

ON THE PACKING/COVERING CONJECTURE OF INFINITE MATROIDS

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ABSTRACT. The Packing/Covering Conjecture was introduced by Bowler and Carmesin motivated by the Matroid Partition Theorem by Edmonds and Fulkerson. A packing for a family $(M_i : i \in \Theta)$ of matroids on the common edge set E is a system $(S_i : i \in \Theta)$ of pairwise disjoint subsets of E where S_i is spanning in M_i . Similarly, a covering is a system $(I_i : i \in \Theta)$ with $\bigcup_{i \in \Theta} I_i = E$ where I_i is independent in M_i . The conjecture states that for every matroid family on E there is a partition $E = E_p \sqcup E_c$ such that $(M_i \upharpoonright E_p : i \in \Theta)$ admits a packing and $(M_i.E_c : i \in \Theta)$ admits a covering. We prove the special case where E is countable and each M_i is either finitary or cofinitary. The connection between packing/covering and matroid intersection problems discovered by Bowler and Carmesin can be established for every well-behaved matroid class. This makes possible to approach the problem from the direction of matroid intersection. We show that the generalized version of Nash-Williams' Matroid Intersection Conjecture holds for countable matroids having only finitary and cofinitary components.

1. INTRODUCTION

Rado asked in 1966 (see Problem P531 in [24]) if it is possible to extend the concept of matroids to infinite without losing duality and minors. Based on the works of Higgs (see [19]) and Oxley (see [22] and [23]) Bruhn, Diestel, Kriesell, Pendavingh and Wollan settled Rado's problem affirmatively in [10] by finding a set of cryptomorphic axioms for infinite matroids, generalising the usual independent set-, bases-, circuit-, closure- and rank-axioms for finite matroids. Higgs named originally these structures B-matroids to distinguish from the original concept. Later this terminology vanished and in the context of infinite combinatorics B-matroids are referred as matroids and the term 'finite matroid' is used to differentiate.

An $M = (E, \mathcal{I})$ is a matroid if $\mathcal{I} \subseteq \mathcal{P}(E)$ with

- (1) $\emptyset \in \mathcal{I}$;
- (2) $\mathcal{I}$ is downward closed;
- (3) For every $I, J \in \mathcal{I}$ where J is $\subseteq$ -maximal in $\mathcal{I}$ but I is not, there exists an $e \in J \setminus I$ such that $I + e \in \mathcal{I}$;
- (4) For every $X \subseteq E$, any $I \in \mathcal{I} \cap \mathcal{P}(X)$ can be extended to a $\subseteq$ -maximal element of $\mathcal{I} \cap \mathcal{P}(X)$.

If E is finite, then (4) is redundant and (1)-(3) is one of the usual axiomatizations of finite matroids. One can show that every dependent set in an infinite matroid contains a

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minimal dependent set which is called a circuit. Before Rado's program was settled, a more restrictive axiom was used as a replacement of (4):

(4') If all the finite subsets of an $X \subseteq E$ are in $\mathcal{I}$, then $X \in \mathcal{I}$.

The implication (4') $\implies$ (4) follows directly from Zorn's lemma thus axioms (1), (2), (3) and (4') describe a subclass $\mathfrak{F}$ of the matroids. This $\mathfrak{F}$ consists of the matroids having only finite circuits and called the class of *finitary* matroids. Class $\mathfrak{F}$ is closed under several important operations like direct sums and taking minors but not under taking duals which was the main motivation of Rado's program for looking for a more general matroid concept. The class $\mathfrak{F}^*$ of the duals of the matroids in $\mathfrak{F}$ consists of the *cofinitary* matroids, i.e. matroid whose all cocircuits are finite. In order to being closed under all matroid operations we need, we work with the class $\mathfrak{F} \oplus \mathfrak{F}^*$ of matroids having only finitary and cofinitary components, equivalently that are the direct sum of a finitary and cofinitary matroid.

Matroid union is a fundamental concept in the theory of finite matroids. For a finite family $(M_i : i \leq n)$ of matroids on a common finite edge set E one can define a matroid $\bigvee_{i \leq n} M_i$ on E by letting $I \subseteq E$ be independent in $\bigvee_{i \leq n} M_i$ if $I = \bigcup_{i \leq n} I_i$ for suitable $I_i \in \mathcal{I}_{M_i}$ (see [11]). This phenomenon fails for infinite families of finitary matroids. Indeed, let E be uncountable and let M_i be the 1-uniform matroid on E for $i \in \mathbb{N}$. Then exactly the countable subsets of E would be independent in $\bigvee_{i \in \mathbb{N}} M_i$ and hence there would be no maximal independent set contradicting (4).¹ Even so, Bowler and Carmesin observed (see section 3 in [8]) that the rank formula in the Matroid Partition Theorem by Edmonds and Fulkerson (Theorem 13.3.1 in [15]), namely:

$$r \left(\bigvee_{i \leq n} M_i \right) = \max_{I_i \in \mathcal{I}_{M_i}} \left| \bigcup_{i \leq n} I_i \right| = \min_{E = E_p \sqcup E_c} |E_c| + \sum_{i \leq n} r_{M_i}(E_p),$$

can be interpreted in infinite setting via the complementary slackness conditions. In the minimax formula above there is equality for the family $(I_i : i \leq n)$ and partition $E = E_p \sqcup E_c$ iff

- I_i is independent in M_i ,
- $\bigcup_{i \leq n} I_i \supseteq E_c$,
- $I_i \cap E_p$ spans E_p in M_i for every i ,
- $I_i \cap I_j \cap E_p = \emptyset$ for $i \neq j$.

Bowler and Carmesin conjectured that for every family $\mathcal{M} := (M_i : i \in \Theta)$ of matroids on a common edge set E there is a family $(I_i : i \in \Theta)$ and partition $E = E_p \sqcup E_c$ satisfying the conditions above. To explain the name ‘‘Packing/Covering Conjecture’’ let us provide an alternative formulation. A *packing* for $\mathcal{M}$ is a system $(S_i : i \in \Theta)$ of pairwise disjoint subsets of E where S_i is spanning in M_i . Similarly, a *covering* for $\mathcal{M}$ is a system $(I_i : i \in \Theta)$ with $\bigcup_{i \in \Theta} I_i = E$ where I_i is independent in M_i .

Conjecture 1.1 (Packing/Covering, Conjecture 1.3 in [8]). *For every family $(M_i : i \in \Theta)$ of matroids on a common edge set E there is a partition $E = E_p \sqcup E_c$ in such a way that $(M_i \upharpoonright E_p : i \in \Theta)$ admits a packing and $(M_i.E_c : i \in \Theta)$ admits a covering.*

¹For a finite family of finitary matroids the union operation results in a matroid (Proposition 4.1 in [6]).

We shall prove the following special case of the Packing/Covering Conjecture 1.1:

Theorem 1.2. *For every family $(M_i : i \in \Theta)$ of matroids on a common countable edge set E where $M_i \in \mathfrak{F} \oplus \mathfrak{F}^*$, there is a partition $E = E_p \sqcup E_c$ such that $(M_i \upharpoonright E_p : i \in \Theta)$ admits a packing and $(M_i.E_c : i \in \Theta)$ admits a covering.*

It is worth to mention that packings and coverings have a crucial role in other problems as well. For example if $(M_i : i \in \Theta)$ is as in Theorem 1.2 and admits both a packing and a covering, then there is a partition $E = \bigsqcup_{i \in \Theta} B_i$ where B_i is a base of M_i (see [13]). Maybe surprisingly, the failure of the analogous statement for arbitrary matroids is consistent with set theory ZFC (Theorem 1.5 of [13]) which might raise some scepticism about the provability of Conjecture 1.1 for general matroids.

The Packing/Covering Conjecture 1.1 is closely related to the Matroid Intersection Conjecture which has been one of the central open problems in the theory of infinite matroids:

Conjecture 1.3 (Matroid Intersection Conjecture by Nash-Williams, [5]). *If M and N are finitary matroids on the same edge set E , then they admit a common independent set I for which there is a partition $E = E_M \sqcup E_N$ such that $I_M := I \cap E_M$ spans E_M in M and $I_N := I \cap E_N$ spans E_N in N .*

Aharoni proved in [1] based on his earlier works with Nash-Williams and Shelah (see [3] and [4]) that the special case of Conjecture 1.3 where M and N are partition matroids holds. The conjecture is also known to be true if we assume that E is countable but M and N can be otherwise arbitrary (see [21]). Let us call Generalized Matroid Intersection Conjecture what we obtain from 1.3 by extending it to arbitrary matroids (i.e. omitting the word “finitary”). Several partial results has been obtained for this generalization but only for well-behaved matroid classes. The positive answer is known for example if: M is finitary and N is cofinitary [6] or both matroids are singular² and countable [16] or M is arbitrary and N is the direct sum of finitely many uniform matroids [20].

Bowler and Carmesin showed that their Packing/Covering Conjecture 1.1 and the Generalized Matroid Intersection Conjecture are equivalent and they also found an important reduction for both (see Corollary 3.9 in [8]). By analysing their proof it is clear that the equivalence can be established if we restrict both conjectures to a class of matroids closed under certain operations. It allows us to prove Theorem 1.4 by showing the following instance of the Generalized Matroid Intersection Conjecture which itself is a common extension of the singular case by Ghaderi [16] and our previous work [21]:

Theorem 1.4. *If M and N are matroids in $\mathfrak{F} \oplus \mathfrak{F}^*$ on the same countable edge set E , then they admit a common independent set I for which there is a partition $E = E_M \sqcup E_N$ such that $I_M := I \cap E_M$ spans E_M in M and $I_N := I \cap E_N$ spans E_N in N .*

The paper is organized as follows. In the following section we introduce some notation and fundamental facts about matroids that are mostly well-know for finite ones. In Section 3 we collect some previous results and relatively easy technical lemmas in order be able

²A matroid is singular if it is the direct sum of 1-uniform matroids and duals of 1-uniform matroids.

the discuss later the proof of the main results without any distraction. Then in Section 4 we reduce the main results to a key-lemma. After these preparations the actual proof begins with Section 5 by developing and analysing an ‘augmenting path’ type of technique. Our main principle from this point is to handle the finitary and the cofinitary parts of matroid N differently in order to exploit the advantage of the finiteness of the circuits and cocircuits respectively. Equipped with these “mixed” augmenting paths we discuss the proof of our key-lemma in Section 6. Finally, we introduce an application in Section 7 about orientations of a graph with in-degree requirements.

2. NOTATION AND BASIC FACTS

In this section we introduce the notation and recall some basic facts about matroids that we will use later without further explanation. For more details we refer to [7].

An enumeration of a countable set X is an $\mathbb{N} \rightarrow X$ surjection that we write as $\{x_n : n \in \mathbb{N}\}$. We denote the symmetric difference $(X \setminus Y) \cup (Y \setminus X)$ of X and Y by $\mathbf{X} \Delta \mathbf{Y}$. A pair $M = (E, \mathcal{I})$ is a *matroid* if $\mathcal{I} \subseteq \mathcal{P}(E)$ satisfies the axioms (1)-(4). The sets in $\mathcal{I}$ are called *independent* while the sets in $\mathcal{P}(E) \setminus \mathcal{I}$ are *dependent*. An $e \in E$ is a *loop* if $\{e\}$ is dependent. If E is finite, then (1)-(3) is one of the the usual axiomization of matroids in terms of independent sets (while (4) is redundant). The maximal independent sets are called *bases*. If M admits a finite base, then all the bases have the same size which is the rank $\mathbf{r}(M)$ of M otherwise we let $r(M) := \infty$.³ The minimal dependent sets are called *circuits*. Every dependent set contains a circuit. The *components* of a matroid are the components of the hypergraph of its circuits. The *dual* of a matroid M is the matroid M^* with $E(M^*) = E(M)$ whose bases are the complements of the bases of M . For an $X \subseteq E$, $\mathbf{M} \upharpoonright \mathbf{X} := (X, \mathcal{I} \cap \mathcal{P}(X))$ is a matroid and it is called the *restriction* of M to X . We write $\mathbf{M} - \mathbf{X}$ for $M \upharpoonright (E \setminus X)$ and call it the minor obtained by the *deletion* of X . The *contraction* of X in M and the contraction of M onto X are $\mathbf{M}/\mathbf{X} := (M^* - X)^*$ and $\mathbf{M}.\mathbf{X} := M/(E \setminus X)$ respectively. Contraction and deletion commute, i.e., for disjoint $X, Y \subseteq E$, we have $(M/X) - Y = (M - Y)/X$. Matroids of this form are the *minors* of M . The independence of an $I \subseteq X$ in $M.X$ is equivalent with $I \subseteq \text{span}_{M^*}(X \setminus I)$. If I is independent in M but $I + e$ is dependent for some $e \in E \setminus I$ then there is a unique circuit $\mathbf{C}_M(\mathbf{e}, \mathbf{I})$ of M through e contained in $I + e$. We say $X \subseteq E$ *spans* $e \in E$ in matroid M if either $e \in X$ or there exists a circuit $C \ni e$ with $C - e \subseteq X$. By letting $\text{span}_M(\mathbf{X})$ be the set of edges spanned by X in M , we obtain a closure operation $\text{span}_M : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$. An $S \subseteq E$ is *spanning* in M if $\text{span}_M(S) = E$. An $S \subseteq X$ spans X in $M.X$ iff $X \setminus S$ is independent in M^* . If $M_i = (E_i, \mathcal{I}_i)$ is a matroid for $i \in \Theta$ and the sets E_i are pairwise disjoint, then their direct sum is $\bigoplus_{i \in \Theta} \mathbf{M}_i = (E, \mathcal{I})$ where $E = \bigsqcup_{i \in \Theta} E_i$ and $\mathcal{I} = \{\bigsqcup_{i \in \Theta} I_i : I_i \in \mathcal{I}_i\}$. For a class $\mathfrak{C}$ of matroids $\mathfrak{C}(E)$ denotes the subclass $\{M \in \mathfrak{C} : E(M) = E\}$. A matroid is called *uniform* if for ever base B and every edges $e \in B$ and $f \in E \setminus B$ the set $B - e + f$ is also a base. Let $\mathbf{U}_{E,n}$ be the n -uniform matroid on E , formally $\mathbf{U}_{E,n} := (E, [E]^{\leq n})$.

We need some further more subject-specific definitions. From now on let M and N be matroids on a common edge set E . We call a $W \subseteq E$ an (M, N) -*wave* if $M \upharpoonright W$ admits

³It is independent of ZFC that the bases of a fixed matroid have the same size (see [18] and [9]).

an $N.W$ -independent base. Waves in the matroidal context were introduced by Aharoni and Ziv in [5] but it was also an important tool in the proof of the infinite version of Menger's theorem [2] by Aharoni and Berger. We write $\mathbf{cond}(\mathbf{M}, \mathbf{N})$ for the condition: 'For every (M, N) -wave W there is an M -independent base of $N.W$.' A set $I \in \mathcal{I}_M \cap \mathcal{I}_N$ is *feasible* if $\mathbf{cond}(M/I, N/I)$ holds. It is known (see Proposition 3.5) that there exists a $\subseteq$ -largest (M, N) -wave which we denote by $\mathbf{W}(\mathbf{M}, \mathbf{N})$. Let $\mathbf{cond}^+(\mathbf{M}, \mathbf{N})$ be the statement that $W(M, N)$ consists of M -loops and $r(N.W(M, N)) = 0$. As the notation indicates it is a strengthening of $\mathbf{cond}(M, N)$. Indeed, under the assumption $\mathbf{cond}^+(M, N)$, $\emptyset$ is an M -independent base of $N.W$ for every wave W . A feasible I is called *nice* if $\mathbf{cond}^+(M/I, N/I)$ holds. For $X \subseteq E$ let $\mathbf{B}(\mathbf{M}, \mathbf{N}, \mathbf{X})$ be the (possibly empty) set of common bases of $M \upharpoonright X$ and $N.X$.

3. PRELIMINARY LEMMAS AND PREPARATION

We collect those necessary lemmas in this section that are either known from previous papers or follow more or less directly from definitions.

3.1. Classical results. The following two statements were proved by Edmonds' in [12]:

Proposition 3.1. *Assume that I is independent, $e_1, \dots, e_m \in \mathbf{span}(I) \setminus I$ and $f_1, \dots, f_m \in I$ with $f_j \in C(e_j, I)$ but $f_j \notin C(e_k, I)$ for $k < j$. Then*

$$(I \cup \{e_1, \dots, e_m\}) \setminus \{f_1, \dots, f_m\}$$

is independent and spans the same set as I .

Proof. We use induction on m . The case $m = 0$ is trivial. Suppose that $m > 0$. On the one hand, the set $I - f_m + e_m$ is independent and spans the same set as I . On the other hand, $C(e_j, I - f_m + e_m) = C(e_j, I)$ for $j < m$ because $f_m \notin C(e_j, I)$ for $j < m$. Hence by using the induction hypothesis for $I - f_m + e_m$ and $e_1, \dots, e_{m-1}, f_1, \dots, f_{m-1}$ we are done. $\square$

Lemma 3.2 (Edmonds' augmenting path method). *For $I \in \mathcal{I}_M \cap \mathcal{I}_N$, exactly one of the following statements holds:*

- (1) *There is a partition $E = E_M \sqcup E_N$ such that $I_M := I \cap E_M$ spans E_M in M and $I_N := I \cap E_N$ spans E_N in N .*
- (2) *There is a $P = \{x_1, \dots, x_{2n+1}\} \subseteq E$ with $x_1 \notin \mathbf{span}_N(I)$ and $x_{2n+1} \notin \mathbf{span}_M(I)$ such that $I \triangle P \in \mathcal{I}_M \cap \mathcal{I}_N$ with $\mathbf{span}_M(I \triangle P) = \mathbf{span}_M(I + x_{2n+1})$ and $\mathbf{span}_N(I \triangle P) = \mathbf{span}_N(I + x_1)$.*

We will develop in Section 5 a "mixed" augmenting path method which operates differently on the finitary and on the cofinitary part of an $N \in (\mathfrak{F} \oplus \mathfrak{F}^*)(E)$. The phrase 'augmenting path' refers always to our mixed method except in the proof of Lemma 3.14. Note that E_M is an (M, N) -wave witnessed by I_M and E_N is an (N, M) -wave witnessed by I_N .

One can define matroids in the language of circuits (see [10]). The following claim is one of the axioms in that case.

Claim 3.3 (Circuit elimination axiom). *Assume that $C \ni e$ is a circuit and $\{C_x : x \in X\}$ is a family of circuits where $X \subseteq C - e$ and C_x is a circuit with $C \cap X = \{x\}$ avoiding e . Then there is a circuit through e contained in*

$$\left(C \cup \bigcup_{x \in X} C_x\right) \setminus X =: Y$$

Proof. Since $C_x - x$ spans x we have $C - e \subseteq \text{span}(Y - e)$ and therefore $e \in \text{span}(\text{span}(Y - e))$. But then $e \in \text{span}(Y - e)$ because span is a closure operator. $\square$

For finite matroids the axiom above is demanded only in the special case where X is a singleton (known as “Strong circuit elimination”) from which the case of arbitrary X can be derived by repeated application.

Corollary 3.4. *Let I be an independent and suppose that there is a circuit $C \subseteq \text{span}(I)$ with $e \in I \cap C$. Then there is an $f \in C \setminus I$ with $e \in C(f, I)$.*

Proof. For every $x \in C \setminus I$ we pick a circuit C_x with $C_x \setminus I = \{x\}$. If $e \in C_x$ for some x , then $f := x$ is as desired. Suppose for a contradiction that there is no such an x . Then by Circuit elimination (Claim 3.3) we obtain a circuit through e which is contained entirely in I contradicting the independence of I . $\square$

The following statement was shown by Aharoni and Ziv in [5] using a slightly different terminology.

Proposition 3.5. *The union of arbitrary many waves is a wave.*

Proof. Suppose that W_β is a wave for $\beta < \kappa$ and let $W_{<\alpha} := \bigcup_{\beta < \alpha} W_\beta$ for $\alpha \leq \kappa$. We fix a base $B_\beta \subseteq W_\beta$ of $M \upharpoonright W_\beta$ which is independent in $N.W_\beta$. Let us define $B_{<\alpha}$ by transfinite recursion for $\alpha \leq \kappa$ as follows.

$$B_{<\alpha} := \begin{cases} \emptyset & \text{if } \alpha = 0 \\ B_{<\beta} \cup (B_\beta \setminus W_{<\beta}) & \text{if } \alpha = \beta + 1 \\ \bigcup_{\beta < \alpha} B_{<\beta} & \text{if } \alpha \text{ is limit ordinal.} \end{cases}$$

First we show by transfinite induction that $B_{<\alpha}$ is spanning in $M \upharpoonright W_{<\alpha}$. For $\alpha = 0$ it is trivial. For a limit α it follows directly from the induction hypothesis. If $\alpha = \beta + 1$, then by the choice of B_β , the set $B_\beta \setminus W_{<\beta}$ spans $W_\beta \setminus W_{<\beta}$ in $M/W_{<\beta}$. Since $W_{<\beta}$ is spanned by $B_{<\beta}$ in M by induction, it follows that $W_{<\beta+1}$ is spanned by $B_{<\beta+1}$ in M .

The independence of $B_{<\alpha}$ in $N.W_{<\alpha}$ can be reformulated as “ $W_{<\alpha} \setminus B_{<\alpha}$ is spanning in $N^* \upharpoonright W_{<\alpha}$ ”, which can be proved the same way as above. $\square$

3.2. Some more recent results and basic facts.

Theorem 3.6 (Aigner-Horev, Carmesin and Frölich; Theorem 1.5 in [6]). *If $M \in \mathfrak{F}(E)$ and $N \in \mathfrak{F}^*(E)$, then there is an $I \in \mathcal{I}_M \cap \mathcal{I}_N$ and a partition $E = E_M \sqcup E_N$ such that $I_M := I \cap E_M$ spans E_M in M and $I_N := I \cap E_N$ spans E_N in N .*

Corollary 3.7. *If $M \in \mathfrak{F}(E)$ and $N \in \mathfrak{F}^*(E)$ satisfy $\text{cond}^+(M, N)$, then there is an M -independent N -base.*

Proof. Let E_M, E_N, I_M and I_N be as in Theorem 3.6. Then E_M is a wave witnessed by I_M thus by $\text{cond}^+(M, N)$ we know that E_M consists of M -loops and $r(N.E_M) = 0$. But then $I_M = \emptyset$ and I_N is a base of N which is independent in M . $\square$

Observation 3.8. *If $\text{cond}(M, N)$ holds and L is a set of the M -loops, then $r(N.L) = 0$ which means $L \subseteq \text{span}_N(E \setminus L)$.*

Corollary 3.9. *If I is feasible, then $r(N.(\text{span}_M(I) \setminus I)) = 0$.*

Observation 3.10. *If W_0 is an (M, N) -wave and W_1 is an $(M/W_0, N - W_0)$ -wave, then $W_0 \cup W_1$ is an (M, N) -wave.*

Corollary 3.11. *For $W := W(M, N)$, the largest $(M/W, N - W)$ -wave is $\emptyset$. In particular, $\text{cond}^+(M/W, N - W)$ holds.*

Observation 3.12. *$\text{cond}^+(M, N)$ implies $\text{cond}^+(M \upharpoonright X, N.X)$ for every $X \subseteq E$.*

Proposition 3.13. *If $I_0 \in \mathcal{I}_N \cap \mathcal{I}_M$ and I_1 is feasible with respect to $(M/I_0, N/I_0)$, then $I_0 \cup I_1$ is feasible with respect to (M, N) . If in addition I_1 is a nice feasible in regards to $(M/I_0, N/I_0)$, then so is $I_0 \cup I_1$ to (M, N) .*

Proof. By definition the feasibility of I_1 w.r.t. $(M/I_0, N/I_0)$ means that the condition $\text{cond}(M/(I_0 \cup I_1), N/(I_0 \cup I_1))$ holds. The feasibility of $I_0 \cup I_1$ w.r.t. (M, N) means the same also by definition. For ‘nice feasible’ the argument is similar, only cond must be replaced by cond^+ . $\square$

The following lemma was introduced in [21].

Lemma 3.14. *Condition $\text{cond}^+(M, N)$ implies that whenever W is an $(M/e, N/e)$ -wave for some $e \in E$ witnessed by $B \subseteq W$, then $B \in B(M/e, N/e, W)$, i.e. B is spanning in $N.W$.*

Proof. Let W be an $(M/e, N/e)$ -wave. Note that $(N/e).W = N.W$ by definition. Pick a $B \subseteq W$ which is an $N.W$ -independent base of $(M/e) \upharpoonright W$. We may assume that $e \in \text{span}_M(W)$ and e is not an M -loop. Indeed, otherwise $(M/e) \upharpoonright W = M \upharpoonright W$ holds and hence W is also an (M, N) -wave. Thus by $\text{cond}^+(M, N)$ we may conclude that W consists of M -loops and hence $B = \emptyset$, moreover, $r(N.W) = 0$ and therefore $\emptyset$ is a base of $N.W$.

Then B is not a base of $M \upharpoonright W$ but “almost”, namely $r(M/B \upharpoonright W) = 1$. We apply the augmenting path Lemma 3.2 by Edmonds with B in regards to $M \upharpoonright W$ and $N.W$. An augmenting path P cannot exist. Indeed, if P were an augmenting path then $B \Delta P$ would show that W is an (M, N) -wave which does not consist of M -loops, contradiction. Thus we get a partition $W = W_0 \sqcup W_1$ instead where W_0 is an $(M \upharpoonright W, N.W)$ -wave witnessed by $B \cap W_0$, and therefore also an (M, N) -wave, and W_1 is an $(N.W, M \upharpoonright W)$ -wave showed by $B \cap W_1$. Then W_0 must consist of M -loops by $\text{cond}^+(M, N)$ and therefore $B \subseteq W_1$ by the M -independence of B . We need to show that B is spanning not just in $N.W_1$ but also in $N.W$. To do so let B' be a base of $N - W$. Then $B \cup B'$ spans $E \setminus W_0$ in N because $B = B \cap W_1$ is a base $N.W_1$. But then $B \cup B'$ is spanning in N because $r(N.W_0) = 0$ by Observation 3.8. We conclude that B is spanning in $N.W$ as desired. $\square$

3.3. Technical lemmas in regards to $B(M, N, W)$.

Proposition 3.15. *For $W := W(M, N)$, the elements of $B(M, N, W)$ are nice feasible sets.*

Proof. Let $B \in B(M, N, W)$. Clearly $B \in \mathcal{I}_M \cap \mathcal{I}_N$ because $B \in \mathcal{I}_M \cap \mathcal{I}_{N.W}$ by definition and $\mathcal{I}_{N.W} \subseteq \mathcal{I}_N$. We know that $W \setminus B$ is an $(M/B, N/B)$ -wave consisting of M/B loops and $r(N.(W \setminus B)) = 0$ because B is a base of $N.W$. In order to show $\text{cond}^+(M/B, N/B)$, it is enough to prove that $W \setminus B$ is actually $W(M/B, N/B) =: W'$. Suppose for a contradiction that $W' \supsetneq W \setminus B$. Let B' be an $N.W'$ -independent base of $M/B \upharpoonright W'$. Note that $B \cup B'$ is a base of $M \upharpoonright (W \cup W')$ and $B' \cap (W \setminus B) = \emptyset$. Since B is N^* -spanned by $W \setminus B \subseteq (W \cup W') \setminus (B \cup B')$ and B' is N^* -spanned by $W' \setminus B' \subseteq (W \cup W') \setminus (B \cup B')$, we may conclude that $B \cup B'$ is independent in $N.(W \cup W')$. Thus $W \cup W'$ is an (M, N) -wave witnessed by $B \cup B'$ which contradicts the maximality of W . $\square$

Corollary 3.16. *Assume that $I \in \mathcal{I}_M \cap \mathcal{I}_N$ and $B \in B(M/I, N/I, W)$ where $W := W(M/I, N/I)$. Then $I \cup B$ is a nice feasible set.*

Proof. Combine Propositions 3.13 and 3.15. $\square$

Observation 3.17. *If $\text{cond}^+(M, N)$ holds and L is a set of M -loops, then $\text{cond}^+(M - L, N - L)$ also holds.*

Observation 3.18. *Let W be a wave and let L be a set of common loops of M and N . Then $W \setminus L$ is also a wave and $B(M, N, W) = B(M, N, W \setminus L)$.*

Lemma 3.19. *Let W be a wave and $L \subseteq W$ such that L consists of M -loops with $r(N.L) = 0$. Then $W \setminus L$ is an $(M - L, N - L, W \setminus L)$ -wave with*

$$B(M, N, W) = B(M - L, N - L, W \setminus L).$$

Proof. A set B is a base of $M \upharpoonright W$ iff it is a base of $M \upharpoonright (W \setminus L)$ because L consists of M -loops. A $B \subseteq W \setminus L$ is $N.W$ -independent iff $B \subseteq \text{span}_{N^*}(W \setminus B)$. This holds if and only if $B \subseteq \text{span}_{N^*/L}(W \setminus (B \cup L))$ i.e. B is independent in $(N - L).(W \setminus L)$. Note that $r(N.L) = 0$ is equivalent with the N^* -independence of L . Thus for $B \subseteq W \setminus L$, $W \setminus B$ is N^* -independent iff $W \setminus (B \cup L)$ is N^*/L -independent. It means that B is spanning in $N.W$ iff it is spanning in $(N - L).(W \setminus L)$.

Thus the sets that are witnessing that $W \setminus L$ is an $(M - L, N - L, W \setminus L)$ -wave are exactly those that are witnessing that W is an (M, N) -wave, moreover, $B(M, N, W) = B(M - L, N - L, W \setminus L)$ holds. $\square$

Lemma 3.20. *Assume that $X_j, Y_j \subseteq E$ for $j \in \{0, 1\}$ where $X_j \sqcup Y_j = Z$ for $j \in \{0, 1\}$, furthermore $\text{span}_M(X_0) = \text{span}_M(X_1)$ and $\text{span}_{N^*}(Y_0) = \text{span}_{N^*}(Y_1)$. Then for every $X \subseteq E \setminus Z$ we have*

$$B(M/X_0 - Y_0, N/X_0 - Y_0, X) = B(M/X_1 - Y_1, N/X_1 - Y_1, X).$$

Proof. The matroids $M/X_0 - Y_0$ and $M/X_1 - Y_1$ are the same as well as the matroids $N/X_0 - Y_0$ and $N/X_1 - Y_1$. $\square$

4. REDUCTIONS

We repeat here our main results for convenience:

Theorem 1.2. *For every family $(M_i : i \in \Theta)$ of matroids on a common countable edge set E where $M_i \in \mathfrak{F} \oplus \mathfrak{F}^*$, there is a partition $E = E_p \sqcup E_c$ such that $(M_i \upharpoonright E_p : i \in \Theta)$ admits a packing and $(M_i.E_c : i \in \Theta)$ admits a covering.*

Theorem 1.4. *If M and N are matroids in $\mathfrak{F} \oplus \mathfrak{F}^*$ on the same countable edge set E , then they admit a common independent set I for which there is a partition $E = E_M \sqcup E_N$ such that $I_M := I \cap E_M$ spans E_M in M and $I_N := I \cap E_N$ spans E_N in N .*

4.1. Matroid Intersection with a finitary M . As we mentioned, the method by Bowler and Carmesin used to prove Corollary 3.9 in [8] works not only for the class of all matroids but can be adapted for every class closed under certain operations. We apply their technique to obtain the following reduction:

Lemma 4.1. *Theorems 1.2 and 1.4 are implied by the special case of Theorem 1.4 where $M \in \mathfrak{F}$.*

Proof. First we show that one can assume without loss of generality in the proof of Theorem 1.2 that Θ is countable. To do so let

$$E' := \{e \in E : |\{i \in \Theta : \{e\} \in \mathcal{I}_{M_i}\}| \leq \aleph_0\}$$

and

$$\Theta' := \{i \in \Theta : (\exists e \in E')(\{e\} \in \mathcal{I}_{M_i})\}.$$

We apply Theorem 1.2 with E' and with the countable family $(M_i \upharpoonright E' : i \in \Theta')$. Then we obtain a partition $E' = E'_p \sqcup E'_c$ such that $(M_i \upharpoonright E'_p : i \in \Theta')$ admits a packing $(S_i : i \in \Theta')$ and $(M_i \upharpoonright E'.E_c : i \in \Theta')$ admits a covering $(I_i : i \in \Theta')$. Let $E_p := E'_p$ and $E_c := E \setminus E_p = E'_c \cup (E \setminus E')$. By construction $r_{M_i}(E') = 0$ for $i \in \Theta \setminus \Theta'$. Thus by letting $S_i := \emptyset$ for $i \in \Theta \setminus \Theta'$ the family $(S_i : i \in \Theta)$ is a packing w.r.t. $(M_i \upharpoonright E_p : i \in \Theta)$. Let $g : E \setminus E' \rightarrow \Theta \setminus \Theta'$ be injective. For $i \in \Theta \setminus \Theta'$, we take $I_i := \{g^{-1}(i)\}$ if $i \in \text{ran}(g)$ and $I_i := \emptyset$ otherwise. Then $(I_i : i \in \Theta)$ is a covering for $(M_i.E_c : i \in \Theta)$ and we are done.

We proceed with the proof of Theorem 1.2 assuming that Θ is countable. For $i \in \Theta$, let M'_i be the matroid on $E \times \{i\}$ that we obtain by “copying” M_i via the bijection $e \mapsto (e, i)$. Then for

$$M := \bigoplus_{i \in \Theta} M'_i, \quad \text{and} \quad N := \bigoplus_{e \in E} U_{\{e\} \times \Theta, 1}$$

we have $M \in (\mathfrak{F} \oplus \mathfrak{F}^*)(E \times \Theta)$ and $N \in \mathfrak{F}(E \times \Theta)$ where $E \times \Theta$ is countable. Thus by assumption there is a partition $E \times \Theta = E_M \sqcup E_N$ and an $I \in \mathcal{I}_M \cap \mathcal{I}_N$ such that $I_M := I \cap E_M$ spans E_M in M and $I_N := I \cap E_N$ spans E_N in N . The M -independence of I ensures that $J_i := \{e \in E : (e, i) \in I\}$ is M_i -independent. The N -independence of I guarantees that the sets J_i are pairwise disjoint. Let $E_c := \{e \in E : (\exists i \in \Theta)(e, i) \in E_N\}$. Then for each $e \in E_c$ there must be some $i \in \Theta$ with $(e, i) \in I_N$ because $E_N \subseteq \text{span}_N(I_N)$. Thus the sets J_i cover E_c and so do the sets $I_i := J_i \cap E_c$. It is enough to show that $S_i := J_i \setminus I_i$ spans $E_p := E \setminus E_c$ in M_i for every $i \in \Theta$. Let $f \in E_p$ and $i \in \Theta$ be given.

Then $\{f\} \times \Theta \subseteq E_M$ follows directly from the definition of E_p , in particular $(f, i) \in E_M$. We know that $(f, i) \in \text{span}_M(I_M)$ and hence $f \in \text{span}_{M_i}(\{e \in E : (e, i) \in I_M\})$. Suppose for a contradiction that $h \in E_c \cap \{e \in E : (e, i) \in I_M\}$. Then for some $j \in \Theta$ we have $(h, j) \in E_N$. Since $(h, j) \in \text{span}_N(I_N)$, we have $(h, k) \in I_N$ for some $k \in \Theta$. But then $(h, i), (h, k) \in I$ are distinct elements thus $i \neq k$ which contradict the N -independence of I . Therefore $E_c \cap \{e \in E : (e, i) \in I_M\} = \emptyset$. Since $\{e \in E : (e, i) \in I_M\} \subseteq J_i$ by the definition of J_i we conclude $\{e \in E : (e, i) \in I_M\} = S_i$. Therefore $(S_i : i \in \Theta)$ is a packing for $(M_i \upharpoonright E_p : i \in \Theta)$ and $(I_i : i \in \Theta)$ is a covering for $(M_i.E_c : i \in \Theta)$ as desired.

Now we derive Theorem 1.4 from Theorem 1.2. To do so, we take a partition $E = E_p \sqcup E_c$ such that (S_M, S_N) is a packing for $(M \upharpoonright E_p, N^* \upharpoonright E_p)$ and (R_M, R_N) is a covering for $(M.E_c, N^*.E_c)$. Let $I_M \subseteq S_M$ be a base of $M \upharpoonright E_p$ and we define $I_N := R_M$. By construction $E_p \subseteq \text{span}_M(I_M)$ and $I_N \in \mathcal{I}_{M.E_c}$. We also know that

$$I_M \subseteq \text{span}_{N^*}(S_N) \subseteq \text{span}_{N^*}(E_p \setminus I_M)$$

which means $I_M \in \mathcal{I}_{N.E_p}$. Finally, $R_N \in \mathcal{I}_{N^*.E_c}$ means that $E_c \setminus R_N$ spans E_c in N and therefore so does $I_N = R_M \supseteq E_c \setminus R_N$. $\square$

4.2. Finding an M -independent base of N . The following reformulation of the matroid intersection problem was introduced by Aharoni and Ziv in [5] but its analogue by Aharoni was already an important tool to attack (and eventually solve in [2]) the Erdős-Menger Conjecture.

Claim 4.2. *Assume that $M \in \mathfrak{F}(E)$ and $N \in (\mathfrak{F} \oplus \mathfrak{F}^*)(E)$ such that E is countable and $\text{cond}^+(M, N)$ holds. Then there is an M -independent base of N .*

Lemma 4.3. *Claim 4.2 implies our main results Theorems 1.2 and 1.4.*

Proof. By Lemma 4.1 it is enough to show that the special case of Theorem 1.4 where $M \in \mathfrak{F}$ follows from Claim 4.2. To do so, let $E_M := W(M, N)$ and let $I_M \subseteq E_M$ be a witness that E_M is a wave. For $E_N := E \setminus E_M$, we have $M/E_M \in \mathfrak{F}(E_N)$ and $N - E_M \in (\mathfrak{F} \oplus \mathfrak{F}^*)(E_N)$, furthermore, $\text{cond}^+(M/W, N - W)$ holds (see Corollary 3.11). By Claim 4.2, there is an M/E_M -independent base I_N of $N - E_M$. Then $I \in \mathcal{I}_M \cap \mathcal{I}_N$ and $E = E_M \sqcup E_N$ such that $I_M := I \cap E_M$ spans E_M in M and $I_N := I \cap E_N$ spans E_N in N as desired. $\square$

4.3. Reduction to a key-lemma. From now on we assume that $M \in \mathfrak{F}(E)$ and $N \in (\mathfrak{F} \oplus \mathfrak{F}^*)(E)$ where E is countable. Let $\mathbf{E}_0$ be the union of the finitary components of N and let $\mathbf{E}_1 := E \setminus E_0$. Note that $N \upharpoonright E_0$ is finitary, $N \upharpoonright E_1$ is cofinitary and no N -circuit meets both E_0 and E_1 .

Lemma 4.4 (key-lemma). *If $\text{cond}^+(M, N)$ holds, then for every $e \in E_0$ there is a nice feasible I with $e \in \text{span}_N(I)$.*

Lemma 4.5. *Lemma 4.4 implies our main results Theorems 1.2 and 1.4.*

Proof. It is enough to show that Lemma 4.4 implies Claim 4.2 because of Lemma 4.3. Let us fix an enumeration $\{e_n : n \in \mathbb{N}\}$ of E_0 . We build an $\subseteq$ -increasing sequence (I_n) of nice feasible sets starting with $I_0 := \emptyset$ (who is nice feasible by $\text{cond}^+(M, N)$) in such a

way that $e_n \in \text{span}_N(I_{n+1})$. Suppose that I_n is already defined. If $e_n \notin \text{span}_N(I_n)$, then we apply Lemma 4.4 with $(M/I_n, N/I_n)$ and e_n and take the union of the resulting I with I_n to obtain I_{n+1} (see Observation 3.13), otherwise let $I_{n+1} := I_n$. The recursion is done. Now we construct an M -independent $I_n^+ \supseteq I_n$ with $E_1 \subseteq \text{span}_N(I_n^+)$ for $n \in \mathbb{N}$. The matroid $M/I_n \upharpoonright (E_1 \setminus I_n)$ is finitary and $N.(E_1 \setminus I_n) = (N \upharpoonright E_1)/(I_n \cap E_1)$ is cofinitary, moreover, by Observation 3.12

$$\text{cond}^+(M/I_n, N/I_n) \implies \text{cond}^+(M/I_n \upharpoonright (E_1 \setminus I_n), N.(E_1 \setminus I_n)).$$

Thus by Corollary 3.7 there is an M/I_n -independent base B_n of $(N \upharpoonright E_1)/(E_1 \cap I_n)$ and $I_n^+ := I_n \cup B_n$ is as desired.

Let $\mathcal{U}$ be a free ultrafilter on $\mathcal{P}(\mathbb{N})$ and we define

$$S := \{e \in E : \{n \in \mathbb{N} : e \in I_n^+\} \in \mathcal{U}\}.$$

Then S is M -independent and N -spanning and therefore we are done. Indeed, suppose for a contradiction that S contains an M -circuit C . For $e \in C$, we pick a $U_e \in \mathcal{U}$ with $e \in I_n^+$ for $n \in U_e$. Since M is finitary, C is finite, thus $U := \cap\{U_e : e \in C\} \in \mathcal{U}$. But then for $n \in U$ we have $C \subseteq I_n^+$ which contradicts the M -independence of I_n^+ . Clearly, $I_{n+1} \subseteq S$ for every $n \in \mathbb{N}$ and therefore $E_0 \subseteq \text{span}_N(S)$. Finally, suppose for a contradiction that there is some $N^* \upharpoonright E_1$ -circuit C' with $S \cap C' = \emptyset$. For $e \in C'$, we can pick a $U'_e \in \mathcal{U}$ with $e \notin I_n^+$ for $n \in U'_e$. Since $N^* \upharpoonright E_1$ is finitary, C' is finite, thus $U' := \cap\{U'_e : e \in C'\} \in \mathcal{U}$. But then for $n \in U'$ we have $I_n^+ \cap C' = \emptyset$ which contradicts $E_1 \subseteq \text{span}_N(I_n^+)$. $\square$

5. MIXED AUGMENTING PATHS

In the section we introduce an ‘augmenting path’ type of method and analyse it in order to show some properties we need later. On E_0 the definition will be the same as in the proof of the Matroid Intersection Theorem by Edmonds [12] but on E_1 we need to define it in a different way considering that $N \upharpoonright E_1$ is cofinitary. For brevity we write $\mathring{\text{span}}_M(F)$ for $\text{span}_M(F) \setminus F$ and F^j for $F \cap E_j$ where $F \subseteq E$ and $j \in \{0, 1\}$. We call an $F \subseteq E$ *dually safe* if F^1 is spanned by $\mathring{\text{span}}_M(F)$ in N^* .

Lemma 5.1. *If $I \in \mathcal{I}_M \cap \mathcal{I}_n$ is dually safe and $B \in B(M/I, N/I, W)$ for $W := W(M/I, N/I)$, then $I \cup B$ is a nice dually safe feasible set.*

Proof. We already know by Corollary 3.16 that $I \cup B$ is a nice feasible set. By using that I is dually safe and $\mathring{\text{span}}_M(I) \subseteq \mathring{\text{span}}_M(I \cup B)$ we get

$$I^1 \subseteq \text{span}_{N^*}(\mathring{\text{span}}_M(I)) \subseteq \text{span}_{N^*}(\mathring{\text{span}}_M(I \cup B)).$$

Since $B \in B(M/I, N/I, W)$ we have $W \setminus B \subseteq \mathring{\text{span}}_M(I \cup B)$ and B is a base of $N.W$. The latter can be rephrased as ‘ $W \setminus B$ is a base of $N^* \upharpoonright W$ ’. By combining these $\mathring{\text{span}}_M(I \cup B)$ spans B in N^* . Therefore $(I \cup B) \cap E_1 \subseteq \text{span}_{N^*}(\mathring{\text{span}}_M(I \cup B))$, which means that $I \cup B$ is dually safe. $\square$

Proposition 5.2. *For a dually safe feasible I , $\mathring{\text{span}}_M(I)^1$ is a base of $N^* \upharpoonright \text{span}_M(I)^1$.*

Proof. By the definition of ‘dually safe’, $\mathring{\text{span}}_M(I)^1$ spans $N^* \upharpoonright \text{span}_M(I)^1$. Furthermore, $r(N.\mathring{\text{span}}_M(I)) = 0$ by Corollary 3.9, which is equivalent with the N^* -independence of $\mathring{\text{span}}_M(I)$. $\square$

For a dually safe feasible I , we define an auxiliary digraph $D(I)$ on E . Let xy be an arc of $D(I)$ iff one of the following possibilities occurs:

- (1) $x \in E \setminus I$ and $I + x$ is M -dependent with $y \in C_M(x, I) - x$,
- (2) $x \in I^0$ and $C_N(y, I)$ is well-defined and contains x ,
- (3) $x \in I^1$ and $y \in C_{N^*}(x, \mathring{\text{span}}_M(I)^1) - x$ (see Proposition 5.2).

An augmenting path for a nice dually safe feasible I is a $P = \{x_1, \dots, x_{2n+1}\}$ where

- (i) $x_1 \in E_0 \setminus \text{span}_N(I)$,
- (ii) $x_{2n+1} \in E_0 \setminus \text{span}_M(I)$,
- (iii) $x_k x_{k+1} \in D(I)$ for $1 \leq k \leq 2n$,
- (iv) $x_k x_\ell \notin D(I)$ if $k + 1 < \ell$.

Proposition 5.3. *If I is a dually safe feasible set and $P = \{x_1, \dots, x_{2n+1}\}$ is an augmenting path for I , then $I \triangle P$ is a dually safe element of $\mathcal{I}_M \cap \mathcal{I}_N$ with*

- (A) $\text{span}_M(I \triangle P) = \text{span}_M(I + x_{2n+1})$,
- (B) $\text{span}_N(I \triangle P) \cap E_0 = \text{span}_N(I + x_1) \cap E_0$,
- (C) $\text{span}_{N^*}(\mathring{\text{span}}_M(I)^1) = \text{span}_{N^*}(\mathring{\text{span}}_M(I)^1 \triangle P^1)$ and $\mathring{\text{span}}_M(I)^1 \triangle P^1 \in \mathcal{I}_{N^*}$.

Proof. The set $I + x_{2n+1}$ is M -independent by the definition of P . Property (iv) ensures that we can apply Proposition 3.1 with $I + x_{2n+1}$, $e_j = x_{2j-1}$, $f_j = x_{2j}$ ($1 \leq j \leq n$) and M and conclude that $I \triangle P \in \mathcal{I}_M$ and (A) holds.

To prove $(I \triangle P) \cap E_0 \in \mathcal{I}_N$ and (B) we proceed similarly. We start with the N -independent set $(I + x_1) \cap E_0$. In order to satisfy the premisses of Proposition 3.1 via property (iv), we need to enumerate the relevant edge pairs backwards. Namely, for $j \leq |I^0 \cap P|$ let $e_j := x_{i_j+1}$ where i_j is the j th largest index with $x_{i_j} \in I^0$ and $f_j := x_{i_j}$. We conclude that $(I \triangle P) \cap E_0 \in \mathcal{I}_N$ and (B) holds.

Finally, we let $e_j := x_{i_j}$ for $j \leq |I^1 \cap P|$ where i_j is the j th smallest index with $x_{i_j} \in I^1$ and $f_j := x_{i_j+1}$. Recall that $\mathring{\text{span}}_M(I)^1$ is N^* -independent (see Proposition 5.2). We apply Proposition 3.1 with $\mathring{\text{span}}_M(I)^1$, e_j , f_j and N^* to conclude (C). This means that $\mathring{\text{span}}_M(I)^1 \triangle P^1$ is a base of $N^* \upharpoonright \text{span}_M(I)^1$ because $\mathring{\text{span}}_M(I)^1$ was a base of it by Proposition 5.2. By (A) and by the definition of P we know that

$$(I \triangle P) \cap E_1 \subseteq I^1 \cup P^1 \subseteq \text{span}_M^1(I).$$

By combining these we obtain

$$(I \triangle P) \cap E_1 \subseteq \text{span}_{N^*}(\mathring{\text{span}}_M(I)^1 \triangle P^1).$$

The set $I \triangle P$ is disjoint from $\mathring{\text{span}}_M(I) \triangle P$ because I is disjoint from $\mathring{\text{span}}_M(I)$, moreover, $\mathring{\text{span}}_M(I) \triangle P$ contained in $\text{span}_M(I \triangle P)$. Hence $\mathring{\text{span}}_M(I \triangle P)$ contains $\mathring{\text{span}}_M(I)^1 \triangle P^1$ and therefore N^* -spans $(I \triangle P) \cap E_1$, i.e. $I \triangle P$ is dually safe. It means that $(I \triangle P) \cap E_1$ is independent in $N.\text{span}_M(I \triangle P)^1$. Thus uniting $(I \triangle P) \cap E_0 \in \mathcal{I}_N$ with $(I \triangle P) \cap E_1$ preserves N -independence, i.e. $I \triangle P \in \mathcal{I}_N$. $\square$

Lemma 5.4. *If I is a nice dually safe feasible set and $P = \{x_1, \dots, x_{2n+1}\}$ is an augmenting path for I , then $I \triangle P$ can be extended to a nice dually safe feasible set.*

Proof. Let $M' := M/(I \triangle P)$ and $N' := N/(I \triangle P)$. By Lemma 5.1 it is enough to show that $B(M', N', W) \neq \emptyset$ for $W := W(M', N')$. Statements (A) and (B) of Proposition 5.3 ensure that the elements of $L := \{x_1, x_3, \dots, x_{2n-1}\} \cap E_0$ are common loops of M' and N' . Statement (C) tells that $Y_0 := \mathring{\text{span}}_M(I)^1$ and $Y_1 := \mathring{\text{span}}_M(I)^1 \triangle P^1$ have the same N^* -span, furthermore, Y_1 is N^* -independent, i.e. $r(N.Y_1) = 0$. Note that Y_1 consists of M' -loops by (A). Thus by applying Observation 3.18 with W and L and then Lemma 3.19 with $W \setminus L$ and Y_1 we can conclude that $W \setminus (L \cup Y_1)$ is an $(M' - Y_1, N' - Y_1)$ -wave, furthermore,

$$B(M', N', W) = B(M' - Y_1, N' - Y_1, W \setminus (L \cup Y_1)).$$

The sets $X_0 := I + x_{2n+1}$ and $X_1 := I \triangle P$ have the same M -span (see Proposition 5.3/(A)). Recall that $M' = M/X_1$ and $N' = N/X_1$ by definition. Hence Lemma 3.20 ensures that $W \setminus (Y_1 \cup L)$ is also an $(M/X_0 - Y_0, M/X_0 - Y_0)$ -wave with

$$B(M/X_1 - Y_1, N/X_1 - Y_1, W \setminus (Y_1 \cup L)) = B(M/X_0 - Y_0, M/X_0 - Y_0, W \setminus (Y_1 \cup L)).$$

We have $\text{cond}^+(M/I, N/I)$ because I is a nice feasible set by assumption. Then by Observation 3.17 $\text{cond}^+(M/I - Y_0, N/I - Y_0)$ also holds. Applying Lemma 3.14 with $M/I - Y_0$, $N/I - Y_0$, $W \setminus (Y_0 \cup L)$ and x_{2n+1} tells $B(M/X_0 - Y_0, M/X_0 - Y_0, W \setminus (Y_1 \cup L)) \neq \emptyset$ which completes the proof. $\square$

Lemma 5.5. *If $P = \{x_1, \dots, x_{2n+1}\}$ is an augmenting path for I which contains neither x nor any of its out-neighbours in $D(I)$, then $xy \in D(I)$ implies $xy \in D(I \triangle P)$.*

Proof. Suppose that $xy \in D(I)$. First we assume that $x \notin I$. Then the set of the out-neighbours of x is $C_M(x, I) - x$. By assumption $P \cap C_M(x, I) = \emptyset$ and therefore $C_M(x, I) \subseteq I \triangle P$ thus $C_M(x, I) = C_M(x, I \triangle P)$. This means by definition that x has the same out-neighbours in $D(I)$ and $D(I \triangle P)$.

If $x \in I^1$, then we can argue similarly. The set of the out-neighbours of x in $D(I)$ is $C_{N^*}(x, \mathring{\text{span}}_M(I)^1) - x$. By assumption $P \cap C_{N^*}(x, \mathring{\text{span}}_M(I)^1) = \emptyset$ and therefore $C_{N^*}(x, \mathring{\text{span}}_M(I)^1) \cap (I \triangle P) = \{x\}$. Since $\mathring{\text{span}}_M^1(I \triangle P) \supseteq \mathring{\text{span}}_M^1(I)$ because of Proposition 5.3/(A), we also have $C_{N^*}(x, \mathring{\text{span}}_M(I)^1) \subseteq \mathring{\text{span}}_M^1(I \triangle P)$. By combining these we conclude

$$C_{N^*}(x, \mathring{\text{span}}_M(I \triangle P)^1) = C_{N^*}(x, \mathring{\text{span}}_M(I)^1).$$

This means by definition that x has the same out-neighbours in $D(I)$ and $D(I \triangle P)$.

We turn to the case where $x \in I^0$. By definition $C_N(y, I)$ is well-defined and contains x , in particular $y \in E_0$. For $k \leq n$, let us denote $I + x_1 - x_2 + x_3 - \dots - x_{2k} + x_{2k+1}$ by I_k . Note that $I_n = I \triangle P$. We show by induction on k that I_k is N -independent and $x \in C_N(y, I_k)$. Since $I + x_1$ is N -independent by definition and $x_1 \neq y$ because $y \notin P$ by assumption, we obtain $C_N(y, I) = C_N(y, I_0)$, thus for $k = 0$ it holds. Suppose that $n > 0$ and we already know the statement for some $k < n$. We have $C_N(x_{2k+3}, I_k) = C_N(x_{2k+3}, I) \ni x_{2k+2}$ because there is no “jumping arc” in the augmenting path by property (iv). It follows via the N -independence of I_k that I_{k+1} is also N -independent. If $x_{2k+2} \notin C_N(y, I_k)$ then

$C_N(y, I_k) = C_N(y, I_{k+1})$ and the induction step is done. Suppose that $x_{2k+2} \in C_N(y, I_k)$. Then $x_{2k+2}, x_{2k+3} \in E_0$, moreover, $x \notin C_N(x_{2k+3}, I)$ since otherwise P would contain the out-neighbour x_{2k+3} of x in $D(I)$. We apply circuit elimination (Claim 3.3) with $C = C_N(y, I_k)$, $e = x$, $X = \{x_{2k+2}\}$, $C_{x_{2k+2}} = C_N(x_{2k+3}, I_k)$. The resulting circuit $C' \ni x$ can have at most one element out of I_{k+1} , namely y . Since I_{k+1} is N -independent, there must be at least one such an element and therefore $C' = C_N(y, I_{k+1})$. $\square$

Observation 5.6. *If $xy \in D(I)$ and $J \supseteq I$ is a dually safe feasible set with $\{x, y\} \cap J = \{x, y\} \cap I$, then $xy \in D(J)$ (the same circuit is the witness).*

6. PROOF OF THE KEY-LEMMA

Lemma 4.4 (key-lemma). *If $\text{cond}^+(M, N)$ holds, then for every $e \in E_0$ there is a nice feasible I with $e \in \text{span}_N(I)$.*

Proof. It is enough to build a sequence (I_n) of nice dually safe feasible sets such that $(\text{span}_N(I_n) \cap E_0)$ is an ascending sequence exhausting E_0 . We fix a well-order $\prec$ of type $|E_0|$ on E_0 . Let $I_0 = \emptyset$, which is a nice dually safe feasible set by $\text{cond}^+(M, N)$. Suppose that I_n is already defined. If there is no augmenting path for I_n , then we let $I_m := I_n$ for $m > n$. Otherwise we pick an augmenting path P_n for I_n in such a way that its first element is as $\prec$ -small as possible. Then we apply Lemma 5.3 to extend $I \triangle P$ to a nice dually safe feasible set which we define to be I_{n+1} . The recursion is done.

Let $\mathbf{X} := E \setminus \bigcup_{n \in \mathbb{N}} \text{span}_N(I_n)$ and for $x \in X$, let $\mathbf{E}(x, \mathbf{n})$ be the set of elements that are reachable from x in $D(I_n)$ by a directed path. We define $\mathbf{n}_x$ to be the smallest natural number such that for every $y \in E \setminus X$ with $y \prec x$ we have $y \in \text{span}_N(I_{n_x})$. We shall prove that

$$\mathbf{W} := \bigcup_{x \in X} \bigcup_{n \geq n_x} E(x, n)$$

is a wave.

Lemma 6.1. *For every $x \in X$ and $\ell \geq m \geq n_x$,*

- (1) $I_m \cap E(x, m) = I_\ell \cap E(x, m)$,
- (2) $C_M(y, I_\ell) = C_M(y, I_m) \subseteq E(x, m)$ for every $y \in E(x, m) \setminus I_m$,
- (3) $C_{N^*}(y, \text{span}_M(I_\ell)^1) = C_{N^*}(y, \text{span}_M(I_m)^1)$ for every $y \in E(x, m) \cap I_m^1$,
- (4) If $yz \in D(I_m)$ with $y, z \in E(x, m)$, then $yz \in D(I_\ell)$,
- (5) $E(x, m) \subseteq E(x, \ell)$.

Proof. Suppose that there is an $n \geq n_x$ such that we know already the statement whenever $m, \ell \leq n$. For the induction step it is enough to show that the claim holds for n and $n+1$. We may assume that P_n exists, i.e. $I_n \neq I_{n+1}$, since otherwise there is nothing to prove.

Proposition 6.2. $P_n \cap E(x, n) = \emptyset$.

Proof. A common element of P_n and $E(x, n)$ would show that there is also an augmenting path in $D(I_n)$ starting at x which is impossible since $x \in X$ and $n \geq n_x$. $\square$

Corollary 6.3. $I_n \cap E(x, n) = (I_n \triangle P_n) \cap E(x, n)$.

Proposition 6.4. $(I_n \triangle P_n) \cap E(x, n) = I_{n+1} \cap E(x, n)$.

Proof. If $y \in E(x, n) \setminus I_n$, then its out-neighbours in $D(I_n)$ are in $E(x, n) \cap I_n$ and span y in M . Thus $I_{n+1} \setminus (I_n \triangle P_n)$ cannot contain any edge from $E(x, n)$. $\square$

Corollary 6.5. $I_n \cap E(x, n) = I_{n+1} \cap E(x, n)$ and for every $y \in E(x, n) \setminus I_n$ we have $C_M(y, I_n) = C_M(y, I_{n+1}) \subseteq E(x, n)$.

Corollary 6.6. For $y \in E(x, n) \setminus I_n$, y has the same out-neighbours in $D(I_n)$ and in $D(I_{n+1})$ and they span y in N^* . More concretely:

$$C_{N^*}(y, \mathring{\text{span}}_M(I_{n+1})^1) = C_{N^*}(y, \mathring{\text{span}}_M(I_n)^1).$$

Finally, for $y \in E(x, n)$, P_n does not contain y or any of its out-neighbours with respect to $D(I_n)$ because $P_n \cap E(x, n) = \emptyset$. Hence by applying Lemma 5.5 with P_n, y and I_n (and then Observation 5.6) we may conclude that $yz \in D(I_{n+1})$ whenever $yz \in D(I_n)$. This implies $E(x, n) \subseteq E(x, n+1)$ since reachability from x is witnessed by the same directed paths. $\square$

Let

$$B := \bigcup_{m \in \mathbb{N}} \bigcap_{n > m} W \cap I_n.$$

We are going to show that B witnesses that W is a wave. Since M is finitary the M -independence of the sets $I_n \cap W$ implies the M -independence of B . Similarly B^0 is independent in N because $N \upharpoonright E_0$ is finitary. Statements (1) and (2) of Claim 6.1 ensure $W \subseteq \text{span}_M(B)$, while (1) and (3) guarantee $B^1 \subseteq \text{span}_{N^*}(W \setminus B)$. The latter means that B^1 is independent in $N.(W^1 \setminus B^1)$. Suppose for a contradiction that B^0 is not independent in $N.W^0$. Then there exists an N -circuit $C \subseteq E_0$ that meets B but avoids $W \setminus B$. We already know that B^0 is N -independent thus C is not contained in B . Hence $C \setminus B = C \setminus W \neq \emptyset$. Let us pick some $e \in C \cap B$. Since C is finite, for every large enough n we have $C \cap B \subseteq C \cap I_n$ and I_n spans C in N (for the latter we use $X \subseteq W \setminus B$). Applying Corollary 3.4 with I_n, N, C and e tells that $e \in C_N(f, I_n)$ for some $f \in C \setminus W$ whenever n is large enough. Then by (5) of Claim 6.1 we can take an $x \in X$ and an $n \geq n_x$ such that $e \in E(x, n) \cap C_N(f, I_n)$ for some $f \in C \setminus W$. Then by definition $f \in E(x, n) \subseteq W$ which contradicts $f \in C \setminus W$. Thus B^0 is indeed independent in $N.W^0$ and hence B in $N.W$ as well therefore W is a wave.

By $\text{cond}^+(M, N)$ we know that W consists of M -loops and $r(N.W) = 0$. It implies $r(N.X) = 0$ because $X \subseteq W$ by definition. This means $X \subseteq \text{span}_N(E \setminus X)$. Since $X \subseteq E_0$ and E_0 is the union of the finitary N -components, $X \subseteq \text{span}_N(E_0 \setminus X)$ follows. Thus for every $x \in X$ there is a finite N -circuit $C \subseteq E_0$ with $C \cap X = \{x\}$. The sequence $(\text{span}_N(I_n) \cap E_0)$ is ascending by construction and exhausts $E_0 \setminus X$ by the definition of X . As $C - x \subseteq E_0 \setminus X$ is finite, this implies that for every large enough n , I_n spans $C - x$ in N and hence spans x itself as well. But then by the definition of X , we must have $X = \emptyset$. Therefore $(\text{span}_N(I_n) \cap E_0)$ exhausts E_0 and the proof of Lemma 4.4 is complete. $\square$

7. AN APPLICATION: DEGREE-CONSTRAINED ORIENTATIONS OF INFINITE GRAPHS

Matroid intersection is a powerful tool in graph theory and in combinatorial optimization. Our generalization Theorem 1.4 extends the scope of its applicability to infinite graphs.

To illustrate this, let us consider a classical problem in combinatorial optimization. A graph is given with degree-constraints and we are looking for either an orientation that satisfies it or a certain substructure witnessing the non-existence of such an orientation (see [17]).

Let a (possibly infinite) graph $G = (V, E)$ be fixed through this section. We denote the set of edges incident with v by $\delta(v)$. Let $o : V \rightarrow \mathbb{Z}$ with $|o(v)| \leq d(v)$ for $v \in V$ which we will treat as ‘lower bounds’ for in-degrees in orientations in the following sense. We say that the orientation D of G is *above* o at v if either $o(v) \geq 0$ and v has at least $o(v)$ ingoing edges in D or $o(v) < 0$ and all but at most $-o(v)$ edges in $\delta(v)$ are oriented towards v by D . We say *strictly above* if we forbid equality in the definition. Orientation D is *above* o if it is above o at every $v \in V$. We say that D is (strictly) *bellow* o at v if the reverse of D is (strictly) above $-o(v)$. Finally, D is (strictly) *bellow* o if the reverse of D is strictly above $-o$.

Theorem 7.1. *Let $G = (V, E)$ be a countable graph and let $o : V \rightarrow \mathbb{Z}$. If there is no orientation of G above o , then there is a $V' \subseteq V$ and an orientation D of G such that*

- *D is bellow o at every $v \in V'$;*
- *There exists a $v \in V'$ such that D is strictly bellow o at v ;*
- *Every edge between V' and $V \setminus V'$ is oriented by D towards V' .*

Proof. Without loss of generality we may assume that G is loopless. We define the digraph $\vec{G} = (V, A)$ by replacing each $e \in E$ by back and forth arcs a_e, a'_e between the end-vertices of e . Let $\delta^+(v)$ be the set of the ingoing edges of v in $\vec{G}$. For $v \in V$, let M_v be $U_{\delta^+(v), o(v)}$ if $o(v) \geq 0$ and $U_{\delta^+(v), -o(v)}^*$ if $o(v) < 0$. We define N_e to be $U_{\{a_e, a'_e\}, 1}$ for $e \in E$. Let

$$M := \bigoplus_{v \in V} M_v \text{ and } N := \bigoplus_{e \in E} N_e.$$

Since $M, N \in (\mathfrak{F} \oplus \mathfrak{F}^*)(A)$, Theorem 1.4 guarantees that there exists an $I \in \mathcal{I}_M \cap \mathcal{I}_N$ and a partition $A = A_M \sqcup A_N$ such that $I_M := I \cap A_M$ spans A_M in M and $I_N := I \cap A_N$ spans A_N in N . Note that $|I \cap \{a_e, a'_e\}| \leq 1$ by the N -independence of I . We define D by taking the orientation a_e of e if $a_e \in I$ and a'_e otherwise. Let V'' consists of those vertices v for which I_M contains a base of M_v and let $V' := V \setminus V''$. We claim that whenever an edge $e \in E$ is incident with some $v \in V'$, then I contains one of a_e and a'_e . Indeed, if I contains none of them then they cannot be N -spanned by I_N thus they are M -spanned by M which implies that both end-vertices of e belong to V'' , contradiction. Thus if e is incident some $v \in V'$, then all ingoing arcs of v in D must be in I . Then the M -independence of I ensures that D is bellow o at v .

Suppose for a contradiction that a_e is an arc in D from a $v \in V'$ to a $w \in V''$. As we have already shown, we must have $a_e \in I$. By $w \in V''$ we know that I_M contains a base of M_w thus $a_e \notin I_N$ by the M -independence of I and therefore $a_e \in I_M$. But then a'_e cannot be spanned by I_N in N hence $a'_e \in \text{span}_M(I_M)$, which means that I_M contains a base of M_v contradicting $v \in V'$. We conclude that all the edges between V'' and V' are oriented towards V' in D . By the definition of V'' , D is above o at every $w \in V''$. If D is also above o for every $v \in V'$, then D is above o . Otherwise there exists a $v \in V'$ such that D is strictly bellow o at v , but then V' is as desired. $\square$

Easy calculation shows that if G is finite, then the existence of a V' described in Theorem 7.1 implies the non-existence of an orientation above o . Indeed, the total demand by o on V' is more than the number of all the edges that are incident with a vertex in V' . That is why for finite G , “if” can be replaced by “if and only if” in Theorem 7.1. For an infinite G it is not always the case. Indeed, let G be the one-way infinite path $v_0, v_1, \dots$ and let $o(v_n) = 1$ for $n \in \mathbb{N}$. Then orienting edge $\{v_n, v_{n+1}\}$ towards v_{n+1} for each $n \in \mathbb{N}$ and taking $V' := V$ satisfies the three points in Theorem 7.1. However, taking the opposite orientation is above o .

A natural next step is to introduce upper bounds $p : V \rightarrow \mathbb{Z}$ beside the lower bounds $o : V \rightarrow \mathbb{Z}$. To avoid trivial obstructions we assume that o and p are *consistent* which means that for every $v \in V$ there is an orientation D_v which is above o at bellow p at v .

Question 7.2. *Let G be a countable graph and let $o, p : V \rightarrow \mathbb{Z}$ be a consistent pair of bounding functions. Suppose that there are orientations D_o and D_p that are above o and bellow p respectively. Is there always a single orientation D which is above o and bellow p ?*

The positive answer for finite graphs is not too hard to prove, as far we know its first appearance in the literature is [14].

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