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EXACT RENORMALIZATION GROUP AS A SCHEME FOR CALCULATIONS* **

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0. INTRODUCTION

In this lecture I would like to report on recent work to use exact renormalization group methods to construct a scheme for calculations in quantum field theory and classical statistical mechanics on the continuum. "Exact" shall mean that use is made of convergent expansions only, without any uncontrolled approximations. "Scheme for calculations" shall mean that quantities of interest - such as correlation functions (Greens functions) - are obtained in the form of convergent series which are computable in the sense that the n-th terms is given by O(n)-dimensional integrals (as in Feynman perturbation theory). "Quantum field theory" shall mean quantum field theory models in Euclidean formulation on continuous space time without UV- or finite volume cutoffs. Some possible applications in other fields of physics (or engineering) will also be mentioned.

I will begin by recalling Wilson's renormalization group philosophy, followed by a brief review of recent results, by several groups of authors, on the question of existence of local quantum field theory models (without cutoffs). These results are based on use of convergent

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expansions, and so is our own work. I will therefore continue with some explanation of the nature of expansion methods in statistical mechanics, and recall the formalism of polymer systems. Then I will discuss how to reformulate a Euclidean quantum field theory on continuous space time as a theory living on a "staggered lattice", and how to apply expansion methods for lattice systems to it in order to obtain computable expansions for correlation functions of weakly coupled continuum theories. This part of the talk is based on joint work with A. Pordt [1]. In joint work with K. Pinn [2], techniques have been developped to combine the analytical techniques presented here with Monte Carlo methods, in order to apply them to theories that are asymptotically free but not weakly coupled.

An example of a quantum field theory model that has been extensively studied is $\lambda\phi^4$-theory. It requires to give meaning to, and compute, functional integrals of the form

$$\langle e^{J\phi}\rangle_{\mathcal{H}} = \mathcal{N}\int \mathcal{D}\phi\, e^{-\mathcal{H}(\phi)+\int J(x)\phi(x)d^\nu x} \equiv Z^{-1}\int d\mu_\nu(\phi)\, e^{-V(\phi)+\int J\phi\, d^\nu x}$$

with $\mathcal{D}\phi = \prod_{x\in\mathbb{R}^\nu} d\phi(x)$ and

$$\mathcal{H}(\phi) = \int d^\nu x \left[\tfrac{1}{2}(\nabla_\mu\phi(x))^2 + \lambda\phi(x)^4 + \tfrac{m}{2}\phi(x)^2 + \text{counter terms}\right] = \mathcal{H}_0(\phi) + V(\phi)$$

$d\mu_\nu(\phi)$ is the free field measure associated with $\mathcal{H}_0(\phi)$. The corresponding free propagator is the ν-dimensional Coulomb potential v.

1. K. WILSONS RENORMALIZATION GROUP [3]

(block spin method or "real space renormalization group")

A crucial notion is that of

Effective Hamiltonian (or action) & observables

Given a classical statistical mechanical system on the continuum

(or on a lattice $(a\mathbb{Z})^\nu$ with arbitrarily small lattice spacing a) - for instance a Euclidean field theory or a ferromagnet - with fields (spins) $\varphi(x)$, Hamiltonian $\mathcal{H}(\varphi)$, observables $O_i = O_i(\varphi)$, it

gets mapped into a lattice theory

on a lattice of lattice spacing a' with lattice fields $\Phi(x')$, Hamiltonian $\mathcal{H}'(\Phi) = \mathcal{H}_{eff}(\Phi)$, and observables $O_i'(\Phi)$, such that the expectation values

$$\langle O_i \rangle_{\mathcal{H}} = \langle O_i' \rangle_{\mathcal{H}'} \quad \text{exactly, and for all observables.} \qquad (1.0)$$

This is achieved by a

Blockspin transformation

for instance [4]

$$\Phi(x') = \underset{x \in x'}{av}\, \varphi(x) \quad \text{(average over hypercubes x' of side length a')} \qquad (1.1)$$

One regards x' as points of a lattice Λ of lattice spacing a'. Typically one is ultimately interested in a' = 0 (correlation length). The effective Hamiltonian $\mathcal{H}'$ is defined by

$$e^{-\mathcal{H}'(\Phi)} = \int \mathcal{D}\varphi\, e^{-\mathcal{H}(\varphi)} \prod_{x'} \delta\Big(\Phi(x') - \underset{x \in x'}{av}\, \varphi(x)\Big) \qquad (1.2)$$

Similarly, effective observables O_i' are defined by

$$O_i'(\Phi) e^{-\mathcal{H}'(\Phi)} = \int \mathcal{D}\varphi\, O_i(\varphi)\, e^{-\mathcal{H}(\varphi)} \prod_{x'} \delta \quad \text{(as above)} \qquad (1.2')$$

It is worth emphasizing [5], that the block spin variables need not be fields of the same kind as those in the original theory. For instance, in the case of an Ising model with variables $\sigma(z) = \pm 1$ one may work with real block spins. For the 2-dimensional planar rotator with variables s(z) one should include among the block fields topological variables n(x) which count the winding number of s(z) as z runs around the boundary of a square x. An illustrative example can be found in the proof of confinement for 3-dimensional U(1) lattice gauge theory by Göpfert and the author [6]. This example shows also that it is crucial to verify that a particular choice of block spin leads

to effective Hamiltonians that are local up to exponential tails. Otherwise one can get qualitatively wrong results. In the exact approach, such verification is part of the convergence proof. It is important for some applications that the block spin method is able to produce exact expressions for original continuum correlation functions and not only their long distance behavior. The prize to pay for this is that one must determine effective observables in addition to the effective Hamiltonian.

Question i) How can one calculate the (complicated) $\mathcal{H}_{eff}$?

ii) How could one do Monte Carlo simulations for the resulting lattice theory with such complicated $\mathcal{H}_{eff}$ (when necessary)?

Renormalization group (RG)

A block spin transformation a → a' (which maps a theory on a lattice spacing a into one with lattice spacing a') can be carried out through a sequence of block spin transformations as above. For instance

$$a \equiv a_N \to a_{N-1} \to a_{N-2} \to \cdots \to a_1 \to a_0 \equiv a' = L^N a \,; \quad \frac{a_j}{a_{j-1}} = L$$

This is so because the average of an average is an average,

$$\underset{x \in x''}{av}\, \varphi(x) = \underset{x' \in x''}{av}\; \underset{x \in x'}{av}\, \varphi(x) \,.$$

In this way a sequence of Hamiltonians is introduced

$$\mathcal{H} \equiv \mathcal{H}_N \to \mathcal{H}_{N-1} \to \cdots \to \mathcal{H}_1 \to \mathcal{H}_0 \equiv \mathcal{H}_{eff}$$

They live on a sequence of lattices Λ_j. By doing simple dimensional analysis, we may also regard them as living on lattices of lattice spacing 1 instead. The resulting sequence of Hamiltonians is called a (discrete) RG-flow.

A single RG-step $\mathcal{H}_j \to \mathcal{H}_{j-1}$, $\Lambda_j \to \Lambda_{j-1}$ is manageable,

at least if L is not too large and $\mathcal{H}_j$ admits a split

$$\mathcal{H}_j(\varphi) = \mathcal{H}_{free}(\varphi) + V_j(\varphi) \,, \qquad (1.3)$$

with a "sufficiently small" perturbation V_j - i.e. if the running coupling constants are not too big (yet) at length scale a_j. The field φ lives on Λ_j. Split φ into a term that is determined by φ's block averages Φ, and a remainder ζ called the "fluctuation field" [4]

$$\varphi(x) = \sum_{x' \in \Lambda_{j-1}} A(x,x') \Phi(x') + \zeta(x)$$
$$\equiv A\Phi(x) + \zeta(x) \quad , \quad x \in \Lambda_j \ .$$

The split is made unique by requiring that ζ has zero block average, and choosing the kernel A so that $\langle \Phi(x') \zeta(x) \rangle_{\mathcal{H}_{free}} = 0$. Integration over φ becomes integration over Φ and ζ. The defining equation of $\mathcal{H}_{j-1}$ has a δ-function in the integrand. Therefore the Φ-integral can be done, with the result [4]

$$e^{-\mathcal{H}_{j-1}(\Phi)} = \int d\mu_\Gamma(\zeta)\, e^{-V_\Phi(\zeta)}$$
$$= \text{partition function of auxiliary lattice field theory} \quad (1.4)$$
$$V_\Phi(\zeta) = V_j(A\Phi + \zeta) \ ,$$

The Gaussian measure

$$d\mu_\Gamma(\zeta) = \text{free field measure determined by propagator } \Gamma .$$

The fluctuation field propagator Γ is determined by A and the propagator ν associated with $\mathcal{H}_{free}$, its crucial property is that it decays exponentially [4] over distance a_{j-1}. As a result

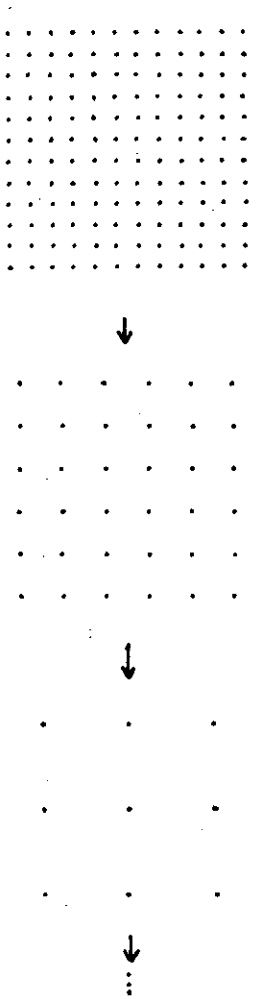


Figure 1

the auxiliary lattice field theory has short range correlations ($\approx$ L lattice spacing if $a_{j-1}/a_j = L$), at least if the system is weakly coupled at scale a_j - i.e. if V in eq. (1.3) is "small". This is the most fundamental ingredient of Wilsons philosophy. It is basic for us because weakly coupled lattice theories whose free propagator has correlation length of a few lattice spacings can be handled (including nonperturbative contributions) by expansion methods of classical statistical mechanics. To use such exact renormalization group transformations by way of expansions as a scheme for calculations will require

i) Iteration of renormalization group steps (∞ many for a continuum theory)

ii) The expansions for the individual steps must be formulated and combined so that the leading terms in the combined expansions are computable as low dimensional integrals.

More on that later ...

Existence of quantum field theory models [5] (on the continuum).

To obtain a theory on continuous space time, the limit $N \to \infty$ has to be taken so that the initial lattice spacing (UV-cutoff $^{-1}$)

$$a \equiv a_N = L^{-N} a_0 \longrightarrow 0 .$$

The initial Hamiltonian $\mathcal{H}_N$ depends on, and is determined by, bare coupling constants $g_u \equiv g(N)$. Therefore also $\mathcal{H}_j$ will depend on $g(N)$ viz. $\mathcal{H}_j = \mathcal{H}_j(N, g(N) | \varphi)$.

Renormalizability holds nonperturbatively if N-dependent $g(N)$ exists such that

$$\lim_{N \to \infty} \mathcal{H}_0(N, g(N) | \varphi) = \mathcal{H}_{eff}^{continuum}(\varphi)$$

exists and is nontrivial (not quadratic in φ). Evidence is accumulating that perturbative renormalizability is neither necessary nor sufficient for this.

Here is a list of models that have been shown to exist and obey nonperturbative renormalizability

pert. superrenormalizable [6-11] φ_2^4, φ_3^4, Yukawa$_2$, Abelian Higgs$_2$, Abelian Higgs$_3$ (and probably soon to come: Nonabelian pure Yang Mills$_3$ [Balaban])

pert. renormalizable: Gross-Neveu$_2$ [12, 13], $-\lambda\varphi_4^4$ [14]

pert. nonrenormalizable: Gross-Neveu$_{2+\varepsilon}$ [15]

The subcripts on the models indicate space time dimensionality. $-\lambda\varphi_4^4$ is φ^4-theory in 4 dimensions with the "wrong" sign of the coupling constant. It can be given meaning as a Euclidean field theory but lacks Osterwalder Schrader positivity so that analytic continuation to a bona fide quantum field theory in Minkowski space is not possible. This model is asymptotically free in the ultraviolet. A comparable achievement is the control of the infrared behavior of

massless φ_4^4-theory on a lattice [16, 17]

This model is asymptotically free in the infrared. In contrast, Gross-Neveu$_{2+\varepsilon}$ is not asymptotically free, but its ultraviolet behavior is governed by a nontrivial RG fixed point which approaches the trivial (Gaussian) one as $\varepsilon \to 0$. This fixed point is believed to be associated with a nontrivial conformal invariant QFT, and its existence was expected on the basis of the conformal bootstrap. [18]

Construction of the pert. renormalizable and nonrenormalizable models mentioned above represents last years main progress. There is hope that realistic models like 4-dimensional Yang Mills will also come under control in the not too distant future. [19] The most active groups of constructive field theorists working on these questions of existence are Balaban, Imbrie and Jaffe; Battle and Federbush; Feldmann, Magnen, Rivasseau and Seneor; Gallavotti and Nicoló, Kupiainen and Gawedzki. Gallavotti's review article will be helpful to the reader [20]. Federbush's approach has some similarities in spirit with our own [21]. Methods developped for disordered systems [22] are also relevant here because they help treating the socalled "large field problem". The pioneering work

of Benfatto at al. [23] on this was very influential [4], but we will follow another route. Concerning the confinement problem, see Ito's contribution [24].

2. POSSIBLE APPLICATIONS OUTSIDE QUANTUM FIELD THEORY AND CLASSICAL STATISTICAL MECHANICS

The need to convert continuum problems into lattice problems is ubiquitous because only lattice problems are suited for analysis by computer. We will mention two examples of problems where our method could be applied, thanks to the existence of a functional integral formulation.

i) Solution of classical Maxwell equations with boundary conditions. This is important e.g. in the design of microwave cavities, or of elementary particle accelerators. Let us for simplicity consider an electrostatic problem and look for the fundamental solution $G(x,y)$ of the potential equation with Dirichlet boundary conditions on the boundary Γ of a domain D (and Neumann boundary conditions for D')

$$(-\Delta_x + i\omega)\, G(x,y) = \delta(x-y) \quad \text{away from } \Gamma$$

$$G(x,y) = 0 \quad \text{for } x \in \Gamma,\ y \in D$$

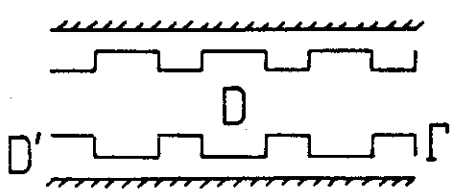


Figure 2

G admits a functional integral representation

$$G(x,y) = \langle \varphi(x)\varphi(y)\rangle_G = Z^{-1}\int \varphi(x)\varphi(y)\, e^{-\mathcal{H}(\varphi)}\, \mathcal{D}\varphi$$

$$\mathcal{H}(\varphi) = \tfrac{1}{2}\int d^\nu z\ \varphi(z)(-\Delta^\Gamma + i\omega)\varphi(z) \equiv \tfrac{1}{2}(\varphi, [-\Delta^\Gamma + i\omega]\varphi),$$

Δ^Γ = Laplacian with the specified boundary condition on Γ. Now split into a translation invariant free part $\mathcal{H}_0$ (e.g. on a torus) and an

"interaction" V that is concentrated on Γ [35]

$$\mathcal{H}(\varphi) = \mathcal{H}_0(\varphi) + V(\varphi) \quad \text{with} \quad \mathcal{H}_0(\varphi) = \tfrac{1}{2}(\varphi, -\Delta\varphi)$$

$$V(\varphi) = -\lim_{\varepsilon \to 0} \int_\Gamma d\sigma \left[\varphi(x+n\varepsilon)\, n\cdot\nabla\varphi(x)\right]$$

To facilitate the reader's understanding, let us mention the corresponding forulae in case we were to start from a lattice rather than continuum. In this case, Γ may be regarded as a set of directed links $\langle x,y\rangle$ (= pairs of nearest neighbours) normal to the "surface" Γ.

$$\mathcal{H}_0(\varphi) = a^{\nu-2} \sum_{\langle x,y\rangle} \frac{1}{4}\left[\varphi(x) - \varphi(y)\right]^2$$

(sum over ordered pairs of nearest neighbours)

$$V(\varphi) = a^{\nu-2} \sum_{\langle x,y\rangle \in \Gamma} \varphi(y)\left[\varphi(x) - \varphi(y)\right]$$

in ν dimensions, for lattice spacing a. $\langle x,y\rangle \in \Gamma$ only if $x \in D$, $y \in D'$.

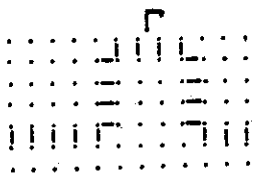


Figure 3

Let us return to the continuum theory. The effective action formalism will reexpress the exact Greens function G(x,y) as expectation value of a lattice theory.

$$G(x,y) = Z^{-1} \int O'_x(\Phi)\, O'_y(\Phi)\, e^{-\mathcal{H}'_0(\Phi) - V_{\text{eff}}(\Phi)} \prod_{x'} d\Phi(x')$$

compare the discussion in section 1. The idea is to handle this effective theory by computer, using for instance the method developped by K. Pinn and the author [2].

ii) Quantum statistical lattice models in ν dimensions. Such models can often be mathematically reformulated as classical statistical mechanics in $\nu+1$ dimensions with continuous periodic (Euclidean) "time" [26]. We propose to use our methods to discretize time by way of an exact transformation (as opposed to a discrete time approximation) and to handle the resulting effective theory by computer.

3. ON EXPANSION METHODS

Question: What do expansion methods of (classical) statistical mechanics really do?

Answer: They express observable quantities (e.g. free energy as function of external fields or sources) of an infinite system in terms of properties of small subsystems.

Expansion = sequence of approximate answers that are determined by partition functions of ("small") subsystems of increasing size.

Example: Virial expansion for a real gas [27]. The leading term (ideal gas) is known if the partition function for a single particle is known. The 1-st order correction involves 2-particle clusters and is known if the partition function of a system of 1 or 2 particles is known.

Different expansions are obtained by different choices of subsystems. For instance, in a lattice gas the subsystem might either consist of n = 1, 2, 3, ... particles, or it might consist of 1, 2, 3 ... sites that could be occupied.

A systematic procedure is provided by the theory of polymer systems [28]. Before proceeding let us emphasize again that we are only interested in convergent expansions, so that also nonperturbative effects are properly treated.

Polymer systems

The polymers considered here are a mathematical abstraction. To specify a polymer system one must, first of all, specify a set Λ of sites which may be occupied by the polymers - for instance the squares of a chessboard. Certain finite nonempty subsets P of Λ are declared to be polymers - for instance unions of squares that can be cut out of card board without falling apart. A polymer $P = \{x\}$ with only a single site x is called a monomer. Finally, a (real) activity A(P) is assigned to each polymer P - for instance $A(P) = \exp(-\beta n^4)$, n = no. sites in P.

Once a polymer system is defined in this way, partition functions of finite subsets X of Λ are defined*

*Here and everywhere, the symbol Σ is used for union of disjoint sets.

$$Z(X) = \sum_{X = \Sigma P} \prod_{P} A(P) \qquad (3.1)$$

Summation is over partitions of X into mutually compatible polymers P. In our applications, mutually compatible shall mean nonoverlapping.

It is important to note that the activity A(P) is uniquely determined by the partition functions Z(X) for $X \subsetneq P$. Proof: Suppose true for (no. of sites in P) $\leq n$. Consider a polymer X with n+1 sites. Then A(X) = Z(X) − Σ Π(activities of polymers with $\leq n$ sites) is determined by partition functions Z(Y) for $Y \subsetneq X$. □

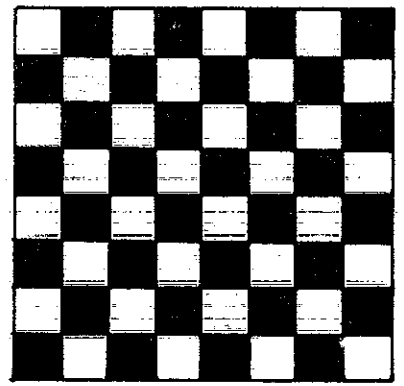
There exists a formula for the free energies

$$\ln Z(X) = \sum_{x \in X} \ln A(\{x\}) + \sum_{Q} a(Q) \prod_{P \in Q} \bar{A}(P) \, . \qquad (3.2)$$

The first term involves monomer activities. Sum over Q is over collections of not necessarily distinct polymers P such that the following graph is connected: Draw a vertex for each P and link two such vertices when the polymers are incompatible. $a(Q)$ are combinatorial coeffients [1, 29] and

$$\bar{A}(P) = A(P) / \prod_{x \in P} A(\{x\}) \, .$$

Convergence of the infinite series for $\ln Z(X)$ is assured if the activities A(P) satisfy suitable bounds [28, 29]. The convergence conditions are very restrictive - for instance they may be violated because the monomer activities $A(\{x\})$ are too small - and it is therefore a good idea to avoid taking logarithms by way of expansion (3.2) whenever possible. Instead one may proceed as follows [1].



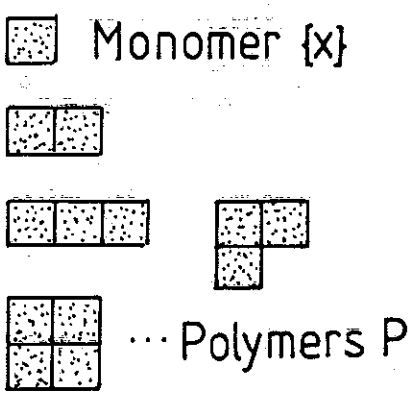


Figure 4

It is customary and convenient to consider partition functions as functions of external fields or sources ψ such that desired correlation functions are obtained as derivatives of Z

$$G(z_1, \ldots, z_n) = Z(\Lambda|0)^{-1} \frac{\delta}{\delta\psi(z_1)} \cdots \frac{\delta}{\delta\psi(z_n)} Z(\Lambda|\psi)\Big|_{\psi=0} \qquad (3.3)$$

By including in the definition of $Z(X|\psi)$ suitable (X-dependent but ψ-independent) normalization factors, it may be enforced that

$$Z(X|\psi=0) = 1 \qquad \text{for all } X \, .$$

This implies

$$A(P|\psi=0) = \delta_{1,n} \qquad (n = \text{no. sites in } P)$$

It follows now from the polymer representation (3.1) that

$$G(z_1, z_2) = \sum_{P} \frac{\delta}{\delta\psi(z_1)} \frac{\delta}{\delta\psi(z_2)} A(P|\psi=0) + \sum_{P_1, P_2 \text{ compat.}} \frac{\delta}{\delta\psi(z_1)} A(P_1|\psi=0) \frac{\delta}{\delta\psi(z_2)} A(P_2|\psi=0) \, . \qquad (3.4)$$

etc. Often $Z(X|\psi)$ depends on $\psi(z)$ only if $z \in X$. In this case the same is true of $A(X|\psi)$. It follows that the series for $G(z_1, z_2)$ converges on an infinite lattice if the series

$$\sum_{P \,:\, z_1 \in P} \frac{\delta}{\delta\psi(z_1)} A(P|\psi=0)$$

converges absolutely, together with its derivative with respect to $\psi(z_2)$.

Finally we need to mention one rather trivial generalization of a polymer system. One may admit different polymers that occupy the same set of sites, e.g.

Figure 5

The partition functions will only depend on the sum of their activities.

4. COMPUTABLE CONVERGENT EXPANSIONS FOR EUCLIDEAN FIELD THEORY ON CONTINUOUS SPACE TIME

The program to obtain them is as follows [1]

- Remember: There exist convergent weak coupling expansions for lattice field theories.
- Transform continuum field theory exactly into a field theory on a staggered lattice. The staggered lattice Λ is composed of an ∞ sequence of hypercubic lattices Λ_j of decreasing lattice spacing $a_j = L^{-j}a_o$ $(L > 1)$, $j = 0, 1, 2, \ldots$.
- In case of a weakly coupled model treat this lattice theory on Λ by expansion methods of classical statistical mechanics.
- For theories that are not weakly coupled but asymptotically free we envisage calculation of the Boltzmann-factor of an effective lattice theory on the (ordinary) lattice Λ_0 in the form of a polymer representation by use of the expansion methods of classical statistical mechanics. The further analysis of the lattice theory on Λ_0 would have to proceed by computers. Methods to do Monte Carlo simulations for such polymer systems were developped by K. Pinn and the author [2] but will not be described here.

The expansions are not power series expansions in the coupling constant λ . But they will have the following properties, for the weakly coupled $\lambda\phi^4$-theory-models of section 1.

- Computability: The n-th term is given by an O(n)-dimensional integral.
- Relation with perturbation theory: The remainder after the (n-1)-st term is $O(\lambda^{n\varepsilon})$, $\varepsilon = \frac{1}{4}$ or $\frac{1}{8}$.

The expansions are computable versions of phase cell cluster expansions. In principle, phase cell cluster expansions are obtained by iterating expansions for doing individual renormalization group steps. In practise they were independently introduced and developped in parallel with the renormalization group technology, by Glimm and Jaffe [7], Magnen and Seneor [8], Feldmann and Rivasseau [13], Federbush [19] and Battle [21], and others. Typically the phase cell cluster expansions used in work on existence questions consist of terms that are ∞ -dimensional functional integrals. For computability we insist on having instead O(n)-dimensional integrals for the n-th term [1]. Federbush's approach has some similarity in spirit with our own, but does not use block spin variables.

A single renormalization group step $a_j \to a_{j-1}$ involves expansions on the lattice Λ_j . Compounding all the steps, one is naturally led to expansions on the staggered lattice $\Lambda = \Lambda_0 + \Lambda_1 + \Lambda_2 + \ldots$

5. THE STAGGERED LATTICE [1]

Pedagogical prelude

Consider divisions of the real interval 0 ...1 into intervals $n \cdot 2^{-k} \ldots (n+1)2^{-k}$ of length 2^{-k} $(n = 0 \ldots k)$. Do so for $k = 0, 1, 2, \ldots$ Sufficiently good functions F(z) of the real variable $z \in [0 \ldots 1]$ admit a (unique) representation as sums of functions f_k that are constant on the intervals of length 2^{-k}, and have zero average over the next larger intervals (of length 2^{-k+1}) if $k \neq 0$. Let us reserve the letter x to label intervals, and let Λ be the countable set of all such intervals x (of arbitrary length). The intervals of length 2^{-k} are said to form the layer Λ_j of the staggered lattice Λ . Set $f(x) = f_j(x)$ when $x \in \Lambda_j$, and write $\chi_x(z)$ for the characteristic function of interval x. Then the decomposition looks as follows

$$F(z) = \sum_{x \in \Lambda} \chi_x(z) f(x) \ .$$

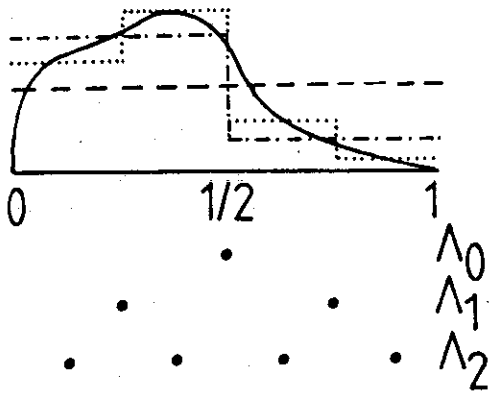


Figure 6

The figure on top shows successive approximations to F obtained by restricting the sum over x to layers Λ_j with $j \leq k = 0, 1, 2, \ldots$. The characteristic function $\chi_x(z)$ of interval x could be replaced by other kernels $A(z,x)$, e.g. by smooth approximations to step function. This construction is used in the theory of the Fourier transform, and the block spin method of the renormalization group appears to have an ancestor there in Littlewood Paley theory [30].

The staggered lattice

Consider now a ν-dimensional continuum $\mathbb{R}^\nu$, called the base space (base). Choose a_0 and an integer $L > 1$, and decompose base space into hypercubes of sidelength $L^{-k}a_0$ similarly as before. Do so for $k = 0, 1, 2, \ldots$. Let Λ be the countable set of all hypercubes x obtained in this way. Regard the hypercubes x as points of a staggered lattice Λ whose layers Λ_j are made of the cubes of side length $a_j = L^{-j}a_0$. The same construction can be carried through for a base space that is a lattice of lattice spacing $a = a_N = L^{-N}a_0$. In this case there is a finite number N of layers. $N \to \infty$ in the continuum limit.

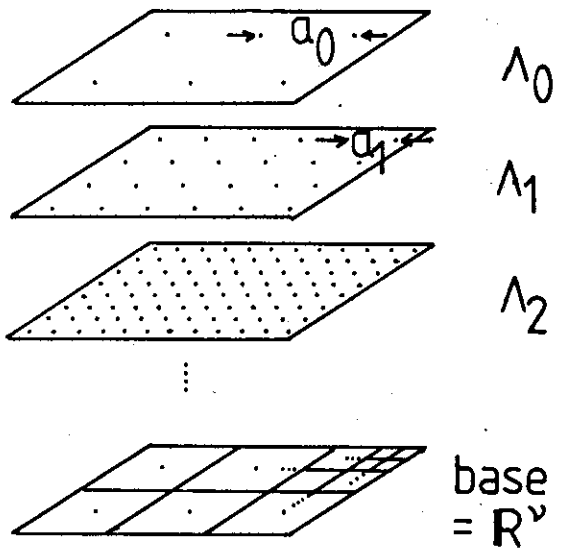


Figure 7

It is convenient to introduce an "integration" over the staggered lattice

$$\int_{x\in\Lambda} (\cdots) = \sum_{j=0}^{N} a_j^\nu \sum_{x\in\Lambda_j} (\cdots) \qquad \text{in } \nu \text{ dimensions.} \tag{5.1}$$

Now we may decompose fields and propagators on base space. Points on base space shall be denoted by z, while x is reserved for points on the staggered lattice.

$$\phi(z) = \sum_{j=0}^{N} \int_{x\in\Lambda_j} A(z,x)\varphi^j(x) \equiv \int_{x\in\Lambda} A(z,x)\varphi(x) \tag{5.2}$$

The decomposition is unique when it is required that the average of φ^j over hypercubes $y \in \Lambda_{j-1}$ vanishes, for $j \geq 1$. [The measure $d\mu_\nu(\varphi)$ is concentrated on such fields φ because the average $a\mu_{x\in y} v(x,x') = 0$]. Write $\langle , \rangle_\nu$ for expectation value with respect to a free field theory

(on base space) with propagator v. Regard φ as determined by ϕ. The kernels $\mathcal{A}$ can be so chosen that there is no correlation between different layers in a free field theory [1, 4], viz.

$$\langle \varphi(x)\varphi(y)\rangle_v = \delta_{jk}\, v(x,y) \quad \text{for } x\in\Lambda_j,\ y\in\Lambda_k$$

Since $\langle \phi(z_1)\phi(z_2)\rangle_v = v(z_1,z_2)$ it follows that

$$v(z_1,z_2) = \sum_j \iint_{x_1,x_2\in\Lambda_j} \mathcal{A}(z_1,x_1)\, v(x_1,x_2)\, \mathcal{A}(z_2,x_2)$$
$$\equiv \iint_{x_1,x_2\in\Lambda} \mathcal{A}(z_1,x_1)\, v(x_1,x_2)\, \mathcal{A}(z_2,x_2) \qquad (5.3)$$

In shorthand and graphical notation

$$\phi = \mathcal{A}\varphi \quad , \quad v = \mathcal{A} v \mathcal{A}^* \qquad (5.4)$$

z_1 ——— z_2 = z_1 ~~~ x_1 x_2 ~~~ z_2

Semigroup of symmetries.

The staggered lattice Λ admits a semigroup S of symmetries (maps) compounded from

1) translation by $\vec{n}\cdot a_0$
2) rotations by $\frac{\pi}{2}$ around axes of the cubic lattice Λ_0
3) dilations by L^{-k}, $k = 0, 1, 2, \ldots$

One might want to call Λ a "semicrystal" since it has long range order and a semigroup of symmetries [31].

We shall have occasion to consider polymer systems on the staggered lattice Λ. The elements α of the semigroup S can also act on subsets P (polymers) of Λ. Their dual α^* acts on activities A of polymers on Λ

$$(\alpha^* A)(P) = A(\alpha P)$$

In quantum field theory, the activities A will depend on parameters g (renormalized coupling constants) that determine the theory: $A(P|g)$. The action of α^* associated with dilations α determines the Gell-Mann Low renormalization (semi) group [32] of transformations $g \to g_\alpha$

$$(\alpha^* A)(P|g) = A(P|g_\alpha)$$

The running coupling constants g_α may be regarded as functions of the length scale μ^{-1} determined by α, viz $\mu^{-1} = a_j$ if $\alpha\Lambda_0 = \Lambda_j$.

6. POLYMER SYSTEMS ON THE STAGGERED LATTICE

Finally we are ready to specify our expansions. Consider for instance a weakly coupled $\lambda\phi^4$-theory, on 2- or 3-dimensional continuous space time ("superrenormalizable case") or on a 4-dimensional lattice but with zero physical mass ("renormalizable case") - cp. section 1. The free propagator is denoted by v. The free-propagator-amputated Greens functions

$$G(z_1,\ldots,z_n) = Z(0)^{-1} \frac{\delta}{\delta\psi(z_1)} \cdots \frac{\delta}{\delta\psi(z_n)} Z(\psi)\Big|_{\psi=0} \qquad (6.1)$$

are determined by the partition functions $Z(\psi)$ of the theory in the presence of a background field ψ [33]. Normalization constants are understood to be chosen so that $Z(0) = 1$.

We will define partition functions $Z(X|\psi)$ for finite subsets X of the staggered lattice Λ in such a way that

$$Z(\psi) = \lim_{X\nearrow\Lambda} Z(X|\psi) \quad \text{and} \quad Z(X|0) = 1 \,. \qquad (6.2)$$

We know from our earlier discussion of expansion methods that the partition functions $Z(X|\psi)$ with $X \subseteq P$ will determine the activity $A(P|\psi)$ of a polymer system on Λ. The Greens functions $G(z_1,\ldots,z_n)$ can be represented as sums of products of derivatives of (up to n) activities $A(P_i|\psi)$, cp. eq. (3.4). For a finite staggered lattice Λ (finite volume in base space and finite UV cutoff a_N^{-1}) the sum is a finite sum. Its convergence in the no-cutoff limit requires a careful choice of the X-dependent partition functions $Z(X|\psi)$ including "proper renormalization". More on this below. Now we turn to the definition of partition functions $Z(X|\psi)$. Given v, we define X-dependent propagators v_X on the staggered lattice by

$$v_X(x,y) = \begin{cases} v(x,y) & \text{if } x,y \in X \\ 0 & \text{otherwise} \end{cases} \tag{6.3}$$

The Gaussian measure $d\mu_{v_X}(\varphi)$ is supported on fields φ on Λ with $\varphi(x) = 0$ for $x \notin X$. Therefore $\int d\mu_{v_X}(\varphi) f(\varphi)$ is an n-fold integral over n real variables $\varphi(x)$, $x \in X$ when X has n points. [For an explicit formula see eqs. (3.12) - (3.15) of ref. 1.]

Next we define interaction Boltzmannians. For a theory in which only mass and vacuum energy counterterms are needed it takes the form

$$\mathcal{Z}_1(v,\phi) = \exp\left(\delta e(v) + \sum_{z \in \text{base}} \left[-\lambda\phi(z)^4 + \tfrac{1}{2}\delta m^2(v|z)\phi(z)^2\right]\right) . \tag{6.4}$$

In 4 dimensions, wave functions and coupling constant renormalization counter terms have to be added. The counter terms will have to depend on the free propagator v as is the case in Feynman perturbation theory. Therefore also $\mathcal{Z}_1(v,\phi)$ will depend on v. When the propagator is cut off by substitution of v_X for v, the same substitution will have to be made in the counter terms in order to maintain "proper renormalization". If v is not translation invariant, $\delta m^2(v|z)$ may depend on z. The crucial fact is that the <u>propagator</u> $v(x,y)$ <u>on the staggered lattice</u> propagates only "horizontally" and <u>decays exponentially with decay length L lattice spacings</u> in each layer. The decay of correlations in the "vertical" direction (which come from the interaction $\lambda\phi^4$) is achieved by proper renormalization.

Note that the interaction $\phi(z)^4$, when rewritten in terms of fields φ, involves a product of fields $\varphi(x_1) \ldots \varphi(x_4)$ on four layers $\Lambda_{j_1} \cdots \Lambda_{j_4}$ which may be different. So it is nonlocal on Λ. In this respect the situation is different from an ordinary lattice field theory.

The natural choice for an X-dependent partition function might appear to be of the form

$$Z(X|\psi) = \int d\mu_{v_X}(\varphi)\, \mathcal{Z}_1(v_X, \phi+\psi) \tag{6.5}$$

with φ and ϕ related by eq. (5.2), and $v_X = \mathcal{A} v_X \mathcal{A}^*$.

However, this is not a legitimate expression for a partition function of a polymer systen on Λ because the condition $Z(\emptyset|\psi)=1$ is violated when $\psi \neq 0$. There are different ways to remedy this. One could divide by a X-independent normalization factor, as was done in ref. 1. Here we will describe an alternative which leads to a computable variant of the phase cell cluster expansions of Magnen and Seneor [8, 17, 34]. Given a field ϕ on base space with decomposition (5.2), we define ϕ_X so that it vanishes when $X = \emptyset$, as follows.

Let us first consider a slightly simplified $\lambda\phi^4$-model whose free propagator v is such that it admits representation (5.3) with kernels $\mathcal{A}$ that are simple step functions

$$\mathcal{A}(z,x) = a_j^{-\nu} \chi_{[x]}(z) \tag{6.6a}$$

In this case we set

$$\phi_X(z) = \int_{x \in X} \mathcal{A}(z,x)\varphi(x)$$

and (6.6b)

$$v_X = \mathcal{A} v_X \mathcal{A}^*$$

The appropriate definition for the general case will be given below.

The staggered lattice has a hierarchical structure. Since points $x \in \Lambda$ are also cubes in base space, the inclusion relation $\subseteq$ is defined and supplies a partial order relation on Λ. A subset $X \subseteq \Lambda$ will be called $\subseteq$-convex if $x \in X$, $y \in X$, $x \subseteq u \subseteq y$ implies $u \in X$.

We define partition functions $Z(X|\psi)$ for $\subseteq$-convex subsets X of Λ first.

$$Z(X|\psi) = \int d\mu_{v_X}(\varphi)\, \mathcal{Z}_1(v_X, \phi_X + \psi_X) \tag{6.7}$$

The fields φ and ϕ_X are related as above, and the interaction Boltzmannian $\mathcal{Z}_1$ was defined before. We may imagine taking the limit $X \nearrow \Lambda$

through $\subsetneq$-convex X. In this way the Greens functions (6.1) become defined.

The definition of partition functions is extended to arbitrary finite X as follows. A subset $X \subset \Lambda$ will be called $\subsetneq$-connected if the following graph is connected: Draw a vertex for every point $x \in X$, and a directed link from x to y when $x \in \Lambda_j$, $y \in \Lambda_{j-1}$ and $x \subsetneq y$. Arbitrary $X \subset \Lambda$ decompose into connected components X_i. They are $\subsetneq$-convex. For general X with $\subsetneq$-connected components X_i we set

$$\mathcal{Z}(X|v,\phi) = \prod_i \mathcal{Z}_1(v_X, \phi_{X_i}) \quad \text{and} \quad Z(X|\psi) = \int d\mu_{v_X}(\varphi)\, \mathcal{Z}(X|v, \phi+\psi) \tag{6.8}$$

It is easy to verify that this is consistent with the above definition for $\subsetneq$-convex sets X. Evidently $Z(\emptyset|\psi) = 1$ ($\emptyset$ = empty set).

According to the discussion in section 3, definition of partition functions $Z(X|\psi)$ for arbitrary finite $X \subset \Lambda$ specifies the activities $A(P|\psi)$ of a polymer system on Λ such that

$$Z(X|\psi) = \sum_{X = \Sigma P} \prod_P A(P|\psi) \tag{6.9}$$

Arbitrary finite subsets P of X are admitted as polymers. We imposed the condition $Z(X|0) = 1$. It follows that the Greens functions $G(z_1,\dots,z_n)$ have an expansion in sums of products of up to n polymer activities, cp. eq. (3.4) of section 3.

The terms in this expansion are given by finite dimensional integrals. But to fulfill also our second requirement that the n-th term in the expansion is $O(\lambda^{n\varepsilon})$, we will still need to split and reorder the terms in the expansion.

The activities $A(P|\psi)$ are represented as sums of terms that are labelled by collections $(C_1, \dots, C_k)$ $(k \geq 1)$ of subsets C_i of P, called cores, whose convex hulls $\bar{C}_i$ are disjoint and have P as their union

$$A(P|\psi) = \sum_{\Sigma \bar{C}_i = P} \check{A}(C_1,\dots,C_k|\psi) \tag{5.10}$$

We interpret $A(C_1 \dots C_k|\psi)$ as activities of different polymers that

occupy the same set of sites $P = \Sigma \bar{C}_i$, cp. end of section 2. $A(C_1 \dots C_k|\psi)$ will be given by integrals over (up to) n real variables $\varphi(x)$, $x \in C_i$ if $C_1 \dots C_k$ together have n points. Inserting this decomposition, one obtains a representation of the Greens functions $G(z_1 \dots z_n)$ as a sum of products of (up to m) polymer activities $A(C_1 \dots C_k|\psi)$. The cores C_i appearing as arguments in such a product will have nonintersecting convex hulls. We order the terms in the sum in order of total number n of points in the cores. The sum over polymer-cores C_i with given number n of points can be written as n-fold integral over the staggered lattice. These integrals replace the integrals over space time in Feynman perturbation theory and make up the n-th term in the expansion. Step functions are inserted in the integrands to take care of the restriction that polymer cores should have disjoint convex hulls. The somewhat technical explanation of the precise way in which the split (6.10) is performed will be furnished below.

For the usual choice of propagator v, the definition of partition functions etc. can be retained, but the definition of ϕ_X and v_X is adapted as follows.

A subset Y of a layer Λ_j of the staggered lattice determines a subset Y_b of base space. It consists of the union of cubes $x \in Y$. We write $\chi_Y(z)$ for the characteristic function of this set ($\chi_Y(z) = 1$ for $z \in Y_b$, and = 0 otherwise). Using the shorthand notation (5.4) we define

$$\phi_X(z) = \mathcal{A}_X \varphi(z) \equiv \int_{x \in \Lambda} \mathcal{A}_X(z,x)\, \varphi(x)$$

and

$$v_X = \mathcal{A}_X v_X \mathcal{A}_X^* \tag{6.6c}$$

with

$$\mathcal{A}_X(z,x) = \chi_{X \cap \Lambda_j}(z)\, \mathcal{A}(z,x) \quad \text{for } x \in \Lambda_j .$$

The resulting expressions for partition functions $Z(X|\psi)$ have some subtle

aspects. The definition of the propagator v_X involves a characteristic function (step function) on the staggered lattice. This choice is made in order to achieve computability. On the other hand, the definition of ϕ_X involves a characteristic function on base space. This matches with the locality of the interaction $\phi(z)^4$ in the original base space formulation.

In the simplified $\lambda\phi^4$-model with kernels A that are step functions (6.6a), definitions (6.6c) agree with (6.6b). For this slightly simplified $\lambda\phi^4$-model, our expansions reduce to a simplified version of the phase cell cluster expansions of Feldmann, Magnen, Seneor and Rivasseau [17]. Their convergence is known both for the superrenormalizable case and for the renormalizable case.

For the usual choice of free propagator v, A is not exactly a step function, but $A(z,x)$ decays exponentially with the distance of z from hypercube $x \in \Lambda_j$, with decay length on lattice spacing a_j [4]. Work is still in progress at the time of this writing to extend the results on convergence to this case. It is clear, however, that this extension of convergence proof requires no more than handling tedious technicalities.

The same expansions can be obtained by iterated renormalization group transformations. In brief, this goes as follows. The effective interaction Boltzmannian $e^{-V_j(\Phi)}$, which depends on a field on Λ_j, is obtained in the form of a polymer representation on $\Lambda_{\geq j} = \Lambda_N + \ldots + \Lambda_j$. One begins with $\Lambda_{\geq N} \equiv \Lambda_N$ ($N \to \infty$ in the end). Through expansion steps associated with successive renormalization group transformations, the polymers and their activities are constructed by an iterative procedure, layer by layer. The normalization condition $Z(X|0) = 1$ is used to get rid of finished polymers.

Let me finally give the definition of $\check{A}(\ldots)$, assuming eq. (6.6a). Basically, activities A are given by certain integrals performed with Gaussian measures. One performs a Taylor expansion with remainder in the integrands to first order in $\varphi(x), v(x,\cdot)$. Introduce integer

variables $s(x)=0,1$ attached to points $x \in \Lambda$. Given s, let $S = \{x \in \Lambda \mid s(x)=1\}$ Observe that $(v_S)_X = v_{S \cap X}$ and $(\phi_S)_X = \phi_{S \cap X}$. Define auxiliary s-dependent Boltzmannians

$$\mathcal{Z}_s(X|v,\phi) = \mathcal{Z}(X|v_S,\phi_S)$$

There will be corresponding partition functions Z_s and polymer activities A_s.

To construct the polymer activities A associated with the partition functions Z, one may start with a polymer representation for the Boltzmann factors $\mathcal{Z}$

$$\mathcal{Z}_s(X|v,\phi) = \sum_{X=\Sigma M} \prod_M \mathcal{B}_s(M|v,\phi)$$

Integrate this to get an expression for $Z_s = \int d\mu_v \mathcal{Z}_s$. Expressing the activities A_s in terms of the partition functions Z_s produces a representation of the form

$$A_s(P|\phi) = \sum_k \sum_{P=\Sigma_1^k M_i} \hat{A}_s(M_1,\ldots,M_k|\phi)$$

Each term depends on $s(x)$ for $x \in P$ only. For $C \subset \Lambda$, $y \in \Lambda$ set

$$s_C(x) = \begin{cases} s(x) & x \in C \\ 0 & \text{otherwise} \end{cases} \qquad (y^s)(x) = \begin{cases} s(x) & x \neq y \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Delta_y f_s = f_s - f_{y^s}$$

Let $C_i \subseteq M_i$ and $C = \Sigma_1^k C_i$. The formula for the desired quantities $\check{A}$ reads

$$\check{A}(C_1,\ldots,C_k|\phi) = \prod_{x \in C} \Delta_x \hat{A}_{s_C}(M_1,\ldots,M_k|\phi)_{s \equiv 1}$$
when $\bar{C}_i = M_i$

[The right hand side of this expression vanishes, unless either $C = \Sigma C_i$, $\bar{C}_i = M_i$ for all i, or k = 1, M_1 is a monomer, and $C = \phi$. In the last case it equals 1. This comes about because $\mathcal{B}_s(M|v_Y,\phi_Y) = \delta_{1,n}$

if $M \setminus Y$ contains an extremal point, n = No. of points in M. If some M_i are monomers with $C_i = \not\Phi$, the truncated expectation value vanishes unless k = 1. The restriction in the sum (6.10) to k-tuples $C_1 \ldots C_k$ whose convex hulls fill P originates from this fact.]

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