even degree in G.

Theorem 1.2 (Lovász-Cherkassky Theorem [1,2])**.** *Let G be any finite graph, and let $T \subseteq V(G)$ such that G is inner-Eulerian for T. Then the maximum number of pairwise edge-disjoint T-paths in G is equal to*

$$\frac{1}{2}\sum_{t\in T}\lambda_G(t, T \setminus \{t\}).$$

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In this note, we extend the theorem of Lovász and Cherkassky to all locally-finite infinite graphs and their ends. For infinite graphs and their ends, we follow and assume familiarity with the terminology in [8, §8], in particular in §8.6. We allow a graph to have parallel edges, but we do not allow any loops; if a graph has no parallel edges, we call it *simple*. The *degree* of a vertex v in a graph G is the number $d_G(v) \in \mathbb{N} \cup \{\infty\}$ of edges of G incident with v. If all the vertices of G have finite degree, then we say that G is *locally finite*. Note that in a locally finite graph, there can be only finitely many parallel edges between any two vertices. We write $\hat{V}(G)$ for the union of the vertex set $V(G)$ of G and the set $\Omega(G)$ of all ends of G. An arc A in the end compactification $|G|$ of a locally-finite connected graph G is a *T-arc* for a set $T \subseteq \hat{V}(G)$ if A has its endpoints but no inner points in T. An arc A is an *X–Y arc* between two sets X and Y if A intersects X precisely in one endpoint and Y precisely in the other. We call an arc *graphic* if it is defined by a finite graph-theoretic path, a ray, or a double ray with its tails in distinct ends. Two arcs in $|G|$ are *edge-disjoint* if they do not meet in inner points of edges.

A subset $X \subseteq \hat{V}(G)$ *lives in* a subgraph $C \subseteq G$ or a vertex set $C \subseteq V(G)$ if all the vertices of X lie in C and all the rays of ends in X have tails in C or $G[C]$, respectively. A finite cut F of a graph G is an *X–Y cut* for two sets $X, Y \subseteq \hat{V}(G)$ if X and Y live in distinct sides of F. If $Y \subseteq \hat{V}(G)$ is a set of vertices and ends of a locally finite graph G, and $x \in \hat{V}(G)$ is not contained in the closure of Y in $|G|$, then G admits a (finite) x–Y cut, and we denote the least size of such a cut by $\lambda_G(x, Y)$.

If S is a set of vertices of a graph G and ω is an end of G, then by an *S–ω ray* we mean a ray which has precisely its first vertex in S and belongs to ω. An end ω of a locally finite graph is *even*, or *has even degree*, if there is a finite vertex set $S \subseteq V(G)$ such that for every finite set $S' \supseteq S$ of vertices, the maximum number of edge-disjoint S'–ω rays is even. Otherwise, ω is *odd* or *has odd degree*. We refer to [4, Section 3] for a discussion of the parity of ends. A locally finite graph G is *inner-Eulerian* for a set $T \subseteq \hat{V}(G)$ if every vertex and end in $\hat{V}(G) \setminus T$ has even degree in G.

Our main result reads as follows:

Theorem 1. *Let G be any locally-finite connected graph, and let $T \subseteq \hat{V}(G)$ be discrete in $|G|$ such that G is inner-Eulerian for T. Then $|G|$ contains a set $\mathcal{A}$ of pairwise edge-disjoint graphic T-arcs such that for every $t \in T$, the number of t–$(T \setminus \{t\})$ arcs in $\mathcal{A}$ is equal to $\lambda_G(t, T \setminus \{t\})$.*

The assumption that T is a discrete subset of $|G|$ naturally arises here as it is equivalent to asking that there exists a t–$(T \setminus \{t\})$ cut in G for each end $t \in T$, which precisely ensures that $\lambda_G(t, T \setminus \{t\})$ is defined for all $t \in T$. We remark that the assumption that T is discrete is also motivated by the work of Bruhn, Diestel, and Stein [3], which is a generalisation of the Erdős-Menger theorem by Aharoni and Berger [5] from infinite graphs to infinite graphs and their ends, under a similar assumption on the ends which implies discreteness in our setting. (Diestel discusses this assumption in detail in [6, §3].)

We conclude the introduction with two examples that discuss why certain weakenings in the assumptions of Theorem 1 cannot be made.

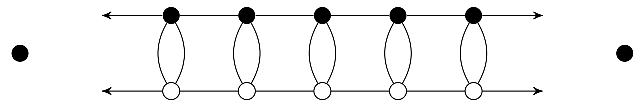

FIGURE 1. T consists of the black vertices and ends

Example 1.3. We claim that it is not possible to drop in Theorem 1 the requirement that T is discrete in $|G|$ and, instead, replace $\lambda_G(t, T \setminus \{t\})$ in the wording of the theorem with the maximum number $\mu_G(t, T \setminus \{t\})$ of pairwise edge-disjoint graphic t–$(T \setminus \{t\})$ arcs in $|G|$. Indeed, let G be obtained from the double ladder by duplicating each rung, and let T consist of both ends of G together with the vertices of one of the two main double rays; see Figure 1. Then

$$\mu_G(t, T \setminus \{t\}) = \begin{cases} 4 & \text{if } t \in T \cap V(G) \\ 1 & \text{if } t \in T \cap \Omega(G). \end{cases}$$

But any t–$(T \setminus \{t\})$ arc for some $t \in T \cap \Omega(G)$ does already preclude the existence of four t'–$(T \setminus \{t'\})$ arcs for some of the $t' \in T \cap V(G)$, and hence there is no desired arc-system for T.

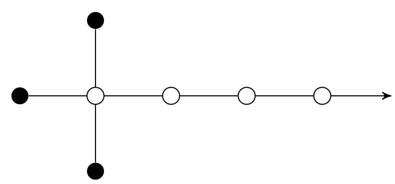

FIGURE 2. T consists of the black vertices

Example 1.4. Theorem 1 requires that all vertices and all ends in $\hat{V}(G) \setminus T$ have even degree in G. We claim that requiring even end degrees in addition to even vertex degrees is really necessary. Indeed, let us consider the tree G in Figure 2 and let T consist of its leaves. All the non-leaves of G have even degree, but the unique end of G has degree 1. Any set $\mathcal{A}$ of T-arcs has size at most one, so $\mathcal{A}$ contains no t–$(T \setminus \{t\})$ arc for at least one $t \in T$, even though $\lambda_G(t, T \setminus \{t\}) = 1$.

This note is organised as follows. In Section 2, we introduce the tools and terminology that we need. In Section 3, we prove Theorem 1.

2. Tools and Terminology

In this section, we recall a number of results that we need in order to prove Theorem 1. Originally, all these results have been proved only for simple graphs. However, all of them extend to graphs with parallel edges: Given a graph with parallel edges, just subdivide each edge once, apply the original result, and then suppress all subdividing vertices.

A cut F is said to lie *on* a set $\mathcal{P}$ of edge-disjoint paths in G if F consists of a choice of exactly one edge from each path in $\mathcal{P}$. For a vertex set $X \subseteq V(G)$, we denote by $d_G(X)$ the number of edges of G between X and its complement $V(G) \setminus X$. Note that this notation is consistent with the above definition of the degree $d_G(v)$ of a vertex v of G in that $d_G(v) = d_G(\{v\})$ for every vertex $v \in G$.

We will need the following generalisation of the Lovász-Cherkassky theorem to countably infinite graphs without ends:

Theorem 2.1 ([9, Theorem 1.3])**.** *Let G be any graph, and let $T \subseteq V(G)$ be a countable vertex set such that there is no $X \subseteq V(G) \smallsetminus T$ for which $d_G(X)$ is an odd natural number. Then G contains a set $\mathcal{P}$ of edge-disjoint T-paths such that for each vertex $t \in T$, the graph G contains a t–$(T \smallsetminus \{t\})$ cut on the set of t–$(T \smallsetminus \{t\})$ paths in $\mathcal{P}$.*

In order to use Theorem 2.1, we will need the following lemma:

Lemma 2.2. *If G is a locally finite connected graph and $T \subseteq \hat{V}(G)$ is given such that every vertex and end in $\hat{V}(G) \smallsetminus T$ has even degree, then there is no vertex set $X \subseteq V(G)$ whose closure in $|G|$ is disjoint from T and for which $d_G(X)$ is an odd natural number.*

This lemma follows directly from the Infinite Handshaking Lemma, as follows.

Lemma 2.3 (Infinite Handshaking Lemma [4, Proposition 15])**.** *The number of odd vertices and ends in a locally finite graph is even or infinite.*

Proof of Lemma 2.2. Assume for a contradiction that there is a vertex set $X \subseteq V(G) \smallsetminus T$ with $\overline{X} \cap T = \emptyset$, but such that $d_G(X)$ is an odd natural number. Then X and $V(G) \smallsetminus X$ are non-empty. Consider the graph H that arises from G by contracting $G - X$ to a single vertex v, keeping parallel edges. Then v has odd degree in H, but no other vertex or end of H has odd degree, which contradicts Lemma 2.3. □

Finally, we need the following lemma about the connectivity between an end of a graph and a finite set of vertices:

Lemma 2.4 ([4, Lemma 10])**.** *Let G be a locally finite connected graph, let ω be an end of G, and let S be a finite set of vertices in G. Then the maximum number of edge-disjoint S–ω rays is equal to the minimum size of a cut that separates S and ω.*

3. Proof of the main result

Proof of Theorem 1. Let us first show that T is countable. Since G is locally finite, $|G|$ is second-countable, meaning that the topology on $|G|$ has some countable base $\mathcal{U}$. Recall that by assumption, T is discrete in $|G|$. Therefore, we find for each $t \in T$ a basic neighbourhood $U_t \in \mathcal{U}$ so that U_t and $T \smallsetminus \{t\}$ are disjoint; in particular, $U_t \neq U_{t'}$ for distinct $t, t' \in T$. Then T is countable because $\mathcal{U} \supseteq \{U_t \colon t \in T\}$ is countable.

Since $T \cap \Omega(G)$ is countable, we may fix an enumeration $(w_n \colon n < \kappa)$ of $T \cap \Omega(G)$ where $\kappa := |T \cap \Omega(G)| \leqslant \aleph_0$. Next, we recursively find for each $n < \kappa$ an ω_n–$(T \smallsetminus \{\omega_n\})$ cut F_n of G such that the component C_n of $G - F_n$ in which ω_n lives is disjoint from the component C_m of

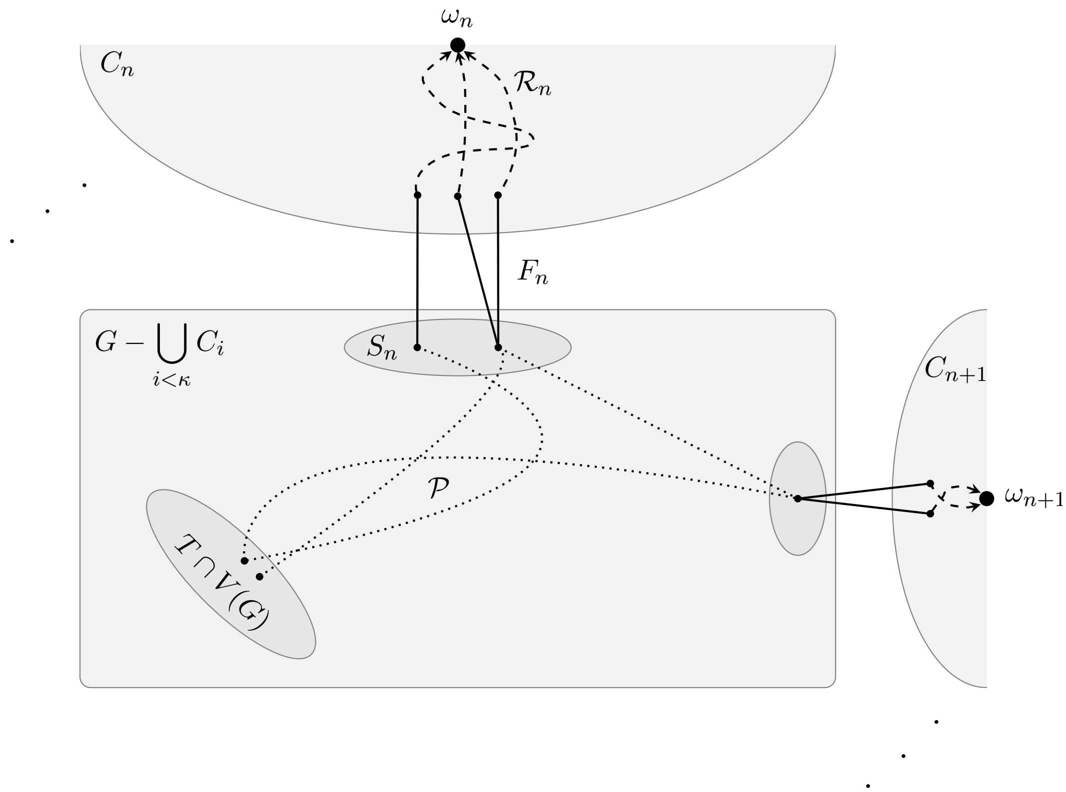

FIGURE 3. Situation in the proof of the main result

$G - F_m$ in which ω_m lives for all $m < \kappa$ other than n (see also Figure 3 for a visualisation of the whole proof).

Given any $n < \kappa$, assume that we have already found suitable finite cuts F_i for all $i < n$. Let G_n be the graph obtained from G by contracting each component C_i for $i < n$ to a single vertex v_i, keeping all the parallel edges that may arise. Since all cuts F_i for $i < n$ are finite, the contraction minor G_n is again locally finite. Let

$$T_n := (T \setminus \{\omega_i \colon i < n\}) \cup \{v_i \colon i < n\} \subseteq \hat{V}(G_n).$$

Note that $|G_n| = |G| \,/\, \{\overline{C_i} \colon i < n\}$, since all the F_i are finite.

Since T is discrete in $|G|$ and the components C_i for $i < n$ are disjoint, T_n is discrete in $|G_n|$. Therefore, the end ω_n is not contained in $\overline{T_n \setminus \{\omega_n\}}$ where the closure is taken in $|G_n|$. Thus, there is a smallest ω_n–$(T_n \setminus \{\omega_n\})$ cut F_n^* in $|G_n|$. This finite cut F_n^* in $|G_n|$ defines a finite cut F_n of $|G|$ of the same size.

Finally, we observe that the component $C_n \subseteq |G_n|$ of $G_n - F_n^*$ in which ω_n lives does not contain any v_i with $i < n$, since F_n^* separates ω_n and $T_n \setminus \{\omega_n\} \supseteq \{v_i \colon i < n\}$. Thus, C_n is also a component of $G - F_n$ and disjoint from each previous C_i. Altogether, this shows that the cut F_n is as desired.

Next, we simultaneously contract each component C_n for $n < \kappa$ to a single vertex v_n, again keeping all the parallel edges that may arise, and obtain a contraction minor G_κ of G. As before, this contraction minor G_κ is locally finite since all the cuts F_n are finite. Let

$$T_\kappa := (T \smallsetminus \Omega(G)) \cup \{v_n : n < \kappa\}\,,$$

and note that $T_\kappa \subseteq V(G_\kappa)$. Moreover, T_κ is countable because it has the same size as T.

By assumption, no vertex or end in $\hat{V}(G) \smallsetminus T$ has odd degree in G. Hence, by Lemma 2.2, there is no vertex set $X \subseteq V(G)$ whose closure in $|G|$ is disjoint from T and for which $d_G(X)$ is an odd natural number. It follows that there is no vertex set $X \subseteq V(G_\kappa)$ whose closure in $|G_\kappa|$ is disjoint from T_κ and for which $d_{G_\kappa}(X)$ is an odd natural number. And since T_κ contains no ends, it further follows that there is no vertex set $X \subseteq V(G_\kappa) \smallsetminus T_\kappa$ for which $d_{G_\kappa}(X)$ is an odd natural number. Therefore, we can apply Theorem 2.1 in G_κ to T_κ to obtain a set $\mathcal{P}$ of edge-disjoint T_κ-paths in G_κ with the following property: For every vertex $t' \in T_\kappa$, there is a cut $F'_{t'}$ of G on the set of t'–$(T_\kappa \smallsetminus \{t'\})$ paths in $\mathcal{P}$. Note that all cuts $F'_{t'}$ are finite, because G_κ is locally finite and the paths in $\mathcal{P}$ are edge-disjoint. It remains to translate the set $\mathcal{P}$ of edge-disjoint T_κ-paths in G_κ into the desired set $\mathcal{A}$ of pairwise edge-disjoint graphic T-arcs in $|G|$.

For every $n < \kappa$, the finite ω_n–$(T \smallsetminus \{\omega_n\})$ cut F_n has smallest size. Let S_n be the set of those endvertices of F_n in $G - C_n$. By definition, F_n is a minimal S_n–ω_n cut in G. So we can apply Lemma 2.4 to S_n and ω_n to find a set $\mathcal{R}_n$ of $|F_n|$ many edge-disjoint S_n–ω_n rays. Note that the rays in $\mathcal{R}_n$ use only vertices of $V(C_n) \cup S_n$ since F_n is a cut. In particular, there is for each edge in F_n precisely one ray in $\mathcal{R}_n$ which starts in this edge.

Now every T_κ-path $P \in \mathcal{P}$ in G_κ uniquely defines a graphic T-arc in $|G|$ as follows: If the first or last edge e of P runs between a contraction vertex v_n and another vertex u, then we replace it with the unique u–ω_n ray in $\mathcal{R}_n$ which traverses the inner points of e and add the end ω_n to it. Here we allow one exception: if P consists of just one edge e between two vertices v_n and v_m in T_κ, then we replace e with the double ray which arises from the two rays in $\mathcal{R}_n$ and $\mathcal{R}_m$ that start in e, and add the ends ω_n and ω_m to it. In either case, let us write $A(P)$ for the graphic arc defined by P in this way. We claim that $\mathcal{A} := \{A(P) \mid P \in \mathcal{P}\}$ is the desired set of graphic T-arcs.

The arcs in $\mathcal{A}$ are edge-disjoint because the paths in $\mathcal{P}$ are edge-disjoint, the components C_n are disjoint, and the rays in each set $\mathcal{R}_n$ are edge-disjoint. It remains to show that for each $t \in T$, the number of arcs in $\mathcal{A}$ that link t to $T \smallsetminus \{t\}$ is equal to $\lambda_G(t, T \smallsetminus \{t\})$. Given any $t \in T$, let $t' \in T_\kappa$ be equal to t if t is a vertex, and let $t' := v_n$ if t is an end ω_n. The finite t'–$(T_\kappa \smallsetminus \{t'\})$ cut $F'_{t'}$ of G_κ witnesses that the set of t'–$(T_\kappa \smallsetminus \{t'\})$ paths in $\mathcal{P}$ form a maximal sized edge-disjoint t'–$(T_\kappa \smallsetminus \{t'\})$ path system in G_κ. Since the t–$(T \smallsetminus \{t\})$ arc system defined by $\mathcal{A}$ arises from this path system by replacing each path P with the arc $A(P)$, the number of arcs in this system is equal to the size of the finite cut $F'_{t'}$. As the t'–$(T_\kappa \smallsetminus \{t'\})$ cut $F'_{t'}$ of G_κ induces a t–$(T \smallsetminus \{t\})$ cut of G of the same size, we have $\lambda_G(t, T \smallsetminus \{t\}) \leqslant |F'_{t'}|$, which completes the proof. $\square$

References

[1] László Lovász, *On some connectivity properties of Eulerian graphs*, Acta Mathematica Academiae Scientiarum Hungarica **28** (1976), no. 1-2, 129–138, DOI 10.1007/BF01902503.

[2] Boris V. Cherkassky, *A solution of a problem on multicommodity flows in a network (English title), Reshenie odnoi zadachi o mnogoproduktovykh potokakh v seti (Russian title)*, Ekonomika i Matematicheskie Metody **13** (1977), no. 1, 143–151. (in Russian).

[3] Henning Bruhn, Reinhard Diestel, and Maya Stein, *Menger's theorem for infinite graphs with ends*, Journal of Graph Theory **50** (2005), no. 3, 199–211, DOI 10.1002/jgt.20108.

[4] Henning Bruhn and Maya Stein, *On end degrees and infinite cycles in locally finite graphs*, Combinatorica **27** (2007), no. 3, 269, DOI 10.1007/s00493-007-2149-0.

[5] Ron Aharoni and Eli Berger, *Menger's theorem for infinite graphs*, Inventiones mathematicae **176** (2009), no. 1, 1–62, DOI 10.1007/s00222-008-0157-3.

[6] Reinhard Diestel, *The countable Erdős–Menger conjecture with ends*, Journal of Combinatorial Theory, Series B **87** (2003), no. 1, 145–161, DOI 10.1016/S0095-8956(02)00032-1.

[7] ______, *Locally finite graphs with ends: A topological approach, I. Basic theory*, Discrete Mathematics **311** (2011), no. 15, 1423–1447, DOI 10.1016/j.disc.2010.05.023.

[8] ______, *Graph Theory*, 5th ed., Graduate texts in mathematics, vol. 173, Springer, 2016.

[9] Attila Joó, *The Lovász-Cherkasskiy theorem in countable graphs*, arXiv preprint, arXiv:2102.04203 (2021). available at `https://arxiv.org/abs/2102.04203`.

Attila Joó: Universität Hamburg, Department of Mathematics, Bundesstrasse 55 (Geomatikum), 20146 Hamburg, Germany and Alfréd Rényi Institute of Mathematics, Set theory and general topology research division, 13-15 Reáltanoda St., Budapest, Hungary

Email address: `attila.joo@uni-hamburg.de, jooattila@renyi.hu`

Raphael W. Jacobs, Paul Knappe, Jan Kurkofka, Ruben Melcher: Universität Hamburg, Department of Mathematics, Bundesstrasse 55 (Geomatikum), 20146 Hamburg, Germany

Email address: `raphael.jacobs@uni-hamburg.de, paul.knappe@uni-hamburg.de`

Email address: `jan.kurkofka@uni-hamburg.de, ruben.melcher@uni-hamburg.de`