DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 85-125
November 1985

A DERIVATION OF THE DERBENEV-KONDRATENKO FORMULA USING SEMI-CLASSICAL ELECTRODYNAMICS

by

S.R. Mane

Physics Dept., Cornell University, Ithaca, USA

ISSN 0418-9833

NOTKESTRASSE 85 · 2 HAMBURG 52

To be sure that your preprints are promptly included in the HIGH ENERGY PHYSICS INDEX , send them to the following address (if possible by air mail) :

DESY
Bibliothek
Notkestrasse 85
2 Hamburg 52
Germany

DESY 85-125
November 1985

ISSN 0418-9833

A Derivation of the Derbenev-Kondratenko Formula Using Semi-Classical Electrodynamics

S. R. MANE[1]

Physics Department, Clark Hall, Cornell University,
Ithaca, New York 14853, USA

Oct. 21, 1985

ABSTRACT. We present a detailed exposition of the mechanism for the build-up of polarization in electron storage rings. A semi-classical approach is used to derive the rate of growth and asymptotic degree of polarization in an electron storage ring (the Derbenev-Kondratenko formula). Statistical mechanical concepts used to obtain as classical an understanding as possible of this phenomenon.

[1]Present address : DESY, Notkestrasse 85, 2 Hamburg 52, Federal Republic of Germany

I. INTRODUCTION

It is well known that electrons in a storage ring become polarized as a result of the emission of synchrotron radiation (the Sokolov-Ternov effect /1/). Derbenev and Kondratenko /2/ examined the mechanism which leads to the polarization of the electron beam and gave expressions for the rate of growth and the asymptotic degree of polarization (called τ_o and P_∞, respectively). However (Montague /3/), " ... the early papers ... were somewhat terse".

We have derived the Derbenev-Kondratenko formula by following Schwinger's /4,5/ semiclassical method of deriving the synchrotron radiation power spectrum. In so doing, we have attempted not only to simplify the mathematics but also to gain a more classical understanding of the mechanism of electron spin polarization in high-energy storage rings. "Classical" in this case requires the use of statistical mechanics, because we have to deal with a large number of particles, and with a concept (polarization) that is only defined macroscopically.

It should be realized that the Derbenev-Kondratenko formula is statistical in character (presumably this explains the title of their paper /2/ : "Polarization Kinetics ... "). In this sense it is very different from the work of Sokolov and Ternov /1/, although it may still predict a value of $8/5\sqrt{3}$ for P_∞ in an idealized storage ring, as will be shown in Section 5.

We use a canonical ensemble to describe the electron beam. The ensemble is stable from turn to turn around the ring (neglecting, as always, loss of particles from the beam, random voltage spikes in the magnet power supply, etc), even though the individual particles undergo Brownian motion. In deriving the Derbenev-Kondratenko formula, photon emission by an individual electron is used only to establish the spin-flip transition rate in a given volume of phase space, which then determines the equilibrium properties of the ensemble. No attempt is made to trace the history of an individual electron in order to add up its spin with all the others to get the degree of polarization. This is a point that does not appear to have been properly appreciated in the literature.

We have tried to clarify such issues, because the aim of this paper is first of all didactic : we seek to provide a clearer understanding of a known formula. In doing so, we discover some features of electron beam polarization that have hitherto not been realized. In particular, the algorithm given by Chao /6/ for numerically evaluating P_∞ suffers from a numerical error. We shall, in this context, go carefully through the calculation of the polarization and its physical origin.

II. General Remarks on Polarization in Electron Storage Rings

It was pointed out many years ago /1/ that electrons in a storage ring become polarized through the emission of synchrotron radiation. If this polarization could be controlled, it would provide a useful experimental tool. In particular, a storage ring called HERA is currently being built at DESY in which electrons of approximately 30 GeV will collide against protons of approximately 820 GeV. It is planned to control the direction of the electron beam polarization so as to obtain longitudinal polarization at the interaction points, thus enabling experimenters to control the helicity of the electrons. In an electron storage ring, for an initially unpolarized beam, the magnitude of polarization $P(t)$ builds up exponentially

$$P(t) = P_\infty(1 - e^{-t/\tau_o}) \tag{2.1}$$

τ_o is called the "polarization time" and P_∞ the "asymptotic degree of polarization". It is obviously useful to be able to calculate P_∞ and τ_o so as to produce a machine design in which P_∞ is as large as possible and τ_o is as small as possible. In practice polarization build-up is a slow process (τ_o is typically in the range of about 10 minutes to hundreds of minutes, which corresponds to many millions of revolutions around the circumference of the storage ring). This makes the attainment of a high degree of polarization a difficult business, since even a small perturbation can destroy the polarization if the electron beam takes millions of turns to recover from the effect of the perturbation.

For our purposes, the polarization can be described by a vector

$$\vec{P}_\infty = P_\infty \, \hat{n}_o(s) \tag{2.2}$$

where s is the arc-length along the ring, say $0 \leq s < L$. $\hat{n}_o(s)$ is a unit vector.

The emission of synchrotron radiation causes the electron spin to (sometimes) flip. Then polarization is produced along $\hat{n}_o$ because the probability per unit time of spin-flip from $-\hat{n}_o$ to $\hat{n}_o$ is not equal to that from $\hat{n}_o$ to $-\hat{n}_o$ (because the electrons radiate in the presence of a static magnetic field, not in a vacuum).

In an ensemble, if there are N_+ electrons with spin projection $1/2$ along $\hat{n}_o$ and N_- electrons with spin projection $-1/2$ along $\hat{n}_o$ ($N_+ + N_- = N_0 =$ constant), then we have

$$P(t) = \frac{N_+ - N_-}{N_0} \tag{2.3}$$

Further, let the probability per unit time for spin-flip from $\hat{n}_o$ to $-\hat{n}_o$ be W_+, and W_- the corresponding quantity for the reverse transition. Then

$$\frac{dN_+}{dt} = -W_+N_+ + W_-N_- \tag{2.4a}$$

$$\frac{dN_-}{dt} = -\frac{dN_+}{dt} \tag{2.4b}$$

We have assumed W_+ and W_- are constants (independent of time, and also of the values of N_+ and N_-, thus yielding linear equations (2.4a,b)).

In equilibrium

$$\frac{dN_+}{dt} = -\frac{dN_-}{dt} = 0 \tag{2.5}$$

whereupon

$$\frac{N_+}{N_-} = \frac{W_-}{W_+} \tag{2.6}$$

which yields the following expressions for P_∞ and τ_o

$$P_\infty = \frac{W_- - W_+}{W_- + W_+} \tag{2.7a}$$

$$\tau_o = \frac{1}{W_+ + W_-} \tag{2.7b}$$

Using (2.3),(2.4) and (2.7), we can write

$$\frac{dP(t)}{dt} = \frac{1}{\tau_o}(P_\infty - P(t)) \tag{2.8}$$

the solution of which is (2.1) if $P(0) = 0$.

We therefore need to calculate the probabilities per unit time W_+ and W_-. In most work referenced in this report W_+ and W_- have been calculated using the "independent-particle approximation". In this approximation one considers the consequences of photon emission by a single "typical" electron in the storage ring. The definition of "typical" will be determined by the ensemble used. One then calculates probabilities per unit time for spin-flip in different directions. The ensemble average converts the spin-flip probabilities of a single particle into transition rates for the ensemble. We shall also employ the independent-particle approximation and take the ensemble average in the way just described.

This approach assumes collective effects in the ensemble to be unimportant. In performing the calculations, multiphoton processes are also neglected, and emission of different photons is assumed to be uncorrelated. We could call this an "independent-electron, single-photon approximation".

Consider the evaluation of W_+. We can write

$$W_+ = \int_0^\infty \frac{dW_+}{d\omega}\, d\omega \tag{2.9}$$

where $\frac{dW_+}{d\omega}\delta\omega$ is the probability per unit time for spin-flip from $\hat{n}_o$ to $-\hat{n}_o$ as a result of photon emission of energy in the range $(\hbar\omega, \hbar(\omega + \delta\omega))$.

In the single-photon approximation, $dW_+/d\omega$ is related to the corresponding one-electron differential power spectrum in frequency of spin-flip synchrotron radiation $d\mathrm{P}_+/d\omega$ by

$$\frac{d\mathrm{P}_+}{d\omega} = \hbar\omega \frac{dW_+}{d\omega} \tag{2.10a}$$

or

$$\frac{dW_+}{d\omega} = \frac{1}{\hbar\omega}\frac{d\mathrm{P}_+}{d\omega} \tag{2.10b}$$

Our procedure in this paper will be to calculate the power spectrum $d\mathrm{P}_+/d\omega$ and to get W_+ by working backwards through (2.10b) to (2.9). We can easily see that the asymptotic degree of polarization will be

$$P_\infty = \frac{\int_0^\infty \frac{1}{\hbar\omega}\left(\frac{d\mathrm{P}_-}{d\omega} - \frac{d\mathrm{P}_+}{d\omega}\right) d\omega}{\int_0^\infty \frac{1}{\hbar\omega}\left(\frac{d\mathrm{P}_-}{d\omega} + \frac{d\mathrm{P}_+}{d\omega}\right) d\omega} \tag{2.11}$$

and the polarization time will be

$$\tau_o = \left(\int_0^\infty \frac{1}{\hbar\omega}\left(\frac{d\mathrm{P}_-}{d\omega} + \frac{d\mathrm{P}_+}{d\omega}\right) d\omega\right)^{-1} \tag{2.12}$$

Sokolov and Ternov /1/ solved the Dirac equation to calculate the synchrotron radiation spectrum of an electron moving in a uniform static magnetic field. They found that the spin sometimes flips when a photon is emitted, and the spin-flip power spectrum depends on the initial spin orientation. Thus, in an ensemble, the spins will preferentially align in a certain direction, leading to polarization in that direction. They found that positrons(electrons) polarize along(against) the direction of the magnetic field $\hat{b}$. They also found that P_∞ is not unity (the polarization is not complete), but is $\pm 8/5\sqrt{3}$.

$$\hat{n}_o(s) = \hat{b} = \text{constant} \tag{2.13a}$$

$$P_\infty = \pm\frac{8}{5\sqrt{3}} = \pm 0.924 \tag{2.13b}$$

$$\tau_o = \left(\frac{5\sqrt{3}}{8}\frac{e^2\hbar\gamma^5}{m^2c^2\rho^3}\right)^{-1} \tag{2.13c}$$

P_∞ is positive(negative) for positrons(electrons) — electrons and positrons become polarized in opposite directions.

Baier and Katkov /7/ generalized the above expressions to include motion in inhomogenous fields. Then $\hat{n}_o$ is not always parallel to the local magnetic field direction $\hat{b} = \hat{b}(s)$. If $\rho = \rho(s)$ now denotes the local radius of curvature,

$$P_\infty = \frac{8}{5\sqrt{3}} \frac{\oint \frac{ds}{|\rho|^3} \hat{b}.\hat{n}_o}{\oint \frac{ds}{|\rho|^3} \left\{1 - \frac{2}{9}(\hat{n}_o.\hat{v})^2\right\}} \tag{2.14a}$$

$$\tau_o = \left(\frac{5\sqrt{3}}{8} \frac{e^2\hbar\gamma^5}{m^2c^2} \frac{1}{L} \oint \frac{ds}{|\rho|^3} \left\{1 - \frac{2}{9}(\hat{n}_o.\hat{v})^2\right\}\right)^{-1} \tag{2.14b}$$

where $\hat{v}$ is a unit vector in the direction of the particle velocity. (2.14a,b) will be derived later, as part of the Derbenev-Kondratenko formula (5.12a), at which point the origin of the various constants in the above equations will be apparent.

It is desirable to have as classical as possible an understanding of the derivation of the above expressions, and thus of the process of electron polarization in storage rings. To this end, Jackson /8/ developed a semiclassical model of electron motion (described in Section 3, and which we shall use with some extensions), and obtained (2.14a,b). Jackson explains why the emission of photons in a static background magnetic field cannot be modelled in this case by a simple magnetic dipole transition (a magnetic dipole transition would lead to complete polarization — the spins would all align in the state of lower energy). Jackson shows that it is not only the interaction of the spin with the magnetic field that matters. The interaction of the spin with the orbital motion of the electron is also very important.

Just as Schwinger /4,5/ showed that the orbital motion may be treated classically in the study of ordinary synchrotron radiation, Jackson shows that the orbital motion may be treated classically in the study of spin-dependent synchrotron radiation. Since spin-flip is a quantum-mechanical phenomenon that has no real classical analogue, Jackson refers to his calculation as "semi-classical". We shall use the same terminology for our work, too.

Jackson does not however go further, to the Derbenev-Kondratenko /2/ formula (5.12a), which is a more detailed expression for P_∞ (there is also a more detailed expression for τ_o, of course). We shall use the "independent-electron,single-photon" approximation to formulate and derive these expressions. Our purpose is to check that they do indeed follow from the above approximation, to obtain a more classical understanding of the formula, and to present a simpler mathematical derivation than given by the original authors /2/.

In the following Sections we shall define more precisely the model used, in particular the assumptions that go into its construction. We shall define the Hamiltonian we use, then the calculational scheme to get the transition rates, and the results.

III. Basic Model of Electron Motion in a Storage Ring

The orbital motion of an electron with charge e, position vector $\vec{r}(t)$ and velocity $c\vec{\beta}(t)$ is given by the Lorentz equation

$$\frac{d\vec{p}}{dt} = \frac{d(mc\gamma\vec{\beta})}{dt} = e(\vec{\mathrm{E}} + \vec{\beta} \times \vec{\mathrm{B}}) \tag{3.1}$$

$\vec{\mathrm{E}}$ and $\vec{\mathrm{B}}$ are the electric and magnetic fields seen by the particle. We care only about orbits which can be split into a series of circular arcs, hence γ can be taken as constant. Hence

$$\frac{d\vec{\beta}}{dt} = \frac{e}{mc\gamma}(\vec{\mathrm{E}} + \vec{\beta} \times \vec{\mathrm{B}}) \tag{3.2}$$

If (3.2) is solved neglecting electromagnetic radiation by the particle, i.e. in the static fields of the accelerator only, then in all cases of interest to us (3.2) will have a periodic solution for $\vec{\beta}$. This means, if τ_{rev} is the time taken to traverse the ring circumference once,

$$\vec{r}(t + \tau_{rev}) = \vec{r}(t) \tag{3.3a}$$

$$\vec{\beta}(t + \tau_{rev}) = \vec{\beta}(t) \tag{3.3b}$$

Such a solution of (3.2), which satisfies (3.3), will be called the particle closed orbit, or periodic solution, for orbital motion. It will depend on energy, in general. If the energy is E_0, the design energy of the accelerator, we shall also call it the equilibrium closed orbit.

Spin motion is described by the Thomas-BMT equation /9/

$$\frac{d\vec{s}}{dt} = \frac{e}{mc}\vec{s} \times \left[(a + \frac{1}{\gamma})\vec{\mathrm{B}} - \frac{a\gamma}{\gamma+1}\vec{\beta}.\vec{\mathrm{B}}\vec{\beta} - (a + \frac{1}{\gamma+1})\vec{\beta} \times \vec{\mathrm{E}}\right] \tag{3.4}$$

where $a = (g-2)/2$. g is the gyromagnetic ratio of the particle. We take the electron spin to be $\hbar\vec{s}$, so $\vec{s}$ is dimensionless. In (3.4) $\vec{s}$ is the spin vector in the electron rest frame, but $\vec{\mathrm{E}}$,$\vec{\mathrm{B}}$ etc are all referred to the laboratory frame. See Jackson /10/ for a more detailed discussion of this curious formulation. An important point to note is that at the classical level, $\vec{s}$ only rotates – it does not flip. Therefore polarization can neither develop nor decay. We shall need to include the possibility of spin-flip in order to get polarization. We shall have more to say about this later in this Section. Notice that we must know the orbital motion before we can calculate the spin motion.

(3.4) can be written in the form

$$\frac{d\vec{s}}{dt} = \vec{s} \times \vec{\Omega} \tag{3.5}$$

which we shall sometimes use for brevity.

The transformation of $\vec{s}$ between two points with arc-length s_o and s_1 is a rotation

$$\vec{s}(s_1) = \mathrm{R}(s_1, s_o)\vec{s}(s_o) \tag{3.6}$$

$$\mathrm{R} \;=\; \text{rotation matrix in } 3-\text{dimensional space}$$

provided no photons are emitted in the interval between s_o and s_1. R will also depend on the orbital trajectory from s_o to s_1. (In fact, we should say not s_o and s_1, but two points of phase space $\vec{\pi}_o$ and $\vec{\pi}_1$. We shall get to the notion of phase space later.)

We follow the definition of $\hat{n}$ given by Derbenev and Kondratenko /11/. For an orbital trajectory specified by action and angle variables $\{I_i, \psi_i, i = 1, 2, \ldots\}$, $\hat{n}$ is defined to be the solution of (3.4) which satisfies the double equality

$$\hat{n}(I_i, \psi_i, s_o) = \hat{n}(I_i, \psi_i + 2\pi, s_o) = \hat{n}(I_i, \psi_i, s_o + L) \tag{3.7a}$$

where "$\psi_i + 2\pi$" means any one (or more) of the ψ_i is/are increased by 2π. The matter will be discussed in greater detail in Section 6, where we shall show how to calculate $\hat{n}$.

Note that $\hat{n}$ depends on s_o, on the orbital trajectory, momentum and energy. If the orbital trajectory is a betatron oscillation (i.e. not periodic), then we shall find

$$\hat{n}(I_i, \psi_i(s_o + L), s_o + L) \neq \hat{n}(I_i, \psi_i(s_o), s_o) \tag{3.7b}$$

because for a betatron oscillation

$$\psi_i(s_o + L) = \psi_i(s_o) + 2\pi Q_i \quad , \quad Q_i \neq \text{integer} \tag{3.7c}$$

$\Rightarrow$

$$\psi_i(s_o + L) \neq \psi_i(s_o)$$

In the special cases when all the Q_i relevant to $\hat{n}$ are integers, $\hat{n}$ will be periodic.

The vector $\hat{n}_o$ is defined to be the periodic solution of (3.4) for an electron which follows the equilibrium closed orbit (given by (3.2) and (3.3)).

$$\hat{n}_o(s_o + L) = \hat{n}_o(s_o) \tag{3.8}$$

Polarization builds up along $\hat{n}_o$. Basically, this is because polarization builds up slowly, over many millions of turns around the ring. Any component of polarization orthogonal to $\hat{n}_o$ tends to average out to zero. Only the component along $\hat{n}_o$ will build up, turn after turn around the machine. If the polarization process could be speeded up so as to take only a few turns (or even less than one turn) around the ring, the polarization would not necessarily point along $\hat{n}_o$. See Montague /3/ for a fuller discussion on this point.

Both (3.1) and (3.4) can be derived from Hamiltonians. (3.1) follows from a Hamiltonian H_1 ($\vec{P}$ is the momentum canonically conjugate to position)

$$H_1 = \sqrt{(\vec{P} - \frac{e}{c}\vec{A}_{static})^2 c^2 + m^2 c^4} + e\Phi_{static} + e(\Phi_{rad} - \vec{\beta}.\vec{A}_{rad}) \tag{3.9a}$$

Φ and $\vec{A}$ are the electromagnetic scalar and vector potentials, respectively, of the electric and magnetic fields seen by the particle. (3.4) follows from a Hamiltonian H_2

$$H_2 = -\frac{e\hbar}{mc}\vec{s}.\left[(a+\frac{1}{\gamma})\vec{B} - \frac{a\gamma}{\gamma+1}\vec{\beta}.\vec{B}\vec{\beta} - (a+\frac{1}{\gamma+1})\vec{\beta}\times\vec{E}\right] \tag{3.9b}$$

and so the total Hamiltonian for orbital and spin motion is

$$H = H_1 + H_2 \tag{3.9c}$$

Note that, strictly, H will give spin-dependent forces on the rhs of (3.2). The effects of such forces are seen in Stern-Gerlach experiments with non-relativistic electrons, but are very weak in high-energy accelerators and will be ignored in (3.2).

We assume the above Hamiltonian remains unchanged in form if we reinterpret the various quantities to be quantum operators. In particular this will give rise to the possibility of spin-flip via photon emission. Let us return to consider the spin in more detail.

The classical vector represented by $\vec{s}$ in (3.4) is a specific matrix element of the quantum-mechanical spin operator $\vec{s}_{op}$. If the electron spin wavefunction is $|\Psi\rangle$, then the expectation value of $\vec{s}_{op}$

$$\vec{s}_{classical} = \langle\Psi|\vec{s}_{op}|\Psi\rangle \tag{3.10}$$

is the vector commonly called the "electron spin" in classical descriptions.

But we can also construct other matrix elements of $\vec{s}_{op}$. If we have states $|\phi\rangle$ and $|\psi\rangle$, we can define

$$\vec{s}_{\phi\psi} = \langle\phi|\vec{s}_{op}|\psi\rangle \tag{3.11}$$

If we interpret H as a quantum-mechanical Hamiltonian, then (3.4) will be the resulting equation of motion of $\vec{s}_{op}$ ($\vec{E}, \vec{B}$ etc will then be operators). In thinking of (3.4) as a classical equation, $\vec{E}, \vec{B}$ are taken to be classical fields, and $\vec{s}_{op}$ is replaced by the matrix element $\vec{s}_{\phi\psi}$. In particular, we can define

$$\begin{aligned} |\phi\rangle &= |-\hat{n}\rangle \\ |\psi\rangle &= |\hat{n}\rangle \\ \vec{s}_{-\hat{n}\hat{n}} &= \langle -\hat{n}|\vec{s}_{op}|\hat{n}\rangle \end{aligned} \tag{3.12}$$

where by $|\vec{a}\rangle$ we mean a normalized eigenstate of $\hbar\vec{s}_{op}.\hat{a}$ with eigenvalue $+\hbar/2$. $\vec{s}_{-\hat{n}\hat{n}}$ is a vector, and it obeys (3.4), but it is not the "classical spin vector".

To get (2.14a,b), the model of electron motion stops here. Jackson /8/ considers radiation from an electron moving along the equilibrium closed orbit and considers spin-flips from $\hat{n}_o$ to $-\hat{n}_o$ and vice-versa. Then he performs a calculation using the Fermi Golden Rule to derive (2.14a,b). The matrix element used is

$$\langle -\hat{n}_o|\mathrm{H_{int}}|\hat{n}_o\rangle = \\ -\frac{e\hbar}{mc}\vec{s}_{-\hat{n}_o\hat{n}_o}\cdot\left[(a+\frac{1}{\gamma})\vec{\mathrm{B}}_{\mathrm{rad}} - \frac{a\gamma}{\gamma+1}\vec{\beta}.\vec{\mathrm{B}}_{\mathrm{rad}}\vec{\beta} - (a+\frac{1}{\gamma+1})\vec{\beta}\times\vec{\mathrm{E}}_{\mathrm{rad}}\right] \quad (3.13)$$

and we replace $\vec{s}_{-\hat{n}_o\hat{n}_o}$ by $\langle\hat{n}_o|\vec{s}_{op}|-\hat{n}_o\rangle$ for transitions from $-\hat{n}_o$ to $\hat{n}_o$. $\vec{\mathrm{E}}_{\mathrm{rad}}$ and $\vec{\mathrm{B}}_{\mathrm{rad}}$ are the radiation fields of the emitted photon. The ensemble average is trivial — all electrons follow the same trajectory. Processes where the spin does not flip also occur, of course (in fact they are much more numerous), but since only spin-flip transitions change the polarization, they are ignored. We shall now proceed to extend this model in order to include effects so far neglected. The major change will be to refine the ensemble used.

IV. The Electron Ensemble

We begin the refinement of the ensemble by discussion of a simple example, not especially related to electron motion in a storage ring, but useful as an illustrative example. Consider the flow of a fluid along a channel (streamline flow). We know that the individual particles execute Brownian motion, and their motion is not deterministic. Yet at a macroscopic level, we can define streamlines or trajectories of flow, and we can speak of a drift velocity, etc. We can talk of a "steady flow". We do not claim that any individual particle actually follows a streamline or has a drift velocity — these are concepts defined at the macroscopic level which emerge as ensemble averaged results of a calculation of the microscopic model. There will always be fluctuations, a consequence of the underlying Brownian motion.

Let us return to the matter of electrons in a storage ring. At the microscopic level, a single electron undergoes Brownian motion because of the random kicks received as a result of photon emission (which is not a deterministic process). At the macroscopic level the effects of radiation damping and random excitation of perturbations cause the electron ensemble to settle down to "equilibrium". It is true that in a real accelerator particles are continuously lost from the beam, hence "equilibrium" does not strictly exist, but we shall ignore this as a slow process — to put it another way, we would be concerned about totally different matters if particles were being rapidly lost from the beam.

In the macroscopic model, we can define such concepts as betatron trajectories and an "equilibrium closed orbit". This is done by solving (3.2) in the presence of only static fields — radiation is neglected. This is of course a phenomenological approach, but at the macroscopic level it is valid. We do not claim that any single electron actually follows a macroscopic trajectory, except between two successive photon emissions, a rather short time interval. Nevertheless, at the macroscopic level, we can imagine the electron ensemble as a sort of fluid which has a drift velocity along these trajectories. In particular, we can define a probability distribution or density in phase space to say what fraction of the electrons in the ensemble we expect to find in a given volume of phase space. We say "phase space" because a macroscopic trajectory is characterized not only by a set of initial coordinates but also by initial momentum and energy. Typically we measure all of the above parameters as offsets from the equilibrium closed orbit. Thus the equilibrium closed orbit defines the origin of phase space, in the notation of accelerator physicists. More details will be given in Section 6.

This brings us to the ensemble of the Derbenev-Kondratenko formula. We do not yet need to specify the probability distribution, but it is clear that we are not going to imagine all the electrons to follow the equilibrium closed orbit.

We have defined $\hat{n}$ earlier for every macroscopic trajectory, but have not used the rather complicated definition until now. We shall now take $\hat{n}$, not $\hat{n}_o$, to be the spin quantization axis on the trajectory associated with $\hat{n}$. Why define the quantization axis in this curious way, a different one for each trajectory? It is possible to choose bases for orbital and spin motion which are completely decoupled. By using bases which are coupled, however, we can find an eigenvector of spin motion which is naturally linked to the periodicities of the orbital motion.

Such a vector will provide a natural quantization axis for motion along the corresponding orbital trajectory. $\hat{n}$ is this vector, being both a solution of the equation of motion (3.4) and having the required periodicities (3.7).

We could stick to $\hat{n}_o$ as the quantization axis in all cases. This would make the ensemble much simpler to visualize, but it has disadvantages away from the equilibrium closed orbit, as mentioned above. This might still be satisfactory, but for now let us stick to the Derbenev-Kondratenko choice of quantization axis. This does not mean we are going to duplicate their work. It means that we wish to spell out clearly the concepts used, and we shall present a simpler mathematical calculation when the time comes to evaluate the necessary integrals. At that point we shall deviate considerably from Ref. 2, introducing a simpler calculational scheme. Part of this paper, however, is also devoted to a clearer exposition of basic concepts, e.g. the nature of the ensemble is never explicitly spelt out in Ref. 2.

When an electron emits a photon, it will move from its original trajectory to a new trajectory, with the same coordinates in space but a lower energy. (Trajectories are characterized partly by energy. Two trajectories can have the same spatial coordinates but different energies.) Then the quantization axis $\hat{n}$ after emission will be the one appropriate to the new trajectory. "Spin-flip" is defined by Derbenev and Kondratenko /11/ to be a change of orientation from $\hat{n}_i$ to $-\hat{n}_f$ (initial to final).

It is convenient to write

$$\hat{n}_f = \hat{n}_i - \frac{\hbar\omega}{E_0} \times \{\text{something}\} \tag{4.1}$$

because we expect the change between $\hat{n}_i$ and $\hat{n}_f$ to be of $O(\frac{\hbar\omega}{E_0})$. Derbenev and Kondratenko do so, but care must be exercised because $\hat{n}_i$ and $\hat{n}_f$ refer to different phase space points, say $\vec{\pi}_i$ and $\vec{\pi}_f$ respectively. We need to express $\hat{n}_f$ in terms of the coordinate system at $\vec{\pi}_i$ in order to compare them, else we could end up defining "spin-flip" in a totally meaningless way. We therefore need to proceed step by step.

$$\begin{aligned} \hat{n}' &= \hat{n}_f(\text{using coordinate system at } \vec{\pi}_i) \\ \hat{n}_i - \hat{n}' &= \frac{\hbar\omega}{E_0}\vec{d} \end{aligned} \tag{4.2}$$

We shall explain all this in more detail later (Section 6). Suffice it for now to say that the result of the comparison is a vector called $\vec{d}$. Derbenev and Kondratenko call it $\gamma(\partial\hat{n}/\partial\gamma)$ instead. $\vec{d}$ is called $\vec{\Gamma}$ by Montague /3/. Notation is therefore not uniform, but the use of the symbol $\vec{d}$ appears to be the most common and may perhaps eventually become standard. In any case, the name for all notations is "spin-orbit coupling vector". We shall show how to calculate $\vec{d}$ in Section 6, when we show how to calculate $\hat{n}$ (obviously they are related problems). As with $\hat{n}$, $\vec{d}$ is a function of phase space coordinates, i.e. $\vec{d} = \vec{d}(\vec{\pi})$.

A word of warning. In writing "$\frac{\hbar\omega}{E_0}\vec{d}$" we have implicitly assumed the magnitude of the perturbations of spin motion to be proportional to $\hbar\omega/E_0$ i.e. we are implicitly (and from now on explicitly) assuming $\vec{d}$ not to depend on the emitted photon energy. This will be valid if $\frac{\hbar\omega}{E_0} \ll 1$. In present day electron storage rings $\frac{\hbar\omega}{E_0} \simeq 10^{-6}$.

From now on in this paper, we shall denote an ensemble average by angular brackets — $\langle \vec{d} \rangle$ will mean the ensemble average of $\vec{d}$, and $\langle \vec{d}^2 \rangle$ will denote the ensemble average (mean-square) of $\vec{d}^2$. Of course, $\langle \vec{d}^2 \rangle$ will not in general equal $\langle \vec{d} \rangle^2$ — beware! The Derbenev-Kondratenko formula contains terms both linear and quadratic in $\vec{d}$, and also an ensemble average over such terms.

Another word of warning. The perturbation excited by the emission of a photon will depend not only on photon energy but also on the photon direction. (The recoil of the electron will be in the direction opposite to that of the photon, and as this direction varies the resulting spin perturbation will vary too.) In practice the photon is emitted in a very narrow cone, of opening angle $1/\gamma$ centred on the tangent to the orbit. Hence the direction of the photon does not significantly matter and so $\vec{d}$ will be assumed not to depend on the direction of photon emission.

We calculate the polarization as follows. We divide phase space into a large number of little cells. Since the rate of emission of synchrotron radiation varies around the ring, we shall initially choose the cells to all have a common value of arc-length s_o. Within each cell we measure the spin projection onto $\hat{n}$. This is given by $g_+(\vec{\pi}) - g_-(\vec{\pi})$, after which we sum over all cells. $g_\pm(\vec{\pi})$ is the fraction of particles in the ensemble in the chosen cell with spin along/against $\hat{n}$ — in equilibrium they will not change on successive turns around the ring.

$$\int \Big[g_+(\vec{\pi}) + g_-(\vec{\pi}) \Big] \, d\vec{\pi} = 1 \tag{4.3a}$$

$$P = \oint \frac{ds}{L} \int \big(g_+(\vec{\pi}) - g_-(\vec{\pi}) \big) \, d\vec{\pi} \tag{4.3b}$$

What remains now is to obtain the probabilities per unit time W_+ and W_-, for each cell in phase space, and as we have pointed out in a previous Section, we shall do this by calculating the power spectrum. Unlike Derbenev and Kondratenko, we shall not evaluate the quantum mechanical matrix element directly. We shall use a simpler technique.

Again we shall divide phase space into little cells $d\vec{\pi}$. Within each cell, we measure the rate of change of spin orientation (spin-flip) produced by emission of synchrotron radiation. Recalling remarks made in conection with (3.4), we can ignore electrons which do not radiate because they do not change the polarization. As pointed out above, we shall have to do some fiddling with $\hat{n}$ to define spin-flip. We then sum the rate over all cells. Combining the concepts in Section 2 with the ensemble average, we can write

$$\begin{aligned} W_\pm &= \left\langle \int d\omega \, \frac{dW_\pm}{d\omega} \right\rangle \\ &= \left\langle \int d\omega \, \frac{1}{\hbar\omega} \frac{d\mathrm{P}_\pm}{d\omega} \right\rangle \end{aligned} \tag{4.4}$$

Let us now turn our attention to the evaluation of this handsome sequence of integrations.

V. Calculation of the Transition Rates

In this Section we shall describe the calculation of the probabilities per unit time W_+ and W_-. From remarks made in a previous Section, we need the power spectra $dP_\pm/d\omega$. To get them, we employ the elegant formulation of Schwinger /4/

$$\frac{dP_\pm(\vec{\pi},t)}{d\omega} = \frac{\omega^2}{4\pi^2 c}\mathrm{Re}\left\{\int\left[\frac{1}{c^2}\vec{j}_\pm(\vec{r},t).\vec{j}^*_\pm(\vec{r}',t') - \rho_\pm(\vec{r},t)\rho^*_\pm(\vec{r}',t')\right]e^{-i\omega(t'-t-\hat{k}.(\vec{r}'-\vec{r})/c)}\,dt'\,d\Omega\right\} \quad (5.1)$$

where $\vec{j}_\pm, \rho_\pm$ are the appropriate spin-flip current and charge densities, with an obvious notation. The use of (5.1) instead of the Fermi Golden Rule is the simplifying mathematical step that renders the calculation much more tractable than Jackson's /8/. (The use of this mathematical approach was suggested by L.N. Hand /12/ in a private communication to the author.)

Certainly by comparison to a quantum field theoretic treatment there is no question that (5.1) is simpler. By comparison to the use of the Fermi Golden Rule, (5.1) automatically includes the sum over photon polarizations, and it also obviates the need to average over the time of emission of the photon — it is t', if we wish to look at (5.1) quantum mechanically, ~~which is integrated~~ over, thus automatically performing the averaging which Jackson has to do explicitly.

In (5.1), we have taken into account the possibility that $\vec{j}$ and ρ might be complex (Schwinger considers only real $\vec{j}$ and ρ) but otherwise the derivation of (5.1) is exactly as given in Ref. 4. Note that $\vec{j}$ and ρ are to be evaluated on the classical trajectory $\vec{r}(t)$. (Schwinger is slightly more general about this, but we are only concerned with a single particle, not an arbitrary charge distribution.) $\hat{k}$ is the direction of propagation of the emitted radiation.

We get the charge and current densities from the Hamiltonian via the relationship (for electromagnetism)

$$\mathrm{H_{int}} = \frac{1}{c}j_\mu \mathrm{A}^\mu = \rho\Phi - \frac{1}{c}\vec{j}.\vec{\mathrm{A}} \quad (5.2)$$

In his calculation of the quantum correction to the classical synchrotron radiation power spectrum, Schwinger /5/ showed how to do this systematically to various powers of $\hbar$. Since we are only going to calculate the leading term in $\hbar$, the ordering of operators in $\mathrm{H_{int}}$ will not concern us, eg. we shall ignore

$$[r_i, p_j] = i\hbar\delta_{ij}$$

because the commutator is of a higher power in $\hbar$.

To get ρ and $\vec{j}$, we must first express $\vec{\mathrm{E}}$ and $\vec{\mathrm{B}}$ in terms of $\vec{\mathrm{A}}$ and Φ. Writing

$$\vec{\mathrm{A}}(\vec{r},t) = \int \vec{\mathrm{A}}(\hat{k},\omega) e^{-i\omega(t-\hat{k}.\vec{r}/c)} \frac{d\Omega}{4\pi} \frac{d\omega}{2\pi} \tag{5.3a}$$

$$\Phi(\vec{r},t) = \int \Phi(\hat{k},\omega) e^{-i\omega(t-\hat{k}.\vec{r}/c)} \frac{d\Omega}{4\pi} \frac{d\omega}{2\pi} \tag{5.3b}$$

and using

$$\vec{\mathrm{E}} = -\frac{1}{c}\frac{\partial \vec{\mathrm{A}}}{\partial t} - \vec{\nabla}\Phi \tag{5.4a}$$

$$\vec{\mathrm{B}} = \vec{\nabla} \times \vec{\mathrm{A}} \tag{5.4b}$$

we get

$$\vec{\mathrm{E}}(\hat{k},\omega) = \frac{i\omega}{c}\vec{\mathrm{A}}(\hat{k},\omega) - \frac{i\omega}{c}\hat{k}\Phi(\hat{k},\omega) \tag{5.5a}$$

$$\vec{\mathrm{B}}(\hat{k},\omega) = \frac{i\omega}{c}\hat{k} \times \vec{\mathrm{A}}(\hat{k},\omega) \tag{5.5b}$$

If $\Phi_{\mathrm{rad}}, \vec{\mathrm{A}}_{\mathrm{rad}}, \vec{\mathrm{E}}_{\mathrm{rad}}, \vec{\mathrm{B}}_{\mathrm{rad}}$ all refer to emission of a single photon of frequency ω' and direction $\hat{k}'$, then

$$\vec{\mathrm{A}}(\hat{k},\omega) = \vec{\mathrm{a}}\delta(\omega-\omega')\delta^{(3)}(\hat{k}-\hat{k}') \tag{5.6a}$$

$$\Phi(\hat{k},\omega) = \quad \delta(\omega-\omega')\delta^{(3)}(\hat{k}-\hat{k}') \tag{5.6b}$$

The matrix element of the interaction Hamiltonian using the Derbenev-Kondratenko choice of quantization axis is

$$\begin{aligned}\langle -\hat{n} + \frac{\hbar\omega}{E_0}\vec{d}|\mathrm{H}_{\mathrm{int}}|\hat{n}\rangle = \langle -\hat{n} + \frac{\hbar\omega}{E_0}\vec{d}|\Big\{ &(-e\vec{\beta}.\vec{\mathrm{A}}_{\mathrm{rad}} + e\Phi_{rad}) \\ &- \frac{e\hbar}{mc}\vec{s}_{op}.\left[(a+\frac{1}{\gamma})\vec{\mathrm{B}}_{\mathrm{rad}} - \frac{a\gamma}{\gamma+1}\vec{\beta}.\vec{\mathrm{B}}_{\mathrm{rad}}\vec{\beta} - (a+\frac{1}{\gamma+1})\vec{\beta}\times\vec{\mathrm{E}}_{\mathrm{rad}}\right]\Big\}|\hat{n}\rangle\end{aligned} \tag{5.7a}$$

To the leading order in $\hbar$, this boils down to

$$\begin{aligned}&(-e\vec{\beta}.\vec{\mathrm{A}}_{\mathrm{rad}} + e\Phi_{rad})\langle -\hat{n} + \frac{\hbar\omega}{E_0}\vec{d}|\hat{n}\rangle \\ &\quad - \frac{e\hbar}{mc}\langle -\hat{n}|\vec{s}_{op}|\hat{n}\rangle.\left[(a+\frac{1}{\gamma})\vec{\mathrm{B}}_{\mathrm{rad}} - \frac{a\gamma}{\gamma+1}\vec{\beta}.\vec{\mathrm{B}}_{\mathrm{rad}}\vec{\beta} - (a+\frac{1}{\gamma+1})\vec{\beta}\times\vec{\mathrm{E}}_{\mathrm{rad}}\right]\end{aligned} \tag{5.7b}$$

For brevity it will be convenient to define

$$\vec{D} = \frac{\hbar\omega}{E_0}\vec{d} \tag{5.8a}$$

and to recall

$$\vec{s}_{-\hat{n}\hat{n}} = \langle -\hat{n}|\vec{s}_{op}|\hat{n}\rangle \tag{5.8b}$$

whence

$$\rho = e\langle -\hat{n} + \vec{D}|\hat{n}\rangle - \frac{ie\hbar\omega}{mc^2}(a + \frac{1}{\gamma+1})(\vec{s}_{-\hat{n}\hat{n}} \times \vec{\beta}).\hat{k} \tag{5.9a}$$

$$\frac{1}{c}\vec{j} = e\vec{\beta}\langle -\hat{n} + \vec{D}|\hat{n}\rangle + \frac{ie\hbar\omega}{mc^2}\left[(a + \frac{1}{\gamma})\vec{s}_{-\hat{n}\hat{n}} \times \hat{k} - \frac{a\gamma}{\gamma+1}\vec{s}_{-\hat{n}\hat{n}}.\vec{\beta}\vec{\beta} \times \hat{k} - (a + \frac{1}{\gamma+1})\vec{s}_{-\hat{n}\hat{n}} \times \vec{\beta}\right] \tag{5.9b}$$

To begin with, we shall neglect a. Then

$$\rho = e\langle -\hat{n} + \vec{D}|\hat{n}\rangle - \frac{ie\hbar\omega}{mc^2}\frac{1}{\gamma+1}(\vec{s}_{-\hat{n}\hat{n}} \times \vec{\beta}).\hat{k} \tag{5.9c}$$

$$\frac{1}{c}\vec{j} = e\vec{\beta}\langle -\hat{n} + \vec{D}|\hat{n}\rangle + \frac{ie\hbar\omega}{mc^2}\left[\frac{1}{\gamma}\vec{s}_{-\hat{n}\hat{n}} \times \hat{k} - \frac{1}{\gamma+1}\vec{s}_{-\hat{n}\hat{n}} \times \vec{\beta}\right] \tag{5.9d}$$

We perform the integral over solid angles Ω first, followed by the integral over t'. The approximations made are the same as in Ref. 4 (or as in Ref. 8, except that we put $a = 0$). We also assume that the trajectory is an arc of a circle, of radius ρ, over the range relevant for the integration. (This is a perfectly standard approximation in synchrotron radiation calculations. It states that the variation of the accelerator magnetic field is sufficiently smooth over the distance traversed during a photon emission that it may be locally taken as constant, thus yielding a circular orbit with local radius $\rho(s)$.) We follow the notation of Jackson /8/

$$\begin{aligned} \omega_o &= \frac{\beta c}{\rho(s)} \\ \omega_c &= \frac{3}{2}\gamma^3\omega_o \\ \nu &= \omega/\omega_c \\ \hat{n}_o(s) &= (\sin\theta_o\cos\phi_o, \sin\theta_o\sin\phi_o, \cos\theta_o) \end{aligned} \tag{5.10a}$$

We must exercise a little care in interpreting Jackson's θ_o and ϕ_o. He begins by stating them to be the initial direction of the spin $\vec{s}_{classical}$ at the time of emission t_o, but subsequently he averages over t_o, which implies an average over θ_o and ϕ_o. He points this out towards the end of his paper, in particular when comparing with the results of Baier and Katkov /7/ (i.e. (2.14,a,b)).

In (2.14a,b) $\hat{n}_o$ unquestionably denotes $\hat{n}_o(s)$, so in Jackson's notation one should say that the θ_o, ϕ_o that appear at the beginning of his paper denote $\hat{n}(t = t_o)$ and the θ_o, ϕ_o that appear at the end mean $\hat{n}_o(s)$. We define θ_o and ϕ_o via (5.10a), i.e. $\hat{n}_o(s)$. We shall in fact be more concerned with

$$\begin{aligned} \hat{n}(\vec{\pi}) &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\ \vec{\pi} &= \text{ phase space point} \end{aligned} \tag{5.10b}$$

The differential power spectrum in frequency is

$$\frac{dP_+}{d\omega} = \frac{1}{2\pi\sqrt{3}} \frac{e^2\hbar^2\omega^3}{m^2c^5\gamma^4} \oint \frac{ds}{L} \left\{ \frac{\langle \sin^2\theta \rangle}{2} \int_\nu^\infty K_{\frac{1}{3}}(\nu')\, d\nu' + \frac{\langle 1+\cos^2\theta \rangle}{2} K_{\frac{2}{3}}(\nu) \right.$$
$$\left. + \langle \cos\theta \rangle K_{\frac{1}{3}}(\nu) + \frac{\langle \vec{d}^2 \rangle}{2} \int_\nu^\infty K_{\frac{5}{3}}(\nu')\, d\nu' - \langle \hat{b}.\vec{d} \rangle K_{\frac{1}{3}}(\nu) \right\} \tag{5.11}$$

To get $dP_-/d\omega$ we replace θ by $\pi - \theta$ and $\vec{d}$ by $-\vec{d}$. The terms linear in $\cos\theta$ and $\hat{b}.\vec{d}$ are antisymmetric under this transformation, hence $dP_-/d\omega - dP_+/d\omega$ will not vanish. The terms independent of $\vec{d}$ in (5.11) will give (2.14a,b). They are the terms that result from using only the interaction Hamiltonian (3.13). With all terms included, we obtain the Derbenev-Kondratenko formula

$$P_\infty = \frac{8}{5\sqrt{3}} \frac{\oint \frac{ds}{|\rho|^3} \langle \hat{b}.(\hat{n} - \vec{d}) \rangle}{\oint \frac{ds}{|\rho|^3} \left\{ 1 - \frac{2}{9} \langle (\hat{n}.\hat{v})^2 \rangle + \frac{11}{18} \langle \vec{d}^2 \rangle \right\}} \tag{5.12a}$$

Strictly, $\oint \frac{ds}{|\rho|^3}$ should be in the ensemble average too, but this is a very minor point. The corresponding polarization time is

$$\tau_o = \left(\frac{5\sqrt{3}}{8} \frac{e^2\hbar\gamma^5}{m^2c^2} \frac{1}{L} \oint \frac{ds}{|\rho|^3} \left\{ 1 - \frac{2}{9} \langle (\hat{n}.\hat{v})^2 \rangle + \frac{11}{18} \langle \vec{d}^2 \rangle \right\} \right)^{-1} \tag{5.12b}$$

Let us specialize to the case of motion in a uniform magnetic field. Then $\hat{n}$ and $\hat{b}$ are both constant, and $\hat{b}.\hat{n} = -1$ (+1 for positrons). It can be shown that for such a magnetic field $\vec{d}$ vanishes identically. Then we obtain the result of Sokolov and Ternov /1/

$$P_{\infty\ ST} = -\frac{8}{5\sqrt{3}} \tag{2.13b}$$

Notice that $P_{\infty\ ST}$ is independent of energy, radius of orbit, etc. This may be traced to the expression for the power spectrum (5.11). All terms appear with the same overall dependence on energy and radius. This dependence cancels out between numerator and denominator in (5.12a), hence the resulting asymptotic degree of polarization is a pure number, a ratio of integrals of Bessel functions

$$P_{\infty\ ST} = -\frac{\int_0^\infty \nu^2 K_{\frac{1}{3}}(\nu)\, d\nu}{\int_0^\infty \nu^2 K_{\frac{2}{3}}(\nu)\, d\nu} = -\frac{8}{5\sqrt{3}} \tag{5.13}$$

The generalization by Baier and Katkov /7/ to the expression (2.14a) multiplies $P_{\infty\ ST}$ by an energy and orbit dependent factor, hence P_∞ will (in general) depend on such information. Actually, however, most electron storage rings are flat, i.e. the design particle orbit lies in a plane. Thus the magnetic fields in the bending magnets are all parallel (vertical, in practice). Very little synchrotron radiation is emitted in other parts of a storage ring, e.g. in quadrupole focussing fields, so we neglect such fields. Then once again $\hat{b}$ and $\hat{n}$ are constant, $\hat{b}.\hat{n} = -1$ everywhere, and also $\vec{d} = 0$ everywhere. This brings us back to the Sokolov-Ternov result, that in most electron storage rings the design value of P_∞ is in fact $-8/5\sqrt{3}$, independent of energy.

In a flat storage ring, it is the inhomogenieties and misalignments in the fields which result in distortions of the equilibrium closed orbit, and coupling of orbital perturbations to spin motion (via the vector $\vec{d}$), that cause the lower degrees of polarization actually observed, in particular in the vicinity of so-called spin resonances. A particularly important source of such distortions, for example, is the misalignments of the quadrupole focussing magnets. Consequently, the value of P_∞ in a real storage ring depends strongly on the value of $\vec{d}$, in particular on $\langle \vec{d}^2 \rangle$. The difference between (2.14) and (5.12) is quite significant in practice. This is one of the reasons we wish to check the Derbenev-Kondratenko formula.

We have so far neglected the consequences of $a \neq 0$. Let us now consider the modifications to the above results due to non-zero $g - 2$. We begin with a simple model of motion in a uniform static vertical magnetic field $\vec{B} = B\hat{z}$. (3.2) and (3.4) reduce to

$$\frac{d\vec{\beta}}{dt} = \frac{eB}{mc\gamma}\vec{\beta} \times \hat{z} \tag{5.14}$$

$$\frac{d\vec{s}}{dt} = \frac{eB}{mc\gamma}(\gamma a + 1)\vec{s} \times \hat{z} \tag{5.15}$$

The equations are very similar, but the spin precession frequency is $\gamma a + 1$ times bigger than the orbital (Larmor) frequency. In existing or proposed high-energy electron storage rings,

$$\begin{aligned} \gamma a &= \frac{E\ [\text{GeV}]}{.440} \\ &\simeq 40 \ \text{ in PETRA} \\ &\simeq 70 \ \text{ in HERA} \\ &\simeq 100 \text{ in LEP} \end{aligned} \tag{5.16}$$

Clearly, then, in modern machines neglect of the value of a is not justified. We recalculate the power spectrum, P_∞ and τ_o to the leading order in a (since $a \simeq 10^{-3}$ we can expand P_∞ etc. in powers of a). The formal definitions of $\hat{n}$ and $\vec{d}$ do not change. We only need to use (5.9a,b) instead of (5.9c,d).

$$\frac{dP_+}{d\omega} = \frac{1}{2\pi\sqrt{3}}\frac{e^2\hbar^2\omega^3}{m^2c^5\gamma^4}\oint\frac{ds}{L}\Big\{\frac{\langle\sin^2\theta\rangle}{2}\Big[\int_\nu^\infty K_{\frac{1}{3}}(\nu')\,d\nu' + a\{\int_\nu^\infty K_{\frac{1}{3}}(\nu')\,d\nu' - K_{\frac{2}{3}}(\nu)\}\Big]$$
$$+\frac{\langle 1+\cos^2\theta\rangle}{2}\Big[K_{\frac{2}{3}}(\nu)+\frac{a}{2}\{9K_{\frac{2}{3}}(\nu)-\int_\nu^\infty K_{\frac{1}{3}}(\nu')\,d\nu'\}\Big]$$
$$+\langle\cos\theta\rangle\Big[K_{\frac{1}{3}}(\nu)+\frac{a}{4}\{16K_{\frac{1}{3}}(\nu)+\frac{16}{3}\frac{1}{\nu}\int_\nu^\infty K_{\frac{1}{3}}(\nu')\,d\nu'\}\Big]$$
$$+\frac{\langle\vec{d}^2\rangle}{2}\int_\nu^\infty K_{\frac{5}{3}}(\nu')\,d\nu' - \langle\hat{b}.\vec{d}\rangle(1+\frac{3}{2}a)K_{\frac{1}{3}}(\nu)\Big\} \quad (5.17)$$

$$P_\infty = \frac{8}{5\sqrt{3}}\frac{\oint\frac{ds}{|\rho|^3}\Big\langle(1+\frac{14}{3}a)\hat{b}.\hat{n}-(1+\frac{3}{2}a)\hat{b}.\vec{d}\Big\rangle}{\oint\frac{ds}{|\rho|^3}\Big\{1+\frac{37}{9}a-(\frac{2}{9}+\frac{39}{9}a)\langle(\hat{n}.\hat{v})^2\rangle+\frac{11}{18}\langle\vec{d}^2\rangle\Big\}} \quad (5.18a)$$

$$\tau_o = \left(\frac{5\sqrt{3}}{8}\frac{e^2\hbar\gamma^5}{m^2c^2}\frac{1}{L}\oint\frac{ds}{|\rho|^3}\Big\{1+\frac{37}{9}a-(\frac{2}{9}+\frac{39}{9}a)\langle(\hat{n}.\hat{v})^2\rangle+\frac{11}{18}\langle\vec{d}^2\rangle\Big\}\right)^{-1} \quad (5.18b)$$

We see that although the actual spin motion is considerably different from the case if $a = 0$, the formal expressions for P_∞ and τ_o are not much affected. The dependence of P_∞ and τ_o is an implicit, rather than explicit, function of a. This justifies the way in which P_∞ is normally calculated — (5.12) is used instead of (5.18), but $\hat{n}$ and $\vec{d}$ are calculated with $a \neq 0$. This way of calculating P_∞ and τ_o is probably motivated by two reasons :

1) Derbenev and Kondratenko /2/ and Jackson /8/ showed that (2.14) does not change much formally when we take into account non-zero a, and

2) the expression (5.18) has not been hitherto published.

Why should the coefficients of the various terms in $\hat{n}$ and $\vec{d}$ not depend strongly on a, if $\hat{n}$ and $\vec{d}$ themselves do? The answer is given by tracing the contributions of individual terms in (5.9) to (5.17), thence to (5.18). Notice that ρ and $\vec{j}$ are of $O(\frac{1}{\gamma})$. Thus, naively, we would expect $dP/d\omega$ to be of $O(\frac{1}{\gamma^2})$, but it is of $O(\frac{1}{\gamma^4})$. This is due to cancellations between the various terms in (5.9) when inserted in the integral (5.1). In particular, $O(\frac{1}{\gamma^4})$ is the result of a cancellation of the form $O((\frac{1}{\gamma}-\frac{1}{\gamma+1})^2)$. When $a \neq 0$ is included in the calculations, the cancellation changes to one of the form $O(((a+\frac{1}{\gamma})-(a+\frac{1}{\gamma+1}))^2)$, which does not depend on a, even though $a \gg \frac{1}{\gamma}$. The leading explicit contribution of a to $dP/d\omega$ is of $O(\frac{a}{\gamma^4})$. The cancellation depends on the assumption of a locally circular orbit.

VI. CALCULATION OF $\hat{n}$ AND $\vec{d}$

We have derived the Derbenev-Kondratenko formula, but we have not yet shown how to calculate the quantities $\hat{n}$ and $\vec{d}$ that go into it. The time has now come to do so. The calculation is perhaps most easily phrased in the matrix formalism developed by A. Chao /6/, hence we shall first spend a few words of introduction on this notation.

We consider only trajectories in the vicinity of the equilibrium closed orbit, so that it is valid to consider perturbations to linear order only. The phase space variables are all measured with respect to the equilibrium closed orbit. Although we have previously used $\vec{\pi}$ to denote a phase-space point, it is more common to call it y, so from now on we shall denote a phase-space point by y. Then

$$y(s_o) = \begin{bmatrix} x \\ p_x \\ z \\ p_z \\ \sigma \\ \delta \end{bmatrix} \tag{6.1}$$

where x, z = horizontal,vertical transverse coordinate displacement and p_x, p_z = canonically conjugate momenta. $\sigma = s - s_o$ is the longitudinal offset along the equilibrium closed orbit and $\delta = \Delta E/E_0$, the relative energy offset, is the variable conjugate to σ. The above phase space is defined with respect to a Hamiltonian H_1', which is H_1 (neglecting radiation) of Section 3, except that the independent variable has been changed from time t to arc-length s_o by means of a canonical transformation. (The product $\sigma\delta$ does not have the dimension of action because of the scaling of δ, but that is a technical detail.)

To linear order in x, p_x, etc the transformation of y from azimuth s_1 to s_2 is given by a matrix, called the transfer matrix

$$y(s_2) = \mathrm{M}_{6\times 6}(s_2, s_1)y(s_1) \tag{6.2}$$

One should note that y is a point in phase space, and does not belong to an electron, and M and y define a macroscopic trajectory, followed by an electron only between successive photon emissions. A particularly important matrix will be the one-turn transfer matrix based at s_o, $\mathrm{M}_{6\times 6}(s_o + L, s_o)$. Given this matrix, each value of $y(s_o)$ will define a trajectory around the ring.

It will be useful to bring spin into the picture at this point. For a given trajectory, we defined $\hat{n}$ in Section 3. We can also find two other solutions of (3.4), say $\hat{e}_1$ and $\hat{e}_2$, such that $(\hat{n}, \hat{e}_1, \hat{e}_2)$ is a right-handed orthonormal system. $\hat{e}_1$ and $\hat{e}_2$ will not satisfy (3.7) in general. Hence, for a given value of s_o and y, they will change, turn to turn, around the ring, i.e. they will be multivalued functions of s_o. To remedy this difficulty, we shall define $\hat{e}_1$ and $\hat{e}_2$ over one turn only, and introduce a discontinuous change in the basis between $s_o = L$ and $s_o = 0$ (the analogue of a branch cut in complex variable theory when dealing with multivalued functions).

The vectors $\hat{n}_o, \hat{e}_{1o}, \hat{e}_{2o}$ merit special attention. They are $(\hat{n}, \hat{e}_1, \hat{e}_2)$ along the equilibrium closed orbit. If $\hat{n}$ does not differ from $\hat{n}_o$ much, we can write

$$\hat{n} = \hat{n}_o + \alpha \hat{e}_{1o} + \beta \hat{e}_{2o} \qquad (\alpha^2 + \beta^2 \ll 1) \tag{6.3}$$

and define two coordinates α and β, which will be sufficient to characterize $\hat{n}$. In this paper, we shall work only to linear order in α and β. Following Chao /6/ we can extend the phase space to eight dimensions by writing

$$Y(s_o) = \begin{bmatrix} y(s_o) \\ \\ \alpha \\ \beta \end{bmatrix} \tag{6.4}$$

Because α and β are not canonically conjugate variables, this 8-dimensional phase space will follow only indirectly from the Hamiltonian H (again neglecting radiation) of Section 3, but this is again a technical matter. The corresponding 8×8 transfer matrix will have the form

$$\mathrm{M}_{8\times 8}(s_o + L, s_o) = \begin{pmatrix} \mathrm{M}_{6\times 6} & 0_{6\times 2} \\ \mathrm{M}_{2\times 6} & \mathrm{M}_{2\times 2} \end{pmatrix} \tag{6.5}$$

The zero in the top right hand rectangle reflects the (very good) approximation that orbital motion is not affected by spin motion. We have immediately written the one-turn transfer matrix based on s_o. Any other matrix $\mathrm{M}_{8\times 8}(s_2, s_1)$ will have the same form, but the one-turn matrix will be of most interest to us.

$\mathrm{M}_{6\times 6}(s_o + L, s_o)$ will have 6 eigenvectors E_k

$$\begin{aligned} \mathrm{M}_{6\times 6} E_k &= e^{i2\pi Q_k} E_k \qquad k = \pm 1, \pm 2, \pm 3 \\ E_{-k} &= E_k^* \\ Q_{-k} &= -Q_k \end{aligned} \tag{6.6}$$

The above properties follow because $\mathrm{M}_{6\times 6}$ is symplectic, but for the present purposes there is no need to go into details. We can extend these eigenvectors to 8 dimensions easily

$$\mathrm{M}_{8\times 8} \begin{bmatrix} E_k \\ \\ E_{7k} \\ E_{8k} \end{bmatrix} = e^{i2\pi Q_k} \begin{bmatrix} E_k \\ \\ E_{7k} \\ E_{8k} \end{bmatrix} \tag{6.7a}$$

$\Rightarrow$

$$\mathrm{M}_{2\times 6} E_k + \mathrm{M}_{2\times 2} \begin{bmatrix} E_{7k} \\ E_{8k} \end{bmatrix} = e^{i2\pi Q_k} \begin{bmatrix} E_{7k} \\ E_{8k} \end{bmatrix} \tag{6.7b}$$

which can be solved for E_{7k} and E_{8k}.

We shall need to know the structure of $M_{2\times 2}$

$$M_{2\times 2} = \begin{pmatrix} \cos 2\pi\nu & \sin 2\pi\nu \\ -\sin 2\pi\nu & \cos 2\pi\nu \end{pmatrix} \tag{6.8}$$

ν is called the spin tune. It depends (very weakly) on the trajectory, but not on s_o i.e. it is a global property of the trajectory. In (6.8) the trajectory is the equilibrium closed orbit.

$M_{8\times 8}$ has two more eigenvectors. They are

$$E_{\pm k} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \\ 1 \\ \pm i \end{bmatrix} \qquad k = 4 \qquad \text{eigenvalues } = e^{\pm i 2\pi\nu} \tag{6.9}$$

but they will not be as important to us as $\{E_k, k = \pm 1, \pm 2, \pm 3\}$.

In terms of action-angle variables and the above phase-space coordinate system, the Hamiltonian H_1' can be written in the form (see, e.g., Ref. 13, and references therein, for a discussion of Hamiltonian mechanics with particular application to accelerator physics)

$$H_1' = \sum_k \frac{2\pi Q_k}{L} I_k \tag{6.10}$$

The equations of motion for an action I and angle ψ are

$$\begin{aligned} \frac{d\psi}{ds} &= \frac{\partial H_1'}{\partial I} = \frac{2\pi Q}{L} \\ \frac{dI}{ds} &= -\frac{\partial H_1'}{\partial \psi} = 0 \end{aligned} \tag{6.11}$$

An example of an orbital oscillation around the equilibrium closed orbit is a betatron oscillation. It is almost a sinusoidal oscillation (in fact it is obtained using a WKB approximation). It has the form

$$x(s) = \sqrt{I\beta(s)}\, exp\Big\{\pm i\Big[\psi(s) - \frac{2\pi Q s}{L} + \int_o^s \frac{ds'}{\beta(s')}\Big]\Big\} \tag{6.12}$$

where $\beta(s)$, called a beta-function, is a periodic function of the accelerator magnetic fields. The phase advance $2\pi Q$ is related to $\beta(s)$.

$$\begin{aligned} \beta(s+L) &= \beta(s) \\ 2\pi Q &= \oint \frac{ds}{\beta(s)} \end{aligned} \tag{6.13}$$

There are numerical programs (e.g. see Chao /6/) to compute $M_{8\times 8}$ and the E_k and Q_k. We shall take the availability of such information as given. In that case, we are now ready to embark on the calculation of $\hat{n}$ and $\vec{d}$.

We begin with $\hat{n}$. Given the approximations we have made we can represent $\hat{n}$ by a point in the α, β plane, i.e. we express $\hat{n}$ as a function of the 8-dimensional "phase-space coordinate system" introduced above. Given an orbital phase space point $y_o(s_o)$, we can expand in eigenvectors

$$y_o(s_o) = \sum_{k=\pm 1,\pm 2,\pm 3} a_k E_k \tag{6.14}$$

Extend this to 8 dimensions and write

$$Y_o = \begin{bmatrix} y_o \\ \\ n_1 \\ n_2 \end{bmatrix}$$

where

$$\hat{n} - \hat{n}_o \equiv \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \sum \left\{ a_k \begin{bmatrix} E_{7k} \\ E_{8k} \end{bmatrix} \right\} + \begin{bmatrix} \alpha_o \\ \beta_o \end{bmatrix} \tag{6.15}$$

defines n_1, n_2, α_o and β_o.

Now $\hat{n}$ must satisfy the periodicity requirements (3.7). It is worth noting that the eigenvectors E_k also satisfy similar constraints

$$E_k(I, \psi, s_o) = E_k(I, \psi + 2\pi, s_o) = E_k(I, \psi, s_o + L) \tag{6.16a}$$

For example, for the betatron oscillation (6.12), using (6.13),

$$\begin{aligned} &\sqrt{I\beta(s_o)}\, exp\Big\{\pm i[\psi(s_o) - \frac{2\pi Q s_o}{L} + \int_o^{s_o} \frac{ds'}{\beta(s')}]\Big\} \\ &\qquad = \sqrt{I\beta(s_o)}\, exp\Big\{\pm i[\psi(s_o) + 2\pi - \frac{2\pi Q s_o}{L} + \int_o^{s_o} \frac{ds'}{\beta(s')}]\Big\} \\ &\qquad = \sqrt{I\beta(s_o + L)}\, exp\Big\{\pm i[\psi(s_o) - \frac{2\pi Q (s_o + L)}{L} + \int_o^{s_o + L} \frac{ds'}{\beta(s')}]\Big\} \end{aligned} \tag{6.16b}$$

(6.12) refers only to a coordinate displacement from equilibrium, whereas the E_k are column vectors that include much more information, but (6.12) is nevertheless useful as an illustrative example to demonstrate the effect of the periodicity conditions (6.16a). The above remarks in fact give us the expression for $\hat{n}$, viz.

$$\hat{n} - \hat{n}_o \equiv \sum \left\{ a_k \begin{bmatrix} E_{7k} \\ E_{8k} \end{bmatrix} \right\} \tag{6.17}$$

i.e. the displacement of $\hat{n}$ from the spin equilibrium closed orbit $\hat{n}_o$ displays the same periodicities as the displacement y_o from the orbital equilibrium closed orbit.

As pointed out by Montague /3/, upon successive turns around the machine $\hat{n}$ will trace a Lissajous figure centred on $\hat{n}_o$ with the periodicities of the various orbital modes of oscillation

$$\hat{n}(s_o + mL) - \hat{n}_o(s_o) \equiv \sum \left\{ a_k \begin{bmatrix} E_{7k} \\ E_{8k} \end{bmatrix}_{s_o} e^{i2\pi m Q_k} \right\} \quad , \quad m = \text{integer} \tag{6.18}$$

As we would also expect from (3.7), the above expression also satisfies

$$\hat{n}(I_i, \psi_i, s_o) = \hat{n}(I_i, \psi_i, s_o + mL) \tag{6.19}$$

This specifies $\hat{n}$ completely, given a_k, which we must now calculate. The conventional orthonormality relations for eigenvectors of symplectic matrices (again it does not concern us why) are

$$E_k^\dagger \mathrm{J}\, E_k = \begin{cases} i & k = 1,2,3 \\ -i & k = -1,-2,-3 \end{cases} \tag{6.20a}$$

$\Rightarrow$

$$a_k = \mp i E_k^\dagger \mathrm{J}\, y_o \qquad \begin{cases} k > 0 \Rightarrow -i \\ k < 0 \Rightarrow i \end{cases} \tag{6.20b}$$

where

$$\mathrm{J} = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix} \tag{6.20c}$$

The blank spaces contain zero, which have been omitted for clarity.

We can go no further until we know y_o, and so we shall now turn to the evaluation of $\vec{d}$. Let us denote the initial phase space point by Y_i and the final one by Y_f, both at azimuth s_o, of course. Let the final state have a relative energy offset $\epsilon = \Delta E / E_0$ with respect to the initial, but no other spatial or momentum displacement. We shall tacitly assume ϵ to be positive in the following calculation. This makes no difference to the physics, but helps to keep track of minus signs. Then define

$$\mathrm{Y}_i(s_o) = \begin{bmatrix} y_i \\ \\ n_{i1} \\ n_{i2} \end{bmatrix} = \sum \left\{ a_k \begin{bmatrix} E_k \\ \\ E_{7k} \\ E_{8k} \end{bmatrix} \right\} \tag{6.21a}$$

and

$$\mathrm{Y}_f(s_o) = \begin{bmatrix} y_f \\ \\ n_{f1} \\ n_{f2} \end{bmatrix} = \sum \left\{ b_k \begin{bmatrix} E_k \\ \\ E_{7k} \\ E_{8k} \end{bmatrix} \right\} \tag{6.21b}$$

where by definition

$$y_f - y_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \epsilon \end{bmatrix} \tag{6.21c}$$

Since $\hat{n}_f$ and $\hat{n}_i$ have been expressed in terms of the same coordinate system, the difference between them is given by

$$\hat{n}_f - \hat{n}_i \equiv \sum \left\{ (b_k - a_k) \begin{bmatrix} E_{7k} \\ E_{8k} \end{bmatrix} \right\} \tag{6.22}$$

This can be simplified, because

$$b_k - a_k = -iE_k^\dagger \mathrm{J} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \epsilon \end{bmatrix} = -iE_{5k}^* \epsilon \qquad , \; k > 0 \tag{6.23a}$$

$$b_{-k} - a_{-k} = (b_k - a_k)^*$$

$\Rightarrow$

$$\hat{n}_f - \hat{n}_i \equiv 2\epsilon \sum_{k=1,2,3} \mathrm{Im} \left\{ E_{5k}^* \begin{bmatrix} E_{7k} \\ E_{8k} \end{bmatrix} \right\} \tag{6.23b}$$

We define $\epsilon \vec{d}(y_i)$ to be this difference. The reader will realize that $\hat{n}$ is in fact a vector field, and $\vec{d}$ is a derivative of $\hat{n}$. The reader will also realize that, in general, $\vec{d}$ will in fact be the covariant derivative of $\hat{n}$, treated as a vector field, and is not just the derivative of a vector. In general, the basis vectors of the coordinate system used to describe the spin part of the trajectories will not be the same at both Y_i and Y_f, and we shall need to take this fact into account when comparing $\hat{n}_i$ and $\hat{n}_f$, hence the covariant derivative. Replacing E by γmc^2,

$$\begin{aligned} \vec{d}(y_i) &= \gamma \frac{\partial \hat{n}}{\partial \gamma}\bigg|_{covariant} \\ &= 2 \sum_{k=1,2,3} \mathrm{Im} \left\{ E_{5k}^* \begin{bmatrix} E_{7k} \\ E_{8k} \end{bmatrix} \right\} \end{aligned} \tag{6.24}$$

Chao's result /6/ is the negative of the above, which is the numerical error alluded to in the Introduction. Because all deviations from equilibrium have been considered only to linear order, $\vec{d}$ turns out to depend only on s_o, not y_i. Higher-order terms in the calculation of either $\hat{n}$ or y (or both) will lead to a dependence of $\vec{d}$ on y_i.

VII. CONCLUSION

We have thus derived the Derbenev-Kondratenko formula. By using semi-classical electrodynamics, we have attempted not only to simplify the mathematics but also to gain a more classical understanding of the mechanism of electron spin polarization in high-energy storage rings.

We reiterate that the Derbenev-Kondratenko formula is statistical in character. Photon emission by individual electrons is used only to establish the spin-flip transition rate in a given volume of phase space, which then determines the equilibrium properties of the ensemble. No attempt is made to trace the history of an individual electron.

As mentioned in the Introduction, our goal is mainly didactic : we seek to provide a clearer understanding of a known formula, and we have tried to clarify various misconceptions about it. For example, it is often stated (e.g. see Montague /3/) that $\vec{d}$ is not γ times the partial derivative of $\hat{n}$ with respect to γ.

But $\vec{d}$ *is* $\gamma(\partial\hat{n}/\partial\gamma)$! It is a covariant derivative of the vector field $\hat{n}$. The reason for this misunderstanding may lie partly in the failure to realize that $\hat{n}$ is more than a vector, it is a vector field. It is therefore erroneous to treat the derivative as merely that of a vector, which has been done in the past. Obviously, $\vec{d}$ is a vector field too. Possibly nomenclature such as the "$\hat{n}$ field" and "$\vec{d}$ field" would be appropriate in this context (just as one speaks of the "$\vec{E}$" and "$\vec{B}$" fields).

Let us repeat the argument of the physical origin of polarization by synchrotron radiation emission. Polarization develops because the probability per unit time for spin-flip is not the same for flips in opposite directions (processes which do not flip spin do not change the magnitude of the polarization). When an electron emits a photon, its position in phase space changes discontinuously, and its spin projection on the new quantization axis may be opposite to its projection on the old. This may happen either because of the presence of spin-dependent terms in the interaction matrix element or because the initial and final quantization axes are not parallel. The covariant derivative $\vec{d}$ expresses the difference between the initial and final quantization axes.

What is the detailed expression for the electron ensemble, as a function of phase-space coordinates? We have not yet specified it. Actually, we do not need a detailed expression for the sake of the derivation of the Derbenev-Kondratenko formula, only for numerical calculation of P_∞ for a specific storage ring. All we need to know is that the ensemble has the same properties turn after turn around the ring, i.e. the distribution function of particles in phase space should not be explicitly time-dependent.

The Derbenev-Kondratenko formula pertains to the "thermodynamic limit", in which there are infinitely many particles in the ensemble. The concept of polarization is still meaningful in such a case. Thus it is assumed there are no serious fluctations in the populations of "spin-up" and "spin-down" particles, even when P_∞ becomes large. Naturally, in a real ring, as the magnitude of polarization increases, one of these two populations will get depleted.

Under what other circumstances does the formula fail? The principal cause of failure is that under certain conditions $|\vec{d}| \to \infty$, at so-called spin resonances (polarization resonances, really). Such resonances have been observed in electron storage rings. (See references given in Montague /3/ and Jackson /8/.) The resonances appear in the expression for $\vec{d}$ (which is not contained in the Baier-Katkov result (2.14), evidence yet again that (5.12) is a much different formula in its practical consequences).

A less stringent constraint on the applicability of the formula is that the radiation damping time must be much shorter than the polarization build-up time. This is because we have assumed the ensemble to be already in equilibrium in the orbital phase space, and that spin motion does not perturb this orbital equilibrium. In reality, of course, the processes leading to equilibrium in both orbital and spin phase space occur simultaneously, so the previous statement can only be valid if the time scale for the establishment of equilibrium in orbital motion is much shorter than the corresponding time scale for polarization. In modern high energy electron storage rings, $\tau_o \sim 10$ minutes to hundreds of minutes, whereas $\tau_{rad} \sim 10$ msec. For a fixed bending radius, $\tau_o \sim \gamma^{-5}$ and $\tau_{rad} \sim \gamma^{-3}$, so in theory we could imagine a storage ring where $\tau_o \ll \tau_{rad}$, but not at energies currently available.

ACKNOWLEDGEMENTS

The author is grateful to L.N. Hand for first bringing his attention to this problem. He is also indebted to R.G. Glasser for much critical questioning and reading of the manuscript during its preparation. Most of this work was done while the author was on a one-year visit to DESY, where this paper was written, and he wishes to thank the DESY staff for their hospitality. This work was supported in part by a grant from the National Science Foundation.

REFERENCES

1. A.A. Sokolov & I.M. Ternov, Sov Phys Doklady **8** (1964), p. 1203.
2. Ya.S. Derbenev & A.M. Kondratenko, Sov Phys JETP **37** (1973), p. 968.
3. B.W. Montague, Physics Reports **113** (1984).
4. J. Schwinger, Physical Review **75** (1949), p. 1912.
5. J. Schwinger, Proc Natl Acad Sci USA **40** (1954), p. 132.
6. A. Chao, Nuc Inst & Methods **29** (1981), p. 180.
7. V.N. Baier & V.M. Katkov, Sov Phys JETP **25** (1967), p. 944.
8. J.D. Jackson, Rev Mod Phys **48** (1976), p. 417.
9. L. Thomas, Phil Mag **3** (1927);
   V. Bargmann, L. Michel, V.L. Telegdi, Phys Rev Letters **2** (1959), p. 435.
10. J.D. Jackson, "Classical Electrodynamics", 2nd Edition, John Wiley & Sons, Inc, New York, 1975.
11. Ya.S. Derbenev & A.M. Kondratenko, Sov Phys JETP **35** (1972), p. 230.
12. L.N. Hand, private communication.
13. E.D. Courant, R.D. Ruth, W.T. Weng, *Stability in Dynamical Systems* I, Invited paper presented at the 3^{rd} Summer School on High Energy Particle Accelerators, Upton, New York, July 6-16, 1983, SLAC-PUB-3415 (1984).

APPENDIX A
EVALUATION OF SPIN-DEPENDENT MATRIX ELEMENTS

For our purposes, we may assume the orbit is locally circular, which is equivalent to assuming the field is locally a uniform, static magnetic field. For a particle with spin $\hbar\vec{s}$, the equation of motion (3.4) then reduces to

$$\frac{d\vec{s}}{dt} = \vec{s} \times \vec{\Omega} \tag{A.1}$$

where

$$\begin{aligned} \vec{\Omega} &= (1 + \gamma a)\vec{\omega}_o \\ \vec{\omega}_o &= \frac{e\vec{B}}{\gamma mc} \\ |\vec{\omega}_o| &= \frac{\beta c}{\rho} \end{aligned} \tag{A.2}$$

and ρ is the radius of the orbit.

For the purposes of integration, we shall use Jackson's /8/ coordinate system, which is $(\hat{x}, \hat{y}, \hat{z})$ with $\hat{x}$ along the direction of motion, $\hat{y}$ along the direction of acceleration and $\hat{z}$ along the direction of the magnetic field (because electrons are negatively charged, this makes $\hat{z} = \hat{x} \times \hat{y}$).

Let us replace t' by $\tau = t' - t$ in the integral (5.1) and express the integrand in terms of τ instead of t'. The limits of integration are $-\infty < \tau < \infty$. Let us define

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{A.3}$$

$$s_z(\tau) = s_z(0) \tag{A.4a}$$

$$s_\pm(\tau) = s_\pm(0)e^{\pm i\Omega\tau} \tag{A.4b}$$

where $\vec{s}$ here means $\vec{s}_{-\hat{n}\hat{n}}$. θ, ϕ are defined by $\hat{n}$. As for $\vec{s}(0)$,

$$\begin{aligned} s_z(0) &= -\frac{1}{2}\sin\theta \\ s_+(0) &= -\frac{1}{4}e^{i\phi}(1 - \cos\theta) \\ s_-(0) &= \frac{1}{4}e^{-i\phi}(1 + \cos\theta) \end{aligned} \tag{A.5}$$

We also need the matrix elements $\langle -\hat{n} + \frac{\hbar\omega}{E_0}\vec{d}|\hat{n}\rangle \ldots$ of (5.7) and (5.9). Since $\frac{\hbar\omega}{E_0}|\vec{d}| \ll 1$, we shall evaluate this matrix element only to linear order in $\vec{d}$.

For brevity, put

$$\vec{D} = \frac{\hbar\omega}{E_0}\vec{d} \tag{A.6}$$

$\hat{n} - \vec{D}$ is also a unit vector (by definition) and so

$$(\hat{n} - \vec{D})^2 = 1$$

$\Rightarrow$

$$\hat{n}.\vec{D} = 0 \qquad \text{(neglecting } O(|\vec{d}|^2)\text{)}$$

$\Rightarrow$

$$(D_x \cos\phi + D_y \sin\phi)\sin\theta + D_z \cos\theta = 0 \tag{A.7}$$

whence

$$\langle -\hat{n} + \vec{D}|\hat{n}\rangle = \frac{1}{2}\Big[-(D_x \cos\phi + D_y \sin\phi)\cos\theta + D_z \sin\theta + i(D_x \sin\phi - D_y \cos\phi)\Big] \tag{A.8a}$$

which can be simplified using (A.7) to

$$\langle -\hat{n} + \vec{D}|\hat{n}\rangle = \frac{1}{2}\left[\frac{D_z}{\sin\theta} + i(D_x \sin\phi - D_y \cos\phi)\right] \tag{A.8b}$$

Note that D_z vanishes at $\theta = 0, \pi$ hence (A.8b) contains no divergent terms.

Now we need the evolution of $\hat{n}$ and $\vec{D}$ along the electron trajectory in order to integrate over τ or t'. Derbenev and Kondratenko adopt a brutally simple solution — they take both $\hat{n}$ and $\vec{d}$ to be constant over the duration of photon emission. Fortunately, if we permit ourselves to peek ahead into the evaluation of (5.1), we find that only $\theta, |\langle -\hat{n} + \vec{D}|\hat{n}\rangle|$ and $\mathrm{Re}\{\langle -\hat{n} + \vec{D}|\hat{n}\rangle\}$ appear in the calculation. In a uniform static field $\vec{\mathrm{B}} = \mathrm{B}\hat{b}$, these are constants.

$$\begin{array}{ccccc} \theta & & & = \text{constant} & \\ |\langle -\hat{n} + \vec{D}|\hat{n}\rangle| & = & \frac{1}{2}|\vec{D}| & = \text{constant} & \\ \mathrm{Re}\{\langle -\hat{n} + \vec{D}|\hat{n}\rangle\} & = & \hat{b}.\vec{D}/2\sin\theta & = \text{constant} & \end{array} \tag{A.9}$$

This result depends crucially on the assumption of a uniform rotation about $\hat{b}$ for spin motion over the course of the interaction.

APPENDIX B
DETAILS OF INTEGRATION

In writing (5.1) we have employed the elegant derivation of Schwinger, who uses arguments of time-reversal symmetry to eliminate irrelevant terms. Schwinger considered only real currents, but we need to generalize to complex ones. (5.1) actually holds for an arbitrary charge and current distribution, but we take ρ and $\vec{j}$ to be for a single particle, and evaluate them on the particle trajectory. Thus, for the radiated power, we start from

$$\mathrm{P} = -\mathrm{Re}\Big\{ \int \vec{j}.\vec{\mathrm{E}}^{*}_{\mathrm{ret}}\, d^3\vec{r}\, \Big\} \tag{B.1}$$

but otherwise we derive (5.1) as in Ref. 4.

Although the spin dependent terms must be calculated at the appropriate phase-space point, quantities such as electron energy and bending radius are (to a good approximation) the same if the orbital trajectory is taken to be the equilibrium closed orbit. A difference in bending radius of a few mm does not matter when $\rho \simeq 100$m. In a quadrupole magnet this is not true, but very little synchrotron radiation comes from non-bending magnets, in practice. For brevity let us introduce some notation

$$\left.\begin{aligned}
\vec{r}' &= \vec{r}(t') \\
\vec{j}' &= \vec{j}^{*}(r', t') \\
\rho' &= \rho^{*}(r', t') \\
j^{\mu} &= (c\rho, \vec{j}) \\
j'^{\mu} &= (c\rho', \vec{j}') \\
\frac{1}{c^2} j_{\mu} j'^{\mu} &= \rho\rho' - \frac{1}{c^2}\vec{j}.\vec{j}' \\
\tau &= t' - t \\
\vec{R} &= \frac{\omega}{c}(\vec{r}' - \vec{r}) \\
R &= |\vec{R}| \\
x &= \frac{\hat{k}.\vec{R}}{R} \qquad (\hat{k} = \text{ photon direction})
\end{aligned}\right\} \tag{B.2}$$

(5.1) can then be rewritten

$$\frac{d\mathrm{P}}{d\omega} = -\frac{\omega^2}{4\pi^2 c^3} \int \Big[\mathrm{Re}(j_{\mu} j'^{\mu}) \cos(\omega t - xR) + \mathrm{Im}(j_{\mu} j'^{\mu}) \sin(\omega t - xR) \Big]\, d\tau\, d\Omega \tag{B.3}$$

The integrand is independent of azimuth around $\hat{k}$, so we take the polar axis of the integral over Ω along $\hat{k}$, in which case

$$d\Omega = 2\pi\, dx \tag{B.4}$$

From the form of $j_\mu j'^\mu$ we find 3 types of angular integrals

$$\int f(\tau,x)\,dx \quad \int \hat{k}_i\, f(\tau,x)\,dx \quad \int \hat{k}_i\hat{k}_j\, f(\tau,x)\,dx$$

which we shall label scalar, vector and tensor integrals respectively. $f(\tau,x)$ is some function invariant under rotation around $\hat{k}$. It is then easy to prove

$$\int \hat{k}_i\, f(\tau,x)\,dx = \hat{R}_i \int x f(\tau,x)\,dx \tag{B.5a}$$

$$\left.\begin{aligned} \int \hat{k}_i\hat{k}_j\, f(\tau,x)\,dx &= A\,\delta_{ij} + B\,\hat{R}_i\hat{R}_j \\ A &= \frac{1}{2}\left[\int f\,dx - \int x^2 f\,dx\right] \\ B &= \frac{1}{2}\left[3\int x^2 f\,dx - \int f\,dx\right] \end{aligned}\right\} \tag{B.5b}$$

To proceed further we need the form of $f(\tau,x)$. At this point it becomes easier to begin with $a = 0$, and to expand in powers of a. In that case $f(\tau,x)$ is always of the form

$$f(\tau,x) = (\omega_o\tau)^n \begin{cases} \cos(\omega\tau - xR) \\ \sin(\omega\tau - xR) \end{cases} \qquad n = 0,1\ldots \tag{B.6}$$

The integral over Ω can then be easily performed. This leaves us with an integral over τ. As usual in synchrotron radiation calculations, we expand $j_\mu j'^\mu$ and R in powers of τ because the rapid oscillation of the exponential kills off contributions from large values of τ. The spin-dependent terms are given Appendix A. As for $\vec{\beta}$ and $\vec{R}$,

$$\begin{aligned} \vec{\beta} &= \beta\left[(1 - \frac{\omega_o^2\tau^2}{2})\,\hat{x} + \omega_o\tau\,\hat{y}\right] \\ \vec{R} &= (\omega_o\tau - \frac{\omega_o^3\tau^3}{6})\,\hat{x} + \frac{\omega_o^2\tau^2}{2}\,\hat{y} \end{aligned} \tag{B.7}$$

When we integrate, we get modified Bessel functions, as did Schwinger. As pointed out by Jackson /8/, the calculation is not different in principle from Schwinger's, only more laborious.