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The differential graded Verlinde Formula and the Deligne Conjecture

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A modular category $\mathcal{C}$ gives rise to a differential graded modular functor which assigns to the torus the Hochschild complex and, in the dual description, the Hochschild cochain complex of $\mathcal{C}$. On both complexes, the monoidal product of $\mathcal{C}$ induces the structure of an E_2-algebra, to which we refer as the *differential graded Verlinde algebra*. At the same time, the modified trace induces on the tensor ideal of projective objects in $\mathcal{C}$ a Calabi-Yau structure so that the cyclic Deligne Conjecture endows the Hochschild cochain and chain complex of $\mathcal{C}$ with a second E_2-structure. Our main result is that the action of a specific element S in the mapping class group of the torus transforms the differential graded Verlinde algebra into this second E_2-structure afforded by the Deligne Conjecture. This result is established for both the Hochschild chain and the Hochschild cochain complex of $\mathcal{C}$. In general, these two versions of the result are inequivalent. In the case of Hochschild chains, we obtain a block diagonalization of the Verlinde algebra through the action of the mapping class group element S. In the semisimple case, both results reduce to the Verlinde formula. In the non-semisimple case, we recover after restriction to zeroth (co)homology earlier proposals for non-semisimple generalizations of the Verlinde formula.

Contents

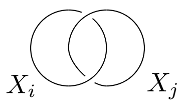
1 Introduction and summary

For any fusion category over an algebraically closed field k of characteristic zero, the k-vector space spanned by the isomorphism classes $[x_0], [x_1] \dots, [x_n]$ of its simple objects becomes an associative and unital algebra by means of the monoidal product: By semisimplicity, we have a decomposition $x_i \otimes x_j \cong \bigotimes_{\ell=0}^n N_{ij}^\ell x_\ell$ of $x_i \otimes x_j$ into a direct sum over the basis of simple objects, in which x_ℓ occurs with multiplicity N_{ij}^ℓ, a non-negative integer. These fusion rules allow us to write the multiplication explicitly as

$$[x_i] \otimes [x_j] = \sum_{\ell=0}^n N_{ij}^\ell [x_\ell] \ . \tag{1.1}$$

By a slight abuse of notation, the symbol $\otimes$ will also used for the multiplication. The class $[I]$ of the monoidal unit I (which by convention is the zeroth object x_0 in the list of simple objects) is the unit of the multiplication. The resulting algebra is called the *Verlinde algebra* of the fusion category (one can also see it as the linearized version of the Grothendieck ring or the K_0-ring of $\mathcal{C}$).

New tools for the computation of the fusion coefficients N_{ij}^ℓ become available when $\mathcal{C}$ is a *semisimple modular category*, i.e. additionally has a non-degenerate braiding and a ribbon structure (we recall the terminology in more detail in a moment, see page 4). Modular categories form an important class of categories in representation theory and conformal field theory [Tur94, KLM01, Hua08a, Hua08b, EGNO17]. In this case, the famous *Verlinde formula* conjectured by Verlinde [Ver88] and proven by Moore and Seiberg [MS90], Witten [Wit89] and Turaev [Tur94] expresses the fusion coefficients N_{ij}^ℓ via the *S-matrix*, an invertible $(n+1) \times (n+1)$-matrix whose (i,j)-entry is given by the evaluation of the graphical calculus of $\mathcal{C}$ [RT90, RT91, Tur94] on the Hopf link labeled by the two simple objects x_i and x_j:

$S_{ij} :=$ 

$\in k$ (1.2)

Now the Verlinde formula asserts

$$N_{ij}^\ell = \sum_{p=0}^n \frac{S_{ip} S_{jp} \left(S^{-1}\right)_{p\ell}}{S_{0p}} \in k \ . \tag{1.3}$$

It should be mentioned that (1.3) is only one of several incarnations of the Verlinde formula.

The Verlinde formula (1.3) relies on semisimplicity. Nonetheless, a lot of the ingredients above can be given sense beyond semisimplicity such that aspects of the Verlinde formula still hold. Proposals in this direction have been given in [FHST04, FGST06, GR19, GR20], see [LO17, GLO18, CGR20] for examples of modular categories which are not semisimple.

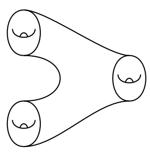
One of the key differences between semisimple and non-semisimple finite tensor categories is that, in the non-semisimple case, the homological algebra of tensor categories enriches the picture: For instance, the (Hochschild) cohomology of finite tensor categories has been studied e.g. in [GK93, EO04, MPSW09, Bic13, NP18, LQ19] (this refers mostly, but not exclusively to the Hopf algebraic case). Multiplicative structures have been investigated in [FS04, Men11, Her16]. More recently, the interaction of this homological algebra with low-dimensional topology has been developed in [LMSS18, SW19, LMSS20, SW20].

The purpose of this article is to understand the content of the Verlinde formula within a differential graded framework. This framework will feature the relevant quantities appearing in the homological algebra of a modular category and the higher structures that they naturally come equipped with.

Since this generalization can be best understood and proven as a topological result, it will be beneficial to recall the topological underpinning of the semisimple Verlinde formula. Indeed, a topological viewpoint already informed [Ver88]. The viewpoint presented here is mostly due to [Wit89, Tur94]: If $\mathcal{C}$ is a semisimple modular category, then $\mathcal{C}$ gives rise to a three-dimensional topological field theory $Z_\mathcal{C}$ by the Reshetikhin-Turaev construction [RT90, RT91]. In fact, semisimple modular categories are equivalent to once-extended three-dimensional topological field theories by a result of Bartlett, Douglas, Schommer-Pries and Vicary [BDSPV15]. The topological field theory $Z_\mathcal{C}$ assigns to the torus $\mathbb{T}^2$ the vector space spanned by the isomorphism classes of simple objects of $\mathcal{C}$;

$$Z_\mathcal{C}(\mathbb{T}^2) \cong k\,[\ [x_0], [x_1], \dots, [x_n]\ ]\ .$$

Since every mapping class group element can be seen as an invertible three-dimensional bordism, the vector space $Z_\mathcal{C}(\mathbb{T}^2)$ comes with an action of the mapping class group $\mathrm{SL}(2, \mathbb{Z})$ of the torus. Generally, the bordism category needs to be twisted by a 2-cocycle to account for the so-called *framing anomaly*; as a result, the mapping class group actions will only be projective instead of linear (for the torus, one can make the action linear, but this would require making choices that would lead to problems elsewhere). The multiplication (1.1) induced by the monoidal product can be obtained by the evaluation of $Z_\mathcal{C}$ on the three-dimensional bordism

$P \times \mathbb{S}^1 =$  $: \ \mathbb{T}^2 \times \mathbb{T}^2 \longrightarrow \mathbb{T}^2$

where $P : \mathbb{S}^1 \sqcup \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is the two-dimensional pair of pants bordism. Note that this treats the two $\mathbb{S}^1$-factors of the torus differently: While on the first factor two copies of the circle are fused together via the pair of pants, the second factor is just a spectator. This disparity turns out to be responsible for the usefulness of the Verlinde formula: As a result of treating the $\mathbb{S}^1$-factors differently, the multiplication

$$Z_\mathcal{C}(P \times \mathbb{S}^1) : Z_\mathcal{C}(\mathbb{T}^2) \otimes Z_\mathcal{C}(\mathbb{T}^2) \longrightarrow Z_\mathcal{C}(\mathbb{T}^2)$$

is maximally incompatible with the action of the mapping class group $\mathrm{SL}(2, \mathbb{Z})$ of the torus on the vector space $Z_\mathcal{C}(\mathbb{T}^2)$ meaning that, except for trivial cases, the mapping class group elements will never act through algebra morphisms. More explicitly, if we pick a mapping class group element $R \in \mathrm{SL}(2, \mathbb{Z})$ and conjugate the multiplication with R, i.e. replace it with

$$Z_\mathcal{C}(R) \circ Z_\mathcal{C}(P \times \mathbb{S}^1) \circ \left(Z_\mathcal{C}(R)^{-1} \otimes Z_\mathcal{C}(R)^{-1}\right)\ , \tag{1.4}$$

the result will generally be different from $Z_\mathcal{C}(P \times \mathbb{S}^1)$. Phrased differently, the mapping class group orbit of $Z_\mathcal{C}(P \times \mathbb{S}^1)$ is very non-trivial. Now the idea is to find within the mapping class group orbit of $Z_\mathcal{C}(P \times \mathbb{S}^1)$ a multiplication which is as easy as possible, preferably diagonal. Then $Z_\mathcal{C}(P \times \mathbb{S}^1)$ may be reconstructed from this easy multiplication and the mapping class group action on $Z_\mathcal{C}(\mathbb{T}^2)$. Verlinde's formula, when understood topologically, tells us that this is indeed possible in the semisimple case: To describe the solution, we identify a mapping class group element of the torus with the element in $\mathrm{SL}(2, \mathbb{Z})$ describing its action on the first homology $H_1(\mathbb{T}^2; \mathbb{Z}) \cong \mathbb{Z}^2$; it is important that here the 'first' circle factor is exactly the 'first' one from the definition of the multiplication, i.e. the one participating in the fusion. Now consider the mapping class

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})\ ,$$

the so-called *S-transformation*. Note that besides an orientation reversal on one of the circles, it exchanges the two circle factors of the torus, which means that we should expect the effect on the multiplication to be especially drastic. Indeed, if we conjugate the multiplication $Z_\mathcal{C}(P \times \mathbb{S}^1)$ with the automorphism $Z_\mathcal{C}(S)$ of $Z_\mathcal{C}(\mathbb{T}^2)$ following (1.4), we transform the multiplication (1.1) coming from the monoidal product into the very simple *diagonal multiplication* which can be shown to be given by

$$[x_i] \star [x_j] = \delta_{i,j} d_i^{-1} \cdot [x_i]\ , \tag{1.5}$$

where $d_i = S_{i,0} = S_{0,i} \in k^\times$ is the *quantum dimension* of x_i. In other words, the automorphism $Z_\mathcal{C}(S)$ diagonalizes the multiplication coming from the monoidal product. In yet another equivalent description, we may say that the map

$$Z_\mathcal{C}(S) : \left(\ Z_\mathcal{C}(\mathbb{T}^2) \ , \ Z_\mathcal{C}(P \times \mathbb{S}^1) \ \right) \xrightarrow{\ \cong\ } \left(\ Z_\mathcal{C}(\mathbb{T}^2) \ , \ \star \ \right) \tag{1.6}$$

is an isomorphism of algebras. In the canonical basis of $Z_\mathcal{C}(\mathbb{T}^2)$ given by the classes of simple objects, the matrix elements of the automorphism $Z_\mathcal{C}(S)$ turn out to be precisely the numbers S_{ij} from (1.2). If we use this matrix presentation of $Z_\mathcal{C}(S)$ and spell out what it means for $Z_\mathcal{C}(S)$ to be an algebra isomorphism of the form (1.6), we arrive at the Verlinde formula (1.3).

When attempting to generalize the topological setup used to describe the Verlinde formula above to the non-semisimple case, one faces — as a first major drawback — the problem that in order to build a once-extended three-dimensional topological field theory in the sense of [RT90, RT91, BDSPV15] from a modular category, semisimplicity is needed. If one is willing to give up the duality of the bordism category, the results in [DRGGPMR19] generalize a substantial part of the Reshetikhin-Turaev construction to the non-semisimple case using work of Lyubashenko [Lyu95a, Lyu95b, Lyu96] and the theory of modified traces [GPT09, GKP11, GKP13, GPV13, GKP21]. These constructions, however, are still insensitive to the homological algebra of the modular category and the higher structures associated with it (which is exactly what we include in this article).

Fortunately, the structures actually needed to describe the topological setup above *do* exist within a homotopy coherent framework, namely in terms of *differential graded modular functors* instead of topological field theories, see [Til98, BK01] for the definition of a modular functor with values in vector spaces. A differential graded modular functor comes very close to a three-dimensional chain complex valued topological field theory, but cannot be evaluated on non-invertible three-dimensional bordisms (although extensions to *some* non-invertible bordisms will exist in relevant cases). In other words, a differential graded modular functor is an assignment of a chain complex (the so-called *conformal block*) to each surface. These complexes will carry a homotopy coherent projective action of the respective mapping class groups and will satisfy excision, i.e. are compatible with gluing. In [SW20], it is proven, as an extension of [LMSS18, SW19, LMSS20], that any not necessarily semisimple modular category gives rise to a differential graded modular functor that in zeroth homology reduces to Lyubashenko's vector space valued modular functor [Lyu95a, Lyu95b, Lyu96].

In order to present our main results on the differential graded Verlinde algebra, let us recall and fix some terminology: For a fixed field k, which will be assumed to be algebraically closed throughout the article (unlike for the discussion of the semisimple case above, we do not assume characteristic zero), a *finite category* is an Abelian category enriched over finite-dimensional k-vector spaces with enough projective objects and finitely many simple objects up to isomorphism; additionally, we require that every object has finite length. A *tensor category* is a linear Abelian rigid monoidal category with simple unit. A *finite tensor category* in the sense of Etingof and Ostrik [EO04] is a tensor category with a finite category as underlying linear category.

A finite tensor category $\mathcal{C}$ with a braiding, i.e. natural isomorphisms $c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$ for $X, Y \in \mathcal{C}$ subject to several coherence conditions, is called a *braided finite tensor category*. From a topological viewpoint, the braiding extends the monoidal product to the structure of an algebra over the little disks operad E_2, see [Fre17] for a textbook reference. An extension to an algebra over the *framed* little disks operad [SW03] amounts to a *balancing*, i.e. a natural automorphism of the identity whose components $\theta_X : X \longrightarrow X$ satisfy $\theta_{X\otimes Y} = c_{Y,X}c_{X,Y}(\theta_X \otimes \theta_Y)$ for $X, Y \in \mathcal{C}$ and $\theta_I = \mathrm{id}_I$, where I is the monoidal unit of $\mathcal{C}$. A *finite ribbon category* is a braided finite tensor category $\mathcal{C}$ with balancing θ that is compatible with the duality $-^\vee$ in the sense that $\theta_{X^\vee} = \theta_X^\vee$ for $X \in \mathcal{C}$. The *Müger center* of a braided finite tensor category $\mathcal{C}$ is the full subcategory of $\mathcal{C}$ given by the *transparent objects*, i.e. those objects $X \in \mathcal{C}$ that satisfy $c_{Y,X}c_{X,Y} = \mathrm{id}_{X\otimes Y}$ for every $Y \in \mathcal{C}$. The braiding c (and then also the braided finite tensor category) is referred to as *non-degenerate* if its Müger center is as small as possible, namely spanned by the monoidal unit under finite direct sums (various equivalent characterizations of non-degeneracy are given in [Shi19a]). A *modular category* is a non-degenerate finite ribbon category.

The main result of [SW20] is that any modular category gives canonically rise to a differential graded modular functor, i.e. a symmetric monoidal functor

$$\mathfrak{F}_\mathcal{C} \ : \ \mathcal{C}\text{-}\mathsf{Surf}^\mathsf{c} \longrightarrow \mathsf{Ch}_k \tag{1.7}$$

from (the central extension of) a category of extended surfaces, whose boundary components are labeled with projective objects in $\mathcal{C}$, to the category of differential graded vector spaces over k (one can also allow non-projective boundary labels). The differential graded modular functor $\mathfrak{F}_\mathcal{C}$ satisfies an excision property which allows us to compute the conformal block $\mathfrak{F}_\mathcal{C}(\Sigma, \underline{X})$ for a surface Σ with boundary label $\underline{X}$ via a pair of pants decomposition and a gluing procedure using homotopy coends. This is a consequence of the fact that, on a given fixed surface, the differential graded modular functor is constructed as a homotopy colimit over a contractible ∞-groupoid of colored markings; this is referred to as *homotopy coherent Lego Teichmüller game* and an extension of the techniques used by Bakalov and Kirillov [BK00], which, in turn, crucially rely on classical results on cut systems of surfaces due to Grothendieck [Gro84], Hatcher and Thurston [HT80] and Harer [Har83].

On the closed torus, the differential graded modular functor produces the Hochschild complex of $\mathcal{C}$. More precisely, the choice of a *certain specific colored marking* on the torus gives us an equivalence from the Hochschild complex of $\mathcal{C}$ to the chain complex $\mathfrak{F}_\mathcal{C}(\mathbb{T}^2)$. Recall that for a finite (tensor) category $\mathcal{C}$, one calls the homotopy coend $\int_\mathbb{L}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ running over the endomorphism spaces of projective objects the *Hochschild complex of* $\mathcal{C}$. Explicitly, it is given by the (normalized) chains on the simplicial vector space

$$\dots \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \bigoplus_{X_0,X_1\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X_1,X_0)\otimes\mathcal{C}(X_0,X_1) \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \bigoplus_{X_0\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X_0,X_0)\ ,$$

where $\mathcal{C}(-,-)$ denotes the morphism vector spaces. When writing $\mathcal{C}$, as a linear category, as the category of finite-dimensional modules over a finite-dimensional algebra A, we recover the Hochschild complex of A. This is a form of the *Agreement Principle* of McCarthy [MCar94] and Keller [Kel99].

In [SW19] it was already established that the Hochschild chain complex of a finite braided tensor category comes with a non-unital E_2-multiplication generalizing the one discussed above in (1.1) for the semisimple case. This multiplication comes directly from the monoidal product. Already in the setting of ordinary linear modular functors, it is a crucial idea for the understanding of Verlinde formulae to consider centers and class functions simultaneously. For an in-depth study of the multiplicative structure on the differential graded conformal block for the torus, this means that the Hochschild chain complex of $\mathcal{C}$ must be treated in tandem with the Hochschild *cochain* complex of $\mathcal{C}$, i.e. the homotopy end $\int^\mathbb{R}_{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$. The latter is the value of the *dual* differential graded modular functor

$$\mathfrak{F}^\mathcal{C} := \mathfrak{F}^*_\mathcal{C} \tag{1.8}$$

on the torus, i.e. the functor obtained by taking point-wise the dual chain complex in (2.1). The fact that this really yields the Hochschild cochain complex makes use of the Calabi-Yau structure on the tensor ideal $\mathsf{Proj}\,\mathcal{C} \subset \mathcal{C}$, see [SW20, Remark 3.12] and also [SW21]. While the Hochschild chain and cochain complex of a modular category are dual as chain complexes, obtaining an E_2-structure on the Hochschild *cochain* complex of a finite tensor category that is induced by the monoidal product is significantly more involved (it is clear that both products cannot be dual because duality would translate a product to a coproduct). We prove the following result for the Hochschild cochain complex of a unimodular braided finite tensor category (unimodularity is implied by modularity and discussed in detail in Section 2.3):

Theorem 3.24. *Let $\mathcal{C}$ be a unimodular braided finite tensor category with chosen trivialization $D \cong I$ of the distinguished invertible object of $\mathcal{C}$. Then the Hochschild cochain complex $\int^\mathbb{R}_{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ inherits from its braided monoidal product the structure of an E_2-algebra.*

We refer to this E_2-algebra as the *differential graded Verlinde algebra* on the Hochschild cochain complex of $\mathcal{C}$ and denote the product by $\otimes$.

If $\mathcal{C}$ is modular, we have, thanks to (1.8), the homotopy coherent mapping class group action on $\int^\mathbb{R}_{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ at our disposal. By acting with the mapping class element $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, we obtain another multiplication — ideally a simpler one which does not depend on the monoidal product. This is exactly the idea behind the Verlinde formula in its formulation (1.6). In fact, there is a natural candidate for an E_2-structure on the Hochschild cochain complex, which very conveniently does not see the monoidal product at all, but only the linear structure, namely the well-known E_2-structure afforded by the *Deligne Conjecture*: Deligne conjectured in 1993 that the Gerstenhaber structure on the Hochschild cohomology of an associative algebra [Ger63] has its origin in an E_2-structure on the Hochschild cochain complex of that algebra (for a suitable model of E_2). By now numerous proofs exist [Tam98, MCS02, BF04], including proofs of the *cyclic Deligne Conjecture* [TZ06, Cos07, Kau08], a refinement for symmetric Frobenius algebras.

As our first main result, we prove that the S-transformation (or rather its inverse because of the dualization in (1.8)) indeed transforms the E_2-algebra induced by the monoidal product (Theorem 3.24) into Deligne's E_2-structure. This means that, as in the semisimple case, the Verlinde algebra lies in the mapping class group orbit of a simpler E_2-algebra structure that just uses the linear structure of $\mathcal{C}$.

Theorem 4.2 *(Verlinde formula for the Hochschild cochain complex)***.** *For any modular category $\mathcal{C}$, the action of the mapping class group element $S^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ on the Hochschild cochain complex of $\mathcal{C}$ yields an equivalence*

$$\mathfrak{F}^{\mathcal{C}}(S^{-1}) : \left(\int^{\mathbb{R}}_{X \in \mathsf{Proj}\, \mathcal{C}} \mathcal{C}(X,X) \ , \ \otimes \right) \simeq \left(\int^{\mathbb{R}}_{X \in \mathsf{Proj}\, \mathcal{C}} \mathcal{C}(X,X) \ , \ \smile \right)$$

of E_2-algebras which are given as follows:

- *On the left hand side, the E_2-structure is the differential graded Verlinde algebra on the Hochschild cochain complex induced by the monoidal product (Theorem 3.24).*
- *On the right hand side, the E_2-structure is the one afforded by Deligne's Conjecture with the underlying multiplication being the cup product $\smile$.*

The proof provides natural models of the homotopy end and the E_2-operad such that $\mathfrak{F}^{\mathcal{C}}(S^{-1})$ is even an isomorphism of E_2-algebras. Moreover, we prove that both E_2-algebras in Theorem 4.2 naturally extend to framed E_2-algebras such that $\mathfrak{F}^{\mathcal{C}}(S^{-1})$ is an equivalence of framed E_2-algebras, see Corollary 4.3 for the definition of these framed E_2-structures and the precise statement.

The effect of S on the non-unital E_2-structure on the Hochschild *chain* complex from [SW19] is different and the subject of our second main result:

Theorem 4.10 *(Verlinde formula for the Hochschild chain complex)***.** *For any modular category $\mathcal{C}$, the action of the mapping class group element $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ yields an equivalence*

$$\mathfrak{F}_{\mathcal{C}}(S) : \left(\int_{\mathbb{L}}^{X \in \mathsf{Proj}\, \mathcal{C}} \mathcal{C}(X,X) \ , \ \otimes \right) \simeq \left(\int_{\mathbb{L}}^{X \in \mathsf{Proj}\, \mathcal{C}} \mathcal{C}(X,X) \ , \ \star \right)$$

of non-unital E_2-algebras whose multiplication, up to homotopy, is concentrated in degree zero.

- *On the left hand side, the E_2-structure is the differential graded Verlinde algebra on the Hochschild chain complex induced the monoidal product [SW19].*
- *On the right hand side, the non-unital E_2-structure is the almost trivial one that is a part of the cyclic version of Deligne's Conjecture applied to the Calabi-Yau structure coming from the modified trace on the tensor ideal of projective objects.*

The product $\star$ was defined and investigated in [SW21] using the *trace field theory* $\Phi_{\mathcal{C}} : \mathsf{OC} \longrightarrow \mathsf{Ch}_k$, an open-closed topological conformal field theory that can be associated to a finite tensor category and a suitable trivialization of the right Nakayama functor of $\mathcal{C}$ as right $\mathcal{C}$-module functor relative to a pivotal structure on $\mathcal{C}$. Therefore, we have the following additional information on $\star$:

- The product $\star$ is block diagonal [SW21, Proposition 5.3]. Hence, Theorem 4.10 implies that the S-transformation 'block diagonalizes' the product $\otimes$.
- The $\star$-product of the identity morphisms id_P and id_Q of two projective objects P and Q is given, up to boundary, by the handle element $\xi_{P,Q} \in \mathcal{C}(P,P)$ of $\Phi_{\mathcal{C}}$ [SW21, Theorem 5.6],

$$\mathrm{id}_P \star \mathrm{id}_Q \simeq \xi_{P,Q} \ , \tag{1.9}$$

a certain central endomorphism $\xi_{P,Q} : P \longrightarrow P$ whose modified trace is given by

$$\mathsf{t}_P \xi_{P,Q} = \dim \mathcal{C}(P,Q) \ .$$

Hence, the modified traces of the handle elements recover the Cartan matrix of $\mathcal{C}$. (We focus in (1.9) on identity morphisms for presentation purposes; a similar formula holds for all endomorphisms.)

Formula (1.9) reduces to (1.5) if P is simple. Then $\xi_{P,Q}$ can be identified with a number and

$$\mathrm{id}_P \star \mathrm{id}_Q \simeq (d_P^{\mathrm{m}})^{-1} \dim \mathcal{C}(P,Q) \cdot \mathrm{id}_P \ ,$$

where $d_P^{\mathrm{m}} \in k^\times$ is the modified dimension of P. Therefore, the product $\star$ extracted from the cyclic Deligne Conjecture generalizes the product $\star$ from (1.5) to the non-semisimple case.

Having stated the two main results, we will now highlight a number of consequences and applications that follow from the main results or the techniques developed in this paper.

Restriction to zeroth (co)homology. Specializing Theorem 4.2 to zeroth cohomology recovers the formula proposed and proven by Gainutdinov and Runkel [GR19] as a generalization of the Verlinde formula to the non-semisimple case (Corollary 4.5). This formula features a complete system of the simple objects in $\mathcal{C}$ and multiplicities in Jordan-Hölder series, see Corollary 4.5. However, the differential graded Verlinde algebra on the Hochschild cochain complex is significantly richer than its restriction to zeroth cohomology. In particular, its product and Gerstenhaber bracket are non-trivial, see Example 4.4. Of course, this can easily been seen thanks to Theorem 4.2 because the usual E_2-structure on Hochschild chains is well understood in a lot of cases.

For Theorem 4.10, the situation is different. Here one only has a statement in zeroth homology. It leads to a formula involving the fusion coefficients in the linearized K_0-ring of $\mathcal{C}$ (Corollary 4.13).

E_2-structures on homotopy invariants of braided commutative algebras. For all of our main statements, we need efficient tools to construct new E_2-structures and to compare them to existing ones. To this end, we develop a construction of E_2-algebras from an algebra $\mathbb{T}$ in a finite tensor category $\mathcal{C}$ together with a lift to a braided commutative algebra $\mathsf{T} \in Z(\mathcal{C})$ in the Drinfeld center $Z(\mathcal{C})$ of $\mathcal{C}$, where the term *lift* means $U\mathsf{T} = \mathbb{T}$ as algebras with $U : Z(\mathcal{C}) \longrightarrow \mathcal{C}$ being the forgetful functor.

Theorem 3.10. *Let $\mathbb{T} \in \mathcal{C}$ be an algebra in a finite tensor category $\mathcal{C}$ together with a lift to a braided commutative algebra $\mathsf{T} \in Z(\mathcal{C})$ in the Drinfeld center. Then the multiplication of $\mathbb{T}$ induces the structure of an E_2-algebra on the complex $\mathcal{C}(I, \mathbb{T}^\bullet)$ of homotopy invariants of $\mathbb{T}$, i.e. the complex of morphisms from I to an injective resolution $\mathbb{T}^\bullet$ of $\mathbb{T}$.*

There is a natural source for such algebras: The right adjoint $R : \mathcal{C} \longrightarrow Z(\mathcal{C})$ to the forgetful functor $U : Z(\mathcal{C}) \longrightarrow \mathcal{C}$ sends the unit I of $\mathcal{C}$ to a braided commutative algebra $\mathsf{A} = R(I) \in Z(\mathcal{C})$ with underlying object $\mathbb{A} = UR(I)$ thanks to a result of Davydov, Müger, Nikshych and Ostrik [DMNO13]. The homotopy invariants $\mathcal{C}(I, \mathbb{A}^\bullet)$ are well-known to be equivalent to the Hochschild cochains of $\mathcal{C}$, see e.g. [Bic13, Section 2.2]:

$$\mathcal{C}(I, \mathbb{A}^\bullet) \simeq \int_{X \in \mathsf{Proj}\, \mathcal{C}}^{\mathbb{R}} \mathcal{C}(X, X) \ .$$

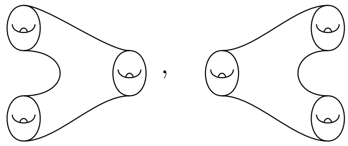
We prove in Theorem 3.12 that this is actually an equivalence of E_2-algebras if the left hand side is equipped with the E_2-structure afforded by Theorem 3.10, i.e. the one induced by the multiplication of $\mathbb{A}$, while the right hand side carries the usual E_2-structure on Hochschild cochains. This result allows us to make the connection between differential graded conformal blocks and the Deligne Conjecture. In particular, we obtain a relatively easy solution to Deligne's Conjecture in presence of a rigid monoidal product.

If we apply Theorem 3.10 to unimodular pivotal finite tensor categories, we obtain a framed E_2-structure on the self-extension algebra and the Hochschild cochains of a unimodular pivotal finite tensor category. This generalizes a result of Menichi [Men11] who previously proved the result at cohomology level for unimodular pivotal Hopf algebras by giving a Batalin-Vilkovisky structure (i.e. the structure of an algebra over the homology of the framed E_2-operad) on the self-extension algebra.

Partial three-dimensional extension for differential graded modular functors. A priori, a modular functor is less than a three-dimensional topological field theory. For a the differential graded modular functor of a modular category, however, we can give the following partial extension result:

Corollary 4.15 *(Partial three-dimensional extension of the differential graded modular functor)***.** *The differential graded modular functor $\mathfrak{F}_\mathcal{C}$ associated to a modular category $\mathcal{C}$ extends to three-dimensional oriented bordisms of the form $\Sigma \times \mathbb{S}^1 : (\mathbb{T}^2)^{\sqcup p} \longrightarrow (\mathbb{T}^2)^{\sqcup q}$, where $\Sigma : (\mathbb{S}^1)^{\sqcup p} \longrightarrow (\mathbb{S}^1)^{\sqcup q}$ is a compact oriented two-dimensional bordism such that every component of Σ has at least one incoming boundary component.*

On the bordisms,



this extension is given by the product $\otimes$ from Theorem 4.10 and the product from Theorem 3.24 dualized via the Calabi-Yau structure, respectively. On the solid torus seen as bordism $\mathbb{T}^2 \longrightarrow \emptyset$, one obtains the modified trace precomposed with the S-transformation. An extension to the solid closed torus as bordism $\emptyset \longrightarrow \mathbb{T}^2$ will generally not exist in the non-semisimple case (Remark 4.16).

Conventions. Plenty of key notions have already been defined in the introduction, and more will follow in the main text. In this additional short list, we want to collect some more technical or notational conventions.

(1) For the entire article, we work over a fixed algebraically closed field k. We do not assume that k has characteristic zero.

(2) We use the notation Ch_k for the symmetric monoidal category of chain complexes over k. Whenever needed, we equip it with its projective model structure in which weak equivalences (for short: equivalences) are quasi-isomorphisms and fibrations are degree-wise surjections. Following standard terminology, we call a (co)fibration which is also an equivalence a *trivial (co)fibration*. By a *(canonical) equivalence* between chain complexes (notation $\simeq$ as opposed to the notation $\cong$ reserved for isomorphisms) we do not necessarily mean a map in either direction, but also allow a (canonical) zigzag. We refer to a category enriched over Vect_k or Ch_k as a *linear* or *differential graded category*, respectively. All functors between linear and differential graded categories will assumed to be enriched.

(3) We follow the duality conventions of [EGNO17]: For every object $X \in \mathcal{C}$ in a *rigid* monoidal category, we denote
   - the *left dual* by $X^\vee$ (it comes with an evaluation $d_X : X^\vee \otimes X \longrightarrow I$ and a coevaluation $b_X : I \longrightarrow X \otimes X^\vee$),
   - and the *right dual* by ${}^\vee X$ (it comes with an evaluation $\widetilde{d}_X : X \otimes {}^\vee X \longrightarrow I$ and a coevaluation $\widetilde{b}_X : I \longrightarrow {}^\vee X \otimes X$).

   Evaluation and coevaluation are subject to the usual zigzag identities. By left and right duality, we obtain the natural adjunction isomorphisms

$$\begin{aligned}\mathcal{C}(X \otimes Y, Z) &\cong \mathcal{C}(X, Z \otimes Y^\vee) \ , \\ \mathcal{C}(Y^\vee \otimes X, Z) &\cong \mathcal{C}(X, Y \otimes Z) \ , \\ \mathcal{C}(X \otimes {}^\vee Y, Z) &\cong \mathcal{C}(X, Z \otimes Y) \ , \\ \mathcal{C}(Y \otimes X, Z) &\cong \mathcal{C}(X, {}^\vee Y \otimes Z)\end{aligned}$$

   for $X, Y, Z \in \mathcal{C}$.

(4) Any finite (tensor) category $\mathcal{C}$ is a module category over the symmetric monoidal category of finite-dimensional k-vector spaces. This means that we have a tensoring $V \otimes X \in \mathcal{C}$ for a finite-dimensional vector space V and $X \in \mathcal{C}$ and also a powering $X^V = V^* \otimes X \in \mathcal{C}$. Here V^* is the dual vector space of V.

(5) For the definition of the S-matrix in (1.2), we have already used the *graphical calculus* for morphisms in (braided) monoidal categories, see e.g. [Kas95]. This graphical calculus will be used throughout

the text whenever the corresponding computations in equations would become too complicated or hardly insightful. Objects are symbolized by vertical lines and the monoidal product by the juxtaposition of lines (the monoidal unit is the empty collection of lines). The braiding and inverse braiding are denoted by an overcrossing and undercrossing, respectively. The evaluation and coevaluation is denoted by a cap and cup, respectively. The morphisms are always to be read from bottom to top. The composition is represented by vertical stacking.

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2 The Hochschild chain complex of a modular category as differential graded conformal block for the torus

The natural starting point for this article is to give the action of the mapping class group element S on the conformal block of the torus, and thereby also on the Hochschild complex of a modular category. More specifically, the goal of this section is to present a model for the Hochschild complex in which the effect of the S-transformation can be explicitly described. While this description is mostly extracted from [SW19, SW20], the main result of this section (Proposition 2.7) is new.

In order to place the results in the proper context, recall that any modular category gives canonically rise to a differential graded modular functor, i.e. a symmetric monoidal functor

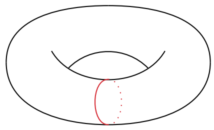
$$\mathfrak{F}_{\mathcal{C}} \ : \ \mathcal{C}\text{-}\mathsf{Surf}^{\mathsf{c}} \longrightarrow \mathsf{Ch}_k \tag{2.1}$$

defined on a category $\mathcal{C}\text{-}\mathsf{Surf}^{\mathsf{c}}$ of extended $\mathcal{C}$-labeled surfaces. Its objects are *$\mathcal{C}$-labeled extended surfaces*: An extended surface is a compact oriented two-dimensional manifold (possibly with boundary) with a marked point on each boundary component. Moreover, each boundary component carries an orientation. If it agrees with the orientation inherited from the surface, the boundary component is referred to as *outgoing*, otherwise as *incoming*. By a *$\mathcal{C}$-labeling* of an extended surface we mean that a projective object in $\mathcal{C}$ is attached to each boundary component. The morphisms in $\mathcal{C}\text{-}\mathsf{Surf}^{\mathsf{c}}$ are generated by mapping classes (defined in this context as isotopy classes of orientation-preserving diffeomorphisms that map marked points to marked points) and sewing morphisms that glue an incoming to an outgoing boundary component provided that they carry the same label (which is then omitted after the gluing). Mapping classes and sewings compose in the expected way except for the fact that relations between mapping classes are twisted by a 2-cocycle coming from the framing anomaly, see e.g. [GM13, Section 3].

The differential graded modular functor $\mathfrak{F}_{\mathcal{C}} \ : \ \mathcal{C}\text{-}\mathsf{Surf}^{\mathsf{c}} \longrightarrow \mathsf{Ch}_k$ is constructed using the homotopy coherent Lego Teichmüller game: Let us briefly sketch the construction for an extended surface Σ that, for simplicity, we assume to be closed. One defines a *category $\widehat{\mathsf{M}}(\Sigma)$ of colored markings on Σ* whose objects are, roughly, a cut system on Σ that prescribes how to cut the surface into a disjoint union of surfaces of genus zero (each cut is either colored or uncolored; the cut system must have at least one colored cut or a boundary component per connected component) plus a certain graph. Morphisms are generated in a relatively technical way by uncolorings and certain moves. The key property of $\widehat{\mathsf{M}}(\Sigma)$ is its contractibility, i.e. the geometric realization $|\widehat{\mathsf{M}}(\Sigma)|$ is homotopy equivalent to a point (in contrast to the usual Lego Teichmüller game, $\widehat{\mathsf{M}}(\Sigma)$ is not equivalent to the point as a category).

Using the algebraic structure of $\mathcal{C}$, one then defines a marked block functor $\mathsf{B}_{\mathcal{C}}^{\Sigma,-} : \widehat{\mathsf{M}}(\Sigma) \longrightarrow \mathsf{Ch}_k$. For the definition on objects, one uses a gluing prescription, where the cuts prescribe the type of gluing: homotopy coends for colored cuts, ordinary coends for uncolored cuts.

We will be content with discussing the example which is most relevant in this article: Consider the following colored marking of the torus with one colored cut (the graph part of the marking is suppressed for simplicity):



(2.2)

The marking describes how the torus is obtained by gluing two boundary circles of the cylinder together. By construction the marked block for a cylinder is given by the morphism spaces of $\mathcal{C}$. As a consequence, the marked block for the marking (2.2), and in fact for *any* marking that cuts the torus into a cylinder, is the homotopy coend $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$, i.e. the Hochschild complex of $\mathcal{C}$.

The differential graded conformal block $\mathfrak{F}_{\mathcal{C}}(\Sigma)$ can be described as the homotopy colimit

$$\mathfrak{F}_{\mathcal{C}}(\Sigma) = \operatorname*{hocolim}_{\Gamma\in\widehat{\mathsf{M}}(\Sigma)} \mathsf{B}_{\mathcal{C}}^{\Sigma,\Gamma}$$

of marked blocks over the category $\widehat{\mathsf{M}}(\Sigma)$ of colored markings. The mapping class group action on $\mathfrak{F}_{\mathcal{C}}(\Sigma)$ arises solely through the mapping class group action on colored markings.

Since the marked block functor sends all morphisms to equivalences, it descends to the contractible ∞-groupoid obtained by localizing $\widehat{\mathsf{M}}(\Sigma)$ at all morphisms. As a consequence, for any colored marking Γ, the canonical map

$$\mathsf{B}_{\mathcal{C}}^{\Sigma,\Gamma} \xrightarrow{\ \simeq\ } \mathfrak{F}_{\mathcal{C}}(\Sigma) \tag{2.3}$$

is an equivalence. This means in particular that the marking (2.2) provides us with an equivalence

$$\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \xrightarrow{\ \simeq\ } \mathfrak{F}_{\mathcal{C}}(\mathbb{T}^2)\ .$$

The problem is that a mapping class group element that acts in a very natural topological way on the right hand side, will at first amount to a zigzag of equivalences on the left hand side that is then converted to a chain map by inverting the equivalences that point the 'wrong way'. If one is just interested in the mapping class group actions, this is no issue at all; after all, $\mathfrak{F}_{\mathcal{C}}(\mathbb{T}^2)$ is a perfectly fine realization of this action that is even strict. When considering multiplicative structures, the homotopy coend $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ is easier to work with and allows to make contact to more 'traditional' algebraic quantities, but the mapping class group action, particularly the effect of the S-transformation, is difficult to describe.

The purpose of this section is to give a model of the conformal block for the torus which is closely related to $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$, but on which the effect of the S-transformation can be described in a convenient way. This model is related to the one used in [SW19], but we need to go beyond that to later prove the main results in later sections.

2.1 The Hochschild complex of a finite tensor category and the Lyubashenko coend

In any finite tensor category $\mathcal{C}$, one may define the *canonical coend* $\mathbb{F} := \int^{X\in\mathcal{C}} X^\vee \otimes X$ and the *canonical end* $\mathbb{A} = \int_{X\in\mathcal{C}} X \otimes X^\vee$ which, due to their appearance in [Lyu95a, Lyu95b], are also called the *Lyubashenko coend and end*, respectively. Both objects are absolutely vital to the description of mapping class group actions, multiplicative structure and Hochschild complexes.

In [SW19, Section 3.2] the module structure of $\mathcal{C}$ over finite-dimensional vector spaces (see Convention (4) on page 8) is used to define, for any exact functor $G : \mathcal{C}^{\mathrm{op}} \boxtimes \mathcal{C} \longrightarrow \mathcal{C}$ with the property that $G(X,Y)$ is projective if X and Y are projective, the objects

$$\bigoplus_{X_0,\dots,X_n\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X_n,X_{n-1})\otimes\dots\otimes\mathcal{C}(X_1,X_0)\otimes G(X_0,X_n)$$

in $\mathcal{C}$ or, strictly speaking, in a completion of $\mathcal{C}$ under infinite direct sums. By means of these objects, one assembles a simplicial object

$$\dots \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \bigoplus_{X_0,X_1\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X_1,X_0)\otimes G(X_0,X_1) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \bigoplus_{X_0\in\mathsf{Proj}\,\mathcal{C}} G(X_0,X_0)\ , \tag{2.4}$$

and it is proven that it is possible to reduce the infinite direct sums appearing here to finite ones, i.e. to restrict to a certain finite collections of projective objects, without changing the simplicial object up to equivalence. The realization of the resulting simplicial object in $\mathcal{C}$ is denoted by $\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} G(X,X)$, where the subscript 'f' stands for 'finite'. Note that this homotopy coend gives us a differential graded object in $\mathcal{C}$. It is different from the homotopy coends with vector space valued integrand, such as the homotopy coend $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ that we use to define the Hochschild complex.

Proposition 2.1 *([SW19, Corollary 3.7 & Theorem 3.9]). Let $\mathcal{C}$ be a pivotal finite tensor category.*

(i) For any projection resolution $\mathbb{F}_\bullet$ of $\mathbb{F}$, we have

$$\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \simeq \mathcal{C}\left(I,\mathbb{F}_\bullet\right) \ .$$

(ii) The (finite) homotopy coend $\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} X^\vee \otimes X$ is a projective resolution of the canonical coend $\mathbb{F} = \int^{X\in\mathcal{C}} X^\vee \otimes X$ and hence allows to write the Hochschild complex of $\mathcal{C}$ up to equivalence as

$$\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \simeq \mathcal{C}\left(I, \int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} X^\vee \otimes X\right) \ .$$

The subscript $\bullet$ will be often used in the sequel to denote a projective resolution.

Remark 2.2. There is a well-known expression for the Hochschild *cohomology* of a Hopf algebra A in terms of Exts in A-modules that goes back to Cartan and Eilenberg [CE56], see [Bic13, Proposition 2.1]. This description generalizes to finite tensor categories and will actually be needed in Section 3.1. The above equivalence, although it might look similar, is different: The complex $\mathcal{C}\left(I,\mathbb{F}_\bullet\right)$ is obviously *not* a derived hom and will generally *not* be canonically equivalent to the derived *dual* hom either.

Remark 2.3 *(Duality conventions for the coend).* As for duality in general, there are no completely standard conventions for the Lyubashenko (co)end: Instead of $\int^{X\in\mathcal{C}} X^\vee \otimes X$ (which is used in the present article because it will turn out to be the more convenient choice later), we could work with $\int^{X\in\mathcal{C}} X \otimes X^\vee$ as we did in [SW19, SW20] (for different reasons). Moreover, if $\mathcal{C}$ is pivotal (which is assumed in Proposition 2.1), both ways to define the canonical (co)end agree up to isomorphism. If one drops pivotality, one would get an equivalence

$$\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \simeq \mathcal{C}\left(I, \int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee\right) \ ,$$

where on the right hand side $\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee$ is a projective resolution of $\int^{X\in\mathcal{C}} X \otimes X^\vee$ and could be replaced by any other projective resolution.

2.2 The Drinfeld center and unimodularity

In order to effectively use the homological algebra from Subsection 2.1, we need to recall some standard facts on the Drinfeld center and unimodularity.

For a finite tensor category $\mathcal{C}$, we denote by $Z(\mathcal{C})$ its *Drinfeld center*, the *braided* tensor category that consists of pairs of an object $X \in \mathcal{C}$ and a *half braiding*, i.e. a natural isomorphism $X\otimes - \cong -\otimes X$ subject to coherence conditions. The Drinfeld center can be seen as the center of $\mathcal{C}$ as E_1-algebra and is therefore an E_2-algebra, i.e. braided (and this braiding is actually the one that one can directly give based on the description of $Z(\mathcal{C})$ in terms of half braidings). It is also a finite tensor category [Shi17a, Theorem 3.8]. The forgetful functor $U : Z(\mathcal{C}) \longrightarrow \mathcal{C}$ is exact and therefore has a left adjoint $L : \mathcal{C} \longrightarrow Z(\mathcal{C})$ and a right adjoint $R : \mathcal{C} \longrightarrow Z(\mathcal{C})$. Since U is strong monoidal, L and R are automatically oplax and lax monoidal, respectively, see [BV12] for a more detailed account on the structure of these adjoint pairs and the (co)monads they give rise to. As a consequence, the images of the monoidal unit $I \in \mathcal{C}$

$$\mathsf{F} := LI \ , \quad \mathsf{A} := RI$$

are a coalgebra and an algebra in $Z(\mathcal{C})$, respectively. The underlying objects in $\mathcal{C}$

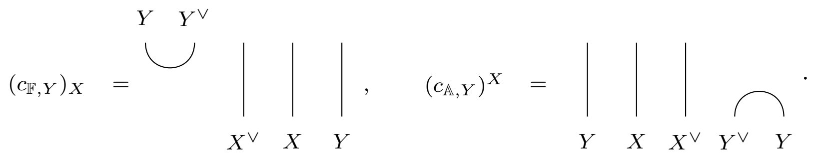
$$U\mathsf{F} = \mathbb{F} = \int^{X\in\mathcal{C}} X^\vee \otimes X \ , \quad U\mathsf{A} = \mathbb{A} = \int_{X\in\mathcal{C}} X \otimes X^\vee \tag{2.5}$$

are the canonical coend and the canonical end, respectively. In order to give the half braiding $c_{\mathbb{F},Y} : \mathbb{F} \otimes Y \longrightarrow Y \otimes \mathbb{F}$ and $c_{\mathbb{A},Y} : \mathbb{A} \otimes Y \longrightarrow Y \otimes \mathbb{A}$ with $Y \in \mathcal{C}$ (that is often referred to as *non-crossing half braiding*), it suffices by the universal property of the (co)end to give the restriction to $X^\vee \otimes X \otimes Y$ and the component $X \otimes X^\vee \otimes Y$, respectively, by

$$X^\vee \otimes X \otimes Y \xrightarrow{(c_{\mathbb{F},Y})_X} Y \otimes Y^\vee \otimes X^\vee \otimes X \otimes Y \cong Y \otimes ((X \otimes Y)^\vee \otimes X \otimes Y) \longrightarrow Y \otimes \mathbb{F} \quad (2.6)$$

$$\mathbb{A} \otimes Y \longrightarrow (Y \otimes X \otimes (Y \otimes X)^\vee) \otimes Y \cong Y \otimes X \otimes X^\vee \otimes Y^\vee \otimes Y \xrightarrow{(c_{\mathbb{A},Y})^X} Y \otimes X \otimes X^\vee \ , \quad (2.7)$$

where the last map in (2.6) and the first map in (2.7) are the structure maps of the coend and the end and where



(2.8)

As stated in our conventions on the graphical calculus on page 9, the cup is the evaluation, and the cap is the coevaluation. As a consequence of (2.5), $\mathbb{F}$ is coalgebra and $\mathbb{A}$ an algebra in $\mathcal{C}$. In fact, the coalgebra structure $\delta : \mathbb{F} \longrightarrow \mathbb{F} \otimes \mathbb{F}$ on $\mathbb{F}$ is induced by the coevaluation

$$X^\vee \otimes X \xrightarrow{X^\vee \otimes b_X \otimes X} X^\vee \otimes X \otimes X^\vee \otimes X$$

while, dually, $\mathbb{A}$ inherits an algebra structure $\gamma : \mathbb{A} \otimes \mathbb{A} \longrightarrow \mathbb{A}$ on $\mathbb{A}$ induced by the evaluation

$$X \otimes X^\vee \otimes X \otimes X^\vee \xrightarrow{X \otimes d_X \otimes X^\vee} X \otimes X^\vee \ . \quad (2.9)$$

The algebra structure on A (and $\mathbb{A}$) and the coalgebra structure on F (and $\mathbb{F}$) correspond to each other under the duality ${}^\vee\mathsf{F} \cong \mathsf{A}$ (and ${}^\vee\mathbb{F} \cong \mathbb{A}$).

The left and the right adjoint to the forgetful functor U turn out to be intimately related to the distinguished invertible object D of $\mathcal{C}$. This objects controls by [ENO04] the quadruple dual of a finite tensor category through the *Radford formula*

$$-^{\vee\vee\vee\vee} \cong D \otimes - \otimes D^{-1}$$

that generalizes the classical result on the quadruple of the antipode of a Hopf algebra [Rad76]. By [Shi17a, Lemma 5.5] D is the socle of the projective cover of the monoidal unit. A finite tensor category is called *unimodular* if $D \cong I$.

Theorem 2.4 *(Shimizu [Shi17a, Lemma 4.7 & Theorem 4.10])***.** *For any finite tensor category $\mathcal{C}$, the left adjoint L and the right adjoint R to the forgetful functor $U : Z(\mathcal{C}) \longrightarrow \mathcal{C}$ are related by canonical natural isomorphisms*

$$L(D \otimes -) \cong R \cong L(- \otimes D) \ , \quad R(D^{-1} \otimes -) \cong L \cong R(- \otimes D^{-1})$$

and $\mathcal{C}$ is unimodular if and only if there is a natural isomorphism $L \cong R : \mathcal{C} \longrightarrow Z(\mathcal{C})$. Moreover, the algebra $\mathsf{A} = RI \in Z(\mathcal{C})$ has the structure a Frobenius algebra if and only if $\mathcal{C}$ is unimodular.

For the moment, we do not need a more concrete description of L and R. Concrete models for these functors will appear later in the proof of Lemma 4.7.

Definition 2.5 *(Radford map)***.** Let $\mathcal{C}$ be a finite tensor category. If $\mathcal{C}$ is unimodular and if a trivialization of D is chosen (the possible choices form a $k^\times$-torsor), we define the I-component of the natural isomorphism $UR \cong UL$ resulting from this choice via Theorem 2.4 as the *Radford map* and denote it by

$$\Psi : \mathbb{A} = UR(I) \xrightarrow{\cong} UL(I) = \mathbb{F} \ .$$

We justify the terminology through the comments after Proposition 2.7.

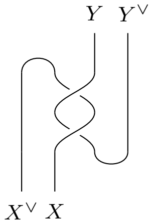
In the case that $\mathcal{C}$ is also pivotal, i.e. equipped with a monoidal isomorphism $-^{\vee\vee} \cong \mathrm{id}_\mathcal{C}$, we record the following for later use:

Proposition 2.6. *For a unimodular pivotal finite tensor category $\mathcal{C}$, the Frobenius algebra $\mathsf{A} \in Z(\mathcal{C})$ is symmetric.*

Proof. We give the proof using the language of pivotal module categories from [Shi19b]: For the canonical algebra $\mathsf{A} \in Z(\mathcal{C})$, denote by $\mathsf{A}-\mathsf{mod}_{Z(\mathcal{C})}$ the category of left A-modules in $Z(\mathcal{C})$. Then A can be recovered as the endomorphisms of the left regular A-module A in $Z(\mathcal{C})$; in short $\mathsf{A} = \mathrm{End}_{\mathsf{A}}(\mathsf{A}, \mathsf{A})$. By [Shi17a, Theorem 6.1 (2)] $\mathsf{A}-\mathsf{mod}_{Z(\mathcal{C})} \simeq \mathcal{C}$ as $Z(\mathcal{C})$-module categories. Since $\mathcal{C}$ is pivotal, $\mathcal{C}$ is also pivotal as a module category over itself. Of course, $\mathcal{C}$ is also a module category over $Z(\mathcal{C})$, and it is in fact a pivotal module category by [FS20, Corollary 38]. Therefore, $\mathsf{A}-\mathsf{mod}_{Z(\mathcal{C})}$ is also a pivotal $Z(\mathcal{C})$-module category. By $\mathsf{A} = \mathrm{End}_{\mathsf{A}}(\mathsf{A}, \mathsf{A})$ the object $\mathsf{A} \in Z(\mathcal{C})$ can be recovered as the endomorphism object of an object in a pivotal module category and hence inherits the structure of a symmetric Frobenius algebra in $Z(\mathcal{C})$ by [Shi19b, Theorem 3.15]. □

2.3 The effect of the S-transformation

A final ingredient is needed to describe the effect of the S-transformation explicitly: For any finite braided tensor category $\mathcal{C}$, the maps $X^\vee \otimes X \longrightarrow Y \otimes Y^\vee$ given by the double braiding



induce a map

$$\mathbb{D} : \mathbb{F} \longrightarrow \mathbb{A} \ ,$$

the so-called *Drinfeld map* [Dri90].

The Drinfeld map can be used to characterize non-degeneracy of the braiding (the definition was given on page 4 in the introduction): By [Shi19a] a braided finite tensor category is non-degenerate if and only if its Drinfeld map is an isomorphism. In particular, a finite ribbon category is modular if and only if its Drinfeld map is an isomorphism.

We may now explicitly describe the effect of the S-transformation on the differential graded conformal block of the torus explicitly as follows:

Proposition 2.7. *Let $\mathcal{C}$ be a modular category. After identification of the Hochschild complex with $\mathcal{C}(I, \mathbb{F}_\bullet)$ for a projective resolution $\mathbb{F}_\bullet$ of $\mathbb{F}$ (Proposition 2.1), the mapping class group element $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ acts by the equivalence*

$$\mathcal{C}(I, \mathbb{F}_\bullet) \xrightarrow{\ \mathbb{D}_\bullet\ } \mathcal{C}(I, \mathbb{A}_\bullet) \xrightarrow{\ \Psi_\bullet\ } \mathcal{C}(I, \mathbb{F}_\bullet) \ ,$$

where the first arrow is induced by the Drinfeld map and the second arrow by the Radford map.

It is standard in the theory of modular functors that the S-transformation acts by a composition of (some form of the) Drinfeld and the Radford map, see e.g. [GT09, Section 3] and also [GR20, Remark 2.14]. For a lot of constructions, this holds by definition. For the construction of the *differential graded* modular functor via the homotopy coherent Lego Teichmüller game from [SW20] it requires a proof, especially because the 'Radford map' from Definition 2.5 was named without a detailed comparison to other definitions. The proof of Proposition 2.7 is an unpacking of the homotopy coherent Lego Teichmüller game combined with a few algebraic results. The reader willing to accept that the S-transformation, within the framework of differential graded modular functors, acts as described in Proposition 2.7 can safely skip ahead to Example 2.9.

For the proof of Proposition 2.7, first recall that by [ENO04, Proposition 4.5] any modular category is unimodular, and we will tacitly assume in the sequel that an isomorphism $D \cong I$ has been fixed for any modular category.

Lemma 2.8. *For any modular category $\mathcal{C}$, the automorphism $\mathbb{S} : \mathbb{F} \longrightarrow \mathbb{F}$ from [Lyu95b, Definition 6.3] satisfies*

$$\mathbb{S} = \Psi \circ \mathbb{D} \ . \tag{2.10}$$

Proof. The choice of an isomorphism $D \cong I$ fixes a non-zero morphism $\mathsf{A} \longrightarrow I$, namely the image of the identity of I under the isomorphism

$$\mathcal{C}(I,I) = \mathcal{C}(I,UI) \cong Z(\mathcal{C})(LI,I) \overset{L\cong R}{\cong} Z(\mathcal{C})(RI,I) = Z(\mathcal{C})(\mathsf{A},I) \ .$$

Explicitly, this is the morphism

$$\mathsf{A} = RI \cong LI = LUI \longrightarrow I \ , \tag{2.11}$$

where the first isomorphism is the component of the isomorphism $R \cong L$ at I and $LUI \longrightarrow I$ is the counit of the adjunction $L \dashv U$ evaluated at the monoidal unit I (this gives us also the counit of the coalgebra $\mathsf{F} = LI$). The morphism (2.11) is the Frobenius form giving us the Frobenius structure for A mentioned in Theorem 2.4, see also [Shi17a, Remark 6.2]. After applying the forgetful functor U to (2.11), we obtain by Definition 2.5 the Frobenius form

$$\lambda : \mathbb{A} \xrightarrow{\ \Psi\ } \mathbb{F} \xrightarrow{\ \varepsilon\ } I \tag{2.12}$$

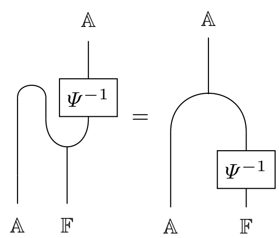
for $\mathbb{A}$, where Ψ is the Radford map and ε the counit of $\mathbb{F}$. From [Shi17a, Theorem 6.1 (3)] we conclude that $\Psi : \mathbb{A} \longrightarrow \mathbb{F}$ is a morphism of right $\mathbb{A}$-modules, where $\mathbb{A}$ is seen as right regular module over itself and $\mathbb{A}$ acts on $\mathbb{F}$ via

$$\mathbb{F} \otimes \mathbb{A} \xrightarrow{\ \delta\otimes\mathbb{A}\ } \mathbb{F} \otimes \mathbb{F} \otimes \mathbb{A} \xrightarrow{\ \mathbb{F}\otimes d_{\mathbb{A}}\ } \mathbb{F} \ .$$

The isomorphism $\Psi : \mathbb{A} \longrightarrow \mathbb{F}$ of right $\mathbb{A}$-modules equivalently describes the Frobenius algebra structure on $\mathbb{A}$ with Frobenius form (2.12) as follows from general statements on Frobenius algebras [FS08, Proposition 8 & 9]. Since any modular category is unimodular and pivotal, A (and then also $\mathbb{A}$) is a *symmetric* Frobenius algebra by Proposition 2.6. One way of describing the symmetry is by the statement $\Psi^\vee = \Psi : \mathbb{A} \longrightarrow \mathbb{F}$, where $\Psi^\vee$ is seen as a map $\mathbb{A} \longrightarrow \mathbb{F}$ again by the pivotal structure. Using [FS08, Lemma 5], one can see that $\Psi : \mathbb{A} \longrightarrow \mathbb{F}$ is also a map of left $\mathbb{A}$-modules, where $\mathbb{A}$ carries the left regular action and the left $\mathbb{A}$-action on $\mathbb{F}$ is given by

$$\mathbb{A} \otimes \mathbb{F} \xrightarrow{\ \mathbb{A}\otimes\delta\ } \mathbb{A} \otimes \mathbb{F} \otimes \mathbb{F} \xrightarrow{\ d_{\mathbb{F}}\otimes\mathbb{F}\ } \mathbb{F} \ .$$

But then $\Psi^{-1} : \mathbb{F} \longrightarrow \mathbb{A}$ is also a map of left $\mathbb{A}$-modules, which in the graphical calculus can be expressed as follows:

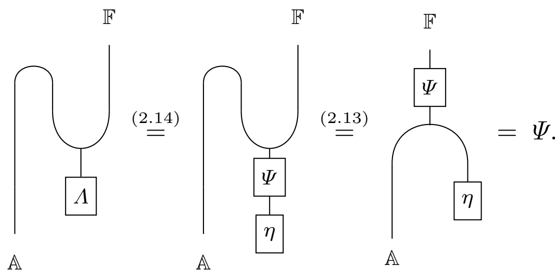


(2.13)

The map $\Lambda := \lambda^\vee : I \longrightarrow \mathbb{F}$ is given by the composition

$$\Lambda : I \xrightarrow{\ \eta\ } \mathbb{A} \xrightarrow{\ \Psi\ } \mathbb{F} \ , \tag{2.14}$$

where $\eta = \varepsilon^\vee : I \longrightarrow \mathbb{A}$ is the unit of the algebra $\mathbb{A}$ (we have used here $\Psi^\vee = \Psi$ which holds by symmetry). This implies



(2.15)

Since $\mathcal{C}$ is braided, $\mathbb{F}$ comes with the structure a Hopf algebra in $\mathcal{C}$ [Lyu95b]. By [Shi17a, Theorem 6.9] Λ is a two-sided integral for the Hopf algebra $\mathbb{F}$.

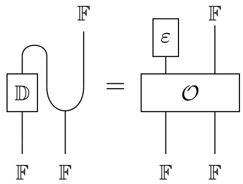
The automorphism $\mathbb{S} : \mathbb{F} \longrightarrow \mathbb{F}$ [Lyu95b, Definition 6.3] in the description of [FSS14, Eq. (2.16)] is given by

$$\mathbb{S} := (\varepsilon \otimes \mathbb{F}) \circ \mathcal{O} \circ (\mathbb{F} \otimes \Lambda) \ , \tag{2.16}$$

where the map $\mathcal{O} : \mathbb{F} \otimes \mathbb{F} \longrightarrow \mathbb{F} \otimes \mathbb{F}$ is induced the double braiding, more precisely by the maps

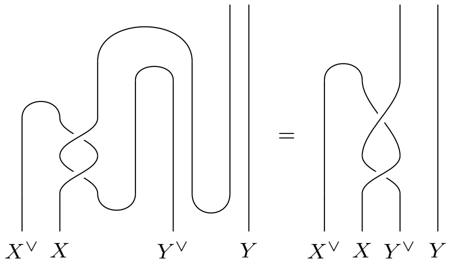
$$X \otimes X^\vee \otimes Y \otimes Y^\vee \xrightarrow{X \otimes (c_{Y,X^\vee} \circ c_{X^\vee,Y}) \otimes Y^\vee} X \otimes X^\vee \otimes Y \otimes Y^\vee \ .$$

As the next step, we verify:



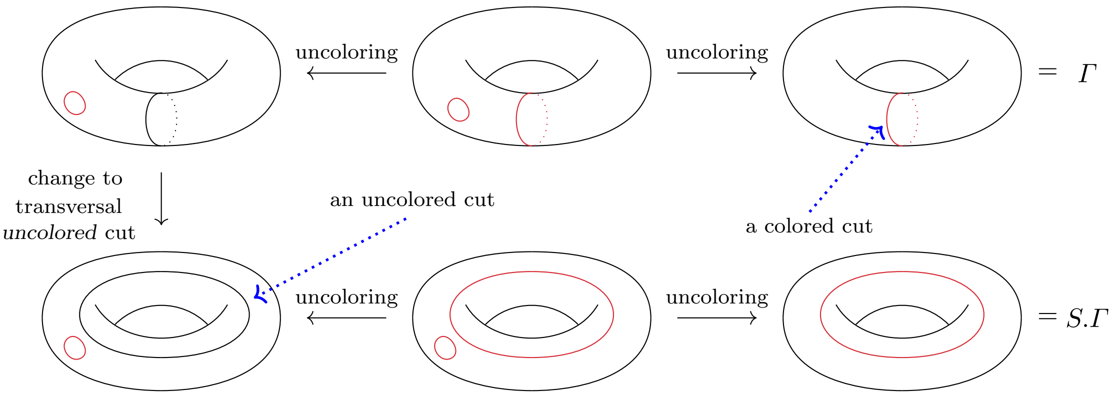
(2.17)

By the universal property of the coend $\mathbb{F}$ it suffices to verify this equality after restriction along $X^\vee \otimes X \otimes Y^\vee \otimes Y \longrightarrow \mathbb{F} \otimes \mathbb{F}$ for $X, Y \in \mathcal{C}$, i.e. after precomposition with the structure maps of the coend. Then the left hand side of (2.17) is



and hence coincides with the restriction of the right hand side of (2.17). This proves (2.17). Now we precompose with the integral Λ in the respective right slot on the left and right hand side of (2.17). On the left hand side, we then find $\Psi \circ \mathbb{D}$ by (2.15); on the right hand side, we find $\mathbb{S}$ by (2.16). This shows (2.10) and concludes the proof. $\square$

Proof of Proposition 2.7. We find the following zigzag in the category of colored markings (or rather colored cut systems because we suppress the graph decoration) on the torus from the colored marking Γ from (2.2) (whose marked block was the Hochschild complex) to its image $S.\Gamma$ under the S-transformation:



By unpacking the homotopy coherent Lego Teichmüller game used in [SW20, Section 4 & 5] it now follows that, under the equivalence $\int_{\mathbb{L}}^{X \in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \simeq \mathcal{C}(I, \mathbb{F}_\bullet)$ from Proposition 2.1, the action of S is given by $\mathcal{C}(I, \mathbb{F}_\bullet) \xrightarrow{\mathbb{S}_\bullet} \mathcal{C}(I, \mathbb{F}_\bullet)$, where $\mathbb{S} : \mathbb{F} \longrightarrow \mathbb{F}$ is Lyubashenko's automorphism (this is because $\mathbb{S}$, by definition, is assigned to the change to a transversal *uncolored* cut). Now the assertion follows from Lemma 2.8. $\square$

Example 2.9. Let A be a ribbon factorizable Hopf algebra. Then the category of finite-dimensional A-modules is modular (see [NTV03] for the semisimple case and e.g. [LMSS20, Section 2.3] for the non-semisimple case). By [KL01, Theorem 7.4.13] the Lyubashenko coend $\mathbb{F}$ is isomorphic to A^*_{coadj}, the dual of A with coadjoint A-action

$$A \otimes A^* \longrightarrow A^* \ , \quad a \otimes \alpha \longmapsto (b \longmapsto \alpha\,(S(a'ba''))\ ,$$

where $\Delta a = a' \otimes a''$ is the Sweedler notation for the coproduct and S is the antipode. The Hochschild complex of A is equivalent to $\mathrm{Hom}_A(k, A^*_{\mathrm{coadj}_\bullet})$, where $A^*_{\mathrm{coadj}_\bullet} \longrightarrow A^*_{\mathrm{coadj}}$ is a projective resolution. If A is the Drinfeld double $D(G)$ of a finite group G (the category will be non-semisimple if the characteristic of k divides $|G|$), the complex $\mathrm{Hom}_A(k, A^*_{\mathrm{coadj}_\bullet})$ is equivalent to $C_*(\mathsf{PBun}_G(\mathbb{T}^2); k)$, the k-chains on the groupoid $\mathsf{PBun}_G(\mathbb{T}^2)$ of G-bundles over the torus [SW19, Lemma 3.2], and the mapping class group action is the obvious geometric one. For an arbitrary ribbon factorizable Hopf algebra A, the mapping class group action comes from an action of the braid group B_3 on three strands on A^*_{coadj} [LMSS18], i.e. it descends along the epimorphism $B_3 \longrightarrow \mathrm{SL}(2,\mathbb{Z})$. This remains even true for arbitrary modular categories [SW19].

3 Little disks algebras from homotopy invariants of braided commutative algebras

One of the new insights of this article is a connection between a generalization of the Verlinde algebra and Deligne's Conjecture by means of the mapping class group action on differential graded conformal blocks. For a finite tensor category $\mathcal{C}$, the Hochschild (co)chain complex and, in the modular case, the mapping class group action on it are to a large extent controlled by the canonical (co)end of the finite tensor category. Therefore, the key step will be to describe the precise relation between the canonical (co)end and Deligne's Conjecture. In more detail, we will use the canonical end $\mathbb{A} \in \mathcal{C}$ of a finite tensor category $\mathcal{C}$ (or rather its lift $\mathsf{A} \in Z(\mathcal{C})$ to the Drinfeld center) to provide a solution to Deligne's Conjecture for the Hochschild cochain complex of a finite tensor category. This connection, at least when viewed superficially, is surprising because the canonical end makes heavy use of the monoidal structure of a finite tensor category while the E_2-structure appearing in Deligne's Conjecture is only sensitive to the linear structure of $\mathcal{C}$.

3.1 The finite homotopy end and the Hochschild cochain complex of a finite tensor category

Let $\mathcal{A}$ be a linear category over k. Then its Hochschild cochain complex $\int^{\mathbb{R}}_{a\in\mathcal{A}} \mathcal{A}(a,a)$ is the homotopy end over the endomorphism spaces of objects in $\mathcal{A}$, i.e. the cochain complex of vector spaces which in cohomological degree $n \geq 0$ is given by

$$\left(\int^{\mathbb{R}}_{a\in\mathcal{A}} \mathcal{A}(a,a) \right)^n = \begin{cases} \prod_{a_0\in\mathcal{A}} \mathcal{A}(a_0,a_0) & \text{for } n=0\ , \\ \prod_{a_0,\dots,a_n\in\mathcal{A}} \mathrm{Hom}_k\left(\mathcal{A}(a_1,a_0)\otimes\dots\otimes\mathcal{A}(a_n,a_{n-1}), \mathcal{A}(a_n,a_0)\right) & \text{for } n\geq 1\ . \end{cases}$$

The differential comes as usual from the composition in $\mathcal{A}$.

On the Hochschild cochain complex, one may define the *cup product* $\smile$: Let φ and ψ be a p-cochain and a q-cochain, respectively, and let $(a_0,\dots,a_{p+q})$ be a $p+q$-tuple of objects in $\mathcal{A}$. Then the $(a_0,\dots,a_{p+q})$-component $(\varphi\smile\psi)_{a_0,\dots,a_{p+q}}$ of the $p+q$-cochain $\varphi\smile\psi$ is given by

$$(\varphi\smile\psi)_{a_0,\dots,a_{p+q}} := \varphi_{a_0,\dots,a_p} \circ_{a_p} \psi_{a_p,\dots,a_{p+q}}\ ,$$

where $\circ_{a_p}$ is the composition in $\mathcal{A}$ over a_p. The cup product induces a graded commutative product on Hochschild cohomology that by work of Gerstenhaber [Ger63] extends to an algebraic structure that today is known as a *Gerstenhaber algebra* (we will review the Gerstenhaber bracket once we need it). As briefly recalled in the introduction, by Deligne's Conjecture this Gerstenhaber structure lifts to the structure of an algebra over the little disks operad E_2 (we refer to the monograph [Fre17] for an introduction to operads).

For any finite tensor category $\mathcal{C}$, $X \in \mathcal{C}$ and a finite-dimensional vector space W, recall that we denote by $X^W = W^* \otimes X$ the powering of X by W. We may now define the objects

$$\prod_{X_0,\dots,X_n\in\mathsf{Proj}\,\mathcal{C}} (X_0\otimes X_n^\vee)^{\mathcal{C}(X_n,X_{n-1})\otimes\dots\otimes\mathcal{C}(X_1,X_0)} \tag{3.1}$$

that we organize into a cosimplicial object

$$\prod_{X_0 \in \mathsf{Proj}\,\mathcal{C}} X_0 \otimes X_0^\vee \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \prod_{X_0, X_1 \in \mathsf{Proj}\,\mathcal{C}} (X_0 \otimes X_1^\vee)^{\mathcal{C}(X_1, X_0)} \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \dots \, . \tag{3.2}$$

which a priori lives in a completion of $\mathcal{C}$ by infinite products. We denote by $\int^{\mathrm{f}\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee$ the differential graded object in $\mathcal{C}$ which is obtained as the totalization of the restriction to any finite collection of projective objects in $\mathcal{C}$ that contains a projective generator.

Lemma 3.1. *Let $\mathcal{C}$ be a finite tensor category. The differential graded object $\int^{\mathrm{f}\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee$ in $\mathcal{C}$ is well-defined up to equivalence and an injective resolution of the canonical end $\mathbb{A} = \int_X X \otimes X^\vee$.*

Proof. We apply the duality functor $-^\vee$ to the simplicial object (2.4) and observe that, after a substitution of dummy variables, we obtain the cosimplicial object (3.2). Since $\mathbb{F}^\vee \cong \mathbb{A}$, we now obtain the well-definedness of $\int^{\mathrm{f}\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee$ and the fact that it is an injective resolution of $\mathbb{A}$ (all of this crucially uses that in a finite tensor category the projective objects are precisely the injective ones). $\square$

Let $X_0, \dots, X_{p+q}$ be a family of projective objects in $\mathcal{C}$ from the finite collection of projective objects used to define $\int^{\mathrm{f}\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee$. The evaluation $d_{X_p} : X_p^\vee \otimes X_p \longrightarrow I$ of X_p induces a map

$$\begin{array}{r} \left(X_0 \otimes X_p^\vee\right)^{\mathcal{C}(X_1,X_0)\otimes \cdots \otimes \mathcal{C}(X_p,X_{p-1})} \\ \otimes \left(X_p \otimes X_{p+q}^\vee\right)^{\mathcal{C}(X_p,X_{p+1})\otimes \cdots \otimes \mathcal{C}(X_{p+q-1},X_{p+q})} \end{array} \longrightarrow \left(X_0 \otimes X_{p+q}^\vee\right)^{\mathcal{C}(X_1,X_0)\otimes \cdots \otimes \mathcal{C}(X_{p+q},X_{p+q-1})} \, . \tag{3.3}$$

We may now define map

$$\gamma^\bullet_{p+q} : \left(\int^{\mathrm{f}\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee \right)^p \otimes \left(\int^{\mathrm{f}\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee \right)^q \longrightarrow \left(\int^{\mathrm{f}\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee \right)^{p+q} \tag{3.4}$$

as follows: By the universal property of the product it suffices to give the component for any family $X_0, \dots, X_{p+q}$ of projective objects in $\mathcal{C}$. We define this component to be the map that projects to the component of $X_0, \dots, X_p$ for the first factor and to $X_p, \dots, X_{p+q}$ for the second factor and applies the map (3.3). A direct computation shows that (3.4) yields a chain map and an associative and unital multiplication $\gamma^\bullet$ on $\int^{\mathrm{f}\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee$. This is a model for a lift of the algebra structure $\gamma : \mathbb{A} \otimes \mathbb{A} \longrightarrow \mathbb{A}$ from (2.9) to the injective resolution as one can directly verify:

Lemma 3.2. *For any finite tensor category $\mathcal{C}$, the coaugmentation $\mathbb{A} \longrightarrow \int^{\mathrm{f}\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee$ is an equivalence*

$$(\ \mathbb{A}\ ,\ \gamma\ ) \xrightarrow{\ \simeq\ } \left(\int^{\mathrm{f}\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee \ ,\ \gamma^\bullet \right)$$

of differential graded algebras.

Proposition 3.3. *For any finite tensor category $\mathcal{C}$, there is a canonical equivalence*

$$\left(\int^{\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)\ ,\ \smile \right) \xrightarrow{\ \simeq\ } \left(\mathcal{C}\left(I, \int^{\mathrm{f}\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee \right)\ ,\ \gamma^\bullet \right)$$

of differential graded algebras.

We refer to $\int^{\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ as the *Hochschild cochain complex of $\mathcal{C}$*. The fact that we use only the subcategory $\mathsf{Proj}\,\mathcal{C} \subset \mathcal{C}$ is always implied here.

Remark 3.4. If we ignore the algebra structure, Proposition 3.3 reduces (on cohomology) to an isomorphism $HH^*(\mathcal{C}) \cong \mathrm{Ext}^*_{\mathcal{C}}(I, \mathbb{A})$ which is well-known for Hopf algebras and goes back to Cartan and Eilenberg [CE56] as reviewed in [Bic13, Proposition 2.1] and formulated for arbitrary finite tensor categories in [Shi20, Corollary 7.5]. The new aspect in Proposition 3.3 is the compatibility with the algebra structures.

Proof of Proposition 3.3. The finite homotopy end $\int^{\mathrm{f}\mathbb{R}}_{X\in \mathsf{Proj}\,\mathcal{C}} X\otimes X^\vee$ runs by definition over a full subcategory of $\mathsf{Proj}\,\mathcal{C}$ consisting of a family of finitely many objects that include a projective generator. For this proof, we denote this subcategory by $\mathcal{F}\subset \mathsf{Proj}\,\mathcal{C}$. Now by definition

$$\int^{\mathrm{f}\mathbb{R}}_{X\in \mathsf{Proj}\,\mathcal{C}} X\otimes X^\vee = \int^{\mathbb{R}}_{X\in\mathcal{F}} X\otimes X^\vee \ .$$

Moreover, we set

$$\int^{\mathrm{f}\mathbb{R}}_{X\in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) := \int^{\mathbb{R}}_{X\in\mathcal{F}} \mathcal{C}(X,X) = \int^{\mathbb{R}}_{X\in\mathcal{F}} \mathcal{F}(X,X) \ . \tag{3.5}$$

(i) We define a map $\int^{\mathbb{R}}_{X\in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)\longrightarrow \mathcal{C}\left(I, \int^{\mathrm{f}\mathbb{R}}_{X\in \mathsf{Proj}\,\mathcal{C}} X\otimes X^\vee\right)$ by the commutativity of the triangle

$$\begin{tikzcd}
\int^{\mathbb{R}}_{X\in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \arrow[rr, "(*)"] \arrow[dr, "\simeq"'] & & \mathcal{C}\left(I, \int^{\mathrm{f}\mathbb{R}}_{X\in \mathsf{Proj}\,\mathcal{C}} X\otimes X^\vee\right) \\
& \int^{\mathrm{f}\mathbb{R}}_{X\in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \ , \arrow[ur, "\cong"'] &
\end{tikzcd}$$

where the maps appearing in the triangle are defined as follows:

- The equivalence on the left comes from the restriction to $\mathcal{F}$. (This is the dual version of the Agreement Principle of McCarthy [MCar94] and Keller [Kel99], see [SW19, Section 2.2] for a review.)
- The isomorphism on the right comes, in each degree, from the isomorphisms

$$\begin{aligned}
&\prod_{X_0,\dots,X_n\in\mathcal{F}} \mathcal{C}(X_n,X_0)^{\mathcal{C}(X_1,X_0)\otimes\cdots\otimes\mathcal{C}(X_n,X_{n-1})} \\
\cong\ & \mathcal{C}\left(I, \prod_{X_0,\dots,X_n\in\mathcal{F}} (X_0\otimes X_n^\vee)^{\mathcal{C}(X_1,X_0)\otimes\cdots\otimes\mathcal{C}(X_n,X_{n-1})}\right)
\end{aligned}$$

  induced by duality.

Now clearly $(*)$ is an equivalence.

(ii) Since $\mathcal{F}\subset\mathsf{Proj}\,\mathcal{C}$ is a full subcategory, the cup product of $\int^{\mathbb{R}}_{X\in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ restricts to the cup product of $\int^{\mathbb{R}}_{X\in\mathcal{F}} \mathcal{F}(X,X)$. With the notation from (3.5), this means that

$$\left(\int^{\mathbb{R}}_{X\in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)\ ,\ \smile\right) \xrightarrow{\ \simeq\ } \left(\int^{\mathrm{f}\mathbb{R}}_{X\in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)\ ,\ \smile\right)$$

is an equivalence of differential graded algebras. Hence, it remains to prove that

$$\left(\int^{\mathrm{f}\mathbb{R}}_{X\in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)\ ,\ \smile\right) \cong \left(\mathcal{C}\left(I, \int^{\mathrm{f}\mathbb{R}}_{X\in \mathsf{Proj}\,\mathcal{C}} X\otimes X^\vee\right)\ ,\ \gamma^\bullet\right)$$

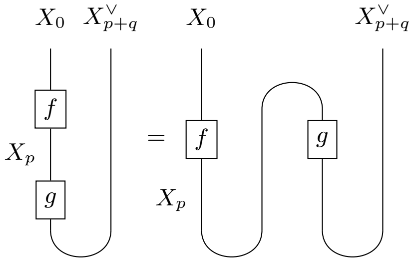
is an isomorphism of algebras. After unpacking the definition of the multiplications on both sides, this means that for $X_0,\dots,X_{p+q}\in\mathcal{F}$ and the abbreviations

$$\begin{aligned}
V &:= \mathcal{C}(X_1,\dots,X_0)\otimes\cdots\otimes\mathcal{C}(X_p,X_{p-1}) \ , \\
W &:= \mathcal{C}(X_{p+1},X_p)\otimes\cdots\otimes\mathcal{C}(X_{p+q},X_{p+q-1}) \ ,
\end{aligned}$$

the square

$$
\begin{tikzcd}
\mathcal{C}(X_p, X_0)^V \otimes \mathcal{C}(X_{p+q}, X_p)^W \arrow[r, "\smile"] \arrow[d, "\cong"] & \mathcal{C}(X_{p+q}, X_0)^{V \otimes W} \arrow[d, "\cong"] \\
\mathcal{C}\left(I, X_0 \otimes {X_p^\vee}^V\right) \otimes \mathcal{C}\left(I, X_p \otimes {X_{p+q}^\vee}^W\right) \arrow[r, "d_{X_p}"] & \mathcal{C}\left(I, X_0 \otimes {X_{p+q}^\vee}^{V\otimes W}\right)
\end{tikzcd} \quad ,
$$

in which the vertical isomorphisms come from duality, commutes. The exponentials are just tensored together in both clockwise and counterclockwise direction. Since the cup product composes over X_p, the commutativity of the square now boils down to the basic equality



of morphisms $I \longrightarrow X_0 \otimes X_{p+q}^\vee$.

$\square$

3.2 The braided operad of a braided commutative algebra

Our goal is to use the canonical end (or its resolution) to endow the complex $\mathcal{C}\left(I, \int^{\mathbb{fR}}_{X \in \mathsf{Proj}\, \mathcal{C}} X \otimes X^\vee\right)$ with the structure of an E_2-algebra *by means of additional structure on* $\int^{\mathbb{fR}}_{X \in \mathsf{Proj}\, \mathcal{C}} X \otimes X^\vee$. In this subsection, we describe a relatively technical step to this end: We construct a certain auxiliary acyclic *braided operad* for a given braided commutative algebra.

First, let us briefly recall the definition of a braided operad from [Fre17, Section 5.1.1]: Let $\mathcal{M}$ be a closed and bicomplete symmetric monoidal category. A *braided operad* $\mathcal{O}$ consists of objects $\mathcal{O}(n) \in \mathcal{M}$ of n-ary operations, where $n \geq 0$, that carry an action of the braid group B_n on n strands (as commonplace in the theory of operads, we will work with *right* actions throughout), a unit $I \longrightarrow \mathcal{O}(1)$ for 1-ary operations (here I is the unit of $\mathcal{M}$) and composition maps

$$\mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n) \longrightarrow \mathcal{O}(m_1 + \cdots + m_n)$$

such that the composition is associative, unital and compatible with the braid group actions. Morphisms of braided operads are defined analogously to the symmetric case.

There is an obvious restriction functor $\mathsf{Res} : \mathsf{SymOp}(\mathcal{M}) \longrightarrow \mathsf{BrOp}(\mathcal{M})$ from symmetric operads in $\mathcal{M}$ to braided operads in $\mathcal{M}$ (it restricts arity-wise along the epimorphisms from braid groups to symmetric groups). This restriction functor has a left adjoint, the *symmetrization*

$$\mathsf{Sym} \ : \ \mathsf{BrOp}(\mathcal{M}) \ \rightleftarrows \ \mathsf{SymOp}(\mathcal{M}) \ : \ \mathsf{Res} \ , \tag{3.6}$$

that takes arity-wise orbits of pure braid group actions, i.e. $(\mathsf{Sym}\mathcal{O})(n) = \mathcal{O}(n)/P_n$ for $n \geq 0$, where P_n is the pure braid group on n strands.

Let $\mathcal{B}$ be an $\mathcal{M}$-enriched braided monoidal category. Then for any $A \in \mathcal{B}$, the objects $\mathrm{End}_A(n) := [A^{\otimes n}, A]$ for $n \geq 0$ (we denote here by $[-,-]$ the $\mathcal{M}$-valued hom) form a braided operad in $\mathcal{M}$, the *braided endomorphism operad of A*. If $\mathcal{O}$ is a braided operad in $\mathcal{M}$, a *braided $\mathcal{O}$-algebra in $\mathcal{B}$* is an object $A \in \mathcal{B}$ and a map $\mathcal{O} \longrightarrow \mathrm{End}_A$ of braided operads.

A braided commutative algebra $\mathsf{T} \in \mathcal{B}$ in a braided monoidal category $\mathcal{B}$ is an algebra whose multiplication $\mu : \mathsf{T} \otimes \mathsf{T} \longrightarrow \mathsf{T}$ satisfies $\mu \circ c_{\mathsf{T},\mathsf{T}} = \mu$, where $c_{\mathsf{T},\mathsf{T}} : \mathsf{T} \otimes \mathsf{T} \longrightarrow \mathsf{T} \otimes \mathsf{T}$ is the braiding. Although we will treat general braided commutative algebras (and braided commutative coalgebras which are defined dually), it is instructive to think of the example of the canonical algebra and the canonical coalgebra in the Drinfeld center that were defined in Section 2.2:

Lemma 3.5 *(Davydov, Müger, Nikshych, Ostrik [DMNO13, Lemma 3.5]).* *For any finite tensor category $\mathcal{C}$, the canonical algebra $\mathsf{A} \in Z(\mathcal{C})$ is braided commutative and the canonical coalgebra $\mathsf{F} \in Z(\mathcal{C})$ is braided cocommutative.*

In [DMNO13] this Lemma is only given in the semisimple case, but the argument can be extended to the non-semisimple case as well, see [Shi20, Appendix A.3] for a formulation in terms of module categories that also covers the situation as given in Lemma 3.5.

For a braided finite tensor category $\mathcal{B}$ and a braided commutative algebra $\mathsf{T} \in \mathcal{B}$, denote by $\iota : \mathsf{T} \longrightarrow \mathsf{T}^\bullet$ an injective resolution of T in $\mathcal{B}$; the map ι will be referred to as *coaugmentation*. For $n \geq 0$, we define the map

$$k \longrightarrow \mathcal{B}(\mathsf{T}^{\otimes n}, \mathsf{T}^\bullet) \tag{3.7}$$

that selects the map $\mathsf{T}^{\otimes n} \xrightarrow{\mu} \mathsf{T} \xrightarrow{\iota} \mathsf{T}^\bullet$ defined as the concatenation of the (n-fold) multiplication μ of T with the coaugmentation ι of the injective resolution. The map itself is not very interesting, but the non-trivial point is that it is actually a map of chain complexes *with B_n-action*, where the B_n-action on k is trivial and the one on $\mathcal{B}(\mathsf{T}^{\otimes n}, \mathsf{T}^\bullet)$ comes by virtue of $\mathcal{B}$ being a braided category. The fact that (3.7) is really B_n-equivariant follows from the assumption that T is braided commutative.

Using again that $\mathcal{B}$ is braided, the mapping complex $\mathcal{B}(\mathsf{T}^{\bullet \otimes n}, \mathsf{T}^\bullet)$ comes with a B_n-action. The precomposition with the map $\iota^{\otimes n} : \mathsf{T}^{\otimes n} \longrightarrow \mathsf{T}^{\bullet \otimes n}$ yields a map

$$\left(\iota^{\otimes n}\right)^* : \mathcal{B}(\mathsf{T}^{\bullet \otimes n}, \mathsf{T}^\bullet) \longrightarrow \mathcal{B}(\mathsf{T}^{\otimes n}, \mathsf{T}^\bullet) \ . \tag{3.8}$$

This map is also B_n-equivariant. We may now define the chain complex $J_\mathsf{T}(n)$ as a pullback

$$\begin{tikzcd} J_\mathsf{T}(n) \arrow[r] \arrow[d] & \mathcal{B}(\mathsf{T}^{\bullet \otimes n}, \mathsf{T}^\bullet) \arrow[d, "(\iota^{\otimes n})^*"] \\ k \arrow[r, "(3.7)"'] & \mathcal{B}(\mathsf{T}^{\otimes n}, \mathsf{T}^\bullet) \ , \end{tikzcd} \tag{3.9}$$

of differential graded k-vector spaces with B_n-action (it will be explained in the proof of Proposition 3.6 below that $(\iota^{\otimes n})^*$ is, in particular, a fibration, so that $J_\mathsf{T}(n)$ is also a homotopy pullback).

Proposition 3.6. *Let $\mathcal{B}$ be a braided finite tensor category and $\mathsf{T} \in \mathcal{B}$ a braided commutative algebra. The chain complexes $J_\mathsf{T}(n)$ defined in* (3.9) *for $n \geq 0$ naturally form a braided operad J_T in differential graded k-vector spaces such that the following holds:*

(i) The operad J_T is acyclic in the sense that it comes with a canonical trivial fibration $J_\mathsf{T} \longrightarrow k$, i.e. J_T is a model for the braided commutative operad.

(ii) The maps $J_\mathsf{T}(n) \longrightarrow \mathcal{B}(\mathsf{T}^{\bullet \otimes n}, \mathsf{T}^\bullet)$ from (3.9) *endow $\mathsf{T}^\bullet$ with the structure of a braided J_T-algebra.*

Proof. We first establish the braided operad structure: The complexes $J_\mathsf{T}(n)$ come with an B_n-action by definition. Moreover, the identity of $\mathsf{T}^\bullet$ seen as map $k \longrightarrow \mathcal{B}(\mathsf{T}^\bullet, \mathsf{T}^\bullet)$ yields a map $k \longrightarrow J_\mathsf{T}(1)$ that we define as a unit. In order to define the operadic composition, let $m_1, \ldots, m_n \geq 0$ be given. By definition of J_{T} we obtain the maps $(*)$ and $(**)$ in the following diagram:

$$\begin{tikzcd}
J_{\mathsf{T}}(n) \otimes \bigotimes_{j=1}^{n} J_{\mathsf{T}}(m_j) \arrow[ddr, dashed, "\exists !"] \arrow[drr, bend left, "(**)"] \arrow[dddr, bend right, "(*)"'] & & \\
& & \mathcal{B}(\mathsf{T}^{\bullet \otimes n}, \mathsf{T}^{\bullet}) \otimes \bigotimes_{j=1}^{n} \mathcal{B}(\mathsf{T}^{\bullet \otimes m_j}, \mathsf{T}^{\bullet}) \arrow[d, "\text{composition in } \mathcal{B}"] \\
& J_{\mathsf{T}}(m_1 + \cdots + m_n) \arrow[r] \arrow[d] & \mathcal{B}(\mathsf{T}^{\bullet \otimes (m_1 + \cdots + m_n)}, \mathsf{T}^{\bullet}) \arrow[d, "\left(\iota^{\otimes(m_1+\cdots+m_n)}\right)^*"] \\
& k \arrow[r] & \mathcal{B}(\mathsf{T}^{\otimes (m_1 + \cdots + m_n)}, \mathsf{T}^{\bullet}) \ .
\end{tikzcd}$$

It is straightforward to see that the outer pentagon commutes. Now by the universal property of the pullback there is unique map $J_{\mathsf{T}}(n) \otimes \bigotimes_{j=1}^{n} J_{\mathsf{T}}(m_j) \longrightarrow J_{\mathsf{T}}(m_1 + \cdots + m_n)$ making the entire diagram commute. We define this to be the needed operadic composition map. The composition can be seen to be equivariant. Since it is induced by composition in $\mathcal{B}$, it is associative and unital with respect to the identity (which we defined as operadic unit). The operad structure on J_{T} is defined in such a way that statement (ii) holds by construction.

It remains to prove (i): First observe that the maps $J_{\mathsf{T}}(n) \longrightarrow k$ are B_n-equivariant by construction and are also compatible with composition. Therefore, we only need to show that $J_{\mathsf{T}}(n) \longrightarrow k$ for fixed $n \geq 0$ is a trivial fibration. Indeed, the exactness of the monoidal product in $\mathcal{B}$ ensures that $\iota^{\otimes n} : \mathsf{T}^{\otimes n} \longrightarrow \mathsf{T}^{\bullet \otimes n}$ is again an injective resolution, i.e. a trivial cofibration in the injective model structure on complexes in $\mathcal{B}$. Since $\mathsf{T}^{\bullet}$ is fibrant in this model structure, the precomposition with $\iota^{\otimes n}$ in (3.8) is a trivial fibration. Now $J_{\mathsf{T}}(n) \longrightarrow k$, as the pullback of a trivial fibration according to its definition in (3.9), is a trivial fibration as well. □

3.3 Homotopy invariants of a braided commutative algebra as an E_2-algebra

The construction from Proposition 3.6 can be used for the construction of differential graded E_2-algebras. In order to see this, let us record the following two straightforward Lemmas.

It is well-known that a symmetric lax monoidal functor preserves operadic algebras. The following Lemma is a braided version of this fact:

Lemma 3.7. *Let $F : \mathcal{B} \longrightarrow \mathcal{B}'$ be an enriched braided lax monoidal functor between braided monoidal categories enriched over $\mathcal{M}$ and let $\mathcal{O}$ be a braided operad in $\mathcal{M}$. Then for any braided $\mathcal{O}$-algebra A in $\mathcal{B}$, the image $F(A)$ naturally comes with the structure of a braided $\mathcal{O}$-algebra.*

Proof. The structure maps that turn $F(A)$ into a braided $\mathcal{O}$-algebra are

$$\mathcal{O}(n) \longrightarrow [A^{\otimes n}, A] \xrightarrow{\ F\ } \left[F\left(A^{\otimes n}\right), F(A)\right] \longrightarrow \left[\left(F\left(A\right)\right)^{\otimes n}, F(A)\right] , \quad n \geq 0 \ .$$

The first map is the structure map of A, the third map precomposes with the maps $(F(A))^{\otimes n} \longrightarrow F\left(A^{\otimes n}\right)$ that are a part of the lax monoidal structure of F. These maps are B_n-equivariant because F is braided. □

Lemma 3.8. *Let $\mathcal{O}$ be a braided operad in $\mathcal{M}$ and A a braided $\mathcal{O}$-algebra in a* symmetric *monoidal category $\mathcal{B}$ enriched over $\mathcal{M}$. Then A induces in a canonical way an algebra over the symmetrization $\mathsf{Sym}\,\mathcal{O}$ of $\mathcal{O}$. More precisely, the structure map $\mathcal{O} \longrightarrow \mathrm{End}_A$ canonically factors through the unit $\mathcal{O} \longrightarrow \mathsf{Res}\,\mathsf{Sym}\,\mathcal{O}$ of the adjunction $\mathsf{Sym} \dashv \mathsf{Res}$ from (3.6).*

Proof. Pure braid group elements act trivially on $[A^{\otimes n}, A]$ because $\mathcal{B}$ is symmetric. As a consequence, the structure maps of A factor as

$$\begin{tikzcd}
\mathcal{O}(n) \arrow[rr] \arrow[dr] & & {[A^{\otimes n}, A]} \ . \\
& \mathsf{Sym}\,\mathcal{O}(n) = \mathcal{O}(n)/P_n \arrow[ur, dashed] &
\end{tikzcd}$$

This implies the assertion. □

Proposition 3.9. *Let $\mathcal{B}$ be a braided finite tensor category. Then for any braided commutative algebra $\mathsf{T} \in \mathcal{B}$ and any braided cocommutative coalgebra $\mathsf{K} \in \mathcal{B}$ the derived morphism space $\mathcal{B}(\mathsf{K}, \mathsf{T}^\bullet)$ is naturally an E_2-algebra.*

Proof. We denote by $\check{J}_\mathsf{T}$ a resolution of the braided operad J_T constructed in Proposition 3.6. By this we mean an arity-wise projective B_n-module, i.e. a braided operad $\check{J}_\mathsf{T}$ with arity-wise projective braid group actions and a trivial fibration $\check{J}_\mathsf{T} \longrightarrow J_\mathsf{T}$. Such a resolution exists: We could, for instance, achieve this by tensoring J_T with the braided operad obtained from universal coverings of classifying spaces of braid groups constructed by Fiedorowicz [Fie96].

Since J_T and thus $\check{J}_\mathsf{T}$ is acyclic thanks to Proposition 3.6 (i) and since $\check{J}_\mathsf{T}$ has a projective braid group action, we obtain

$$(\mathsf{Sym}\,\check{J}_\mathsf{T})(n) = \left(\check{J}_\mathsf{T}(n)\right)/P_n = C_*(BP_n; k) \simeq C_*(E_2(n); k) \ ,$$

where $C_*(-;k)$ is the functor taking k-chains. Consequently, we obtain an equivalence

$$\mathsf{Sym}\,\check{J}_\mathsf{T} \simeq C_*(E_2; k) \tag{3.10}$$

of operads. In other words, $\mathsf{Sym}\,\check{J}_\mathsf{T}$ is a model for E_2. This is an instance of the Recognition Principle for E_2 [Fie96].

If we pull back the J_T-action on $\mathsf{T}^\bullet$ from Proposition 3.6 (ii) along $\check{J}_\mathsf{T} \longrightarrow J_\mathsf{T}$, we turn $\mathsf{T}^\bullet$ into a braided $\check{J}_\mathsf{T}$-algebra.

The functor $\mathcal{B}(\mathsf{K}, -) : \mathsf{Ch}(\mathcal{B}) \longrightarrow \mathsf{Ch}_k$ is

- lax monoidal since K is a coalgebra,
- and also braided since K is braided cocommutative.

This implies by Lemma 3.7 that $\mathcal{B}(\mathsf{K}, \mathsf{T}^\bullet)$ becomes a braided $\check{J}_\mathsf{T}$-algebra, which by Lemma 3.8 induces a $\mathsf{Sym}\,\check{J}_\mathsf{T}$-algebra structure on $\mathcal{B}(\mathsf{K}, \mathsf{T}^\bullet)$ because Ch_k is symmetric. Now (3.10) yields the assertion. □

We now apply Proposition 3.9 to the canonical coalgebra $\mathsf{F} \in Z(\mathcal{C})$ to obtain a source of E_2-algebras:

Theorem 3.10. *Let $\mathbb{T} \in \mathcal{C}$ be an algebra in a finite tensor category $\mathcal{C}$ together with a lift to a braided commutative algebra $\mathsf{T} \in Z(\mathcal{C})$ in the Drinfeld center. Then the multiplication of $\mathbb{T}$ induces the structure of an E_2-algebra on the space $\mathcal{C}(I, \mathbb{T}^\bullet)$ of homotopy invariants of $\mathbb{T}$.*

Proof. By assumption we have $\mathbb{T} = U\mathsf{T}$ as algebras with the forgetful functor $U : Z(\mathcal{C}) \longrightarrow \mathcal{C}$, where T is braided commutative. Now let F be the canonical coalgebra in $Z(\mathcal{C})$, namely the image LI of the unit $I \in \mathcal{C}$ under the oplax monoidal left adjoint $L : \mathcal{C} \longrightarrow Z(\mathcal{C})$ to U. Since F is braided cocommutative by Lemma 3.5, we may apply Proposition 3.9 and find that $Z(\mathcal{C})(\mathsf{F}, \mathsf{T}^\bullet)$ comes with an E_2-structure.

Using the adjunction $L \dashv U$ observe

$$Z(\mathcal{C})(\mathsf{F}, \mathsf{T}^\bullet) = Z(\mathcal{C})(LI, \mathsf{T}^\bullet) \cong \mathcal{C}(I, U\mathsf{T}^\bullet) \ .$$

Therefore, $\mathcal{C}(I, U\mathsf{T}^\bullet)$ inherits the E_2-structure.

It remains to show that $U\mathsf{T}^\bullet$ is an injective resolution of $\mathbb{T}$: Since U is exact, the map $\mathbb{T} = U\mathsf{T} \longrightarrow U\mathsf{T}^\bullet$ is a monomorphism and an equivalence. Finally, $U\mathsf{T}^\bullet$ is also degree-wise injective because U is a right adjoint whose left adjoint L is exact by [Shi17a, Corollary 4.9] and hence preserves injective objects. □

Proposition 3.11 *(Naturality in the braided algebra)*. *Let $\mathcal{C}$ be a finite tensor category with algebras $\mathbb{T}$ and $\mathbb{U}$ in $\mathcal{C}$ with lifts T and $\mathsf{U} \in Z(\mathcal{C})$ to braided commutative algebras. Then any algebra map $\varphi : \mathbb{T} \longrightarrow \mathbb{U}$, which has the property to induce a map $\mathsf{T} \longrightarrow \mathsf{U}$ in the Drinfeld center, gives rise to a map*

$$\varphi^\bullet : \mathcal{C}(I, \mathbb{T}^\bullet) \longrightarrow \mathcal{C}(I, \mathbb{U}^\bullet)$$

of E_2-algebras.

Proof. The assumption says exactly that $\varphi : \mathsf{T} \longrightarrow \mathsf{U}$ gives us a map of algebras in $Z(\mathcal{C})$. We can extend φ to a map $\varphi^\bullet : \mathsf{T}^\bullet \longrightarrow \mathsf{U}^\bullet$ between injective resolutions such that $\varphi^\bullet \circ \iota_\mathsf{T} = \iota_\mathsf{U} \circ \varphi$ for the coaugmentations $\iota_\mathsf{T} : \mathsf{T} \longrightarrow \mathsf{T}^\bullet$ and $\iota_\mathsf{U} : \mathsf{U} \longrightarrow \mathsf{U}^\bullet$. The idea is to see $\varphi^\bullet : \mathsf{T}^\bullet \longrightarrow \mathsf{U}^\bullet$ as map of braided algebras over an acyclic braided operad and then to apply the functor $Z(\mathcal{C})(\mathsf{F}, -)$ as in the proof of Theorem 3.10. This will give us a map of E_2-algebras.

The non-obvious point is *how* to see $\varphi^\bullet : \mathsf{T}^\bullet \longrightarrow \mathsf{U}^\bullet$ as map of braided algebras over an acyclic braided operad because, according to Proposition 3.6, $\mathsf{T}^\bullet$ is a J_T-algebra while $\mathsf{U}^\bullet$ is a J_U-algebra. Both algebras are defined over different operads. Moreover, φ does *not* directly induce a map $J_\mathsf{T} \longrightarrow J_\mathsf{U}$.

Instead, we use the following construction: First we define for $n \geq 0$ the pullback of complexes with B_n-action

$$\begin{tikzcd} K_\varphi(n) \arrow[r] \arrow[d] & Z(\mathcal{C})(\mathsf{T}^{\bullet \otimes n}, \mathsf{U}^\bullet) \arrow[d, "(\iota_\mathsf{T}^{\otimes n})^*"] \\ k \arrow[r, "1 \longmapsto \left(\mathsf{T}^\otimes \xrightarrow{\mu_\mathsf{T}^n} \mathsf{T} \xrightarrow{\varphi} \mathsf{U} \xrightarrow{\iota_\mathsf{U}} \mathsf{U}^\bullet \right)"'] & Z(\mathcal{C})(\mathsf{T}^{\otimes n}, \mathsf{U}^\bullet) \ . \end{tikzcd} \tag{3.11}$$

As in the proof of Proposition 3.6, this is also a homotopy pullback. Since the right vertical map is a trivial fibration, so is $K_\varphi(n) \longrightarrow k$. The map

$$J_\mathsf{T}(n) \longrightarrow Z(\mathcal{C})(\mathsf{T}^{\bullet \otimes n}, \mathsf{T}^\bullet) \xrightarrow{\varphi_*^\bullet} Z(\mathcal{C})(\mathsf{T}^{\bullet \otimes n}, \mathsf{U}^\bullet)$$

induces by construction a map $J_\mathsf{T}(n) \longrightarrow K_\varphi(n)$ that by abuse of notation we just write as φ_*. Similarly, the map

$$J_\mathsf{U}(n) \longrightarrow Z(\mathcal{C})(\mathsf{U}^{\bullet \otimes n}, \mathsf{U}^\bullet) \xrightarrow{(\varphi^{\bullet \otimes n})^*} Z(\mathcal{C})(\mathsf{T}^{\bullet \otimes n}, \mathsf{U}^\bullet)$$

induces a map $\varphi^* : J_\mathsf{U}(n) \longrightarrow K_\varphi(n)$. This uses

$$j \circ \varphi^{\bullet \otimes n} \circ \iota_\mathsf{T}^{\otimes n} = j \circ \iota_\mathsf{U}^{\otimes n} \circ \varphi^{\otimes n} = \iota_\mathsf{U} \circ \mu_\mathsf{U}^n \circ \varphi^{\otimes n} = \iota_\mathsf{U} \circ \varphi \circ \mu_\mathsf{T}^n \quad \text{for} \quad j \in J_\mathsf{U}(n) \ ,$$

where φ being a map of algebras enters in the last step. The maps $\varphi_* : J_\mathsf{T}(n) \longrightarrow K_\varphi(n)$ and $\varphi^* : J_\mathsf{U}(n) \longrightarrow K_\varphi(n)$ are not zero because φ must preserve unit and hence is not zero (we assume that the algebras $\mathbb{T}$ and $\mathbb{U}$ are not zero). Since $J_\mathsf{T}(n)$, $J_\mathsf{U}(n)$ and $K_\varphi(n)$ are acyclic, φ_* and φ^* are equivalences. As a consequence, the homotopy pullback

$$\begin{tikzcd} (J_\mathsf{T} \times_\varphi J_\mathsf{U})(n) \arrow[r] \arrow[d] & J_\mathsf{T}(n) \arrow[d, "\varphi_*"] \\ J_\mathsf{U}(n) \arrow[r, "\varphi^*"'] & K_\varphi(n) \end{tikzcd} \tag{3.12}$$

is also acyclic. The complexes $(J_\mathsf{T} \times_\varphi J_\mathsf{U})(n)$ form an acyclic braided operad $J_\mathsf{T} \times_\varphi J_\mathsf{U}$. By virtue of the projections, $J_\mathsf{T} \times_\varphi J_\mathsf{U} \longrightarrow J_\mathsf{T}$ and $J_\mathsf{T} \times_\varphi J_\mathsf{U} \longrightarrow J_\mathsf{U}$, $\mathsf{T}^\bullet$ and $\mathsf{U}^\bullet$ become braided $J_\mathsf{T} \times_\varphi J_\mathsf{U}$-algebras such that $\varphi^\bullet : \mathsf{T}^\bullet \longrightarrow \mathsf{U}^\bullet$ becomes a $J_\mathsf{T} \times_\varphi J_\mathsf{U}$-algebra map (up to coherent homotopy depending on the model for (3.12)). □

3.4 A solution to Deligne's Conjecture in presence of a rigid monoidal product

In this subsection, we prove that Theorem 3.10 produces a solution to Deligne's Conjecture:

Theorem 3.12. *For any finite tensor category $\mathcal{C}$, the algebra structure on the canonical end $\mathbb{A} = \int_X X \otimes X^\vee$ induces an E_2-algebra structure on the homotopy invariants $\mathcal{C}(I, \mathbb{A}^\bullet)$. Under the equivalence $\mathcal{C}(I, \mathbb{A}^\bullet) \simeq \int^{\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X, X)$ from Proposition 3.3, this E_2-structure provides a solution to Deligne's Conjecture in the sense that it induces the standard Gerstenhaber structure on the Hochschild cohomology of $\mathcal{C}$.*

Outline of the strategy of the proof of Theorem 3.12. In order to obtain the E_2-structure on $\mathcal{C}(I, \mathbb{A}^\bullet)$, we specialize Theorem 3.10 to the algebra $\mathbb{T} = \mathbb{A} \in \mathcal{C}$. This is possible because $\mathbb{A}$ lifts to a braided commutative algebra in $Z(\mathcal{C})$ by Lemma 3.5. One can conclude from Proposition 3.3 that the underlying multiplication of this E_2-structure on $\mathcal{C}(I, \mathbb{A}^\bullet)$ translates to the cup product on the Hochschild cochain complex under the equivalence $\mathcal{C}(I, \mathbb{A}^\bullet) \simeq \int^{\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X, X)$. It remains to prove that the Gerstenhaber bracket that we extract from the E_2-structure on $\mathcal{C}(I, \mathbb{A}^\bullet)$ constructed via Theorem 3.10 yields the standard Gerstenhaber bracket on the Hochschild cohomology $H^*\left(\int^{\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X, X)\right)$.

Unfortunately, this part is relatively involved and will occupy the rest of this subsection: We need to spell out a model for the homotopy h between the multiplication $\gamma^\bullet_*$ on $\mathcal{C}(I, \mathbb{A}^\bullet)$ coming from the product $\gamma^\bullet : \mathbb{A}^\bullet \otimes \mathbb{A}^\bullet \longrightarrow \mathbb{A}^\bullet$ and the opposite multiplication $\gamma^{\bullet\,\mathrm{op}}_*$. Of course, we cannot just exhibit *any* homotopy, but need to compute the specific homotopy that the E_2-structure provided by Theorem 3.10 gives us. Afterwards, we will extract the Gerstenhaber bracket from the homotopy h. Since the Gerstenhaber bracket is an operation on homology, it suffices to compute h up to a higher homotopy.

More concretely, the homotopy h between $\gamma^\bullet_*$ and $\gamma^{\bullet\,\mathrm{op}}_*$ that we need to compute is the evaluation of the map $C_*(E_2(2); k) \otimes \mathcal{C}(I, \mathbb{A}^\bullet)^{\otimes 2} \longrightarrow \mathcal{C}(I, \mathbb{A}^\bullet)$ (that the E_2-structure provides for us) on the 1-chain on $E_2(2)$ given by the path in the configuration space of two disks shown in Figure 1.

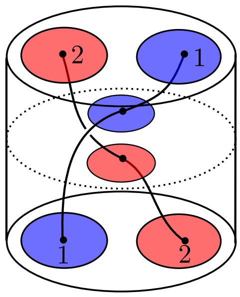


Figure 1: The path in $E_2(2)$ which provides for us the homotopy between multiplication and opposite multiplication.

The steps of the proof are as follows:

(i) Construct explicitly with algebraic tools *some* homotopy (that in hindsight we call h) between the multiplication and the opposite multiplication on $\mathcal{C}(I, \mathbb{A}^\bullet)$.

(ii) Prove that h as constructed in step (i) agrees up to higher homotopy with the topologically extracted homotopy described above.

(iii) Extract the Gerstenhaber bracket from h and prove that it agrees with the standard Gerstenhaber bracket on Hochschild cohomology.

□

Step (i). For step (i), we will choose as the injective resolution for $\mathbb{A}$ the one from Lemma 3.1, i.e. the (finite) homotopy end $\int^{\mathrm{f}\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee$. With this model, the product on $\mathcal{C}(I, \mathbb{A}^\bullet)$ is given by the product $\gamma^\bullet$ of $\mathbb{A}^\bullet$ from (3.4). This has the advantage that $\gamma^\bullet$ translates strictly to the cup product on Hochschild cochains (Proposition 3.3).

Let us now begin with the construction of h: For $p, q \geq 0$ and $0 \leq i \leq p - 1$, we fix an arbitrary family of $p + q$ projective objects

$$C_i = (X_0, \ldots, X_{i-1}, Y_0, \ldots, Y_q, X_{i+2}, \ldots, X_p)$$

(the labeling is chosen in hindsight and will become clear in a moment) and define the vector spaces

$$\begin{aligned} V' &:= \mathcal{C}(X_1, X_0) \otimes \cdots \otimes \mathcal{C}(X_{i-1}, X_{i-2}) \otimes \mathcal{C}(Y_0, X_{i-1}) \ , \\ W &:= \mathcal{C}(Y_1, Y_0) \otimes \cdots \otimes \mathcal{C}(Y_q, Y_{q-1}) \ , \\ V'' &:= \mathcal{C}(X_{i+2}, Y_q) \otimes \cdots \otimes \mathcal{C}(X_p, X_{p-1}) \ . \end{aligned}$$

With this notation, the component $\left(\mathbb{A}^{p+q-1}\right)^{C_i}$ of the product $\mathbb{A}^{p+q-1}$ (this is the $p+q-1$-th term of $\mathbb{A}^\bullet$, not the $p+q-1$-fold monoidal product) indexed by C_i is given by

$$\left(\mathbb{A}^{p+q-1}\right)^{C_i} = \left(X_0 \otimes X_p^\vee\right)^{V' \otimes W \otimes V''} \ , \tag{3.13}$$

see (3.1). Similarly, for the families

$$\begin{aligned} D_i' &:= (X_0, \ldots, X_{i-1}, Y_0, Y_q, X_{i+2}, \ldots, X_p) \ , \\ D_i'' &:= (Y_0, \ldots, Y_q) \ , \end{aligned}$$

we have

$$\begin{aligned} \left(\mathbb{A}^p\right)^{D_i'} &= \left(X_0 \otimes X_p^\vee\right)^{V' \otimes \mathcal{C}(Y_q, Y_0) \otimes V''} \ , \\ \left(\mathbb{A}^q\right)^{D_i''} &= \left(Y_0 \otimes Y_q^\vee\right)^{W} \ . \end{aligned}$$

Next observe

$$\begin{aligned} &\mathcal{C}\left(I, \left(\mathbb{A}^p\right)^{D_i'}\right) \otimes \mathcal{C}\left(I, \left(\mathbb{A}^q\right)^{D_i''}\right) \\ &= \mathrm{Hom}_k(V' \otimes \mathcal{C}(Y_q, Y_0) \otimes V'', \mathcal{C}(I, X_0 \otimes X_p^\vee)) \otimes \mathrm{Hom}_k(W, \mathcal{C}(I, Y_0 \otimes Y_q^\vee)) \\ &\cong \mathrm{Hom}_k(V' \otimes \mathcal{C}(Y_q, Y_0) \otimes V'', \mathcal{C}(I, X_0 \otimes X_p^\vee)) \otimes \mathrm{Hom}_k(W, \mathcal{C}(Y_q, Y_0)) \ . \end{aligned}$$

Composition over $\mathcal{C}(Y_q, Y_0)$ provides a map to $\mathrm{Hom}_k(V' \otimes W \otimes V'', \mathcal{C}(I, X_0 \otimes X_p^\vee))$ which is $\mathcal{C}\left(I, \left(\mathbb{A}^p\right)^{C_i}\right)$ by (3.13). Therefore, we obtain a map

$$\left(h_i^{p,q}\right)^{C_i} : \mathcal{C}\left(I, \left(\mathbb{A}^p\right)^{D_i'}\right) \otimes \mathcal{C}\left(I, \left(\mathbb{A}^q\right)^{D_i''}\right) \longrightarrow \mathcal{C}\left(I, \left(\mathbb{A}^{p+q-1}\right)^{C_i}\right)$$

decreasing degree by one. By the universal property of the product, we may define the linear map

$$h_i^{p,q} : \mathcal{C}(I, \mathbb{A}^p) \otimes \mathcal{C}(I, \mathbb{A}^q) \longrightarrow \mathcal{C}(I, \mathbb{A}^{p+q-1})$$

by the commutativity of the square

$$\begin{tikzcd} \mathcal{C}(I, \mathbb{A}^p) \otimes \mathcal{C}(I, \mathbb{A}^q) \arrow[r, "h_i^{p,q}"] \arrow[d, "\text{projection to components } D_i' \text{ and } D_i''"'] & \mathcal{C}(I, \mathbb{A}^{p+q-1}) \arrow[d, "\text{projection to component } C_i"] \\ \mathcal{C}\left(I, \left(\mathbb{A}^p\right)^{D_i'}\right) \otimes \mathcal{C}\left(I, \left(\mathbb{A}^q\right)^{D_i''}\right) \arrow[r, "\left(h_i^{p,q}\right)^{C_i}"'] & \mathcal{C}\left(I, \left(\mathbb{A}^{p+q-1}\right)^{C_i}\right) \ . \end{tikzcd}$$

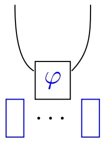
and define the components of h by

$$h^{p,q} := \sum_{i=0}^{p-1} (-1)^{i+(p-1-i)q} h_i^{p,q} : \mathcal{C}\left(I, \mathbb{A}^p\right) \otimes \mathcal{C}\left(I, \mathbb{A}^q\right) \longrightarrow \mathcal{C}\left(I, \mathbb{A}^{p+q-1}\right) \ . \tag{3.14}$$

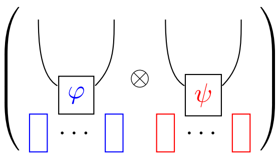
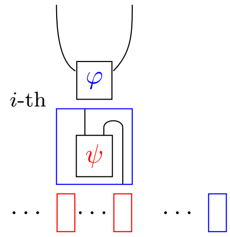
In order to establish a graphical representation for the definition of h, we symbolize a p-cochain in $\mathcal{C}(I, \mathbb{A}^\bullet)$, i.e. a vector in

$$\prod_{X_0, \ldots, X_p \in \mathsf{Proj}\,\mathcal{C}} \mathrm{Hom}_k\left(\underbrace{\mathcal{C}(X_p, X_{p-1}) \otimes \cdots \otimes \mathcal{C}(X_1, X_0)}_{(*)}, \underbrace{\mathcal{C}(I, X_0 \otimes X_p^\vee)}_{(**)}\right)$$

as follows:



The box labeled with φ with attached legs represents the part $(**)$ of the cochain which is a morphism $I \longrightarrow X_0 \otimes X_p^\vee$ (we suppress the labels because the cochains have components running over arbitrary labels). The p blank blue boxes can be filled with composable morphisms and make the cochains multilinearly dependent on p types of morphisms; this is part $(*)$ of the cochain. With this diagrammatic presentation, we arrive at the following description of h:

$h_i^{p,q}$  $=$  (3.15)

On the left, we have p red boxes and q blue boxes; the insertion is made in the i-th one. The total number of boxes on the right is $p+q$.

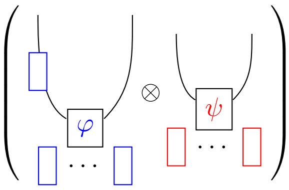
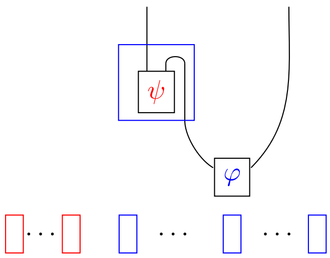
Step (i) is achieved with the following Lemma:

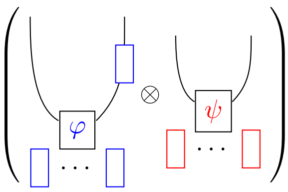
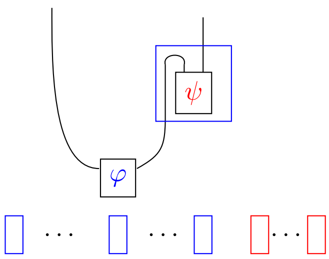
Lemma 3.13. *With the definition* (3.14) *of* h,

$$hd + dh = \gamma_*^{\bullet\,\mathrm{op}} - \gamma_*^\bullet \ , \tag{3.16}$$

i.e. h *is a homotopy from* $\gamma_*^\bullet$ *to* $\gamma_*^{\bullet\,\mathrm{op}}$.

Proof. Let φ and ψ be cochains of degree p and q, respectively. We will write h_i instead of $h_i^{p,q}$ for better readability because the degree can be read off from the cochains that h is being applied to. Thanks to

$h_0(\mathrm{d}_0\varphi \otimes \psi) = h_0$  $=$  $= \gamma_*^\bullet(\psi \otimes \varphi)$,

$h_p(\mathrm{d}_{p+1}\varphi \otimes \psi) = h_p$  $=$  $= \gamma_*^\bullet(\varphi \otimes \psi)$,

we obtain

$$\begin{aligned} h(\mathrm{d}\varphi \otimes \psi) &= (-1)^{pq} h_0(\mathrm{d}_0\varphi \otimes \psi) + \sum_{i=1}^{p} (-1)^{i+(p-i)q} h_i(\mathrm{d}_0\varphi \otimes \psi) \\ &\quad + \underbrace{\sum_{i=0}^{p}\sum_{j=1}^{p} (-1)^{i+j+(p-i)q} h_i(\mathrm{d}_j\varphi \otimes \psi)}_{=:S(\varphi,\psi)} + \sum_{i=0}^{p-1} (-1)^{i+p+1+(p-i)q} h_i(\mathrm{d}_{p+1}\varphi \otimes \psi) \\ &\quad - h_p(\mathrm{d}_{p+1}\varphi \otimes \psi) \\ &= (-1)^{pq}\gamma_*^\bullet(\psi\otimes\varphi) - \mathrm{d}_0 h(\varphi\otimes\psi) + S(\varphi,\psi) \\ &\quad - (-1)^{p+q}\mathrm{d}_{p+q} h(\varphi\otimes\psi) - \gamma_*^\bullet(\varphi\otimes\psi) \ . \end{aligned} \tag{3.17}$$

Next we further compute $S(\varphi,\psi)$: For $0 \le i \le p$ and $1 \le j \le p$, we find

$$h_i(\mathrm{d}_j\varphi \otimes \psi) = \begin{cases} \mathrm{d}_{j-1+q} h_i(\varphi\otimes\psi) \ , & i < j-1 \ , \\ h_i(\varphi \otimes \mathrm{d}_{q+1}\psi) \ , & i = j-1 \ , \\ h_{i-1}(\varphi \otimes \mathrm{d}_0\psi) \ , & i = j \ , \\ \mathrm{d}_j h_{i-1}(\varphi\otimes\psi) \ , & i > j \ . \end{cases}$$

With the dummy variables $\ell = j-1+q$ and $m = i-1$, this leads to

$$\begin{aligned} S(\varphi,\psi) = &- \sum_{\substack{0\le i\le p-1 \\ q\le \ell\le p-1+q \\ \ell > i+q}} (-1)^{i+\ell+(p-1-i)q} \mathrm{d}_\ell h_i(\varphi\otimes\psi) - \sum_{i=0}^{p-1} (-1)^{(p-i)q} h_i(\varphi\otimes \mathrm{d}_{q+1}\psi) \\ &+ \sum_{i=0}^{p-1} (-1)^{(p-1-i)q} h_i(\varphi\otimes\mathrm{d}_0\psi) - \sum_{\substack{0\le m\le p-1 \\ 1\le j\le p \\ m\ge j}} (-1)^{m+j+(p-1-m)q} \mathrm{d}_j h_m(\varphi\otimes\psi) \\ = &\ K(\varphi,\psi) - (-1)^p \big(h(\varphi\otimes(-1)^{q+1}\mathrm{d}_{q+1}\psi) + h(\varphi\otimes\mathrm{d}_0\psi)\big) \end{aligned}$$

$$\text{with} \quad K(\varphi,\psi) := -\sum_{i=0}^{p-1} \left(\sum_{\substack{q\le j\le p-1+q \\ j>i+q}} (-1)^{i+j+(p-1-i)q} \mathrm{d}_j h_i(\varphi\otimes\psi) + \sum_{\substack{1\le j\le p \\ i\ge j}} (-1)^{i+j+(p-1-i)q} \mathrm{d}_j h_i(\varphi\otimes\psi) \right) .$$

As a consequence, we obtain

$$\begin{aligned} S(\varphi,\psi) + (-1)^p h(\varphi\otimes\mathrm{d}\psi) &= K(\varphi,\psi) + (-1)^p \sum_{i=0}^{p-1}\sum_{j=1}^{q} (-1)^{i+j+(p-1-i)(q+1)} h_i(\varphi\otimes\mathrm{d}_j\psi) \\ &= K(\varphi,\psi) + (-1)^p \sum_{i=0}^{p-1}\sum_{j=1}^{q} (-1)^{i+j+(p-1-i)(q+1)} \mathrm{d}_{i+j} h_i(\varphi\otimes\psi) \\ &\quad \text{(because } h_i(\varphi\otimes\mathrm{d}_j\psi) = \mathrm{d}_{i+j}h_i(\varphi\otimes\psi) \text{ for } 0\le i\le p-1 \ , \ 1\le j\le q) \\ &= K(\varphi,\psi) + (-1)^p \sum_{\substack{0\le i\le p-1 \\ i+1\le j\le i+q}} (-1)^{j+(p-1-i)(q+1)} \mathrm{d}_j h_i(\varphi\otimes\psi) \\ &= K(\varphi,\psi) - \sum_{\substack{0\le i\le p-1 \\ i+1\le j\le i+q}} (-1)^{i+j+(p-1-i)q} \mathrm{d}_j h_i(\varphi\otimes\psi) \\ &= -\sum_{i=0}^{p-1}\sum_{j=1}^{p+q-1} (-1)^{i+j+(p-1-i)q} \mathrm{d}_j h_i(\varphi\otimes\psi) \\ &= -\mathrm{d}h(\varphi\otimes\psi) + \mathrm{d}_0 h(\varphi\otimes\psi) + (-1)^{p+q}\mathrm{d}_{p+q}h(\varphi\otimes\psi) \ . \end{aligned} \tag{3.18}$$

In summary,

$$\begin{aligned} h(\mathrm{d}(\varphi\otimes\psi)) &= h(\mathrm{d}\varphi\otimes\psi) + (-1)^p h(\varphi\otimes\mathrm{d}\psi) \\ &\overset{(3.17)}{=} (-1)^{pq}\gamma_*^\bullet(\psi\otimes\varphi) - \mathrm{d}_0 h(\varphi\otimes\psi) + S(\varphi,\psi) \\ &\quad - (-1)^{p+q}\mathrm{d}_{p+q}h(\varphi\otimes\psi) - \gamma_*^\bullet(\varphi\otimes\psi) + (-1)^p h(\varphi\otimes\mathrm{d}\psi) \\ &\overset{(3.18)}{=} (-1)^{pq}\gamma_*^\bullet(\psi\otimes\varphi) - \gamma_*^\bullet(\varphi\otimes\psi) - \mathrm{d}h(\varphi\otimes\psi) \ . \end{aligned}$$

This proves (3.16) and hence the Lemma. □

Step (ii). We now prepare ourselves to prove that the homotopy h constructed algebraically in step (i) agrees with the topologically extracted homotopy: The E_2-structure on $\mathcal{C}(I,\mathbb{A}^\bullet)$ comes by construction from the braided J_{A}-algebra structure on an injective resolution $\mathsf{A}^\bullet$ of $\mathsf{A} \in Z(\mathcal{C})$. In this description, one needs $\mathbb{A}^\bullet = U\mathsf{A}^\bullet$. We will cover afterwards the situation for an injective resolution of $\mathbb{A} \in \mathcal{C}$ which does not lift degree-wise to the Drinfeld center. The construction will now be spelled out:

Lemma 3.14. *For any finite tensor category $\mathcal{C}$, let $\mathsf{A}^\bullet$ be an injective resolution of the canonical algebra $\mathsf{A} \in Z(\mathcal{C})$. Moreover, set $\mathbb{A}^\bullet := U\mathsf{A}^\bullet$.*

(i) Denote by $\Gamma : \mathsf{A}^\bullet \otimes \mathsf{A}^\bullet \longrightarrow \mathsf{A}^\bullet$ the product of $\mathsf{A}^\bullet$ (any extension of the product $\mathsf{A} \otimes \mathsf{A} \longrightarrow \mathsf{A}$ to the injective resolution) and by $c_{\mathsf{A}^\bullet,\mathsf{A}^\bullet} : \mathsf{A}^\bullet \otimes \mathsf{A}^\bullet \longrightarrow \mathsf{A}^\bullet \otimes \mathsf{A}^\bullet$ the braiding of the differential graded object $\mathsf{A}^\bullet$ in $Z(\mathcal{C})$. There is an essentially unique homotopy $\mathsf{H} : \Gamma \simeq \Gamma^{\mathrm{op}} := \Gamma \circ c_{\mathsf{A}^\bullet,\mathsf{A}^\bullet}$ (essentially unique means here up to higher homotopy) with the additional property that the precomposition with the coaugmentation $\iota_{\mathsf{A}}^{\otimes 2} : \mathsf{A}^{\otimes 2} \longrightarrow \mathsf{A}^{\bullet \otimes 2}$

$$Z(\mathcal{C}) \left(\mathsf{A}^{\bullet \otimes 2}, \mathsf{A}^\bullet\right) \xrightarrow{\simeq} Z\left(\mathcal{C}\right)(\mathsf{A}^{\otimes 2}, \mathsf{A}^\bullet)$$

sends H to the zero self-homotopy of the map $\mathsf{A}^{\otimes 2} \longrightarrow \mathsf{A} \xrightarrow{\iota_{\mathsf{A}}} \mathsf{A}^\bullet$ that first applies the product of A (or the opposite product, which is equal) and then the coaugmentation.

(ii) If we apply the forgetful functor $U : Z(\mathcal{C}) \longrightarrow \mathcal{C}$, H yields a homotopy $U\mathsf{H}$ from the multiplication $\gamma^\bullet : \mathbb{A}^\bullet \otimes \mathbb{A}^\bullet \longrightarrow \mathbb{A}^\bullet$ (extending the product $\gamma : \mathbb{A} \otimes \mathbb{A} \longrightarrow \mathbb{A}$) to

$$\bar{\gamma}^\bullet := \gamma^\bullet \circ U(c_{\mathsf{A}^\bullet,\mathsf{A}^\bullet}) : \mathbb{A}^\bullet \otimes \mathbb{A}^\bullet \xrightarrow{U(c_{\mathsf{A}^\bullet,\mathsf{A}^\bullet})} \mathbb{A}^\bullet \otimes \mathbb{A}^\bullet \xrightarrow{\gamma^\bullet} \mathbb{A}^\bullet . \tag{3.19}$$

The multiplications $\gamma^\bullet$ and $\bar{\gamma}^\bullet$ on $\mathbb{A}^\bullet$ induce the multiplications $\gamma_*^\bullet$ and $\gamma_*^{\bullet\mathrm{op}}$ on $\mathcal{C}(I, \mathbb{A}^\bullet)$, respectively. The homotopy h from $\gamma_*^\bullet$ to $\gamma_*^{\bullet\mathrm{op}}$ extracted from the E_2-structure on $\mathcal{C}(I, \mathbb{A}^\bullet)$ is the result of applying $\mathcal{C}(I, -)$ to $U\mathsf{H}$, i.e. it is given by the composition

$$h : \mathcal{C}(I, \mathbb{A}^\bullet) \otimes \mathcal{C}(I, \mathbb{A}^\bullet) \xrightarrow{\text{lax monoidal structure of } \mathcal{C}(I,-)} \mathcal{C}(I, \mathbb{A}^{\bullet \otimes 2}) \xrightarrow{\mathcal{C}(I,U\mathsf{H})} \mathcal{C}(I, \mathbb{A}^\bullet) \tag{3.20}$$

Proof. All of this is consequence of the definition of the braided operad J_{A} (see (3.9)), a careful unpacking of the proofs of Proposition 3.9 and Theorem 3.10, and the fact H, as described in (i), is the result of the evaluation of the J_{A}-action

$$J_{\mathsf{A}}(2) \longrightarrow Z(\mathcal{C}) \left(\mathsf{A}^{\bullet \otimes 2}, \mathsf{A}^\bullet\right)$$

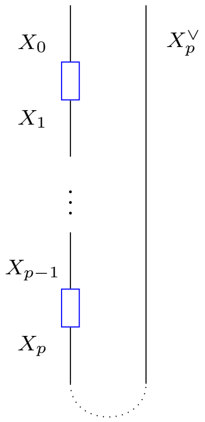
on a 1-chain that is mapped by the epimorphism $J_{\mathsf{A}}(2) \longrightarrow C_*(E_2(2); k)$ to the 1-chain in $E_2(2)$ described in Figure 1. □

One should be a bit careful to see (3.19) as an 'opposite' multiplication because $U(c_{\mathsf{A}^\bullet,\mathsf{A}^\bullet})$ is *not* a part of a braiding in $\mathcal{C}$. Nonetheless, the images of the braiding in $Z(\mathcal{C})$ under U turn $(\mathbb{A}^\bullet)^{\otimes n}$ into a B_n-module.

If we want to use Lemma 3.14 to extract the topologically defined homotopy and compare it with the one concretely given in (3.14), there is a problem: The resolution $\mathbb{A}^\bullet = \int^{\mathrm{f}\mathbb{R}}_{X \in \mathcal{C}} X \otimes X^\vee$ used to write down the homotopy in (3.14) does not lift degree-wise to $Z(\mathcal{C})$, i.e. it does not come with a half braiding. However, it comes with a structure that one could call a *homotopy coherent half braiding*. This means that the half braiding for $\mathbb{A}$ (the non-crossing half braiding from (2.8)) can be extended to $\mathbb{A}^\bullet$ (but without being a half braiding degree-wise). With this homotopy coherent half braiding, Lemma 3.14 remains in principle true, but a little more care is required in some places. In order to provide the details, we will adapt the calculus for monoidal categories such that we can effectively compute with the resolution $\mathbb{A}^\bullet = \int^{\mathrm{f}\mathbb{R}}_{X \in \mathrm{Proj}\, \mathcal{C}} X \otimes X^\vee$. Recall that its p-cochains live in the product

$$\prod_{X_0,\dots,X_p \in \mathrm{Proj}\,\mathcal{C}} \left(X_0 \otimes X_p^\vee\right)^{\mathcal{C}(X_p,X_{p-1}) \otimes \dots \otimes \mathcal{C}(X_1,X_0)}$$

We will write the component of an p-cochain indexed by $(X_0, \ldots, X_p)$ in the graphical calculus by



$$. \tag{3.21}$$

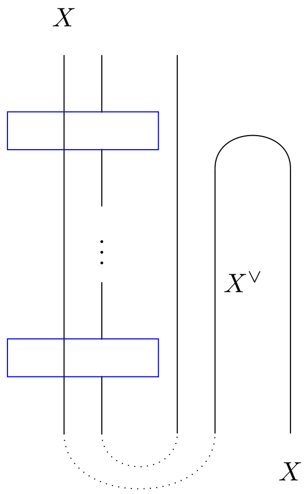
Formally speaking, this picture is to be read as the projection

$$\mathbb{A}^p \longrightarrow \left(X_0 \otimes X_p^\vee\right)^{\mathcal{C}(X_p, X_{p-1}) \otimes \cdots \otimes \mathcal{C}(X_1, X_0)} . \tag{3.22}$$

The blue boxes represent the vector spaces appearing in the exponent of the powering (3.22). More precisely, the blue box between X_{j+1} and X_j for $0 \leq j \leq p-1$ represents a blank argument that can be filled with a morphism $X_{j+1} \longrightarrow X_j$. The dotted line is purely mnemonic: It is not a coevaluation, but symbolizes the constraint that the object on the upper right (here: $X_p^\vee$) must be dual to the one in the left bottom (here: X_p). With this notation, we can actually omit the labeling in (3.21) because all components in the picture run over all labels, with the single constraint implemented through the dotted line.

Now we can give the *homotopy coherent half braiding* of our resolution $\mathbb{A}^\bullet$ with $X \in \mathcal{C}$ by

$$c_{\mathbb{A}^\bullet, X} :=$$



$$: \mathbb{A}^\bullet \otimes X \longrightarrow X \otimes \mathbb{A}^\bullet . \tag{3.23}$$

The lines drawn through the boxes indicate that identities have been inserted. Note that $c_{\mathbb{A}^\bullet, X}$ is determined up to a contractible choice by the fact that the restriction along the coaugmentation $\iota_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathbb{A}^\bullet$ gives us $(X \otimes \iota_{\mathbb{A}}) \circ c_{\mathbb{A}, X}$, where $c_{\mathbb{A}, X}$ is the usual non-crossing half braiding on $\mathbb{A}$. The following is a consequence of this characterization of $c_{\mathbb{A}^\bullet, X}$:

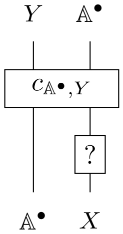
Lemma 3.15. *(i) The homotopy coherent half braiding* (3.23) *endows* ${\mathbb{A}^\bullet}^{\otimes n}$ *with a homotopy coherent action of* B_n *which is the essentially unique* B_n*-action making the* n*-fold coaugmentation*

$$\mathbb{A}^{\otimes n} \xrightarrow{\ \simeq\ } (\mathbb{A}^\bullet)^{\otimes n}$$

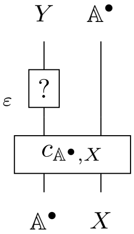
B_n-equivariant up to coherent homotopy.

(ii) There is a unique homotopy $H : \gamma^\bullet \simeq \gamma^\bullet \circ c_{\mathbb{A}^\bullet, \mathbb{A}^\bullet}$ *that becomes trivial if we precompose with the coaugmentation in the first slot.*

(iii) *The homotopy coherent half braiding $c_{\mathbb{A}^\bullet,-}$ is natural up to coherent homotopy: For objects X and Y in $\mathcal{C}$ (that can themselves be differential graded if $c_{\mathbb{A}^\bullet,-}$ is understood degree-wise)*



$\overset{N}{\simeq} \; (-1)^\varepsilon$



$: \mathbb{A}^\bullet \otimes \mathcal{C}(X,Y) \otimes X \longrightarrow Y \otimes \mathbb{A}^\bullet \; ,$

holds up to a coherent homotopy that we denote by N (we suppress the dependence on X and Y in the notation). The box with the question mark can be filled with a morphism $X \longrightarrow Y$. The integer ε is the product of the degree of $?$ and the degree of $\mathbb{A}^\bullet$. This homotopy is the essentially unique one that becomes trivial if we precompose with the coaugmentation $\mathbb{A} \longrightarrow \mathbb{A}^\bullet$.

(iv) *Through the homotopies N, the maps*

$$\mathcal{C}(I,\mathbb{A}^\bullet)^{\otimes n} \longrightarrow \mathcal{C}(I,\mathbb{A}^{\bullet \otimes n})$$

become B_n-equivariant up to coherent homotopy, where on the left hand side the action is the strict action factoring through the permutation group.

Proof. The points (i), (ii) and (iii) follow from the construction because the homotopy coherent half braiding reduces to the usual non-crossing braiding if we precompose with the coaugmentation. One obtains (iv) by specializing (iii) to $X = I$ and $Y = \mathbb{A}^\bullet$. ☐

We cannot only endow the homotopy end $\int^{\mathbb{fR}}_{X \in \mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee$ with a homotopy coherent half braiding, but also an injective resolution $U\mathsf{A}^\bullet$ of $\mathbb{A} \in \mathcal{C}$ that comes from an arbitrary injective resolution of $\mathsf{A} \in Z(\mathcal{C})$. In the latter case, we have of course an actual half braiding, so the statements of Lemma 3.15 hold in a much stricter sense (for all points except (iii), the coherence data are trivial). Now we make two observations:

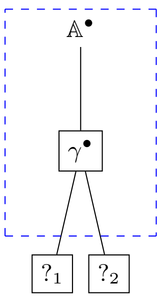
- The injective resolutions $\int^{\mathbb{fR}}_{X \in \mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee$ and $U\mathsf{A}^\bullet$ are homotopy equivalent, and their homotopy coherent half braidings agree, possibly up to higher homotopy. The first part is standard, the second part comes from the fact that the half braiding is by construction determined by its restriction to $\mathbb{A}$. In fact, *all* injective resolutions of $\mathbb{A}$ with homotopy coherent half braiding are equivalent.

- The homotopy $h : \gamma_*^\bullet \simeq \gamma_*^{\bullet\,\mathrm{op}}$ described in Lemma 3.14 depends only on $U\mathsf{A}^\bullet$ as object with half braiding. Indeed, we get the multiplication $\gamma^\bullet$ from the multiplication on $\mathbb{A}$, $\bar\gamma^\bullet$ from the braid group action (3.14) which is a special case of the half braiding, and the needed homotopy from $\gamma^\bullet$ to $\bar\gamma^\bullet$ from Lemma 3.14 (ii) can once again be characterized by the fact that it becomes trivial when precomposed with the coaugmentation. With these ingredients, we can obtain h via (3.20).

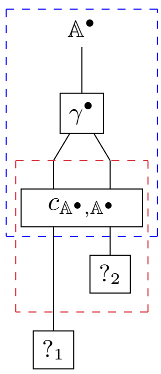
These two observations imply: We can compute h from *any* injective resolution of $\mathbb{A}$ equipped with homotopy coherent half braiding. This will possibly change h by a higher homotopy, but we are only interested in h up to higher homotopy anyway.

Lemma 3.16. *For the resolution $\int^{\mathbb{fR}}_{X \in \mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee$ of $\mathbb{A}$ and the half braiding (3.23), the topologically extracted homotopy $h : \gamma^\bullet \simeq \gamma_*^{\bullet\,\mathrm{op}}$ is given by the homotopy*

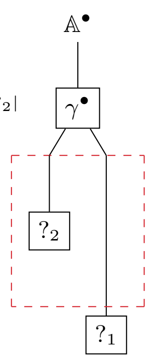
$h \; : \; \gamma_*^\bullet \; =$



$\xrightarrow[\text{Lemma 3.15 (ii)}]{H}$



$\xrightarrow[\text{Lemma 3.15 (iii)}]{N} \; (-1)^{|?_1||?_2|}$



$= \; \gamma_*^{\bullet\,\mathrm{op}} \; , \qquad (3.24)$

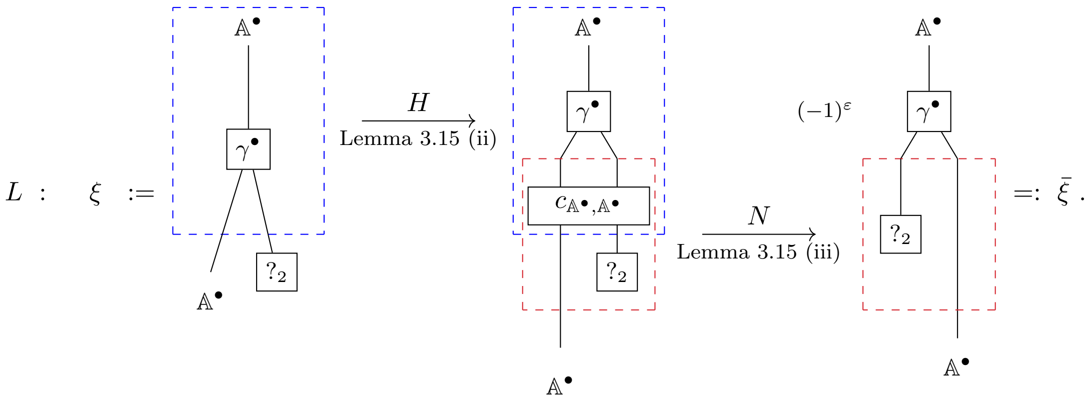
where the colored frames indicate the areas that H and N are applied to.

Proof. For *any* injective resolution $R = (\mathbb{A}^\bullet, c)$ of $\mathbb{A}$ with homotopy coherent half braiding, one obtains the homotopies H and N that we can use to associate to R the homotopy h_R via the formula (3.24). But all of such injective resolutions with homotopy coherent half braidings are equivalent as explained, and hence so are the h_R. Now it remains to verify that h_R with $R = U\mathsf{A}^\bullet$ reduces to the homotopy from (3.20) from Lemma 3.14 (ii). But this is true because the homotopy N, in this case, happens to be trivial since we have an *actual* half braiding. $\square$

With the following Lemma, we complete step (ii):

Lemma 3.17. *The homotopy $h : \gamma_*^\bullet \simeq \gamma_*^{\bullet\,\mathrm{op}}$ from Lemma 3.13 given for the injective resolution $\mathbb{A}^\bullet = \int^{\mathrm{f}\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee$ agrees up to higher homotopy with the topologically extracted homotopy $\gamma_*^\bullet \simeq \gamma_*^{\bullet\,\mathrm{op}}$ of the E_2-algebra $\mathcal{C}(I, \mathbb{A}^\bullet)$.*

Proof. For the entire proof, we fix the injective resolution $\mathbb{A}^\bullet = \int^{\mathrm{f}\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} X \otimes X^\vee$. Rephrasing Lemma 3.16, the topologically extracted homotopy h is obtained by applying $\mathcal{C}(I, -)$ to the homotopy L of maps $\xi, \bar{\xi} : \mathbb{A}^\bullet \otimes \mathcal{C}(I, \mathbb{A}^\bullet) \longrightarrow \mathbb{A}^\bullet$ defined by



In short,

$$h = \mathcal{C}(I, L) \ . \tag{3.25}$$

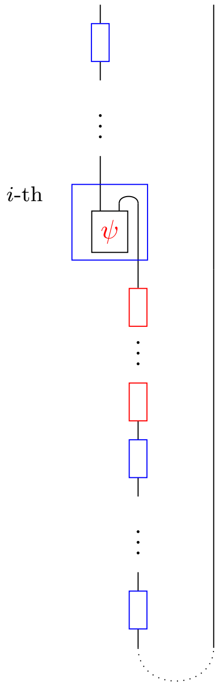
If we precompose with the coaugmentation, L becomes trivial (this follows for H from Lemma 3.15 (ii) and for N by from Lemma 3.15 (iii)). But

$$\mathcal{C}(\mathbb{A}^\bullet \otimes \mathcal{C}(I, \mathbb{A}^\bullet), \mathbb{A}^\bullet) \xrightarrow{(\iota_{\mathbb{A}} \otimes \mathrm{id}_{\mathcal{C}(I,\mathbb{A}^\bullet)})^*} \mathcal{C}(\mathbb{A} \otimes \mathcal{C}(I, \mathbb{A}^\bullet), \mathbb{A}^\bullet) \ .$$

is again a trivial fibration which implies that L is the essentially unique homotopy $\xi \simeq \bar{\xi}$ that becomes trivial when precomposed with the coaugmentation. This allows us to give a model for L: First we define

$L_i^{p,q} : \mathbb{A}^p \otimes \mathcal{C}(I, \mathbb{A}^q) \longrightarrow \mathbb{A}^{p+q-1}$ through

$L_i^{p,q}(- \otimes \psi) =$



for $\quad \psi \in \mathcal{C}(I, \mathbb{A}^q)$.

The operations $L_i^{p,q}$ are binary, and we use blue and red for the boxes associated to the first and second argument, respectively. Next we set

$$L^{p,q} := \sum_{i=0}^{p-1} (-1)^{i+(p-1-i)q} L_i^{p,q} : \mathbb{A}^p \otimes \mathcal{C}(I, \mathbb{A}^q) \longrightarrow \mathbb{A}^{p+q-1} \ . \tag{3.26}$$

Then

$$L\mathrm{d} + \mathrm{d}L = \bar{\xi} - \xi \ ,$$

i.e. L is a homotopy from ξ to $\bar{\xi}$. This can be confirmed with essentially the same computation as for Lemma 3.13. In order to verify that this homotopy really models L, we need to verify that it vanishes when precomposed with the coaugmentation. But this follows from $L^{0,q} = 0$.

Now we can use the model (3.26) to compute h via (3.25). This gives us exactly the formula for h in Lemma 3.13. □

Step (iii). We can now finally compute algebraically the Gerstenhaber bracket of the E_2-algebra $\mathcal{C}(I, \mathbb{A}^\bullet)$. So far, we obtained for the E_2-algebra $\mathcal{C}(I, \mathbb{A}^\bullet)$ the homotopy between multiplication and opposite multiplication coming from the path in $E_2(2)$ given in Figure 1. The key technical ingredient for this is Lemma 3.17 that tells us that the homotopy h concretely defined via (3.14) gives us a model for this homotopy.

Finally, in this step, we want to compute the Gerstenhaber bracket for the E_2-algebra $\mathcal{C}(I, \mathbb{A}^\bullet)$. To this end, we compute the binary degree one operation b corresponding to the fundamental class of $E_2(2) \simeq \mathbb{S}^1$ (the orientation comes here from preferring the braiding over its inverse). From this operation b and an additional sign, we obtain the Gerstenhaber bracket as we will explain in a moment.

We can obtain the loop in $E_2(2) \simeq \mathbb{S}^1$ corresponding to the fundamental class by composing two half circular paths. We have established that the homotopy h is the evaluation of the E_2-algebra $\mathcal{C}(I, \mathbb{A}^\bullet)$ on the first of these half circles, at least up to higher homotopy. As a consequence, the binary degree one operation b is the composition $h + h\tau$ of the homotopy h from $\gamma_*^\bullet$ to $\gamma_*^{\bullet\mathrm{op}}$ and the homotopy $h\tau$ from $\gamma_*^{\bullet\mathrm{op}}$ to $\gamma_*^\bullet$, where τ is the symmetric braiding in Ch_k:

$$b(\varphi, \psi) = h(\varphi, \psi) + (-1)^{pq} h(\psi, \varphi) \quad \text{for} \quad \varphi \in \mathcal{C}(I, \mathbb{A}^p) \ , \ \psi \in \mathcal{C}(I, \mathbb{A}^q) \ .$$

The connection to the Gerstenhaber bracket is $[\varphi, \psi] = (-1)^p b(\varphi, \psi)$ [SW03, page 220] (this additional sign ensures the anti-symmetry of the Gerstenhaber bracket), which leads us to

$$[\varphi, \psi] = (-1)^p h(\varphi, \psi) + (-1)^{pq+p} h(\psi, \varphi) \ . \tag{3.27}$$

Under order to express compactly the Gerstenhaber bracket induced on Hochschild cohomology under the equivalence $\int_{X\in\mathsf{Proj}\,\mathcal{C}}^{\mathbb{R}}\mathcal{C}(X,X)\simeq\mathcal{C}(I,\mathbb{A}^\bullet)$, recall the i-th partial composition operation

$$\circ_i : \left(\int_{X\in\mathsf{Proj}\,\mathcal{C}}^{\mathbb{R}}\mathcal{C}(X,X)\right)^p\otimes\left(\int_{X\in\mathsf{Proj}\,\mathcal{C}}^{\mathbb{R}}\mathcal{C}(X,X)\right)^q\longrightarrow\left(\int_{X\in\mathsf{Proj}\,\mathcal{C}}^{\mathbb{R}}\mathcal{C}(X,X)\right)^{p+q-1}\ ,\quad 0\le i\le p-1$$
$$\alpha\otimes\beta\longmapsto\alpha\circ_i\beta\ ,$$

where

$$(\alpha\circ_i\beta)_{X_0,\dots,X_{i-1},Y_0,\dots,Y_q,X_{i+1},\dots,X_p}:=\alpha_{X_0,\dots,X_{i-1},Y_0,Y_q,X_{i+1},\dots,X_p}(-,\beta_{Y_0,\dots,Y_q},-)\ .$$

The operations $\circ_i$ are used to define the circle product in the sense of [Wit19, Definition 1.4.1]

$$\alpha\circ\beta:=\sum_{i=0}^{p-1}(-1)^{(q-1)i}\alpha\circ_i\beta\ ,\quad|\alpha|=p\ ,\quad|\beta|=q\ .$$

Lemma 3.18. *Under the equivalence, $\int_{X\in\mathsf{Proj}\,\mathcal{C}}^{\mathbb{R}}\mathcal{C}(X,X)\simeq\mathcal{C}(I,\mathbb{A}^\bullet)$ the Gerstenhaber bracket of $\mathcal{C}(I,\mathbb{A}^\bullet)$ translates to the bracket*

$$[\alpha,\beta]=-(-1)^{(p-1)(q-1)}\alpha\circ\beta+\beta\circ\alpha\ ,\tag{3.28}$$

where α and β are in degree p and q, respectively.

With our sign conventions, (3.28) is the 'standard' Gerstenhaber bracket on Hochschild cohomology (we comment on other conventions in Remark 3.19). This finishes the proof that the E_2-algebra $(\ \mathcal{C}(I,\mathbb{A}^\bullet)\ ,\ \gamma_*^\bullet\ )$ is a (very explicit) solution to Deligne's Conjecture.

Proof. Under the equivalence $\int_{X\in\mathsf{Proj}\,\mathcal{C}}^{\mathbb{R}}\mathcal{C}(X,X)\simeq\mathcal{C}(I,\mathbb{A}^\bullet)$, the part $h_i^{p,q}$ of the homotopy h from step (i), (3.15) (here $p,q\ge 0$ and $0\le i\le p-1$) translates to the partial composition operation $\circ_i$. Hence, the homotopy h on $\mathcal{C}(I,\mathbb{A}^\bullet)$ from (3.14) translates to the degree one operation on $\int_{X\in\mathsf{Proj}\,\mathcal{C}}^{\mathbb{R}}\mathcal{C}(X,X)$ sending α in degree p and β in degree q to

$$\sum_{i=0}^{n-1}(-1)^{i+(p-1-i)q}\alpha\circ_i\beta=(-1)^{pq+q}\alpha\circ\beta\ .$$

But then the bracket (3.27) translates to the bracket

$$[\alpha,\beta]=(-1)^{p+pq+q}\alpha\circ\beta+\beta\circ\alpha=-(-1)^{(p-1)(q-1)}\alpha\circ\beta+\beta\circ\alpha\ .$$

□

Remark 3.19. In many places in the literature, including the textbook [Wit19], a different convention for the cup product is used. This alternative cup product $\smile'$ relates to ours by $\alpha\smile'\beta=(-1)^{pq}\alpha\smile\beta$ with $p=|\alpha|$ and $q=|\beta|$. For us, this convention would be a bad choice because it does not turn $\smile'$ into a chain map (only up to a sign), but it can be convenient for other purposes. If we want to obtain the bracket associated to $\smile'$ rather than $\smile$, we need to multiply the homotopy h above also degree-wise with $(-1)^{pq}$. This entails that b on $\mathcal{C}(I,\mathbb{A}^\bullet)$ must also be multiplied with $(-1)^{pq}$, thereby giving us b'. But

$$b'(\varphi,\psi)=(-1)^{pq}b(\varphi,\psi)=b(\psi,\varphi)\quad\text{for}\quad\varphi\in\mathcal{C}(I,\mathbb{A}^p)\ ,\ \psi\in\mathcal{C}(I,\mathbb{A}^q)$$

because b is graded commutative by definition. This changes the bracket that we obtain on $\int_{X\in\mathsf{Proj}\,\mathcal{C}}^{\mathbb{R}}\mathcal{C}(X,X)$ to

$$[\alpha,\beta]'=(-1)^{p+pq+q}\beta\circ\alpha+\alpha\circ\beta=\alpha\circ\beta-(-1)^{(p-1)(q-1)}\beta\circ\alpha\ .$$

This agrees now with [Wit19, Definition 1.4.1].

Theorem 3.12 allows for a generalization to exact module categories: For a finite tensor category $\mathcal{C}$, an *exact (left) module category* $\mathcal{M}$ [EO04] is a finite category together with structure of a left module $\otimes : \mathcal{C} \boxtimes \mathcal{M} \longrightarrow \mathcal{M}$ over $\mathcal{C}$ such that $P \otimes M$ is projective for $P \in \mathsf{Proj}\,\mathcal{C}$ and $M \in \mathcal{M}$. If we denote by $[-,-] : \mathcal{M} \boxtimes \mathcal{M} \longrightarrow \mathcal{C}$ the internal hom of the module category $\mathcal{M}$, one may define the object $\mathbb{A}_\mathcal{M} := \int_{M\in\mathcal{M}} [M, M] \in \mathcal{C}$. This object is an algebra in $\mathcal{C}$ and lifts in fact to a braided commutative algebra $\mathsf{A}_\mathcal{M}$ in $Z(\mathcal{C})$ [Shi20, Theorem 4.9]. The object $\mathbb{A}_\mathcal{M}$ allows to express the Hochschild cochains of $\mathcal{M}$ as $\int^{\mathbb{R}}_{M\in\mathsf{Proj}\,\mathcal{M}} \mathcal{M}(M, M) \simeq \mathcal{C}(I, \mathbb{A}^\bullet_\mathcal{M})$; this is [Shi20, Corollary 7.5] in a slightly different language (this is the point where exactness of $\mathcal{M}$ is needed). After implementing the needed changes to Theorem 3.12 and its proof, we arrive at the following generalization:

Theorem 3.20. *For any exact module category $\mathcal{M}$ over a finite tensor category $\mathcal{C}$, the algebra structure on the canonical end $\mathbb{A}_\mathcal{M} = \int_{M\in\mathcal{M}} [M, M] \in \mathcal{C}$ induces an E_2-algebra structure on the derived morphism space $\mathcal{C}(I, \mathbb{A}^\bullet_\mathcal{M})$. Under the equivalence $\mathcal{C}(I, \mathbb{A}^\bullet_\mathcal{M}) \simeq \int^{\mathbb{R}}_{M\in\mathsf{Proj}\,\mathcal{M}} \mathcal{M}(M, M)$, this E_2-structure provides a solution to Deligne's Conjecture in the sense that it induces the standard Gerstenhaber structure on the Hochschild cohomology of $\mathcal{M}$.*

Considering $\mathcal{C}$ as an exact module category over itself, the above results specializes to Theorem 3.12.

3.5 Construction of the differential graded Verlinde algebra on the Hochschild cochain complex

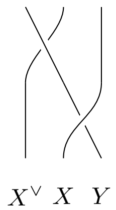
By means of Theorem 3.10, which yields an E_2-structure on the homotopy invariants of braided commutative algebras, we can establish another E_2-structure on the Hochschild cochain complex of a unimodular braided finite tensor category. The multiplication, in contrast to Theorem 3.12, will actually be induced by the *monoidal product and the braiding*. This E_2-structure is one of the two E_2-structures featuring in one of our main results (Theorem 4.2). In order to construct this E_2-structure, recall the following standard result:

Proposition 3.21 *(Lyubashenko [Lyu95b]). Let $\mathcal{C}$ be a braided finite tensor category. Then the maps*

$$X^\vee \otimes X \otimes Y^\vee \otimes Y \xrightarrow{X^\vee \otimes c_{X,Y^\vee\otimes Y}} (Y \otimes X)^\vee \otimes Y \otimes X$$

induce a map $\mu : \mathbb{F}\otimes\mathbb{F} \longrightarrow \mathbb{F}$ that endows the canonical coend $\mathbb{F}$ with the structure of a unital associative algebra in $\mathcal{C}$.

It is also well-known that the maps



$$: \ X^\vee \otimes X \otimes Y \longrightarrow Y \otimes X^\vee \otimes X \tag{3.29}$$

induce a half braiding for $\mathbb{F}$ that is often referred to as *dolphin half braiding* and in other contexts as *field goal transform*. We denote this lift of $\mathbb{F}$ to the Drinfeld center by $\mathbb{F}^{\dolphin}$. Since the multiplication $\mu : \mathbb{F} \otimes \mathbb{F} \longrightarrow \mathbb{F}$ from Proposition 3.21 lifts to a braided commutative multiplication on $\mathbb{F}^{\dolphin} \in Z(\mathcal{C})$ [FGSS18, Lemma 2.8], we obtain an E_2-structure on the homotopy invariants $\mathcal{C}(I, \mathbb{F}^\bullet)$ by Theorem 3.10.

Definition 3.22. For any braided finite tensor category $\mathcal{C}$, we denote $\mathcal{C}(I, \mathbb{F}^\bullet)$ with its E_2-structure coming from the multiplication $\mu : \mathbb{F} \otimes \mathbb{F} \longrightarrow \mathbb{F}$ from Proposition 3.21 and the dolphin half braiding of $\mathbb{F}$ by $\mathfrak{A}_\mathcal{C}^{\dolphin}$ and refer to this E_2-algebra as the *dolphin algebra* of $\mathcal{C}$.

Remark 3.23. We find $H^0(\mathfrak{A}_\mathcal{C}^{\dolphin}) \cong \mathcal{C}(I, \mathbb{F})$, where the multiplication is induced by the product of $\mathbb{F}$ given in Proposition 3.21. On the other hand, $\mathcal{C}(I, \mathbb{F}) \cong Z(\mathcal{C})(\mathsf{F}, \mathsf{F})$, so $\mathcal{C}(I, \mathbb{F})$ comes with a multiplication by being an endomorphism algebra in the center. The vector space $\mathcal{C}(I, \mathbb{F})$ with this product is referred to as the *class functions* $\mathsf{CF}(\mathcal{C})$ of $\mathcal{C}$. By [Shi17b, Proposition 3.13] the algebra structure of the class functions actually agrees with the product on $\mathcal{C}(I, \mathbb{F})$ coming from the multiplication from Proposition 3.21. Hence, we have $H^0(\mathfrak{A}_\mathcal{C}^{\dolphin}) \cong \mathsf{CF}(\mathcal{C})$ as algebras.

Theorem 3.24 *(Verlinde algebra on the Hochschild cochain complex)***.** *Let $\mathcal{C}$ be a unimodular braided finite tensor category with a fixed trivialization $D \cong I$ of the distinguished invertible object. Then the Hochschild cochain complex $\int^{\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ inherits from its braided monoidal product the structure of an E_2-algebra $\left(\int^{\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)\ ,\ \otimes \right)$ whose multiplication we denote again by $\otimes$.*

We refer to this E_2-algebra as the *differential graded Verlinde algebra on the Hochschild cochain complex of $\mathcal{C}$*.

Proof. Thanks to unimodularity and the trivialization $D \cong I$, we have the Radford map $\Psi : \mathbb{A} \longrightarrow \mathbb{F}$. In combination with Proposition 3.3 (exploited here only at the level of complexes), we obtain an equivalence of chain complexes

$$\int^{\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \simeq \mathcal{C}(I, \mathbb{A}^\bullet) \xrightarrow{\ \Psi^\bullet\ } \mathcal{C}(I, \mathbb{F}^\bullet)\ , \tag{3.30}$$

and with suitable models for the homotopy end and a resolution of $\mathbb{A}$, this is actually an isomorphism (see the proof of Proposition 3.3). As a consequence, there is, up to homotopy, a unique E_2-algebra $\left(\int^{\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)\ ,\ \otimes \right)$ such that

$$\left(\int^{\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)\ ,\ \otimes \right) \xrightarrow{\ (3.30)\ } \mathfrak{A}_{\mathcal{C}}$$

is an equivalence (or depending on the model even an isomorphism) of E_2-algebras. Here $\mathfrak{A}_{\mathcal{C}}$ is the dolphin algebra from Definition 3.22. $\square$

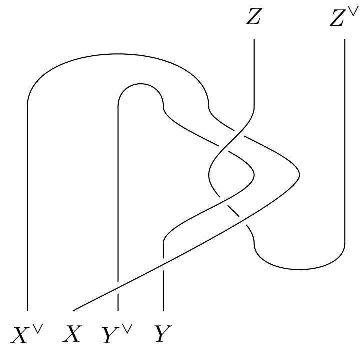
3.6 The effect of non-degeneracy

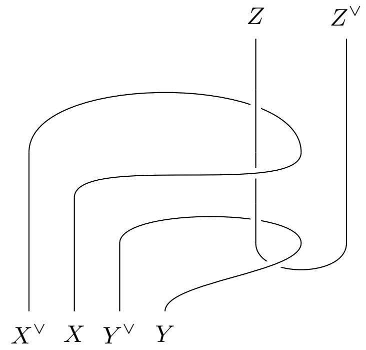
We will later need a technical result on the dolphin algebra from Definition 3.22. This will rely on our construction of E_2-algebras (Theorem 3.10), but also on the following well-known result:

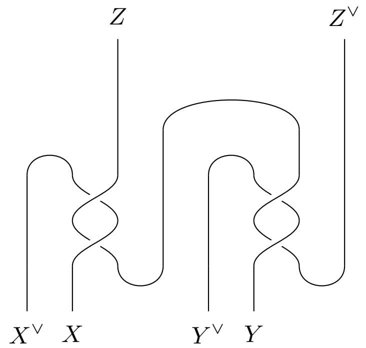
Proposition 3.25. *The Drinfeld map $\mathbb{D} : \mathbb{F} \longrightarrow \mathbb{A}$ of a finite braided tensor category is a map of algebras $(\mathbb{F}, \mu) \longrightarrow (\mathbb{A}, \gamma)$. Moreover, it lifts to a morphism $\mathbb{F} \longrightarrow \mathsf{A}$ of algebras in $Z(\mathcal{C})$, where $\mathbb{F} \in Z(\mathcal{C})$ is the canonical coend of $\mathcal{C}$ equipped with the dolphin half braiding and A is the canonical algebra of $Z(\mathcal{C})$.*

This statement, at least in the Hopf algebraic case, goes back to Drinfeld [Dri90]. If $\mathcal{C}$ is modular, then Proposition 3.25 tells us that $\mathbb{F}$ and $\mathbb{A}$ are isomorphic as algebras. In this situation, a related statement taking also comultiplications into account is given in [Kar19, Theorem 5.16]. For us, however, the version given in Proposition 3.25 is sufficient. Since Proposition 3.25 is quite vital and since the argument behind it is very insightful, we give a short graphical proof.

Proof of Proposition 3.25. In order to prove $\mathbb{D} \circ \mu = \gamma \circ (\mathbb{D} \otimes \mathbb{D})$, it suffices by the universal property of (co)ends to prove that for $X, Y, Z \in \mathcal{C}$ the restriction of the Z-component to $X^\vee \otimes X \otimes Y^\vee \otimes Y$ of both maps agree. We denote this component by $(\mathbb{D} \circ \mu)^Z_{X,Y}$ and $(\gamma \circ (\mathbb{D} \otimes \mathbb{D}))^Z_{X,Y}$, respectively. Now the proof follows from the following computation in the graphical calculus:

$(\mathbb{D} \circ \mu)^Z_{X,Y} \quad =$ 

$=$ 

$=$ 

$= \quad (\gamma \circ (\mathbb{D} \otimes \mathbb{D}))^Z_{X,Y}$

A similar computation proves that $\mathbb{D}$ makes the square

$$\begin{tikzcd}
\mathbb{F} \otimes X \arrow[rr, "\text{half braiding from (3.29)}"] \arrow[d, "\mathbb{D}\otimes X"'] & & X \otimes \mathbb{F} \arrow[d, "X\otimes\mathbb{D}"] \\
\mathbb{A} \otimes X \arrow[rr, "\text{half braiding from (2.8)}"'] & & X \otimes \mathbb{A}
\end{tikzcd}$$

commute. Therefore, $\mathbb{D}$ is also a morphism in the Drinfeld center $Z(\mathcal{C})$. ☐

Proposition 3.26. *For any braided finite tensor category $\mathcal{C}$, the Drinfeld map $\mathbb{D} : \mathbb{F} \longrightarrow \mathbb{A}$ induces a map of E_2-algebras*

$$\mathbb{D}^\bullet : \mathfrak{A}_{\mathcal{C}} \longrightarrow \mathcal{C}(I, \mathbb{A}^\bullet) \tag{3.31}$$

from the dolphin algebra to the homotopy invariants of the canonical end. This map is an equivalence (or isomorphism, for suitable models) if and only if the braiding of $\mathcal{C}$ is non-degenerate.

Proof. The two E_2-algebras in question were constructed in Definition 3.22 and Theorem 3.12. It follows from Proposition 3.25 and Proposition 3.11 that (3.31) is a map of E_2-algebras. If $\mathcal{C}$ is non-degenerate, then (3.31) is an equivalence because the Drinfeld map is an isomorphism [Shi19a]. If conversely (3.31) is an equivalence, then, in particular, the map $\mathcal{C}(I, \mathbb{F}) \longrightarrow \mathcal{C}(I, \mathbb{A})$ induced by $\mathbb{D}$ in zeroth cohomology is an isomorphism. By applying again Shimizu's results [Shi19a] this suffices to ensure non-degeneracy. ☐

3.7 Application I: The self-extension algebra of a finite tensor category and the Farinati-Solotar Gerstenhaber bracket

Our results, in particular Theorem 3.10, allow us to simplify the proofs of some known results in the homological algebra of finite tensor categories and also lead to generalizations. These applications can already be formulated without our main results on the differential graded Verlinde formula in the next section. The reader only interested in the two main results may skip ahead to Section 4. The results of this subsection and the next one will only be needed for Corollary 4.3.

One of the key homological algebra quantities of a finite tensor category $\mathcal{C}$ is the *self-extension algebra* $\mathrm{Ext}^*_{\mathcal{C}}(I,I)$ of the unit I which was studied, in the framework of finite tensor categories, by Etingof and Ostrik [EO04], see [NP18] for an overview. It is known that $\mathrm{Ext}^*_{\mathcal{C}}(I,I)$ is graded commutative. If $\mathcal{C}$ is the category of finite-dimensional representation of a finite group G, $\mathrm{Ext}^*_{\mathcal{C}}(I,I)$ is the group cohomology ring $H^*(G;k)$. For certain small quantum groups, the Ext algebra was computed by Ginzburg and Kumar [GK93].

If $\mathcal{C}$ is given by the category of finite-dimensional modules over a finite-dimensional Hopf algebra, Farinati and Solotar [FS04] have given a Gerstenhaber bracket on the self-extension algebra, see also [Her16] for a discussion of the inclusion $\mathrm{Ext}^*_{\mathcal{C}}(I,I) \longrightarrow HH^*(\mathcal{C})$ of algebras. The appearance of the Farinati-Solotar bracket comes from the fact that under mild conditions the derived endomorphisms $\mathcal{C}(I,I^\bullet)$ of the unit of a tensor category actually form an E_2-algebra. Such an argument is given in a non-linear setting by Kock and Toën in [KT05] in terms of weak 2-monoids and discussed in terms of B_∞-algebras by Lowen and van den Bergh in [LvdB19]. Theorem 3.10 allows us to give a new and easy proof of these results within the framework of finite tensor categories:

Corollary 3.27. *Let $\mathcal{C}$ be a finite tensor category. The self-extension algebra $\mathcal{C}(I,I^\bullet)$ carries the structure of an E_2-algebra that comes with a canonical map*

$$\mathcal{C}(I,I^\bullet) \longrightarrow \int^{\mathbb{R}}_{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \tag{3.32}$$

to the Hochschild complex of $\mathcal{C}$ equipped with the usual E_2-structure (Theorem 3.12). This map is a map of E_2-algebras. After taking cohomology, it is a monomorphism

$$\mathrm{Ext}^*_{\mathcal{C}}(I,I) \longrightarrow HH^*(\mathcal{C}) \tag{3.33}$$

of Gerstenhaber algebras (with suitable models, it is also a monomorphism at chain level), where the left hand side carries an extension of the Farinati-Solotar Gerstenhaber bracket to an arbitrary finite tensor category and the right hand side the usual Gerstenhaber bracket.

Proof. The unit I is trivially an algebra in $\mathcal{C}$ that also lifts to a braided commutative algebra in $Z(\mathcal{C})$. This turns $\mathcal{C}(I,I^\bullet)$ into an E_2-algebra by Theorem 3.10 which by construction is equivalent to the E_2-algebra $Z(\mathcal{C})(\mathsf{F},I^\bullet)$ from Proposition 3.9. By Theorem 3.12 the Hochschild cochain complex $\int^{\mathbb{R}}_{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ is canonically equivalent as an E_2-algebra to $\mathcal{C}(I,\mathbb{A}^\bullet)$ and also to $Z(\mathcal{C})(\mathsf{F},\mathsf{A}^\bullet)$. Therefore, up to equivalence, the map (3.32) is the map

$$Z(\mathcal{C})(\mathsf{F},I^\bullet) \longrightarrow Z(\mathcal{C})(\mathsf{F},\mathsf{A}^\bullet) \tag{3.34}$$

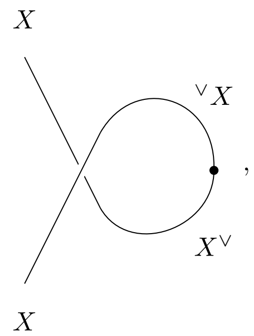
induced by the unit map $I \longrightarrow \mathsf{A}$ of $\mathsf{A} \in Z(\mathcal{C})$. Since this unit map is a morphism of algebras, (3.34) is a map of E_2-algebras by Proposition 3.11. This gives us the morphism of E_2-algebras (3.32). Moreover, $I \longrightarrow \mathsf{A}$ is a monomorphism. Since we can model $\mathsf{A}^\bullet$ as $I^\bullet \otimes \mathsf{A}$ and since the monoidal product is exact, $I^\bullet \longrightarrow \mathsf{A}^\bullet$ is a monomorphism as well. Actually, by definition $I^\bullet$ and $\mathsf{A}^\bullet$ are fibrant in the injective model structure on cochain complexes in $Z(\mathcal{C})$, so the monomorphism $I^\bullet \longrightarrow \mathsf{A}^\bullet$ is split, hence absolute (i.e. preserved by any functor). As a consequence, (3.34) is a monomorphism (this would have also followed since $\mathcal{C}(I,-)$ is left exact) and (3.33) is also a monomorphism. In order to complete the proof, we must compare the structure on cohomology with the one given by Farinati and Solotar [FS04]: By virtue of (3.33) being a monomorphism of Gerstenhaber algebras and Theorem 3.12, the Gerstenhaber structure obtained this way on $\mathrm{Ext}^*_{\mathcal{C}}(I,I)$ is a restriction of the usual Gerstenhaber structure on Hochschild cohomology. The same is true for the Farinati-Solotar Gerstenhaber algebra structure by the construction in [FS04]. Hence, both structures must agree on cohomology. □

By [Her16, Corollary 6.3.17 & Remark 6.3.19] the Gerstenhaber bracket on $\mathrm{Ext}^*_{\mathcal{C}}(I,I)$ vanishes if $\mathcal{C}$ is braided.

3.8 Application II: Generalizing a result of Menichi

Menichi proves in [Men11, Theorem 63] that for a finite-dimensional pivotal and unimodular Hopf algebra A, the inclusion $\mathrm{Ext}^*_A(k,k) \longrightarrow HH^*(A;A)$ is not only a monomorphism of Gerstenhaber algebras, but actually a monomorphism of Batalin-Vilkovisky algebras. We can use Theorem 3.10 to give a generalization of this result to a result at chain level that holds for all unimodular pivotal finite tensor categories, not only those coming from pivotal and unimodular Hopf algebras.

The Drinfeld center $Z(\mathcal{C})$ of a pivotal finite tensor category $\mathcal{C}$ comes with an induced pivotal structure. As usual, a pivotal structure allow us to define a balancing on $X \in Z(\mathcal{C})$ by

$\theta_X \quad =$ 

,

where the black dot is the natural isomorphism $X^\vee \cong {}^\vee X$ given by the pivotal structure. In other words, we extend $Z(\mathcal{C})$ to a framed E_2-algebra.

Lemma 3.28. *For any unimodular pivotal finite tensor category $\mathcal{C}$, the canonical algebra $\mathsf{A} \in Z(\mathcal{C})$ is not only braided commutative, but framed braided commutative in the sense that additionally the balancing of A is trivial, $\theta_{\mathsf{A}} = \mathrm{id}_{\mathsf{A}}$. The same is true for the canonical coalgebra $\mathsf{F} \in Z(\mathcal{C})$.*

Proof. The canonical algebra $\mathsf{A} \in Z(\mathcal{C})$ is a symmetric Frobenius algebra (Proposition 2.6) and braided commutative (Lemma 3.5) (note that the braided commutativity does *not* imply the symmetry because we are not working in a symmetric category). Thanks to [FFRS06, Proposition 2.25 (i)], this implies that the balancing of A is trivial, $\theta_{\mathsf{A}} = \mathrm{id}_{\mathsf{A}}$. The same holds true for the canonical coalgebra $\mathsf{F} \in Z(\mathcal{C})$ because $\mathsf{A} \cong \mathsf{F}$ as objects in $Z(\mathcal{C})$ thanks to unimodularity. $\square$

Theorem 3.29. *Let $\mathbb{T} \in \mathcal{C}$ be an algebra in a unimodular pivotal finite tensor category $\mathcal{C}$ with a lift to a framed braided commutative algebra $\mathsf{T} \in Z(\mathcal{C})$. Then the multiplication of $\mathbb{T}$ induces the structure of a framed E_2-algebra on the space $\mathcal{C}(I, \mathbb{T}^\bullet)$ of homotopy invariants of $\mathbb{T}$.*

Proof. Passing from E_2 to framed E_2 means passing from braid groups to framed braid groups. Therefore, it is straightforward to observe that Proposition 3.9 remains true if we replace the braided category by a balanced braided category and the braided commutative algebra by a framed braided commutative algebra (on the operadic level, the tools for this generalization are provided in [SW03]). We can now proceed as in the proof of Theorem 3.10 because F is *framed* braided cocommutative by Lemma 3.28. $\square$

Using Theorem 3.29 and Lemma 3.28, we can copy the proof of Corollary 3.27 to obtain a generalization of [Men11, Theorem 63]:

Corollary 3.30. *For any unimodular pivotal finite tensor category $\mathcal{C}$, both the self-extension algebra $\mathcal{C}(I, I^\bullet)$ and the Hochschild cochain complex $\int^{\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X, X)$ come equipped with a framed E_2-algebra structure such that*

$$\mathcal{C}(I, I^\bullet) \longrightarrow \int^{\mathbb{R}}_{X \in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X, X)$$

is a map (and with suitable models even a monomorphism) of framed E_2-algebras. After taking cohomology, it induces a monomorphism

$$\mathrm{Ext}^*_{\mathcal{C}}(I, I) \longrightarrow HH^*(\mathcal{C})$$

of Batalin-Vilkovisky algebras.

4 The differential graded Verlinde formula

By Theorem 3.24 the Hochschild *cochain* complex of a unimodular braided finite tensor category carries an E_2-multiplication induced by the braided monoidal product. We referred to this as the differential graded Verlinde algebra on the Hochschild cochain complex. The Hochschild *chain* complex also carries an E_2-multiplication that was already established in [SW19]:

Proposition 4.1 *([SW19, Proposition 3.11]). The Hochschild chain complex of a braided finite tensor category carries a non-unital E_2-structure induced by the braided monoidal product.*

Both the Verlinde algebra on the Hochschild cochain complex (Theorem 3.24) and the Verlinde algebra on the Hochschild chain complex (Proposition 4.1) generalize the Verlinde algebra from the semisimple case. Nonetheless, they behave very differently in the non-semisimple case, which will become clear in this section.

On both the Hochschild chain complex and the Hochschild cochain complex of a modular category, the mapping class group $\mathrm{SL}(2,\mathbb{Z})$ acts up to coherent homotopy because they are equivalent to the differential graded conformal block for the torus and its dual, respectively, see Section 2. Our two main results describe the action of the mapping class group element $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ on the two above-mentioned E_2-algebras.

The Verlinde algebra on the Hochschild cochain complex is transformed into Deligne's E_2-structure through the mapping class group action. In other words, both E_2-algebras lie in the same mapping class group orbit. Note that this requires $\mathcal{C}$ to be modular. For the Hochschild chain complex, a similar statement can be formulated using the cyclic Deligne Conjecture applied to the Calabi-Yau structure coming from the modified trace.

We prove that in the semisimple case, both results — the one for the Hochschild chains and the Hochschild cochains — reduce to the same statement, namely the semisimple Verlinde formula recalled in the introduction. We also spell out the statements obtained by restriction to zeroth (co)homology in the non-semisimple case and compare them to proposals for the generalization of the Verlinde formula (1.3) to the non-semisimple case [GR19].

4.1 The differential graded Verlinde formula on the Hochschild cochain complex of a modular category

We consider the Hochschild cochains first.

Theorem 4.2 *(Differential graded Verlinde formula for the Hochschild cochain complex). For any modular category $\mathcal{C}$, the action of the mapping class group element $S^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ on the Hochschild cochain complex of $\mathcal{C}$ yields an equivalence*

$$\mathfrak{F}^{\mathcal{C}}(S^{-1}) : \left(\int_{X \in \mathsf{Proj}\,\mathcal{C}}^{\mathbb{R}} \mathcal{C}(X,X) \ , \ \otimes \right) \simeq \left(\int_{X \in \mathsf{Proj}\,\mathcal{C}}^{\mathbb{R}} \mathcal{C}(X,X) \ , \ \smile \right)$$

of E_2-algebras which are given as follows:

- *On the left hand side, the E_2-structure is the differential graded Verlinde algebra on the Hochschild cochains of $\mathcal{C}$ induced by the monoidal product (Theorem 3.24).*
- *On the right hand side, the E_2-structure is the one afforded by Deligne's Conjecture with the underlying multiplication being the cup product $\smile$.*

Proof. According to the definition (1.8) of the *dual* differential graded modular functor, the mapping class group element S^{-1} acts on the dual conformal block by acting with S on the chain version of the differential graded conformal block and dualization. From Proposition 2.7 one can conclude that the action of S^{-1} on $\int_{X \in \mathsf{Proj}\,\mathcal{C}}^{\mathbb{R}} \mathcal{C}(X,X)$ is given by the composition of equivalences

$$\int_{X \in \mathsf{Proj}\,\mathcal{C}}^{\mathbb{R}} \mathcal{C}(X,X) \simeq \mathcal{C}(I,\mathbb{A}^\bullet) \xrightarrow{\Psi^\bullet} \mathcal{C}(I,\mathbb{F}^\bullet) \xrightarrow{\mathbb{D}^\bullet} \mathcal{C}(I,\mathbb{A}^\bullet) \simeq \int_{X \in \mathsf{Proj}\,\mathcal{C}}^{\mathbb{R}} \mathcal{C}(X,X) \ ,$$

where both unlabeled equivalences are the canonical one from Proposition 3.3 (recall that it can be turned into an isomorphism for suitable models of the homotopy end). The isomorphism $\Psi : \mathbb{A} \longrightarrow \mathbb{F}$ is the Radford map and $\mathbb{D} : \mathbb{F} \longrightarrow \mathbb{A}$ is the Drinfeld map.

We can now consider the following diagram in which the vertices are E_2-algebras:

$$\begin{tikzcd}
\left(\int_{X \in \mathsf{Proj}\, \mathcal{C}}^{\mathbb{R}} \mathcal{C}(X,X) \ , \ \otimes \right) \arrow[r, "\simeq"] \arrow[dd, "\mathfrak{F}^{\mathcal{C}}(S^{-1})"'] & (\ \mathcal{C}(I, \mathbb{A}^\bullet) \ , \ \otimes \ ) \arrow[d, "\Psi^\bullet"] \\
 & \mathfrak{A}_{\mathcal{C}}^{\dolphin} \arrow[d, "\mathbb{D}^\bullet"] \\
\left(\int_{X \in \mathsf{Proj}\, \mathcal{C}}^{\mathbb{R}} \mathcal{C}(X,X) \ , \ \cup \right) \arrow[r, "\simeq"'] & (\ \mathcal{C}(I, \mathbb{A}^\bullet) \ , \ \gamma^\bullet \ )
\end{tikzcd}$$

The description of S^{-1} that we have just extracted from Proposition 2.7 means that the diagram commutes as a diagram of chain complexes. It remains to be shown that all of the maps in the diagram, except $\mathfrak{F}^{\mathcal{C}}(S^{-1})$, are not only chain maps, but maps of E_2-algebras because then $\mathfrak{F}^{\mathcal{C}}(S^{-1})$ is also a map of E_2-algebras. For all of the maps appearing in the diagram, this has been established previously in the text; we just have to tie everything together: The upper horizontal map and $\Psi^\bullet$ are maps of E_2-algebras as follows from the construction of $\left(\int_{X \in \mathsf{Proj}\, \mathcal{C}}^{\mathbb{R}} \mathcal{C}(X,X) \ , \ \otimes \right)$ in Theorem 3.24. For $\mathbb{D}^\bullet$, this is a consequence of Proposition 3.26 and Proposition 3.11. Finally, for the lower horizontal map, it follows from Theorem 3.12. This finishes the proof. $\square$

Actually, we can promote $\mathfrak{F}^{\mathcal{C}}(S^{-1})$ to an equivalence of framed E_2-algebras:

Corollary 4.3 *(Framed extension)***.** *If $\mathcal{C}$ is a unimodular finite ribbon category and if a trivialization $D \cong I$ has been fixed, the differential graded Verlinde algebra $\left(\int_{X \in \mathsf{Proj}\, \mathcal{C}}^{\mathbb{R}} \mathcal{C}(X,X) \ , \ \otimes \right)$ naturally extends to a framed E_2-algebra. If additionally we fix for $\left(\int_{X \in \mathsf{Proj}\, \mathcal{C}}^{\mathbb{R}} \mathcal{C}(X,X) \ , \ \smile \right)$ the framed E_2-algebra afforded by Corollary 3.30, and if $\mathcal{C}$ is modular, the map*

$$\mathfrak{F}^{\mathcal{C}}(S^{-1}) : \left(\int_{X \in \mathsf{Proj}\, \mathcal{C}}^{\mathbb{R}} \mathcal{C}(X,X) \ , \ \otimes \right) \simeq \left(\int_{X \in \mathsf{Proj}\, \mathcal{C}}^{\mathbb{R}} \mathcal{C}(X,X) \ , \ \smile \right) \tag{4.1}$$

is an equivalence of framed E_2-algebras.

Proof. One can show that $\theta_{\mathbb{F}^{\dolphin}} = \mathrm{id}_{\mathbb{F}^{\dolphin}}$ (a proof is given in [FGSS18, Lemma 2.10 (i)] under slightly stronger assumptions, but the argument applies here as well; it uses that $\mathcal{C}$ is actually ribbon and not just balanced). Now we can conclude from Theorem 3.29 that the dolphin algebra of $\mathcal{C}$ extends to a framed E_2-algebra. By the construction of $\left(\int_{X \in \mathsf{Proj}\, \mathcal{C}}^{\mathbb{R}} \mathcal{C}(X,X) \ , \ \otimes \right)$ in Theorem 3.24, it follows that $\left(\int_{X \in \mathsf{Proj}\, \mathcal{C}}^{\mathbb{R}} \mathcal{C}(X,X) \ , \ \otimes \right)$ becomes a framed E_2-algebra as well. The fact that (4.1), in the modular case, respects also the *framed* E_2-structures follows from the framed version of Proposition 3.11. $\square$

For a cohomology class $[\varphi]$ in the Hochschild cochain complex of a modular category $\mathcal{C}$, we write the action by the mapping class group element as $S[\varphi]$ (instead of $\mathfrak{F}^{\mathcal{C}}(S)[\varphi]$). Then Theorem 4.2 tells us in particular

$$S[\varphi] \otimes S[\psi] = S([\varphi] \cup [\psi]) \ ,$$

where $\otimes$, by slight abuse of notation, denotes the multiplication induced by the monoidal product in the sense of Theorem 3.24. If we denote by $[-,-]_\otimes$ the Gerstenhaber bracket associated to $\otimes$ and by $[-,-]$ the usual Gerstenhaber bracket on Hochschild cohomology, then

$$[S[\varphi], S[\psi]]_\otimes = S\left[[\varphi], [\psi]\right] \ . \tag{4.2}$$

Example 4.4. Consider the modular category $\mathsf{Mod}_k D(G)$ of finite-dimensional modules over the Drinfeld double of a finite group G (see also Example 2.9). Then the differential graded modular functor for $\mathsf{Mod}_k D(G)$ can be seen as a differential graded version of the Dijkgraaf-Witten modular functor as explained in [SW20, Example 3.13]. Dualizing Example 2.9, the dual differential graded conformal block for the torus, i.e. the Hochschild cochain complex of $D(G)$, is equivalent to the complex $C^*(\mathsf{PBun}_G(\mathbb{T}^2);k)$ of cochains on the groupoid of G-bundles over the torus. Now the cohomology of the differential graded Verlinde algebra of $\mathsf{Mod}_k D(G)$, seen as Batalin-Vilkovisky algebra, is determined by the Batalin-Vilkovisky structure on the Hochschild cohomology of group algebras and the mapping class group action on $C^*(\mathsf{PBun}_G(\mathbb{T}^2);k)$ (which is the geometric one).

An example for the non-triviality of the Gerstenhaber bracket of the differential graded Verlinde algebra can be obtained as follows: Over an algebraically closed field of characteristic $p>0$, the differential graded Verlinde algebra of modules over $D(\mathbb{Z}_p)$

$$\left(\int^{\mathbb{R}}_{X\in \mathsf{Proj}\,\mathsf{Mod}_k D(\mathbb{Z}_p)} \mathrm{Hom}_{D(\mathbb{Z}_p)}(X,X)\ ,\ \otimes \right)$$

has a non-zero Gerstenhaber bracket. In order to see this, observe that the linear category of modules over $D(\mathbb{Z}_p)$ is equivalent to modules over the action groupoid $\mathbb{Z}_p//\mathbb{Z}_p$ of the conjugation action of $\mathbb{Z}_p$ on itself, which is trivial here, of course. Therefore, $\mathbb{Z}_p//\mathbb{Z}_p \simeq \sqcup_{\mathbb{Z}_p} \star //\mathbb{Z}_p$. Now the statement follows from (4.2) and the computation of the Gerstenhaber bracket on $HH^*(k[\mathbb{Z}_p])$ in [LZ13], where it is shown in particular that the Gerstenhaber bracket is non-trivial.

Thanks to Theorem 4.2, the statement that the cohomology of the differential graded Verlinde algebra can be obtained through the Hochschild cohomology (which can be seen as a Gerstenhaber algebra or Batalin-Vilkovisky algebra) and the $\mathrm{SL}(2,\mathbb{Z})$-action remains true beyond Drinfeld doubles. Here, however, obtaining the needed ingredients is much more involved. At least in the general Hopf-algebraic case, the mapping class group action is explicitly given in [LMSS18] (see also the comments in Example 2.9). The Hochschild cohomology, at least as graded ring, is known e.g. for certain small quantum groups [LQ19]. A further investigation of this class of examples is beyond the scope of this article.

Spelling out the Verlinde formula on Hochschild cochains (Theorem 4.2) in zeroth cohomology, we recover a formula that Gainutdinov and Runkel have proposed and proven in [GR19] as a non-semisimple generalization of the Verlinde formula. Their result is partly phrased in terms of the linear Grothendieck ring: Recall from [EGNO17, Definition 4.5.2] that for a finite tensor category $\mathcal{C}$, the *Grothendieck ring* $\mathsf{Gr}\,\mathcal{C}$ of $\mathcal{C}$ is the free Abelian group generated by a complete set of representatives $(X_i)_{i=0,\dots,n}$ for its finitely many isomorphism classes of simple objects (we denote the generator corresponding to X_i by $[X_i]$), where the ring structure is given by

$$[X_i]\cdot[X_j] := \sum_{\ell=0}^n N_{ij}^\ell [X_\ell]\ ,$$

with $N_{ij}^\ell := [X_i\otimes X_j : X_\ell]\in\mathbb{N}_0$ being the multiplicity of the simple object X_ℓ in the Jordan-Hölder series of the tensor product $X_i\otimes X_j$ (the numbers N_{ij}^ℓ generalize the fusion coefficients used in the semisimple case). Let now $\mathcal{C}$ be pivotal and unimodular. Then by [Shi17b, Theorem 4.1 & Corollary 4.3] the *internal character map*

$$\mathsf{ch}: \mathsf{Gr}_k\mathcal{C} = k\otimes_{\mathbb{Z}} \mathsf{Gr}\,\mathcal{C} \longrightarrow \mathsf{CF}(\mathcal{C})\ ,\quad [X_i]\longmapsto \left(I \xrightarrow{\ \widetilde{b}_X\ } {}^\vee X\otimes X \xrightarrow{\ \text{pivotal structure}\ } X^\vee\otimes X \longrightarrow \mathbb{F}\right)$$

exhibits the linear Grothendieck ring of $\mathcal{C}$ as a subalgebra of the algebra $\mathsf{CF}(\mathcal{C}) = \mathcal{C}(I,\mathbb{F})$ of class functions.

If $\Psi:\mathbb{A}\longrightarrow\mathbb{F}$ is again the Radford map, the family

$$(\phi_i)_{i=0,\dots,n}\ ,\quad \text{where}\quad \phi_i := \Psi^{-1}\circ \mathsf{ch}(X_i) : I\longrightarrow \mathbb{A} \tag{4.3}$$

is linear independent in $\mathcal{C}(I,\mathbb{A})$. For the next statement, we will denote the automorphism of $\mathcal{C}(I,\mathbb{A})$ corresponding to the action of S^{-1} on $HH^0(\mathcal{C})\cong \mathcal{C}(I,\mathbb{A})$ by $\mathfrak{S}$ (we do this to match the slightly different conventions in [GR19]). Moreover, we will denote the multiplication on $\mathcal{C}(I,\mathbb{A})$ coming from the cup product by $\circ$ because it amounts to the composition of natural endotransformations of the identity functor of $\mathcal{C}$.

Corollary 4.5 *(Gainutdinov-Runkel [GR19, Theorem 3.9]). Let $\mathcal{C}$ be a modular category and $(\phi_i)_{i=0,\dots,n}$ the linear independent family associated to a complete set of representatives of the finitely many isomorphism classes of simple objects via* (4.3)*. Then*

$$\mathfrak{S}^{-1}\left(\mathfrak{S}(\phi_i)\circ\mathfrak{S}(\phi_j)\right)=\sum_{\ell=0}^{n}N_{ij}^{\ell}\phi_\ell \ .$$

In the semisimple case, this statement reduces to the ordinary Verlinde formula.

Proof. Theorem 4.2, when spelled out in zeroth cohomology, states that the zeroth cohomology restriction of the action of S^{-1} (which we agreed to denote by $\mathfrak{S}$)

$$\mathfrak{S}:\mathcal{C}(I,\mathbb{A})\xrightarrow{\varPsi_*}\mathcal{C}(I,\mathbb{F})\xrightarrow{\mathbb{D}_*}\mathcal{C}(I,\mathbb{A})$$

induced by the Radford map $\varPsi:\mathbb{A}\longrightarrow\mathbb{F}$ and the Drinfeld map $\mathbb{D}:\mathbb{F}\longrightarrow\mathbb{A}$ is an isomorphism of algebras if we endow

- the vector space $\mathcal{C}(I,\mathbb{A})$ on the left hand side with the multiplication from Theorem 3.24,
- and the vector space $\mathcal{C}(I,\mathbb{A})$ on the right hand side with the multiplication coming from the cup product on zeroth Hochschild cohomology which here is just the multiplication coming from the usual algebra structure $\gamma:\mathbb{A}\otimes\mathbb{A}\longrightarrow\mathbb{A}$ from (2.9).

Now the map $\mathcal{C}(I,\mathbb{F})\xrightarrow{\varPsi_*^{-1}}\mathcal{C}(I,\mathbb{A})\xrightarrow{\mathfrak{S}}\mathcal{C}(I,\mathbb{A})$ is an isomorphism of algebras if $\mathcal{C}(I,\mathbb{F})$ is endowed with the product coming from the multiplication $\mu:\mathbb{F}\otimes\mathbb{F}\longrightarrow\mathbb{F}$ defined using the braiding of $\mathcal{C}$, see Proposition 3.21. Recall that by Remark 3.23 the algebra $\mathcal{C}(I,\mathbb{F})$ actually agrees with the algebra of class functions of $\mathcal{C}$. In summary, Theorem 4.2, when evaluated in zeroth cohomology, states that

$$\mathfrak{S}\circ\varPsi_*^{-1}:\mathsf{CF}(\mathcal{C})\xrightarrow{\varPsi_*^{-1}}\mathcal{C}(I,\mathbb{A})\xrightarrow{\mathfrak{S}}(\mathcal{C}(I,\mathbb{A}),\gamma_*)=(\mathcal{C}(I,\mathbb{A}),\circ) \tag{4.4}$$

is an isomorphism of algebras. (Of course, when considering the composition (4.4), we can actually cancel $\varPsi$, so that the statement that (4.4) is an isomorphism of algebras will alternatively follow from Proposition 3.25, but we actually need the factorization (4.4) to compare to [GR19].)

With the definition of the family $(\phi_i)_{i=0,\dots,n}$ in (4.3), we find:

$$\begin{aligned}\mathfrak{S}(\phi_i)\circ\mathfrak{S}(\phi_j)=\mathfrak{S}(\phi_i\otimes\phi_j)&\overset{(4.3)}{=}\mathfrak{S}\circ\varPsi_*^{-1}(\mathsf{ch}X_i\cdot\mathsf{ch}X_j)\\&=\mathfrak{S}\circ\varPsi_*^{-1}\left(\sum_{\ell=0}^{n}N_{ij}^{\ell}\,\mathsf{ch}X_\ell\right)\quad\text{(because ch is an algebra map)}\\&=\mathfrak{S}\left(\sum_{\ell=0}^{n}N_{ij}^{\ell}\phi_\ell\right)\ .\end{aligned}$$

□

4.2 The trace field theory and the block diagonal product on Hochschild chains

In order to prove the Verlinde formula for the Hochschild *chain* complex, we will need a few more tools that we provide in this subsection.

Following [FSS20], the *(right) Nakayama functor* $\mathsf{N}^{\mathsf{r}}:\mathcal{C}\longrightarrow\mathcal{C}$ of a finite category $\mathcal{C}$ can be described in a Morita-invariant way by

$$\mathsf{N}^{\mathsf{r}}X:=\int^{Y\in\mathcal{C}}\mathcal{C}(X,Y)^*\otimes Y\quad\text{for}\quad X\in\mathcal{C}\ . \tag{4.5}$$

For finite tensor category $\mathcal{C}$, there is a natural isomorphism [FSS20, Theorem 3.18]

$$\mathsf{N}^{\mathsf{r}}\cong D^{-1}\otimes -^{\vee\vee}$$

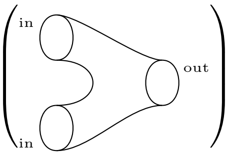
turning N^{r} into an equivalence from $\mathcal{C}$ as regular right $\mathcal{C}$-module category to $\mathcal{C}$ as regular $^{\vee\vee}$-twisted right $\mathcal{C}$-module category. A trivialization of N^{r} as right $\mathcal{C}$-module functor relative to a pivotal structure on $\mathcal{C}$ is referred to as *symmetric Frobenius structure* in [SW21]; it amounts to a pivotal structure on $\mathcal{C}$ and a trivialization $D\cong I$ of the distinguished invertible object.

Theorem 4.6 *([SW21, Theorem 3.6], see also [SS21]). For any finite tensor category $\mathcal{C}$ with symmetric Frobenius structure, the tensor ideal $\mathsf{Proj}\,\mathcal{C}$ canonically comes with a Calabi-Yau structure. The associated trace functions form a right modified trace on $\mathsf{Proj}\,\mathcal{C}$.*

A modified trace on $\mathsf{Proj}\,\mathcal{C}$ is a cyclic, non-degenerate trace satisfying the partial trace property, see [GPT09, GKP11, GKP13, GPV13, GKP21] for details. Under the above assumptions, it is unique up to invertible scalar. Note that in Theorem 4.6 *one specific* modified trace is obtained through the trivialization of the Nakayama functor; no other choice is made.

If $\mathcal{C}$ is a finite tensor category with symmetric Frobenius structure, the *trace field theory* of $\mathcal{C}$ [SW21] is defined as the open-closed topological conformal field theory $\Phi_{\mathcal{C}} : \mathsf{OC} \longrightarrow \mathsf{Ch}_k$ that the Calabi-Yau structure on $\mathsf{Proj}\,\mathcal{C}$ coming from the fixed trivialization of N^{r} gives rise to by a result of Costello [Cos07]. Depending on the characteristic, we need here additionally the results of Egas Santander [ES15] and Wahl and Westerland [WW16]. Here OC is a differential graded version of the open-closed two-dimensional bordism category with the projective objects of $\mathcal{C}$ as label set aka set of 'D-branes', see [Cos07] for the definition and [SW21, Section 4] for a very brief review. By evaluation of $\Phi_{\mathcal{C}}$ on the pair of pants, one obtains the *block diagonal $\star$-product* of the finite tensor category $\mathcal{C}$ with symmetric Frobenius structure:

$$\star := \Phi_{\mathcal{C}}$$



$$: \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \otimes \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \longrightarrow \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)\,. \tag{4.6}$$

By construction this is a non-unital E_2-multiplication (non-unital because the bordism without incoming boundary that would usually provide the unit is not admissible in OC). It is the multiplication afforded by the cyclic Deligne Conjecture applied to the Calabi-Yau structure on $\mathsf{Proj}\,\mathcal{C}$ coming from the modified trace, where the last connection to the modified trace is a consequence of Theorem 4.6. Wahl and Westerland [WW16] prove that a product extracted in the way (4.6) from the open-closed topological conformal field theory of a symmetric Frobenius algebra (or, more generally, a Calabi-Yau category) is, up to homotopy, supported in homological degree zero. Specifically for finite tensor categories, this product is further investigated in [SW21]. In particular, it is shown in [SW21, Proposition 5.3] that the product $\star$ is block diagonal; this will be spelled out in more detail on page 48.

The description (4.6) of the block diagonal product $\star$ is entirely topological. We will make use of this fact later, but we need additionally a description in terms of the canonical coend of our category: For any finite tensor category $\mathcal{C}$ (we do not assume a trivialization of N^{r} for the moment), the maps $Y \otimes \mathcal{C}(Y,X) \longrightarrow X$ for $X, Y \in \mathcal{C}$ induce maps

$$X^\vee \otimes X \longrightarrow (Y \otimes \mathcal{C}(Y,X))^\vee \otimes X \cong Y^\vee \otimes \mathcal{C}(Y,X)^* \otimes X \longrightarrow Y^\vee \otimes \mathsf{N}^{\mathsf{r}} Y\ , \tag{4.7}$$

where we have used the definition of the Nakayama functor in (4.5). These maps descend to the coend $\mathbb{F} = \int^{X\in\mathcal{C}} X^\vee \otimes X$ and factor through the end $\int_{Y\in\mathcal{C}} Y^\vee \otimes \mathsf{N}^{\mathsf{r}} Y$; in other words, they yield a map

$$\mathbb{F} \xrightarrow{\ \cong\ } \int_{Y\in\mathcal{C}} Y^\vee \otimes \mathsf{N}^{\mathsf{r}} Y \tag{4.8}$$

which is in fact an isomorphism because it can be obtained by applying the duality functor and the monoidal product to the isomorphism

$$\int^{X\in\mathcal{C}} X \boxtimes X \cong \int_{X\in\mathcal{C}} X \boxtimes \mathsf{N}^{\mathsf{r}} X \quad \text{in} \quad \mathcal{C}^{\mathrm{op}} \boxtimes \mathcal{C} \tag{4.9}$$

from [FSS20, equation (3.52)].

If $\mathcal{C}$ comes with a symmetric Frobenius structure, we obtain an isomorphism

$$\Omega : \mathbb{F} \xrightarrow{(4.8)} \int_{Y\in\mathcal{C}} Y^\vee \otimes \mathsf{N}^{\mathsf{r}} Y \xrightarrow{\mathsf{N}^{\mathsf{r}}\cong \mathrm{id}_{\mathcal{C}}} \int_{Y\in\mathcal{C}} Y^\vee \otimes Y \cong \int_{Y\in\mathcal{C}} Y \otimes Y^\vee = \mathbb{A} \quad \text{in}\ \mathcal{C}\ , \tag{4.10}$$

where in the last step we relabel the dummy variable and use the pivotal structure.

Lemma 4.7. *For any finite tensor category $\mathcal{C}$ with symmetric Frobenius structure, the isomorphism $\Omega : \mathbb{F} \longrightarrow \mathbb{A}$ is the inverse of the Radford map $\Psi : \mathbb{A} \longrightarrow \mathbb{F}$ from Definition 2.5.*

Proof. Thanks to $\mathsf{N}^{\mathsf{r}} \cong D^{-1} \otimes -^{\vee\vee}$, we can obtain Ω by applying the monoidal product to

$$\int^{X\in\mathcal{C}} X^\vee \boxtimes X \overset{(4.9)}{\cong} \int_{X\in\mathcal{C}} X^\vee \boxtimes D^{-1} \otimes X^{\vee\vee} \cong \int_{X\in\mathcal{C}} X \boxtimes D^{-1} \otimes X^\vee \quad \text{in} \quad \mathcal{C}\boxtimes\mathcal{C} \tag{4.11}$$

and using afterwards the isomorphism $D \cong I$ that is part of the symmetric Frobenius structure.

We now need to relate this to the Radford map $\Psi : \mathbb{A} \longrightarrow \mathbb{F}$ that we had defined using Shimizu's result [Shi17a] (cited as Theorem 2.4 above) on the relation between the left adjoint $L : \mathcal{C} \longrightarrow Z(\mathcal{C})$ and the right adjoint $R : \mathcal{C} \longrightarrow Z(\mathcal{C})$ to the forgetful functor $U : Z(\mathcal{C}) \longrightarrow \mathcal{C}$. In order to understand the relation to Ω, denote by $\mathcal{C}^{\mathrm{rev}}$ the finite tensor category obtained by reversing the monoidal product of $\mathcal{C}$. Then $\mathcal{C}^{\mathrm{env}} := \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$ is a finite tensor category, and the monoidal product of $\mathcal{C}$ turns $\mathcal{C}$ into a $\mathcal{C}^{\mathrm{env}}$-module category (the action is $(X \otimes Y).Z := X \otimes Z \otimes Y$ for $X, Y, Z \in \mathcal{C}$). The comparison of L and R is based on the algebra $B \in \mathcal{C}^{\mathrm{env}}$ which is the internal endomorphism object of $I \in \mathcal{C}$ for the $\mathcal{C}^{\mathrm{env}}$-action on $\mathcal{C}$; explicitly $B = \int^{X\in\mathcal{C}} X \boxtimes X^\vee \in \mathcal{C}^{\mathrm{env}}$. Denote by ${}_B B$ and B_B the object $B \in \mathcal{C}^{\mathrm{env}}$ as regular left and right module over itself, respectively. The crucial observation used in [Shi17a] is now that $Z(\mathcal{C})$ can be identified with the category ${}_B(\mathcal{C}^{\mathrm{env}})_B$ of B-bimodules. Under this identification, the left and the right adjoint of U take the form

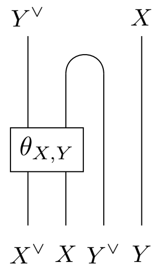
$$L : \mathcal{C} \longrightarrow {}_B(\mathcal{C}^{\mathrm{env}})_B \ , \quad Y \longmapsto {}_B B \otimes (Y \boxtimes I) \otimes B_B \overset{(4.11)}{\cong} \left(\int_{X\in\mathcal{C}} X \boxtimes D^{-1} \otimes X^\vee \right) \otimes (Y \boxtimes I) \otimes B_B \ ,$$
$$R : \mathcal{C} \longrightarrow {}_B(\mathcal{C}^{\mathrm{env}})_B \ , \quad Y \longmapsto B_B^\vee \otimes (Y \boxtimes I) \otimes B_B = \left(\int_{X\in\mathcal{C}} X \boxtimes X^\vee \right) \otimes (Y \boxtimes I) \otimes B_B \ ,$$

thereby leading us to $L \cong R(- \otimes D^{-1})$. In order to obtain Ψ^{-1}, we have to postcompose with the forgetful functor U, evaluate the resulting isomorphism $UL \cong UR(- \otimes D^{-1})$ at I and use the isomorphism $D \cong I$, but this is Ω by the description through (4.11). □

Up to the use of the pivotal structure in the last step in (4.10), the components of Ω are the maps

$$\theta_{X,Y} : X^\vee \otimes X \xrightarrow{(4.7)} Y^\vee \otimes \mathsf{N}^{\mathsf{r}} Y \xrightarrow{\mathsf{N}^{\mathsf{r}} \cong \mathrm{id}_{\mathcal{C}}} Y^\vee \otimes Y \ .$$

which will provide us with maps



$$: \quad X^\vee \otimes X \otimes Y^\vee \otimes Y \longrightarrow Y^\vee \otimes Y \ . \tag{4.12}$$

We have used here that left and right duality coincide thanks to the pivotal structure.

Proposition 4.8. *For any finite tensor category $\mathcal{C}$ with symmetric Frobenius structure, there is a unique algebra structure $\circledast : \mathbb{F} \otimes \mathbb{F} \longrightarrow \mathbb{F}$ characterized by any of the following equivalent descriptions:*

(i) The product $\circledast$ is induced by the maps (4.12).

(ii) The product $\circledast$ is the unique product on $\mathbb{F}$ turning

$$\Psi : (\ \mathbb{A}\ ,\ \gamma\ ) \xrightarrow{\cong} (\ \mathbb{F}\ ,\ \circledast\ )$$

into an isomorphism of algebras, where $\Psi : \mathbb{A} \longrightarrow \mathbb{F}$ is the Radford map.

It is implied by (2.15) that the product $\circledast$, in its description (ii), can also be expressed through the (co)integral of $\mathbb{F}$. Equivalently, it is the multiplication that exists on the dual $\mathbb{F}$ of $\mathbb{A}$ by virtue of $(\mathbb{A}, \gamma)$ being a Frobenius algebra (Theorem 2.4). We will, however, not need these other descriptions.

Proof of Proposition 4.8. Thanks to $\Psi^{-1} = \Omega$ by Lemma 4.7, it suffices to *define* $\circledast$ via (4.12) and verify

$$\gamma \circ (\Omega \otimes \Omega) = \Omega \circ \circledast \ . \tag{4.13}$$

This will in particular prove that $\circledast$ actually yields the structure of an algebra on $\mathbb{F}$ (which, just from (4.12), would not be clear).

By definition the components of $\Omega : \mathbb{F} \longrightarrow \mathbb{A}$ are given by

$$\widetilde{\theta}_{X,Y} : X^\vee \otimes X \xrightarrow{\ \theta_{X,Y^\vee}\ } Y^{\vee\vee} \otimes Y^\vee \xrightarrow{\text{pivotal structure } \omega} Y \otimes Y^\vee .$$

We will now describe the components in terms of the Calabi-Yau structure on $\mathsf{Proj}\,\mathcal{C}$. For this, we may assume that X and Y are projective, which is justified by [KL01, Proposition 5.1.7]. Now $\widetilde{\theta}_{X,Y}$ is explicitly given by the composition

$$X^\vee \otimes X \longrightarrow (Y^\vee \otimes \mathcal{C}(Y^\vee, X))^\vee \otimes X \overset{\omega}{\cong} Y \otimes \mathcal{C}(Y^\vee, X)^* \otimes X \overset{(*)}{\cong} Y \otimes \mathcal{C}(X, Y^\vee) \otimes X \longrightarrow Y \otimes Y^\vee ,$$

where in step $(*)$ we use the isomorphism $\mathcal{C}(Y^\vee, X)^* \cong \mathcal{C}(X, Y^\vee)$ afforded by the Calabi-Yau structure (this uses that X and Y are assumed to be projective). The isomorphism $\mathcal{C}(Y^\vee, X)^* \cong \mathcal{C}(X, Y^\vee)$ can be expressed through the coproducts and the unit of the Calabi-Yau category $\mathsf{Proj}\,\mathcal{C}$. Here by coproduct we mean the map

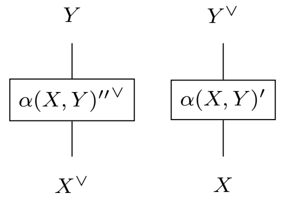
$$\Delta_{X,Y^\vee} : \mathcal{C}(X, X) \longrightarrow \mathcal{C}(X, Y^\vee) \otimes \mathcal{C}(Y^\vee, X)$$

obtained by dualizing the composition over $Y^\vee$ via the Calabi-Yau structure. This means that the isomorphism $\mathcal{C}(Y^\vee, X)^* \cong \mathcal{C}(X, Y^\vee)$ is given by the commuting square

$$\begin{tikzcd}
\mathcal{C}(Y^\vee, X)^* \arrow[rr, "\mathcal{C}(Y^\vee, X)^* \otimes \mathrm{id}_X"] \arrow[d, "\cong"'] & & \mathcal{C}({}^\vee Y, X)^* \otimes \mathcal{C}(X, X) \arrow[d, "\mathcal{C}(Y^\vee, X)^* \otimes \Delta_{X,Y^\vee}"] \\
\mathcal{C}(X, Y^\vee) & & \mathcal{C}(Y^\vee, X)^* \otimes \mathcal{C}(X, Y^\vee) \otimes \mathcal{C}(Y^\vee, X) . \arrow[ll, "\text{evaluation}"]
\end{tikzcd}$$

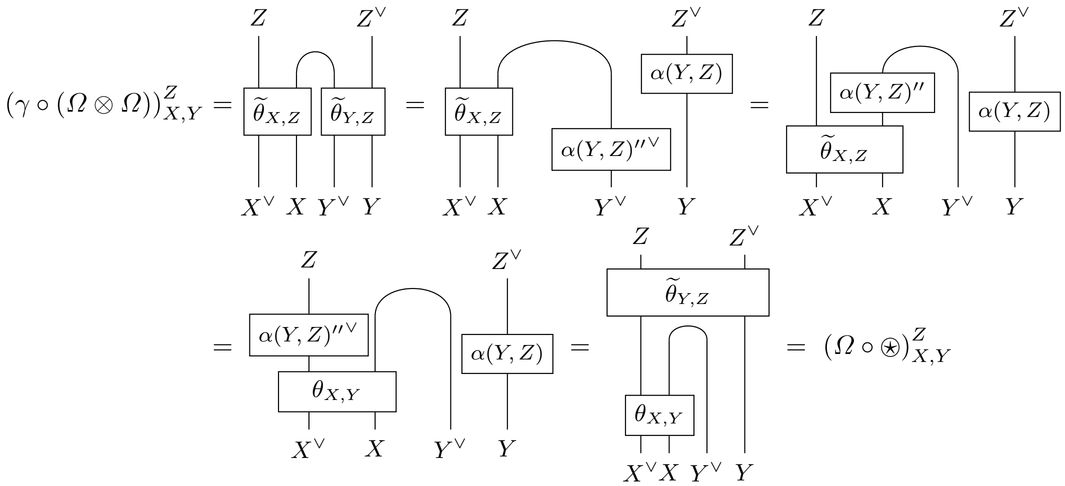
In order to be even more explicit, we use Sweedler notation $\Delta_{X,Y^\vee}(\mathrm{id}_X) = \alpha(X,Y)' \otimes \alpha(X,Y)'' \in \mathcal{C}(X, Y^\vee) \otimes \mathcal{C}(Y^\vee, X)$. With this notation,

$$\widetilde{\theta}_{X,Y} =$$



$$: \quad X^\vee \otimes X \longrightarrow Y \otimes Y^\vee , \qquad (4.14)$$

where, by slight abuse of notation, we see $\alpha(X,Y)''^\vee$ as a map $X^\vee \longrightarrow Y$ via the pivotal structure. Now denote by $(\gamma \circ (\Omega \otimes \Omega))^Z_{X,Y}$ the Z-component of the restriction of $\gamma \circ (\Omega \otimes \Omega)$ to $X^\vee \otimes X \otimes Y^\vee \otimes Y$ (again, we assume $X, Y, Z \in \mathsf{Proj}\,\mathcal{C}$). The computation



now proves (4.13) and hence finishes the proof of the Proposition. Note that in the last line the tilde on the θ disappears because the pivotal structure is absorbed into the dual of the map $\alpha(Y,Z)''$ which, by abuse of notation, we see as a map $\alpha(Y,Z)''^\vee : Y^\vee \longrightarrow Z$. $\square$

The product $\circledast$ induces the block diagonal product $\star$ on the Hochschild chains in the following sense:

Theorem 4.9. *Let $\mathcal{C}$ be a finite tensor category with symmetric Frobenius structure. Then the equivalence $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \simeq \mathcal{C}(I,\mathbb{F}_\bullet)$ of differential graded vector spaces from Proposition 2.1 yields an equivalence*

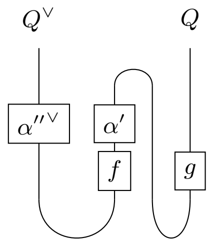
$$\left(\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)\ ,\ \star \right) \simeq (\ \mathcal{C}(I,\mathbb{F}_\bullet)\ ,\ \circledast_\bullet\ )$$

of non-unital E_2-algebras.

Proof. We already know that the product on the left hand side, up to homotopy, is supported in degree zero. In fact, this can also be directly seen for $(\ \mathcal{C}(I,\mathbb{F}_\bullet)\ ,\ \circledast_\bullet\ )$. Hence, it remains to confirm the compatibility of $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \simeq \mathcal{C}(I,\mathbb{F}_\bullet)$ with the algebra structure in degree zero. For this purpose, let us denote the equivalence $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \simeq \mathcal{C}(I,\mathbb{F}_\bullet)$ by Z.

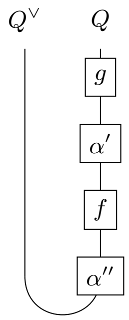
If we choose $\int_{\mathrm{f}\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} X^\vee \otimes X$ as the resolution of $\mathbb{F}$ (Proposition 2.1) and endomorphisms $f : P \longrightarrow P$ and $g : Q \longrightarrow Q$ for $P,Q \in \mathsf{Proj}\,\mathcal{C}$, we can obtain with Sweedler notation $\Delta_{P,Q}(\mathrm{id}_P) = \alpha' \otimes \alpha'' \in \mathcal{C}(P,Q) \otimes \mathcal{C}(Q,P)$

$Z(f) \circledast Z(g) =$



(see the proof of Proposition 4.8)

$=$



$= Z(g \star f)$ ([SW21, Lemma 5.2] in Sweedler notation)

$\simeq Z(f \star g)$.

□

4.3 The differential graded Verlinde algebra on the Hochschild chain complex of a modular category

We now prove our second main result. It is concerned with the effect of the S-transformation on the products on the Hochschild *chain* complex. As in the case of Theorem 4.2, the mapping class group action comes from the differential graded modular functor that $\mathcal{C}$ gives rise to.

Theorem 4.10 *(Differential graded Verlinde formula for the Hochschild chain complex)***.** *For any modular category $\mathcal{C}$, the action of the mapping class group element $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ yields an equivalence*

$$\mathfrak{F}_{\mathcal{C}}(S) : \left(\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)\ ,\ \otimes \right) \simeq \left(\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)\ ,\ \star \right)$$

of non-unital E_2-algebras whose multiplication, up to homotopy, is concentrated in degree zero.

- *On the left hand side, the E_2-structure is the differential graded Verlinde algebra on the Hochschild chains of $\mathcal{C}$ induced the monoidal product [SW19], see Proposition 4.1.*
- *On the right hand side, the non-unital E_2-structure is the almost trivial one that is a part of the cyclic version of Deligne's Conjecture applied to the Calabi-Yau structure coming from the modified trace on the tensor ideal of projective objects.*

Proof. Similarly to the proof of Theorem 4.2, all ingredients have been established, and we just tie them together: The effect of the mapping class group element S was computed in Proposition 2.7. After the canonical identification $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \simeq \mathcal{C}(I,\mathbb{F}_\bullet)$, it acts as the equivalence

$$\mathcal{C}(I,\mathbb{F}_\bullet) \xrightarrow{\mathbb{D}_\bullet} \mathcal{C}(I,\mathbb{A}_\bullet) \xrightarrow{\Psi_\bullet} \mathcal{C}(I,\mathbb{F}_\bullet) .$$

In fact, these maps are morphisms of non-unital E_2-algebras

$$(\ \mathcal{C}(I,\mathbb{F}_\bullet)\ ,\ \otimes\ ) \xrightarrow{\mathbb{D}_\bullet} (\ \mathcal{C}(I,\mathbb{A}_\bullet)\ ,\ \gamma_\bullet\ ) \xrightarrow{\Psi_\bullet} (\ \mathcal{C}(I,\mathbb{F}_\bullet)\ ,\ \circledast\ ) .$$

This is a consequence of Proposition 3.25 for $\mathbb{D}_\bullet$. For $\Psi_\bullet$, it follows from Proposition 4.8.

It remains to confirm that under the equivalence $\int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \simeq \mathcal{C}(I,\mathbb{F}_\bullet)$, the non-unital E_2-algebra $(\ \mathcal{C}(I,\mathbb{F}_\bullet)\ ,\ \circledast\ )$ translates into the non-unital E_2-algebra afforded by the cyclic Deligne Conjecture applied to the modified trace on the tensor ideal of projective objects. Indeed, this follows from $(\ \mathcal{C}(I,\mathbb{F}_\bullet)\ ,\ \circledast\ ) \simeq \left(\ \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)\ ,\ \star\ \right)$ (Theorem 4.9) and the fact that $\star$ is actually the non-unital E_2-multiplication coming from the cyclic Deligne Conjecture applied to the modified trace. The latter is a consequence of the results of [SW21] and in particular Theorem 4.6 from above. ☐

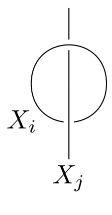
Remark 4.11 *(Products versus coproducts)***.** There seems to be an asymmetry between Theorem 4.2 on Hochschild cochains, where two rather rich higher multiplicative structures are compared, and Theorem 4.10 on Hochschild chains, which is concerned with an almost trivial product. This asymmetry, however, is mostly a consequence of our way of presenting the results: In both cases, we relate the products on the Verlinde algebra on the Hochschild cochains and chains via the S-transformation to the ones afforded by the cyclic Deligne Conjecture — and the partial triviality of one of the products is simply a feature which is already visible for the product coming from the Deligne Conjecture. The asymmetry comes from preferring product over coproducts: If we consider coproducts rather than products (which we can equivalently do thanks to the Calabi-Yau structure), the situation is reversed in the sense that the Hochschild cochains carry an almost trivial non-unital coproduct and the Hochschild chains a rather interesting coproduct.

Corollary 4.12. *For a semisimple modular category $\mathcal{C}$, the statements of Theorem 4.2 and 4.10 are equivalent and both amount precisely to the semisimple Verlinde formula.*

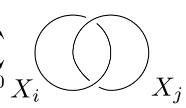
Proof. We choose a complete system $x_0 = I, x_1, \dots, x_n$ of simple objects of $\mathcal{C}$ and denote by $[x_i] \in HH_0(\mathcal{C})$ the element corresponding to the identity on x_i in zeroth Hochschild homology. Then $HH_0(\mathcal{C})$ has $[x_0], \dots, [x_n]$ as its basis thanks to $HH_0(\mathcal{C}) \cong \bigoplus_{i=0}^n \mathcal{C}(X_i,X_i) \cong \bigoplus_{i=0}^n k \cdot \mathrm{id}_{X_i}$. We denote by Ψ_* the isomorphism $HH_0(\mathcal{C}) \longrightarrow HH_0(\mathcal{C})$ induced by the Radford map $\Psi : \mathbb{A} \longrightarrow \mathbb{F}$ (here we fix $\mathbb{A} = \mathbb{F} = \bigoplus_{i=0}^n X_i^\vee \otimes X_i$ as a model for both the canonical coend and canonical end). From the concrete description of the inverse Ω of Ψ (in particular (4.14) in the proof of Lemma 4.7), we extract $\Psi_*[x_i] = d_i[x_i]$, where d_i is the usual quantum dimension of x_i (because the modified trace reduces to the quantum trace in the semisimple case).

Now we can compute the effect of the S-transformation by

$$\mathfrak{F}_{\mathcal{C}}(S)[x_i] \overset{\text{Proposition 2.7}}{=} \Psi_*\mathbb{D}_*[x_i] = \sum_{j=0}^{n} d_j \cdot$$



$$= \sum_{j=0}^{n}$$



$$\cdot [X_j] ,$$

i.e. it reduces to the S-matrix from the introduction. The product on zeroth Hochschild homology induced by the monoidal product in the sense of Proposition 4.1 is just given by $[x_i] \otimes [x_j] = \sum_{\ell=0}^n N_{ij}^\ell [x_\ell]$ if $x_i \otimes x_j \cong \bigoplus_{\ell=0}^n N_{ij}^\ell x_\ell$; this can be observed directly or concluded from the description of this product in [SW19]. Therefore, Theorem 4.10 is equivalent to the semisimple Verlinde formula in the formulation (1.6) if we can show that the multiplication $\star$ in Theorem 4.10 (which was defined in (4.6)) agrees with the $\star$-product

$$[x_i] \star [x_j] = d_i^{-1} \delta_{i,j} [x_i] \tag{4.15}$$

from (1.5). Indeed, this follows from [SW21, Theorem 5.6 (iii)] (again because the modified dimension agrees with the quantum dimension in the semisimple case).

In order to see that Theorem 4.2 (the cochain version) also reduces to the semisimple Verlinde formula, we can perform a similar computation. Alternatively, we can observe that in the semisimple case, Theorem 4.2 completely reduces to the statement extracted from it in Corollary 4.5, where we reproduced the result from [GR19]. This statement, on the other hand, is equivalent to the usual Verlinde formula in the semisimple case as explained in [GR19]. □

In the semisimple case, the S-transformation transforms the multiplication induced by the monoidal product into a diagonal product $\star$ given in (4.15), where diagonal means $[X_i] \star [X_j] = 0$ if X_i and X_j are non-isomorphic simple objects, i.e. if $\mathcal{C}(X_i, X_j) = 0$. In the non-semisimple case, Theorem 4.10 achieves at least a *block diagonalization* because the product $\star$ is block diagonal [SW21, Proposition 5.3].

In order to be more explicit, denote by $P_0, \dots, P_n$ a complete system of mutually non-isomorphic indecomposable projective objects of $\mathcal{C}$ and set $G := \bigoplus_{i=0}^n P_j$. The object G is a projective generator. We now define $B_1, \dots, B_m$ as the equivalence classes of the equivalence relation $P_i \simeq P_j :\Leftrightarrow \mathcal{C}(P_i, P_j) \neq 0$ on $\{P_0, \dots, P_n\}$. We refer to these equivalence classes as *blocks*. The endomorphism algebra $A := \mathcal{C}(G, G)$ allows us to write $\mathcal{C}$, as a linear category, as finite-dimensional modules over A. Moreover, A becomes a symmetric Frobenius algebra via the modified trace. In the same way, the endomorphism algebras $A_\ell := \mathcal{C}(G_\ell, G_\ell)$ of $G_\ell := \bigoplus_{P_i \in B_\ell} P_i$ for $1 \le \ell \le m$ become symmetric Frobenius algebras, and we find

$$A \cong A_1 \oplus \cdots \oplus A_m \tag{4.16}$$

as symmetric Frobenius algebras. The Hochschild complex $\int_{\mathbb{L}}^{X \in \mathsf{Proj}\, \mathcal{C}} \mathcal{C}(X, X)$ is equivalent to the ordinary Hochschild complex of A. In degree zero, i.e. on A, the product $\star$ from (4.6) is given by

$$a \star b = a' b a'' \quad \text{for} \quad a, b \in A \tag{4.17}$$

with Sweedler notation $\Delta a = a' \otimes a''$ (note that $\star$ yields only a commutative associative multiplication on $HH_0(\mathcal{C}) = A/[A, A]$, but not on A); this follows from [WW16] or also [SW21, Lemma 5.1]. The operation being block diagonal now means exactly that it preserves the decomposition (4.16) in the sense $A_\ell \star A_\ell \subset A_\ell$ and $A_\ell \star A_{\ell'} = 0$ for $\ell \neq \ell'$.

A tensor product of the indecomposable projective objects $P_0, \dots, P_n$ may be decomposed:

$$P_i \otimes P_j \cong \bigoplus_{\ell=0}^n P_\ell^{\oplus M_{ij}^\ell} \ ,$$

where the multiplicities $M_{ij}^\ell \in \mathbb{N}_0$ are the structure constants of the ring $K_0(\mathcal{C})$ that as an Abelian group is generated by $[P_0], \dots, [P_n]$. Via the map

$$K_0(\mathcal{C}) \otimes_{\mathbb{Z}} k \xrightarrow{[P_i] \longmapsto \mathrm{id}_{P_i}} \bigoplus_{j=0}^n \mathcal{C}(P_i, P_i) \longrightarrow HH_0(\mathcal{C}) \ ,$$

$[P_i]$ gives rise to a class in $HH_0(\mathcal{C})$ that we denote by $h_i \in HH_0(\mathcal{C}) \cong A/[A, A]$. If we act with the S-transformation on h_i, we may represent the result by an element $\mathfrak{s}_i \in A$; the choice we are making here is unique up to commutator.

Corollary 4.13. *With the above notation,*

$$\mathfrak{s}_i' \mathfrak{s}_j \mathfrak{s}_i'' = \sum_{\ell=0}^n M_{ij}^\ell \mathfrak{s}_\ell \quad \mathit{mod} \quad [A, A] \ ,$$

where $\Delta \mathfrak{s}_i = \mathfrak{s}_i' \otimes \mathfrak{s}_i''$ is the Sweedler notation for the coproduct of the Frobenius structure on A coming from the modified trace.

Proof. We find

$$\begin{aligned}
\sum_{\ell=0}^{n} M_{ij}^{\ell}\mathfrak{s}_\ell \mod [A,A] &= S\left(\sum_{\ell}^{n} M_{ij}^{\ell} h_\ell\right)\\
&= S(h_i \otimes h_j)\\
&= \mathfrak{s}_i \star \mathfrak{s}_j \mod [A,A] \quad \text{(Theorem 4.2)}\\
&= \mathfrak{s}_i'\mathfrak{s}_j\mathfrak{s}_i'' \mod [A,A] \quad \text{([SW21, Lemma 5.1], or see (4.17))} .
\end{aligned}$$

□

4.4 Partial three-dimensional extension and dimensional reduction

The two main Theorems 4.2 and 4.10 can be combined as follows: Let $\mathcal{C}$ be a modular category. By Proposition 4.1 the monoidal product induces a non-unital E_2-multiplication on the Hochschild chain complex

$$\otimes : \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \otimes \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \longrightarrow \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$$

supported, up to equivalence, in homological degree zero. In degree zero, however, it is relatively complicated and can be described by the S-transformation and the modified trace, see Theorem 4.10. The E_2-multiplication on the Hochschild cochain complex

$$\int^{\mathbb{R}}_{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \otimes \int^{\mathbb{R}}_{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \longrightarrow \int^{\mathbb{R}}_{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$$

induced by the monoidal product and unimodularity (Theorem 3.24) behaves totally differently; it will generally have a non-trivial Gerstenhaber bracket and is unital. By means of the Calabi-Yau structure, we can dualize it to an E_2-coproduct on the Hochschild chain complex

$$\Delta : \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \longrightarrow \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \otimes \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) ,$$

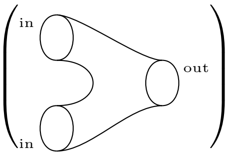
see Remark 4.11. Finally, we can define the following S-twisted version of the modified trace

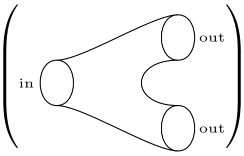
$$\tau : \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \xrightarrow{\quad S\text{-transformation}\quad} \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \xrightarrow{\quad\text{modified trace}\quad} k .$$

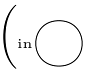
The three maps $\otimes, \Delta$ and τ combine into a closed topological conformal field theory.

Theorem 4.14. *Let $\mathcal{C}$ be a modular category, then the following assignments extend in a canonical way to a closed topological conformal field theory $\Lambda_{\mathcal{C}} : \mathsf{C} \longrightarrow \mathsf{Ch}_k$:*

$$\Lambda_{\mathcal{C}}(\mathbb{S}^1) := \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) ,$$

$\Lambda_{\mathcal{C}}$ () $:= \otimes : \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \otimes \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \longrightarrow \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) ,$

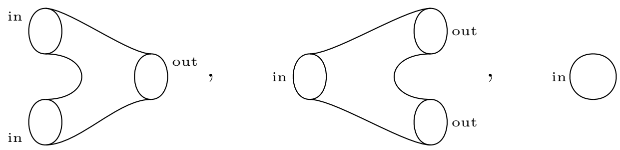
$\Lambda_{\mathcal{C}}$ () $:= \Delta : \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \longrightarrow \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \otimes \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) ,$

$\Lambda_{\mathcal{C}}$ () $:= \tau : \int_{\mathbb{L}}^{X\in\mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X) \longrightarrow k .$

Proof. Denote by $\Phi_{\mathcal{C}} : \mathsf{OC} \longrightarrow \mathsf{Ch}_k$ the trace field theory of $\mathcal{C}$ [SW21] and by $j : \mathsf{C} \longrightarrow \mathsf{OC}$ the inclusion of the closed part of OC into OC. We write $j^*\Phi_{\mathcal{C}} = \Phi_{\mathcal{C}} \circ j$ for the restriction of $\Phi_{\mathcal{C}}$ to the closed part of OC. The assertion follows if we can show

$$\Lambda_{\mathcal{C}}(\Sigma) = \left(\mathfrak{F}_{\mathcal{C}}(S^{-1})\right)^{\otimes q} \circ j^*\Phi_{\mathcal{C}}\,(\Sigma) \circ \mathfrak{F}_{\mathcal{C}}(S)^{\otimes p} \tag{4.18}$$

(p and q the number incoming and outgoing boundary components of Σ, respectively), where Σ is one of the bordisms



and $\Lambda_\mathcal{C}(\Sigma)$ is defined as above. We now prove (4.18) in the three relevant cases:

- If Σ is the pair of pants, the statement (4.18) is exactly Theorem 4.10, i.e. the Verlinde formula for Hochschild chains.
- If Σ is the opposite pair of pants, the statement (4.18) follows from Theorem 4.2, i.e. the Verlinde formula for Hochschild cochains, because the evaluation of $\Phi_\mathcal{C}$ on the reversed pair of pants is the usual E_2-structure on Hochschild cochains, but dualized via the Calabi-Yau structure (the latter is a part of Costello's result [Cos07]).
- If Σ is the disk, the statement (4.18) follows because the evaluation of the trace field theory on the disk is induced by the evaluation of $\Phi_\mathcal{C}$ on labeled disks, where it is given by the modified trace [SW21, Theorem 4.9].

□

Corollary 4.15 *(Partial three-dimensional extension of the differential graded modular functor)***.** *The differential graded modular functor $\mathfrak{F}_\mathcal{C}$ associated to a modular category $\mathcal{C}$ extends to three-dimensional oriented bordisms of the form $\Sigma \times \mathbb{S}^1 : \left(\mathbb{T}^2\right)^{\sqcup p} \longrightarrow \left(\mathbb{T}^2\right)^{\sqcup q}$, where $\Sigma : \left(\mathbb{S}^1\right)^{\sqcup p} \longrightarrow \left(\mathbb{S}^1\right)^{\sqcup q}$ is a compact oriented two-dimensional bordism such that every component of Σ has at least one incoming boundary component.*

More precisely, we get chain maps

$$\begin{aligned} C_*(\mathcal{M}_{p,q};k) &\longrightarrow \left[\mathfrak{F}_\mathcal{C}(\mathbb{T}^2)^{\otimes p}, \mathfrak{F}_\mathcal{C}(\mathbb{T}^2)^{\otimes q}\right] \\ \Sigma &\longmapsto \mathfrak{F}_\mathcal{C}(\Sigma \times \mathbb{S}^1)\ , \end{aligned}$$

where $\mathcal{M}_{p,q}$ is the moduli space of compact oriented surfaces with p incoming and q outgoing boundary circles. These are compatible with gluing along tori.

Proof of Corollary 4.15. For a compact oriented two-dimensional bordism such that every component of Σ has at least one incoming boundary component, we set

$$\mathfrak{F}_\mathcal{C}(\Sigma \times \mathbb{S}^1) := \Lambda_\mathcal{C}(\Sigma)\ :\ \left(\int_\mathbb{L}^{X \in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)\right)^{\otimes p} \longrightarrow \left(\int_\mathbb{L}^{X \in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)\right)^{\otimes q}\ , \tag{4.19}$$

but this will only give us the desired extension we can, in this situation, *canonically* identify the Hochschild complex $\int_\mathbb{L}^{X \in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ with $\mathfrak{F}_\mathcal{C}(\mathbb{T}^2)$. Generally, such an identification does *not* exist (and this is extremely crucial for the existence of the mapping class group actions), but it exists here: The bordism $\Sigma \times \mathbb{S}^1 : \left(\mathbb{T}^2\right)^{\sqcup p} \longrightarrow \left(\mathbb{T}^2\right)^{\sqcup q}$, by its very definition, distinguishes a circle direction in each of the tori, namely the one corresponding to the circle that we cross Σ with; we call this the *spectator direction*. Now each of the tori has a unique colored cut system (see Section 2 for this terminology) with one cut transversal to the spectator direction. By means of (2.3), this distinguished colored cut system provides for us the identification of $\int_\mathbb{L}^{X \in \mathsf{Proj}\,\mathcal{C}} \mathcal{C}(X,X)$ and $\mathfrak{F}_\mathcal{C}(\mathbb{T}^2)$, which for the evaluation on bordisms of the form $\Sigma \times \mathbb{S}^1$ becomes canonical. □

Remark 4.16. Corollary 4.15 does *not* include an extension to bordisms of the form $\Sigma \times \mathbb{S}^1$ if Σ has no incoming boundary components. If we included this, we would have admitted enough bordisms such that $\mathbb{T}^2$ comes with an evaluation and a coevaluation (given by bent cylinders over $\mathbb{T}^2$, as usual in the bordism category). But this would imply that $H_*\mathfrak{F}_\mathcal{C}(\mathbb{T}^2)$ is a dualizable, hence finite-dimensional graded vector space, and this will generally not be the case. In fact, since $\operatorname{Ext}^*_\mathcal{C}(I,I)$ is a direct summand of $HH^*(\mathcal{C})$ and $HH^*(\mathcal{C}) \cong (HH_*(\mathcal{C}))^*$ thanks to the Calabi-Yau structure, $\dim \operatorname{Ext}^*_\mathcal{C}(I,I) = \infty$ will already imply $\dim HH_*(\mathcal{C}) = \infty$ and hence rule out the possibility to extend to the solid torus as bordism $\emptyset \longrightarrow \mathbb{T}^2$. Examples with $\dim \operatorname{Ext}^*_\mathcal{C}(I,I) = \infty$ can be easily obtained from Drinfeld doubles in positive characteristic. In fact, we are not aware of *any* case where $\dim \operatorname{Ext}^*_\mathcal{C}(I,I) < \infty$ holds in the non-semisimple situation.

Corollary 4.15 does *not* turn the differential graded modular functor $\mathfrak{F}_\mathcal{C}$ into a three-dimensional topological field theory with values on chain complexes (and Remark 4.16 discussed concrete obstructions to such an extension). Instead, it offers a *partial three-dimensional extension* to bordisms of the form $\Sigma \times \mathbb{S}^1$ subject to the condition that each component of Σ has at least one incoming boundary component. Although the extension is not complete, it is exactly substantial enough for the *dimensional reduction*

$$\mathsf{Red}_{\mathbb{S}^1}\mathfrak{F}_\mathcal{C} := \mathfrak{F}_\mathcal{C}(\mathbb{S}^1 \times -) : \mathsf{OC} \longrightarrow \mathsf{Ch}_k$$

to exist (where the requirements on the numbers of boundary components are still implicit). Then (4.18) and (4.19) immediately imply the following compact reformulation of our results that comprises simultaneously the Verlinde formula for the Hochschild chains and cochains:

Corollary 4.17 *(Higher genus Verlinde formula).* *For any modular category $\mathcal{C}$, the dimensional reduction of the partial extension of the differential graded modular functor $\mathfrak{F}_\mathcal{C}$ to non-invertible three-dimensional bordisms from Corollary 4.15 is equivalent, via the S-transformation, to the trace field theory $\Phi_\mathcal{C}$ of $\mathcal{C}$ (the topological conformal field theory associated to the modified trace);*

$$\mathsf{Red}_{\mathbb{S}^1}\mathfrak{F}_\mathcal{C} \overset{S}{\simeq} \Phi_\mathcal{C}. \tag{4.20}$$

This is an equivalence of closed topological conformal field theories.

Since the partial three-dimensional extension was set up precisely to the extent that a dimensional reduction makes sense, one could naïvely think that one could *define* the partial three-dimensional extension via (4.20), thereby making the above Corollary a tautology, but this does not work: The non-trivial point is that Corollary 4.17 is not a statement about *some* partial extension of $\mathfrak{F}_\mathcal{C}$, but *the one obtained from Corollary 4.15*, for which we have given a concrete description *independent* of (4.20). Hence, if one used (4.20) as a definition, one would still need the Verlinde formula for both Hochschild chains and cochains to arrive at Corollary 4.17.

As yet another caveat in connection to Corollary 4.17, it is important to stress that the trace field theory $\Phi_\mathcal{C}$ only knows about the dimensional reduction of the partial extension of $\mathfrak{F}_\mathcal{C}$, but very little about $\mathfrak{F}_\mathcal{C}$ itself; in particular, the dimensional reduction loses practically all information on mapping class group actions on differential graded conformal blocks. Instead, (4.20) describes the multiplicative structures on $\mathfrak{F}_\mathcal{C}(\mathbb{T}^2)$ in terms of the linear category $\mathcal{C}$ and the modified trace (this is very much in the spirit of the original Verlinde formula). Moreover, (4.20) is not helpful as an abstract equivalence; the S-transformation which absorbs the information on the braiding needs to be remembered.

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