

The Trace Field Theory of a Finite Tensor Category

Christoph Schweigert ^a and Lukas Woike ^b

^a *Fachbereich Mathematik
Universität Hamburg
Bereich Algebra und Zahlentheorie
Bundesstraße 55
D-20146 Hamburg*

^b *Institut for Matematiske Fag
Københavns Universitet
Universitetsparken 5
DK-2100 København Ø*

Given a finite tensor category \mathcal{C} , we prove that a modified trace on the tensor ideal of projective objects can be obtained from a suitable trivialization of the Nakayama functor as right \mathcal{C} -module functor. Using a result of Costello, this allows us to associate to any finite tensor category equipped with such a trivialization of the Nakayama functor a chain complex valued topological conformal field theory, the *trace field theory*. The trace field theory topologically describes the modified trace, the Hattori-Stallings trace, and also the structures induced by them on the Hochschild complex of \mathcal{C} . In this article, we focus on implications in the *linear* (as opposed to differential graded) setting: We use the trace field theory to define a non-unital homotopy commutative product on the Hochschild chains in degree zero. This product is block diagonal and can be described through the handle elements of the trace field theory. Taking the modified trace of the handle elements recovers the Cartan matrix of \mathcal{C} .

1 Introduction and summary

A *finite category* over a fixed field k , that we will assume to be algebraically closed throughout, is an Abelian category enriched over finite-dimensional k -vector spaces which has enough projective objects and finitely many isomorphism classes of simple objects; moreover, one requires that every object has finite length. Any finite category \mathcal{C} comes with a right exact endofunctor

$$\mathbf{N}^r : \mathcal{C} \longrightarrow \mathcal{C}, \quad X \longmapsto \mathbf{N}^r X := \int^{Y \in \mathcal{C}} \mathcal{C}(X, Y)^* \otimes Y,$$

the (*right*) *Nakayama functor*, where \otimes denotes the tensoring of objects in \mathcal{C} with finite-dimensional vector spaces. This Morita invariant description was given in [FSS20], and it reduces to the usual definition of the (*right*) Nakayama functor for the category of finite-dimensional modules over a finite-dimensional k -algebra. As a consequence of the coend description of \mathbf{N}^r , we obtain in Corollary 2.3 natural isomorphisms

$$\mathcal{C}(P, X) \cong \mathcal{C}(X, \mathbf{N}^r P)^* \quad \text{for } X \in \mathcal{C}, \quad P \in \text{Proj } \mathcal{C}$$

turning the subcategory $\text{Proj } \mathcal{C} \subset \mathcal{C}$ into an \mathbf{N}^r -twisted Calabi-Yau category. Through the correspondence between (twisted) Calabi-Yau structures and (twisted) traces, one obtains the trace

$$\mathfrak{t}_P : \mathcal{C}(P, \mathbf{N}^r P) \longrightarrow k \quad \text{for } P \in \text{Proj } \mathcal{C}. \quad (1.1)$$

It is now an obvious task to relate this relatively generically constructed trace (or rather family of traces) to the modified trace [GPT09, GKP11, GKP13, GPV13, GKP21, BBG18]. Modified traces are not only a concept of independent algebraic interest, but can also be used for the construction of invariants of

closed three-dimensional manifolds, see e.g. [CGPM14, BCGPM16]. In such constructions, they serve as a non-semisimple replacement for quantum traces.

In this article, we prove in Theorem 3.6 that, for a finite tensor category in the sense of Etingof-Ostrik [EO04], i.e. a finite category with rigid monoidal product and simple unit, the trace (1.1) on the tensor ideal of projective objects, indeed produces a twisted modified trace.

THEOREM 3.6. *For any finite tensor category \mathcal{C} , the twisted trace $(\mathfrak{t}_P : \mathcal{C}(P, \mathbf{N}^r P) \rightarrow k)_{P \in \text{Proj } \mathcal{C}}$ from (1.1) is (twisted) cyclic, non-degenerate and satisfies a generalized partial trace property. Under the additional assumption that on the finite tensor category \mathcal{C} a pivotal structure has been chosen, the twisted trace $(\mathfrak{t}_P : \mathcal{C}(P, \mathbf{N}^r P) \rightarrow k)_{P \in \text{Proj } \mathcal{C}}$ can be naturally identified with a right modified D -trace, where $D \in \mathcal{C}$ is the distinguished invertible object of \mathcal{C} .*

This uses crucially that by [FSS20, Theorem 4.26] the Nakayama functor of a finite tensor category can be expressed as

$$\mathbf{N}^r \cong D^{-1} \otimes -^{\vee\vee}$$

using the distinguished invertible object $D \in \mathcal{C}$ [ENO04] and the double dual functor $-^{\vee\vee}$. Note that we do not require a pivotal structure to define our traces because the double dual functor can be conveniently absorbed into the Nakayama functor. If \mathcal{C} is pivotal, however, we recover the usual definitions.

Our motivation for unraveling the connection between the Nakayama functor and the modified trace is topological, but does not directly come from invariants of closed three-dimensional manifolds. Instead, we are motivated by two-dimensional topological conformal field theory, a certain type of differential graded two-dimensional open-closed topological field theory: Suppose that we are given a finite tensor category \mathcal{C} and a *symmetric Frobenius structure*, by which we mean a certain trivialization of the right Nakayama functor as right \mathcal{C} -module functor relative to a pivotal structure (we give the details in Definition 4.6; it will amount to a pivotal structure and a trivialization of the distinguished invertible object). Then the trace coming from *this* particular trivialization of \mathbf{N}^r produces, as discussed above, a Calabi-Yau structure on the tensor ideal $\text{Proj } \mathcal{C} \subset \mathcal{C}$. To this Calabi-Yau structure on $\text{Proj } \mathcal{C}$, Costello's Theorem [Cos07] associates a topological conformal field theory $\Phi_{\mathcal{C}}$ that we refer to as the *trace field theory* of the finite tensor category \mathcal{C} with symmetric Frobenius structure. On a technical level, $\Phi_{\mathcal{C}} : \text{OC} \rightarrow \text{Ch}_k$ is a symmetric monoidal functor from a certain differential graded version of the open-closed bordism category to chain complexes, we recall the details in Section 4.

If we evaluate $\Phi_{\mathcal{C}}$ on the *open* part of the two-dimensional bordism category, $\Phi_{\mathcal{C}}$ provides topological tools to compute with traces, but only captures information that one could have obtained by hand. This is drastically different for the *closed* part of the two-dimensional bordism category: On a closed boundary component, i.e. on the circle, we obtain, following again [Cos07], the Hochschild complex of \mathcal{C} , i.e. the homotopy coend $\int_{\mathbb{L}}^{X \in \text{Proj } \mathcal{C}} \mathcal{C}(X, X)$ over the endomorphism spaces of projective objects. On this complex, we have an action of the prop provided by the chains on moduli spaces of Riemann surfaces with closed boundary components. Phrased differently, the trace field theory $\Phi_{\mathcal{C}}$ captures the higher structures induced by the modified trace on the Hochschild complex of \mathcal{C} while, at the same time, being very accessible through the tools available for computations with Nakayama functors. These higher structures will be developed in detail elsewhere. For the present article, no homotopy theory is needed, and we focus entirely on the purely linear consequences, i.e. on structures induced in homological degree zero.

THEOREM 4.9. *Let \mathcal{C} be a finite tensor category with symmetric Frobenius structure and $\Phi_{\mathcal{C}} : \text{OC} \rightarrow \text{Ch}_k$ its trace field theory. The evaluation of $\Phi_{\mathcal{C}}$ on the disk with one incoming open boundary interval whose complementing free boundary carries the label $P \in \text{Proj } \mathcal{C}$*

$$\Phi_{\mathcal{C}} \left(\text{in} \left(\text{circle} \right) P \right) : \mathcal{C}(P, P) \rightarrow k$$

is a right modified trace, while the evaluation of $\Phi_{\mathcal{C}}$ on the cylinder with one incoming open boundary interval with complementing free boundary label $P \in \text{Proj } \mathcal{C}$ and one outgoing closed boundary circle

$$\Phi_{\mathcal{C}} \left(\text{in} \left(\text{cylinder} \right) P \right) : \mathcal{C}(P, P) \rightarrow \int_{\mathbb{L}}^{P \in \text{Proj } \mathcal{C}} \mathcal{C}(P, P)$$

agrees, after taking zeroth homology, with the Hattori-Stallings trace of \mathcal{C} .

Note. While finalizing this manuscript, the preprint [SS21] appeared which provides a proof for a connection between the Nakayama functor and modified traces very similar to the one afforded by Theorem 3.6.

2 Traces on finite categories

For any finite category \mathcal{C} , the (right) Nakayama functor $\mathbf{N}^r : \mathcal{C} \rightarrow \mathcal{C}$ is given by

$$\mathbf{N}^r X := \int^{Y \in \mathcal{C}} \mathcal{C}(X, Y)^* \otimes Y . \quad (2.1)$$

This is the Morita invariant description given in [FSS20] for the usual Nakayama functor for finite-dimensional modules over a finite-dimensional algebra A which is given by

$$\mathbf{N}^r X = \text{Hom}_A(X, A)^* \cong A^* \otimes_A X \quad (2.2)$$

for any finite-dimensional A -module X . The Nakayama functor sends projective objects to injective objects.

PROPOSITION 2.1. *For any finite category \mathcal{C} , there is a canonical isomorphism of chain complexes*

$$\mathcal{C}(X, Y_\bullet)^* \cong \mathcal{C}(Y_\bullet, \mathbf{N}^r X)$$

natural in objects $X, Y \in \mathcal{C}$, where Y_\bullet is a projective resolution of Y .

Proof. Since every finite category can be written as finite-dimensional modules over a finite-dimensional algebra, we conclude from the comparison of (2.1) and (2.2) that the right hand side of (2.1) can be modeled as a *finite* colimit.

Since $\mathcal{C}(Y_\bullet, -)$ is exact, the finite colimit used to define $\mathbf{N}^r X$ is preserved, which leads to

$$\mathcal{C}(Y_\bullet, \mathbf{N}^r X) \cong \int^{Z \in \mathcal{C}} \mathcal{C}(Y_\bullet, \mathcal{C}(X, Z)^* \otimes Z) \cong \int^{Z \in \mathcal{C}} \mathcal{C}(Y_\bullet, Z) \otimes \mathcal{C}(X, Z)^* \cong \mathcal{C}(X, Y_\bullet)^* .$$

All coends are computed degree-wise here. In the last step, we have used the Yoneda Lemma. □

The isomorphisms from Proposition 2.1 can be used to obtain a twisted Calabi-Yau structure on $\text{Proj } \mathcal{C}$.

DEFINITION 2.2. An (F, G) -twisted Calabi-Yau category is a linear category \mathcal{A} with endofunctors $F, G : \mathcal{A} \rightarrow \mathcal{A}$ and isomorphisms $\mathcal{A}(F(X), Y) \cong \mathcal{A}(Y, G(X))^*$ natural in $X, Y \in \mathcal{A}$.

In order to avoid overloaded notation, we call a twisted Calabi-Yau category *left F -twisted* and *right G -twisted* if the twist datum (F, G) is given $(F, \text{id}_{\mathcal{A}})$ and $(\text{id}_{\mathcal{A}}, G)$, respectively. By a *Calabi-Yau category* (without any mention of twists) we will understand an untwisted Calabi-Yau category in the sense that $F = G = \text{id}_{\mathcal{A}}$. A Calabi-Yau category with one object is a symmetric Frobenius algebra.

COROLLARY 2.3. *For a finite category \mathcal{C} , there are canonical isomorphisms*

$$\mathcal{C}(P, X) \cong \mathcal{C}(X, \mathbf{N}^r P)^* \quad (2.3)$$

natural in $X \in \mathcal{C}$ and $P \in \text{Proj } \mathcal{C}$. In particular, $\text{Proj } \mathcal{C}$ is a right \mathbf{N}^r -twisted Calabi-Yau category.

Proof. For $X \in \mathcal{C}$ and $P \in \text{Proj } \mathcal{C}$, we find (we denote equivalences of chain complexes aka quasi-isomorphisms by \simeq and isomorphisms, as before, by \cong)

$$\begin{aligned} \mathcal{C}(X, \mathbf{N}^r P)^* &\simeq \mathcal{C}(X_\bullet, \mathbf{N}^r P)^* && \text{(because } \mathbf{N}^r P \text{ is injective)} \\ &\cong \mathcal{C}(P, X_\bullet) && \text{(Proposition 2.1)} \\ &\simeq \mathcal{C}(P, X) && \text{(because } P \text{ is projective)} . \end{aligned}$$

□

DEFINITION 2.4 (*Trace of a finite category*). For any finite category \mathcal{C} , we define the pairings

$$\langle -, - \rangle : \mathcal{C}(P, X) \otimes \mathcal{C}(X, \mathbf{N}^r P) \xrightarrow{(2.3)} \mathcal{C}(X, \mathbf{N}^r P)^* \otimes \mathcal{C}(X, \mathbf{N}^r P) \xrightarrow{\text{evaluation}} k \quad (2.4)$$

for $X \in \mathcal{C}$, $P \in \text{Proj } \mathcal{C}$

and, by considering the case $X = P$ in (2.4), the *twisted trace*

$$\mathfrak{t}_P : \mathcal{C}(P, \mathbf{N}^r P) \xrightarrow{\langle \text{id}_P, - \rangle} k \quad \text{for } P \in \text{Proj } \mathcal{C} \quad (2.5)$$

on $\mathcal{C}(P, \mathbf{N}^r P)$. We refer to the family of maps (2.5), where P runs over all projective objects, as the *twisted trace on \mathcal{C}* . By an *untwisting* of the twisted trace, we mean a trivialization $\mathbf{N}^r \cong \text{id}_{\mathcal{C}}$ of \mathbf{N}^r (if there exists any) and the resulting identification of the maps (2.5) with maps $\mathcal{C}(P, P) \rightarrow k$ that we then refer to as *untwisted trace*, or just *trace* for brevity.

It is important to note that the twisted trace is canonical while the untwisting (if possible) will involve choices. An untwisting of the trace is equivalent to an untwisting of the twisted Calabi-Yau structure from Corollary 2.3.

The usual correspondence between Calabi-Yau structures and traces can be adapted to the present situation and leads to the following:

LEMMA 2.5. *For any finite category \mathcal{C} , the twisted trace*

$$\mathfrak{t}_P : \mathcal{C}(P, \mathbf{N}^r P) \rightarrow k \quad \text{for } P \in \text{Proj } \mathcal{C}$$

has the following properties:

(i) *Cyclicity: For $P, Q \in \text{Proj } \mathcal{C}$, $f : P \rightarrow \mathbf{N}^r Q$ and $g : Q \rightarrow P$, we have*

$$\mathfrak{t}_Q(fg) = \mathfrak{t}_P(\mathbf{N}^r(g)f) .$$

(ii) *Non-degeneracy: The trace is non-degenerate in the sense that the pairings*

$$\mathcal{C}(P, X) \otimes \mathcal{C}(X, \mathbf{N}^r P) \rightarrow k, \quad f \otimes g \mapsto \mathfrak{t}_P(gf) \quad (2.6)$$

are non-degenerate. In fact, they agree with the pairings (2.4).

Proof. Let $X, Y \in \mathcal{C}$ and $P, Q \in \text{Proj } \mathcal{C}$. Naturality of (2.3) in X means for $a : P \rightarrow X$, $b : Y \rightarrow \mathbf{N}^r P$ and $c : X \rightarrow Y$

$$\langle a, bc \rangle = \langle ca, b \rangle . \quad (2.7)$$

Naturality of (2.3) in P means for $a : Q \rightarrow X$, $b : P \rightarrow Q$ and $c : X \rightarrow \mathbf{N}^r P$

$$\langle a, \mathbf{N}^r(b)c \rangle = \langle ab, c \rangle . \quad (2.8)$$

This implies for $f : P \rightarrow \mathbf{N}^r Q$ and $g : Q \rightarrow P$

$$\mathfrak{t}_Q(fg) \stackrel{(2.5)}{=} \langle \text{id}_Q, fg \rangle \stackrel{(2.7)}{=} \langle g, f \rangle \stackrel{(2.8)}{=} \langle \text{id}_P, \mathbf{N}^r(g)f \rangle \stackrel{(2.5)}{=} \mathfrak{t}_P(\mathbf{N}^r(g)f) .$$

This proves cyclicity. Non-degeneracy holds by construction because it follows easily from (2.7) that the pairing (2.6) agrees with $\langle -, - \rangle$. \square

3 Traces on finite tensor categories and connection to modified traces

We will now turn to finite *tensor* categories and connect the construction from Definition 2.4 to modified traces [GKP11, GKP13, GPV13, GKP21]. A (twisted, right) modified trace on the tensor ideal of projective objects in a pivotal finite tensor category is a cyclic, non-degenerate trace that satisfies the right partial trace property as we will discuss in detail below. The first two properties hold very generally for traces constructed from linear trivializations of the Nakayama functor thanks to Lemma 2.5.

The partial trace property makes use of the monoidal structure. In order to understand when we can formulate and prove such a property for the trace from Definition 2.4, one needs to understand the Nakayama functor of a finite tensor category: Let \mathcal{C} be any finite tensor category. We denote by $\mathcal{C}_{\mathcal{C}}$ the finite category \mathcal{C} as regular right module category over itself and by $\mathcal{C}^{\vee\vee}$ the finite category \mathcal{C} as \mathcal{C} -right module with action given by $X.Y := X \otimes Y^{\vee\vee}$ for $X, Y \in \mathcal{C}$.

THEOREM 3.1 ([FSS20, Theorem 3.26]). For any finite tensor category \mathcal{C} , the (right) Nakayama functor is an equivalence $\mathbf{N}^r : \mathcal{C}_{\mathcal{C}} \xrightarrow{\cong} \mathcal{C}^{\vee\vee}$ of right \mathcal{C} -module categories; in particular, comes with canonical isomorphisms $\mathbf{N}^r(- \otimes X) \cong \mathbf{N}^r(-) \otimes X^{\vee\vee}$ for $X \in \mathcal{C}$. Moreover, $\mathbf{N}^r I \cong D^{-1}$, where $D \in \mathcal{C}$ is the distinguished invertible object and D^{-1} its dual, and hence

$$\mathbf{N}^r \cong D^{-1} \otimes -^{\vee\vee}$$

by a canonical isomorphism.

Together with Proposition 2.1, this implies:

COROLLARY 3.2. For any finite tensor category \mathcal{C} , there are canonical isomorphisms

$$\mathcal{C}(X, Y_{\bullet})^* \cong \mathcal{C}(Y_{\bullet}, D^{-1} \otimes X^{\vee\vee})$$

natural in objects $X, Y \in \mathcal{C}$, where Y_{\bullet} is a projective resolution of Y . In particular, any pivotal structure on \mathcal{C} provides canonical isomorphisms

$$\mathcal{C}(X, Y_{\bullet})^* \cong \mathcal{C}(Y_{\bullet}, D^{-1} \otimes X).$$

We now propose a generalization of the partial trace property that does not need a pivotal structure (from our perspective, this will turn out to be more natural): Let \mathcal{C} be a finite tensor category. For $X \in \mathcal{C}$ and $P \in \text{Proj } \mathcal{C}$, we may use Theorem 3.1 to define a map

$$\mathcal{C}(P \otimes X, \mathbf{N}^r(P) \otimes X^{\vee\vee}) \longrightarrow \mathcal{C}(P, \mathbf{N}^r P) \quad (3.1)$$

sending $f : P \otimes X \longrightarrow \mathbf{N}^r(P) \otimes X^{\vee\vee}$ to

$$P \xrightarrow{P \otimes b_X} P \otimes X \otimes X^{\vee} \xrightarrow{f \otimes X^{\vee}} \mathbf{N}^r(P) \otimes X^{\vee\vee} \otimes X^{\vee} \xrightarrow{\mathbf{N}^r(P) \otimes d_{X^{\vee}}} \mathbf{N}^r P.$$

DEFINITION 3.3. Let \mathcal{C} be a finite tensor category, $P \in \text{Proj } \mathcal{C}$ and $X \in \mathcal{C}$. Then we define the *right partial trace* as the composition

$$\text{tr}_r^X : \mathcal{C}(P \otimes X, \mathbf{N}^r(P \otimes X)) \xrightarrow{\text{Theorem 3.1}} \mathcal{C}(P \otimes X, \mathbf{N}^r(P) \otimes X^{\vee\vee}) \xrightarrow{(3.1)} \mathcal{C}(P, \mathbf{N}^r(P)).$$

All of this crucially uses that $P \otimes X$ (and also $X \otimes P$) is projective if P is, i.e. the ideal property property of $\text{Proj } \mathcal{C}$.

REMARK 3.4. A pivotal structure is not needed for the definition given here because the double dual is absorbed into the Nakayama functor. In presence of a pivotal structure $\omega : -^{\vee\vee} \cong \text{id}_{\mathcal{C}}$, however, our definition specializes to the usual partial trace property for a *right D -trace* in the terminology of [GKP21] in the sense that the composition

$$\mathcal{C}(D \otimes P \otimes X, P \otimes X) \cong \mathcal{C}(P \otimes X, \mathbf{N}^r(P \otimes X)) \xrightarrow{\text{tr}_r^X} \mathcal{C}(P, \mathbf{N}^r(P)),$$

where the first isomorphism uses duality, Theorem 3.1 and the (inverse) pivotal structure, is the usual partial trace.

PROPOSITION 3.5. For any finite tensor category \mathcal{C} , the twisted trace

$$\mathfrak{t}_P : \mathcal{C}(P, \mathbf{N}^r P) \longrightarrow k \quad \text{for } P \in \text{Proj } \mathcal{C}$$

from Definition 2.4 satisfies the *right partial trace property*: For $X \in \mathcal{C}$ and $P \in \text{Proj } \mathcal{C}$ and any morphism $f : P \otimes X \longrightarrow \mathbf{N}^r(P \otimes X)$

$$\mathfrak{t}_P \text{tr}_r^X(f) = \mathfrak{t}_{P \otimes X}(f). \quad (3.2)$$

Proof. For $X, Y \in \mathcal{C}$, $P \in \text{Proj } \mathcal{C}$ and a projective resolution Y_{\bullet} of Y , consider the following diagram in which all maps are isomorphisms (we explain all parts of the diagram and its commutativity afterwards):

$$\begin{array}{ccc} \mathcal{C}(P \otimes X, Y_{\bullet})^* & \xleftarrow{\text{YL}} \int^{Z \in \mathcal{C}} \mathcal{C}(Y_{\bullet}, Z) \otimes \mathcal{C}(P \otimes X, Z)^* & \xrightarrow{(\diamond)} \mathcal{C}(Y_{\bullet}, \mathbf{N}^r(P \otimes X)) \\ \downarrow \vee & \downarrow \vee & \downarrow \mathbf{N}^r(- \otimes X) \cong \mathbf{N}^r(-) \otimes X^{\vee\vee} \\ \mathcal{C}(P, Y_{\bullet} \otimes X^{\vee})^* & \xleftarrow{\text{YL}} \int^{Z \in \mathcal{C}} \mathcal{C}(Y_{\bullet}, Z) \otimes \mathcal{C}(P, Z \otimes X^{\vee})^* & \\ & \swarrow \text{YL} & \\ & \int^{Z' \in \mathcal{C}} \mathcal{C}(Y_{\bullet} \otimes X^{\vee}, Z') \otimes \mathcal{C}(P, Z')^* & \xrightarrow{(\diamond)} \mathcal{C}(Y_{\bullet} \otimes X^{\vee}, \mathbf{N}^r P) \xrightarrow{\vee} \mathcal{C}(Y_{\bullet}, \mathbf{N}^r P \otimes X^{\vee\vee}). \end{array}$$

The isomorphisms labeled ‘YL’ and ‘ \vee ’ come from the Yoneda Lemma and duality, respectively. The isomorphisms (\diamond) pull the coend and the tensoring with vector spaces out of the hom functor using exactness of $\mathcal{C}(Y_\bullet, -)$ (they follow essentially from the definition (2.1) of the Nakayama functor). The ‘relabeling’ isomorphism $\int^{Z \in \mathcal{C}} \mathcal{C}(Y_\bullet, Z) \otimes \mathcal{C}(P, Z \otimes X^\vee)^* \rightarrow \int^{Z' \in \mathcal{C}} \mathcal{C}(Y_\bullet \otimes X^\vee, Z') \otimes \mathcal{C}(P, Z')^*$ sends $f \otimes \alpha \in \mathcal{C}(Y_\bullet, Z) \otimes \mathcal{C}(P, Z \otimes X^\vee)^*$ living in the summand indexed by Z of the coend $\int^{Z \in \mathcal{C}} \mathcal{C}(Y_\bullet, Z) \otimes \mathcal{C}(P, Z \otimes X^\vee)^*$ to $(f \otimes X^\vee) \otimes \alpha$ living in the summand indexed by $Z \otimes X^\vee$ of the coend $\int^{Z' \in \mathcal{C}} \mathcal{C}(Y_\bullet \otimes X^\vee, Z') \otimes \mathcal{C}(P, Z')^*$. The vertical isomorphism on the very right uses Theorem 3.1.

In fact, the isomorphism $\mathbf{N}^r(- \otimes X) \cong \mathbf{N}^r(-) \otimes X^{\vee\vee}$ can be *obtained* by extracting the isomorphism $\mathcal{C}(Y_\bullet, \mathbf{N}^r(P \otimes X)) \rightarrow \mathcal{C}(Y_\bullet, \mathbf{N}^r P \otimes X^{\vee\vee})$ by going in counterclockwise direction in the hexagon on the right (this follows from an analysis of the proof of [FSS20, Theorem 3.18]). As a consequence, the hexagon on the right commutes.

A direct computation shows that the square and the triangle on the left commute. Therefore, the entire diagram commutes.

After taking the linear dual of the entire diagram and remembering that the isomorphisms ‘YL’ and (\diamond) combine into the isomorphisms from Proposition 2.1, we see that the diagram

$$\begin{array}{ccc} \mathcal{C}(P \otimes X, Y_\bullet) & \xrightarrow{\text{Proposition 2.1}} & \mathcal{C}(Y_\bullet, \mathbf{N}^r(P \otimes X))^* \cong \mathcal{C}(Y_\bullet, \mathbf{N}^r P \otimes X^{\vee\vee})^* \\ \downarrow \vee & & \downarrow \vee \\ \mathcal{C}(P, Y_\bullet \otimes X^\vee) & \xrightarrow{\text{Proposition 2.1}} & \mathcal{C}(Y_\bullet \otimes X^\vee, \mathbf{N}^r P)^* \end{array}$$

commutes. Since P is projective (and hence $\mathbf{N}^r P$ injective — in fact, the projective objects in \mathcal{C} even coincide with the injective ones), this reduces to the commutative diagram

$$\begin{array}{ccc} \mathcal{C}(P \otimes X, Y) & \xrightarrow{\text{Corollary 2.3}} & \mathcal{C}(Y, \mathbf{N}^r(P \otimes X))^* \cong \mathcal{C}(Y, \mathbf{N}^r P \otimes X^{\vee\vee})^* \\ \downarrow \vee & & \downarrow \vee \\ \mathcal{C}(P, Y \otimes X^\vee) & \xrightarrow{\text{Corollary 2.3}} & \mathcal{C}(Y \otimes X^\vee, \mathbf{N}^r P)^* \end{array}$$

in which the horizontal maps have specialized to the ones from Corollary 2.3. If we spell out the commutativity of this diagram in equations for morphisms $g : P \otimes X \rightarrow Y$ and $h : Y \otimes X^\vee \rightarrow \mathbf{N}^r P$, we obtain with the bracket notation from Definition 2.4 (we use here additionally the graphical calculus for morphisms in a monoidal category — to be read from bottom to top; we refer to [Kas95] for a textbook treatment)

$$\left\langle \begin{array}{c} Y \\ \downarrow \\ \boxed{g} \\ \downarrow \downarrow \\ P \quad X \end{array}, \begin{array}{c} \mathbf{N}^r P \quad X^{\vee\vee} \\ \downarrow \\ \boxed{h} \\ \downarrow \downarrow \\ Y \end{array} \right\rangle = \left\langle \begin{array}{c} Y \quad X^\vee \\ \downarrow \\ \boxed{g} \\ \downarrow \downarrow \\ P \end{array}, \begin{array}{c} \mathbf{N}^r P \\ \downarrow \\ \boxed{h} \\ \downarrow \downarrow \\ Y \quad X^\vee \end{array} \right\rangle. \quad (3.3)$$

As another preparation, recall that the double dual functor $-^{\vee\vee} : \mathcal{C} \rightarrow \mathcal{C}$ is monoidal, hence it preserves the duality pairing $d_{\vee X} : X \otimes \vee X \rightarrow I$ (we use here the canonical identification $\vee(X^\vee) \cong X$) and therefore sends it to $d_{X^\vee} : X^{\vee\vee} \otimes X^\vee \rightarrow I$. Using Theorem 3.1 we find the equality of morphisms

$$\mathbf{N}^r(P \otimes d_{\vee X}) = \mathbf{N}^r P \otimes d_{X^\vee} : \mathbf{N}^r(P) \otimes X^{\vee\vee} \otimes X^\vee \rightarrow \mathbf{N}^r P,$$

which implies for a morphism $f : P \otimes X \rightarrow \mathbf{N}^r(P \otimes X)$

$$\mathrm{tr}_r^X f = \begin{array}{c} \mathbb{N}^r P \\ | \\ \text{---} \\ | \\ \boxed{f} \\ | \\ P \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ X \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ X^{\vee\vee} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ X^\vee \end{array} = \begin{array}{c} \mathbb{N}^r P \\ | \\ \boxed{\mathbb{N}^r(P \otimes d_{\nu_X})} \\ | \\ \boxed{f} \\ | \\ P \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ X \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ X^{\vee\vee} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ X^\vee \end{array} . \quad (3.4)$$

The desired equality (3.2) now follows from:

$$\begin{aligned} \mathfrak{t}_P \mathrm{tr}_r^X f &= \begin{array}{c} \mathbb{N}^r P \\ | \\ \boxed{\mathbb{N}^r(P \otimes d_{\nu_X})} \\ | \\ \boxed{f} \\ | \\ P \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ P \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ X^{\vee\vee} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ X^\vee \end{array} \\ &\stackrel{(2.8)}{=} \begin{array}{c} \mathbb{N}^r P \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ P \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ X^\vee X \end{array} \begin{array}{c} \mathbb{N}^r P \\ | \\ \boxed{f} \\ | \\ P \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ X^{\vee\vee} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ X^\vee \end{array} \\ &\stackrel{(3.3)}{=} \begin{array}{c} X \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ P \end{array} \begin{array}{c} \mathbb{N}^r P \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ P \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ X^{\vee\vee} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ X \end{array} \begin{array}{c} \boxed{f} \\ | \\ P \end{array} \\ &= \begin{array}{c} \mathbb{N}^r P \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ P \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ X \end{array} \begin{array}{c} \mathbb{N}^r P \\ | \\ \boxed{f} \\ | \\ P \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ X^{\vee\vee} \end{array} \\ &= \mathfrak{t}_{P \otimes X}(f) \end{aligned}$$

□

THEOREM 3.6. For any finite tensor category \mathcal{C} , the twisted trace $(\mathfrak{t}_P : \mathcal{C}(P, \mathbb{N}^r P) \rightarrow k)_{P \in \mathrm{Proj} \mathcal{C}}$ from Definition 2.4 is cyclic, non-degenerate and satisfies the partial trace property in the sense of Proposition 3.5. Under the additional assumption that on the finite tensor category \mathcal{C} a pivotal structure has been chosen, the twisted trace $(\mathfrak{t}_P : \mathcal{C}(P, \mathbb{N}^r P) \rightarrow k)_{P \in \mathrm{Proj} \mathcal{C}}$ from Definition 2.4 can be naturally identified with a right modified D -trace, where $D \in \mathcal{C}$ is the distinguished invertible object of \mathcal{C} .

REMARK 3.7. More precisely, the twisted trace from Definition 2.4 yields a canonical right modified D -trace and thereby trivializes the k^X -torsor of right modified D -traces in a canonical way.

Proof of Theorem 3.6. We use the pivotal structure $\omega : -^{\vee\vee} \cong \mathrm{id}_{\mathcal{C}}$ to obtain isomorphisms

$$\mathcal{C}(P, \mathbb{N}^r P) \stackrel{\text{Theorem 3.1}}{\cong} \mathcal{C}(P, D^{-1} \otimes P^{\vee\vee}) \stackrel{\omega \text{ and duality}}{\cong} \mathcal{C}(D \otimes P, P) \quad \text{for } P \in \mathrm{Proj} \mathcal{C} .$$

As a consequence, the twisted trace from Definition 2.4 gives us maps $\mathcal{C}(D \otimes P, P) \rightarrow k$ which are cyclic and non-degenerate (Lemma 2.5). Moreover, Proposition 3.5 combined with Remark 3.4 gives us the usual partial trace property in presence of a pivotal structure.

Note that one needs really a monoidal isomorphism $-^{\vee\vee} \cong \mathrm{id}_{\mathcal{C}}$ to get the desired maps $\mathcal{C}(D \otimes P, P) \rightarrow k$. If ω is just linear, one would get maps $\mathcal{C}(D \otimes P, P) \rightarrow k$, but they would not necessarily satisfy the partial trace property: The proof of the partial trace property in its \mathbb{N}^r -twisted version (Proposition 3.5) relies on the monoidal structure of $-^{\vee\vee}$. The partial trace property only transfers along the isomorphisms $\mathcal{C}(P, D^{-1} \otimes P^{\vee\vee}) \cong \mathcal{C}(P, D^{-1} \otimes P)$ if $-^{\vee\vee}$ is replaced by $\mathrm{id}_{\mathcal{C}}$ as monoidal functor. □

4 The trace field theory

We now introduce the topological conformal field theory induced by the Calabi-Yau structure appearing in the previous section. To this end, let us recall from [Cos07] the definition of the (differential graded) open-closed two-dimensional cobordism category, see [ES15, WW16] for models of this symmetric monoidal differential graded category in terms of fat graphs. An *open-closed Riemann surface* is a Riemann surface with the following data:

- A subset of its boundary components, the so-called *closed boundary components*. They are parametrized and labeled as incoming or outgoing.
- A finite number of embedded intervals in the remaining boundary components, the so-called *open boundary intervals*. They are also parametrized and labeled as incoming or outgoing.

The free boundary components are defined as the complement (in the boundary) of the closed boundary components and the open boundary intervals. It will be required that each connected component of the Riemann surface has at least one free boundary component or at least one incoming closed boundary. An example (that additionally contains certain labels that will be discussed in a moment) is depicted in Figure 1.

One can now define the symmetric monoidal differential graded category \mathcal{OC} of *open-closed cobordisms* for a set Λ of labels (that we will fix later and that will be suppressed in the notation; the set of labels is sometimes referred to as set of ‘D-branes’): The objects are pairs of finite sets O and C (that in a moment will play the role of open boundary intervals and closed boundary components of Riemann surfaces) and two maps $s, t : O \rightarrow \Lambda$ (that attach a ‘start’ and an ‘end’ label to any open boundary). The chain complex of morphisms from (O, C, s, t) to (O', C', s', t') is given by the k -chains on the moduli space of Riemann surfaces Σ with

- an identification of its set of incoming open and incoming closed boundary components with (O, C) , an identification of its set of outgoing open and outgoing closed boundary components with (O', C') ,
- a label in the set Λ of D-branes for each free boundary component

subject to the following requirement: First observe that any incoming open boundary interval $o \in O$ inherits a label for its start point and its end point, namely the label of the free boundary component that it is bounded by. We require that this label agrees with $(s(o), t(o))$; the analogous requirement is imposed for outgoing open boundary intervals. Explicitly, for the objects $X = (O, C, s, t)$ and $X' = (O', C', s', t')$, the morphism complexes are given, up to equivalence, by

$$\mathcal{OC}(X, X') \simeq \bigoplus_{S: X \rightarrow X'} C_*(B\mathrm{Map}(S); k),$$

where the direct sum is running over all topological types of compact oriented open-closed bordisms S with incoming and outgoing boundary described by X and X' , respectively, and $\mathrm{Map}(S)$ is the mapping class group of S ; we refer to [ES15, WW16] for a description of these morphism complexes by means of classifying spaces of categories of fat graphs. Composition in \mathcal{OC} is by gluing. Disjoint union provides a symmetric monoidal structure.

DEFINITION 4.1 (Costello [Cos07] following Getzler [Get94] and Segal [Seg04]). For a fixed set Λ of D-branes, an *open-closed topological conformal field theory* is a symmetric monoidal functor $\mathcal{P} : \mathcal{OC} \rightarrow \mathrm{Ch}_k$. An *open topological conformal field theory* is a symmetric monoidal functor $\mathcal{O} \rightarrow \mathrm{Ch}_k$ defined only on the subcategory $\mathcal{O} \subset \mathcal{OC}$ of open bordisms.

Open-closed topological conformal field theories are a differential graded generalization of ordinary vector space-valued two-dimensional (open-closed) topological field theories. The latter can be constructed and classified in terms of symmetric and commutative Frobenius algebras, see [Koc03, LP08] for the precise statements.

In [Cos07], Costello proves that one may construct an *open* topological conformal field theory from a (linear) Calabi-Yau category (Costello actually considers differential graded Calabi-Yau categories, but we just need the linear case). By homotopy left Kan extension, one obtains an open-closed topological conformal field theory:

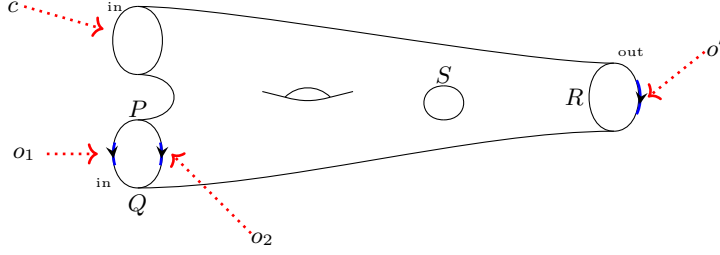


Figure 1: An open-closed surface with D-brane labels. As a morphism in OC, we read the surface from left to right, i.e. with the source object (constituted by the incoming boundary components) on the left and the target object (constituted by the outgoing boundary components) on the right. We will, however, deviate from the left-to-right drawing convention at times if it simplifies the surface; for this reason, we also indicate by ‘in’ and ‘out’ whether a boundary component is incoming or outgoing. The source object is given by $\{c, o_1, o_2\}$ (where the identification with boundary components is through the dotted arrows in the picture) plus the assignment $s(o_1) = s(o_2) = P$ and $t(o_1) = t(o_2) = Q$. The target object is $\{o'\}$ plus the assignments $s(o') = t(o') = R$.

THEOREM 4.2 (Costello [Cos07]). (i) Any linear Calabi-Yau category \mathcal{A} gives rise to an open-closed topological conformal field theory $\text{OC} \rightarrow \text{Ch}_k$ with the object set of \mathcal{A} as the set of D-branes.

(ii) The value of this field theory on the circle is equivalent to the Hochschild complex of \mathcal{A} .

In particular, if $\mathcal{M}_{p,q}$ is the moduli space of Riemann surfaces with p incoming closed ($p \geq 1$), q outgoing closed and no open and no free boundary components, there are maps

$$C_*(\mathcal{M}_{p,q}; k) \otimes \left(\int_{\mathbb{L}}^{a \in \mathcal{A}} \mathcal{A}(a, a) \right)^{\otimes p} \rightarrow \left(\int_{\mathbb{L}}^{a \in \mathcal{A}} \mathcal{A}(a, a) \right)^{\otimes q}.$$

We refer to [SW19] for details on the homotopy coends appearing here. In the present text, this is only needed to a very limited extent. It suffices to know that in degree zero the complex $\int_{\mathbb{L}}^{a \in \mathcal{A}} \mathcal{A}(a, a)$ is given by $\bigoplus_{a \in \mathcal{A}} \mathcal{A}(a, a)$.

REMARK 4.3. In [Cos07], it is actually required that k has characteristic zero, but an extension to fields of arbitrary characteristic is given in [ES15, WW16].

From Corollary 2.3 and Costello’s result, we immediately obtain:

COROLLARY 4.4. For a finite category \mathcal{C} , any trivialization of the Nakayama functor $\mathbf{N}^f : \mathcal{C} \rightarrow \mathcal{C}$ yields a Calabi-Yau structure on $\text{Proj } \mathcal{C}$ and hence gives rise to a topological conformal field theory $\Phi_{\mathcal{C}} : \text{OC} \rightarrow \text{Ch}_k$ with set of D-branes given by the set of projective objects of \mathcal{C} .

REMARK 4.5. The evaluation of the field theory $\Phi_{\mathcal{C}}$ on the disk with one incoming open boundary interval whose complementing free boundary carries the D-brane label $P \in \text{Proj } \mathcal{C}$

$$\Phi_{\mathcal{C}} \left(\text{in} \begin{array}{c} \text{---} \\ \downarrow \\ \text{---} \end{array} \bigcirc \begin{array}{c} \text{---} \\ \uparrow \\ \text{---} \end{array} P \right) : \mathcal{C}(P, P) \rightarrow k$$

is exactly the trace function of the Calabi-Yau structure from Definition 2.4 (this follows directly from Costello’s construction), while for $P, Q \in \text{Proj } \mathcal{C}$, the map

$$\Phi_{\mathcal{C}} \left(\begin{array}{c} \text{---} \\ \text{in} \\ \downarrow \\ \text{---} \\ \text{in} \\ \downarrow \\ \text{---} \\ \text{out} \\ \uparrow \\ \text{---} \\ \text{---} \end{array} \bigcirc \begin{array}{c} P \\ Q \\ P \end{array} \right) : \mathcal{C}(P, Q) \otimes \mathcal{C}(Q, P) \rightarrow \mathcal{C}(P, P)$$

is the composition over Q .

The construction of Corollary 4.4 does not do much: It just translates a trivialization of \mathbf{N}^f to a Calabi-Yau structure and then a topological conformal field theory, and in fact, this construction will only be of limited use to us since we want to treat finite tensor categories and not just linear categories. Fortunately, we can give a natural refinement: The construction from Corollary 4.4 becomes more meaningful in the context of finite tensor categories if \mathbf{N}^f is trivialized not just as a linear functor, but as a right \mathcal{C} -module functor relative to a pivotal structure. Let us define what we mean by that:

DEFINITION 4.6. For any finite tensor category \mathcal{C} and a pivotal structure $\omega : -^{\vee\vee} \cong \text{id}_{\mathcal{C}}$, denote by $(\text{id}_{\mathcal{C}}, \omega)$ the identity functor endowed with the structure of a right \mathcal{C} -module functor $\mathcal{C}_{\mathcal{C}} \rightarrow \mathcal{C}^{\vee\vee}$ by means of ω . We refer to an isomorphism $\mathbf{N}^r \cong (\text{id}_{\mathcal{C}}, \omega)$ of right \mathcal{C} -module functors as a trivialization of \mathbf{N}^r as a right \mathcal{C} -module functor relative to ω . We define a *symmetric Frobenius structure* on a finite tensor category \mathcal{C} as a trivialization of \mathbf{N}^r as right \mathcal{C} -module functor relative to a pivotal structure, where the pivotal structure is part of the data.

REMARK 4.7. Thanks to Theorem 3.1, a symmetric Frobenius structure on a finite tensor category is a pivotal structure plus a trivialization of D . We use the term symmetric Frobenius structure not only as convenient shorthand for the rather clumsy description of ‘pivotal unimodular finite tensor category with a trivialization of the distinguished invertible object as part of the data’, but also for a deeper reason: The symmetric Frobenius algebra structure on a finite tensor category allows us to write \mathcal{C} , as a linear category, as modules over a symmetric Frobenius algebra. This can be read off from the Corollary 3.2 because it provides canonical natural isomorphisms

$$\mathcal{C}(X, Y_{\bullet})^* \cong \mathcal{C}(Y_{\bullet}, X),$$

where $X \in \mathcal{C}$ and Y_{\bullet} is a projective resolution of $Y \in \mathcal{C}$. However, a finite tensor category with symmetric Frobenius structure requires a compatibility of the monoidal structure with the trivialization of \mathbf{N}^r . It is not just an identification of \mathcal{C} , as *linear category*, with modules over a symmetric Frobenius algebra. In the latter sense, the notion is used in [Shi20].

DEFINITION 4.8. Let \mathcal{C} be a finite tensor category with symmetric Frobenius structure. For the trivialization of the right Nakayama functor \mathbf{N}^r (that \mathcal{C} by Definition 4.6 comes equipped with), we refer to the topological conformal field theory $\Phi_{\mathcal{C}} : \text{OC} \rightarrow \text{Ch}_k$ built from *this particular* trivialization in the sense of Corollary 4.4 as the *trace field theory of \mathcal{C}* .

The name is chosen because $\Phi_{\mathcal{C}}$ does not only recover the trace of the Calabi-Yau structure by Remark 4.5, but can also be recovered from the trace itself.

THEOREM 4.9. *Let \mathcal{C} be a finite tensor category with symmetric Frobenius structure and $\Phi_{\mathcal{C}} : \text{OC} \rightarrow \text{Ch}_k$ its trace field theory. The evaluation of $\Phi_{\mathcal{C}}$ on the disk with one incoming open boundary interval whose complementing free boundary carries the label $P \in \text{Proj } \mathcal{C}$*

$$\Phi_{\mathcal{C}} \left(\text{in} \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} P \right) : \mathcal{C}(P, P) \rightarrow k$$

is a right modified trace, while the evaluation of $\Phi_{\mathcal{C}}$ on the cylinder with one incoming open boundary interval with complementing free boundary label $P \in \text{Proj } \mathcal{C}$ and one outgoing closed boundary circle

$$\Phi_{\mathcal{C}} \left(\text{in} \begin{array}{c} \text{out} \\ \text{in} \end{array} P \right) : \mathcal{C}(P, P) \rightarrow \int_{\mathbb{L}}^{P \in \text{Proj } \mathcal{C}} \mathcal{C}(P, P) \quad (4.1)$$

agrees, after taking zeroth homology, with the Hattori-Stallings trace of \mathcal{C} .

Proof. The statement about the modified trace can be extracted from Theorem 3.6 and Remark 4.5.

For the second statement, we first observe that by the construction of $\Phi_{\mathcal{C}}$ the map (4.1) is just the inclusion of $\mathcal{C}(P, P)$ into the direct sum $\bigoplus_{P \in \text{Proj } \mathcal{C}} \mathcal{C}(P, P)$, which is the degree zero term of the Hochschild complex. After taking homology, we get the natural map $\mathcal{C}(P, P) \rightarrow HH_0(\mathcal{C})$, i.e. the quotient map projecting to zeroth Hochschild homology, and hence the Hattori-Stallings trace for \mathcal{C} . The connection to the traditional Hattori-Stallings trace [Hat65, Sta65] uses that by writing \mathcal{C} , as a linear category, as finite-dimensional modules over a finite-dimensional algebra A (which we can always do), the zeroth homology $HH_0(\mathcal{C})$ is isomorphic to the zeroth Hochschild homology $HH_0(A) = A/[A, A]$ of A . This is a consequence of the Agreement Principle of McCarthy [MCar94] and Keller [Kel99], see also [SW19, Section 3.2] for this principle in the context of finite tensor categories. \square

We formulate the result in Theorem 4.9 topologically (although the Theorem 3.6 that it relies on is purely algebraic) because, instead of traces, we will in Section 5 use the trace field theory as an efficient tool for computations.

REMARK 4.10. The reader should appreciate that the trace field $\Phi_{\mathcal{C}}$ is defined through a specific trivialization of the Nakayama functor and not by choosing a modified trace (although Theorem 4.9 tells us that we could have done that). This has the advantage that, through the closed formula for \mathbf{N}^r , the trace field theory $\Phi_{\mathcal{C}}$ becomes very accessible. In fact, we will rely on the particular definition of $\Phi_{\mathcal{C}}$ given above in future work.

5 The block diagonal product on Hochschild chains

We will now see how we can profit from the topological description of traces. First we define a multiplication by evaluation on the pair of pants:

DEFINITION 5.1. Let \mathcal{C} be a finite tensor category with symmetric Frobenius structure and $\Phi_{\mathcal{C}} : \text{OC} \rightarrow \text{Ch}_k$ the associated trace field theory. Then we define the *block diagonal \star -product* on the Hochschild complex $\int_{\mathbb{L}}^{X \in \text{Proj } \mathcal{C}} \mathcal{C}(X, X)$ by

$$\star := \Phi_{\mathcal{C}} \left(\begin{array}{c} \text{in} \\ \text{in} \end{array} \left. \begin{array}{c} \text{out} \end{array} \right) : \int_{\mathbb{L}}^{X \in \text{Proj } \mathcal{C}} \mathcal{C}(X, X) \otimes \int_{\mathbb{L}}^{X \in \text{Proj } \mathcal{C}} \mathcal{C}(X, X) \longrightarrow \int_{\mathbb{L}}^{X \in \text{Proj } \mathcal{C}} \mathcal{C}(X, X).$$

The sense in which the product \star is block diagonal will be discussed in Proposition 5.3.

The results of Wahl and Westerland on the product obtained from a topological conformal field theory in [WW16, Section 6] imply that, up to homotopy, the multiplication is concentrated in degree zero (they prove it for symmetric Frobenius algebras, but their proof carries over to our situation). They also give a formula for the degree zero part of the homotopy commutative multiplication. We will below give a slightly different formula which, when working with a Calabi-Yau category instead of a symmetric Frobenius algebra, is a little more convenient.

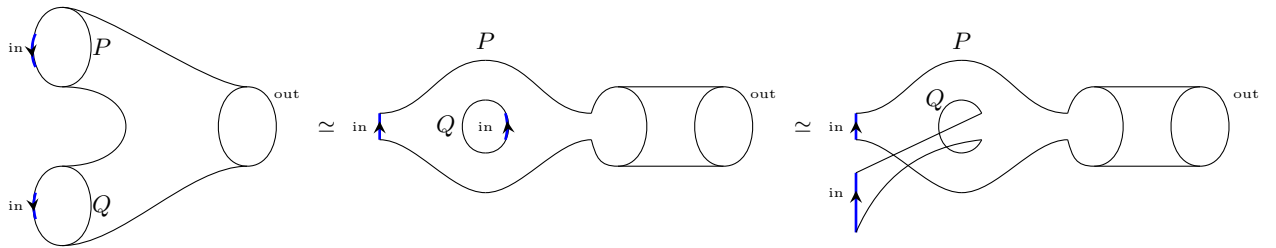
LEMMA 5.2. *The degree zero part*

$$\star : \bigoplus_{P, Q \in \text{Proj } \mathcal{C}} \mathcal{C}(P, P) \otimes \mathcal{C}(Q, Q) \longrightarrow \bigoplus_{P \in \text{Proj } \mathcal{C}} \mathcal{C}(P, P)$$

of the product from Definition 5.1 is given on the summand $\mathcal{C}(P, P) \otimes \mathcal{C}(Q, Q)$, up to boundary, by the linear map

$$\Phi_{\mathcal{C}} \left(\begin{array}{c} \text{in} \\ \text{in} \end{array} \left. \begin{array}{c} \text{out} \end{array} \right) : \mathcal{C}(P, P) \otimes \mathcal{C}(Q, Q) \longrightarrow \mathcal{C}(P, P). \quad (5.1)$$

Proof. Let P and Q be projective objects in \mathcal{C} . The following morphisms in OC can be deformed into each other and hence represent homologous zero chains:



THEOREM 5.6. Let \mathcal{C} be a finite tensor category with symmetric Frobenius structure.

- (i) For $P, Q \in \text{Proj } \mathcal{C}$, the \star -product of id_P and id_Q is the handle element $\xi_{P,Q}$ of P and Q , up to boundary in the Hochschild complex of \mathcal{C} ;

$$\text{id}_P \star \text{id}_Q \simeq \xi_{P,Q} .$$

- (ii) All handle elements in the sense of Definition 5.4 are central in the endomorphism algebras of \mathcal{C} .
(iii) The modified trace of the handle element is given by

$$\text{t}_P \xi_{P,Q} = \dim \mathcal{C}(P, Q) . \quad (5.2)$$

Of course, the numbers $\dim \mathcal{C}(P, Q)$ on the right hand side of (5.2) are the entries of the Cartan matrix of \mathcal{C} , considered here as elements in k .

If P is simple, the handle element is the number

$$\xi_{P,Q} = \frac{\dim \mathcal{C}(P, Q)}{d^{\text{m}}(P)} \in k ,$$

where $d^{\text{m}}(P) := \text{t}_P(\text{id}_P) \in k^\times$ is the modified dimension of P (note that $\text{t}(\text{id}_P) \neq 0$ is a consequence of the non-degeneracy of the trace).

Proof of Theorem 5.6. In order to compute $\text{id}_P \star \text{id}_Q$ for $P, Q \in \text{Proj } \mathcal{C}$, we use Lemma 5.2 and the functoriality of $\Phi_{\mathcal{C}}$:

$$\text{id}_P \star \text{id}_Q \simeq \Phi_{\mathcal{C}} \left(\text{in} \left(\begin{array}{c} P \\ \text{---} \\ Q \\ \text{---} \\ P \end{array} \right) \right) \circ \Phi_{\mathcal{C}} \left(\begin{array}{c} P \text{---} \text{out} \\ Q \text{---} \text{out} \end{array} \right) = \Phi_{\mathcal{C}} \left(\begin{array}{c} P \\ \text{---} \\ Q \end{array} \right) \text{---} \text{out} = \xi_{P,Q}$$

This proves (i).

For the proof of (ii), recall that in the setting of symmetric Frobenius algebras, it is shown in [Koc03, page 128] that the handle element is central. A straightforward computation by means of the trace field theory $\Phi_{\mathcal{C}}$ shows that this holds still true in our more general situation: In fact, one can directly see that both the map $\xi_{P,Q} \circ - : \mathcal{C}(P, P) \rightarrow \mathcal{C}(P, P)$ and the map $- \circ \xi_{P,Q} : \mathcal{C}(P, P) \rightarrow \mathcal{C}(P, P)$ are given by

$$\Phi_{\mathcal{C}} \left(\begin{array}{c} \text{out} \\ \text{---} \\ P \\ \text{---} \\ Q \\ \text{---} \\ \text{in} \end{array} \right)$$

in terms of the trace field theory.

For the proof of (iii), first observe

$$\Phi_{\mathcal{C}} \left(\begin{array}{c} \text{in} \\ \text{---} \\ P \end{array} \right) \circ \Phi_{\mathcal{C}} \left(\begin{array}{c} P \\ \text{---} \\ Q \end{array} \right) \text{---} \text{out} = \Phi_{\mathcal{C}} \left(\begin{array}{c} P \\ \text{---} \\ Q \end{array} \right)$$

(This is the generalization of the fact that the counit evaluated on the handle element on a symmetric Frobenius algebra is the linear dimension [Koc03, page 129] to Calabi-Yau categories.) Now we use Theorem 4.9 which asserts that the evaluation of $\Phi_{\mathcal{C}}$ on the disk with one incoming open boundary interval is actually the modified trace. This proves (5.2). \square

Let us also formulate this on the level of homology and denote for an endomorphism $f : P \rightarrow P$ of $P \in \text{Proj } \mathcal{C}$ by $\text{HS}(f) \in HH_0(\mathcal{C})$ the Hattori-Stallings trace. Then Theorem 4.9 and Theorem 5.6 imply:

COROLLARY 5.7. *For any finite tensor category \mathcal{C} with symmetric Frobenius structure,*

$$\mathfrak{t}(\text{HS}(\text{id}_P) \star \text{HS}(\text{id}_Q)) = \dim \mathcal{C}(P, Q) \quad \text{for } P, Q \in \text{Proj } \mathcal{C} .$$

Here, by slight abuse of notation, we denote the map on $HH_0(\mathcal{C})$ induced by the modified trace again by \mathfrak{t} .

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