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CUTTING A CAKE FOR INFINITELY MANY GUESTS

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Abstract. Fair division with unequal shares is an intensively studied recourse allocation problem. For $i \in [n]$, let μ_i be an atomless probability measure on the measurable space $(C, \mathcal{S})$ and let t_i be positive numbers (entitlements) with $\sum_{i=1}^{n} t_i = 1$. A fair division is a partition of C into sets $S_i \in \mathcal{S}$ with $\mu_i(S_i) \geq t_i$ for every $i \in [n]$.

We introduce new algorithms to solve the fair division problem with irrational entitlements. They are based on the classical Last diminisher technique and we believe that they are simpler than the known methods. Then we show that a fair division always exists even for infinitely many players.

1. Introduction

Cake cutting is a metaphor of the distribution of some inhomogeneous continuos goods and is intensively investigated by not just mathematicians but economists and political scientists as well. The preferences of the players P_i involved in the sharing are usually represented as atomless probability measures μ_i defined on a common σ-algebra $\mathcal{S} \subseteq \mathcal{P}(C)$ of the possible 'slices' of the 'cake' C. One option of how a division can be "good" is *proportionality*. This means that each of the n players gets at least one nth of the cake according to their own measurement, i.e. $C = \bigsqcup_{i=1}^{n} S_i$ with $\mu_i(S_i) \geq \frac{1}{n}$. The division is called *strongly proportional* if all these inequalities are strict. For $n = 2$ a proportional division can be found by the so called "Cut and choose" procedure. This was used by Abraham and Lot in the Bible to share Canaan. Abraham divided Canaan into two parts which have equal value for him and then Lot chose his favourite among these two parts leaving Abraham the other one. For a general n, Steinhaus challenged his students Banach and Knaster to find a solution that they successfully accomplished by developing the so called "Last Diminisher" procedure (see [12]). In this method P_1 picks a slice T_1 with $\mu_1(T_1) = \frac{1}{n}$. If $\mu_2(T_1) > \frac{1}{n}$, then P_2 diminishes T_1 in the sense that he takes an $T_2 \subseteq T_1$ with $\mu_2(T_2) = \frac{1}{n}$, otherwise he lets $T_2 := T_1$. They proceed similarly and slice T_n is allocated to the player who lastly diminished or to P_1 if nobody did so. Then the remaining cake worth at least $\frac{n-1}{n}$ for each of the remaining $n-1$ players and they can continue using the same protocol.

A natural extension of the concept of proportional division is the so called "fair division with unequal shares". In this variant there are entitlements $t_i > 0$ associated to the players satisfying $\sum_i t_i = 1$. A division is called (strongly) *fair* if the slice S_i given to player P_i worths for him at least (more than) t_i, i.e. $\mu_i(S_i) \geq t_i$ ($\mu_i(S_i) > t_i$) holds for each i. If all

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of these entitlements are rational numbers, say $\frac{p_1}{q}, \dots, \frac{p_n}{q}$, then a fair division according to them can be reduced to a proportional division problem for $\sum_{i=1}^n p_i$ players where measure μ_i is "cloned" to p_i copies. In the presence of irrational entitlements such a "player-cloning" argument is no more applicable.

Several finite procedures were developed to find a (strongly) fair division allowing irrational entitlements (see [2, 5, 11]). Our first contribution (Section 2) is two such proceeders which we believe are simpler than the known methods. Both of them are based on Last Diminisher-type of principles. In the first one we use rational approximation of irrationals while in the second one not even that is necessary.

It was shown in [6] based on Lyapunov's theorem that if not all the measures are identical, then a strongly proportional division exists. A constructive proof was obtained later in [13] which was then further developed for the case of unequal shares (i.e. strong fairness) in [2]. We point out in Section 3 that the strongly fair division problem (for potentially infinitely many players) can be actually reduced to the fair division problem in a completely elementary way, which reduction we need later.

In the last section (Section 4) we consider the (strongly) fair division problem for infinitely many players. Rational entitlements do not make this problem easier since representing them with a common denominator is impossible in general. Since the entitlements sum up to 1, they must converge to 0, thus extension of protocols in which one need to start with the smallest positive entitlement (like the one given in [5]) is problematic. By Last Diminisher-type of methods we are facing in addition the difficulty that diminishing infinitely often might be necessary in which case no "last diminisher" exists, moreover, we may end up with the empty set as a limit of the iterated trimmings. Eliminating one player and using induction for the rest is also not applicable for obvious reasons. Although the so called Fink protocol (see [7]) can be considered as such a player-eliminating recursive algorithm, it inspired our procedure that finds a fair division for infinitely many players.

Theorem 1.1. *Assume that $(C, \mathcal{S})$ is a measurable space and for $i \in \mathbb{N}$, μ_i is an atomless probability measure defined on $\mathcal{S}$ and t_i is a positive number such that $\sum_{i=0}^{\infty} t_i = 1$. Then there is a partition $C = \bigsqcup_{i=0}^{\infty} S_i$ such that $S_i \in \mathcal{S}$ with $\mu_i(S_i) \geq t_i$ for each $i \in \mathbb{N}$. Furthermore, if not all the μ_i are identical, then '$\mu_i(S_i) \geq t_i$' can be strengthened to '$\mu_i(S_i) > t_i$' for every $i \in \mathbb{N}$.*

Let us mention that cake cutting problems have a huge literature and this particular model and notion of fairness that we consider is only a tiny fragment of it. About the so called exact, envy-free and equitable divisions (none of which are extendable to infinitely many players for obvious reasons) and the corresponding existence results a brief but informative survey can be found in [4]. For a more general picture about this field, including completely different mathematical models of the problem, we refer to [1], [3], [9] and [10].

2. 'Last Diminisher'-type of procedures for fair division with irrational entitlements

Our aim is to find a fair division $S_1, \dots, S_n$ for players $P_1, \dots, P_n$ with respective atomless probability measures $\mu_1, \dots, \mu_n$ and (potentially irrational) entitlements $0 < t_1 \leq t_2 \leq$

$\cdots \leq t_n < 1$ where $\sum_{i=1}^n t_i = 1$. An algorithm was given in Section 7 of [5] that reduces this problem to two sub-problems in one of which the number of players is smaller by one while in the other all the entitlements are rational and the number of players remains the same. We introduce two algorithms both of which solves the problem in finitely many steps and based on 'Last diminisher'-type of ideas. The first one reduces the problem to another one in which either the number of players is smaller by one or all the entitlements are rationals and the number of players is the same. In this algorithm we need to pick rational numbers from non-degenerate intervals (which was used in the algorithms given in [5, 13]). In the second algorithm no such rational approximation is needed.

As it is standard in the cake cutting literature, algorithms use certain queries. We allow the following operations.

- The four basic arithmetical operations and comparison on $\mathbb{R}$.
- The set operations on $\mathcal{S}$.
- Computing $\mu_i(S)$ for some $i \in [n]$ where slice S is obtained in a previous step.
- Cutting a slice $S' \subseteq S$ with $\mu_i(S) = \alpha$ for an $i \in [n]$ and $\alpha \in [0, \mu_i(S)]$ where either $S = C$ or S is obtained in a previous step.[1]

2.1. **Algorithm I.** Player P_1 picks some T_1 with $\mu_1(T_1) = t_1$. If T_i is already defined for some $i < n$, we let $T_{i+1} := T_i$ if $\mu_{i+1}(T_i) \leq t_1$ and we define T_{i+1} to be a subset of T_i with $\mu_{i+1}(T_{i+1}) = t_1$ if $\mu_{i+1}(T_i) > t_1$. After the recursion is done, $\mu_i(T_n) \leq t_1$ holds for each i and there is equality for at least one index.

If $\mu_1(T_n) = t_1$, then we let $S_1 := T_n$ and remove player P_1 from the process. Since the rest of the cake worth at least $1 - t_1$ for all the players, dividing it fairly with respect to the entitlements $\frac{t_i}{1-t_1}$ for $1 < i \leq n$ leads to a fair division. Thus we invoke the algorithm for this sub-problem with less players.

If $\mu_1(T_n) < t_1$, then there must be a player who diminished the slice during the recursion. Let k be the largest index for which P_k is such a player. We allocate T_n to P_k but we do not remove P_k from the process unless $t_1 = t_k$. In order to satisfy P_k, he needs to get at least the $t'_k := \frac{t_k - t_1}{\mu_k(C \setminus T_n)}$ fraction of the rest of the cake $C \setminus T_n$ according to his measure μ_k, while for $i \neq k$ player P_i should get at least the fraction $t'_i := \frac{t_i}{\mu_i(C \setminus T_n)}$ of $C \setminus T_n$ w.r.t. μ_i. As we already noticed $\mu_i(T_n) \leq t_1$ and hence $\mu_i(C \setminus T_n) \geq 1 - t_1$ for every i, furthermore, the inequality is strict for $i = 1$ in this branch of the case distinction. Therefore

$$\sum_{i=1}^n t'_i < \frac{t_k - t_1}{1 - t_1} + \sum_{i \neq k} \frac{t_i}{1 - t_1} = \frac{(\sum_{i=1}^n t_i) - t_1}{1 - t_1} = 1.$$

Thus we can pick rational numbers $t''_i > t'_i$ with $\sum_{i \leq n} t''_i = 1$. Finally, we use a subroutine to divide $C \setminus T_n$ fairly among the players w.r.t. the rational entitlements t''_i to obtain a strongly fair division for the original problem.

2.2. **Algorithm II.** In this algorithm no 'rounding up to rationals' is necessary. We shall make several rounds and in each of them allocate a slice chosen in a 'Last diminisher' manner. The satisfied players are dropping out of the process. The algorithm itself is quite simple in this case as well but the proof of the correctness is somewhat more involved.

[1] It is well-defined because μ_i is atomless (see [8, Theorem 5]).

For $1 \leq i \leq n$, we denote by S_i^m the portion allocated to player P_i at the beginning of round m. We set $S_i^0 = \emptyset$ for every i. The rest of the cake is $C_m := C \setminus \bigcup_{i=1}^n S_i^m$. We also have improved entitlements t_i^m where $t_i^0 := t_i$. Let us define the set of indices of the players that are unsatisfied at the beginning of round m as

$$I_m := \{i \in [n] : \ t_i > \mu_i(S_i^m)\}.$$

If I_m is a singleton, $I_m = \{i\}$ say, then we allocate C_m to player P_i and the algorithm terminates. As long as I_m is not a singleton, the algorithm does the following. It considers the smallest $i_m \in I_m$ that minimizes $\frac{t_i^m - \mu_i(S_i^m)}{\mu_i(C_m)}$. Then player P_{i_m} takes a $T_1^m \subseteq C_m$ with $\mu_{i_m}(T_1^m) = t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)$. After that players P_i for $i \in I_m \setminus \{i_m\}$ diminish or keep unchanged the actual slice depending on if the value of their normed measures $\frac{\mu_i}{\mu_i(C_m)}$ exceed the constant $\frac{t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)}{\mu_{i_m}(C_m)}$ on it or not. Eventually they obtain a $T_{|I_m|}^m =: R_m$ such that

$$\frac{\mu_i(R_m)}{\mu_i(C_m)} \leq \frac{t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)}{\mu_{i_m}(C_m)} \tag{1}$$

for every $i \in I_m$ and there is equality for at least one index. Let $j_m := i_m$ if there is equality at (1) for i_m and let j_m be the smallest index in I_m for which we have equality if the inequality is strict for i_m. We allocate R_m to player P_{j_m}, formally $S_{j_m}^{m+1} := S_{j_m}^m \cup R_m$ and $S_i^{m+1} := S_i^m$ for $i \in \{1, \ldots, n\} \setminus \{j_m\}$. For $i \in I_{m+1}$ let

$$t_i^{m+1} := \mu_i(S_i^{m+1}) + \frac{t_i^m - \mu_i(S_i^{m+1})}{\sum_{j \in I_{m+1}} \frac{t_j^m - \mu_j(S_j^{m+1})}{\mu_j(C_{m+1})}},$$

which completes the description of the general step of the algorithm.

We turn to the proof of the correctness. First, we show by induction that the steps described above can be done, the algorithm maintains the equation

$$\sum_{i \in I_m} \frac{t_i^m - \mu_i(S_i^m)}{\mu_i(C_m)} = 1 \tag{2}$$

and t_i^m is an increasing function of m for every i. For $m = 0$, (2) says $\sum_{i=1}^n t_i = 1$ which we assumed. Suppose we know the statement up to some m. If I_m is a singleton, then the algorithm terminates after round m and there is nothing to prove. Suppose that $|I_m| > 1$. Since the summands at (2) are all positive, we have $\frac{t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)}{\mu_{i_m}(C_m)} < 1$. Therefore $t_{i_m}^m - \mu_{i_m}(S_{i_m}^m) < \mu_{i_m}(C_m)$, thus there is indeed a $T_1^m \subseteq C_m$ with $\mu_{i_m}(T_1^m) = t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)$. By subtracting both sides of (1) from 1 and taking the reciprocates we obtain

$$\frac{\mu_i(C_m)}{\mu_i(C_{m+1})} \leq \frac{1}{1 - \frac{t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)}{\mu_{i_m}(C_m)}}. \tag{3}$$

Note that player P_{i_m} will be satisfied after round m if $j_m = i_m$ because in that case we have equality at (1) for i_m. We claim that

$$\sum_{i \in I_{m+1}} \frac{t_i^m - \mu_i(S_i^{m+1})}{\mu_i(C_{m+1})} \leq 1$$

and therefore $t_i^{m+1} \geq t_i^m$ by the definition of t_i^{m+1}, moreover, both of these inequalities are strict if $j_m \neq i_m$. Indeed

$$\sum_{i \in I_{m+1}} \frac{t_i^m - \mu_i(S_i^{m+1})}{\mu_i(C_{m+1})} \leq \sum_{i \in I_m} \frac{t_i^m - \mu_i(S_i^{m+1})}{\mu_i(C_{m+1})} = \sum_{i \in I_m} \frac{t_i^m - \mu_i(S_i^{m+1})}{\mu_i(C_m)} \cdot \frac{\mu_i(C_m)}{\mu_i(C_{m+1})} \overset{(3)}{\leq}$$

$$\sum_{i \in I_m} \frac{t_i^m - \mu_i(S_i^{m+1})}{\mu_i(C_m)} \cdot \frac{1}{1 - \frac{t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)}{\mu_{i_m}(C_m)}} \leq \left[\left(\sum_{i \in I_m} \frac{t_i^m - \mu_i(S_i^m)}{\mu_i(C_m)}\right) - \frac{\mu_{j_m}(R_m)}{\mu_{j_m}(C_m)}\right] \cdot \frac{1}{1 - \frac{t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)}{\mu_{i_m}(C_m)}}$$

$$\overset{(2)}{=} \left[1 - \frac{t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)}{\mu_{i_m}(C_m)}\right] \frac{1}{1 - \frac{t_{i_m}^m - \mu_{i_m}(S_{i_m}^m)}{\mu_{i_m}(C_m)}} = 1,$$

where the overestimation of $\frac{\mu_{i_m}(C_m)}{\mu_{i_m}(C_{m+1})}$ via (3) was strict if $i_m \neq j_m$.

Suppose for a contradiction that the algorithm does not terminate for $\mu_1, \ldots, \mu_n$ and $t_1, \ldots, t_n$. Let k be the smallest number for which $I_k = I_m$ for every $m > k$. Then $j_k \neq i_k$ since otherwise we had $I_{k+1} = I_k \setminus \{i_k\} \subsetneq I_k$. As we have already seen, this implies $t_i^{k+1} > t_i^k \geq t_i$ for every $i \in I_k$. Let $(m_\ell)_{\ell \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers with $m_0 > k$ such that there are $i^*, j^* \in I_k$ with $i_{m_\ell} = i^*$ and $j_{m_\ell} = j^*$ for every ℓ. There cannot be a $\varepsilon > 0$ such that $\mu_{j^*}(R_{m_\ell}) \geq \varepsilon$ for infinitely many ℓ because then P_{j^*} would be eventually satisfied and removed from the process, contradicting the definition of k. Thus $\lim_{\ell \to \infty} \mu_{j^*}(R_{m_\ell}) = 0$. Since there is equality for j^* at (1) for each m_ℓ, we know that

$$\mu_{j^*}(R_{m_\ell}) = [t_{i^*}^{m_\ell} - \mu_{i^*}(S_{i^*}^{m_\ell})] \frac{\mu_{j^*}(C_{m_\ell})}{\mu_{i^*}(C_{m_\ell})}.$$

If $\lim_{\ell \to \infty} t_{i^*}^{m_\ell} - \mu_{i^*}(S_{i^*}^{m_\ell}) = 0$, then $\mu_{i^*}(S_{i^*}^{m_\ell}) \geq t_{i^*}$ for a large enough ℓ because $t_{i^*}^{m_0} > t_{i^*}$ and $t_{i^*}^{m_\ell}$ is increasing in ℓ, a contradiction. Therefore we must have $\lim_{\ell \to \infty} \frac{\mu_{j^*}(C_{m_\ell})}{\mu_{i^*}(C_{m_\ell})} = 0$. Since $\mu_{i^*}(C_{m_\ell}) \leq \mu_{i^*}(C_{m_0})$, this implies $\lim_{\ell \to \infty} \mu_{j^*}(C_{m_\ell}) = 0$. But then it follows from (2) that $\lim_{\ell \to \infty} t_{j^*}^{m_\ell} - \mu_{j^*}(S_{j^*}^{m_\ell}) = 0$. As earlier with i^*, this implies that player P_{j^*} will be eventually satisfied, which is a contradiction.

Finally, if $I_m = \{i\}$ for some $m \in \mathbb{N}$ and $i \in [n]$, then (2) ensures $\mu_i(C_m) = t_i^m - \mu_i(S_i^m)$ and therefore the inequality $t_i^m \geq t_i$ combined with the definition of I_m guarantee that all the players are satisfied when the algorithm terminates after round m.

3. From fairness to strong fairness, an elementary approach

Lemma 3.1. *Assume that $(C, \mathcal{S})$ is a measurable space, I is a countable index set, and for $i \in I$, μ_i is an atomless probability measure defined on $\mathcal{S}$ and t_i is a positive number such not all the μ_i are identical and $\sum_{i \in I} t_i = 1$. Then there is a partition $C = C' \sqcup C''$ and $t_i', t_i'' > 0$ with $\sum_{i \in I} t_i' = \sum_{i \in I} t_i'' = 1$ such that $t_i' \cdot \mu_i(C') + t_i'' \cdot \mu_i(C'') > t_i$ for each $i \in I$.*

Proof. Suppose that $j, k \in I$ and $C' \in \mathcal{S}$ such that $\mu_j(C') < \mu_k(C')$. It is enough to find $s_i', s_i'' > 0$ with $\sum_{i \in I} s_i', \sum_{i \in I} s_i'' < 1$ and $s_i' \cdot \mu_i(C') + s_i'' \cdot \mu_i(C'') = t_i$ for every $i \in I$ because then

$$t_i' := \frac{s_i'}{\sum_{\ell \in I} s_\ell'} \text{ and } t_i'' := \frac{s_i''}{\sum_{\ell \in I} s_\ell''}$$

are as desired. We are looking for $\varepsilon, \delta > 0$ for which the definitions

- $s_j' := t_j - \varepsilon$
- $s_j'' := t_j + \varepsilon \cdot \frac{\mu_j(C')}{\mu_j(C'')}$
- $s_k' := t_k + \delta \cdot \frac{\mu_k(C'')}{\mu_k(C')}$
- $s_k'' := t_k - \delta$
- $s_i'' := s_i' := t_i$ for $i \in \mathbb{N} \setminus \{j, k\}$

are suitable. Note that whatever ε and δ we choose, $s_i' \cdot \mu_i(C') + s_i'' \cdot \mu_i(C'') = t_i$ will hold for each $i \in \mathbb{N}$. Thus the requirements $s_i', s_i'' > 0$ and $\sum_{i \in I} s_i', \sum_{i \in I} s_i'' < 1$ mean for ε and δ that they satisfy

$$\varepsilon \in (0, t_j)$$
$$\delta \in (0, t_k)$$
$$\varepsilon > \delta \cdot \frac{\mu_k(C'')}{\mu_k(C')}$$
$$\delta > \varepsilon \cdot \frac{\mu_j(C')}{\mu_j(C'')}$$

If $\mu_j(C') = 0$, then the last inequality is redundant and the existence of a solution is straightforward. Otherwise the last two inequalities demand

$$\frac{\mu_k(C'')}{\mu_k(C')} < \frac{\varepsilon}{\delta} < \frac{\mu_j(C'')}{\mu_j(C')}.$$

Since $\frac{\mu_k(C'')}{\mu_k(C')} < \frac{\mu_j(C'')}{\mu_j(C')}$ follows from $\mu_j(C') < \mu_k(C')$, the desired ε and δ exist in this case as well. □

Let μ_i' be the restriction of $\frac{\mu_i}{\mu_i(C')}$ to $\mathcal{S} \cap \mathcal{P}(C')$ if $\mu_i(C') \neq 0$ and an arbitrary atomless probability measure on $\mathcal{S} \cap \mathcal{P}(C')$ if $\mu_i(C') = 0$. We define μ_i'' analogously with respect to C''.

Corollary 3.2. *Assume the settings of Lemma 3.1. If $\{S_i' : i \in I\}$ is a fair division with respect to μ_i', t_i' $(i \in I)$ and $\{S_i'' : i \in I\}$ is a fair divisions with respect to μ_i'', t_i'' $(i \in I)$, then for $S_i := S_i' \cup S_i''$, $\{S_i : i \in I\}$ is a strongly fair division with respect to μ_i, t_i $(i \in I)$.*

Proof. We have $\mu_i(S_i') \geq t_i' \cdot \mu_i(C')$ and $\mu_i(S_i'') \geq t_i'' \cdot \mu_i(C'')$ by fairness, thus by Lemma 3.1

$$\mu_i(S_i) = \mu_i(S_i' \sqcup S_i'') = \mu_i(S_i') + \mu_i(S_i'') \geq t_i' \cdot \mu_i(C') + t_i'' \cdot \mu_i(C'') > t_i.$$

□

4. Existence of a fair division for infinitely many players

We repeat the theorem here for convenience.

Theorem 1.1. *Assume that $(C, \mathcal{S})$ is a measurable space and for $i \in \mathbb{N}$, μ_i is an atomless probability measure defined on $\mathcal{S}$ and t_i is a positive number such that $\sum_{i=0}^{\infty} t_i = 1$. Then there is a partition $C = \bigsqcup_{i=0}^{\infty} S_i$ such that $S_i \in \mathcal{S}$ with $\mu_i(S_i) \geq t_i$ for each $i \in \mathbb{N}$. Furthermore, if not all the μ_i are identical, then '$\mu_i(S_i) \geq t_i$' can be strengthened to '$\mu_i(S_i) > t_i$' for every $i \in \mathbb{N}$.*

Proof. Without loss of generality we may looking for a sub-partition instead of a partition, i.e. we can relax '$C = \bigsqcup_{i=0}^{\infty} S_i$' to '$C \supseteq \bigsqcup_{i=0}^{\infty} S_i$' since the remaining surplus part of the cake can be given to anybody. The last sentence of Theorem 1.1 follows from the rest of it via Corollary 3.2.

For $n \in \mathbb{N}$, we let $t_0^n, t_1^n, \dots, t_n^n$ to be the first $n+1$ entitlements scaled to sum up to 1, i.e.

$$t_i^n := \frac{t_i}{\sum_{j=0}^{n} t_j}.$$

Observation 4.1. $(1 - t_{n+1}^{n+1})t_i^n = t_i^{n+1}$ *and* $\lim_{n\to\infty} t_i^n = t_i$.

Proof.

$$\frac{t_i^{n+1}}{t_i^n} = \frac{\sum_{j=0}^{n} t_j}{\sum_{j=0}^{n+1} t_j} = \frac{\sum_{j=0}^{n+1} t_j - t_{n+1}}{\sum_{j=0}^{n+1} t_j} = 1 - t_{n+1}^{n+1},$$

$$\lim_{n\to\infty} t_i^n = \lim_{n\to\infty} \frac{t_i}{\sum_{j=0}^{n} t_j} = \frac{t_i}{\lim_{n\to\infty}\sum_{j=0}^{n} t_j} = t_i.$$

□

We shall define recursively $S_i^n \in \mathcal{S}$ for $i, n \in \mathbb{N}$ with $i \leq n$ in such a way that

(i) $C = \bigsqcup_{i \leq n} S_i^n$ for every n;
(ii) $\mu_i(S_i^n) \geq t_i^n$;
(iii) For every fixed $i \in \mathbb{N}$ the sequence $(S_i^n)_{n \geq i}$ is $\subseteq$-decreasing.

Observe that conditions (i) and (ii) say that for each fixed n the sets $S_0^n, S_1^n, \dots, S_n^n$ form a fair division with respect to the measures μ_i and entitlements t_i^n. Although such a fair division can be found for every particular n, it cannot be guaranteed without condition (iii) that they have a meaningful "limit" which provides a fair division in the original settings.

We let $S_0^0 := C$ which obviously satisfies the conditions. Suppose that $S_0^n, S_1^n \dots, S_n^n$ are already defined for some $n \in \mathbb{N}$. We need to find for each $i \leq n$ an $S_i^{n+1} \subseteq S_i^n$ with $\mu_i(S_i^{n+1}) \geq t_i^{n+1}$ in such a way that for

$$S_{n+1}^{n+1} := C \setminus \bigcup_{i \leq n} S_i^{n+1}$$

we have $\mu_{n+1}(S_{n+1}^{n+1}) \geq t_{n+1}^{n+1}$. For the last inequality it is enough to ensure that

$$\mu_{n+1}(S_i^n \setminus S_i^{n+1}) \geq \mu_{n+1}(S_i^n) \cdot t_{n+1}^{n+1} \text{ for } i \leq n. \tag{4}$$

Indeed, since

$$S_{n+1}^{n+1} = \bigsqcup_{i \leq n} S_i^n \setminus S_i^{n+1},$$

the inequalities (4) imply

$$\begin{aligned}\mu_{n+1}(S_{n+1}^{n+1}) &= \mu_{n+1}\left(\bigsqcup_{i \leq n} S_i^n \setminus S_i^{n+1}\right) = \sum_{i=0}^{n} \mu_{n+1}(S_i^n \setminus S_i^{n+1}) \geq \sum_{i=0}^{n} \mu_{n+1}(S_i^n) \cdot t_{n+1}^{n+1} \\ &= t_{n+1}^{n+1} \cdot \sum_{i=0}^{n} \mu_{n+1}(S_i^n) = t_{n+1}^{n+1} \cdot \mu_{n+1}(C) = t_{n+1}^{n+1} \cdot 1 = t_{n+1}^{n+1},\end{aligned}$$

where we used (i) combined with the fact that μ_{n+1} is a probability measure. Therefore it is enough to find for every $i \leq n$ an $S_i^{n+1} \subseteq S_i^n$ such that

$$\mu_i(S_i^{n+1}) \geq t_i^{n+1} \tag{5}$$

$$\mu_{n+1}(S_i^n \setminus S_i^{n+1}) \geq \mu_{n+1}(S_i^n) \cdot t_{n+1}^{n+1}. \tag{6}$$

Let $i \leq n$ be fixed. If $\mu_{n+1}(S_i^n) = 0$, then we let $S_i^{n+1} := S_i^n$ which is clearly appropriate since $t_i^n \geq t_i^{n+1}$ (see Observation 4.1). Suppose that $\mu_{n+1}(S_i^n) > 0$ and note that $\mu_i(S_i^n) \geq t_i^n > 0$ by assumption. We claim that choosing S_i^{n+1} to be the slice corresponding to i in a fair division of S_i^n between P_i and P_{n+1} with respect to the restrictions of $\frac{\mu_i}{\mu_i(S_i^n)}$ and $\frac{\mu_{n+1}}{\mu_{n+1}(S_i^n)}$ to $\mathcal{S} \cap \mathcal{P}(S_i^n)$ and respective entitlements $1 - t_{n+1}^{n+1}$ and t_{n+1}^{n+1} is suitable. Indeed, by the fairness of the obtained bipartition $\{S_i^{n+1},\ S_i^n \setminus S_i^{n+1}\}$ of S_i^n we have

$$\frac{\mu_i(S_i^{n+1})}{\mu_i(S_i^n)} \geq 1 - t_{n+1}^{n+1},$$

$$\frac{\mu_{n+1}(S_i^n \setminus S_i^{n+1})}{\mu_{n+1}(S_i^n)} \geq t_{n+1}^{n+1}.$$

Here the second inequality is equivalent with (6) and the first one implies (5) since

$$\mu_i(S_i^{n+1}) \geq (1 - t_{n+1}^{n+1})\mu_i(S_i^n) \geq (1 - t_{n+1}^{n+1})t_i^n = t_i^{n+1},$$

where we used $\mu_i(S_i^n) \geq t_i^n$ and Observation 4.1. The recursion is done.

We define $S_i := \bigcap_{n \geq i} S_i^n$ for $i \in \mathbb{N}$. Then for $i < j$ we have $S_i \cap S_j = \emptyset$ because $S_i \subseteq S_i^j,\ S_j \subseteq S_j^j$ and $S_i^j \cap S_j^j = \emptyset$ by (i), furthermore,

$$\mu_i(S_i) = \mu_i\left(\bigcap_{n \geq i} S_i^n\right) = \lim_{n\to\infty} \mu_i(S_i^n) \geq \lim_{n\to\infty} t_i^n = t_i$$

by (iii), (ii) and Observation 4.1. This completes the proof of Theorem 1.1. □

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