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Abstract

Expectation values at an arbitrary point of the 3-dimensional coupling parameter space in the lattice-regularized SU(2) Higgs-model with a doublet scalar field are expressed by a series of expectation values at infinite self-coupling ($\lambda = \infty$). Questions of convergence of this "strong self-coupling expansion" (SSCE) are investigated. The SSCE is a potentially useful tool for the study of the λ-dependence at any value (zero or non-zero) of the bare gauge coupling.

1 Introduction

The standard $SU(3) \otimes SU(2) \otimes U(1)$ model has turned out to be very successful up to the 100 GeV energy range. The extension to still higher energies is, obviously, one of the central issues in our field. The simplest extension is, of course, to assume that the standard model is valid in its presently known simple form up to some very high cut-off, say, the Planck scale. The basic question is, whether this extrapolation over roughly 17 orders of magnitude is possible at all, at least in principle. In other words, is the standard model mathematically consistent with such a high cut-off? The SU(3) colour factor is asymptotically free, therefore there seems to be no problem with a high cut-off in this case. The U(1) electromagnetic coupling is not asymptotically free, therefore there could be a problem, at least in perturbation theory, but the value of the coupling can still be very small at the Planck scale, therefore the problem is not really serious. The least understood and most problematic piece is the Higgs-sector, which in its simplest form is a four-component (complex scalar doublet) ϕ^4 model coupled to the weak SU(2) gauge field. A simple and important question is, whether this "standard Higgs model" is mathematically consistent with a very high cut-off. Going to the extreme, and assuming lattice regularization, the question is, whether the standard Higgs model on the lattice has a non-trivial continuum limit.

Other important problems for the lattice-regularized $SU(3) \otimes SU(2) \otimes U(1)$ theory are connected to the non-perturbative features of spontaneous symmetry breaking occurring in the Higgs-sector of

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the SU(2) ⊗ U(1) electroweak component. The primary question in this respect is the phase structure of the Higgs-sector. In fact, the main concern of the first numerical Monte Carlo investigations [1,2] in the SU(2) Higgs model with scalar doublet field was the study of the phase transitions. In these papers, as a first approximation, the coupling to fermions and the $U(1)_{hypercharge}$-factor was neglected. The inclusion of the U(1)-factor is, in principle, not very difficult. (For a first study see Ref. [3].) The coupling to fermions (through the Yukawa- and gauge-couplings) is much harder to include in a numerical approach but, clearly, one has to start at the easier end.

Concerning the existence of a non-trivial continuum limit in the standard SU(2) Higgs model, the distinctive feature, compared to the SU(3)-colour gauge sector, is the appearance of the non-asymptotically-free scalar ϕ^4 self-coupling λ. According to perturbation theory, such couplings have the tendency to grow for increasing cut-off (or decreasing lattice spacing). This growth could, in principle, be stopped, if there were a limiting fixed point at some finite λ-value. In the pure ϕ^4-model (neglecting the gauge coupling) the non-existence of such a non-trivial fixed point is, however, almost rigorously proven [4]. Consequently, the bare λ coupling reaches infinity for some finite cut-off, and the continuum limit of the ϕ^4-model has to be trivial (non-interacting).

The inclusion of the SU(2) gauge coupling can, however, induce a non-trivial continuum limit. Such a limit could, in principle, exist for finite gauge coupling $\beta < \infty$, somewhere in the interior of the three-dimensional coupling parameter space. A first numerical study of the scaling properties of correlations and static energies in Ref. [5] showed, however, that this is not probable. All the numerical information obtained up to now on these quantities [5,6,7,8] is consistent with a possibly non-trivial continuum limit at $\beta = \infty$, and the critical value of the hopping parameter $\kappa = \kappa_{cr}(\lambda)$ in the pure ϕ^4-model, for any fixed positive λ. This critical point in the ϕ^4-model is very probably trivial i.e. non-interacting for every λ. Therefore, *any interaction in the coupled gauge-Higgs system is induced by the gauge coupling.* In particular, due to the asymptotic freedom of the gauge coupling, the induced ϕ^4-coupling has to vanish, too, at infinite energy. Considering the simplest possible gauge-quantum exchange graphs, one can infer, that the physical ϕ^4-coupling has to go to zero by some power of the high energy scale.

The triviality implies that the pure ϕ^4-model becomes approximately λ-independent, if the cut-off is high enough. In fact, this approximate λ-independence seems to set in quite early, as it is shown, for instance, by the figures in recent papers on numerical Monte Carlo studies of the ϕ^4-model [9,10]. After the inclusion of the gauge coupling the approximate λ-independence remains true. This is shown by recent numerical results [5,8] in the wide range $0.1 \leq \lambda \leq \infty$, for relatively large gauge coupling ($\beta = 2.3$). It is quite amazing, that the λ-independence looks qualitatively so similar, if one compares the behaviour of the correlation lengths at $\beta = 2.3$ and $\beta = \infty$ [5].

Of course, the λ-independence of the continuum limit in the Higgs model implies, that the continuum theory has one free parameter less than the regularized theory. Since this is an essentially non-perturbative phenomenon, it can be called *non-perturbative parameter reduction*[1]. A direct consequence of it is, that the value of the Higgs-boson mass can be predicted from the W-boson mass and the renormalized gauge coupling. The question of the λ-dependence in the standard Higgs model is thus very interesting both from the principal and practical points of view.

Numerical Monte Carlo studies can, on their own, only provide indications but never proofs for such mathematical features like the exact λ-independence or the existence of a continuum limit. Therefore, it is useful to combine the numerical work with analytical techniques. In this context one of us suggested [6] to use the expansion around $\lambda = \infty$ for the study of λ-dependence in the standard Higgs model (and also in other types of Higgs models).

The purpose of the present paper is to study such "strong self-coupling expansions" (SSCE's) in the standard Higgs model. After elaborating on the general technique of SSCE, the convergence will be proven for a finite lattice in Section 2. The convergence in the practical sense will be questioned in Section 3 by numerically determining the $\lambda = \infty$ correlations, which appear in a low order SSCE

[1] We thank Barbara Schrempp for proposing us this suggestive terminology.

of some relevant quantities. Section 4. contains some concluding remarks. Useful formulas for SSCE calculations are collected in the Appendix.

2 General framework

2.1 Lattice action

The standard Higgs model is described on the lattice by the SU(2) link-variables $U(x,\mu) \in \mathrm{SU}(2)$ for the gauge field and by the length- ($\rho_x \geq 0$) and angular- ($\alpha_x \in \mathrm{SU}(2)$) variables of the Higgs-field. x denotes lattice points, $\mu = \pm1, \pm2, \pm3, \pm4$ are link directions and (x,μ) is the link from the point x to the neighbouring point $(x+\hat{\mu})$ in direction μ. The lattice action in these variables can be written like

$$S_{\lambda,\beta,\kappa} = \beta \sum_P \left(1 - \frac{1}{2} Tr\, U_P\right)$$

$$+ \sum_x \left\{ \rho_x^2 - 3\log\rho_x + \lambda(\rho_x^2 - 1)^2 - \kappa \sum_{\mu>0} \rho_{x+\hat{\mu}}\rho_x Tr\, \left(\alpha_{x+\hat{\mu}}^+ U(x,\mu)\alpha_x\right) \right\} \tag{1}$$

Here $\sum_P$ stands for a summation over positively oriented plaquettes. The first term is the familiar Wilson-action [12] for the gauge field proportional to the bare gauge coupling $\beta \equiv 4/g^2$. The bare coupling parameters for the Higgs-field are: the self-coupling λ and the hopping parameter κ. The integration measure corresponding to Eq. (1) is $d\rho_x\, d^3\alpha_x\, d^3U(x,\mu)$ (where d^3g denotes the Haar-measure in SU(2)). The peculiarity of the SU(2) doublet scalar field is, that its angular part is equivalent to the local gauge degree of freedom. Therefore, at any finite β it is possible to introduce, instead of the SU(2) link- and site-variables, a gauge invariant link variable

$$V(x,\mu) \equiv \alpha_{x+\hat{\mu}}^+ U(x,\mu)\alpha_x \tag{2}$$

In terms of this, the lattice action is

$$S_{\lambda,\beta,\kappa} = \beta \sum_P \left(1 - \frac{1}{2} Tr\, V_P\right) + \sum_x \left\{ \rho_x^2 - 3\log\rho_x + \lambda(\rho_x^2 - 1)^2 - \kappa \sum_{\mu>0} \rho_{x+\hat{\mu}}\rho_x Tr\, V(x,\mu) \right\} \tag{3}$$

After performing the trivial integration over α_x, the integration measure for Eq. (3) is $d\rho_x\, d^3V(x,\mu)$.

In the limit $\beta \to \infty$ the variable change in Eq. (2) is inappropriate, because the gauge part of the action vanishes (the link-variables become gauge equivalent to unity). Therefore, one has to use for the $\beta \to \infty$ action

$$S_{\lambda,\beta=\infty,\kappa} = \sum_x \left\{ \rho_x^2 - 3\log\rho_x + \lambda(\rho_x^2 - 1)^2 - \kappa \sum_{\mu>0} \rho_{x+\hat{\mu}}\rho_x Tr\, \left(\alpha_{x+\hat{\mu}}^+ \alpha_x\right) \right\} \tag{4}$$

This defines an $\mathrm{SU}(2) \otimes \mathrm{SU}(2)$- (or $O(4)$-) symmetric four-component ϕ^4-model corresponding to the lattice version of the Gell-Mann-Lévy linear σ-model [13].

In this paper we shall often consider the $\lambda \to \infty$ limit. In this case the length of the Higgs-field is frozen to $\rho_x = 1$, and the action in Eq. (3) goes over into

$$S_{\lambda=\infty,\beta,\kappa} = \beta \sum_P \left(1 - \frac{1}{2} Tr\, V_P\right) - \kappa \sum_{x,\mu>0} Tr\, V(x,\mu) \tag{5}$$

The corresponding limit of the action in Eq. (4) defines the $\mathrm{SU}(2) \otimes \mathrm{SU}(2)$-symmetric non-linear σ-model on the lattice:

$$S_{\lambda=\infty,\beta=\infty,\kappa} = -\kappa \sum_{x,\mu>0} Tr\, \left(\alpha_{x+\hat{\mu}}^+ \alpha_x\right) \tag{6}$$

2.2 Expansion of the partition function

The aim of the SSCE is to express the expectation values in an arbitrary point (λ, β, κ) of the coupling parameter space in terms of a series of expectation values at $\lambda = \infty$. The gauge coupling will be kept fixed, and the hopping parameter will be changed by an arbitrary (positive) scale factor s^2 to $\bar{\kappa} \equiv s^2\kappa$. The motivation for this will become clear later. According to Eqs. (4), (6) the relation between the actions in these points is:

$$S_{\lambda,\beta,\kappa} = S_{\lambda=\infty,\beta,\bar{\kappa}} + \sum_x \left\{ -\frac{1}{4\lambda} - 3\log\rho_x + \lambda\left(\rho_x^2 - 1 + \frac{1}{2\lambda}\right)^2 - \kappa\sum_{\mu>0}(\rho_{x+\hat{\mu}}\rho_x - s^2)Tr\,V(x,\mu)\right\} \quad (7)$$

Therefore, the partition function $Z_{\lambda,\beta,\kappa}$ in the point (λ, β, κ) can be written like

$$Z_{\lambda,\beta,\kappa} = \int [dV]\,[d\rho]\,\exp(-S_{\lambda,\beta,\kappa})$$

$$= C\int[dV]\,\exp(-S_{\lambda=\infty,\beta,\bar{\kappa}})\left\langle \exp\left[\kappa\sum_l T_l(\rho_{x+\hat{\mu}}\rho_x - s^2)\right]\right\rangle_\lambda \quad (8)$$

where C is an unessential (β, $\bar{\kappa}$)-independent constant, l is an abbreviation for the positively-oriented link (x,μ), T_l stands for $Tr\,V(x,\mu)$ and the λ-dependent expectation value $\langle\cdots\rangle_\lambda$ is defined, for an arbitrary ρ-dependent function $f(\rho)$, by

$$\langle f\rangle_\lambda \equiv \frac{\int[d\rho]\,f(\rho)\exp\left\{-\sum_x\left[-3\log\rho_x + \lambda(\rho_x^2 - 1 + \frac{1}{2\lambda})^2\right]\right\}}{\int[d\rho]\,\exp\left\{-\sum_x\left[-3\log\rho_x + \lambda(\rho_x^2 - 1 + \frac{1}{2\lambda})^2\right]\right\}} \quad (9)$$

In the following expressions an important rôle will be played by the λ-dependent expectation value of the powers of a single length variable ρ_y (for an arbitrary site y):

$$\langle\rho_y^k\rangle_\lambda = \frac{\int_0^\infty d\rho_y\,\rho_y^{3+k}\exp\left[-\lambda(\rho_y^2 - 1 + \frac{1}{2\lambda})^2\right]}{\int_0^\infty d\rho_y\,\rho_y^3\exp\left[-\lambda(\rho_y^2 - 1 + \frac{1}{2\lambda})^2\right]} \equiv \frac{I_k(\lambda)}{I_0(\lambda)} \equiv i_k \quad (10)$$

For an explicit form of these integrals see Appendix A.1.

The λ-dependent expectation value in Eq. (8) can be exponentiated in the well-known way, as

$$\left\langle \exp\left[\kappa\sum_l T_l(\rho_{x+\hat{\mu}}\rho_x - s^2)\right]\right\rangle_\lambda$$

$$= \exp\left\{\kappa\sum_l T_l(i_1^2 - s^2) + \sum_{n=2}^\infty \frac{\kappa^n}{n!}\sum_{l_1\ldots l_n} T_{l_1}\ldots T_{l_n}c_n(l_1\ldots l_n)\right\} \quad (11)$$

where the "connected i-product" c_n is defined by

$$c_n(l_1\ldots l_n) \equiv c(l_1\ldots l_n) \equiv \langle(\rho_{x_1+\hat{\mu}_1}\rho_{x_1})\cdots(\rho_{x_n+\hat{\mu}_n}\rho_{x_n})\rangle_\lambda^c \quad (12)$$

The superscript c on $\langle\cdots\rangle_\lambda^c$ denotes "connected" (λ-dependent) expectation value. In the definition of the connected parts, the products within parentheses have to be considered as a single entity. (This kind of notation will be used throughout this paper, also in connection with other sorts of expectation values.)

The basic relation (11) makes it possible to obtain the expectation value of any function $F(V)$ depending only on the link variables $V(x,\mu)$:

$$\langle F(V)\rangle_{\lambda,\beta,\kappa} \equiv \frac{\int[dV]\,[d\rho]\,\exp(-S_{\lambda,\beta,\kappa})F(V)}{\int[dV]\,[d\rho]\,\exp(-S_{\lambda,\beta,\kappa})}$$

$$= Z^{-1}_{\lambda,\beta,\kappa} \int [dV] \exp(-S_{\lambda=\infty,\beta,\bar\kappa}) F(V) \left\langle \exp\left[\kappa \sum_l T_l (\rho_{x+\hat\mu}\rho_x - s^2)\right]\right\rangle_\lambda =$$

$$Z^{-1}_{\lambda,\beta,\kappa} \int [dV] \exp(-S_{\lambda=\infty,\beta,\bar\kappa}) F(V) \exp\left\{\kappa \sum_l T_l (i_1^2 - s^2) + \sum_{n=2}^{\infty} \frac{\kappa^n}{n!} \sum_{l_1 \ldots l_n} T_{l_1} \ldots T_{l_n} c_n(l_1 \ldots l_n)\right\} \quad (13)$$

This is a typical relation which will be the starting point for the SSCE of correlation functions, considered below. In order to obtain the SSCE series from Eq. (13), one has to expand the right hand side in powers of κ, (or, equivalently, in the number of link-traces T_l, or in half of the total number of powers of Higgs-field lengths contained in the i-factors of Eqs. (10), (12).)

Up to now, the framework for SSCE was kept quite general. Relations like (13) can connect, for fixed β, an arbitrary point to an arbitrary $\lambda = \infty$ point. Some special cases are, however, of particular interest. If, for instance, the $\lambda = \infty$ point is chosen at ($\beta = \infty$, $\bar\kappa = 0$), then the resulting expansion is equivalent to the "high temperature expansion" of the ϕ^4 model in statistical physics [11]. In this case the $\lambda = \infty$ action $S_{\lambda=\infty,\beta=\infty,\bar\kappa=0}$ is trivial, and the right hand side of Eq. (13) does not contain any further integration. By choosing $\bar\kappa = 0$ for any $\beta \leq \infty$ one obtains a more general "hopping parameter expansion", where the terms of the SSCE series are given by expectation values of correlations in the pure SU(2) gauge theory.

For the study of the λ-dependence at any fixed (finite or infinite) β, a favourable choice of the κ-rescaling factor

$$s^2 \equiv s^2(\lambda) = \frac{\bar\kappa(\lambda = \infty)}{\kappa(\lambda)} \quad (14)$$

is such, that along the curve $\kappa(\lambda) = \bar\kappa / s^2(\lambda)$ the link expectation value is constant:

$$\langle Tr\, V(x,\mu)\rangle_{\lambda,\beta,\kappa(\lambda)} = const. \quad (15)$$

The reason is the approximate universality of the physical expectation values along the curves ($\beta = const.$, $\langle T_l\rangle = const.$).

2.3 The convergence of SSCE on a finite lattice

Let us now sketch the proof of the convergence of the SSCE on the example of the link-variable dependent expectation value in Eq. (13). It is convenient to introduce an integral $E_F(\alpha)$, which depends on an extra parameter α,

$$E_F(\alpha) = \int [dV] \exp(-S_{\lambda=\infty,\beta,s^2\kappa}) F(V) \left\langle \exp\left[\alpha\kappa \sum_l T_l (\rho_{x+\hat\mu}\rho_x - s^2)\right]\right\rangle_\lambda \quad (16)$$

The expectation value in Eq. (13) is then given by

$$\langle F(V)\rangle_{\lambda,\beta,\kappa} = \left.\frac{E_F(\alpha)}{E_1(\alpha)}\right|_{\alpha=1} \quad (17)$$

The SSCE is equivalent to the power series expansion of the right hand side in α, taken at $\alpha = 1$. We shall see that, for any bounded F, $E_F(\alpha)$ is an entire function of α.

The power series expansion of $E_F(\alpha)$ is

$$E_F(\alpha) = \int [dV] \exp(-S_{\lambda=\infty,\beta,s^2\kappa}) F(V) \cdot$$

$$\sum_{n=0}^{\infty} \frac{(\alpha\kappa)^n}{n!} \sum_{l_1 \ldots l_n} T_{l_1} \ldots T_{l_n} \left\langle (\rho_{x_1+\hat\mu_1}\rho_{x_1} - s^2) \ldots (\rho_{x_n+\hat\mu_n}\rho_{x_n} - s^2)\right\rangle_\lambda \quad (18)$$

The interesting piece of this series satisfies

$$\left|\sum_{l_1\ldots l_n} T_{l_1}\ldots T_{l_n}\left\langle(\rho_{x_1+\hat\mu_1}\rho_{x_1}-s^2)\ldots(\rho_{x_n+\hat\mu_n}\rho_{x_n}-s^2)\right\rangle_\lambda\right|$$
$$\leq 2^n\sum_{l_1\ldots l_n}\left\langle(\rho_{x_1+\hat\mu_1}\rho_{x_1}+s^2)\ldots(\rho_{x_n+\hat\mu_n}\rho_{x_n}+s^2)\right\rangle_\lambda \tag{19}$$

In addition, the power series expansion

$$\left\langle\exp\left[2\alpha\kappa\sum_l(\rho_{x+\hat\mu}\rho_x+s^2)\right]\right\rangle_\lambda=\sum_{n=0}^{\infty}\frac{(2\alpha\kappa)^n}{n!}\sum_{l_1\ldots l_n}\left\langle(\rho_{x_1+\hat\mu_1}\rho_{x_1}+s^2)\ldots(\rho_{x_n+\hat\mu_n}\rho_{x_n}+s^2)\right\rangle_\lambda \tag{20}$$

has positive coefficients and is, obviously, convergent for every α. It follows, that the series under the $[dV]$ integral in Eq. (18) is uniformly convergent, hence it can be integrated term by term. In view of Eq. (20), the resulting series is convergent for an arbitrary α, therefore $E_F(\alpha)$ is an entire function of α, as stated before.

The integral in the denominator $E_1(\alpha)$ is, of course, also an entire function, and for real α it is positive definite, hence the function $E_F(\alpha)/E_1(\alpha)$ is analytic at least in a strip along the real axis. The isolated zeros of $E_1(\alpha)$ produce singularities. Therefore, the SSCE series in Eq. (17) can be convergent only if the nearest zero α_0 of $E_1(\alpha)$ satisfies $|\alpha_0|\geq 1$. The convergence of the SSCE can, however, always be achieved by a simple resummation. Namely, instead of α, one can always introduce a new variable $\bar\alpha$, such that the relation

$$\alpha=\sum_{k=1}^{\infty}a_k\bar\alpha^k \tag{21}$$

defines a conformal mapping of the unit circle in $\bar\alpha$ into the finite strip of analyticity along the real α-axis between, say, $\mathrm{Re}\,\alpha=\pm1$. An explicit class of such mappings can be given, for instance, by the integral representation

$$\alpha=c\int_0^{id(1-i\bar\alpha)/(1+i\bar\alpha)}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}-i\delta \tag{22}$$

With a suitable choice of the real parameters $0<k<1$ and $c,d,\delta>0$ one can achieve, that the conformal image of the unit circle in $\bar\alpha$ is a rectangle within the region of analyticity in α.

The reordering of the Taylor series of $E_F(\alpha)/E_1(\alpha)$ according to the powers of $\bar\alpha$ defines a convergent reordering of the SSCE series (at $\alpha=1$). This reordering is such, that the k^{th} term of the new series is a combination of the first k terms of the old one. Since $E_1(\alpha)$ appears in every expectation value, the reordering is the same for every quantity. In fact, at a given finite order k in $\bar\alpha$, the resummation is equivalent to the multiplication of the terms of SSCE by some number obtained from the partial sums of the expansions of α^i ($i\leq k$) at $\alpha=1$. (For every fixed i, the multiplicator of the i^{th} term tends to 1 if $k\to\infty$.) Such a resummation can have some practical importance only if very long series are at disposal. The pragmatic question of convergence is, of course, whether a relatively small number of terms gives a good approximation or not.

In the above proof, the finiteness of the lattice was essential, because the summation over the links $\sum_l$ could otherwise cause problems. In an extension to an infinite lattice, one has first to control the link sums. This can be done at such points of the parameter space ($\lambda=\infty$, β, κ), where the correlation length is finite, since then the connected parts vanish exponentially with the distance of links. In general, the non-convergence of SSCE on an infinite lattice can come from two sources: first, the individual terms of the expansion can diverge because of an infinite correlation length, second, the sum can diverge, even if the terms are finite, because of an unfavourable choice of the κ-rescaling factor s^2 in Eq. (14).

2.4 Expansion of the correlation functions

The connected correlation functions are of particular interest for the study of quantum field theories. For instance, the mass gap in a given channel can be determined from the exponential decay of an appropriate 2-point function at large distances. In the following we consider in detail the SSCE of some simple types of correlation functions. The expansion of other, more complicated types of correlations can be performed in a similar way.

As a first example, let us take the correlation function of m link variables $T_l \equiv TrV(x,\mu)$, which is defined by

$$\langle T_{l_1}T_{l_2}\ldots T_{l_m}\rangle^c_{\lambda,\beta,\kappa} \equiv \left.\frac{\partial^m}{\partial j_{l_1}\ldots\partial j_{l_m}}\right|_{j=0} \log\left\langle \exp\left[\sum_l j_l T_l\right]\right\rangle_{\lambda,\beta,\kappa} \tag{23}$$

For the expectation value on the right hand side one can apply Eq. (13). But first it is convenient to introduce the generating function of the expectation values of arbitrary composite T_l-products by

$$Z[K]_\infty \equiv \log\left\langle \exp\left[\sum_{n=0}^{\infty}\sum_{l_1\ldots l_n} K_n(l_1\ldots l_n)T_{l_1}\ldots T_{l_n}\right]\right\rangle_{\lambda=\infty,\beta,\kappa} \tag{24}$$

The subscript ∞ refers to the fact, that $Z[K]_\infty$ is an expectation value in the $\lambda = \infty$ point, where SSCE is done. The functions $K_n(l_1\ldots l_n)$ $(n = 0,1,\ldots)$ are, for the moment, arbitrary composite currents. From Eq. (13) one obtains

$$\log\left\langle \exp\left[\sum_l j_l T_l\right]\right\rangle_{\lambda,\beta,\kappa} = Z[K^{(j)}]_\infty - Z[K^{(0)}]_\infty \tag{25}$$

where the composite currents $K_n^{(j)}$ are given by

$$K_0^{(j)} = 0; \quad K_1^{(j)}(l_1) = j_{l_1} + \kappa(i_1^2 - s^2);$$

$$K_n^{(j)}(l_1\ldots l_n) = \frac{\kappa^n}{n!}c_n(l_1\ldots l_n) \quad (n \geq 2) \tag{26}$$

It is clear from Eqs. (24-26), that $Z[K]_\infty$ can be written as a sum over the "partitions" of n. By the partition $n\{n_1\ldots n_k\}$ of the positive integer n we understand a set of positive integers $n_1,\ldots,n_k$ satisfying

$$n = n_1 + 2n_2 + \ldots + kn_k \tag{27}$$

Let us define the factor $f^n_{\{n_1\ldots n_k\}}$ belonging to the partition $n\{n_1\ldots n_k\}$ by

$$f^n_{\{n_1\ldots n_k\}} = (i_1^2 - s^2)^{n_1}\prod_{i=1}^{k}\frac{1}{n_i!(i!)^{n_i}} \tag{28}$$

Then the SSCE of the n-link correlation function in Eq. (23) can be written as

$$\langle T_{l_1}T_{l_2}\ldots T_{l_m}\rangle^c_{\lambda,\beta,\kappa} = \sum_{n=0}^{\infty}\kappa^n\left\langle F^{(n)}T_{l_1}\ldots T_{l_m}\right\rangle^c_{\lambda=\infty,\beta,\kappa} \tag{29}$$

Here $F^{(n)}$ is a sum over partitions:

$$F^{(n)} = \sum_{n\{n_1\ldots n_k\}} f^n_{\{n_1\ldots n_k\}}T^n_{\{n_1\ldots n_k\}} \tag{30}$$

The function $T^n_{\{n_1 \dots n_k\}}$ belonging to the partition $n\{n_1 \dots n_k\}$ is the product of n_1 single-link variables, n_2 two-link variables, ..., n_k k-link variables, summed over all the links. The first few $F^{(n)}$ in Eq. (30) are given explicitly by

$$F^{(1)} = (i_1^2 - s^2) \sum_{l_1} T_{l_1}$$

$$F^{(2)} = \sum_{l_1 l_2} \left[\frac{1}{2}(i_1^2 - s^2)^2 T_{l_1} T_{l_2} + \frac{1}{2} c(l_1 l_2)\, (T_{l_1} T_{l_2}) \right]$$

$$F^{(3)} = \sum_{l_1 l_2 l_3} \left[\frac{1}{6}(i_1^2 - s^2)^3 T_{l_1} T_{l_2} T_{l_3} + \frac{1}{2}(i_1^2 - s^2) c(l_2 l_3) T_{l_1}\, (T_{l_2} T_{l_3}) + \frac{1}{6} c(l_1 l_2 l_3)\, (T_{l_1} T_{l_2} T_{l_3}) \right]$$

$$F^{(4)} = \sum_{l_1 l_2 l_3 l_4} \left[\frac{1}{24}(i_1^2 - s^2)^4 T_{l_1} T_{l_2} T_{l_3} T_{l_4} + \frac{1}{4}(i_1^2 - s^2)^2 c(l_3 l_4) T_{l_1} T_{l_2}\, (T_{l_3} T_{l_4}) + \frac{1}{6}(i_1^2 - s^2) \cdot \right.$$
$$\left. c(l_2 l_3 l_4) T_{l_1}\, (T_{l_2} T_{l_3} T_{l_4}) + \frac{1}{8} c(l_1 l_2) c(l_3 l_4)\, (T_{l_1} T_{l_2})\, (T_{l_3} T_{l_4}) + \frac{1}{24} c(l_1 l_2 l_3 l_4)\, (T_{l_1} T_{l_2} T_{l_3} T_{l_4}) \right] \quad (31)$$

where we recall that in the connected expectation values of Eq. (29), the variables put in parentheses in Eq. (31) count as a single entity.

As a second example, let us consider the m-point correlations of the site variables $\rho_{y_1} \dots \rho_{y_m}$, which are defined by

$$\langle \rho_{y_1} \rho_{y_2} \dots \rho_{y_m} \rangle^c_{\lambda,\beta,\kappa} \equiv \left. \frac{\partial^m}{\partial r_{y_1} \dots \partial r_{y_m}} \right|_{r=0} \log \left\langle \exp \left[\sum_y r_y \rho_y \right] \right\rangle_{\lambda,\beta,\kappa} \quad (32)$$

The derivation of SSCE consists of similar steps as before. Instead of Eq. (25) we have now

$$\log \left\langle \exp \left[\sum_y r_y \rho_y \right] \right\rangle_{\lambda,\beta,\kappa} = Z[K^{(r)}]_\infty - Z[K^{(0)}]_\infty \quad (33)$$

where the composite current $K^{(r)}$ is given by

$$K^{(r)}_n(l_1 \dots l_n) = -\delta_{1n} \kappa s^2 + \frac{\kappa^n}{n!} \sum_{i=\delta_{n0}}^{\infty} \frac{1}{i!} \sum_{z_1 \dots z_i} r_{z_1} \dots r_{z_i} c_{n,i}(l_1 \dots l_n)_{z_1 \dots z_i} \quad (34)$$

The λ-dependent connected expectation values $c_{n,i}$ are defined, in analogy with Eq. (12), by

$$c_{n,i}(l_1 \dots l_n)_{z_1 \dots z_i} \equiv c(l_1 \dots l_n)_{z_1 \dots z_i} \equiv \langle (\rho_{x_1} + \hat{\mu}_1 \rho_{x_1}) \dots (\rho_{x_n} + \hat{\mu}_n \rho_{x_n}) \rho_{z_1} \dots \rho_{z_i} \rangle^c_\lambda \quad (35)$$

The final result for the m-point correlation function is:

$$\langle \rho_{y_1} \dots \rho_{y_m} \rangle^c_{\lambda,\beta,\kappa} = \delta_{y_1 y_2} \dots \delta_{y_1 y_m} c_{0,m} + \sum_{N=1}^{\infty} \kappa^N \left\langle \sum_{n=1}^{N} F^{(N-n)} G^{(n)}_{y_1 \dots y_m} \right\rangle^c_{\lambda=\infty,\beta,\kappa} \quad (36)$$

The link variable dependent combinations $F^{(n)}$ are the same as in Eqs. (30-31), supplied with $F^{(0)} \equiv 1$. The new combinations $G^{(n)}_{y_1 \dots y_m}$ appearing in Eq. (36) are again sums over partitions of the points $y_1 \dots y_m$. The first few of them are:

$$G^{(n)}_{y_1} = \frac{1}{n!} \sum_{l_1 \dots l_n} c(l_1 \dots l_n)_{y_1}\, (T_{l_1} \dots T_{l_n})$$

$$G^{(n)}_{y_1 y_2} = \sum_{l_1 \dots l_n} \left[\frac{1}{n!} c(l_1 \dots l_n)_{y_1 y_2}\, (T_{l_1} \dots T_{l_n}) \right.$$

$$+\sum_{m=1}^{n-1}\frac{1}{m!(n-m)!}c(l_1\ldots l_m)_{y_1}c(l_{m+1}\ldots l_n)_{y_2}\left(T_{l_1}\ldots T_{l_m}\right)\left(T_{l_{m+1}}\ldots T_{l_n}\right)\Bigg]$$

$$G^{(1)}_{y_1\ldots y_m}=\sum_{l_1}c(l_1)_{y_1\ldots y_m}T_{l_1} \tag{37}$$

The whole expression for the n^{th} order in Eq. (36) can also be considered as a sum over the partitions of n links. The difference compared to Eq. (29) is, that the additional points $y_1\ldots y_m$ are distributed among the connected link groups in all possible ways.

An alternative way to derive the SSCE for the connected ρ-correlation functions is to use a generalization of the relation in Eq. (11):

$$\left\langle \exp\left[\sum_y r_y\rho_y+\kappa\sum_l T_l(\rho_{x+\hat\mu}\rho_x)\right]\right\rangle_\lambda$$

$$=\exp\left\{\sum_{n+i\geq 1}\frac{\kappa^n}{n!\,i!}\sum_{l_1\ldots l_n}T_{l_1}\ldots T_{l_n}\sum_{y_1\ldots y_i}r_{y_1}\ldots r_{y_i}c_{n,i}(l_1\ldots l_n)_{y_1\ldots y_i}\right\} \tag{38}$$

This implies identities between the expectation values of ρ-products and infinite series of link-variable expectation values. The simplest examples of such identities are:

$$\langle\rho_{y_1}\rangle_{\lambda,\beta,\kappa}=\sum_{n=0}^{\infty}\frac{\kappa^n}{n!}\sum_{l_1\ldots l_n}\langle T_{l_1}\ldots T_{l_n}\rangle_{\lambda,\beta,\kappa}\,c(l_1\ldots l_n)_{y_1} \tag{39}$$

and

$$\langle\rho_{y_1}\rho_{y_2}\rangle^c_{\lambda,\beta,\kappa}=\sum_{n=0}^{\infty}\frac{\kappa^n}{n!}\sum_{l_1\ldots l_n}\langle T_{l_1}\ldots T_{l_n}\rangle_{\lambda,\beta,\kappa}\,c(l_1\ldots l_n)_{y_1y_2}$$

$$+\sum_{n=2}^{\infty}\sum_{m=1}^{n-1}\frac{\kappa^n}{m!\,(n-m)!}\sum_{l_1\ldots l_n}\langle(T_{l_1}\ldots T_{l_m})(T_{l_{m+1}}\ldots T_{l_n})\rangle^c_{\lambda,\beta,\kappa}\,c(l_1\ldots l_m)_{y_1}c(l_{m+1}\ldots l_n)_{y_2} \tag{40}$$

Here the right hand sides are combinations of expectation values depending only on the link-variables T_l. These can be expanded by using Eq. (13). The results are, of course, identical to Eqs. (36,37).

Up to now the fixed β-value for SSCE was always considered to be finite. This is, however, not essential, because every formula is valid without change also for the limit $\beta=\infty$. The reason is, that in the $\beta=\infty$ actions (4,6) the rôle of $T_l\equiv Tr\,V(x,\mu)$ is taken over by

$$T_l\equiv Tr\,\left(\alpha^+_{x+\hat\mu}\alpha_x\right) \tag{41}$$

The above SSCE formulas are valid also for $\beta=\infty$, if for T_l this expression is used.

3 Monte Carlo calculation of the SSCE coefficients

In order to have a feeling about the qualitative behaviour of low order SSCE, we performed a Monte Carlo calculation of the $\lambda=\infty$ correlations needed in the 3^{rd} order SSCE of some simple quantities. The numerical calculation was done on an 8^4 lattice, at the point ($\lambda=\infty$, $\beta=2.3$, $\kappa=0.4$). The choice of the β-value is motivated by the existence of 8^4 Monte Carlo data obtained earlier at $\beta=2.3$. The value of the hopping parameter is such that the point is in the Higgs-phase, in the neighbourhood of but not too close to the confinement-Higgs phase transition. At this point, on the 8^4 lattice, the link expectation value is $\langle\frac{1}{2}TrV\rangle=0.2933(7)$, the W-mass in lattice units $am_W=0.66(7)$, and the Higgs boson mass $am_H=0.58(6)$. We collected our statistics in about $2\cdot10^5$ sweeps, where the first

10000 sweeps were omitted for equilibration. The updating was done by the Metropolis method with 6 hits per link, using the gauge invariant variables in the action (5).

The quantities we have studied are the average Higgs field length $\langle\rho\rangle$, the average link $\langle\frac{1}{2}TrV\rangle$, the link-link correlation $\langle\frac{1}{2}TrV \cdot \frac{1}{2}TrV\rangle^c$, the length-link correlation $\langle\rho \cdot \frac{1}{2}TrV\rangle^c$ and length-length correlation $\langle\rho\rho\rangle^c$. In the SSCE of these quantities, Eqs. (29, 36), the numerically difficult contributions come from multi-link connected correlation functions summed over the link positions. As one can see from Eq. (31), the number of such contributions can be minimized by choosing the free parameter s in the expansions to be $s = i_1$. (In this case, in Eq. (30), only those partitions contribute, where $n_1 = 0$.) In order to have reasonable statistical errors in the SSCE coefficients, we had to restrict ourselves to third order, or sometimes even less.

Table I.

SSCE predictions for ($\lambda = 1.0,\ \beta = 2.3,\ \kappa = 0.375$) from numerical data at ($\lambda = \infty,\ \beta = 2.3,\ \kappa = 0.4$), corresponding to $s = i_1$.

quantity	0^{th} order	1^{st} order	2^{nd} order	3^{rd} order	data
$\langle\rho\rangle$	1.0325	1.1554(4)	1.1674(5)	1.30(1)	1.29
$\langle\frac{1}{2}TrV\rangle$	0.292(1)	0.292(1)	0.61(3)	0.89(4)	0.57
$\langle\hat{\rho}\rangle$	1.0325	1.0325	1.0445(1)	1.0420(1)	1.05
$C_{\rho\rho}(4)$	0	0	0.03(1)	0.04(1)	
$C_{\rho L}(1)$	0	0.22(2)	0.24(2)		
$C_{\rho L}(2)$	0	0.12(2)	0.13(2)		
$C_{\rho L}(3)$	0	0.08(2)	0.09(2)		
$C_{\rho L}(4)$	0	0.07(2)	0.08(2)		
$C_{LL}(1)$	0.50(3)	0.50(3)			
$C_{LL}(2)$	0.28(3)	0.28(3)			
$C_{LL}(3)$	0.19(3)	0.19(3)			
$C_{LL}(4)$	0.16(3)	0.16(3)			

The obtained numerical results for $s = i_1$ are collected in Table I. Besides the average length and average link we also included the combination

$$\langle\hat{\rho}\rangle \equiv \langle\rho\rangle - 16\kappa i_1(i_2 - i_1^2)\langle\frac{1}{2}TrV\rangle \tag{42}$$

In this quantity many of the difficult contributions cancel and, as one can see from Table I, the third order expansion gives a very good and convergent result. The correlations given in the Table are built up from the time-slice variables

$$\tilde{\rho}(t) \equiv \frac{1}{N_3}\sum_{\mathbf{x}}\rho(\mathbf{x},t) \qquad \tilde{L}(t) \equiv \frac{1}{3N_3}\sum_{\mathbf{x}}\sum_{m=1}^{3}\frac{1}{2}TrV(\mathbf{x},t,m) \tag{43}$$

Here N_3 is the number of lattice points in a time-slice. The total number of lattice points will be denoted by N_4. The definition of the time-slice correlations for time distance t is:

$$C_{\rho\rho}(t) \equiv N_4\langle\tilde{\rho}(0)\tilde{\rho}(t)\rangle^c$$
$$C_{\rho L}(t) \equiv N_4\langle\tilde{\rho}(0)\tilde{L}(t)\rangle^c$$
$$C_{LL}(t) \equiv N_4\langle\tilde{L}(0)\tilde{L}(t)\rangle^c \tag{44}$$

In the last column of the Table some numerical results obtained from a linear interpolation of Monte Carlo data, measured at the corresponding parameter values, are shown.

The dependence of third order SSCE for the average length $\langle\rho\rangle$ on the variable s is illustrated in Figs. 1 and 2 for $\lambda = 1.0,\ 0.1$, respectively. In these figures the existing numerical Monte Carlo data are also included, partly from Ref. [5] and partly some new data obtained in more recent runs. As one can see from the figures, the third order SSCE describes the Monte Carlo data qualitatively well in a wide range of κ, but the errors are large and the agreement is not quantitative. The connected contributions shift the approximate second order, given in Ref. [6], away from the Monte Carlo data. In [6] all connected contributions were neglected and s was chosen according to Eq. (15). In this way one obtains a curve, which almost coincides with the dashed-dotted lines on the figures (the non-connected second and third order contributions are very small). As one can see, this agrees rather well with the Monte Carlo data for s given by Eq. (15). Shifting the expansion point with κ according to Eq. (15), one obtains good agreement in a wider range of κ (see Ref. [6]). Unfortunately, the connected contributions are non-negligible. For the moment we have no good explanation for the good agreement obtained without the connected parts. (The neglection of connected correlations corresponds to some sort of mean-field approximation.) The complete third order seems to indicate, although with large errors, that for the choice (15), the connected contributions in SSCE have the tendency to alternate in sign. This is, of course, disadvantageous for a very fast convergence. The choice $s = i_1$ seems to give a better convergence, but even at this point for a quantitative description of all the Monte Carlo data more orders are needed than the third order calculated here.

4 Discussion

Having in mind the numerical difficulties in the calculation of higher order SSCE coefficients, one can conclude that a low order SSCE presumably cannot replace a direct numerical study of the λ-dependence in the standard Higgs model. However, the qualitatively good description of the λ-dependence is, in our opinion, still useful. Moreover, the closed form of the expansion can be a valuable analytical tool. Since SSCE is, apart from an eventually necessary trivial reordering, convergent on finite lattices, a very interesting possibility is to use it for the study of the infinite volume limit (for questions like the order of the confinement-Higgs phase transition etc.).

A possibly very interesting application of the SSCE is to use it near the continuum limit at $\beta = \infty$. In fact, SSCE is in principle very similar to the ordinary perturbation theory at $g = 0$ (vanishing gauge coupling): in both cases one is doing an expansion about a point on the boundary of the coupling parameter space. The usual weak coupling perturbation theory is a double expansion in the point $(g = 0,\ \lambda = 0)$. It is very well possible, that a double expansion at $(g = 0,\ \lambda = \infty)$ is at least as useful. At the first sight, this combined strong-weak coupling expansion seems to be much more difficult, because the expansion point still has a non-trivial lattice action. Going to the critical point in the third variable κ, however, implies that the expansion point is approaching a free theory in disguise. Therefore, using the known triviality of the critical point at $(\lambda = \infty,\ \beta = \infty,\ \kappa = \kappa_{cr}(\infty))$, one arrives at the same level of simplicity as in the starting point of weak coupling perturbation theory. An important advantage of the strong-weak coupling expansion is that it has a much better chance for incorporating the λ-independence of the continuum limit.

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A Appendix

A.1 Integrals over the Higgs-field length

The λ-dependence in the SSCE formulas are given by the integrals defined in Eq. (10). The basic integral is ($k = 0, 1, 2, \ldots$):

$$
\begin{aligned}
I_k(\lambda) &\equiv \int_0^\infty d\rho\, \rho^{3+k} \exp\left[-\lambda\left(\rho^2 - 1 + \frac{1}{2\lambda}\right)^2\right] \\
&= \frac{1}{2\sqrt{\lambda}} \int_{-\sqrt{\lambda}+1/(2\sqrt{\lambda})}^{\infty} dx\, \exp(-x^2) \left(1 + \frac{x}{\sqrt{\lambda}} - \frac{1}{2\lambda}\right)^{1+\frac{k}{2}} \\
&= \frac{\Gamma(2+\frac{k}{2})}{2(2\lambda)^{1+\frac{k}{4}}} \exp\left[-\frac{1}{4}\left(\frac{1}{\sqrt{2\lambda}} - \sqrt{2\lambda}\right)^2\right] D_{-2-\frac{k}{2}}\left(\frac{1}{\sqrt{2\lambda}} - \sqrt{2\lambda}\right)
\end{aligned}
\tag{45}
$$

Here $D_n(z)$ denotes the parabolic cylinder function [14]. The relevant ratio i_k is, therefore, given by

$$
i_k \equiv \frac{I_k(\lambda)}{I_0(\lambda)} = \frac{\Gamma\left(2+\frac{k}{2}\right)}{(2\lambda)^{\frac{k}{4}}} \frac{D_{-2-\frac{k}{2}}\left(\frac{1}{\sqrt{2\lambda}} - \sqrt{2\lambda}\right)}{D_{-2}\left(\frac{1}{\sqrt{2\lambda}} - \sqrt{2\lambda}\right)}
\tag{46}
$$

The asymptotic behaviour of i_k for very large and very small λ is [14] :

$$
i_k \to \begin{cases} (\lambda \to \infty) & 1 + k(k-2)/(16\lambda) + o(\lambda^{-2}) \\ (\lambda \to 0) & \Gamma(2+k/2)(1 - \lambda k(k+6)/4 + o(\lambda^2)) \end{cases}
\tag{47}
$$

The integrals in Eq. (45) can easily be performed numerically. The result for some characteristic values of λ is shown for i_k ($k = 1, 2, \ldots, 10$) in Table A1.

Table A1.

The values of $i_1, \ldots, i_{10}$ for some typical λ.

$\lambda =$	10.0	1.0	0.1	0.01
i_1	0.99516	1.03253	1.20569	1.30832
i_2	1.00263	1.13373	1.60794	1.92884
i_3	1.02178	1.30652	2.31982	3.12745
i_4	1.05250	1.56687	3.56824	5.48675
i_5	1.09509	1.94393	5.79188	10.2943
i_6	1.15027	2.48403	9.84615	20.4756
i_7	1.21915	3.25837	17.4293	42.8816
i_8	1.30326	4.37575	31.9803	94.0450
i_9	1.40459	6.00302	60.6000	215.031
i_{10}	1.52566	8.39795	118.233	510.693

In the SSCE of simple quantities the i_k-factors always appear in combinations which are much smaller than the individual terms. Such typical combinations are:

$$
\begin{aligned}
j_2 &\equiv i_2 - i_1^2 \\
j_3 &\equiv i_3 - 3i_1 i_2 + 2i_1^3 \\
j_4 &\equiv i_4 - 4i_1 i_3 - 3i_2^2 + 12 i_1^2 i_2 - 6 i_1^4 \\
j_{23} &\equiv i_2 i_3 - i_1 i_2^2 - 2 i_1^3 i_2 + 2 i_1^5
\end{aligned}
$$

$$
\begin{aligned}
j_{33}^{(1)} &\equiv i_3^2 - 3i_1^2 i_2^2 + 2i_1^6 \\
j_{33}^{(2)} &\equiv i_3^2 - 2i_1 i_2 i_3 - 3i_1^2 i_2^2 + 8i_1^4 i_2 - 4i_1^6 \\
j_{24} &\equiv i_2 i_4 - 2i_1 i_2 i_3 - 2i_1^3 i_3 - i_2^3 + 10 i_1^4 i_2 - 6i_1^6 \\
j_{34} &\equiv i_3 i_4 - i_1 i_3^2 - 3i_1^2 i_2 i_3 - 3i_1 i_2^3 + 6i_1^3 i_2^2 + 6i_1^5 i_2 - 6i_1^7 \\
j_{44} &\equiv i_4^2 - 4i_1^2 i_3^2 - 3i_2^4 + 12 i_1^4 i_2^2 - 6i_1^8 \\
j_{224} &\equiv i_2^2 i_4 - 4i_1^3 i_2 i_3 - i_2^4 + 2i_1^4 i_2^2 + 8i_1^6 i_2 - 6i_1^8
\end{aligned}
\tag{48}
$$

The values of these combinations are given in Table A2.

Table A2.

The values of $j_2, \ldots, j_{224}$ for the λ-values in Table A1.

$\lambda =$	10.0	1.0	0.1	0.01
j_2	1.2287E-2	6.7607E-2	1.5425E-1	2.1715E-1
j_3	-4.4984E-4	-3.7077E-3	9.1900E-3	3.5718E-2
j_4	4.4786E-5	-6.1618E-4	-6.0460E-3	-1.6840E-3
j_{23}	-1.5053E-4	5.2351E-3	7.2149E-2	1.9228E-1
$j_{33}^{(1)}$	-2.2367E-5	1.9535E-2	2.5007E-1	7.0642E-1
$j_{33}^{(2)}$	-2.1800E-5	-1.0215E-3	6.9209E-3	4.1866E-2
j_{24}	3.7613E-5	-5.9822E-4	1.0363E-3	3.7526E-2
j_{34}	7.9730E-6	-1.3041E-3	3.2030E-2	2.0466E-1
j_{44}	8.5451E-6	-7.6604E-4	1.5568E-1	9.1146E-1
j_{224}	3.7705E-5	7.1154E-4	3.9172E-2	2.1669E-1

A.2 Summation formulas

The SSCE of correlation functions are built up from sums over links. The terms in the sum are proportional to some "connected i-products", like $c(l_1 \ldots l_n)$ defined in Eq. (12). The basic task is to perform sums like

$$
S_n \equiv \sum_{l_1 \ldots l_n} c(l_1 \ldots l_n) T_{l_1} \ldots T_{l_n}
\tag{49}
$$

The simplest non-trivial case is given by

$$
S_2 = \sum_{l_1 l_2} c(l_1 l_2) T_{l_1} T_{l_2} = j_2 \left[(i_2 + i_1^2) \sum_{l_1} T_{l_1}^2 + i_1^2 \sum_{(l_1 l_2)} T_{l_1} T_{l_2} \right]
\tag{50}
$$

Here the second term is a sum over link-pairs with one common point. It can easily be shown that also in the general case, the sum $\sum_{l_1 \ldots l_n}$ in Eq. (49) gets non-zero contributions only from the totally connected link configurations. For the more complicated cases let us agree on the convention that if some links are put in parentheses, then they have a single common point. (For instance, a closed path along a plaquette is in this notation $(l_1 l_2)(l_2 l_3)(l_3 l_4)(l_4 l_1)$.) In this case we have

$$
\begin{aligned}
S_3 = {} & j_{33}^{(1)} \sum_{l_1} T_{l_1}^3 + 3 i_1 j_{23} \sum_{(l_1 l_2)} T_{l_1}^2 T_{l_2} \\
& + i_1^3 j_3 \sum_{(l_1 l_2 l_3)} T_{l_1} T_{l_2} T_{l_3} + 3 i_1^2 j_2^2 \sum_{(l_1 l_2)(l_2 l_3)} T_{l_1} T_{l_2} T_{l_3}
\end{aligned}
\tag{51}
$$

and

$$S_4 = j_{44}\sum_{l_1} T_{l_1}^4 + 4i_1 j_{34} \sum_{(l_1 l_2)} T_{l_1}^3 T_{l_2}$$

$$+3j_{224} \sum_{(l_1 l_2)} T_{l_1}^2 T_{l_2}^2 + 6i_1^2 j_{33}^{(2)} \sum_{(l_1 l_2)(l_2 l_3)} T_{l_1} T_{l_2}^2 T_{l_3}$$

$$+12 i_1 j_2 j_{23} \sum_{(l_1 l_2)(l_2 l_3)} T_{l_1}^2 T_{l_2} T_{l_3} + 6i_1^2 j_{24} \sum_{(l_1 l_2 l_3)} T_{l_1}^2 T_{l_2} T_{l_3}$$

$$+i_1^4 j_4 \sum_{(l_1 l_2 l_3 l_4)} T_{l_1} T_{l_2} T_{l_3} T_{l_4} + 12 i_1^3 j_2 j_3 \sum_{(l_1 l_2)(l_2 l_3 l_4)} T_{l_1} T_{l_2} T_{l_3} T_{l_4}$$

$$+ 12 i_1^2 j_2^3 \sum_{(l_1 l_2)(l_2 l_3)(l_3 l_4)} T_{l_1} T_{l_2} T_{l_3} T_{l_4} + 12 j_2^3 (i_2 + 3i_1^2) \sum_{(l_1 l_2)(l_2 l_3)(l_3 l_4)(l_4 l_1)} T_{l_1} T_{l_2} T_{l_3} T_{l_4} \tag{52}$$

In the SSCE of ρ-dependent correlations, like for instance Eq. (36), sums with the other type of λ-dependent "connected i-products" $c_{n,i}(l_1 \ldots l_n)_{y_1 \ldots y_i}$ appear. (For the definition of $c_{n,i}$ see Eq. (35).) The notation for these link-sums will be:

$$S_{n(y_1 \ldots y_i)} \equiv \sum_{l_1 \ldots l_n} c(l_1 \ldots l_n)_{y_1 \ldots y_i} T_{l_1} \ldots T_{l_n} \tag{53}$$

The simplest sums over links with a few site variables can be decomposed in a similar way like Eqs. (50-52). In analogy with the above notations, let us put coinciding points in curly brackets, like for instance $\{y_1 y_2\}$. A point on a link will be enclosed in square brackets together with the link: $[yl]$. Touching links will be put in parentheses, as before. Then the simplest link-sums with a single point y_1 are given by

$$S_{1(y_1)} = i_1 j_2 \sum_{[y_1 l_1]} T_{l_1} \tag{54}$$

$$S_{2(y_1)} = j_{23} \sum_{[y_1 l_1]} T_{l_1}^2 + i_1^2 j_3 \sum_{([y_1 l_1][y_1 l_2])} T_{l_1} T_{l_2} + 2 i_1 j_2^2 \sum_{([y_1 l_1] l_2)} T_{l_1} T_{l_2} \tag{55}$$

$$S_{3(y_1)} = j_{34} \sum_{[y_1 l_1]} T_{l_1}^3 + 3 i_1 j_{24} \sum_{([y_1 l_1][y_1 l_2])} T_{l_1} T_{l_2}^2$$

$$+3 j_2 j_{23} \sum_{([y_1 l_1] l_2)} T_{l_1} T_{l_2}^2 + 3 i_1 j_{33}^{(2)} \sum_{(l_1 [y_1 l_2])} T_{l_1} T_{l_2}^2$$

$$+i_1^3 j_4 \sum_{([y_1 l_1][y_1 l_2][y_1 l_3])} T_{l_1} T_{l_2} T_{l_3} + 3 i_1^2 j_2 j_3 \sum_{([y_1 l_1] l_2 l_3)} T_{l_1} T_{l_2} T_{l_3}$$

$$+ 6 i_1 j_2^3 \sum_{([y_1 l_1] l_2)(l_2 l_3)} T_{l_1} T_{l_2} T_{l_3} + 6 i_1^2 j_2 j_3 \sum_{([y_1 l_1][y_1 l_2])(l_2 l_3)} T_{l_1} T_{l_2} T_{l_3} \tag{56}$$

The similar sums with two points y_1, y_2 are:

$$S_{1(y_1 y_2)} = i_1 j_3 \sum_{[\{y_1 y_2\} l_1]} T_{l_1} + j_2^2 \sum_{[y_1 y_2 l_1]} T_{l_1} \tag{57}$$

$$S_{2(y_1 y_2)} = j_{24} \sum_{[\{y_1 y_2\} l_1]} T_{l_1}^2 + j_{33}^{(2)} \sum_{[y_1 y_2 l_1]} T_{l_1}^2 + i_1^2 j_4 \sum_{([\{y_1 y_2\} l_1][\{y_1 y_2\} l_2])} T_{l_1} T_{l_2}$$

$$+ 2 i_1 j_2 j_3 \left\{ \sum_{([\{y_1 y_2\} l_1] l_2)} + \sum_{([y_1 y_2 l_1][y_2 l_2])} + \sum_{([y_1 y_2 l_1][y_1 l_2])} \right\} T_{l_1} T_{l_2} + 2 j_2^3 \sum_{([y_1 l_1][y_2 l_2])} T_{l_1} T_{l_2} \tag{58}$$

Figure captions

Fig. 1. Comparison of 3^{rd} order SSCE of the average Higgs field length $\langle\rho\rangle$ to the Monte Carlo data at ($\beta = 2.3$, $\lambda = 1.0$). The crosses are data points from Ref. [5] and the results of some more recent runs. The dashed line is zero'th order, the dashed-dotted one is first order. The almost full line with error bars is 2^{nd} order, the full line with error bars is 3^{rd} order. The lines are drawn to guide the eye. The dashed-dotted line practically coincides with 3^{rd} order, if the connected contributions are set equal to zero. Two specific values of s are shown by arrows: $s = i_1$ and $s \equiv s_1$ given by Eq. (15).

Fig. 2. The same as Fig. 1., for $\lambda = 0.1$

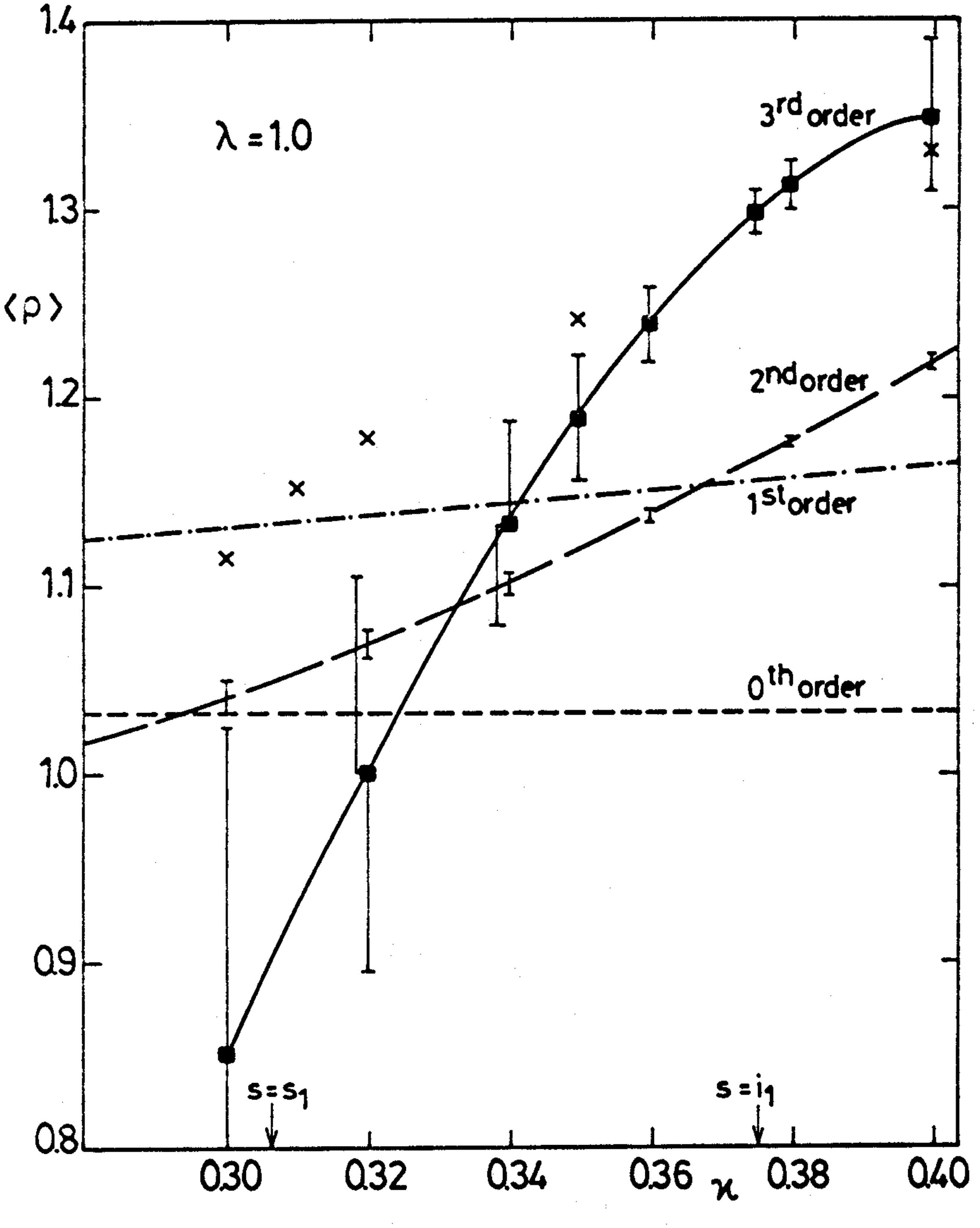
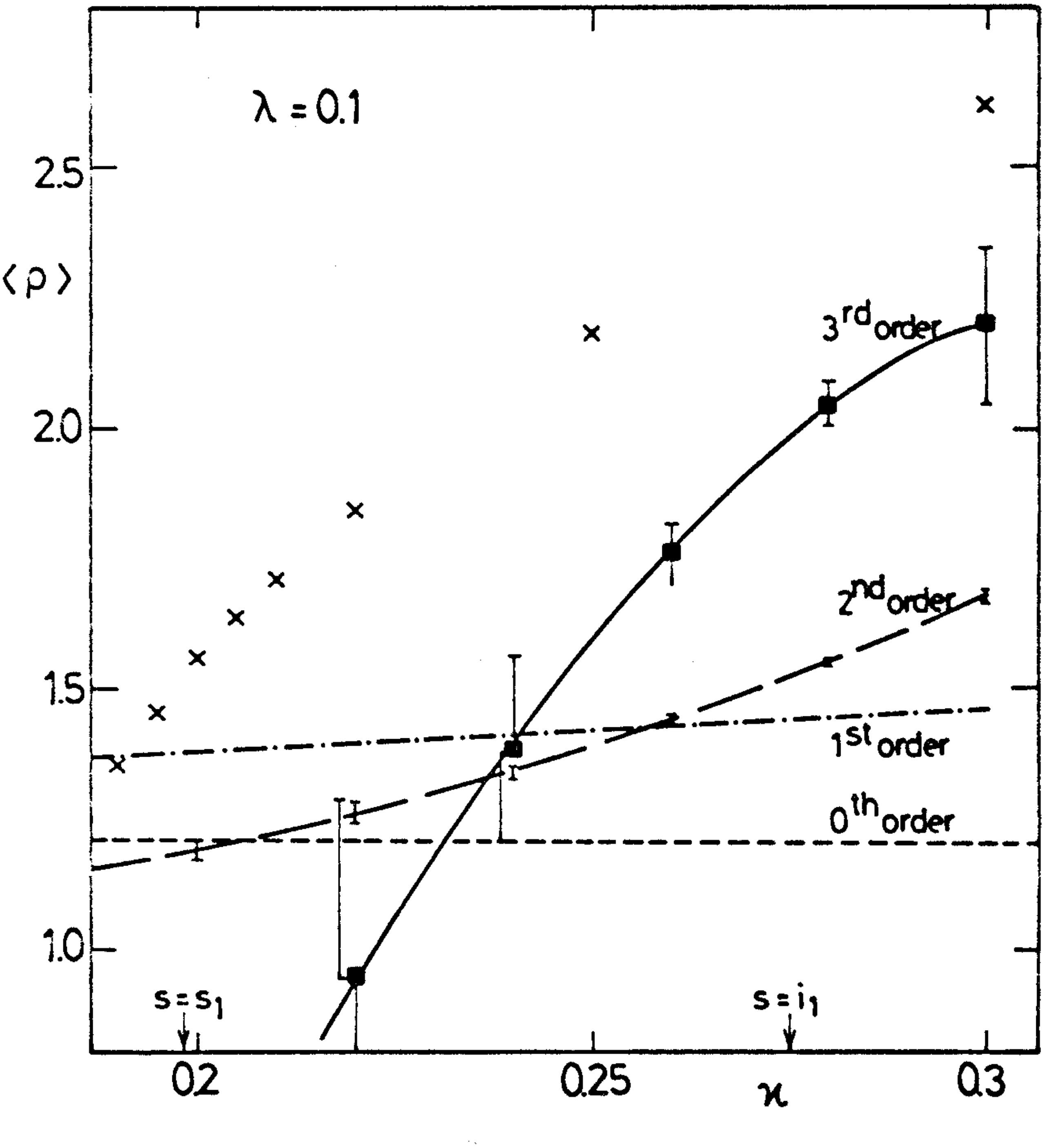

1.4
1.3
1.2
1.1
1.0
0.9
0.8
⟨ρ⟩
λ = 1.0
3rd order
2nd order
1st order
0th order
s = s1
s = i1
0.30
0.32
0.34
0.36
0.38
0.40
κ

Fig.1



λ = 0.1
2.5
2.0
1.5
1.0
⟨ρ⟩
3rd order
2nd order
1st order
0th order
s = s1
s = i1
0.2
0.25
0.3
κ

Fig. 2