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Particle Structure of Gauge Theories (*)

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Abstract: The implications of the principles of quantum field theory for the particle structure of gauge theories are discussed. The general structure which emerges is compared with that of the $\mathbb{Z}_2$ Higgs model on a lattice. The discussion leads to several confinement criteria for gauge theories with matter fields.

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PARTICLE STRUCTURE OF GAUGE THEORIES

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1. INTRODUCTION

Gauge theories are formulated in terms of gauge fields A_μ and matter fields ψ which are not directly connected with physical particles, although the use of notations like "quark" and "gluons" suggests such an interpretation. In fact, the structure of the set of particle states depends strongly on the dynamics, as you all know from the discussion on quark confinement, Higgs mechanism, charge screening and so on. A particular problem is the occurence of "charged" particles, i.e. particles which are separated from the vacuum by some superselection rule, the classical example being particles with half integer spin [1]. By the very definition of superselection rules there cannot exist an observable field which generates states of such a particle out of the vacuum.

In theories which have only global gauge symmetries this problem is often ignored. In these theories one has available besides the observable fields non observable fields which obey local commutation of anticommutation relations. These fields generate a set of superselection sectors which often contain all particle states. It may happen, however, that one has not sufficiently many fields at the beginning. As an example let me mention the Sine-Gordon theory; there the fields creating the soliton states from the vacuum are not contained in the original formulation of the theory (see e.g. [2]). It may also happen, that a non gauge invariant field does not create a new superselection sector out of the vacuum; this occurs in the case of spontaneous breakdown of gauge symmetry.

In gauge theories particles may exist which carry a charge related to the local gauge symmetry. Such a charge can be measured, according to Gauss' law, by the corresponding electric flux through an arbitrarily large surface surrounding the particle. There can never exist a local field creating such a particle from the vacuum, as may be seen by the following (standard) heuristic argument:

Let φ be a local field and Ω the vector representing the vacuum. The charge Q is the limit of the electric fluxes ϕ_R through a sphere with radius R around the origin. Then

$$Q\,\varphi(x)\Omega = \lim_{R\to\infty} \phi_R\,\varphi(x)\Omega = \lim_{R\to\infty} \varphi(x)\,\phi_R\Omega = \varphi(x)Q\Omega \tag{1.1}$$

hence if Ω is an eigenvector of Q $\varphi(x)\Omega$ is an eigenvector with the same eigenvalue.

Formally one may write down nonlocal fields creating charged particles. An example is the string field

$$\psi_{\mathcal{C}} = \psi(x)\, P\, e^{i\int_{\mathcal{C}} dy\, A(y)} \tag{1.2}$$

where $\mathcal{C}$ is a path from x to spacelike infinity and the symbol P denotes path ordering of the exponential. Another example is the electron field of quantum electrodynamics in the Coulomb gauge,

$$\psi_c(x) = \psi(x)\exp\{ie\int d^3\underline{y}\ \underline{A}(x^0,\underline{y})\cdot(\underline{y}-\underline{x})|\underline{y}-\underline{x}|^{-3}\} \quad . \tag{1.3}$$

Unfortunately, it is very difficult to give a precise meaning to these nonlocal expressions. (cf. however [3]).

For avoiding nonlocal quantities one treats gauge theories usually in a formalism where the fundamental fields are local and act as operators in a vector space W which is equipped with an indefinite metric. There is a subspace V, containing the vacuum and being invariant under the application of gauge invariant operators (observables), on which the scalar product is nonnegative. The subspace V_o of V of vectors with length zero is also invariant under observables, hence there is a natural representation of observables by operators in the factor space V/V_o, whose completion is the space of physical states $\mathcal{H}_{phys}$.

A very difficult question is whether $\mathcal{H}_{phys}$ contains besides the vacuum

sector of the theory also the charged sectors. This depends on the existence of certain elements in W which cannot be created by local fields out of the vacuum. One would like to derive their existence from a completeness property of W, but the absence of a natural notion of convergence obscures this possibility. The general structure of the indefinite metric formalism has been studied by Strocchi and Wightman [4] and later by Morchio and Strocchi [5]. I refer to the lectures of Prof. Strocchi for more details.

I want to start from a more general point of view. I consider the indefinite metric approach or the Euclidean functional integral approach as methods to compute the vacuum expectation values of gauge invariant quantities. I want to use only this information for a construction of the set of particle states. Actually, the explicit formulas for the observables in terms of the fundamental fields are never used. The only structure which is exploited is the association of regions $\mathcal{O}$ of Minkowski space to algebras $\mathfrak{A}(\mathcal{O})$ of Hilbert space operators; e.g.

$$\sum_{a,b} \bar{\psi}_a(x)\psi_b(y)\,(P\,e^{i\int_{\mathcal{C}} A})_{ab} \quad , \mathcal{C} \text{ path from x to y,} \tag{1.4}$$

is an observable which is localized in all regions $\mathcal{O}$ containing $\mathcal{C}$. For avoiding technical complications with domains of definition we restrict ourselves to bounded operators. For a quantum mechanical observable this can always be achieved by a suitable change of scale. This leads to the so-called algebraic framework of quantum field theory which has been proposed by Haag and Kastler [6].

2. THE ALGEBRAIC FRAMEWORK

According to Haag and Kastler [6], the basic object of a quantum field theory is an assignement of finitely extended space time regions $\mathcal{O}$ to operator algebras $\mathfrak{A}(\mathcal{O})$. Each algebra $\mathfrak{A}(\mathcal{O})$ is isomorphic to an algebra of bounded Hilbert space operators which is invariant under taking the adjoint (*-operation). $\mathfrak{A}(\mathcal{O})$ contains the unit operator and is closed with respect to the weak operator topology, i.e.

$$\lim\,(\Phi, A_\lambda\Psi) = (\Phi, A\Psi) \tag{2.1}$$

for all vectors Φ, Ψ and $A_\lambda \in \mathfrak{A}(\mathcal{O})$ for all λ imply $A \in \mathfrak{A}(\mathcal{O})$. Weakly closed *-invariant operator algebras have first been investigated by v. Neumann and are therefore called v. Neumann algebras. (For the mathematics of operator algebras see e.g. [7].)

The assignement $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ is called a local net. It has the following properties:

(1) Isotony: If $\mathcal{O}_1 \subset \mathcal{O}_2$ then $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$, and the unit operators of $\mathfrak{A}(\mathcal{O}_1)$ and $\mathfrak{A}(\mathcal{O}_2)$ coincide.

This property is obvious from the interpretation of $\mathfrak{A}(\mathcal{O})$ as well as from its construction. It enables us to consider the algebra of all local observables,

$$\mathfrak{A}_0 = \bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O}) \quad . \tag{2.2}$$

Also $\mathfrak{A}_0$ can be considered as an operator algebra on some Hilbert space, e.g. the vacuum Hilbert space. Due to the existence of superselection sectors there are representations (*) of $\mathfrak{A}_0$ by operators in other Hilbert spaces which are not unitarily equivalent to the identical representation in the vacuum Hilbert space. The weak operator topologies in inequivalent representations are different; the operator norm, however, and therefore also the closure of $\mathfrak{A}_0$ with respect to this norm

$$\mathfrak{A} = \overline{\mathfrak{A}_0}$$

are independent of the choice of the representation provided the representation is faithful (i.e. injective). $\mathfrak{A}$ is called the algebra of (quasi-local) observables (*-invariant normclosed algebras of Hilbert space operators are called C*-algebras). For more details see [6].

(2) Locality: If $\mathcal{O}_1$ is spacelike separated from $\mathcal{O}_2$ then, from Einstein causality, measurements in $\mathcal{O}_1$ and $\mathcal{O}_2$ cannot disturb each other, hence $[A,B] = 0$ for $A \in \mathfrak{A}(\mathcal{O}_1)$, $B \in \mathfrak{A}(\mathcal{O}_2)$.

(3) Covariance: Let $A \in \mathfrak{A}(\mathcal{O})$ be an observable and $L = (a, \Lambda)$ a Poincaré transformation in the identity component $P_+^\uparrow$ of the Poincaré group. There is a prescription assigning to A an observable $A_L \in \mathfrak{A}(L\mathcal{O})$. The mapping $\alpha_L : A \longrightarrow A_L$ is a symmetry transformation, i.e. it preserves all intrinsic properties of $\mathfrak{A}$, hence α_L is an automorphism of $\mathfrak{A}$:

(*) A representation π of a *-algebra $\mathfrak{A}$ is a linear mapping from $\mathfrak{A}$ into the algebra of bounded operators $B(\mathcal{H}_\pi)$ in some Hilbert space $\mathcal{H}_\pi$ such that
(i) $\pi(AB) = \pi(A)\pi(B)$
(ii) $\pi(A)^* = \pi(A^*)$

$$\begin{array}{ll} \text{(i)} & \alpha_L(\lambda A) = \lambda \alpha_L(A) \\ \text{(ii)} & \alpha_L(A+B) = \alpha_L(A) + \alpha_L(B) \\ \text{(iii)} & \alpha_L(AB) = \alpha_L(A)\alpha_L(B) \\ \text{(iv)} & \alpha_L(A^*) = \alpha_L(A)^* \end{array} \tag{2.4}$$

Moreover, if $L = L_1 L_2$ we have

$$\alpha_L = \alpha_{L_1} \alpha_{L_2} \tag{2.5}$$

hence $L \longrightarrow \alpha_L$ is a representation of $P_+^\uparrow$ by automorphisms of $\mathfrak{A}$ such that

$$\alpha_L(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(L\mathcal{O}) \quad . \tag{2.6}$$

(4) Stability: The systems we encounter in physics have in general certain stability properties. Whether this is merely our inability to prepare unstable systems in reproducible experiments or whether it is a fundamental physical law, in any case it deeply influences the mathematical structure of the relevant models. Stability may be thought of as the existence of a state with "lowest energy". Unfortunately, the known ways of making precise the condition of stability need more technical input whose physical meaning is not fully clarified. I shall come back to this point later.

After having discussed the general properties of the set of observables we now have to consider the notion of a state. In the quantum mechanics of finitely many particles states are described by unit vectors in the Hilbert space of square integrable wave functions. In quantum field theory there is no a priori given Hilbert space of wave functions. The basic object is the algebra of observables $\mathfrak{A}$. If $\mathfrak{A}$ is realized by operators in some Hilbert space $\mathcal{H}$, each unit vector $\Psi \in \mathcal{H}$ describes a state. Let us consider the mapping

$$\mathfrak{A} \ni A \longrightarrow (\Psi, A\Psi) =: \omega_\Psi(A) \tag{2.7}$$

which associates to each observable A its expectation value in the state described by Ψ. The expectation values of all powers of A already fix the whole probability distribution of measured values of A since A is bounded. Hence we may identify a state by its expectation functional. This leads to the following definition:

Definition: A state on a C*-algebra $\mathfrak{A}$ (with unit) is a linear functional ω with

$$\begin{array}{lll} \text{(i)} & \omega(A^*A) \geq 0 \quad , A \in \mathfrak{A} & \text{(positivity)} \\ \text{(ii)} & \omega(1) = 1 & \text{(normalization)} \end{array} \tag{2.8}$$

Examples for states are the expectation functionals induced by unit vectors or density matrices in some Hilbert space representation of $\mathfrak{A}$. Actually every state is the expectation functional of some unit vector in a suitable representation of $\mathfrak{A}$:

Theorem 2.1 (GNS-construction) [7]
Let ω be a state on a C*-algebra $\mathfrak{A}$. Then there exists a Hilbert space $\mathcal{H}$, a representation π of $\mathfrak{A}$ by operators in $\mathcal{H}$ and a unit vector $\Omega \in \mathcal{H}$ such that

(i) $\{\pi(A)\Omega, A \in \mathfrak{A}\}$ is dense in $\mathcal{H}$

(ii) $(\Omega, \pi(A)\Omega) = \omega(A)$.

It is instructive to illustrate this theorem on the example of a state of the form $\omega(A) = \mathrm{Tr}\, \rho A$ with a density matrix ρ in a Hilbert space $\mathcal{H}_o$ where $\mathfrak{A} = B(\mathcal{H}_o)$ is the algebra of bounded operators in $\mathcal{H}_o$. Let $\mathcal{H}$ be the Hilbert space of Hilbert-Schmidt operators in $\mathcal{H}_o$ with the scalar product $(S,T) = \mathrm{Tr}\, S^*T$. Since the algebra of Hilbert-Schmidt operators is an ideal in $B(\mathcal{H}_o)$, $A \in B(\mathcal{H}_o)$ acts by left multiplication as an operator on $\mathcal{H}$,

$$\pi(A)T = AT \quad , A \in B(\mathcal{H}_o), T \in \mathcal{H} \quad . \tag{2.9}$$

π is a representation of $B(\mathcal{H}_o)$. The square root $\rho^{1/2}$ of the density matrix ρ is a Hilbert-Schmidt operator, hence an element of $\mathcal{H}$, and

$$\begin{aligned} (\rho^{1/2}, \pi(A)\rho^{1/2}) &= (\rho^{1/2}, A\rho^{1/2}) = \mathrm{Tr}\, \rho^{1/2} A \rho^{1/2} \\ &= \mathrm{Tr}\, \rho A = \omega(A) \end{aligned}$$

thus ω is a vector state in the representation π .

The advantage of the algebraic notion of a state is a larger flexibility in describing different physical situations. As an example let us look at a free theory of a charged scalar particle. We want to approximate charged states by chargeless states describing particle-antiparticle pairs by shifting the antiparticle "behind the moon". Let Φ_{xy} denote a unit vector describing a particle-antiparticle pair where the particle is near

to the point x and the antiparticle near to y. In the limit y $\longrightarrow$ spacelike infinity the sequence $(\Phi_{x,y})_y$ does not converge strongly, and the weak limit is zero. Local measurements, however, are not influenced by the antiparticle at spacelike infinity, thus the expectation values of local observables converge, and the sequence of states $\omega_{xy}(A) = (\Phi_{xy}, A\ \Phi_{xy})$ converges pointwise to a functional on $\mathfrak{A}$ which is linear, positive and normalized and hence a state on $\mathfrak{A}$. The corresponding GNS construction gives a Hilbert space $\mathcal{H}$ and a representation π of $\mathfrak{A}$ in $\mathcal{H}$. It is not possible to identify $\mathcal{H}$ with the charge zero Hilbert space $\mathcal{H}_o$ by some unitary operator U such that

$$U A U^* = \pi(A) \tag{2.11}$$

(π is not unitarily equivalent to the identical representation on $\mathcal{H}_o$).

This follows (modulo some technicalities) from the fact that the local charge operators

$$Q_R = \int_{|\underline{x}|<R} d^3\underline{x}\ j_o(0,\underline{x}) \tag{2.12}$$

converge weakly to zero in $\mathcal{H}_o$ and to one in $\mathcal{H}$.

The set of all states of $\mathfrak{A}$ is very large, and it is difficult to determine the structure of the whole state space. For the purposes of particle physics, however, only those states must be considered which describe an incoming or outgoing configuration of finitely many particles. Such states should be vector states in a positive energy representation of $\mathfrak{A}$ [8].

Definition: A representation π of $\mathfrak{A}$ is called a positive energy representation if there is a unitary, strongly continuous representation U of the translation group in the representation space $\mathcal{H}_\pi$, implementing the translations on $\mathfrak{A}$,

$$U(x)\,\pi(A)\,U(x)^{-1} = \pi\alpha_x(A) \tag{2.13}$$

such that the generators $P = (P_o, \underline{P})$ of U,

$$U(x) = e^{ixP} \quad , \quad xP = x^o P_o - \underline{x}.\underline{P} \quad , \tag{2.14}$$

fulfil the relativistic spectrum condition

$$\mathrm{sp}\, P \subset \{p \in \mathbb{R}^4, p^2 \geq 0, p_o \geq 0\} \equiv \overline{V_+} \quad . \tag{2.15}$$

The existence of a faithful positive energy representation is a specific form of the stability requirement. Borchers [9] has shown that in a positive energy representation π it is always possible to modify U such that $\{U(x)\}$ is contained in the weak closure $\pi(\mathfrak{A})^-$ of the observable algebra $\pi(\mathfrak{A})$. This justifies the interpretation of P as energy-momentum. If U can be extended to a representation of the Poincaré group implementing the Poincaré transformations α_L, the energy-momentum spectrum is Lorentz invariant. However, as is well known, there are positive energy representations (e.g. coherent infrared representations of the free photon field [10, 11] and representations describing electrically charged states in quantum electrodynamics [12]) where such an extension is not possible. It is remarkable that nevertheless also in the general case there is a natural definition of the normalization of energy-momentum such that sp P is Lorentz invariant [13, 14]. We shall always use this definition of energy-momentum for positive energy representations.

3. CHARGED SINGLE PARTICLE STATES (*)

In gauge theories particles may occur which carry a gauge charge, i.e. a charge which is measurable in the spacelike complement $\mathcal{O}'$ of an arbitrarily large finitely extended region $\mathcal{O}$. Such particles can never be created by local fields. One may conjecture that there exist stringlike localized fields creating such particles; but it is difficult to guess properties of these nonlocal objects. Therefore we do not assume any a priori knowledge of the localization of particles. Instead of this we characterize particles by their energy-momentum properties. Let $H_m = \{p \in \mathbb{R}^4, p^2 = m^2, p_0 \geq 0\}$ be the single particle hyperboloid. We say that a positive energy representation π of $\mathfrak{A}$ contains charged single particle states if

$$H_m \subset \text{sp}\, P \subset H_m \cup \{p \in \mathbb{R}^4, p^2 \geq M^2\} \tag{3.1}$$

for some $0 < m < M$.

The states in a representation may be partially classified by their charge quantum numbers. In the abstract framework considered here charge operators restricted to the representation π are simply those elements of the weak closure $\pi(\mathfrak{A})^-$ of $\pi(\mathfrak{A})$ which commute with each element in $\pi(\mathfrak{A})$ i.e. the elements of the center of $\pi(\mathfrak{A})^-$.

(*) The results in this section rely essentially on joint work with D. Buchholz [15, 16].

A positive energy representation π containing charged single particle states is called a single particle representation if all vector states in the representation space $\mathcal{H}_\pi$ have the same charge quantum numbers, i.e. the center of $\pi(\mathfrak{A})^-$ is trivial,

$$\pi(\mathfrak{A})^- \cap \pi(\mathfrak{A})' = \{\lambda 1, \lambda \in \mathbb{C}\} \quad . \tag{3.2}$$

Here for a subset $\mathcal{M}$ of the set B($\mathcal{H}$) of all bounded operators in a Hilbert space $\mathcal{H}$ $\mathcal{M}'$ denotes the commutant of $\mathcal{M}$,

$$\mathcal{M}' = \{B \in \mathcal{B}(\mathcal{H}), [B, A] = 0 \quad \text{for all} \quad A \in \mathcal{M}\} \quad . \tag{3.3}$$

Representations with a trivial center are called factorial. Note that by (3.1) $0 \notin P$, hence there is no vacuum state in the representation π. By (3.2) a vacuum state in π would have the same charge quantum numbers as the single particle states, so the particle would be "chargeless".

Instead of requiring sharp charge quantum numbers, i.e. factoriality of the representation π, we could use the stronger assumption that π is irreducible; by Schur's Lemma, this means that the commutant of $\pi(\mathfrak{A})$ is trivial [7]. As a matter of fact, it turns out, under one additional assumption, the so called duality condition (3.27) discussed below, that each single particle representation is a multiple of an irreducible single particle representation [15], i.e. the representation space $\mathcal{H}_\pi$ of a single particle representation π can be written as a tensor product $\mathcal{H}_\pi = \mathcal{H}_1 \otimes \mathcal{H}_2$ of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and there is an irreducible single particle representation π_1 on $\mathcal{H}_1$ such that $\pi(A) = \pi_1(A) \otimes 1_{\mathcal{H}_2}$ for all $A \in \mathfrak{A}$.

The mentioned spectral properties of single particle states are typical for particles in a theory without massless particles. In the presence of massless particles more general situations occur; the single particle mass shell H_m will not be isolated from the rest of the spectrum, and it may also be that there is no discrete weight for H_m, as it is the case for infraparticles. The charged particles of quantum electrodynamics are probably examples for such a situation [17]. In these cases the spectral conditions admit many inequivalent representations of the observable algebra describing situations which are from the experimentalist's point of view indistinguishable. Buchholz [18] has proposed the concept of a charge class in which a lot of representations which differ only by some practically unobservable infrared cloud are kept together (cf. the lectures of Prof. Wightman and Prof. Strocchi). In spite of the considerable progress which has been achieved crucial problems, e.g. the scattering theory for infraparticles, are far from being solved.

If no massless particles are present, the particle definition in (3.1), (3.2) seems to be suitable. On a first sight one might believe that gauge symmetry requires the existence of massless particles corresponding to the gauge field. In fact, the associated term in the Lagrangian has no mass term. However, by a Higgs phenomenon or a similar effect it may happen that in the physical mass spectrum no massless particles exist. This is for instance expected in quantum chromodynamics.

Another possibility is , that the charged particles in gauge theories can only exist in the presence of massless particles. Swieca has investigated this question in a gauge theory with gauge group U(1) [19]. In such a theory the charge is the integral of the zeroth component of a conserved current which is the divergence of the field strength,

$$Q = \int d^3\underline{x} \;\; j_0(0,\underline{x}) \qquad , \; j_\mu = \partial^\nu F_{\mu\nu} \quad . \tag{3.4}$$

j_μ and $F_{\mu\nu}$ are observable fields which should therefore obey local commutation relations,

$$[F_{\mu\nu}(x), F_{\varrho\sigma}(y)] = 0 \quad , \; (x-y)^2 < 0 \quad . \tag{3.5}$$

For a scalar particle fulfilling the spectral condition (3.1) the form factor of $F_{\mu\nu}$ has the form

$$\langle \underline{q} \,|\, F_{\mu\nu}(0) \,|\, \underline{p} \rangle = 2i\,(q_\mu p_\nu - q_\nu p_\mu)\, \frac{f((q-p)^2)}{(q-p)^2} \tag{3.6}$$

with $p_o = (\underline{p}^2 + m^2)^{1/2}$, $q_o = (\underline{q}^2 + m^2)^{1/2}$. $f(0)$ is proportional to the charge of the particle. If the charge of the particle is nonzero, the form factor of $F_{\mu\nu}$ would be singular at zero momentum transfer. Swieca has given an argument that this singularity is incompatible with locality (3.5) and the spectral properties (3.1).

Swieca's argument may be sketched as follows. Let g be a strongly decreasing function with a Fourier transform $\tilde{g}$ such that $(m,\underline{0})$ + supp $\tilde{g}$ and $(m,\underline{0})$ - supp $\tilde{g}$ intersect the energy momentum spectrum only on the mass shell, and let $|\underline{p}\rangle$ denote the improper single particle state with spatial momentum $\underline{p}$ and the normalization

$$\langle \underline{p} \,|\, \underline{q} \rangle = 2\omega_{\underline{p}}\, \delta^3(\underline{p}-\underline{q}) \qquad , \; \omega_{\underline{p}} = (\underline{p}^2 + m^2)^{1/2} \, . \tag{3.7}$$

Let $F_{oi}(g) = \int d^4x \; g(x)\, F_{oi}(x)$. Then $F_{oi}(g)\,|\underline{0}\rangle$ and $F_{oi}(g)^*\,|\underline{0}\rangle$ are single particle states. From the locality of F_{oi} and the strong decrease of g the expectation value of the commutator in the zero momentum state $|\underline{0}\rangle$,

$$\langle \underline{0} \,|\, [F_{oi}(\underline{x}), F_{oi}(g)] \,|\, \underline{0} \rangle \equiv h(\underline{x}) \tag{3.8}$$

is rapidly decreasing for large $\underline{x}$,and therefore the Fourier transform $\tilde{h}(\underline{p})$ is a smooth function. Since only single states contribute as intermediate states in (3.8) we find from (3.6)

$$\begin{aligned} \tilde{h}(\underline{p}) = \{ &\langle \underline{0} \,|\, \tilde{F}_{oi}(\underline{p}) \,|\, -\underline{p} \rangle \langle -\underline{p} \,|\, F_{oi}(g) \,|\, \underline{0} \rangle \\ &- \langle \underline{0} \,|\, F_{oi}(g) \,|\, \underline{p} \rangle \langle \underline{p} \,|\, \tilde{F}_{oi}(\underline{p}) \,|\, \underline{0} \rangle \} \\ = -(2\pi)^3\, 8\, \omega_{\underline{p}}\, m^2\, &(f(t)/t)^2\, [\tilde{g}((\omega_{\underline{p}} - m), \underline{p}) - \tilde{g}(-(\omega_{\underline{p}} - m), \underline{p})] \end{aligned} \tag{3.9}$$

where $t = (p - (m,\underline{0}))^2 = 2m(m - \omega_{\underline{p}})$ and $g((\omega_{\underline{p}} - m), \underline{p}) - g(-(\omega_{\underline{p}} - m), \underline{p}) = t\, G(\underline{p})$ with a smooth function G which does not vanish for $\underline{p} = 0$ for suitable g.

Eq. (3.9) implies that f(0) must vanish since otherwise $\tilde{h}$ would not be a smooth function. Thus the particle has zero charge.

Unfortunately, the argument of Swieca is not completely rigorous, due to the use of improper particle states with sharp momentum. It could be improved if it would be known that f is continuous at zero. One might think that this missing point in the proof is of minor importance. There is a widespread belief among theoretical physicists that "high precision" mathematical arguments have no physical meaning. On the contrary it is generally accepted that high precision measurements often lead to the detection of completely new phenomena. I would like to convince you that the search for a "high precision" argument might equally well provide completely new insights, and I think that Swieca's theorem on the absence of charged particles in a massive U(1) gauge theory is a good example.

Actually, the missing information on the continuity of f is of the same character as the information on the singularity of f(t)/t. Swieca's argument shows how one can proceed from the weaker property to the stronger.

In terms of position space both properties refer to the localization of the particle. Thus we may ask the more general question: how well are particles localized?

To find an answer we try to exhibit particle states which are as well localized as possible. The idea is the following one. Let $p \in H_m$ and let $\Phi \in \mathcal{H}_\pi$ with energy momentum spectrum $sp_P \Phi$ in a small neighbourhood of p. Let $A \in \mathfrak{A}(\mathcal{O})$. The ensemble described by $\pi(A)\Phi$ consists of particles in the original ensemble Φ which have been influenced by A, so they must have passed through $\mathcal{O}$, of particles which have not been influenced by A and keep therefore their momentum unchanged, and of components where additional particles have been created and whose momenta are therefore not on the mass shell H_m. Thus the components of $\pi(A)\Phi$ with momenta on the mass shell H_m but not in $sp_P\Phi$ should be localized near $\mathcal{O}$. To filter out only these components we choose a test function $f \in \mathcal{S}(\mathbb{R}^4)$ with $\operatorname{supp}\tilde{f} + sp_P\Phi \cap sp_P\Phi = \emptyset$, $\operatorname{supp}\tilde{f} + sp_P\Phi \cap sp\, P \subset H_m$ and set

$$B = \int d^4x\, f(x)\, \alpha_x(A) \tag{3.10}$$

From the heuristic argument given before $\pi(B)\Phi$ is a candidate for a vector representing a well localized state.

There are different possibilities for a precise definition of the term "well localized". Here the appropriate notion of localization is concentration of the energy momentum content. I want to indicate the argument.

The argument relies essentially on a suitable utilization of almost local operators with a sharp momentum transfer which map single particle states onto single particle states. Here an operator B is called almost local, if there are observables $A_R \in \mathfrak{A}(\mathcal{O}_R)$, $\mathcal{O}_R = \{x \in \mathbb{R}^4, |x^0| + |\underline{x}| < R\}$ such that

$$\|B - A_R\|\, R^n \longrightarrow 0 \;,\; R \longrightarrow \infty \tag{3.11}$$

for all $n \in \mathbb{N}$. The operator B in (3.10) is an example for an almost local operator.

Now let B_1 and B_0 be almost local operators with momentum transfer such that $\pi(B_0)\Phi$ and $\pi(\alpha_x(B_1)B_0)\Phi$ are single particle states and $\pi\,\alpha_x(B_1)\Phi = 0$ for all $x \in \mathbb{R}^4$ (Fig. 3.1). Let $\tilde{B}_1(t,\underline{q}) = \int d^3\underline{x}\; e^{-i\underline{q}\underline{x}}\, \alpha_{(t,\underline{x})}(B_1)$. Then $\pi(\tilde{B}_1(t,\underline{q}))\Phi = 0$ and

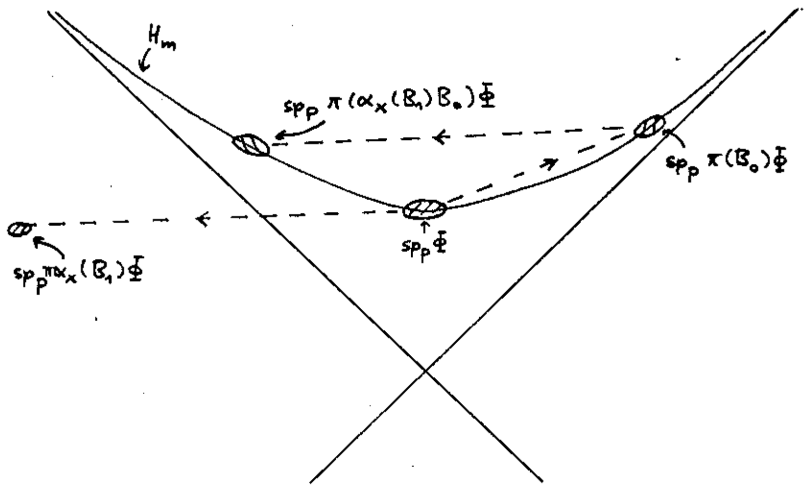

Figure 3.1: The choice of B_o and B_1

$$\begin{aligned}\pi([\tilde{B}_1(t,\underline{q}), B_0])\Phi &= \pi(\tilde{B}_1(t,\underline{q})B_0)\Phi \\ &= f(\underline{P})\,\pi(\tilde{B}_1(0,\underline{q})B_0)\,\Phi = f(\underline{P})\,\pi([\tilde{B}_1(0,\underline{q}),B_0])\Phi\,,\end{aligned} \tag{3.12}$$

$$f(\underline{P}) = \exp\{it[(\underline{P}^2+m^2)^{1/2} - ((\underline{P}+\underline{q})^2+m^2)^{1/2}]\} .$$

Now $B = [\tilde{B}_1(0,\underline{q}), B_0] = \int d^3\underline{x}\; e^{-i\underline{q}\underline{x}}\,[\alpha_{(0,\underline{x})}(B_1), B_0]$ and $B' = [\tilde{B}_1(t,\underline{q}), B_0]$ are almost local operators. Thus we have found a relation

$$\pi(B')\Phi = f(\underline{P})\,\pi(B)\Phi \tag{3.13}$$

with almost local operators B and B'. If B would be invertible, the measurement of the function $f(\underline{P})$ of the momentum operator in the state $\pi(B)\Phi$ could be replaced by the almost local operator $\pi(B'B^{-1})$, so in a sense, the momentum content of the particle in the state $\pi(B)\Phi$ is essentially localized in a finite region.

By using multiple commutators and by smearing over the time variables one can establish the relation (3.13) for an arbitrary smooth function f such that the almost local operator on the right hand side does not depend on f. It is not clear whether one can choose an invertible $B^{(*)}$. The precise theorem which can be derived is the following one:

Theorem 3.1 [14]. Let $p \in H_m$ and let Δ be a neighbourhood of p. There exists some $\Phi \in \mathcal{H}_\pi$ with $\mathrm{sp}_P \Phi \subset \Delta$ and almost local operators B, B_μ, $\mu = 0, \ldots, 3$ such that

(i) $\pi(B)\Phi \neq 0$

(ii) $P_\mu \pi(B)\Phi = \pi(B_\mu)\Phi$

(iii) $\pi(B_\mu^* B)\Phi = \pi(B^* B_\mu)\Phi$

Now let ω be the state on $\mathfrak{A}$ induced by a vector $\pi(B)\Phi$, $\pi(B)$, Φ chosen according to Thm 3.1 with $\|\pi(B)\Phi\| = 1$. Then for $A \in \mathfrak{A}(\mathcal{O})$

$$\begin{aligned}\partial_\mu \omega\alpha_x(A) &= i(\pi(B)\Phi, [P_\mu, \alpha_x(A)]\,\pi(B)\Phi) \\ &= i\{(\pi(B_\mu)\Phi, \pi\alpha_x(A)\pi(B)\Phi) - (\pi(B)\Phi, \pi\alpha_x(A)\pi(B_\mu)\Phi)\} \\ &= i\{(\Phi, \pi[B_\mu^*, \alpha_x(A)]\,\pi(B)\Phi) - (\Phi, \pi[B^*, \alpha_x(A)]\,\pi(B_\mu)\Phi) \\ &\quad + (\Phi, \pi\alpha_x(A)\,\pi(B_\mu^* B - B^* B_\mu)\Phi)\}\end{aligned} \tag{3.14}$$

where the last term disappears because of Thm 3.1 (iii). Hence

$$|\partial_\mu \omega\alpha_x(A)| \le \mathrm{const}\{\|[B_\mu^*, \alpha_x(A)]\| + \|[B^*, \alpha_x(A)]\|\} . \tag{3.15}$$

Thus the derivative of $x \longrightarrow \omega\alpha_x(A)$ decreases rapidly in spacelike directions. $x \longrightarrow \omega_x(A)$ will tend therefore to a constant for x tending to spacelike infinity. This constant is independent of the direction in more than two dimensions. In two space time directions there may be different limits for x tending to the right and tending to the left; the particle is then called a soliton [16]. In any case the limit ω_0 is a translation invariant state which can be interpreted as the vacuum. The corresponding GNS representation π_0 is a positive energy representation with a unique (up to a phase) translation invariant unit vector Ω and a mass gap:

$$\mathrm{sp}\,P \subset \{0\} \cup \{p \in \mathbb{R}^4, p^2 \ge (M-m)^2, p_0 \ge 0\} \tag{3.16}$$

We have found the following theorem:

Theorem 3.2. For all $A \in \mathfrak{A}_0$, $\omega\alpha_x(A) - \omega_0(A)$ is rapidly decreasing in spacelike direction ($|\underline{x}| - |x^0| \longrightarrow \infty$).

(*) Note that no assumption on the multiplicity of spP on H_m, i.e. on the number of components of single particle wave functions has been made.

This result does not depend on the use of bounded operators. It holds equally well in the framework of Wightman fields. Specializing to the case of a U(1) gauge theory, it means that the expectation values of the electric field decrease rapidly in spatial direction, hence the electric charge of the state ω is zero. Since all particle states in the representation π have the same charge, this proves Swieca's theorem [16].

In the Haag-Kastler framework a much stronger result holds. From (3.15) one concludes that

$$|\partial_\mu \omega\alpha_x(A)| \le \|A\|\, h(R) \tag{3.17}$$

with a rapidly decreasing function h, where R is chosen such that $\alpha_x(A)$ commutes with $\mathfrak{A}(\mathcal{O}_R)$. This shows that $\omega\alpha_x$ converges to ω_0 uniformly on large subalgebras of $\mathfrak{A}$ corresponding to certain unbounded regions. Let G be a region containing a path x(s) to spacelike infinity with uniformly bounded tangent vectors $\dot{x}^\mu(s)$ such that for some $\varepsilon > 0$

$$\mathcal{O}_{s\varepsilon} + x(s) \subset G \tag{3.18}$$

Roughly speaking, G may be though of as a string which fattens.

Let $A \in \mathfrak{A}^c(G) = \{A \in \mathfrak{A}, [A,B] = 0$ for all $B \in \mathfrak{A}(\mathcal{O}), \mathcal{O} \subset G\}$.
Then

$$\begin{aligned}|(\omega\alpha_{-x(s)} - \omega_0)(A)| &= \left|\int_s^\infty ds'\, \dot{x}^\mu(s')\, \partial_\mu(\omega\alpha_{-x(s')}(A))\right| \\ &\le \|A\|\, H(s)\end{aligned} \tag{3.19}$$

with the strongly decreasing function $H(s) = \sup|\dot{x}^\mu(s')| \int_s^\infty h(s'\varepsilon)\, ds'$. Thus the convergence $\omega\alpha_{-x(s)} \longrightarrow \omega_0$ is uniform on $\mathfrak{A}^c(G)$. Therefore ω_0 can be extended to a normal (*) state ω_G on $\pi(\mathfrak{A}^c(G))^-$. Thus there is a density matrix ρ_G in $\mathcal{H}_\pi$ such that

$$\omega_0(A) = \mathrm{Tr}\, \rho_G\, \pi(A) , \quad A \in \mathfrak{A}^c(G) . \tag{3.20}$$

(*) A state is called normal if it is weakly continuous on uniformly bounded subsets.

Using more advanced methods of the theory of v. Neumann algebras (Araki's theory of natural cones [7]) one can even find a unit vector $\Psi_G \in \mathcal{H}_\pi$ with

$$\begin{aligned}
&(i) \quad (\Psi_G, \pi(A)\Psi_G) = \omega_0(A) \quad , A \in \mathfrak{A}^c(G)\\
&(ii) \quad \| \Psi_G - \pi(B)\Phi \|^2 \leq \| (\omega_0 - \omega) \restriction \mathfrak{A}^c(G) \| \\
&(iii) \quad \overline{\pi(\mathfrak{A}^c(G))\Psi_G} = \overline{\pi(\mathfrak{A})\pi(B)\Phi} \quad .
\end{aligned} \tag{3.21}$$

Hence there is a vector state in the single particle representation π which looks like the vacuum in the spacelike complement of G. Moreover, this state differs from ω on the whole algebra $\mathfrak{A}$ not more than on the algebra $\mathfrak{A}^c(G)$. If we further assume that $\pi(B)\Phi$ is cyclic for $\pi(\mathfrak{A})$, i.e. $\pi(\mathfrak{A})\pi(B)\Phi$ is dense in $\mathcal{H}_\pi$, we can use Ψ_G for a definition of a unitary charge generating operator. Let V_G be an operator from $\mathcal{H}_0$ to $\mathcal{H}_\pi$ with

$$V_G \pi_0(A)\Omega = \pi(A)\Psi_G \quad , A \in \mathfrak{A}^c(G) \quad . \tag{3.22}$$

$\pi_0(\mathfrak{A}^c(G))\Omega$ is dense in $\mathcal{H}_0$ according to the Reeh-Schlieder Theorem [8], $\pi(\mathfrak{A}^c(G))\Psi_G$ is dense in $\mathcal{H}_\pi$ according to (3.21) and the assumed cyclicity of $\pi(B)\Phi$ for $\pi(\mathfrak{A})$. Moreover

$$\begin{aligned}
\| V_G\pi_0(A)\Omega\|^2 &= \| \pi(A)\Psi_G \|^2 = (\Psi_G, \pi(A^*A)\Psi_G) \\
= \omega_0(A^*A) &= (\Omega, \pi_0(A^*A)\Omega) = \| \pi_0(A)\Omega \|^2 \quad ,
\end{aligned} \tag{3.23}$$

hence V_G is a unitary operator from $\mathcal{H}_0$ onto $\mathcal{H}_\pi$. V_G intertwines the representations π_0 and π , restricted to $\mathfrak{A}^c(G)$,

$$V_G \pi_0(A) = \pi(A) V_G \quad , A \in \mathfrak{A}^c(G) \quad . \tag{3.24}$$

V_G may be interpreted as an operator which generates a charge within the region G.

V_G has similar localization properties as the formal operators $\psi_{\mathcal{C}}$ of (1.2), restricted to the vacuum Hilbert space $\mathcal{H}_0$, and may be considered as a mathematically rigorous version of such an operator. If local fields exist which create the particle states from the vacuum, one can even find unitary operators $V_\mathcal{O}$ associated to bounded regions $\mathcal{O}$ such that

$$V_\mathcal{O} \pi_0(A) = \pi(A) V_\mathcal{O} \quad , A \in \mathfrak{A}(\mathcal{O}') \tag{3.25}$$

where $\mathcal{O}'$ is the spacelike complement of $\mathcal{O}$ and $\mathfrak{A}(\mathcal{O}')$ is the C*-algebra generated by the algebras $\mathfrak{A}(\mathcal{O}_1)$ with $\mathcal{O}_1 \subset \mathcal{O}'$.

Superselection sectors corresponding to representations π fulfilling relation (3.25) are called locally generated. The structure of locally generated superselection sectors has been analyzed in general by Doplicher, Haag and Roberts (DHR) [20]. Their analysis extends and partially corrects an earlier analysis of Borchers [21]. Using the "duality assumption"

$$\pi_0(\mathfrak{A}(\mathcal{O}'))' = \pi_0(\mathfrak{A}(\mathcal{O})) \tag{3.26}$$

they showed that there is a composition law of sectors, corresponding to the idea that charges can be added. Furthermore, they prove that (in more than 2 space time dimensions) there is an intrinsic notion of statistics leaving only the possibilities of (para-) Bose, (para-) Fermi and infinite statistics. The pathological case of infinite statistics has been ruled out for single particle representations by an application of Thm 3.1 [22, 15]. In the case of finite statistics, Doplicher, Haag and Roberts derived the existence of antiparticles and of multiparticle scattering states. Very recently [23], Doplicher and Roberts showed that there is always a compact group (the "global gauge group") whose irreducible representations label the locally generated superselection sectors, and they construct a C*-algebra $\mathcal{F} \supset \mathfrak{A}$ (the "field algebra") on which the gauge group acts by automorphisms such that $\mathfrak{A}$ is the invariant part of $\mathcal{F}$, and which contains enough local charged fields to create all locally generated superselection sectors out of the vacuum.

By analogous methods one can perform a similar analysis for representations fulfilling relation (3.24), and one finds stringlike localized field operators and (in more than 3 space time dimensions) a full set of particle states, including antiparticle states and all incoming and outgoing multiparticle scattering states [15]. Instead of the duality assumption (3.26) one uses in this analysis the assumption

$$\pi_0(\mathfrak{A}^c(G))^- = \pi_0(\mathfrak{A}(G))' \tag{3.27}$$

which can be proven for a sufficiently large set of stringlike regions in the Wightman framework of field theory [24].

One may ask whether a more clever analysis of single particle states in massive theories will always lead to the DHR type of localization. This would mean that gauge charges can never occur in massive theories and would be a very strong generalization of Swieca's Theorem. Such a possibility cannot be ruled out in the moment; the fact, however, that the weaker property (3.24) already leads to the usual structure in the set of particle states shows that there is no intrinsic inconsistency in the stringlike localization. There are, on the other hand, severe dynamical restrictions imposed by this kind of localization. These will be discussed in the next section.

4. DYNAMICAL IMPLICATIONS OF THE EXISTENCE OF GAUGE CHARGES

We have derived localization properties of single particle states which show that there always exist field operators generating the particle states from the vacuum which are localized in a stringlike region G. We shall now concentrate on the case where this localization cannot be improved to a finite localization of the DHR-type.

In such a case there are no nonzero operators V_G fulfilling relation (3.21). The commutant of $\pi_0 \oplus \pi(\mathfrak{A}(\mathcal{O}'))$,

$$\pi_0 \oplus \pi(\mathfrak{A}(\mathcal{O}'))' = \{ B \in \mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}_\pi), [B, \pi_0 \oplus \pi(A)] = 0 \ \forall A \in \mathfrak{A}(\mathcal{O}')\} \quad (4.1)$$

consists therefore only of operators $C_0 \oplus C$, $C_0 \in \pi_0(\mathfrak{A}(\mathcal{O}'))'$ and $C \in \pi(\mathfrak{A}(\mathcal{O}'))'$. The bicommutant, i.e. the commutant of the commutant, is then

$$\pi_0 \oplus \pi(\mathfrak{A}(\mathcal{O}'))'' = \pi_0(\mathfrak{A}(\mathcal{O}'))'' \oplus \pi(\mathfrak{A}(\mathcal{O}'))'' \quad . \quad (4.2)$$

Since the bicommutant coincides with the weak closure (v. Neumann's bicommutant theorem) we find that the "charge operator"

$$Q = 1 \oplus -1 \quad (4.3)$$

is contained in the weak closure of $\pi_0 \oplus \pi(\mathfrak{A}(\mathcal{O}'))$ for all $\mathcal{O}$. This may be considered as an abstract version of Gauss' law.

Now from the localization properties of charged states derived in Sect.3 Q cannot be a sum of operators which are localized in double cones in $\mathcal{O}'$. Thus Q cannot be the sum of partial electric fluses as it is the case for U(1) gauge theories. This might be interpreted as a generalized Swieca theorem:

In massive gauge theories only multiplicative charges can occur.

If we think of the total charge operator Q to be the product of the electric fluxes E_1 and E_2 through two opposite halfspheres, $Q = E_1 E_2$, we have

$$|\omega(E_i) - \omega_0(E_i)| \approx 0 \quad , i = 1, 2 \quad (4.4)$$

and

$$\omega_0(E_1 E_2) = 1 \quad , \quad \omega(E_1 E_2) = -1 \quad . \quad (4.5)$$

Now (4.4) and (4.5) are only compatible if the fluxes E_1 and E_2 are strongly correlated already in the vacuum. Hence the nonvanishing of the correlation

$$\omega_0(E_1 E_2) - \omega_0(E_1)\,\omega_0(E_2) \quad (4.6)$$

is a necessary condition for the existence of a charged particle corresponding to the charge $E_1 E_2$. We shall see that (4.6) will be the basis for a confinement criterion.

Let us investigate the dependence of the charge generator V_G on the stringlike region G. Let G_1 be another stringlike region. Then from (3.18) (ii)

$$\| V_G \Omega - V_{G_1} \Omega \|^2 \le 2 \{ \| (\omega - \omega_0) \restriction \mathfrak{A}^c(G) \| + \| (\omega - \omega_0) \restriction \mathfrak{A}^c(G_1) \| \} \quad (4.7)$$

Now from (3.16), if $G \cap G_1 \supset \mathcal{O}_R$ we have

$$\| V_G \Omega - V_{G_1} \Omega \|^2 \le 4 H(R) \quad (4.8)$$

where H is the rapidly decreasing function in (3.19), hence the asymptotic direction of the string is not visible. If we interpret $V_{G_1}^{-1} V_G$ as an operator which shifts a charge within G to infinity and brings it back in G_1 we see that such a charge transfer on a closed loop has an expectation value near to 1,

$$|(\Omega, V_{G_1}^{-1} V_G \Omega) - 1| \le \| (V_G - V_{G_1}) \Omega \| \le 2\, H(R)^{1/2} \quad . \quad (4.9)$$

If, on the other hand, $G_1 \subset \mathcal{O}_R'$, the states induced by $V_{G_1}\Omega$ will converge weakly to ω_0 in the limit $R \to \infty$; one can show that this implies that the scalar product with $V_G \Omega$ tends to zero,

$$(V_{G_1}\Omega, V_G\Omega) \longrightarrow 0 \quad , R \longrightarrow \infty \quad . \tag{4.10}$$

If $G_1 = G \cap \mathcal{O}_R'$, $V_{G_1}^{-1} V_G$ may be interpreted as a charge transfer inside of G into the spacelike complement of $\mathcal{O}_R$. The vacuum expectation values of such quantities are small,

$$(\Omega, V_{G_1}^{-1} V_G \Omega) \longrightarrow 0 \quad , R \longrightarrow \infty \tag{4.11}$$

In Section 6 we shall see that the comparison of the behaviour of charge transfers on open strings (4.11) with that on closed loops (4.9) leads to another confinement criterion.

5. THE $\mathbb{Z}_2$ HIGGS MODEL

We now want to confront the results of the general analysis with the structure of a lattice gauge theory; as a simple example we take the $\mathbb{Z}_2$ Higgs model which is a gauge theory with gauge group $\mathbb{Z}_2$ coupled to a $\mathbb{Z}_2$ valued Higgs field. This model has first been introduced by Wegner [25]. The present analysis relies mainly on joint work with M. Marcu [26].

Let us first look at the associated classical statistical mechanical system, i.e. the Euclidean theory from the point of view of quantum field theory. On a hypercubic lattice $\mathbb{Z}^{d+1}$, $d \geq 2$ we have a gauge field $\tau(b) = \pm 1$, which is defined on the lattice bonds b of $\mathbb{Z}^{d+1}$, and a Higgs field $\sigma(x)$ which is defined on the sites $x \in \mathbb{Z}^{d+1}$. The Hamilton function (the Euclidean action) is

$$\mathcal{H}(\tau,\sigma) = \beta_g \sum_p \delta\tau(p) + \beta_h \sum_b \tau(b)\,\delta\sigma(b) \tag{5.1}$$

where $\beta_g, \beta_h > 0$ are coupling constants, p runs over the plaquettes and b over the bonds of the lattice and δ denotes the "exterior derivative"

$$\delta\tau(p) = \prod_{b \in \partial p} \tau(b) \quad , \quad \delta\sigma(b) = \prod_{x \in \partial b} \sigma(x) \quad . \tag{5.2}$$

The phase diagrams of the model is expected to have the form shown in Fig. 5.1. It consists out of two phases, the screening/confinement phase (I) and the free charge phase (II) [27]. In both phases there exist subregions (the shaded regions in Fig. 1) where convergent expansions are known (see [28] for the screening/confinement phase and [29] for the free charge phase).

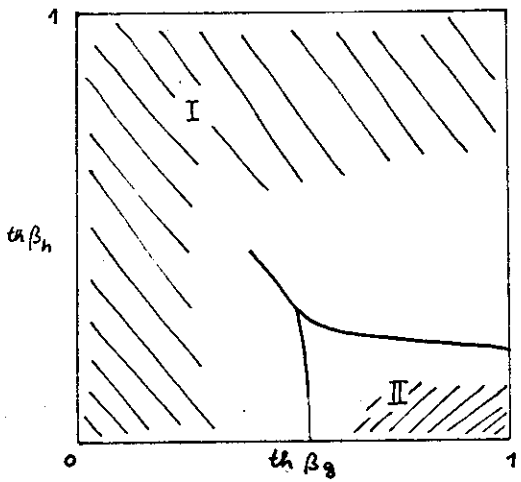

Figure 5.1: The phase diagram of the $\mathbb{Z}_2$ Higgs model (conjectured)

The corresponding quantum system in the temporal gauge is defined on a lattice $\mathbb{Z}^d$ which represents the space. The time is continuous in the quantum system. On each lattice bond $\underline{b}$ there are Pauli matrices $\tau_3(\underline{b})$ and $\tau_1(\underline{b})$ representing the gauge field and the electric field, respectively. On each lattice point $\underline{x}$ there are Pauli matrices $\sigma_3(\underline{x})$ and $\sigma_1(\underline{x})$, representing the Higgs field and its canonical momentum, resp. . One has the "canonical commutation relations"

$$\begin{aligned} &\sigma_i(\underline{x})^2 = \tau_i(\underline{b})^2 = 1 \;,\; \sigma_i(\underline{x})\,\tau_j(\underline{b}) = \tau_j(\underline{b})\,\sigma_i(\underline{x}) \;, \\ &\sigma_i(\underline{x})^* = \sigma_i(\underline{x}) \;,\; \tau_j(\underline{b})^* = \tau_j(\underline{b}) \quad , \; i,j = 1,3 \quad , \\ &\sigma_3(\underline{x})\,\sigma_1(\underline{x}) = -\sigma_1(\underline{x})\,\sigma_3(\underline{x}) \;,\; \tau_3(\underline{b})\,\tau_1(\underline{b}) = -\tau_1(\underline{b})\,\tau_3(\underline{b}) \;. \end{aligned} \tag{5.3}$$

Fields at different points or bonds commute.

Let $\mathcal{F}$ be the *-algebra which is generated by these fields. Gauge transformations on $\mathcal{F}$ are implemented by the operators

$$q(\underline{x}) = \sigma_1(\underline{x})\,\delta^*\tau_1(\underline{x}) \tag{5.4}$$

where $\delta^*\tau_1(\underline{x}) = \prod_{\underline{b}:\underline{x}\in\partial\underline{b}} \tau_1(\underline{b})$ denotes the "divergence" of τ_1. $q(\underline{x})$ can be interpreted as the external charge at the point $\underline{x}$. In a U(1) theory in the continuum (5.4) corresponds to the difference between the charge density and the divergence of the electric field.

Observables A are defined to be gauge invariant elements of $\mathcal{F}$, i.e.

$$A\, q(\underline{x}) = q(\underline{x})\, A \quad , \ \underline{x} \in \mathbb{Z}^d \quad . \tag{5.5}$$

Let $\mathfrak{A}$ denote the algebra of observables. $\mathfrak{A}$ has a nontrivial center which is generated by the external charges. If in a representation π of $\mathfrak{A}$ Gauss' law holds,

$$\pi(\sigma_1(\underline{x})) = \pi(\delta^* \tau_1(\underline{x})) \quad , \ \underline{x} \in \mathbb{Z}^d \quad , \tag{5.6}$$

one has $\pi(q(\underline{x})) = 1$ for all external charges. Let I denote the ideal in $\mathfrak{A}$ which is generated by the operators $q(\underline{x}) - 1$. For representations fulfilling Gauss' law the relevant algebra of observables is

$$\mathcal{B} = \mathfrak{A}/I \quad . \tag{5.7}$$

$\mathcal{B}$ is generated by the rest classes modulo I of $\tau_1(\underline{b})$ and $\tau_3(\underline{b})\,\delta\sigma_3(\underline{b})$ which may be called $u_1(\underline{b})$ and $u_3(\underline{b})$, respectively. $u_1(\underline{b})$ and $u_3(\underline{b})$ have the algebraic properties of Pauli matrices (5.3). For a set $\underline{L}$ of bonds let $\mathcal{B}(\underline{L})$ denote the algebra generated by $u_i(\underline{b})$, $i = 1, 3$, $\underline{b} \in \underline{L}$.

It is interesting to note that the quantum system described by $\mathcal{B}$ has no locally generated charges. This fact holds independently of the dynamics. To see this we first observe that the relative commutant $\mathcal{B}(\underline{L})^c$ of $\mathcal{B}(\underline{L})$,

$$\mathcal{B}(\underline{L})^c = \{ B \in \mathcal{B} \mid [B, B_1] = 0 \quad \forall\, B_1 \in \mathcal{B}(\underline{L}) \} \quad , \tag{5.8}$$

coincides with the algebra $\mathcal{B}(\underline{L}^c)$ where $\underline{L}^c$ denotes the complement of $\underline{L}$ in the set of all lattice bonds. This property would be absent for instance in d = 2 dimensions for the algebra generated by $u_1(\underline{b})$ and $\delta u_3(\underline{p})$. There $\prod_{\underline{b} \in \underline{M}} u_3(\underline{b})$ for some closed curve $\underline{M}$ commutes with all operators $u_1(\underline{b}')$ with $\underline{b}' \notin M$ but is a product of all plaquette operators $\delta u_3(\underline{p})$ where $\underline{p}$ is surrounded by $\underline{M}$.

The next step is more abstract. Let π_0 and π be representations of $\mathcal{B}$ which are disjoint, i.e. the "charge operator" $Q = 1 \oplus -1$ is contained in the weak closure of $\pi_0 \oplus \pi(\mathcal{B})$. Thus there is a sequence $A_n \in \mathcal{B}$ with $\|A_n\| \le 1$ such that $\pi_0(A_n)$ converges weakly to 1 and $\pi(A_n)$ to -1 (*). Let $\underline{L}$ be a finite set of bonds and let $G_{\underline{L}}$ denote the (finite)

(*) Each element of the weak closure of a *-algebra of bounded operators on a separable Hilbert space is the weak limit of a bounded sequence according to Kaplanski's density theorem (see e.g. [7]).

group which is generated by $u_1(\underline{b})$ and $u_3(\underline{b})$, $\underline{b} \in \underline{L}$. Let $m_{\underline{L}}$ denote the "conditional expectation"

$$m_{\underline{L}}(B) = |G_{\underline{L}}|^{-1} \sum_{U \in G_{\underline{L}}} U B U^{-1} \tag{5.9}$$

$m_{\underline{L}}(B)$ commutes with $G_{\underline{L}}$ and therefore also with $\mathcal{B}(\underline{L})$. Thus $m_{\underline{L}}(A_n) \in \mathcal{B}(\underline{L}^c)$ and

$$\pi_0(m_{\underline{L}}(A_n)) \longrightarrow 1 \quad , \quad \pi(m_{\underline{L}}(A_n)) \longrightarrow -1 \tag{5.10}$$

since $m_{\underline{L}}$ is weakly continuous. Thus

$$Q \in \pi_0 \oplus \pi(\mathcal{B}(\underline{L}^c))^- \tag{5.11}$$

which is the abstract version of Gauss' law discussed in Section 4.

Whereas there are no locally generated superselection sectors there is an uncountable number of mutually disjoint representations. We are interested in the question whether there are, besides the vacuum, other positive energy representations of $\mathcal{B}$. This question cannot be answered in the kinematical framework described above, instead we have to introduce a dynamics, and the answer will strongly depend on the dynamics.

A convenient way of introducing a dynamics in a lattice model is the Euclidean method. There the local Hamiltonians $H_{\underline{\Lambda}}$ (*) are defined implicitely as $(-\ln T_{\underline{\Lambda}})$ where the local transfer matrices $T_{\underline{\Lambda}}$ are positive, invertible operators in $\mathcal{F}$. For the gauge invariant Ising model with the Hamilton function (5.1) the transfer matrix is (in the temporal gauge)

$$\begin{aligned} T_{\underline{\Lambda}} &= e^{\frac{1}{2}A_{\Lambda}}\, e^{B_{\Lambda}}\, e^{\frac{1}{2}A_{\Lambda}} \quad , \\ A_{\underline{\Lambda}} &= \beta_h \sum_{\underline{b} \subset \underline{\Lambda}} \delta\sigma_3(\underline{b})\,\tau_3(\underline{b}) + \beta_g \sum_{\underline{p} \subset \underline{\Lambda}} \delta\tau_3(\underline{p}) \quad , \\ B_{\underline{\Lambda}} &= \beta_h^* \sum_{\underline{x} \in \underline{\Lambda}} \sigma_1(\underline{x}) + \beta_g^* \sum_{\underline{b} \subset \underline{\Lambda}} \tau_1(\underline{b}) \quad , \quad \beta^* = -\tfrac{1}{2} \ln \operatorname{th} \beta \quad . \end{aligned} \tag{5.12}$$

The local time evolution on $\mathcal{F}$ is defined by

$$\alpha_z^{\underline{\Lambda}}(A) = e^{izH_{\Lambda}}\, A\, e^{-izH_{\underline{\Lambda}}} \quad , z \in \mathbb{C} \quad . \tag{5.13}$$

(*) Here and in the following $\underline{\Lambda}$ will denote a box.

For integer imaginary values of z the locality properties of T_Λ imply that $\alpha_z^\Lambda(A)$ becomes independent of Λ for Λ sufficiently large. Thus

$$\alpha_{in}(A) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \alpha_{in}^\Lambda(A) \tag{5.14}$$

exists for all $n \in \mathbb{Z}$. α_i is an algebraic automorphism of $\mathcal{F}$ which is not compatible with the *-operation,

$$\alpha_i(A)^* = \alpha_{-i}(A^*) \tag{5.15}$$

and $\alpha_{in} = (\alpha_i)^n$, $n \in \mathbb{Z}$.

It is not known whether α_z^Δ converges for other values of z (in some sense). Under certain conditions one can construct α_t for real t from α_i (essentially by using the "symmetric resolvent" $(e^\lambda \alpha_i + e^{-\lambda}\alpha_{-i})^{-1}$, $\lambda \in \mathbb{R}$; α_i is then called the analytic generator of α_t [30]). It seems, however, that these conditions are not satisfied in the present model.

Instead of defining the real time evolution directly we study the problem in a "vacuum" representation of $\mathcal{B}$. Unfortunately, also the term "vacuum" = "ground state" is usually defined by means of the time evolution. It turns out, however, that one can characterize ground state also in terms of α_i.

Definition: A state ω_o on $\mathcal{F}$ is called a ground state with respect to α_i if

$$0 \leq \omega_o(A^* \alpha_i(A)) \leq \omega_o(A^* A) \tag{5.16}$$

for all $A \in \mathcal{F}$.

A simple consequence of the definition is the α_i-invariance of ω_o . In fact

$$\langle B, A \rangle = \omega_o(B^* \alpha_i(A)) \tag{5.17}$$

is a positive sesquilinear form on $\mathcal{F}$ and therefore hermitean,

$$\overline{\langle B, A \rangle} = \langle A, B \rangle \quad . \tag{5.18}$$

Thus, for B = 1

$$\omega_o(\alpha_i(A)) = \langle 1, A \rangle = \overline{\langle A, 1 \rangle} = \overline{\omega_o(A^*)} = \omega_o(A) \quad . \tag{5.19}$$

Let $(\pi_o, \mathcal{H}_o, \Omega)$ denote the GNS representation of $\mathcal{F}$ induced by ω_o, i.e.

$$\mathcal{H}_o = \overline{\pi_o(\mathcal{F})\Omega} \quad ,$$
$$(\Omega, \pi_o(A)\Omega) = \omega_o(A) \quad , \quad A \in \mathcal{F} \quad . \tag{5.20}$$

In $\mathcal{H}_o$ one can define the global transfer matrix T_o by

$$T_o \pi_o(A)\Omega = \pi_o(\alpha_i(A))\Omega \quad , A \in \mathcal{F} . \tag{5.21}$$

We have

$$(\pi_o(A)\Omega, T_o \pi_o(A)\Omega) = \omega_o(A^* \alpha_i(A)) \tag{5.22}$$

hence from the definition of a ground state

$$0 \leq T_o \leq 1 \quad . \tag{5.23}$$

Moreover, T_o has a densely defined inverse,

$$T_o^{-1} \pi_o(A)\Omega = \pi_o \alpha_{-i}(A)\Omega \quad , \ A \in \mathcal{F} \quad . \tag{5.24}$$

Thus we can define the global Hamiltonian by

$$H_o = -\ln T_o \tag{5.25}$$

and the real time translations by

$$\hat{\alpha}_t(\pi_o(A)) = e^{itH_o} \, \pi_o(A) \, e^{-itH_o} \quad . \tag{5.26}$$

If we insert t = in in (5.26) we find

$$\hat{\alpha}_{in} \circ \pi_o = \pi_o \circ \alpha_{in} \tag{5.27}$$

thus (5.26) is consistent with (5.14).

It is an open problem whether the time translations $\hat{\alpha}_t$ leave the norm closure of $\pi_o(\mathcal{F})$ or at least its weak closure invariant. We therefore have also to consider the probably larger algebra $\hat{\mathcal{F}}$ which is generated by $\hat{\alpha}_t \pi_\circ(\mathcal{F}),\ t \in \mathbb{R}$, and the analogously defined algebras $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$.

If one introduces the dynamics by the Hamiltonian method, i.e. by choosing a local expression for $H_{\underline{\Lambda}}$, e.g.

$$H_{\underline{\Lambda}} = A_{\underline{\Lambda}} + B_{\underline{\Lambda}} \tag{5.28}$$

one can construct the time evolution directly in the algebra. Moreover, this time evolution fulfils a relativistic causality condition (with maximal signal velocity) up to exponential tails [7]. Unfortunately, in this framework it is much more difficult to construct the ground state than in the Euclidean case. Therefore we prefer the Euclidean method.

A ground state of $\mathcal{F}$ with respect to α_i can be defined in terms of a Gibbs state of the Euclidean model. Let $\langle\ \rangle$ denote a Gibbs state of the gauge invariant Ising model in the temporal gauge and assume that $\langle\ \rangle$ fulfils Osterwalder-Schrader positivity (reflection positivity) for hyperplanes containing a lattice hyperplane or lying half between two neighbouring lattice hyperplanes. Define a state $\omega_\circ$ on $\mathcal{F}$ by

$$\omega_\circ\Big(\prod_n \alpha_{i_n}\Big(\prod_{\underline{x}\in \underline{M}_n} \sigma_3(\underline{x}) \prod_{\underline{b}\in \underline{L}_n} \tau_3(\underline{b})\Big)\Big) = \Big\langle \prod_{x\in M} \sigma(x) \prod_{b\in L} \tau(b)\Big\rangle \tag{5.29}$$

where $M = \bigcup_n \{n\}\times \underline{M}_n$, $L = \bigcup_n \{n\}\times \underline{L}_n$, $\underline{M}_n, \underline{L}_n$ being finite sets of sites and bonds in $\mathbb{Z}^d$, respectively. Then $\omega_\circ$ is a ground state of $\mathcal{F}$ with respect to α_i [31].

A Gibbs state with the properties mentioned above may be obtained as the limit of local Gibbs states with free boundary conditions. This limit always exists as a consequence of Griffith inequalities [32].

6. CHARGED STATES OF THE $\mathbb{Z}_2$ THEORY

We now want to find charged states of the model. By definition, a charged state is a state which cannot be represented by a vector in the vacuum Hilbert space and which has finite energy. The latter property means more precisely that in the GNS representation π induced by this state there is a positive bounded operator T - the transfer matrix in the representation π - such that

$$T\pi(A) = \pi\alpha_i(A)\,T \quad , \quad A \in \mathcal{F} \quad . \tag{6.1}$$

The idea for the construction of a charged state is simple. One creates a charge at some point $\underline{x}$ together with a compensating charge and transports the compensating charge to infinity. In a gauge theory the charges are connected by electric flux lines, so one has to arrange these flux lines in such a way that the limit state has finite energy.

A convenient choice of the flux lines is obtained in the following way (*). Let $\underline{x}_r = (2r, 0, \ldots, 0) \in \mathbb{Z}^d$, $r \in \mathbb{N}$, and Let $\underline{L}_r$ be the path along the 1-axis from the origin to $\underline{x}_r$. Let

$$\Phi_r = \sigma_3(\underline{0})\, \sigma_3(\underline{x}_r)\, T_\circ^{\,r}\, \tau_3(\underline{L}_r)\,\Omega \tag{6.2}$$

with $\tau_3(\underline{L}_r) = \prod_{\underline{b}\in \underline{L}_r} \tau_3(\underline{b})$.

The application of the r-th power of the transfer matrix to the string state vector $\tau_3(\underline{L}_r)\Omega$ suppresses its high energy components. Now consider the state

$$\omega_r(A) = \frac{(\Phi_r, A\,\Phi_r)}{\|\Phi_r\|^2} \quad , \quad A \in \mathcal{F} \quad . \tag{6.3}$$

ω_r is interpreted as a state where the charges have been separated by a distance 2r such that the energy remains bounded, independently of r. In fact, for $n \in \mathbb{N}$

$$\begin{aligned}\omega_r(e^{nH_\circ}) &\le \text{const}\ \frac{\|T_\circ^{\,r-n}\tau_3(\underline{L}_r)\Omega\|}{\|T_\circ^{\,r}\tau_3(\underline{L}_r)\Omega\|} \\ &= \text{const}\ \frac{\langle \tau(M_{2r,2(r-n)})\rangle^{1/2}}{\langle \tau(M_{2r,2r})\rangle^{1/2}}\end{aligned} \tag{6.4}$$

where $M_{k,l}$ denotes the rectangular loop in the (0-1)-plane in $\mathbb{Z}^{d+1}$ with side k in the 0-direction and l in the 1-direction. The perimeters of the loops in the numerator and the denominator differ by 4n, so it is plausible that the perimeter law for the Wilson loop which is known to hold in the whole region $\beta_h > 0$ because of Griffiths inequalities implies that $\omega_r(e^{nH_\circ})$ is bounded uniformly in r. (For a proof see [26] .)

(*) Another method which also leads to the construction of charged states has been invented by Szlachányi [33].

Using the convergent cluster expansions one can show that the sequence ω_r converges to a state ω in the free charge phase as well as in the screening/confinement phase. Let us first look at the free charge phase. The local charge operator is

$$Q_{\underline{\Lambda}} = \prod_{\underline{b} \in \partial^* \underline{\Lambda}} \tau_1(\underline{b}) \tag{6.5}$$

where $\partial^*\Lambda$ is the set of bonds with exactly one endpoint in $\underline{\Lambda}$. We find

$$\frac{\omega(Q_{\underline{\Lambda}})}{\omega_0(Q_{\underline{\Lambda}})} \longrightarrow -1 \quad , \quad \underline{\Lambda} \nearrow \mathbb{Z}^d \tag{6.6}$$

whereas for each local excitation of ω_0 ,

$$\omega_F(A) = \frac{\omega_0(F^* A F)}{\omega_0(F^* F)} \quad , F \in \mathcal{F} \quad , \tag{6.7}$$

one has

$$\frac{\omega_F(Q_{\underline{\Lambda}})}{\omega_0(Q_{\underline{\Lambda}})} \longrightarrow 1 \quad . \tag{6.8}$$

This supports the interpretation of ω as a charged state. A second indication that ω is not in the vacuum sector comes from the weak convergence of $\Phi_r \|\Phi_r\|^{-1}$ to zero. This can be shown for all (β_g, β_h) such that the pure gauge theory with coupling β_g satisfies the perimeter law and such that β_h is sufficiently small (dependent on the parameter in the perimeter law). The third property of the free charge phase which indicates the presence of charges is the existence of large vacuum fluctuations between electrical fluxes as discussed in Sect. 4. Let $\underline{\Lambda}_R$ be a cube with side length R and let S_ℓ and S_r denote the left and right half of the boundary $\partial^*\underline{\Lambda}_R$ of $\underline{\Lambda}_R$. Let

$$E_\ell = \prod_{\underline{b} \in S_\ell} \tau_1(b) \tag{6.9}$$

be the electric flux through S_ℓ and E_r the electric flux through S_r. Then $Q_{\Lambda_R} = E_\ell E_r$ and

$$\frac{\omega_0(E_\ell)\,\omega_0(E_r)}{\omega_0(Q_{\underline{\Lambda}_R})} \sim e^{-\alpha|\partial^*\underline{\Lambda}_R|} \quad , \tag{6.10}$$

$$|\partial^*\underline{\Lambda}_R| \sim R^{d-1} \quad .$$

Let us compare these results with the corresponding properties in the screening/confinement phase. There one finds a vector Φ in the vacuum Hilbert space $\mathcal{H}_0$ such that

$$(\Phi, A\,\Phi) = \omega(A) \quad . \tag{6.11}$$

Φ is obtained as the limit of $\sigma_3(\underline{o})\, T^r \sigma_3(\underline{o})\, \Omega \, \| T^r \sigma_3(\underline{o})\, \Omega \|^{-1}$ for $r \to \infty$. Thus ω is certainly not a charged state. For the charge operator one finds

$$\frac{\omega(Q_{\underline{\Lambda}})}{\omega_0(Q_{\underline{\Lambda}})} \longrightarrow 1 \quad , \quad \underline{\Lambda} \nearrow \mathbb{Z}^d \quad . \tag{6.12}$$

The sequence $\Phi_r \|\Phi_r\|^{-1}$ converges weakly to $(\Omega, \Phi)\Phi$ with $(\Omega, \Phi) \neq 0$. Especially

$$\frac{(\Omega, \Phi_r)}{\|\Phi_r\|} \longrightarrow (\Omega, \Phi)^2 \quad . \tag{6.13}$$

Last not least, the correlations of electric fluses are much weaker; one finds

$$\frac{\omega_0(E_\ell)\,\omega_0(E_r)}{\omega_0(Q_{\underline{\Lambda}_R})} \sim e^{-\alpha|\partial^* S_r|} \quad , \tag{6.14}$$

$$|\partial^* S_r| \sim R^{d-2} \quad .$$

If we formulate these results in the framework of the Euclidean theory we find three order parameters which seem to be suitable for the distinction of the free charge phase from the screening/confinement phase. The first one is the expectation value of the charge operator in the state ω :

$$\rho_1 = \lim_{R \to \infty} \frac{\langle \square_R \, \square'_R \rangle}{\langle \square_R \rangle \langle \square'_R \rangle} \tag{6.15}$$

where $\Box R$ means a square like Wilson loop with side length R and $\tilde{\Box} R$ means a dual loop (a loop in the dual theory in d+1 = 3 dimensions and a closed surface in the dual theory in d+1 = 4 dimensions) with side length R. We have $\varrho_1 = 1$ in the screening/confinement region and $\varrho_1 = -1$ in the free charge phase.

The second one measures the overlap of the vacuum with the approximate charged state $\Phi_r \|\Phi_r\|^{-1}$ in the limit $r \to \infty$:

$$\varrho_2 = \lim_{r \to \infty} \frac{\langle \overset{2r}{\sqcap} \rangle}{\langle \underset{2r}{\Box} \rangle^{1/2}} \tag{6.16}$$

We have $\varrho_2 > 0$ in the screening/confinement phase and $\varrho_2 = 0$ in the free charge phase.

The third one is sensitive to the correlations of electric fluxes:

$$\varrho_3(R) = \frac{\langle {}^{R}\tilde{\sqcap} \rangle \langle \tilde{\Box} R \rangle}{\langle \tilde{\Box} R \rangle} \tag{6.17}$$

If behaves like $e^{-\text{const } R^{d-1}}$ in the free charge phase and like $e^{-\text{const } R^{d-2}}$ in the screening/confinement phase. Note that in d+1 = 3 dimensions where the theory is selfdual, $\varrho_3(\infty)$ is the dual of ϱ_2.

All these order parameters may be used as confinement criteria in gauge theories with matter fields. For ϱ_2 this has been proposed in some detail in [34].

The order parameter ϱ_2 has also been tested in Monte Carlo simulations [35]. It shows the expected behaviour beyond the region of convergence of cluster expansions. There is, however, a region in the screening/confinement phase where the results are not yet conclusive. This on the first sight unpleasant fact has an interesting explanation. It is connected probably with the following behaviour of $\varrho_2(r)$ for finite r. For small r $\varrho_2(r)$ decreases in a similar way as in the free charge phase. Then, at a certain r it starts to increase again up to some finite value. This turning point r_f may be interpreted as the distance where fragmentation of the string sets in. It coincides with the transition from the area law to the perimeter law for the Wilson loop. Rough estimates indicate that r_f is very large in this region, thus one cannot see the asymptotic value of ϱ_2 on a relatively small lattice (22^{d+1} lattice points).

There have been several other attempts to find an order parameter which distinguishes the free charge phase from the screening/confinement phase [36, 37, 38]. In general they do not reproduce the known phase diagram; most of them indicate an artificial transition between the screening and the confinement region. There is one order parameter proposed by Bricmont and Fröhlich [36] which looks very similar as the order parameter ϱ_2. Bricmont and Fröhlich argue that the expectation value of a straight string

$$a(r) = \langle \overset{r}{\longrightarrow} \rangle \tag{6.18}$$

behaves like

$$a(r) \sim e^{-\text{const } r} \tag{6.19}$$

in the screening/confinement phase and like

$$a(r) \sim r^{-d/2} e^{-\text{const } r} \tag{6.20}$$

in the free charge phase. As a test which behaviour is present they propose to look whether the limit

$$\varrho_{BF} = \lim_{r \to \infty} \frac{a(r)^2}{a(2r)} \tag{6.21}$$

vanishes.
In the language of the quantum model, ϱ_{BF} is

$$\varrho_{BF} = \lim_{r \to \infty} \frac{(\Omega, \sigma(\underline{0}) T_0^{r} \sigma(\underline{0}) \Omega)^2}{\| T_0^{r} \sigma(\underline{0}) \Omega \|^2} = (\Omega, \Phi)^2 \tag{6.22}$$

with Φ from (6.11). Hence in the screening/confinement region from (6.13) ϱ_{BF} coincides with ϱ_2. In the free charge region, however, this seems unlikely. Namely, ϱ_{BF} vanishes if and only if the highest spectral value of the transfer matrix in the sector with external charge at the origin is an eigenvalue [26]. The corresponding eigenstate may be considered as a bound state of a dynamical charge with the external charge, i.e. it is the "hydrogen-atom" of this model. The existence of such bound states does not exclude in general the existence of isolated charged particles, hence the transition indicated by ϱ_{BF} probably does not coincide with the transition from the free charge phase to the screening/confinement phase. It would be very interesting to verify this conjecture. Some work in this direction has been done by Bricmont and Fröhlich [39].

7. PARTICLE STRUCTURE IN THE CHARGED SECTOR

We now want to investigate the particle structure of the present model more closely. Isolated particle shells in the joint spectrum of the transfer matrix and the translation operators have been found in the vacuum sector of several models. Schor proved their existence in strongly coupled pure gauge theories [40]. By his methods actually a rich class of stable particles was found [41, 42, 43, 44]. Another method has been developed by Bricmont and Fröhlich. They compute power corrections to the exponential decay of 2-point functions and derive the existence of particles from there [39].

A necessary condition for a corresponding proof in the charged sector is the construction of a transfer matrix and of translation operators in the charged representation. Let $(\mathcal{H},\pi,\Phi)$ be the GNS representation induced by ω (Thm. 2.1). Let $\Phi_{\underline{o}} = \pi(\sigma_3(\underline{o}))\Phi$ and let

$$\omega_{\underline{o}}(A) = (\Phi_{\underline{o}}, \pi(A)\Phi_{\underline{o}}) \quad , A \in \mathcal{F} \quad . \tag{7.1}$$

$\omega_{\underline{o}}$ is a state with an external charge at the origin. Moreover, $\omega_{\underline{o}}$ is invariant under α_i. The transfer matrix T in $\mathcal{H}$ is now defined by

$$T\pi(A)\Phi_{\underline{o}} = \pi\alpha_i(A)\Phi_{\underline{o}} \quad , A \in \mathcal{F} \quad . \tag{7.2}$$

T satisfies the relation

$$T\pi(A) = \pi\alpha_i(A)T \quad , A \in \mathcal{F} \tag{7.3}$$

and has the densely defined inverse

$$T^{-1}\pi(A)\Phi_{\underline{o}} = \pi\alpha_{-i}(A)\Phi_{\underline{o}} \quad , \quad A \in \mathcal{F} \, . \tag{7.4}$$

Moreover

$$0 \leq T \leq e^{\alpha} \tag{7.5}$$

where α is the parameter occuring in the perimeter law of the Wilson loop [26].

The lattice translations $\underline{x}$ act as automorphisms $\alpha_{\underline{x}}$ of the algebra $\mathcal{F}$,

$$\begin{aligned} \alpha_{\underline{x}}\sigma_i(\underline{y}) &= \sigma_i(\underline{y}+\underline{x}) \quad , \; i = 1,3 \\ \alpha_{\underline{x}}\tau_i(\underline{b}) &= \tau_i(\underline{b}+\underline{x}) \quad , \; i = 1,3 \end{aligned} \tag{7.6}$$

Let $\omega_{\underline{x}} = \omega_{\underline{o}} \circ \alpha_{\underline{x}}$. We have the following theorem:

<u>Theorem 7.1</u> [26] There is a unique vector $\Phi_{\underline{x}} \in \mathcal{H}$ such that

(i) $T\Phi_{\underline{x}} = \Phi_{\underline{x}}$

(ii) $(\Phi_{\underline{x}}, \pi(A)\Phi_{\underline{x}}) = \omega_{\underline{x}}(A)$

(iii) $(\pi(\sigma_3(\underline{x}))\Phi, \Phi_{\underline{x}}) > 0$

Now the translation operators can be defined by

$$U(\underline{x})\pi(A)\Phi_{\underline{o}} = \pi\alpha_{\underline{x}}(A)\Phi_{\underline{o}} \quad , A \in \mathcal{F} \quad . \tag{7.7}$$

They have the following properties:

<u>Theorem 7.2</u> [26]

(i) $U(\underline{x})U(\underline{y}) = U(\underline{x}+\underline{y})$

(ii) $U(\underline{x})\pi(A)U(-\underline{x}) = \pi\alpha_{\underline{x}}(A)$

(iii) $[U(\underline{x}), T] = 0$

(iv) $(\Psi, U(\underline{x})\Psi) \xrightarrow[|\underline{x}|\to\infty]{} 0$ for all $\Psi \in \mathcal{H}$

Thm 7.2 (iv) shows that the charged representation is really different from the vacuum representation. Namely, for the translation operators $U_o(\underline{x})$ in the vacuum Hilbert space $\mathcal{H}_o$, defined by

$$U_o(\underline{x})\pi_o(A)\Omega = \pi_o\alpha_{\underline{x}}(A)\Omega \tag{7.8}$$

one has instead of Thm 7.2 (iv)

$$(\Psi, U_o(\underline{x})\Psi) \xrightarrow[|\underline{x}|\to\infty]{} |(\Omega,\Psi)|^2 \tag{7.9}$$

for all $\Psi \in \mathcal{H}_o$. (For a more precise discussion see [26, Sect. 7].)

There are many open questions. The first one concerns the existence of an isolated mass shell in the joint spectrum of T and $U(\underline{x})$. It is conceivable that there are methods similar to those used by Schor or by Bricmont and Fröhlich by which one can show the existence of charged particles. Provided these single particle states exist one would like to construct multiparticle scattering states, i.e. to develop a lattice version of the Haag-Ruelle scattering theory [45, 46]. Here the lack of locality of the real time translation in the Euclidean lattice theory will cause some problems, and it may be easier to work in the Hamiltonian formalism (but there one would have to show first the existence of charged states).

The next question is whether these particles will have a well defined statistics, whether antiparticles exist and whether there is a global gauge group labeling the charge sectors. As mentioned in Section 3 these questions have a positive answer in the general framework of quantum field theory in continuous space time.

A very important question is whether the continuum limit exists and whether the charge structure survives in this limit. In this respect it is interesting that the $\mathbb{Z}_2$ Higg model in d+1 = 4 dimensions seems to have a second order phase transition between the free charge and the screening phase [47]. The existence of a second order phase transition is a necessary condition for the existence of a continuum limit.

There are many other lattice models where similar questions could be investigated. Some work has been done on the U(1) Higgs model. Barata and Wreszinski have shown that in some part of the screening/confinement region the expectation value of the charge operator in the state ω_r defined in Eq. (6.3) vanishes in the limit of large r [48, 49]. There is also some recent work by Brydges and Seiler [50] and of Kennedy and King [51] on the noncompact U(1) Higgs model which has been mentioned in the lecture of Prof. Wightman.

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