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CONTENTS

1. INTRODUCTION

Mayer expansions in Euclidean quantum field theory on the lattice lead to convergent expansions and to the existence of the thermodynamical limit of the generating functional for connected amputated Greens functions for sufficiently weak coupling. It is essential for convergence that the mass m in units of inverse lattice spacings a^{-1} is nonzero. The region of convergence for the coupling constant shrinks to zero if m or a goes to zero. Moreover, terms of the expansion may become infrared or ultraviolet divergent. For handling these problems methods of the renormalization group are necessary. We shall only regard massive models on the lattice in this paper. However the results of this paper are useful for single renormalization group steps. The convergence condition of Gruber and Kunz [4] furnishes a condition for the existence of the thermodynamical limit and for the convergence of the Mayer expansion of the generating function for connected free-propagator-amputated Greens functions if the external field ψ is in a (coupling constant and ma-dependent) bounded complex neighborhood of $\psi = 0$. Moreover, the thermodynamical limit of the connected free-propagator-amputated Greens functions exists. Using the tree estimate with extra factors n! of Battle [14] the proof of convergence will be easy for simple Mayer expansions. We will show that this estimate is in fact an immediate corollary of the tree estimate used and derived by M. Göpfert and G. Mack [8]. The extra factors n! permit to absorb the factors n! due to the Cauchy formula for the n-th derivative of a holomorph function. It will be shown that the condition of convergence is fulfilled for the $\lambda\phi^4$–theory, the discrete Gaussian model and the nonlinear σ–model in a (coupling constant and ma–dependent) real neighborhood of $\psi = 0$ for sufficiently small coupling constants.

The activity of a polymer equals the sum of all "point connected" Feynman diagrams whose vertex positions occupy all points of the polymer. A Feynman diagram with given positions of its vertices is called *point connected* if it is connected or becomes connected after all vertices that are positioned at the same points of space are identified. It will be shown how to express Mayer amplitudes by Feynman amplitudes. For $\lambda\phi^4$–theory it will be shown that the perturbation expansion of the activities is Borel summable in λ (on the lattice).

For renormalization and Mayer expansion it will be useful to introduce counterterms which are dependent on subsets of the lattice. The Mayer expansion for the Boltzmannian factor with X-dependent counterterms will be done and it will be shown that the molecular activities are of order $\lambda^{|X|}$ ($|X|$ = number of points in X) if some renormalization conditions are fulfilled.

After splitting the propagator into pieces of increasing range and decreasing strength one gets effective actions in the sense of Wilson's renormalization group approach [21]. We will derive a tree formula for activities corresponding to the iterated Mayer expansion ([8], [10]). The asymptotic expansion in λ of this formula is the Gallavotti Nicoló tree formula of the effective action [11]. The effective action is nonlocal. Appendix B presents a decoupling expansion for nonlocal interactions (corresponding to the tree graph formula for local interactions). It is a modified version of the expansion derived by Brydges [22].

1.1. MAYER-AND FEYNMAN DIAGRAM EXPANSION

J.E. Mayer [1] introduced the method of Mayer expansions for statistical mechanics in the fourties. Mayer considered real gases and their condensation. The essential trick of Mayer in treating the partition function of real gases is to introduce the factor

$$f(r) = e^{-\beta v(r)} - 1 \tag{1.1}$$

instead of the Boltzmannian factor $e^{-\beta v(r)}$, where β is inversely proportional to the temperature and v is the pair potential of the molecules. n molecules form a cluster. The partition function is a sum of products of cluster integrals. The cluster integral is n-dimensional if n is the number of particles in the cluster. Particles of a cluster are connected by bonds such that the resulting graph is connected and two vertices of this graph are connected by only one line (Mayer graph). The integrand of the cluster integral consists of a sum over all Mayer graphs.

Likewise in the fourties, Schwinger introduced the generating functional for Greens functions and their perturbative treatment for quantum field theory. The ***generating functional for Greens functions*** in ν-dimensional Minkowski space is defined by

$$T[J] = \frac{1}{\mathcal{N}} \int \cdots \int [\prod_{x \in \mathbb{R}^\nu} d\phi(x)] \exp\{i \int_{x\in\mathbb{R}^\nu} d^\nu x\ \mathcal{L}(\phi(x)) + \int_{x\in\mathbb{R}^\nu} d^\nu x\ J(x)\phi(x)\} \tag{1.2}$$

where $\mathcal{N}$ is a normalization constant such that $T[0] = 1$. The Lagrange density $\mathcal{L}$ consists of a free part $\mathcal{L}_F$ and an interaction part (perturbative term) $\lambda\mathcal{L}_I$

$$\mathcal{L} = \mathcal{L}_F + \lambda\mathcal{L}_I. \tag{1.3}$$

λ labels the coupling constant of the model. According to R.P.Feynman [2] terms of the perturbation expansion are represented by graphs. These graphs are called Feynman diagrams. The "interaction Boltzmannian"

$$e^{i\lambda \int d^\nu x\ \mathcal{L}_I(\phi(x))}$$

has to be developped for the perturbation expansion. It is essential for the representation in Feynman diagrams that the free term $\mathcal{L}_F$ of the interaction is quadratically in ϕ, i.e.

$$\int d^\nu x\ \mathcal{L}_F(\phi(x)) = \frac{1}{2}\int d^\nu x\ d^\nu y\ \phi(x)K(x,y)\phi(y). \tag{1.4}$$

$K(x,y)$ is the integral kernel of an invertible positive operator K. The inverse operator $v = K^{-1}$ is called free propagator of the model. The perturbation expansion is a formal power series in the coupling constant λ. All terms of the perturbation expansion of order λ^n are represented by Feynman diagrams with n vertices. The line (xy) connecting the points x and y in the Feynman diagram corresponds to the propagator $v(x,y)$.

The two above described expansions for statistical mechanics resp. for the quantum field theory were independent methods that were succesfully applied to different problems. After Wick rotation ($t \to it$) the quantum field theory on Minkowski space will turn into the Euclidean quantum field theory. For Euclidean quantum field theory it is possible to apply methods of statistical mechanics. In the mid-sixties K.Symanzik [3] introduced the method of Mayer expansion for Euclidean quantum field theory (in the form of iterative solutions of Kirkwood Salsburg equations). In this connexion the partition function in statistical mechanics corresponds to the generating functional for Greens functions in Euclidean quantum field theory. The Mayer expansion in statistical mechanics is an expansion in the number of particles and corresponds to an expansion in the number of points in Euclidean quantum field theory. The generating functional for connected Greens functions $\ln T[J]$ corresponds to the free energy $\ln Z$ in statistical mechanics. Terms in the perturbation expansion for Greens functions can be ultraviolet divergent. This ultraviolet divergence arises from non integrable singularities (not well defined products of distributions) of the integrand in the Feynman integral. To circumvent this problem only quantum field theories on the lattice $(a\mathbb{Z})^\nu$ will be considered here. In the following section we will introduce some notations and definitions for the lattice.

1.2. LATTICE NOTATIONS AND DEFINITIONS

Consider a ν-dimensional cubic lattice $(a\mathbb{Z})^\nu$ with lattice spacing a. Differentiation and integration on the lattice are defined as follows

$$\nabla_\mu f(x) = a^{-1}[f(x+e_\mu) - f(x)], \qquad \mu \in \{-\nu, ..., \nu\} \tag{1.5a}$$

$$\int_{x\in(a\mathbb{Z})^\nu} = a^\nu \sum_{x\in(a\mathbb{Z})^\nu}, \qquad e_{-\mu} \equiv e_\mu. \tag{1.5b}$$

Here e_μ is a vector of length a in μ-direction. The negative Laplacian operator is

$$-\Delta = \sum_{\mu=1}^{\nu} \nabla_{-\mu}\nabla_{\mu}. \tag{1.6}$$

Operating with the Laplacian on a function $f : (a\mathbb{Z})^\nu \to \mathbb{C}$ gives

$$\Delta f = a^{-2} \sum_{\substack{y \\ y\ nn\ x}} [f(y) - f(x)] \tag{1.7}$$

where the sum is over all nearest neighbors of $x \in (a\mathbb{Z})^\nu$. The scalar product of two functions f, g on the lattice is defined by

$$(f, g) = \int_{x\in(a\mathbb{Z})^\nu} f(x)g(x) \tag{1.8}$$

Summation by parts

$$(\nabla_\mu f, g) = (f, \nabla_{-\mu} g) \tag{1.9}$$

shows that

$$(f, -\Delta f) = \sum_{\mu=1}^{\nu}(\nabla_\mu f, \nabla_\mu f) = \sum_{(xy)}[f(x) - f(y)]^2 a^{-2}. \tag{1.10}$$

The sum $\sum_{(xy)}$ is over all links (xy) on the lattice. Because of (1.10) the operator $-\Delta$ is positive. If we replace differentiation and integration in the Lagrange density on the continuum by the above defined differentiation and integration on the lattice we get the lattice approximation of the Lagrange density. The Dirac distribution $\delta(x-y)$ corresponds to $a^{-\nu}\delta_{xy}$ on the lattice, where

$$\delta_{xy} = \begin{cases} 1, & \text{if x=y} \\ 0, & \text{otherwise} \end{cases} \tag{1.11}$$

is the Kronecker symbol. The functional derivative $\dfrac{\delta}{\delta\psi(x)}$ becomes the ordinary derivative $a^{-\nu}\dfrac{\partial}{\partial\psi(x)}$ on the lattice. Dimensionsless variables are introduced by

$$\phi_x = a^d\phi(x), \qquad v_{xy} = a^{2d}v(x,y), \qquad j_x = a^{\nu-d}J(x), \tag{1.12}$$

where $d = \frac{1}{2}(\nu - 2)$.

The (normalized) *Gaussian measure* $d\mu_v(\phi)$ is defined by its Fourier transform which is given by the following *Gaussian integral*

$$\int d\mu_v(\phi)e^{i(q,\phi)} = e^{-\frac{1}{2}(q,vq)} \tag{1.13}$$

for a positive semidefinite operator v. For positive definite operators we obtain

$$d\mu_v(\phi) = \det(2\pi v)^{-\frac{1}{2}} \prod_{x\in(a\mathbb{Z})^\nu} d\phi(x)\, e^{-\frac{1}{2}(\phi, v^{-1}\phi)}. \tag{1.14}$$

The field ϕ with Gaussian measure $d\mu_v(\phi)$ may be interpreted as Gaussian distributed random variable. In this propability theoretic interpretation ϕ is called process of covariance v. The moments of the Gaussian distribution are simple to calculate using the defining relation (1.13).

Expectation values of observables $O(\phi)$ with respect to the Gaussian distribution are defined by

$$\langle O\rangle = \int d\mu_v(\phi)O(\phi) \tag{1.15}$$

The *support* of an observable $O(\phi)$ is defined by

$$supp\ \ O = \{x \in (a\mathbb{Z})^\nu |\ O \text{ depends on } \phi_x\}. \tag{1.16}$$

Suppose that *supp* O is finite and the Fourier transform $\widetilde{O}(\phi)$ defined by

$$O(\phi) = \int [\prod_{x \in supp\ O} dq_x]\ \widetilde{O}(q) e^{i \sum_{x \in supp\ O} q_x \phi_x} \tag{1.17}$$

exists. Then $\widetilde{O}(q)$ depends only on q_x, $x \in supp\ O$, and the expectation value of $O(\phi)$ is an n-dimensional integral ($n = |supp\ O|$)

$$\langle O(\phi) \rangle = \int [\prod_{x \in \Lambda} dq_x] e^{-\frac{1}{2} \sum_{x,y \in \Lambda} q_x v_{xy} q_y} \widetilde{O}(q), \qquad \Lambda = supp\ O. \tag{1.18}$$

This follows easily from the defining relation (1.13). With the characteristic function

$$\chi_\Lambda(x) = \begin{cases} 1 & \text{if } x \in \Lambda \\ 0 & \text{otherwise} \end{cases} \tag{1.19}$$

$\Lambda \subseteq (a\mathbb{Z})^\nu$ and the abbreviation

$$v_\Lambda = \chi_\Lambda v \chi_\Lambda \tag{1.20}$$

follows that

$$\langle O \rangle = \int d\mu_{v_{supp\ O}}(\phi) O(\phi). \tag{1.21}$$

We see that the propagator can be restricted to the support of O.

1.3. Euclidean Quantum Field Theory on the Lattice and Statistical Mechanics

After Wick rotation and with the lattice notations of section 1.2. the generating functional of Greens functions on Minkowski space will be replaced by the generating function for Euclidean Greens functions on the lattice

$$T[J] = \frac{1}{\mathcal{N}} \int d\mu_v(\phi) F(\phi) e^{(J,\phi)}, \tag{1.22}$$

where $\mathcal{N}$ is fixed by the condition $T[0] = 1$. The Gaussian measure depends on the free part and the function $F(\phi)$ depends on the interaction part of the Lagrange density. Definitions for Greens functions are given in the following. Euclidean Greens functions are defined by

$$G(x_1, \ldots, x_n) = \frac{\delta^n}{\delta J(x_1) \ldots \delta J(x_n)} T[J]|_{J=0}. \tag{1.23a}$$

The connected Euclidean Greens functions are defined by

$$G_c(x_1, \ldots, x_n) = \frac{\delta^n}{\delta J(x_1) \ldots \delta J(x_n)} \ln\ T[J]|_{J=0}. \tag{1.23b}$$

The *connected free-propagator-amputated Euclidean Greens functions* are defined by

$$G_c(\underline{x}_1, \ldots, \underline{x}_n) = \int_{y_1, \ldots, y_n} v^{-1}(x_1, y_1) \ldots v^{-1}(x_n, y_n) \frac{\delta^n}{\delta J(y_1) \ldots \delta J(y_n)} \ln\ T[J]|_{J=0}. \tag{1.23c}$$

It will be shown in appendix C that

$$G_c(x_1, \ldots, x_n) = \frac{\delta^n}{\delta \psi(x_1) \ldots \delta \psi(x_n)} \ln \left[\frac{Z(\psi)}{Z(\psi = 0)} \right] |_{\psi=0}, \tag{1.24a}$$

where

$$Z(\psi) = \int d\mu_v(\phi)\, F(\phi + \psi). \tag{1.24b}$$

Therefore *the generating functional for the free-propagator-amputated Euclidean Greens functions* is given by

$$\ln\ Z(\psi) - \ln\ Z(\psi = 0).$$

The connected free-propagator-amputated Euclidean Greens functions are not necessarily 1-particle irreducible.

In the following we will consider local interactions. For local interactions the function F in the generating function on the lattice has the following form

$$F(\phi) = \prod_{x\in(a\mathbb{Z})^\nu} F_x(\phi_x). \tag{1.25}$$

For finite $\Lambda \subset (a\mathbb{Z})^\nu$ let

$$Z(\Lambda|\psi) = \int d\mu_v(\phi) \prod_{x\in\Lambda} F_x(\phi_x + \psi_x), \tag{1.26}$$

i.e. the interaction is switched off outside Λ. By (1.13) and the definition of the Fourier transform $\widetilde{F}_x(q_x)$ of $F_x(\phi_x)$

$$F_x(\phi_x) = \int dq_x \widetilde{F}_x(q_x)\, e^{iq_x\phi_x} \tag{1.27}$$

we obtain

$$Z(\Lambda|\psi) = \int [\prod_{x\in\Lambda} dq_x\, e^{-\frac{1}{2}q_x v_{xx} q_x}\, e^{iq_x\psi_x}\, \widetilde{F}_x(q_x)] \prod_{(xy)} e^{-q_x v_{xy} q_y}. \tag{1.28}$$

The product is over all (unordered) pairs (xy) with $x \neq y,\ x, y \in \Lambda$. The (non-normalized) generating function for free-propagator-amputated Greens functions in the form (1.28) can be interpreted as a partition function for a generalized gas with pair potential $q_x v_{xy} q_y$, complex fugacity and continuous charge q (see section 2.1.). The Mayer expansion of the partition function $Z(\Lambda|\psi)$ is based on the following *polymer representation*

$$Z(\Lambda|\psi) = \sum_{\Lambda=\sum X} \prod_X A(X|\psi). \tag{1.29}$$

The sum is over all disjoint partitions of Λ. Finite non empty subsets of the lattice $(a\mathbb{Z})^\nu$ are called *polymers*. $A(X|\psi)$ is called the *activity* of the polymer X. For $|X| = 1$ the activity $A(X|\psi)$ is called *monomer activity*. The activity $A(X|\psi)$ is uniquely determined by $Z(Y|\psi)$ for all $Y \subset X$. This follows from

$$A(X|\psi) = \sum_{n\geq 1} \sum_{X=\sum_{i=1}^n Y_i} (-1)^{n-1}(n-1)! \prod_{i=1}^n Z(Y_i|\psi) \tag{1.30}$$

(proof see app. A). (1.30) is the inverse formula of (1.29). Conversely, the partition function $Z(X|\psi)$ is obviously uniquely determined by $A(Y|\psi)$ for all $Y \subset X$.

The partition functions

$$Z(X|\psi) = \sum_{X=\sum Y} \prod_Y A(Y|\psi) \tag{1.31}$$

are the iterative solutions of the *Kirkwood Salsburg equations*

$$Z(X|\psi) = \sum_{\substack{Y \\ x\in Y\subseteq X}} A(Y|\psi) Z(X - Y|\psi) \tag{1.32}$$

with arbitrary $x \in X$ and initial condition

$$Z(\emptyset|\psi) = 1. \tag{1.33}$$

The Kirkwood Salsburg equations in statistical mechanics correspond to the Schwinger Dyson equations in quantum field theory.

1.4. Tree graph formula, estimates for activities and convergence of Mayer expansion

The tree graph formula leads to estimates for activities $A(X|\psi)$, where $|X| \geq 2$ (see Theorem 2.5.1. or [5]). A tree graph with n vertices is defined by the following function

$$\eta : \{2, \dots, n\} \longrightarrow \{1, \dots, n-1\} , \ \eta(i) < i. \tag{1.34}$$

Fig. 2.2 shows a graphical representation of a tree graph. The number of tree graphs with n vertices is (n-1)! (proof see section 2.5., p.35). Labellings of polymers X are defined by bijective maps

$$\tilde{x} : \{1, \dots, n\} \longrightarrow X , \ \tilde{x}(i) = x_i. \tag{1.35}$$

Given $n-1$ real variables $s_i \in [0,1]$, $i \in \{1, \dots, n-1\}$ we will use the following abbreviation

$$f(\eta|s) = \prod_{a=2}^{n} [s_{a-2} s_{a-3} \dots s_{\eta(a)}]. \tag{1.36}$$

The interpolating covariance $v[s]$ is defined by

$$v[s]_{x_i x_j} = \begin{cases} s_i s_{i+1} \dots s_{j-1} v_{x_i x_j}, & \text{if } i < j \\ s_j s_{j+1} \dots s_{i-1} v_{x_i x_j}, & \text{if } i > j \\ v_{x_i x_i}, & \text{if } i = j. \end{cases} \tag{1.37}$$

We assume that $F_x(\phi_x)$ is C^∞ for all $x \in X$. Let $x \in X$ be an arbitrary point. Then the tree graph formula for the activity $A(X|\psi)$ reads

$$A(X|\psi) = \sum_{\eta} \sum_{\substack{\tilde{x} \\ \tilde{x}(1)=x \in X}} \int_0^1 ds_1 \dots ds_{n-1} \, f(\eta|s) \int d\mu_{v_x[s]}(\phi) \left\{ \prod_{a=2}^{n} [\frac{\partial}{\partial \phi_{x_a}} v_{x_a x_{\eta(a)}} \frac{\partial}{\partial \phi_{x_{\eta(a)}}}] \prod_{b=1}^{n} F_{x_b}(\phi_{x_b} + \psi_{x_b}) \right\}. \tag{1.38}$$

The derivative can be estimated by the Cauchy inequality. Let F be a holomorph function in $\{z \in \mathbb{C} \mid |z| \leq \kappa\}, \kappa > 0$. Then the n-th derivative of F is bounded by

$$|\frac{d^n}{dz^n} F(z)| \leq \frac{n!}{\kappa^n} \max_{|z|=\kappa} |F(z)|. \tag{1.39}$$

The faculties on the rhs of the Cauchy inequality can be dominated by using the following Lemma of Battle (see [14] or Lemma 3.1.5.).

$$\sum_{\eta} \int_0^1 ds_1 \dots ds_{n-1} \, f(\eta|s) \prod_{l=1}^{n} d_l(\eta)! \leq \frac{8^{n-1}}{2}. \tag{1.40}$$

Here $d_l(\eta)$ labels the number of links in the tree graph η having their origin in the vertex l. For fixed labelling $\tilde{x}$ the number of derivatives of F_{x_l} at ϕ_{x_l} in the tree graph formula (1.38) equals $d_l(\eta)$. The Lemma of Battle is an immediate corollary of the following tree estimate ([6], [8])

$$\sum_{\eta} \int_0^1 ds_1 \dots ds_{n-1} \, f(\eta|s) \prod_{l=2}^{n} [\mu(l) \mu(\eta(l))] \leq \prod_{l=2}^{n} [\mu(l) e^{\mu(l-1)}] \tag{1.41}$$

with $\mu(l) \geq 0,\ l \in \{1, \ldots, n\}$ (see Lemma 3.1.4.). For the $\lambda\phi^4$–theory one chooses $\kappa = O(\lambda^{-\frac{1}{4}})$. The Cauchy inequality leads then to a factor of order $\lambda^{\frac{1}{4}}$ for every derivative. Since in the tree graph formula there are $2(n-1)$ derivatives and $(n-1)!$ labellings with $\tilde{x}(1) = x,\ n \geq 2$, we obtain, using

$$\sum_{y \in (a\mathbb{Z})^\nu} v_{xy} = \frac{1}{(ma)^2} \tag{1.42}$$

the bounds

$$|A(X|\psi)| \leq (n-1)!\ O(\lambda^{\frac{n-1}{2}}) \tag{1.43}$$

and

$$\sum_{\substack{X,\ |X|=n \\ x \in X \subset (a\mathbb{Z})^\nu}} |A(X|\psi)| \leq O(\left[\frac{\lambda}{(ma)^4}\right]^{\frac{n-1}{2}}) \tag{1.43'}$$

for *all* real external fields ψ. For general estimates of activities we suppose that $F_x(\phi_x)$ is a holomorphic and bounded function in the complex strip $\{\phi_x \in \mathbb{C} \mid |Im\ \phi_x| \leq \kappa\}$ with $\kappa > 0$ for all $x \in (a\mathbb{Z})^\nu$ (see Theorem 3.2.1. and generalization to N-component theories see Theorem 3.2.2.). Notice that these estimates are independent of real external fields ψ.

We can get better estimates for bounded external fields ψ. Let us change the assumptions for F slightly. For all $x \in (a\mathbb{Z})^\nu$ let $F_x \in C^\infty$ and ϵ, c be constants with

$$\epsilon v \leq c < 1 \tag{1.44}$$

and ϵ-dependent constants $\mathcal{C}_\epsilon,\ h_\epsilon$ such that

$$|e^{-\frac{\epsilon}{4}\phi_x^2} \frac{\partial^d}{\partial \phi_x^d} F_x(\phi_x)| \leq (d-1)!\ \mathcal{C}_\epsilon^d h_\epsilon \tag{1.45}$$

is fulfilled. The Gaussian expectation values are estimated by

$$|\langle G(\phi+\psi)\rangle| \leq |\langle e^{\frac{\epsilon}{2}\sum_{x\in\Lambda}\phi_x^2}\rangle| \sup_{\substack{\phi_x \in \mathbb{R}^\nu \\ x \in \Lambda}} |G(\phi+\psi)\ e^{-\frac{\epsilon}{2}\sum_{x\in\Lambda}\phi_x^2}| \tag{1.46}$$

for finite $\Lambda = supp\ G$ (suppose $G(\phi+\psi) \geq 0$ or ≤ 0 for all ϕ, ψ). We obtain for the expectation value of the rhs

$$|\langle e^{\frac{\epsilon}{2}\sum_{x\in\Lambda}\phi_x^2}\rangle| \leq (1-c)^{-\frac{|\Lambda|}{2}} \tag{1.47}$$

(see proof of Theorem 3.4.1.). The estimates obtained by this method are represented in Theorem 3.4.1. for 1-component theories and in Theorem 3.4.2. for N-component theories. This method yields for $\lambda\phi^4$–theory without counterterms

$$|A(X|\psi)| \leq O(\lambda^n (ma)^{-2(n+1)}) e^{const(ma)^2 \sum_{x\in X}|\psi_x|^2} \tag{1.48}$$

for $\lambda \leq O((ma)^4)$, $|X| \geq 2$ and complex external fields ψ (see Corollary 3.4.5.).

Gruber and Kunz [4] have stated with the help of the Kirkwood Salsburg equations a sufficient convergence condition for the existence of the thermodynamical limit $\Lambda \nearrow (a\mathbb{Z})^\nu$ (in the sense of van Hove) of the reduced correlation functions

$$\rho_\Lambda(X|\psi) = Z(\Lambda - X|\psi)/Z(X|\psi). \tag{1.49}$$

The convergence condition of Gruber and Kunz is fulfilled, if for some $\xi > 1$

$$B(\xi,\psi) < 1, \tag{1.50a}$$

where

$$B(\xi,\psi) = \frac{1}{\xi}\left[1+|M(\{x\}|\psi)|\xi^2 + \sup_{x\in(a\mathbb{Z})^\nu}\left\{\sum_{n\geq 2}\frac{a^{-\nu(n-1)}}{(n-1)!}\int_{\substack{x_2,\ldots,x_n \in (a\mathbb{Z})^\nu \\ x,x_2,\ldots,x_n\ distinct}} |M(\{x,x_2,\ldots,x_n\}|\psi)|\xi^{2n}\right\}\right] \tag{1.50b}$$

and

$$M(X|\psi) = -\delta_{1,|X|} + A(X|\psi). \tag{1.50c}$$

For theories defined by the partition function

$$Z(\Lambda|\psi) = \int d\mu_v(\phi) \prod_{x\in\Lambda} F_x(\phi_x + \psi_x) \tag{1.51}$$

with holomorph and bounded functions F_x in the complex strip

$$S_\kappa = \{\phi_x \in \mathbb{C} \mid |Im\phi_x| \leq \kappa\}, \qquad \kappa > 0 \tag{1.52}$$

we obtain the following estimate for the terms in the series of (1.50b)

$$\frac{a^{\nu(n-1)}}{(n-1)!} \int_{x_2,\ldots,x_n \in (a\mathbb{Z})^\nu} |M(\{x, x_2, \ldots, x_n\}|\psi)| \leq const * (ma\kappa)^{-2(n-1)} b_\kappa^n \tag{1.53a}$$

with the abbreviation

$$b_\kappa = \min_{c\in\mathbb{R}^\nu} \sup_{x\in(a\mathbb{Z})^\nu} \sup_{\substack{\phi_x \in \mathbb{C} \\ |Im\phi_x|=\kappa}} |F_x(\phi_x) - c| \tag{1.53b}$$

(see Theorem 3.2.1.). Especially for the $\lambda\phi^4$–theory without counterterms

$$b_\kappa \leq O(1) \tag{1.54}$$

for $\kappa = O(\lambda^{-\frac{1}{4}})$ (cf. Lemma 3.2.3.). (1.53a) and (1.54) implies

$$\frac{a^{-\nu(n-1)}}{(n-1)!} \int_{x_2,\ldots,x_n \in (a\mathbb{Z})^\nu} |M(\{x, x_2, \ldots, x_n\}|\psi)| \leq O(\lambda^{\frac{n-1}{2}}) \tag{1.55}$$

for $n \geq 2$. Therefore the series in definition (1.48b) will be estimated by a geometrical series, which is small for small coupling constant. For theories, which fulfill

$$F_x(\phi_x + \psi_x) \to 0 \qquad \text{for } |\psi_x| \to 0 \tag{1.56}$$

for all $x \in (a\mathbb{Z})^\nu$, we get

$$|M(\{x\}|\psi)| = |\langle F_x(\phi_x + \psi_x) - 1\rangle| \to 1 \qquad \text{for } |\psi_x| \to \infty. \tag{1.57}$$

Obviously, the convergence condition of Gruber and Kunz is not fulfilled for large external fields (in the renormalization group context: "large field problem"). For bounded external fields we get

$$|M(X|\psi)| \to 0 \qquad \text{for } \lambda \to 0. \tag{1.58}$$

With (1.58) the convergence condition of Gruber and Kunz is fulfilled for small coupling constant and *bounded* external field ψ.

Suppose that the convergence condition of Gruber and Kunz is fulfilled and the support of the external field ψ is finite. It will be shown (in section 2.2.) that with these assumptions the thermodynamical limit $\Lambda \nearrow (a\mathbb{Z})^\nu$ (in the sense of van Hove) exists for connected free-propagator-amputated Greens functions $G_c(x_1, \ldots, x_n)$ and the generating function

$$\ln\ Z(\Lambda|\psi) - \ln\ Z(\Lambda|\psi = 0).$$

Moreover, the expansion

$$\lim_{\Lambda\nearrow(a\mathbb{Z})^\nu} \ln \left[\frac{Z(\Lambda|\psi)}{Z(\Lambda|\psi = 0)}\right] = \sum_Q a(Q)\ [M(Q|\psi) - M(Q|\psi = 0)] \tag{1.59}$$

is convergent in a small complex neighborhood of $\psi = 0$. For notations and definitions of the combinatorial coefficient a(Q) and cluster Q see section 2.2., p.21-22.

The ordinary perturbation expansion is not convergent in general. E.g. the perturbation expansion in λ of the partition function of the $\lambda\phi^4$–theory on a lattice $\Lambda = \{x\}$ that consists of a single point is only an asymptotic expansion

$$Z(\{x\}|\psi) = \left(\frac{2\pi}{m^2}\right)^{-\frac{1}{2}} \int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}m^2\phi^2-\lambda(\phi+\psi)^4} \asymp \sum_{n\geq 0} a_n(-\lambda)^n. \tag{1.60}$$

The series $\sum_{n\geq 0} a_n(-\lambda)^n$ is not convergent for $\lambda \neq 0$, because the integral is divergent for $\lambda < 0$ and therefore the convergence radius of the series is zero. In the same way the following perturbation expansion

$$\ln\left[\frac{Z(\Lambda|\psi)}{Z(\Lambda|\psi=0)}\right] = \sum_{n\geq 1} \frac{(-\lambda)^n}{n!} a^{-\nu n} \int_{x_1,\ldots,x_n\in\Lambda} [\langle \mathcal{V}(\phi_{x_1}+\psi_{x_1}); \ldots; \mathcal{V}(\phi_{x_n}+\psi_{x_n})\rangle - \langle \mathcal{V}(\phi_{x_1}); \ldots; \mathcal{V}(\phi_{x_n})\rangle] \tag{1.61}$$

with

$$Z(\Lambda|\psi) = \int d\mu_v(\phi) \prod_{x\in\Lambda} e^{-\lambda\mathcal{V}(\phi_x+\psi_x)} \tag{1.62}$$

is divergent. $\langle \ldots; \ldots\rangle$ denotes the truncated expectation value (for definition see app. A). Instead of the expansion (1.61) we use the convergent (for ψ in a small complex neighborhood of $\psi = 0$ and small λ) Mayer expansion (1.59) in the following form

$$\ln\left[\frac{Z(\Lambda|\psi)}{Z(\Lambda|\psi=0)}\right] = \sum_{n\geq 1} \frac{a^{-\nu n}}{n!} \int_{x_1,\ldots,x_n\in\Lambda} [\tilde{\mathcal{M}}(x_1,\ldots,x_n|\psi) - \tilde{\mathcal{M}}(x_1,\ldots,x_n|\psi=0)] \tag{1.63}$$

with the definition of the *augmented Mayer amplitude*

$$\tilde{\mathcal{M}}(x_1,\ldots,x_n|\psi) = \sum_{\substack{Q \\ supp\ Q=\{x_1,\ldots,x_n\}}} a(Q) \prod_{\substack{distinct \\ x\in\{x_1,\ldots,x_n\}}} n(x)! \prod_{P\in Q} [-\delta_{1,|P|} + A(P|\psi)]. \tag{1.64}$$

The clusters Q consist of points $x_1,\ldots,x_n$, where x appears in Q with multiplicity $n(x)$. Therefore

$$n(x) = |\{P \in Q|\ x \in P\}|. \tag{1.65}$$

From the polymer representation (1.29) we obtain the following expansion of the partition function in the number of points

$$Z(\Lambda|\psi) = 1 + \sum_{n=1}^{|\Lambda|} \sum_{\substack{n_i \\ \sum_{i=1}^k n_i = n}} \left[\frac{1}{\prod_{j=1}^n m_j(n_i)!}\right] \int_{\substack{y_1,\ldots,y_n\in\Lambda \\ distinct}} \mathcal{M}(y_1,\ldots,y_n|\psi)\mathcal{M}(y_{n_1+1},\ldots,y_{n_1+n_2}|\psi)\cdots \cdots\mathcal{M}(y_{n_1+\ldots n_{k-1}},\ldots,y_{n_k}|\psi), \tag{1.66}$$

where $m_j(\{n_i\}) = |\{r|n_r = j\}|$ and the *Mayer amplitudes* $\mathcal{M}$ are defined by

$$\mathcal{M}(x_1,\ldots,x_n|\psi) = \frac{a^{-\nu n}}{n!}[-\delta_{1,n} + A(\{x_1,\ldots,x_n\}|\psi)] \tag{1.67}$$

for n different points $x_1,\ldots,x_n \in (a\mathbb{Z})^\nu$. The *Feynman amplitude* is defined in terms of Gaussian expectation values by

$$\mathcal{F}(y_1,\ldots,y_n|\psi) = \frac{1}{n!}\langle\prod_{j=1}^n[-\lambda\mathcal{V}(\phi_{y_j}+\psi_{y_j});\ ]\rangle. \tag{1.68}$$

The Feynman amplitude is the sum of all connected Feynman diagrams with n vertices positioned on n distinct points of the lattice. Written as a truncated expectation value the Mayer amplitude reads

$$\mathcal{M}(x_1,\ldots,x_n|\psi) = \frac{a^{-\nu n}}{n!}\langle e^{-\lambda\mathcal{V}(\phi_{x_1}+\psi_{x_1})} - 1; \ldots; e^{-\lambda\mathcal{V}(\phi_{x_n}+\psi_{x_n})} - 1\rangle. \tag{1.69}$$

The Mayer amplitudes may be expressed in terms of Feynman amplitudes (Theorem 2.4.4.,p.33). From this representation we see that the Feynman diagrammatic expansion of the Mayer amplitude consists only of point connected Feynman diagrams. The essential difference between ordinary perturbation expansion and Mayer expansion is the maintainance of the stability condition (boundedness) for the ***interaction Boltzmannian factor***

$$\prod_{x\in(a\mathbb{Z})^\nu} e^{-\lambda \mathcal{V}(\phi_x+\psi_x)}$$

for $\lambda > 0$ in the Mayer expansion. The ordinary perturbation expansion is obtained by developping the e-function in the Boltzmannian factor. The terms in this expansion are not uniformly bounded in ϕ, and this leads to a divergent perturbation series (see example for the lattice with a single point,p.15). Mayer expansions leave the e-functions uneffected. The Mayer amplitudes (1.69) remain bounded for arbitrarily large external fields ψ (*stability*). On the other hand the Feynman amplitudes are not bounded for large external fields ψ.

The formal power series in λ of the Mayer amplitude

$$\mathcal{M}(X|\psi) = \sum_{n\geq|X|} c_n(\psi)\lambda^n, \tag{1.70}$$

where $c_n(\psi) = O(n!)$, is also divergent. One can write

$$\mathcal{M}(X|\psi) = \frac{1}{\lambda}\int_0^\infty B(t)e^{-t/\lambda}dt, \tag{1.71}$$

using the integral representation of the faculty

$$n! = \int_0^\infty t^n e^{-t}dt \tag{1.72}$$

and the definition of the *Borel transform*

$$B(t) = \sum_{n\geq|X|} \frac{c_n(\psi)}{n!}t^n. \tag{1.73}$$

If the series of the Borel transform is convergent the series in (1.70) is called Borel summable. We will show that for small coupling constants λ the perturbation expansion (1.70) for $\lambda\phi^4-$theory without counterterms on the lattice is Borel summable (see Theorem 4.1.4., p.60).

1.5. Renormalization and Mayer expansion; Renormalization group and iterated Mayer expansion

Estimates of the form (1.52) are useless for the continuum limit $a \to 0$. In particular, as in ordinary perturbation theory the problem of ultraviolet divergence appears. As a remedy counterterms are introduced in the action. In perturbation theory the counterterms are determined, so that some renormalization conditions are fulfilled and the resulting Feynman diagrams are finite for all orders in λ (renormalization). Theories, where this renormalization procedure is possible with a ***finite*** number of counterterms, are called ***renormalizable***. The ***degree of convergence*** C is defined by

$$C = 2I - \nu L, \tag{1.74}$$

where I is the number of internal lines and L is the number of loops in the Feynman diagram. The Feynman integral of the Feynman diagram is convergent for $C > 0$. The theory is called ***super renormalizable*** if the minimal degree of convergence of the subdiagrams increases with the number of vertices. E.g. the $\lambda\phi^4-$theory is super renormalizable for dimension $\nu \leq 3$, renormalizable for $\nu = 4$ and non renormalizable for $\nu \geq 5$. Two counterterms are sufficient for the $\lambda\phi^4$–theory in $\nu = 3$ dimensions and the partition function is of the following form

$$Z(\Lambda|\psi) = \int d\mu_v(\phi) e^{-\mathcal{V}(\phi+\psi)-\delta\mathcal{V}(\phi+\psi)}, \tag{1.75a}$$

where

$$\mathcal{V}(\phi) = \lambda \int_{x\in(a\mathbb{Z})^\nu} \phi(x)^4 \tag{1.75b}$$

$$\delta\mathcal{V}(\phi) = -\int_{x\in(a\mathbb{Z})^\nu} [\delta m^2 \phi(x)^2 + \delta e]. \tag{1.75c}$$

The coefficient δm^2 describes the mass renormalization and the coefficient δe describes the vacuum energy renormalization. Perturbation theory yields an expansion in λ for δm^2 and δe. For small lattice spacings (theory near to the continuum limit $a \to 0$) the coefficient δm^2 is positive. The mass counterterm must be dominated by the quartic interaction for maintainance of stability. Because of $\delta m^2 = O(\lambda)$ and $\delta e = O(\lambda)$, we get

$$-\lambda\phi_x^4 + \delta m^2 \phi_x^2 + \delta e \leq \frac{(\delta m^2)^2}{4\lambda} + \delta e \leq O(\lambda). \tag{1.76}$$

For interactions on a finite sublattice we obtain an upper bound for the renormalized action

$$\int_{x\in\Lambda} [-\lambda\phi_x^4 + \delta m^2\phi_x^2 + \delta e] \leq O(\lambda) * |\Lambda| \tag{1.77}$$

($|\Lambda|$ = number of points in Λ). To exploit maintainance of stability we apply the Mayer expansion instead of ordinary perturbation expansion for the partition function with renormalized action (cf. discussion of stability in section 1.4., p.16). For that purpose we introduce counterterms depending on finite subsets $X \subset (a\mathbb{Z})^\nu$. So we consider the partition functions

$$Z(X|\psi) = \int d\mu_v(\phi) [\prod_{x\in X} e^{-\lambda\mathcal{V}(\phi_x+\psi_x)}] e^{-\delta\mathcal{V}_X(\phi+\psi)} \tag{1.78a}$$

with

$$\delta\mathcal{V}_X(\phi) = -\sum_{\substack{P \\ \emptyset\neq P\subseteq X}} [\delta m^2(P) \sum_{x\in P} \phi_x^2 + \delta e(P)] \tag{1.78b}$$

for all finite $X \subset (a\mathbb{Z})^\nu$. For the interaction Boltzmannian factor we obtain the following polymer representation

$$[\prod_{x\in X} e^{-\lambda\mathcal{V}(\phi_x+\psi_x)}] e^{-\lambda\mathcal{V}_X(\phi+\psi)} = \sum_{X=\sum P} \prod_P B(P|\psi). \tag{1.79}$$

The functions $B(P|\psi)$ are called ***molecular activities***. Counterterms and molecular activities $B(P|\psi)$ are fixed by the renormalization conditions

$$\ln Z(X|\psi)|_{\psi=0} = 0 \tag{1.80a}$$

$$\frac{\partial^2}{\partial\psi^2} \ln Z(X|\psi)|_{\psi=0} = 0 \tag{1.80b}$$

for all finite $X \subset (a\mathbb{Z})^\nu$. In the renormalization condition (1.80b) the external field is supposed to be constant on the lattice. The number of renormalization conditions equals the number of counterterms. Renormalization conditions (1.80a,b) may be replaced by the following ones

$$A^{ren}(X|\psi)|_{\psi=0} = \begin{cases} 1 & \text{if } |X|=1 \\ 0 & \text{otherwise} \end{cases} \tag{1.81a}$$

$$\frac{\partial^2}{\partial\psi^2} A^{ren}(X|\psi)|_{\psi=0} = 0 \tag{1.81b}$$

with renormalized activities $A^{ren}(X|\psi)$ defined by

$$\langle \sum_{X=\sum P} \prod_P B(P|\psi)\rangle = \sum_{X=\sum Y} \prod_Y A^{ren}(Y|\psi). \tag{1.82}$$

This formulation of renormalization conditions is appropriate for theories, which are symmetrical under the transformation $\psi \to -\psi$, i.e.

$$Z(X|\psi) = Z(X|-\psi) \tag{1.83}$$

(proof see app. D). It will be shown that the molecular activities $B(P|\psi)$ are uniquely determined by the renormalization conditions (1.80a,b) or (1.81a,b) and are of order $\lambda^{|P|}$ (Theorem 5.1.1.). Therefore the order of the following renormalized activity

$$M^{ren}(X|\psi) = \sum_{X=\sum P} \langle \prod_P [B(P|\psi);\ ]\rangle \tag{1.84a}$$

with

$$M^{ren}(X|\psi) = -\delta_{1,|X|} + A^{ren}(X|\psi) \tag{1.84b}$$

is $\lambda^{|X|}$. In this way the existence of suitable counterterms, and the consistency of the renormalization procedure with X-dependent counterterms is shown.

To obtain estimates for activities we use the basic inequality

$$\langle F(\phi)\rangle \le \sup_\phi |F(\phi)|. \tag{1.85}$$

Suppose that the maximum of $|F(\phi)|$ is at $\phi = 0$. The Gaussian measure with mean value ϕ_0 may be used if the maximum is at $\phi_0 \neq 0$. (1.85) yields

$$\langle F(\phi)\rangle \le F(0). \tag{1.86}$$

The Gaussian measure with covariance $v = 0$ is the Dirac measure

$$d\mu_{v=0}(\phi) = \prod_x d\phi(x)\delta(\phi(x)). \tag{1.87}$$

So we see that the estimate (1.85) is suitable for small propagators v. For small (ma) the propagator $(-\Delta+m^2)^{-1}$ is large and the estimate will become poor. In particular, estimates based on inequality (1.85) are not sufficient to handle the continuum limit and/or massless theories. The same problem exists for the convergence of the Mayer expansion for Yukawa gases at low temperatures in statistical mechanics. The propagator corresponds in statistical mechanics to the product of β and a pair potential. Since β is inverse proportional to the temperature, this product will be large for low temperatures and estimates of the form (1.85) are unsatisfactory. A procedure for handling this problem in statistical mechanics is (for a large region of applications) the method of *iterated Mayer expansion* (see [8], [9], [10]). The corresponding method for euclidean quantum field theory is *renormalization group approach* [21]. For this method the propagator v will be split in N propagators $v,\ i = 1, \dots, N$

$$v = v^1 + \cdots + v^N. \tag{1.88}$$

For the propagators v^i the range decreases and the strength increases if the index i increases. By the *convolution formula of Gaussian measures* (see Lemma 3.1.2. for N=2)

$$\int d\mu_v(\phi)F(\phi) = \int d\mu_{v^1}(\phi^1)\dots d\mu_{v^N}(\phi^N)F(\phi^1 + \dots\phi^N) \tag{1.89}$$

the expectation value $\langle F(\phi)\rangle$ may be computed or estimated successively. For this the Gaussian measure $d\mu_{v^N}(\phi^N)$ will be computed resp. estimated first, then $d\mu_{v^{N-1}}(\phi^{N-1})$ etc. . Every integration over the Gaussian measure in this procedure will be called a *renormalization group step*. So we obtain for the partition function

$$Z(\Lambda|\psi) = \int d\mu_v(\phi)e^{-\mathcal{V}(\phi+\psi)} \tag{1.90}$$

after k renormalization group steps

$$Z(\Lambda|\psi) = \int d\mu_{v^1+\ldots v^{N-k}}(\phi^1 + \cdots + \phi^{N-k}) e^{-\mathcal{V}^{N-k}(\phi+\psi)} \tag{1.91}$$

with the *effective action*

$$\mathcal{V}^{N-k} = -\ln\langle e^{-\mathcal{V}}\rangle_{v^{N-k+1}+\cdots+v^N}. \tag{1.92}$$

We see that after k renormalization group steps the propagator is $v^1 + \ldots v^{N-k}$ and the action is $\mathcal{V}^{N-k}$. For *Pauli-Villars regularized propagators*

$$v = (-\Delta)^{-1} - (-\Delta + M^2)^{-1} \tag{1.93}$$

with *Pauli-Villars cutoff* M and the partition

$$v = v^1 + \cdots + v^N, \qquad v^i = (-\Delta + m_i^2)^{-1} - (-\Delta + m_{i+1}^2)^{-1}, \tag{1.94}$$

where $m_1 = 0 \leq m_2 \leq \cdots \leq m_{N-1} \leq m_N \leq m_{N+1} = M$, we obtain after k renormalization group steps the propagator

$$v^1 + \ldots v^{N-k} = (-\Delta)^{-1} - (-\Delta + m_{N-k+1}^2)^{-1}. \tag{1.95}$$

Thus the Pauli-Villars cutoff M is decreased to m_{N-k+1} after k renormalization group steps. A perturbative representation for the effective action is the *Gallavotti Nicoló tree formula* [11] (see Corollary 5.3.3.). Therein the effective action is the sum of tree graphs of depth k. The trees stand for truncated expectation values and the order in λ equals the number of maximal vertices (= degree of the tree). We obtain a tree formula for the activities (Corollary 5.3.4.) with the help of a partition formula for truncated expectation values (Lemma 5.3.1.). The trees of this formula correspond to partitions of partitions.... of partitions of polymers. The k-fold iterated partitions will be called k-cluster (cf. [8]). They correspond to the polymers in the simple Mayer expansion.

2. SIMPLE MAYER EXPANSION AND THEIR RELATION WITH PERTURBATION THEORY

We will consider here quantum field theories without derivative couplings on the ν-dimensional lattice $\Lambda \subset \Lambda_{tot} = (a\mathbb{Z})^\nu$. The generating function for free-propagator-amputated Greens functions is

$$Z(\Lambda|\psi) = \int d\mu_v(\phi) \prod_{x\in\Lambda} F_x(\phi_x + \psi_x) \tag{2.1}$$

(see app. C,p.94). F_x is a function or distribution. Examples for theories described by $Z(\Lambda|\psi)$ are

a) $\lambda\phi^4$–theory with counterterms:

ϕ is a real scalar field,

$F_x(\phi_x) = e^{-\lambda\mathcal{V}(\phi_x)}$, $\mathcal{V}(\phi_x) = \phi_x^4 - \delta m^2\phi_x^2 + e_0$

$v(x,y) =$ kernel of $(-\Delta + m^2)^{-1}$, $\lambda = \lambda_0 a^{\nu-4}$ dimensionsless coupling constant.

b) discrete Gaussian model:

ϕ is a real scalar field,

$F_x(\phi_x) = \sum_{n\in\mathbb{Z}} \delta(\phi_x - 2\pi n)$

$v = \beta v_{Cb}$, $v_{Cb} =$ kernel of $(-\triangle)^{-1}$.

c) nonlinear σ–model:

ϕ is an N-component real field,

$F_x(\phi_x) = \delta(\phi_x^2 - 1)$

$v = \frac{f_0}{N} v_{Cb}$, $v_{Cb} =$ kernel of $(-\triangle)^{-1}$.

2.1. Quantum field theory and polymer systems

We obtain from the Gaussian integral (1.13) and the definition of the Fourier transform $\tilde{F}_x(q_x)$ of $F_x(\phi_x)$

$$F_x(\phi_x) = \int dq_x \; \tilde{F}_x(q_x) e^{iq_x\phi_x} \tag{2.2}$$

the following relation

$$Z(\Lambda|\psi) = \int \left[\prod_{x\in\Lambda} dq_x \hat{F}(q_x)\right] e^{-\frac{1}{2}\sum_{x,y\in\Lambda} q_x v_{xy} q_y}, \tag{2.3}$$

where

$$\hat{F}(q_x) = \tilde{F}_x(q_x) e^{iq_x\psi_x}. \tag{2.4}$$

The representation (2.3) of the generating function $Z(\Lambda|\psi)$ is called *gas picture* (cf. [11]). The generating function $Z(\Lambda|\psi)$ may be interpreted as a partition function of a generalized gas, whose particles sit on lattice sites $x \in \Lambda$ and carry (not necessarily discrete) charge $q_x \neq 0$. Lattice sites are not occupied by particles if $q_x = 0$. The pair potential of the lattice gas is given by the propagator v. Per definitionem different particles sit on different sites. The charge dependent fugacity is $\hat{F}_x(q_x)$. The notions generating function and partition function are synonym in this context. With this interpretation of a quantum field theoretic model as a model of statistical mechanics the methods of statistical mechanics may be applied to problems in quantum field theory. A model on the lattice Λ described by the partition function $Z(\Lambda|\psi)$ may be considered as a polymer system, where the activities are derived by simple Mayer expansion (without use of the renormalization group).

Theorem 2.1.1. *The polymer representation of the partition function $Z(\Lambda|\psi)$ is*

$$Z(\Lambda|\psi) = \sum_{\Lambda=\sum X} \prod_X A(X|\psi) \tag{2.5}$$

with

$$A(X|\psi) = \sum_{G\in\mathcal{G}_X} \int \left[\prod_{x\in X} dq_x \tilde{F}_x(q_x) e^{iq_x\psi_x} e^{-\frac{1}{2}q_x v_{xx} q_x}\right] \prod_{(xy)\in G} [e^{-q_x v_{xy} q_y} - 1]. \tag{2.6}$$

$\tilde{F}_x(q_x)$ is the Fourier tranform of $F_x(\phi_x)$ (see Eq. (2.2)). The sum $\sum_{\Lambda=\sum X}$ is over all partitions of Λ into disjoint non empty subsets. $\mathcal{G}_X$ is the set of all connected graphs (Mayer graphs) with vertices in Λ and two vertices are linked by only one line.

Remark: The functions $A(X|\psi)$ may be interpreted as (not necessarily positive) *activities* of a *polymer system*. The *polymers* are non empty subsets of the lattice Λ. The activity for *monomers* (=polymers with only one constituent) $\{x\}$

$$A(\{x\}|\psi) = \int dq_x \; \tilde{F}_x(q_x) e^{iq_x\psi_x} e^{-\frac{1}{2}q_x v_{xx} q_x} \tag{2.7}$$

is called *monomer activity.*

PROOF: Splitting the representation (2.4) for the partition function $Z(\Lambda|\psi)$ in point and line dependent factors gives

$$Z(\Lambda|\psi)\int[\prod_{x\in\Lambda}\mathcal{D}q_x][\prod_{(xy)\in\Lambda^*}e^{-q_x v_{xy} q_y}], \tag{2.8}$$

where $\mathcal{D}q_x = dq_x\tilde{F}_x(q_x)e^{iq_x\psi_x}e^{-\frac{1}{2}q_x v_{xx} q_x}$ and Λ^* is the set of all unordered pairs (xy), $x, y \in \Lambda$. By the definition

$$e^{-q_x v_{xy} q_y} = 1 + f_{xy}(q) \tag{2.9}$$

we obtain

$$Z(\Lambda|\psi) = \sum_{\substack{B \\ B\subseteq\Lambda^*}}[\prod_{x\in B}\mathcal{D}q_x]\prod_{(xy)\in B}f_{xy}(q). \tag{2.10}$$

B is a disjoint union of Mayer graphs $G_i \in \mathcal{G}_{X_i}$, $\sum_i X_i = \Lambda$. The q-integrations factorizes and we obtain

$$Z(\Lambda|\psi) = \sum_{\Lambda=\sum X}\prod_X A(X|\psi) \tag{2.11}$$

with

$$A(X|\psi) = \sum_{G\in\mathcal{G}_X}[\prod_{x\in X}\mathcal{D}q_x]\prod_{(xy)\in G}f_{xy}(q). \qquad \surd \tag{2.12}$$

The partition functions for subsets $Y \subseteq \Lambda$ are defined by

$$Z(Y|\psi) = \int d\mu_v(\phi)\prod_{x\in Y}F_x(\phi_x+\psi_x) \tag{2.13}$$

and the polymer representation is

$$Z(Y|\psi) = \sum_{Y=\sum X}\prod_X A(X|\psi) \qquad \text{for all } Y,\ \emptyset \neq Y \subseteq \Lambda. \tag{2.14}$$

Empty products are 1. This yields $Z(\emptyset|\psi) = 1$ for the empty set $\emptyset$. The activities $A(X|\psi)$ are uniquely determined by $Z(Y|\psi)$, $\emptyset \neq Y \subseteq X$ (see section 1.3., p.11).

2.2. Expansion of the free energy $\ln Z(\Lambda|\psi)$ and the Gruber Kunz convergence condition

The free energy $\ln Z(\Lambda|\psi)$ may be represented by a sum of products of activities $A(X|\psi)$, $\emptyset \neq X \subseteq \Lambda$. For that we will define a *cluster* Q of polymers $P \subseteq \Lambda$ (cf. [13]). $Q = (P_1^{n_1}, \ldots, P_k^{n_k})$ is a collection of polymers P_i with multiplicities n_i. In the following we adjoin a graph $\gamma(Q)$ for each cluster Q. The vertices of $\gamma(Q)$ are the polymers $P_1, \ldots, P_k$. P_i is n_i times represented in $\gamma(Q)$ for all $i \in \{1, \ldots, k\}$. Polymers P_i and P_j are *not admissible* if $P_i \cap P_j \neq \emptyset$. P_i is not admissible with itself. Not admissible polymers are connected by a line in $\gamma(Q)$. *Reduced activities* are defined by

$$\overline{A}(P|\psi) = A(X|\psi)/\prod_{x\in P}A(\{x\}|\psi). \tag{2.15}$$

For a cluster $Q = (P_1^{n_1}, \dots, P_k^{n_k})$ we use the notation

$$\overline{A}(Q|\psi) = \prod_{i=1}^{k} \overline{A}(P_i|\psi)^{n_i}. \tag{2.16}$$

The expansion for $\ln Z(\Lambda|\psi)$ is

$$\ln Z(\Lambda|\psi) = \sum_{x\in\Lambda} \ln A(\{x\}|\psi) + \sum_{\substack{Q \\ \text{with } |P_i|\geq 2}} a(Q)\overline{A}(Q|\psi), \tag{2.17a}$$

where

$$a(Q) = \begin{cases} 0 & \text{if } \gamma(Q) \text{ is not connected} \\ \sum_{C\subseteq\gamma(Q)}(-1)^{l(C)}/\prod_{i=1}^{k} n_i! & \text{if } \gamma(Q) \text{ is connected.} \end{cases} \tag{2.17b}$$

The sum in (2.17b) is over all connected subgraphs C of $\gamma(Q)$ with the same set of vertices as $\gamma(Q)$. $l(C)$ is the number of lines in C. A theory described by the partition function $Z(\Lambda|\psi)$ is translation invariant if $v(x,y) = v(x-y)$ and the functions F_x are not x-dependent. For translation invariant theories the expansion for the density of the free energy on the lattice $\Lambda_{tot} = (a\mathbb{Z})^\nu$ is (if the limit exists)

$$\lim_{\Lambda\nearrow\Lambda_{tot}} \frac{1}{|\Lambda|} \ln Z(\Lambda|\psi) = \sum_{\substack{X \\ x\in X\subseteq\Lambda_{tot}}} \frac{V_X}{|X|} \tag{2.18}$$

with

$$V_X = \begin{cases} \ln A(\{x\}|\psi) & \text{if } |X| = 1,\ X = \{x\} \\ \sum_Q a(Q)\overline{A}(Q|\psi) & \text{otherwise.} \end{cases} \tag{2.19}$$

$\lim_{\Lambda\nearrow\Lambda_{tot}}$ denotes the thermodynamical limit (in the sense of van Hove).

Let $F_x(\phi_x)$ (for all $x \in (a\mathbb{Z})^\nu$) be holomorph functions in the complex strip

$$S_\kappa = \{\phi_x \in \mathbb{C} \mid |Im\ \phi_x| \leq \kappa\}, \qquad \kappa > 0 \tag{2.20}$$

(i.e. it exists an open neighborhood U of S_κ, such that F_x is holomorph in U). Furthermore, let F_x be bounded in S_κ. It follows from the convergence of the integral $\int d\mu_v(\phi)\prod_{x\in X} F_x(\phi_x)$ for finite $X \subset (a\mathbb{Z})^\nu$, $\psi_x \in S_\kappa$, that $Z(X|\psi)$ is holomorph in S_κ. Assume that

$$\exists\xi > 1:\ B(\xi,\psi) < K_\xi < 1, \tag{2.21a}$$

where

$$B(\xi,\psi) = \frac{1}{\xi}\left[1 + \sup_{x\in\Lambda_{tot}} \sum_{\substack{X \\ x\in X\subset\Lambda_{tot}\ |X|\geq 2}} |\overline{A}(X|\psi)|\xi^{|X|}\right]. \tag{2.21b}$$

Gruber and Kunz [4] have shown that (2.21a,b) is a sufficient condition for the existence of the thermodynamical limit (in the sense of van Hove) $X \nearrow \Lambda_{tot}$ for the *reduced correlation functions*

$$\rho_X(Y|\psi) = Z(X - Y|\psi)/Z(X|\psi), \tag{2.22}$$

which fulfill the bounds

$$|\rho_X(Y|\psi)| \leq [1 - K_\xi]^{-1} \prod_{x\in Y} A(\{x\}|\psi)^{-1}. \tag{2.23}$$

Furthermore, the thermodynamical limit in (2.18) exists and the expansion (2.17) is convergent for finite Λ (this assertion is non trivial, because there are infinite summands in (2.17)). The following Theorem shows the existence of the thermodynamical limit and the holomorphy of the generating function for free-propagator-amputated Greens functions if the convergence condition of Gruber and Kunz (2.21a,b) is fulfilled.

Theorem 2.2.1. *Let $F_x(\phi_x)$ be holomorph functions for all $x \in (a\mathbb{Z})^\nu$ in the complex strip*

$$S_\kappa = \{\phi_x \in \mathbb{C} \mid |Im\phi_x| \leq \kappa\} \tag{2.20}$$

and let ψ be an external field with finite support supp ψ. Furthermore let the condition (2.21a,b) be fulfilled. Then the thermodynamical limit (in the sense of van Hove) $\Lambda \nearrow \Lambda_{tot}$ exists for the function

$$F_\Lambda(\psi) = \ln[Z(\Lambda|\psi)/Z(\Lambda|\psi = 0)] \tag{2.24}$$

and $\lim_{\Lambda \nearrow \Lambda_{tot}} F_\Lambda(\psi) = F(\psi)$ *is holomorph in a neighborhood of $\psi = 0$.*

PROOF: From (2.21a,b) follows the existence of the thermodynamical limit $\Lambda \nearrow \Lambda_{tot}$ for the reduced correlation functions $\rho_\Lambda(X|\psi)$ (cf.[4]) and the estimates (2.23) are valid. Therefore $\rho_\Lambda(X|\psi)$ is uniformly bounded in S_κ. Since $\rho_\Lambda(X|\psi)$ is holomorph in S_κ for finite Λ, it follows from Vitali's Theorem that $\lim_{\Lambda \nearrow \Lambda_{tot}} \rho_\Lambda(X|\psi)$ is holomorph in S_κ. Let X be a finite subset of Λ_{tot}, such that

$$X \supseteq supp\ \psi. \tag{2.25}$$

By this assumption follows

$$Z(\Lambda - X|\psi) = Z(\Lambda - X|\psi = 0) \tag{2.26}$$

and with the definition of the reduced correlation function (2.22) follows

$$\rho_\Lambda(X|\psi) Z(\Lambda|\psi) = \rho_\Lambda(X|\psi = 0) Z(\Lambda|\psi = 0). \tag{2.27}$$

The function $\rho_\Lambda(X|\psi = 0)/\rho_\Lambda(X|\psi)$ is $\neq 0$ in a suitable neighborhood of $\psi = 0$ and we have

$$\ln[Z(\Lambda|\psi)/Z(\Lambda|\psi = 0)] = \ln[\rho_\Lambda(X|\psi = 0)/\rho_\Lambda(X|\psi)]. \tag{2.28}$$

The thermodynamical limit exists for the rhs of (2.28) and the function $\lim_{\Lambda \nearrow \Lambda_{tot}} \rho_\Lambda(X|\psi = 0)/\rho_\Lambda(X|\psi)$ is in a suitable neighborhood of $\psi = 0$ holomorph and $\neq 0$. Therefore $\lim_{\Lambda \nearrow \Lambda_{tot}} \ln[\rho_\Lambda(X|\psi = 0)/\rho_\Lambda(X|\psi)]$ is holomorph in a neighborhood of $\psi = 0$ and the assertion follows from (2.28).$\surd$

From Theorem 2.2.1. follows immediately

Corollary 2.2.2. *Let $F_x(\phi_x)$, $x \in (a\mathbb{Z})^\nu$, be holomorph functions in S_κ and let the convergence condition of Gruber and Kunz (2.21a,b) be fulfilled. Then the thermodynamical limit $\Lambda \nearrow \Lambda_{tot}$ (in the sense of van Hove) exists for the free-propagator-amputated Greens functions*

$$G_c(\underline{x}_1, \ldots, \underline{x}_n) = \frac{\delta^n}{\delta\psi(x_1)\ldots\delta\psi(x_n)} \ln \frac{Z(\Lambda|\psi)}{Z(\Lambda|\psi = 0)}|_{\psi=0} \tag{2.29}$$

for all $n \in \mathbb{N}^*$ and $x_1, \ldots, x_n \in \Lambda$.

Corollary 2.2.3. *Let the convergence condition of Gruber and Kunz (2.21a,b) be fulfilled. Let ψ be an external field with finite support supp ψ. The following expansion for the generating function for the free-propagator-amputated Greens functions is convergent for translation invariant systems and in a small neighborhood of $\psi = 0$:*

$$\lim_{\Lambda \nearrow \Lambda_{tot}} \ln \frac{Z(\Lambda|\psi)}{Z(\Lambda|\psi = 0)} = \sum_{x \in supp\ \psi} [\ln A(\{x\}|\psi) - A(\{x\}|\psi = 0)] + \sum_Q a(Q)[\overline{A}(Q|\psi) - \overline{A}(Q|\psi = 0)]. \tag{2.30}$$

PROOF: Because of Theorem 2.2.1. the thermodynamical limit exists. It follows from the convergence condition of Gruber and Kunz (2.21a,b) that the reduced correlation functions $\rho_\Lambda(X)$ are analytic in A(Y), $|Y| \geq 2$, $Y \subset \Lambda$ for finite Λ (cf. [4]). For a finite subset $X \supseteq supp\ \psi$ we have

$$\ln[Z(\Lambda|\psi)/Z(\Lambda|\psi = 0)] = \ln[\rho_\Lambda(X|\psi = 0)/\rho_\Lambda(X|\psi)]. \tag{2.28}$$

If ψ is in a sufficient small neighborhood of $\psi = 0$, then

$$\ln \frac{Z(\Lambda|\psi)}{Z(\Lambda|\psi=0)} = \sum_{x \in supp\ \psi} [\ln A(\{x\}|\psi) - A(\{x\}|\psi=0)] + \sum_Q a(Q)[\overline{A}(Q|\psi) - \overline{A}(Q|\psi=0)] \tag{2.31}$$

and the series in the rhs is convergent. For translation invariant polymer systems exist a positive monotone decreasing function $\epsilon(\lambda)$, such that

$$\lim_{\lambda\to\infty} \epsilon(\lambda) = 0 \tag{2.32}$$

and

$$|\rho_\Lambda(X) - \rho_{\Lambda_{tot}}(X)| \leq \xi^{|X|}\epsilon(dist\ (X, \partial\Lambda)) \tag{2.33}$$

with $dist(X,\partial\Lambda) = \inf\{||x-y|||x \in X, y \in \partial\Lambda\}$, $\partial\Lambda =$ boundary of Λ. Therefore we obtain

$$\ln \frac{Z(\Lambda_{tot}|\psi)}{Z(\Lambda_{tot}|\psi=0)} = \ln \frac{Z(\Lambda|\psi)}{Z(\Lambda|\psi=0)} + R(\Lambda) \tag{2.34}$$

with

$$\lim_{\Lambda\nearrow\Lambda_{tot}} R(\Lambda) = \lim_{\Lambda\nearrow\Lambda_{tot}} \ln \left[\frac{1+\frac{\rho_{\Lambda_{tot}}(X|\psi=0)-\rho_\Lambda(X|\psi=0)}{\rho_\Lambda(X|\psi=0)}}{1+\frac{\rho_{\Lambda_{tot}}(X|\psi)-\rho_\Lambda(X|\psi)}{\rho_\Lambda(X|\psi)}}\right] = 0. \tag{2.35}$$

The assertion follows from (2.31), (2.34) and (2.35).$\surd$

2.3. Expansion in the number of lattice points

The polymer representation (2.5) for $Z(\Lambda|\psi)$ may be reformulated as an expansion in the number of lattice points:

Lemma 2.3.1. *Let $\Lambda \subset \Lambda_{tot}$ be finite. Then*

$$Z(\Lambda|\psi) = 1 + \sum_{n=1}^{|\Lambda|} \frac{1}{n!} \int_{\substack{y_1,\dots y_n \in\Lambda \\ distinct}} \sum_{\{y_1,\dots y_n\}=\sum X} \prod_X \{|X|!\ \mathcal{M}(X|\psi)\} \tag{2.36a}$$

with

$$\mathcal{M}(X|\psi) = \frac{a^{-\nu n}}{n!}[-\delta_{1,n} + A(X|\psi)],\ |X| = n. \tag{2.36b}$$

$\mathcal{M}(X|\psi)$ is called Mayer amplitude for the polymer X.

PROOF: By the polymer representation (2.5) we obtain

$$Z(\Lambda|\psi) = \sum_{\Lambda=\sum X} \prod_X A(X|\psi) = \sum_{Y\subseteq\Lambda} \sum_{\substack{Y=\sum X \\ |X|\leq 2}} [\prod_{x\in\Lambda-Y} A(\{x\}|\psi)][\prod_X A(X|\psi)] =$$

$$= 1 + \sum_{\emptyset\neq Y\subseteq\Lambda} \sum_{Y=\sum X} \prod_X [-\delta_{1,|X|} + A(X|\psi)] =$$

$$= 1 + \sum_{n=1}^{|\Lambda|} \frac{a^{-\nu n}}{n!} \int_{\substack{y_1,\dots y_n\in\Lambda \\ distinct}} \sum_{\{y_1,\dots y_n\}=\sum X} \prod_X [-\delta_{1,|X|} + A(X|\psi)] =$$

$$= 1 + \sum_{n=1}^{|\Lambda|} \frac{1}{n!} \int_{\substack{y_1,\dots y_n\in\Lambda \\ distinct}} \sum_{\{y_1,\dots y_n\}=\sum X} \prod_X \{|X|!\ \mathcal{M}(X|\psi)\}.\surd \tag{2.37}$$

In the proof of Lemma 2.3.1. we have shown

$$Z(\Lambda|\psi) = 1 + \sum_{\emptyset\neq Y\subseteq\Lambda}\sum_{Y=\sum X}\prod_X M(X|\psi) \tag{2.38}$$

with

$$M(X|\psi) = -\delta_{1,|X|} + A(X|\psi). \tag{2.39}$$

The expansion (2.38) may be interpreted as a polymer representation of a new polymer system. The lattice sites of the original lattice are split into two sites. Polymers, which contain more than one site, consist only of double sites. Monomers consist of only one of the doubled sites. The monomer activities are set to one. The activity of a polymer P' is M(P) if P' emerge from P by the doubling procedure. We obtain

$$Z(\Lambda|\psi) = \sum_{\tilde{\Lambda}=\sum X}\prod_X M(X|\psi). \tag{2.40}$$

The sum is over disjoint partitions of the doubled lattice $\tilde{\Lambda}$ in polymers X of the new polymer system. The sufficient condition for convergence for the new polymer system is

$$\exists\xi > 1 \; : \; \frac{1}{\xi}\left[1 + \sup_{x\in\Lambda}\sum_{\substack{X\\ x\in X\subseteq\Lambda}} |M(X|\psi)|\xi^{2|X|}\right] < 1. \tag{2.41}$$

The expansion of the free energy $\ln Z(\Lambda|\psi)$ for the new polymer system is

$$\ln Z(\Lambda|\psi) = \sum_Q a(Q)M(Q|\psi). \tag{2.42}$$

The series is convergent for finite Λ if (2.41) is fulfilled. In the follwing we reformulate (2.42) as an expansion in the number of (not necessarily distinct) lattice points. Let *supp* Q be the disjoint union of polymers in the cluster Q. The point x in *supp* Q has the multiplicity $n(x)$. $n(x)$ equals the number of polymers $P \in Q$ with $x \in P$. Let $\mathcal{X}$ be a set of points with multiplicities. The expansion (2.42) is reordered by

$$\ln Z(\Lambda|\psi) = \sum_{\mathcal{X}} \widetilde{M}(\mathcal{X}|\psi) \tag{2.43}$$

with

$$\widetilde{M}(\mathcal{X}|\psi) = \sum_{\substack{Q\\ supp\; Q=\mathcal{X}}} a(Q)\prod_{P\in Q} M(P|\psi). \tag{2.44}$$

By Eq. (2.43) we obtain

$$\ln Z(\Lambda|\psi) = \sum_{n\geq 1}\prod_{x_1,\ldots,x_n\in\Lambda} \widetilde{\mathcal{M}}(x_1,\ldots,x_n|\psi) \tag{2.45}$$

with the definition of the *augmented Mayer amplitude*

$$\widetilde{\mathcal{M}}(x_1,\ldots,x_n|\psi) = \widetilde{M}(\mathcal{X}|\psi)[\prod_{\substack{distinct\\ x\in\mathcal{X}}} n(x)!]/(n!\, a^{\nu n}) \tag{2.46}$$

for $\mathcal{X} = \{x_1,\ldots,x_n\}$. If the condition of convergence (2.41) is fulfilled, the series

$$\lim_{\Lambda\nearrow\Lambda_{tot}} \frac{Z(\Lambda|\psi)}{Z(\Lambda|\psi=0)} = \sum_Q a(Q)[M(Q|\psi) - M(Q|\psi=0)] \tag{2.47}$$

is convergent for finite *supp* ψ in a small neighborhood of $\psi = 0$ (cf. Corollary 2.2.3.). For translation invariant systems the augmented Mayer amplitude is translation invariant and with the notation

$$\widetilde{\widetilde{\mathcal{M}}}(x_1,\ldots,x_n|\psi) = a^\nu\,\widetilde{\mathcal{M}}(x_1 - x_{n+1},\ldots,x_n - x_{n+1},0|\psi) \tag{2.48}$$

we obtain for the density of the free energy in the whole lattice $\Lambda_{tot} = (a\mathbb{Z})^\nu$

$$\lim_{\Lambda\nearrow\Lambda_{tot}} \frac{1}{|\Lambda|} \ln Z(\Lambda|\psi) = 1 + \sum_{n\geq 1} \int_{x_1,\ldots,x_n\in\Lambda_{tot}} \tilde{\tilde{\mathcal{M}}}(x_1,\ldots,x_n|\psi). \tag{2.49}$$

This series is convergent, if (2.41) is fulfilled.

2.4. Connection of perturbation and Mayer expansion

In this section we will carry the perturbation expansion in the form (2.40) by formal resummation and we will show how Mayer amplitudes (rsp. activities) are represented by Feynman diagrams. The perturbation expansion of the Mayer amplitude is not convergent, but it is an asymptotic expansion (cf. chapter 4.). We suppose that the functions F_x in the definition of the partition function (2.1) are of the following form

$$F_x(\phi_x) = e^{-\lambda \mathcal{V}(\phi_x)}. \tag{2.50}$$

λ is a dimensionsless coupling constant. Per substraction of a constant in the propagator the distributions F_x of examples b) and c) (p. 19-20) are transformed in the form (2.50) (cf. chapter 3.). We consider the expansion in λ of the partition function

$$Z(\Lambda|\psi) = \int d\mu_v(\phi) \prod_{x\in\Lambda} e^{-\lambda\mathcal{V}(\phi_x+\psi_x)}. \tag{2.51}$$

In the following we abbreviate $\mathcal{V}(x)$ for $\mathcal{V}(\phi_x + \psi_x)$. Formal expansion in power series yields

$$Z(\Lambda|\psi) = 1 + \sum_{n\geq 1} \frac{(-\lambda)^n}{n!} a^{-\nu n} \int_{x_1,\ldots,x_n\in\Lambda} \langle \mathcal{V}(x_1)\ldots\mathcal{V}(x_n)\rangle. \tag{2.52}$$

This perturbation series is generally not convergent. For example the radius of convergence for the $\lambda\phi^4$–theory on the lattice with single site is zero, since the integral $\int d\mu_v(\phi) e^{-\lambda(\phi_x+\psi_x)^4}$ is divergent for $\lambda < 0$. With partially formal resummation of the perturbation expansion we can get convergent expansions for non vanishing coupling constants λ. With the help of the relation (Wick-theorem)

$$\langle \prod_{x\in X} \mathcal{V}(\phi_x + \psi_x)\rangle = e^{\frac{1}{2}(\frac{\partial}{\partial\phi}, v\frac{\partial}{\partial\phi})} \prod_{x\in X} \mathcal{V}(\phi_x + \psi_x)|_{\phi_x=0} \tag{2.53}$$

the partition function $Z(\Lambda|\psi)$ is represented by (not necessarily connected) Feynman diagrams.

Proof of (2.53): Let the Fourier transform $\tilde{\mathcal{V}}(q_x)$ for $\mathcal{V}(\phi_x)$ be defined by

$$\mathcal{V}(\phi_x) = \int dq_x \tilde{\mathcal{V}}(q_x) e^{iq_x\phi_x}. \tag{2.54}$$

With the help of the Gaussian integral (1.13) we obtain

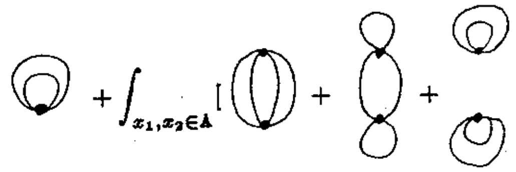
$$\begin{aligned}\langle \prod_{x\in X} \mathcal{V}(\phi_x + \psi_x)\rangle &= \int \prod_{x\in X} [dq_x \tilde{\mathcal{V}}(q_x) e^{iq_x\psi_x}] e^{-\frac{1}{2}(q,vq)} = \\ &= \int \prod_{x\in X} [dq_x \tilde{\mathcal{V}}(q_x) e^{iq_x\psi_x}] e^{\frac{1}{2}(\frac{\partial}{\partial\phi}, v\frac{\partial}{\partial\phi})} e^{iq_x\phi_x}|_{\phi_x=0} = \\ &= e^{\frac{1}{2}(\frac{\partial}{\partial\phi}, v\frac{\partial}{\partial\phi})} \prod_{x\in X} \mathcal{V}(\phi_x + \psi_x)|_{\phi_x=0}. \qquad \checkmark\end{aligned}$$

We get

$$Z(\Lambda|\psi) = 1 + \sum_{n\geq 1} (\text{Feynman diagrams with n vertices}). \tag{2.55a}$$

For example a) the $\lambda\phi^4$–theory without counterterms and external field $\psi = 0$

$$Z(\Lambda|\psi) = 1 + \int_{x_1\in\Lambda} \ldots + \int_{x_1,x_2\in\Lambda} [\ \ldots\ ] + \ldots . \tag{2.55b}$$



The Feynman diagrams are related to algebraic expressions by familiar rules. If the connected Feynman diagrams $F_l^{(x)}$ appear m_l times in the Feynman diagram, we obtain a combinatorial factor $1/\prod_l m_l!$. The number of vertices in the Feynman diagram equals the order in λ. The perturbation series (2.52) will be reordered, such that the integration is over distinct points

$$Z(\Lambda|\psi) = 1 + \sum_{n\geq 1} \sum_{\substack{b\in\mathbb{N}^{|\Lambda|} \\ |supp\ b|=n}} \frac{(-\lambda)^{|b|}}{|b|!} a^{-\nu n} \int_{\substack{y_1,\ldots,y_n\in\Lambda \\ distinct}} \langle \mathcal{V}(y_1)^{b_1} \ldots \mathcal{V}(y_n)^{b_n} \rangle. \tag{2.56}$$

We have used the function

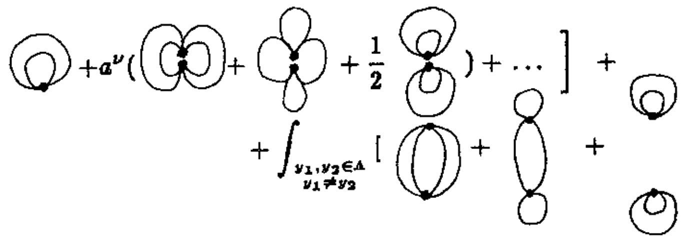
$$b\ :\ \begin{cases} \Lambda \to \mathbb{N} = \{0,1,2,\ldots\} \\ y \mapsto b_y \end{cases}$$

the notations

$$supp\ b = \{x \in \Lambda |\ b_x \neq 0\}, \qquad |b| = \sum_{x\in supp\ b} b_x \tag{2.57}$$

and the abbreviation $b_{y_i} = b_i$. $b \in \mathbb{N}^\Lambda$ is called *occupation function* with point set Λ. For the $\lambda\phi^4$–theory we have

$$Z(\Lambda|\psi = 0) = 1 + \int_{y_1\in\Lambda} \Big[\ \ldots + a^\nu(\ldots + \ldots + \tfrac{1}{2} \ldots) + \ldots \Big] + \int_{\substack{y_1,y_2\in\Lambda \\ y_1\neq y_2}} [\ \ldots + \ldots + \ldots + \ldots\ ] + \ldots \tag{2.58}$$



The Feynman diagrams will be put on the lattice $\Lambda \times \mathbb{N}^*$, $\mathbb{N}^* = \mathbb{N} - \{0\}$. The Feynman diagram F with occupation function b, i.e. the vertices occupy the point set $supp\ b$ and the point y is covered by vertices of the Feynman diagram F, will be put on the lattice

$$I = \sum_{y\in supp\ b} \{y\} \times \{1,2,\ldots,b_y\}. \tag{2.59}$$

The lattice $\Lambda \times \mathbb{N}^*$ is called *index lattice with base* Λ and $I \subset \Lambda \times \mathbb{N}^*$ defined above is called *index set* for the occupation function b. Conversely, it exists for every index set I with $|I \cap \{x\} \times \mathbb{N}^*| < \infty$ for all $x \in \Lambda$ an occupation function b. We can rewrite the expansion (2.54)

$$Z(\Lambda|\psi) = \sum_{\substack{I\subseteq\Lambda\times\mathbb{N}^* \\ I\ index\ set}} (\text{Feynman diagrams } F \in \mathcal{F}_I). \tag{2.60}$$

$\mathcal{F}_I$ is the set of all Feynman diagrams with the set of vertices $= I$. The *canonoical projection* p is defined by

$$p\ :\ \begin{cases} \Lambda \times \mathbb{N}^* \to \Lambda \\ (y,n) \mapsto y\ . \end{cases} \tag{2.61}$$

The Feynman diagrams $F \in \mathcal{F}_I$ on the index lattice $\Lambda \times \mathbb{N}^*$ are related to graphs $p(F)$ with vertices on Λ by the canonical projection p. The set of vertices for the graph $p(F)$ is $p(I)$ and vertices $x, y \in p(I)$ are connected by a line if v_{xy} is in the algebraic expression for F. Two vertices in $p(F)$ are connected by at most one line. The graph $p(F)$ emerge from F by omitting self lines and replacing lines which connect the same vertex by only one line. The Feynman diagram $F \in \mathcal{F}_I$ is called *point connected*, if the projected graph $p(F)$ is connected. Therefore

$$F \in \mathcal{F}_I \text{ point connected} \Longleftrightarrow p(F) \in \mathcal{G}_X \text{ with } X = p(I). \tag{2.62}$$

$\mathcal{G}_X$ is the set of all Mayer graphs with vertices in X and all points of X are vertices. Fig. 2.1 shows examples for point connected vacuum Feynman diagrams for the $\lambda\phi^4$-theory.

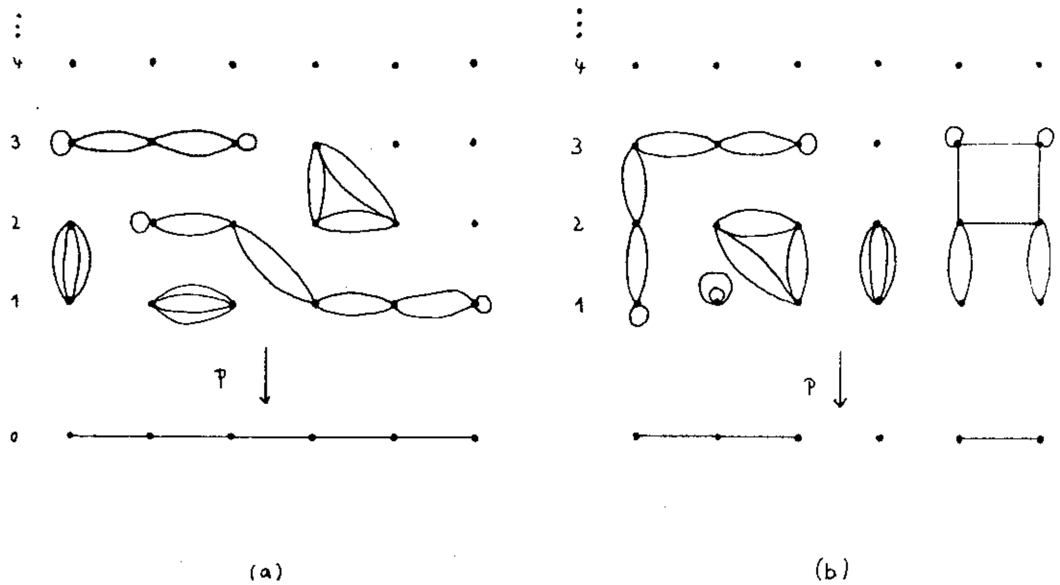


Fig. 2.1 Example of a point connected Feynman diagram (a) and not point connected Feynman diagram (b) and their related Mayer graphs for the $\lambda\phi^4$-theory.

The Mayer graphs $G \in \mathcal{G}_X$ stand for the following algebraic expressions. Every vertex $x \in \Lambda$ of the Mayer graph $G \in \mathcal{G}_X$ stands for

$$\tilde{F}_x(q_x)e^{iq_x\psi_x}e^{-\frac{1}{2}q_xv_{xx}q_x}.$$

Lines in the Mayer graph G, which connect x and y, stand for the "super propagator"

$$e^{-q_xv_{xy}q_y} - 1.$$

There is no combinatorial factor. After integration over q_x, $x \in X$ we get the algebraic expression $I(G)$ for the Mayer graph $G \in \mathcal{G}_X$. Because of Theorem 2.1.1., Eq. (2.6), we obtain

$$A(X|\psi) = \sum_{G \in \mathcal{G}_X} I(G). \tag{2.63}$$

Splitting the expansion (2.60) for $Z(\Lambda|\psi)$ in point connected Feynman diagrams we obtain

$$Z(\Lambda|\psi) = \sum_{Y \subset \Lambda} \sum_{Y = \sum X} \prod_X [\sum_{F \in \mathcal{F}_X^{(pc)}} I(F)]. \tag{2.64}$$

$\mathcal{F}_X^{(pc)}$ is the set of all point connected Feynman diagrams with vertex set X. $I(F)$ is the corresponding algebraic expression for the Feynman diagram F. As in the proof for (2.38) we have

$$Z(\Lambda|\psi) = \sum_{\Lambda = \sum X} \prod_X [\delta_{1,|X|} + \sum_{F \in \mathcal{F}_X^{(pc)}} I(F)]. \tag{2.65}$$

If the coupling constant λ is replaced by the point dependent coupling constant ($Y \subset \Lambda$)

$$\lambda_Y(x) = \begin{cases} \lambda & \text{if } y \in Y \\ 0 & \text{otherwise} \end{cases} \tag{2.66}$$

(the interaction is switched off outside Y), then the partition function $Z(\Lambda|\psi)$ equals $Z(Y|\psi)$ and the representation (2.64) is also correct, if Λ is replaced by Y. From (2.63), (2.65) and the uniqueness of the polymer representation, follows

$$A(X|\psi) = \delta_{1,|X|} + \sum_{F \in \mathcal{F}_X^{(pc)}} I(F) \tag{2.67}$$

and for Mayer amplitudes

$$\mathcal{M}(X|\psi) = \frac{a^{-\nu|X|}}{|X|!} \sum_{F \in \mathcal{F}_X^{(pc)}} I(F). \tag{2.68}$$

The representation (2.36) for the partition function is equivalent to

$$Z(\Lambda|\psi) = 1 + \sum_{n\geq 1} \sum_{k \geq 1} \sum_{\substack{\{n_l\} \\ \sum_{l=1}^{k} n_l = n}} \frac{1}{\prod_{j=1}^{n} m_j(\{n_l\})!} \int_{\substack{y_1,\dots,y_n \in \Lambda \\ distinct}} \mathcal{M}(y_1,\dots,y_{n_1})$$
$$\mathcal{M}(y_{n_1+1},\dots,y_{n_1+n_2}) \dots \mathcal{M}(y_{n_1+\dots+n_{k-1}+1},\dots,y_n) \tag{2.69}$$

where $m_j(\{n_l\}) = |\{r|n_r = j\}|$ is the number of $n_r = j$ in the partition $\{n_l\}$ and $\mathcal{M}(Y|\psi) = \mathcal{M}(y_1,\dots,y_n|\psi)$for $\{y_1,\dots y_n\} = Y$. The Mayer amplitude may be represented as a formal power series. This (divergent) expansion is Borel summable in λ (see ch. 4.). We obtain the following Theorem.

Theorem 2.4.1. *The Mayer amplitudes $\mathcal{M}$ are represented by Feynman diagrams*

$$\mathcal{M}(X|\psi) = \frac{a^{-\nu|X|}}{|X|!} \sum_{F \in \mathcal{F}_X^{(pc)}} I(F). \tag{2.70}$$

$\mathcal{F}_X^{(pc)}$ is the set of point connected Feynman diagrams with vertex set X. The representation by Mayer graphs is

$$\mathcal{M}(X|\psi) = \frac{a^{-\nu|X|}}{|X|!} \begin{cases} -1 + \int dq_x\, \tilde{F}_x(q_x) e^{iq_x\psi_x} e^{-\frac{1}{2}q_x v_{xx} q_x} & \text{if } X = \{x\} \\ \sum_{G \in \mathcal{G}_X} \int [\prod_{x\in X} dq_x\, \tilde{F}_x(q_x) e^{iq_x\psi_x} e^{-\frac{1}{2}q_x v_{xx} q_x}] \prod_{(xy)\in G} [e^{-q_x v_{xy} q_y} - 1] & \text{if } |X| \geq 2. \end{cases} \tag{2.71}$$

From the expansion of the e-function in the representation(2.6) for $A(X|\psi)$ we obtain an explicit expression for the expansion of the activity $A(X|\psi)$ in the number of lines

$$A(X|\psi) = \sum_{n=|X|-1}^{\infty} (-1)^n \sum_{\substack{k \in \mathbb{N}^{X^*},\ s \in \mathbb{N}^X\ supp\ k \text{ connected} \\ |k|+|s|=n,\ P(k)=X\ |s|\neq 0 \text{ if } |X|=1}} [\prod_{(xy)\in X^*} \frac{v_{xy}^{k_{xy}}}{k_{xy}!}][\prod_{x\in X} \frac{(v_{xx}/2)^{s_x}}{s_x!} F_x^{(2s_x+k_x)}(\psi_x)]. \tag{2.72}$$

X^* denotes the set of all lines in $X \subset \Lambda$, $supp\ k = \{(xy) \in X^* |\ k_{xy} \neq 0\}$, $P(k) = \{x \in \Lambda|\ \exists b \in supp\ k :$ x is point of $b\}$, k_{xy} labels the number of lines in the Feynman diagram, which connect x and y , s_x labels the number of self lines in z. $k_x = \sum_{y\in X} k_{yx}$ is the number of lines, which emanate from x and connect a point different from x. The number of lines, which connect different points is $|k| = \sum_{(xy)\in X^*} k_{xy}$ and the number of self lines is $|s| = \sum_{x\in X} s_x$. $F_x^{(k)}(\psi_x)$ denotes the k-th derivative at ψ_x. The possible numbers of lines, which emanate from x, are fixed by the derivatives of F_x at $\psi_x = 0$. For example for the $\lambda\phi^4$–theory $4n$ lines, $n \in \mathbb{N}^*$, can emanate from a point.

PROOF FOR (2.72): Expansion of the e-function in (2.6) gives

$$A(X|\psi) = \sum_{G \in \mathcal{G}_X} \int \prod_{x \in X} [dq_x \tilde{F}_x(q_x) e^{iq_x\psi_x} \sum_{s_x=0}^{\infty} (-1)^{s_x} \frac{(q_x v_{xx} q_x/2)^{s_x}}{s_x!}]$$
$$\prod_{(xy)\in G} [\sum_{k_{xy}=1}^{\infty} (-1)^{k_{xy}} \frac{(q_x v_{xy} q_y)^{k_{xy}}}{k_{xy}!}] \tag{2.73}$$

and from distributivity

$$A(X|\psi) = \sum_{n=|X|-1}^{\infty} (-1)^n \sum_{\substack{k\in\mathbb{N}^{X^*},\ s\in\mathbb{N}^X \\ \mathrm{supp}\ k\ \mathrm{connected}}\ |k|+|s|=n,\ P(k)=X} [\prod_{(xy)\in X^*} \frac{v_{xy}^{k_{xy}}}{k_{xy}!}] [\prod_{x\in X} \frac{(v_{xx}/2)^{s_x}}{s_x!} \int dq_x q_x^{2s_x+k_x} \tilde{F}_x(q_x) e^{iq_x\psi_x}]. \quad (2.74)$$

After inverse Fourier transformation we obtain (2.72). √

The perturbation expansion for $\ln Z(\Lambda|\psi)$ is a sum of truncated expectation values

$$\ln Z(\Lambda|\psi) = \sum_{n\geq 1} \frac{(-\lambda)^n}{n!} a^{-\nu n} \int_{x_1,\ldots,x_n\in\Lambda} \langle \mathcal{V}(x_1); \ldots; \mathcal{V}(x_n)\rangle \quad (2.75)$$

(proof see app. A). The *Feynman amplitude*

$$\mathcal{F}(x_1,\ldots,x_n|\psi) = \frac{(-\lambda)^n}{n!} a^{-\nu n} \langle \mathcal{V}(x_1); \ldots; \mathcal{V}(x_n)\rangle \quad (2.76)$$

is a sum of connected Feynman diagrams. If same arguments occur in the Feynman amplitude we will write

$$\mathcal{F}(y_1^{n_1},\ldots,y_k^{n_k}|\psi) = \mathcal{F}(\underbrace{y_1,\ldots,y_1}_{n_1 \text{ arguments}},\ldots,\underbrace{y_k,\ldots,y_k}_{n_k \text{ arguments}}|\psi). \quad (2.77)$$

After partial formal resummation we obtain for the perturbation expansion (2.75)

$$\ln Z(\Lambda|\psi) = \sum_{n\geq 1} \sum_{\substack{b\in\mathbb{N}^\Lambda \\ |\mathrm{supp}\ b|=n}} \frac{(-\lambda)^{|b|}}{|b|!} \int_{\substack{y_1,\ldots,y_n\in\Lambda \\ \mathrm{distinct}}} \langle \mathcal{V}(y_1)^{b_1}; \ldots; \mathcal{V}(y_n)^{b_n}\rangle =$$

$$= \sum_{n\geq 1} \sum_{\substack{b\in\mathbb{N}^\Lambda \\ |\mathrm{supp}\ b|=n}} \int_{\substack{y_1,\ldots,y_n\in\Lambda \\ \mathrm{distinct}}} \mathcal{F}(y_1^{b_1},\ldots,y_n^{b_n}|\psi) \quad (2.78)$$

with $|b| \stackrel{\mathrm{def}}{=} \sum_{x\in\Lambda} b_x$ and definition(2.57) for *supp* b. Reformulation of the integration over distinct points in a summation over subsets of the lattice Λ gives

$$\ln Z(\Lambda|\psi) = \sum_{\substack{X \\ \emptyset\neq X\subseteq\Lambda}} |X|!\ a^{\nu|X|} \sum_{b\in\mathbb{N}^{*X}} \mathcal{F}(x_1^{b_1},\ldots,x_n^{b_n}|\psi) \quad (2.79)$$

with $X = \{x_1,\ldots,x_n\}$ and $\mathbb{N}^* = \{1,2,3,\ldots\}$. We have shown that

$$\ln Z(\Lambda|\psi) = \sum_{\substack{X \\ \emptyset\neq X\subseteq\Lambda}} V_X \quad (2.80)$$

with the definition (2.19) for V_X. (2.79) and (2.80) are also fulfilled for arbitrary Y, $\emptyset \neq Y \subseteq \Lambda$, instead of Λ. By the following Lemma 2.4.2. we obtain

$$V_X = |X|\ a^{\nu|X|} \sum_{b\in\mathbb{N}^{*X}} \mathcal{F}(x_1^{b_1},\ldots,x_n^{b_n}|\psi). \quad (2.81)$$

Lemma 2.4.2. (Möbius inversion formula). *Let* $Q(Y)$*,* $\emptyset \neq Y \subseteq \Lambda$*, be defined by*

$$Q(Y) = \sum_{\substack{X \\ \emptyset\neq X\subseteq Y}} L(X). \quad (2.82)$$

Then we have

$$L(X) = \sum_{\substack{Y \\ \emptyset \neq Y \subseteq X}} (-1)^{|X|-|Y|} Q(Y) \tag{2.83}$$

and the representation (2.83) is the unique solution of Eq. (2.82).

PROOF (CF. [13], [16]): *Uniqueness of representation (2.83):* By definition (2.82) we have $L(\{x\}) = Q(\{x\})$. Let $L(Z)$ be uniquely determined by Q if $|Z| \leq n$. By Eq. (2.82) we get for X with $|X| = n$ and $x \notin X$

$$L(X + \{x\}) = Q(X + \{x\}) - \sum_{\substack{Z \\ \emptyset \neq Z \subseteq X}} L(Z). \tag{2.84}$$

So we see that $L(X + \{x\})$ is uniquely determined by Q. The uniqueness of representation (2.83) follows by induction.

Proof of (2.83) We have to show

$$Q(Y) = \sum_{\substack{X \\ \emptyset \neq X \subseteq Y}} \sum_{\substack{X' \\ \emptyset \neq X' \subseteq X}} (-1)^{|X|-|X'|} Q(X'). \tag{2.85}$$

(2.85) is fulfilled if

$$\sum_{\substack{X \\ X' \subseteq X \subseteq Y}} (-1)^{|X|-|X'|} = \begin{cases} 1 & \text{if } X' = Y \\ 0 & \text{otherwise.} \end{cases} \tag{2.86}$$

Let $n = |X|$, $s = |X'|$, $t = |Y|$ be the number of elements in X, X', Y. X is fixed by the choice of $n - s$ elements from the $t - s$ elements of $Y - X'$. This can be done in $\binom{t-s}{n-s}$ ways. Therefore by the binominial Theorem follows

$$\sum_{\substack{X \\ X' \subseteq X \subseteq Y}} (-1)^{|X|-|X'|} = \sum_{\substack{n \\ s \leq n \leq t}} (-1)^{n-s} \binom{t-s}{n-s} = \sum_{k=0}^{t-s} (-1)^k \binom{t-s}{k} =$$

$$= (1-1)^{t-s} = \begin{cases} 1 & \text{if } t = s \\ 0 & \text{otherwise.} \end{cases} \quad \checkmark$$

From (2.80) and the Möbius inversion formula follows

$$V_X = \sum_{\substack{Y \\ \emptyset \neq Y \subseteq X}} (-1)^{|X|-|Y|} \ln Z(Y|\psi). \tag{2.87}$$

Because of (2.81) and

$$a^{\nu n} \mathcal{F}(x_1, \ldots, x_n|\psi) = \sum_{\substack{\text{connected Feynman diagrams } F \\ \text{with vertices } x_1, \ldots, x_n}} I(F) \tag{2.88}$$

we have

$$V_X = |X|! \sum_{F \in \mathcal{F}_X^{(c)}} I(F). \tag{2.89}$$

$\mathcal{F}_X^{(c)}$ labels the set of all connected Feynman diagrams, whose vertices occupy the set $X \subseteq \Lambda$. The activities $A(X|\psi)$ in the polymer representation for $Z(\Lambda|\psi)$ may be represented by point connected Feynman diagrams (see Theorem 2.4.1.) and the functions V_X are represented by connected Feynman diagrams. Since all connected Feynman diagrams are point connected, there are less Feynman diagrams required for V_X than for $A(X|\psi)$.

We have only to consider polymers P with $|P| \leq n$ for n-th order perturbation theory. Because of $M(X|\psi) = O(\lambda^{|X|})$, we obtain

$$Z(\Lambda|\psi) = \sum_{\substack{\Lambda = \sum X \\ |X| \leq n}} \prod_X M(X) + O(\lambda^{n+1}). \tag{2.90}$$

Let us remark that the first term on the rhs of (2.90) contains terms of all orders in λ.

The Mayer amplitude M is a truncated expectation value of the following form

$$\mathcal{M}(x_1,\ldots,x_n|\psi) = \frac{a^{-\nu n}}{n!}\langle e^{-\lambda\mathcal{V}(x_1)}-1;\ldots;e^{-\lambda\mathcal{V}(x_n)}-1\rangle \tag{2.91}$$

(proof see app. A). The following Lemma express partial truncated expectation values by complete truncated expectation values. For that we will need the following definition:

Definition: Every matrix can be brought to block form by permutations of rows and/or columns. A matrix is called *irreducible* if it consists of only one block and no row or column is identically zero.

Lemma 2.4.3. *Let $F_i(\phi_{y_i})$, $i=1,\ldots,n$, be functions and $n_i \in \mathbb{N}^* = \{1,2,\ldots\}$ positive integers. Then we have*

$$\langle\prod_{i=1}^{n}[F_i(\phi_{y_i})^{n_i};\ ]\rangle = \sum_{l\geq 1}\sum_{\substack{k^{(l)}\\ \sum_{j=1}^{l}k_{ij}^{(l)}=n_i}} \frac{\prod_{i=1}^{n}n_i!}{(\prod_{i=1}^{n}\prod_{j=1}^{l}k_{ij}^{(l)}!)l!}\prod_{j=1}^{l}\langle\prod_{i=1}^{n}[F_i(\phi_{y_i});\ ]^{k_{ij}^{(l)}}\rangle. \tag{2.92}$$

The sum is over all irreducible $n\times l$–matrices $k^{(l)} = (k_{ij}^{(l)})_{\substack{i=1,\ldots,l\\ j=1,\ldots,l}}$ with $k_{ij}^{(l)} \in \mathbb{N}$ and $\sum_{j=1}^{l}k_{ij}^{(l)} = n_i$.

PROOF: For positive integers $n_i \in \mathbb{N}^*$, $i=1,\ldots,n$, let us define the following index set

$$I = \sum_{i=1}^{n}\{y_i\}\times\{1,2,\ldots,n_i\} \subset \{y_1,\ldots y_n\}\times\mathbb{N}^*. \tag{2.93}$$

With the notation

$$F(x) = F(\phi_{y_i}) \qquad \text{for } p(x) = y_i \tag{2.94}$$

(p = projection map, see p. 27) we obtain from the definition of the truncated expectation value (cf. app. A)

$$\langle F_1(y_1)^{n_1}\ldots F_n(y_n)^{n_n}\rangle = \sum_{I=\sum J}\prod_{J}\langle\prod_{x\in J}[F(x);\ ]\rangle \tag{2.95}$$

or equivalently

$$\langle F_1(y_1)^{n_1}\ldots F_n(y_n)^{n_n}\rangle = \sum_{l=1}^{m}\sum_{I=\sum_{j=1}^{l}I_j}\prod_{j=1}^{l}\langle\prod_{x\in I_j}[F(x);\ ]\rangle \tag{2.96}$$

with $m = |I| = \sum_{i=1}^{n}n_i$. For every partition $I = \sum_{j=1}^{l}I_j$ of the index set I we define an $n\times l$–matrix $(k_{ij}^{(l)})_{\substack{i=1,\ldots,n\\ j=1,\ldots,n}}$ by

$$k_{ij}^{(l)} = |\{x\in I_j \mid p(x) = y_i\}|. \tag{2.97}$$

With this notation it follows from (2.96)

$$\langle F_1(y_1)^{n_1}\ldots F_n(y_n)^{n_n}\rangle = \sum_{l=1}^{m}\sum_{I=\sum_{j=1}^{l}I_j}\prod_{j=1}^{l}\langle\prod_{i=1}^{n}[F_i(y_i);\ ]^{k_{ij}^{(l)}}\rangle. \tag{2.98}$$

For different partitions $\sum I_j$ we can get the same matrix $k_{ij}^{(l)}$. We have

$$\prod_{i=1}^{n}n_i!/[(\prod_{i=1}^{n}\prod_{j=1}^{l}k_{ij}!)\ l!] \tag{2.99}$$

partitions $\sum I_j$ with matrix $(k_{ij}^{(l)})$, defined by (2.97). Since permutations of the index j do not lead to new partitions, we get a factor $1/l!$ and we obtain from (2.98)

$$\langle F_1(y_1)^{n_1}\ldots F_n(y_n)^{n_n}\rangle = \sum_{l=1}^{m}\sum_{\substack{k^{(l)}\\ \sum_{j=1}^{l}k_{ij}^{(l)}=n_i}} \frac{\prod_{i=1}^{n}n_i!}{(\prod_{i=1}^{n}\prod_{j=1}^{l}k_{ij}^{(l)}!)l!}\prod_{j=1}^{l}\langle\prod_{i=1}^{n}[F_i(y_i);\ ]^{k_{ij}^{(l)}}\rangle. \tag{2.100}$$

The matrix $(k_{ij}^{(l)})$ can be brought to block form by permutations of rows or columns. Every block form of $(k_{ij}^{(l)})$ defines a partition $\sum N = \{1,\ldots,n\}$, where N is a set of indices for rows of a block in $(k_{ij}^{(l)})$. So we obtain from (2.100)

$$\langle F_1(y_1)^{n_1}\ldots F_n(y_n)^{n_n}\rangle = \sum_{\sum N=\{1,\ldots,n\}} \prod_N \Bigg\{ \sum_{l(N)\geq 1} \sum_{\substack{(k_{ij}^{l(N)}) \text{ irreducible} \\ \sum_{j=1}^{l(N)} k_{ij}^{l(N)}=n_i}} \frac{\prod_{i\in N} n_i!}{(\prod_{i\in N}\prod_{j=1}^{l(N)} k_{ij}^{l(N)}!)l!}$$

$$\prod_{j=1}^{l(N)} \langle \prod_{i\in N} [F_i(y_i);\ ]^{k_{ij}^{l(N)}}\rangle \Bigg\}. \tag{2.101}$$

From the definition of the truncated expectation value we get

$$\langle \prod_{i=1}^{n} [F_i(y_i)^{n_i};\ ]\rangle = \sum_{l\geq 1} \sum_{\substack{k^{(l)} irreducible \\ \sum_{j=1}^{l} k_{ij}^{(l)}=n_i}} \frac{\prod_{i=1}^n n_i!}{(\prod_{i=1}^n \prod_{j=1}^l k_{ij}^{(l)}!)l!} \prod_{j=1}^{l} \langle \prod_{i=1}^{n} [F_i(y_i);\ ]^{k_{ij}^{(l)}}\rangle . \surd \tag{2.102}$$

Expansion of the e-functions in Eq. (2.91) for the Mayer amplitude gives

$$\mathcal{M}(x_1,\ldots,x_n|\psi) = \frac{a^{-\nu n}}{n!} \sum_{\substack{\{n_i\}_{i=1,\ldots,n} \\ n_i\geq 1}} \frac{(-\lambda)^{\sum_{i=1}^n n_i}}{\prod_{i=1}^n n_i!} \langle \mathcal{V}(x_1)^{n_1};\ldots;\mathcal{V}(x_n)^{n_n}\rangle . \tag{2.103}$$

With the help of Lemma 2.4.3. it follows from (2.103)

$$\mathcal{M}(x_1,\ldots,x_n|\psi) = \frac{a^{-\nu n}}{n!} \sum_{l\geq 1} \sum_{k^{(l)} \; irreducible \; n\times l \; matrix} \frac{1}{(\prod_{j=1}^l \prod_{i=1}^n k_{ij}^{(l)}!)l!} \prod_{j=1}^{l} [(-\lambda)^{\sum_{i=1}^n k_{ij}^{(l)}} \langle \prod_{i=1}^{n} [\mathcal{V}(x_i);\ ]^{k_{ij}^{(l)}}\rangle]. \tag{2.104}$$

With the definition of the Feynman amplitude F (see (2.76)) we obtain the following Theorem 2.4.4. for representation of the Mayer amplitudes by Feynman amplitudes.

Theorem 2.4.4. *The formal power series in λ for the Mayer amplitude is*

$$\mathcal{M}(x_1,\ldots,x_n|\psi) = \frac{a^{-\nu n}}{n!} \sum_{l\geq 1} \sum_{k^{(l)}} \frac{\prod_{j=1}^l (\sum_{i=1}^n k_{ij}^{(l)})!}{(\prod_{j=1}^l \prod_{i=1}^n k_{ij}^{(l)}!)l!} \prod_{j=1}^{l} \mathcal{F}(x_1^{k_1},\ldots,x_n^{k_{n_j}}) \tag{2.105}$$

The sum is over all irreducible matrices $k^{(l)}$ whose entries are non negative integers.

Remark : Feynman amplitudes F are represented by connected Feynman diagrams. The condition of irreducibility for the matrices $k^{(l)}$ in (2.105) corresponds to the fact that Mayer amplitudes are represented by point connected Feynman diagrams. Eq. (2.105) is an explicit expression for the representation of Mayer amplitudes by connected Feynman diagrams.

2.5. The Tree Graph Formula

To obtain estimates for the absolute value of activities $A(X|\psi)$ the tree graph formula is more useful than the representation (2.6) by Mayer graphs in conjugated space. The activities can be expressed by ***tree graphs***, i.e. Mayer graphs without loops. We will need some notations and definitions to formulate the tree graph formula. An *n-tree* is defined by the following function

$$\eta \ : \ \{2,\ldots,n\} \to \{1,\ldots,n-1\} \qquad \text{with } \eta(i) < i \tag{2.106}$$

$1,\ldots,n$ are called the *vertices* of the n-tree η. Vertex 1 is called the *root* of the n-tree η. The links of η are the pairs $(\eta(i), i)$, $i = 2, \ldots n$. For the $n-1$ real parameters $s_1,\ldots,s_{n-1}$ we define

$$f(\eta|s) = \prod_{a=2}^{n} s_{a-2}s_{a-3}\cdots s_{\eta(a)}. \tag{2.107}$$

Empty products are 1 as usual. For example $f(\eta|s) = 1$ if $n = 2$. Fig. 2.2 shows an example for an n-tree, $n = 8$.

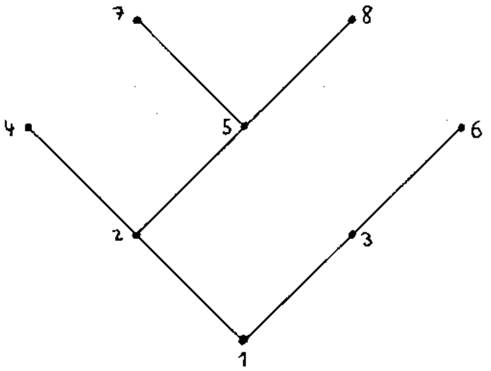


Fig. 2.2 n-tree defined by η:$\{2,\ldots,8\}\to\{1,\ldots,7\}$, $\eta(2)=\eta(3)=1$, $\eta(4)=\eta(5)=2$, $\eta(6)=3$, $\eta(7)=\eta(8)=5$. 1 is the root of the n-tree η. The vertices 4,6,7,8 are called maximal vertices of η. For $f(\eta|s)$ we obtain $f(\eta|s)=s_1 s_2^2 s_3^2 s_4 s_5^2 s_6$.

A tree T with point set X, $|X| = n$, is an n-tree η together with a bijective map (*labelling*)

$$\tilde{x} \ : \ \begin{cases} \{1,\ldots,n\} \to X \\ i \mapsto x_i \ . \end{cases} \tag{2.108}$$

For integration over the parameters $s_1,\ldots,s_{n-1} \in [0,1]$ we abbreviate

$$\int d\sigma_{n-1} = \int_0^1 ds_{n-1} \cdots \int_0^1 ds_1. \tag{2.109}$$

We will define a propagator $v[s]$ modified by the parameters $s_1,\ldots,s_n$

$$v_{x_i x_j}[s] = \begin{cases} s_i s_{i+1} \ldots s_{j-1} v_{x_i x_j} & \text{if } i < j \\ s_j s_{j+1} \ldots s_{i-1} v_{x_i x_j} & \text{if } i > j \\ v_{x_i x_i} & \text{if } i = j. \end{cases} \tag{2.110}$$

The number of links in an n-tree η, which emerge from vertex l is denoted by $d_\eta(l)$. With the definitions and notations given above we have the following

Theorem 2.5.1. (Tree graph formula). *Let $A(X|\psi)$ be the activities in the polymer representation for $Z(\Lambda|\psi)$ and let $x \in X$ be an arbitrary point. Then we have*

$$A(X|\psi) = \sum_{\eta} \sum_{\tilde{x}(1)=x\in X}^{\tilde{x}} \int d\sigma_{n-1} f(\eta|s) \int d\mu_{v[s]}(\phi) \prod_{a=2}^{n} \left[\frac{\partial}{\partial \phi_{\tilde{x}(a)}} v_{\tilde{x}(a)\tilde{x}(\eta(a))} \frac{\partial}{\partial \phi_{\tilde{x}(\eta(a))}} \right] \prod_{b=1}^{n} F_{\tilde{x}(b)}(\phi_{\tilde{x}(b)} + \psi_{\tilde{x}(b)}) \tag{2.111a}$$

and equivalently

$$A(X|\psi) = \sum_{\eta} \sum_{\tilde{x}(1)=x\in X}^{\tilde{x}} [\prod_{(ij)\in\eta} v_{x_i x_j}] \int d\sigma_{n-1} f(\eta|s) \int d\mu_{v[s]}(\phi) \left[\prod_{i=1}^{n} \frac{\partial^{d_i(\eta)}}{\partial \phi_{x_i}^{d_i(\eta)}} F_{x_i}(\phi_{x_i} + \psi_{x_i})\right] \tag{2.111b}$$

with $n = |X| \geq 2$. The sum is over all n-trees η and labellings $\tilde{x}$: $\{1,\ldots,n\} \to X$, $\tilde{x}$ bijective with $\tilde{x}(1) = x$.

PROOF: see app. B.

An n-tree η with labelling $\tilde{x}$ is related to an algebraic expression $I(\eta,\tilde{x})$ by the following rules. For all vertices x_k we set $F_{x_k}(\phi_{x_k} + \psi_{x_k})$ and for all links (ij) in η we set $\frac{\partial}{\partial\phi_{x_i}} v_{x_i x_j} \frac{\partial}{\partial\phi_{x_j}}$. The differential operators acts on the product of the F-terms. After integration by $\int d\sigma_{n-1} f(\eta|s) \int d\mu_{v[s]}(\phi)$ we obtain $I(\eta,\tilde{x})$. The activity $A(X|\psi)$ is

$$A(X|\psi) = \sum_{\eta} \sum_{\tilde{x}(1)=x\in X}^{\tilde{x}} I(\eta,\tilde{x}). \tag{2.112}$$

Remark: There exists $(n-1)!$ n-trees η $(n \geq 2)$.

PROOF (BY INDUCTION): For $n = 2$ there exists one tree with link (12). Let the assertion be true for n. A new vertex $(n+1)$ can be connected to an n-tree η in n ways. We obtain an (n+1)-tree. The number of (n+1)-trees η is therefore $n(n-1)! = [(n+1)-1]!$.$\checkmark$

3. ESTIMATES FOR SIMPLE MAYER EXPANSIONS

In this chapter we consider estimates for the absolute value of the activity

$$M(X|\psi) = -\delta_{1,|X|} + A(X|\psi) = \langle \prod_{x\in X} [F_x(\phi_x + \psi_x) - 1;\ ] \rangle \tag{3.1}$$

where X is a polymer with $|X| \geq 2$. From (3.1) we see that estimates for the activity are at the same time estimates for truncated expectation values and Mayer amplitudes.

3.1. SUBTRACTION TRICK AND TREE ESTIMATE

We need an assumption to obtain estimates with the help of the Cauchy inequality for the functions $F_x(\phi_x)$, $x \in (a\mathbb{Z})^\nu$, in the definition (2.1) of the partition function.

Assumption: Let $F_x(\phi_x)$ be holomorph and bounded in

$$S_\kappa = \{\phi_x \in \mathbb{C} \mid |Im\ \phi_x| \leq \kappa\} \tag{2.20}$$

for all $x \in (a\mathbb{Z})^\nu$.

F_x, $x \in (a\mathbb{Z})^\nu$, is a distribution for the discrete Gaussian model and the nonlinear σ-model (see examples b) and c) in ch. 2., p.19-20). Nevertheless the above assumption can be fulfilled in this examples, if the propagator v is positive definite. For that we have the following Lemma

Lemma 3.1.1. (subtraction trick). *Let ϕ be an N-dimensional field with propagator v. If*

$$v - \delta \mathbb{1} \geq 0 \tag{3.2}$$

for a positive constant $\delta > 0$ we have

$$\int d\mu_v(\phi) \prod_{x\in\Lambda} F_x(\phi_x + \psi_x) = \int d\mu_{v-\delta\mathbb{1}}(\Phi) \prod_{x\in\Lambda} \left\{ (2\pi\delta)^{-\frac{N}{2}} \int d\xi_x e^{-\frac{1}{2\delta}(\xi_x - \Phi_x - \psi_x)^2} F_x(\xi_x) \right\}. \tag{3.3}$$

PROOF: From the convolution formula (Lemma 3.1.2.) we obtain

$$\int d\mu_v(\phi) \prod_{x\in\Lambda} F_x(\phi_x + \psi_x) = \int d\mu_{v-\delta\mathbb{1}}(\Phi) d\mu_{\delta\mathbb{1}}(\xi) \prod_{x\in\Lambda} F_x(\Phi_x + \xi_x + \psi_x) \tag{3.4}$$

(3.3) follows from (3.4) and the definition of the Gaussian measure with covariance $\delta\mathbb{1}$.√

Remark: $F_x(\phi_x + \psi_x)$ is replaced by $(2\pi\delta)^{-\frac{N}{2}} \int d\xi_x e^{-\frac{1}{2}(\xi_x - \phi_x - \psi_x)^2} F_x(\xi_x)$, if v is replaced by $v - \delta\mathbb{1}$. The integral $\int d\xi_x e^{-\frac{1}{2}(\xi_x - \phi_x - \psi_x)^2} F_x(\xi_x)$ is holomorph for $F_x(\xi_x) = \sum_{n\in\mathbb{N}} \delta(\xi_x - 2\pi n)$ (discrete Gaussian model) and for $F_x(\xi_x) = \delta(\xi_x^2 - 1)$ (nonlinear σ-model). Lemma 3.1.3. shows an explicit expression for the constant δ for propagator $v = (-\Delta + m^2)^{-1}$.

The convolution formula used in the proof of Lemma 3.1.1. is

Lemma 3.1.2. (convolution formula).

$$\int d\mu_{v_1+v_2}(\phi + \psi) \prod_{x\in\Lambda} F_x(\phi_x + \psi_x) = \int d\mu_{v_1}(\phi) \int d\mu_{v_2}(\psi) \prod_{x\in\Lambda} F_x(\phi_x + \psi_x). \tag{3.5}$$

PROOF: Let the Fourier transform $\tilde{F}_x(q_x)$ of $F_x(\phi_x)$ be defined by

$$F_x(\phi_x) = \int dq_x\, \tilde{F}_x(q_x) e^{iq_x\phi_x}. \tag{2.2}$$

From the Gaussian integral

$$\int d\mu_v(\phi) \prod_{x\in\Lambda} e^{iq_x\phi_x} = e^{-\frac{1}{2}(q,vq)_\Lambda}$$

with $(q, vq)_\Lambda = \sum_{x,y\in\Lambda} q_x v_{xy} q_y$ follows

$$\int d\mu_{v_1+v_2}(\phi + \psi) \prod_{x\in\Lambda} F_x(\phi_x + \psi_x) = \int \prod_{x\in\Lambda} [dq_x \tilde{F}_x(q_x)] e^{-\frac{1}{2}(q,(v_1+v_2)q)_\Lambda} =$$

$$= \int \prod_{x\in\Lambda} [dq_x \tilde{F}_x(q_x)] \int d\mu_{v_1}(\phi) \int d\mu_{v_2}(\psi) \prod_{x\in\Lambda} e^{iq_x(\psi_x+\phi_x)} =$$

$$= \int d\mu_{v_1}(\phi) \int d\mu_{v_2}(\psi) \prod_{x\in\Lambda} F_x(\phi_x + \psi_x).\ \surd \tag{3.6}$$

Lemma 3.1.3. *Let $v = (-\Delta + M^2)^{-1} a^{-2}$. Δ is the Laplacian operator on the ν-dimensional lattice $(a\mathbb{Z})^\nu$. Then we have*

$$v - \delta\mathbb{1} \geq 0 \tag{3.2}$$

for $\delta = (4\nu + (aM)^2)^{-1}$ and

$$a^{-\nu} \int_{x\in(a\mathbb{Z})^\nu} v_{xy} - \delta = [(aM)^2(1 + \frac{(aM)^2}{4\nu})]^{-1}. \tag{3.7}$$

PROOF: From the definition of v_{xy} follows

$$a^2(-\Delta+M^2)^{-1}v_{xy}=\delta_{xy}. \tag{3.8}$$

The Fourier transform $\tilde{v}(k)$ of v_{xy} is defined by

$$v_{xy}=\frac{a^\nu}{(2\pi)^\nu}\int_{k_\mu\in[-\frac{\pi}{a},\frac{\pi}{a}]}d^\nu k\ \tilde{v}(k)e^{ik(x-y)} \tag{3.9}$$

From $a^2(-\Delta)f(x)=\sum_{y\ nn\ x}^{\nu}[f(x)-f(y)]$ (the sum is over all nearest neighbors y of x) we obtain

$$a^2(-\Delta+M^2)e^{ik(x-y)}=[2\sum_{\mu=1}^{\nu}(1-\cos k_\mu a)+(Ma)^2]e^{ik(x-y)}. \tag{3.10}$$

From (3.10) and

$$\frac{a^\nu}{(2\pi)^\nu}\int_{k_\mu\in[-\frac{\pi}{a},\frac{\pi}{a}]}d^\nu k\ e^{ik(x-y)}=\delta_{xy} \tag{3.11}$$

follows

$$\tilde{v}(k)=\frac{1}{2\sum_{\mu=1}^{\nu}(1-\cos k_\mu a)+(aM)^2}. \tag{3.12}$$

Insertion of (3.12) in (3.9) gives

$$v_{xy}=\frac{a^\nu}{(2\pi)^\nu}\int_{k_\mu\in[-\frac{\pi}{a},\frac{\pi}{a}]}d^\nu k\frac{e^{ik(x-y)}}{2\sum_{\mu=1}^{\nu}(1-\cos k_\mu a)+(aM)^2}. \tag{3.13}$$

Because of

$$\frac{1}{2\sum_{\mu=1}^{\nu}(1-\cos k_\mu a)+(aM)^2}\geq\frac{1}{4\nu+(aM)^2} \tag{3.14}$$

and Eq. (3.13) we obtain

$$v_{xy}-\delta\delta_{xy}\geq 0 \qquad \text{for } \delta=[4\nu+(aM)^2]^{-1}. \tag{3.15}$$

Since δ_{xy} is the kernel of the operator $\mathbb{I}$, we have

$$v-\delta\mathbb{I}\geq 0. \tag{3.2}$$

From $\sum_{x\in(a\mathbb{Z})^\nu}v_{xy}=\tilde{v}(0)=\frac{1}{(aM)^2}$ and (3.15) we get (3.7). ✓

The sum over the trees in the tree graph formula can be estimated by the following tree estimate ([8], [6]):

Lemma 3.1.4. (tree estimate). *We have the following inequality for the sum over all n-trees η with vertices $l\in\{1,2,\ldots,n\}$ and non negative $\mu(l)\geq 0$*

$$\sum_\eta\int_0^1 d\sigma_{n-1}f(\eta|s)\prod_{l=2}^{n}[\mu(l)\mu(\eta(l))]\leq\prod_{l=2}^{n}[\mu(l)e^{\mu(l-1)}]. \tag{3.16}$$

PROOF ([8]): We have to find an upper bound for

$$S(\mu,n)=\sum_\eta\int_0^1 ds_1\ldots ds_{n-1}\prod_{l=2}^{n}[\mu(l)s_{l-2}s_{l-3}\ldots s_{\eta(l)}\mu(\eta(l))]. \tag{3.17}$$

The summation over n-trees η can be replaced by summation over $k=\eta(l)$ from 1 to $l-1$. Therefore

$$S(\mu,n)=\int_0^1 ds_1\ldots ds_{n-1}\prod_{l=2}^{n}[\sum_{k=1}^{l-1}\mu(l)s_{l-2}s_{l-3}\ldots s_k\mu(k)]. \tag{3.18}$$

Because of $\int_0^1 dsue^{su} \leq e^u$ we obtain

$$S'(\mu,n) \leq \mu(n)e^{\mu(n-1)}S'(\mu,n-1) \tag{3.19}$$

with

$$S'(\mu,n) = \int_0^1 ds_1 \dots ds_{n-1} \prod_{l=2}^{n}[\sum_{k=1}^{l-1} \mu(l)s_{l-2}\dots s_k\mu(k)]e^{\sum_{k=1}^{n-1} s_{n-1}\dots s_k\mu(k)} \tag{3.20}$$

From (3.19) follows

$$S'(\mu,n) \leq \prod_{l=2}^{n}[\mu(l)e^{\mu(l-1)}]S'(\mu,1) = \prod_{l=2}^{n}[\mu(l)e^{\mu(l-1)}] \tag{3.21}$$

and the assertion (3.16) follows from $S(\mu,n) \leq S'(\mu,n)$. $\surd$

The following generalization of the Lemma of Battle [14] is a corollary of Lemma 3.1.4. .

Lemma 3.1.5. *We have for all $n_l \in \mathbb{N} \cup \{-1\}$, $l = 1,\dots,n$*

$$\sum_\eta \int_0^1 d\sigma_{n-1} f(\eta|s) \prod_{l=1}^{n}[d_l(\eta)+n_l]! \leq \frac{(n_n+1)!}{(n_1+2)} \prod_{l=1}^{n-1}[2^{n_l+3}(n_l+1)!]. \tag{3.22}$$

The sum is over all n-trees η and $d_l(\eta)$ = number of links in η, which emerge from vertex l.

PROOF: From Lemma 3.1.4. follows ($\mu(l) = t_l$)

$$\sum_\eta \int_0^1 d\sigma_{n-1} f(\eta|s) \prod_{l=1}^{n} t_l^{d_l(\eta)} \leq \prod_{l=1}^{n-1}(t_{l+1}e^{t_l}). \tag{3.23}$$

Multiplication of inequality (3.23) with $t^{n_1+1}[\prod_{l=1}^n t_l^{n_l}e^{-2t_l}]e^{t_n}$ yields

$$\sum_\eta \int_0^1 d\sigma_{n-1} f(\eta|s) t_1^{d_1(\eta)+n_1+1}e^{-2t_1} \prod_{l=2}^{n-1}(t_l^{d_l(\eta)+n_l}e^{-2t_l}) t_n^{d_n(\eta)+n_n}e^{-t_n} \leq \prod_{l=1}^{n} t_l^{n_l+1}e^{-t_l}. \tag{3.24}$$

Because of

$$\int_0^\infty dt\; t^n e^{-t} = n!\,, \qquad \int_0^\infty dt\; t^n e^{-2t} = \frac{n!}{2^{n+1}} \tag{3.25}$$

we obtain from integrating (3.24) over $t_1,\dots,t_n$ from 0 to ∞

$$\sum_\eta \int_0^1 d\sigma_{n-1} f(\eta|s) \frac{(d_1(\eta)+n_1+1)!}{2^{d_1(\eta)+n_1+2}} \prod_{l=2}^{n-1}[\frac{(d_l(\eta)+n_l)!}{2^{d_l(\eta)+n_l+1}}](d_1(\eta)+n_1)! \leq \prod_{l=1}^{n}(n_l+1)!\,. \tag{3.26}$$

From the relation

$$\sum_{l=1}^{n-1} d_l(\eta) = 2(n-1)-1 \tag{3.27}$$

follows

$$\sum_\eta \int_0^1 d\sigma_{n-1} f(\eta|s)(d_1(\eta)+n_1+1)! \prod_{l=2}^{n}[d_l(\eta)+n_l]! \leq \prod_{l=1}^{n-1}[2^{n_l+3}(n_l+1)!](n_n+1)!\,. \tag{3.28}$$

The assertion (3.22) follows from (3.28), since $d_1(\eta) \geq 1$.$\surd$

The following special cases of Lemma 3.1.5. will be used

$$\sum_\eta \int_0^1 d\sigma_{n-1} f(\eta|s) \prod_{l=1}^{n}[d_l(\eta)]! \leq \frac{8^{n-1}}{2} \tag{3.29}$$

and

$$\sum_{\eta}\int_0^1 d\sigma_{n-1} f(\eta|s)\prod_{l=1}^{n}[d_l(\eta)-1]! \leq 4^{n-1}. \tag{3.30}$$

3.2. Estimates with the help of the Cauchy inequality

For the following Theorem 3.2.1. we will define the distance $L(x_1, x_2, \ldots, x_n)$ of n points $x_1, x_2, \ldots, x_n \in (a\mathbb{Z})^\nu$. $L(x_1, \ldots, x_n)$ is the length of the shortest polygon, which connects $x_1, x_2, \ldots, x_n$. Thus

$$L(X) = L(x_1, \ldots, x_n) = \min_T \sum_{(ij)\in T} \|x_i - x_j\|, \qquad X = \{x_1, \ldots, x_n\}. \tag{3.31}$$

$\|.\|$ is the euclidean norm on the lattice. T denotes trees with n vertices $x_1, \ldots, x_n$.

Theorem 3.2.1. (Estimates for truncated expectation values). *Let $F_x(\phi_x)$ be holomorph and bounded functions in the complex strip $|Im\phi_x| \leq \kappa$ for $x \in X = \{x_1, \ldots, x_n\} \subset (a\mathbb{Z})^\nu, n \geq 2$. The constants b_κ and b_κ^X are defined by*

$$b_\kappa \stackrel{\text{def}}{=} \min_{c\in\mathbb{R}} \sup_{x\in(a\mathbb{Z})^\nu} \sup_{\substack{\phi_x\in\mathbb{C}\\ |Im\ \phi_x|=\kappa}} |F_x(\phi_x) - c| \tag{3.32a}$$

and

$$b_\kappa^X \stackrel{\text{def}}{=} \min_{c\in\mathbb{R}} \sup_{x\in X} \sup_{\substack{\phi_x\in\mathbb{C}\\ |Im\ \phi_x|=\kappa}} |F_x(\phi_x) - c| \tag{3.32b}$$

For the truncated expectation value

$$M(X|\psi) = \langle \prod_{x\in X} [F_x(\phi_x + \psi_x);\ ]\rangle \tag{3.33}$$

real ψ_x, we have the following estimates

(i) *Let the propagator v be exponentially decreasing and*

$$|v_{xy}| \leq De^{-m\|x-y\|}, \qquad x, y \in (a\mathbb{Z})^\nu. \tag{3.34}$$

Then we have

$$|M(X|\psi)| \leq (n-1)!\frac{\kappa^2 e^{-mL(X)}}{16D}\left[\frac{8Db_\kappa^X}{\kappa^2}\right]^n. \tag{3.35}$$

(ii) *If*

$$a^{-\nu}\int_{x\in(a\mathbb{Z})^\nu} |v_{xy}| = \frac{1}{(ma)^2} < \infty, \tag{3.36}$$

then we have

$$a^{-\nu(n-1)}\int_{x_2,\ldots,x_n\in(a\mathbb{Z})^\nu} |M(X|\psi)| \leq (n-1)!\frac{(ma\kappa)^2}{16}\left[\frac{8b_\kappa}{(ma\kappa)^2}\right]^n. \tag{3.37}$$

PROOF: Because of

$$\min_{c\in\mathbb{R}} \sup_{x\in X} \sup_{\substack{\phi_x\in\mathbb{C}\\ |Im\ \phi_x|=\kappa}} |(F_x(\phi_x) - 1) - c| = \min_{c\in\mathbb{R}} \sup_{x\in X} \sup_{\substack{\phi_x\in\mathbb{C}\\ |Im\ \phi_x|=\kappa}} |F_x(\phi_x) - c| = b_\kappa^X \tag{3.38}$$

the constants b_κ and b_κ^X are independent of $\psi_x \in \mathbb{R}$. Thus we can suppose $\psi = 0$. Let us set $M(X) = M(X|\psi = 0)$.

(i) From Theorem 2.5.1. (tree graph formula) and (3.34) follows

$$|M(X)| \leq D^{n-1} e^{-mL(X)} \sum_\eta \sum_{\substack{\bar{x} \\ \bar{x}(1)=x\in X}} \int d\sigma_{n-1} f(\eta|s) \int d\mu_{v[s]}(\phi) [\prod_{l=1}^n |\frac{\partial^{d_l(\eta)}}{\partial \phi_{x_l}^{d_l(\eta)}} F_{x_l}(\phi_{x_l})|]. \quad (3.39)$$

The Cauchy inequality reads

$$|\frac{\partial^{d_l(\eta)}}{\partial \phi_{x_l}^{d_l(\eta)}} F_{x_l}(\phi_{x_l})| \leq \frac{d_l(\eta)!}{\kappa^{d_l(\eta)}} b_\kappa^X. \quad (3.40)$$

The Gaussian expectation values are estimated by

$$|\langle F(\phi) \rangle| \leq \sup_\phi |F(\phi)|. \quad (3.41)$$

From (3.39), (3.40) and (3.41) follows

$$|M(X)| \leq (n-1)! D^{n-1} e^{-mL(X)} \sum_\eta \int_0^1 d\sigma_{n-1} f(\eta|s) \prod_{l=1}^n \left[\frac{d_l(\eta)!\, b_\kappa^X}{\kappa^{d_l(\eta)}}\right]. \quad (3.42)$$

The relation $\sum_{l=1}^n d_l(\eta) = 2(n-1)$ yields

$$|M(X)| \leq (n-1)!\, D^{n-1} e^{-mL(X)} (b_\kappa^X)^n \frac{1}{\kappa^{2(n-1)}} \sum_\eta \int_0^1 d\sigma_{n-1} f(\eta|s) \prod_{l=1}^n d_l(\eta)!\,. \quad (3.43)$$

From the special case (3.29) of Lemma 3.1.5. we obtain

$$|M(X)| \leq (n-1)!\, D^{n-1} e^{-mL(X)} \frac{(b_\kappa^X)^n 8^{n-1}}{2\kappa^{2(n-1)}}. \quad (3.44)$$

(ii) The tree graph formula for $M(X)$ reads

$$M(X) = \sum_\eta \sum_{\substack{\bar{x} \\ \bar{x}(1)=x\in X}} \int d\sigma_{n-1} f(\eta|s) \int d\mu_{v[s]}(\phi) (\prod_{(ij)\in\eta} v_{x_i x_j}) \left[\prod_{l=1}^n \frac{\partial^{d_l(\eta)}}{\partial \phi_{x_l}^{d_l(\eta)}} F_{x_l}(\phi_{x_l})\right]. \quad (3.45)$$

Analog to the proof for (i) we obtain from the Cauchy inequality (3.40) and the estimate (3.41) for Gaussian expectation values

$$|M(X|\psi)| \leq \frac{(b_\kappa)^n}{\kappa^{2(n-1)}} \sum_\eta \sum_{\substack{\bar{x} \\ \bar{x}(1)=x\in X}} \int d\sigma_{n-1} f(\eta|s) (\prod_{(ij)\in\eta} v_{x_i x_j}) \prod_{l=1}^n d_l(\eta)!\,. \quad (3.46)$$

Integration of inequality (3.46) over $x_2, \ldots, x_n \in (a\mathbb{Z})^\nu$ gives a factor $(ma)^{-2(n-1)}$ on the rhs. Thus

$$a^{-\nu(n-1)} \int_{x_2,\ldots,x_n \in (a\mathbb{Z})^\nu} |M(X|\psi)| \leq (n-1)! \frac{b_\kappa)^n}{(ma\kappa)^{2(n-1)}} \sum_\eta \int_0^1 d\sigma_{n-1} f(\eta|s) \prod_{l=1}^n d_l(\eta)!\,. \quad (3.47)$$

From the special case (3.29) of Lemma 3.1.5. follows the assertion (ii). $\surd$

Remark : For translation invariant theories the constant b_κ^X is independent from X. If $b_\kappa^X (b_\kappa)$ is replaced by $b_{\kappa+\kappa'}^X (b_{\kappa+\kappa'})$ in the inequalities (3.35) and (3.37) the assertion of Theorem 3.2.1. is valid for all $\psi_x \in S_{\kappa'}$.

Theorem 3.2.2. (Estimates for truncated expectation values of fields with N components). *Let the functions*

$$F_x \ : \ \begin{cases} \mathbb{C}^N \to \mathbb{C} \\ (\phi_{x,1}, \ldots, \phi_{x,N}) \mapsto F_x(\phi_{x,1}, \ldots, \phi_{x,N}) \end{cases} \tag{3.48}$$

be holomorph and bounded in $S_\kappa^N = S_\kappa \times \cdots \times S_\kappa$ *with* $S_\kappa = \{\phi \in \mathbb{C} \mid |Im\ \phi| \leq \kappa\}$ *for all* $x \in (a\mathbb{Z})^\nu$. *The constants* b_κ, b_κ^X *are defined by*

$$b_\kappa \stackrel{\text{def}}{=} \min_{c\in\mathbb{R}} \sup_{x\in(a\mathbb{Z})^\nu} \sup_{\substack{\phi_{x,i}\in\mathbb{C}\ ,\ |Im\ \phi_{x,i}|=\kappa \\ i\in\{1,\ldots,N\}}} |F_x(\phi_{x,1}, \ldots, \phi_{x,N}) - c| \tag{3.49a}$$

and

$$b_\kappa^X \stackrel{\text{def}}{=} \min_{c\in\mathbb{R}} \sup_{x\in X} \sup_{\substack{\phi_{x,i}\in\mathbb{C}\ ,\ |Im\ \phi_{x,i}|=\kappa \\ i\in\{1,\ldots,N\}}} |F_x(\phi_{x,1}, \ldots, \phi_{x,N}) - c|. \tag{3.49b}$$

Let the kernel of the propagator be defined by

$$v_{xy,ij} = \delta_{ij} v_{xy}, \qquad i,j \in \{1, \ldots N\}. \tag{3.50}$$

We have for the truncated expectation value

$$M(X|\psi) = \langle \prod_{x\in X} [F_x(\phi_x + \psi_x);\ ]\rangle \tag{3.51}$$

where $X = \{x_1, \ldots, x_n\} \subset (a\mathbb{Z})^\nu$, $n \geq 2$, and $\langle\ .\ \rangle = \int d\mu_v(\phi)[\ .\ ]$ *the following estimates*

(i) *Let the propagator* v *be exponentially decreasing and*

$$|v_{xy}| \leq De^{-m\|x-y\|}, \qquad x,y \in (a\mathbb{Z})^\nu. \tag{3.34}$$

Then we have

$$|M(X|\psi)| \leq (n-1)!\ \frac{\kappa^2 e^{-mL(X)}}{16\gamma DN^2} \left[\frac{8D\gamma N^2 b_\kappa^X}{\kappa^2}\right]^n. \tag{3.52}$$

(ii) *Let*

$$a^{-\nu} \int_{x\in(a\mathbb{Z})^\nu} |v_{xy}| = \frac{1}{(ma)^2} < \infty. \tag{3.36}$$

Then we have

$$a^{-\nu(n-1)} \int_{x_2,\ldots,x_n\in(a\mathbb{Z})^\nu} |M(X|\psi)| \leq (n-1)!\ \frac{(ma\kappa)^2}{16\gamma N^2} \left[\frac{8b_\kappa\gamma N^2}{(ma\kappa)^2}\right]^n. \tag{3.53}$$

PROOF: We can suppose $\psi = 0$ (see proof of Theorem 3.2.1.). Abbreviate $M(X) = M(X|\psi = 0)$.

(i) From the tree graph formula (Theorem 2.5.1.) and (3.34) we obtain

$$|M(X)| \leq (\gamma D)^{n-1} e^{-mL(X)} \sum_\eta \sum_{\substack{\bar{x} \\ \bar{x}(1)=x\in X}} \int d\sigma_{n-1} f(\eta|s) \int d\mu_{\gamma v[s]}(\phi)$$
$$\{|\prod_{l=2}^n (\sum_{i=1}^N \frac{\partial}{\partial\phi_{x_l,i}} \frac{\partial}{\partial\phi_{x_{\eta(l)},i}}) \prod_{l=1}^n F_{x_l}(\phi_{x_l})|\}. \tag{3.54}$$

From the multinominial Theorem follows for the bracket $\{\ldots\}$ in (3.54)

$$\{\ldots\} \leq \prod_{l=1}^n \sum_{\substack{m_1,\ldots,m_N \\ \sum m_j = d_l(\eta)}} \frac{d_l(\eta)!}{m_1! \ldots m_N!} |(\prod_{i=1}^N \frac{\partial^{m_i}}{\partial\phi_{x_{l,i}}^{m_i}}) F_{x_l}(\phi_{x_l})|. \tag{3.55}$$

Because of

$$\sum_{\substack{m_1,\ldots,m_N \\ \sum m_j = d_l(\eta)}} 1 \le N^{d_l(\eta)} \tag{3.56}$$

and the Cauchy inequality

$$|(\prod_{i=1}^{N} \frac{\partial^{m_i}}{\partial \phi_{x_l,i}^{m_i}}) F_{x_l}(\phi_{x_l})| \le \frac{m_1! \ldots m_N!}{\kappa^{d_l(\eta)}} b_\kappa^X \tag{3.57}$$

we obtain from (3.55)

$$\{\ldots\} \le \frac{d_l(\eta)!}{\kappa^{d_l(\eta)}} N^{d_l(\eta)} b_\kappa^X . \tag{3.58}$$

Inserting (3.58) in (3.54) yields

$$|M(X)| \le (\gamma D)^{n-1} (\frac{N^2}{\kappa^2})^{n-1} (b_\kappa^X)^n (n-1)!\; e^{-mL(X)} \sum_\eta \int_0^1 d\sigma_{n-1} f(\eta|s) \prod_{l=1}^{n} d_l(\eta)! \,. \tag{3.59}$$

From the special case (3.29) of Lemma 3.1.5. follows the assertion (3.52).

(ii) Integration of the tree graph formula over $x_2, \ldots, x_n \in (a\mathbb{Z})^\nu$ gives

$$a^{-\nu(n-1)} \int_{x_2,\ldots,x_n \in (a\mathbb{Z})^\nu} |M(X|\psi)| \le (\frac{\gamma}{(ma)^2})^{n-1} \sum_\eta \sum_{\substack{\bar{x} \\ \bar{x}(1)=x \in X}} \int d\sigma_{n-1} f(\eta|s) \int d\mu_{\gamma v[s]}(\phi)$$

$$\left\{ |\prod_{l=2}^{n} (\sum_{i=1}^{N} \frac{\partial}{\partial \phi_{x_l,i}} \frac{\partial}{\partial \phi_{x_{\eta(l)},i}}) \prod_{l=1}^{n} F_{x_l}(\phi_{x_l})| \right\} . \tag{3.60}$$

The proof of (3.53) is analog to the proof of (i) if we replace

$$e^{-mL(X)} \to 1, \qquad D \to \frac{1}{(ma)^2} . \;\checkmark \tag{3.61}$$

Remarks:

(i) The remark after Theorem 3.2.1. is also valid for fields with N components.

(ii) If we replace κ^2 by $\kappa^2/(\gamma N^2)$ we get the assertion of Theorem 3.2.2. from the assertion of Theorem 3.2.1. .

(iii) The assertions of Theorems 3.2.1. and 3.2.2. are also valid after substracting the propagator v by $\delta \mathbb{1}$.

We will now present upper bounds for the constant

$$b_\kappa = \min_{c \in \mathbb{R}} \sup_{x \in (a\mathbb{Z})^\nu} \sup_{\substack{\phi_x \in \mathbb{C} \\ |Im\ \phi_x| = \kappa}} |F_x(\phi_x) - c| . \tag{3.62}$$

In the case of the $\lambda\phi^4$–theory without counterterms, the discrete Gaussian model and the nonlinear σ–model :

Lemma 3.2.3.

(i) *$\lambda\phi^4$–theory without counterterms:*

For $F_x(\phi_x) = e^{-\lambda \phi_x^4}$ we have

$$b_\kappa \le e^{8\lambda\kappa^4} \tag{3.63}$$

(ii) *discrete Gaussian model:*

For $F_x^{(\delta)}(\phi_x) = (2\pi\delta)^{-\frac{1}{2}} \sum_{n \in (a\mathbb{Z})^\nu} \int d\xi_x e^{-\frac{1}{2}(\xi_x - \phi_x)^2} \delta(\xi_x - 2\pi n)$ we have

$$b_\kappa \le \frac{e^{\kappa^2}}{(2\pi\delta)^{\frac{1}{2}}} \tag{3.64}$$

(iii) ***nonlinear σ-model:***

For $F_x^{(\delta)}(\phi_x) = (2\pi\delta)^{-\frac{N}{2}} \int_{S^{n-1}} d\xi_x e^{-\frac{1}{2}(\xi_x-\phi_x)^2}$ ($S^{N-1} = (N-1)-$ dimensional unit sphere) we have

$$b_\kappa \leq \frac{2e^{N\kappa^2/2\delta}}{(2\delta)^{N/2}\Gamma(N/2)} \tag{3.65}$$

PROOF:

(i)

$$b_\kappa \leq \max_{\phi_x \in \mathbb{R}} |e^{-\lambda(\phi_x+i\kappa)^4}| = \max_{\phi_x \in \mathbb{R}} \{e^{-\lambda\phi_x^4 + 6\lambda\kappa^2\phi_x^2 - \lambda\kappa^4}\}. \tag{3.66}$$

Since $\phi_x^2 - 6\kappa^2\phi_x^2 \geq -9\kappa^4$, we have

$$b_\kappa \leq e^{8\lambda\kappa^4}. \tag{3.67}$$

(ii) From the relation ([15])

$$\frac{1}{2\pi} \sum_{m \in \mathbb{Z}} e^{im\xi_x} = \sum_{n \in \mathbb{Z}} \delta(\xi_x - 2\pi n) \tag{3.68}$$

follows

$$F_x^{(\delta)}(\phi_x \pm i\kappa) = \frac{e^{\kappa^2/2\delta}}{2\pi} \sum_{m \in \mathbb{Z}} (2\pi\delta)^{-\frac{1}{2}} \int d\xi_x e^{-\frac{1}{2\delta}(\xi_x-\phi_x)^2 + i(m\pm\kappa)\xi_x} =$$

$$= \frac{e^{\kappa^2/2\delta}}{2\pi} \sum_{m \in \mathbb{Z}} e^{-\frac{\delta}{2}(m+\kappa)^2} e^{im\phi_x}. \tag{3.69}$$

Therefore

$$b_\kappa \leq |F_x^{(\delta)}(\phi_x \pm i\kappa) - \frac{e^{\frac{\kappa^2}{2}(\frac{1}{\delta}-\delta)}}{2\pi}| \leq \frac{e^{\kappa^2/2\delta}}{2\pi} \sum_{m \in \mathbb{Z}-\{0\}} e^{-\frac{\delta}{2}m^2} =$$

$$= \frac{e^{\kappa^2/2\delta}}{\pi} \sum_{m=1}^{\infty} e^{-\frac{\delta}{2}m^2} \leq \frac{e^{\kappa^2/2\delta}}{\pi} \int_0^\infty dx\, e^{-\frac{\delta}{2}x^2} = \frac{e^{\kappa^2/2\delta}}{(2\pi\delta)^{\frac{1}{2}}}. \tag{3.70}$$

(iii) We obtain for the nonlinear σ-model

$$b_\kappa \leq (2\pi\delta)^{-\frac{N}{2}} \int_{S^{N-1}} d\xi_x |e^{-\frac{1}{2\delta}(\xi_x-\phi_x+i\kappa)^2}| \leq (2\pi\delta)^{-\frac{N}{2}} O_{N-1} e^{N\kappa^2/2\delta} \tag{3.71}$$

$O_{N-1} = 2\pi^{\frac{N}{2}}\Gamma(\frac{N}{2})^{-1}$ is the surface of the $(N-1)$-dimensional unit sphere. √

3.3. Estimates for activities and existence of the thermodynamical limit for the $\lambda\phi^4$-theory, the discrete Gaussian model and the nonlinear σ-model on the lattice

We can obtain conditions for the constants b_κ, (ma) and κ, such that the convergence condition (see section 2.2.)

$$\exists \xi > 1 \ : \ \frac{1}{\xi}\left[1 + \sup_{x \in (a\mathbb{Z})^\nu} \sum_{\substack{X \\ x \in X \subset (a\mathbb{Z})^\nu}} |M(X|\psi)| \xi^{2|X|}\right] < 1 \tag{3.72}$$

is fulfilled, if we use the estimates for the activities given in the Theorems 3.2.1. and 3.2.2. .

For the following Lemma we need estimates for the monomer activity $M(\{x\}|\psi)$. We suppose

$$F_x(0) = 1 \tag{3.73a}$$

for all $x \in (a\mathbb{Z})^\nu$ and we consider only theories, which obey the following symmetry

$$F_x(\phi_x) = F_x(-\phi_x). \tag{3.73b}$$

With this assumptions we obtain by Taylor expansion

$$|M(\{x\}|\psi)| = |\langle F_x(\phi_x+\psi_x) - 1\rangle| = |\langle \frac{1}{2}\sum_{i=1}^{N}(\phi_{x,i}+\psi_{x,i})^2 \frac{\partial^2}{\partial \varsigma_i^2} F_x(s\varsigma)|_{\varsigma=\phi_x+\psi_x}\rangle| \tag{3.74}$$

with some $s \in [0,1]$. The derivative in (3.74) will be estimated with the help of the Cauchy inequality. For that we use the notation

$$b_\kappa = \min_{c\in\mathbb{R}} \sup_{x\in(a\mathbb{Z})^\nu} \sup_{\substack{\phi_{x,i}\in\mathbb{C} \\ |Im\ \phi_{x,i}|=\kappa}} |F_x(\phi_{x,1},\ldots,\phi_{x,N}) - c|. \tag{3.49a}$$

From (3.74) follows

$$|M(\{x\}|\psi)| \le \frac{b_\kappa}{\kappa^2}|\langle\sum_{i=1}^{N}(\phi_{x,i}+\psi_{x,i})^2\rangle| = \frac{b_\kappa}{\kappa^2}(Nv_{xx} + |\psi_x|^2)$$

with $\psi_x^2 = \sum_{i=1}^N \psi_{x,i}^2$. If $v = \gamma(-\Delta+m^2)^{-1}$, then $v_{xx} \le \frac{\gamma}{(am)^2}$. Therefore

$$|M(\{x\}|\psi)| \le \frac{b_\kappa}{\kappa^2}\left[\frac{N\gamma}{(am)^2} + |\psi_x|^2\right]. \tag{3.75}$$

Estimate (3.75) is useful for not too large external fields ψ.

Lemma 3.3.1. *Let $\psi_x \in \mathbb{R}^N$ be an external field for a model with N components and propagator $\gamma(-\Delta + m^2)^{-1}$, which fulfills*

$$\frac{b_\kappa}{\kappa^2}\psi_x^2 \le \frac{1}{20} \tag{3.76a}$$

for all $x \in (a\mathbb{Z})^\nu$. The convergence condition (3.72) is fulfilled, if

$$(16b_\kappa + 1)\frac{128N^2\gamma b_\kappa}{(ma\kappa)^2} < 1. \tag{3.76b}$$

PROOF: The condition (3.72) is expressed by integration over the lattice

$$(3.72) \Longleftrightarrow \exists\xi > 1 \ :\ \frac{1}{\xi}\left[1 + |M(\{x\}|\psi)|\xi^2 + \sum_{n\ge 2}\frac{a^{-\nu(n-1)}}{(n-1)!}\int_{x_2,\ldots,x_n\in(a\mathbb{Z})^\nu}|M(x,x_2,\ldots,x_n|\psi)|\xi^{2n}\right] < 1.$$

From (3.75) and (3.76a,b) follows

$$|M(\{x\}|\psi)| \le \frac{1}{16}.$$

With the help of inequality (3.53) (Theorem 3.2.2.) follows that the convergence condition (3.72) is fulfilled, if

$$\exists\xi > 2 \ :\ \frac{1}{\xi}\left\{1 + \frac{\xi^2}{16} + \sum_{n\ge 2}\frac{(ma\kappa)^2}{16N^2\gamma}[\frac{8b_\kappa\xi^2N^2\gamma}{(ma\kappa)^2}]^n\right\} < 1.$$

This is equivalent to

$$\exists\xi > 2 \ :\ \frac{1}{\xi}\left\{1 + \frac{\xi^2}{16} + \frac{b_\kappa\xi^2}{2}\frac{\frac{8b_\kappa\xi^2}{(ma\kappa)^2}N^2\gamma}{1 - \frac{8b_\kappa\xi^2}{(ma\kappa)^2}N^2\gamma}\right\} < 1.$$

Especially for $\xi = 4$

$$16b_\kappa \frac{\frac{128b_\kappa}{(ma\kappa)^2}N^2\gamma}{1-\frac{128b_\kappa}{(ma\kappa)^2}N^2\gamma} < 1.$$

This is equivalent to

$$(16b_\kappa + 1)\frac{128N^2\gamma b_\kappa}{(ma\kappa)^2} < 1. \checkmark$$

Remark : From Lemma 3.2.3. follows that the condition

$$\frac{b_\kappa}{\kappa^2}\psi_x^2 \le \frac{1}{20} \tag{3.76a}$$

has the following form for the $\lambda\phi^4$–theory without counterterms , the discrete Gaussian model (with extra mass m) and the nonlinear σ–model :

(a) *$\lambda\phi^4$–theory without counterterms*

$$\lambda\psi_x^4 \le \frac{1}{80^2 e} \tag{3.77a}$$

(b) *discrete Gaussian model*

$$\psi_x^2 \le \frac{(2\pi)^{\frac{1}{2}}}{20e}\delta^{\frac{3}{2}} \tag{3.77b}$$

(c) *nonlinear σ–model*

$$\psi_x^2 \le \frac{\Gamma(\frac{N}{2})2^{\frac{N}{2}}}{20eN}\delta^{\frac{N}{2}+1}. \tag{3.77c}$$

For an optimal choice of the constant κ we have to determine the minimum of the function $f(x) = x^{-2}e^{ax^2}$, $a > 0$, $x \in \mathbb{R}_+^* = \{x \in \mathbb{R} |\ x > 0\}$. We have

$$\min_{x>0}(x^{-2}e^{ax^n}) = f(x = (\frac{2}{na})^{\frac{1}{n}}) = (\frac{na}{2})^{\frac{2}{n}}e^{\frac{2}{n}} \tag{3.78}$$

PROOF:

$$f'(x) = [-2x^{-3} + nax^{n-3}]e^{ax^n} = 0 \Longrightarrow x = (\frac{2}{na})^{\frac{1}{n}}. \checkmark$$

We will show the following assertion **B** for the $\lambda\phi^4$–theory with/ without counterterms, the nonlinear σ–model and the discrete Gaussian model with extra mass:

B : Let ψ be a real external field, which fulfills the inequality

$$\frac{b_\kappa}{\kappa^2}\psi_x^2 \le \frac{1}{20} \tag{3.76a}$$

for all $x \in (a\mathbb{Z})^\nu$. The thermodynamical limit (in the sense of van Hove) exists for the free-propagator-amputated Greens functions $G_c(\underline{x}_1, \dots, \underline{x}_n)$ and the free energy $\frac{1}{|\Lambda|}\ln Z(\Lambda|\psi)$. If the support *supp* ψ of the external field ψ is finite, the thermodynamical limit (in the sense of van Hove) exists for the generating function

$$\ln Z(\Lambda|\psi) - \ln Z(\Lambda|\psi = 0)$$

for the free-propagator-amputated Greens functions and the expansion

$$\lim_{\Lambda \nearrow \Lambda_{tot}} \ln\left[\frac{Z(\Lambda|\psi)}{Z(\Lambda|\psi = 0)}\right] = \sum_Q a(Q)\ [M(Q|\psi) - M(Q|\psi = 0)] \tag{2.47}$$

is convergent in some neighborhood of $\psi = 0$.

Corollary 3.3.2.. *Let the partition function for the $\lambda\phi^4-$theory without counterterms on the lattice $\Lambda \subseteq (a\mathbb{Z})^\nu$ be*

$$Z(\Lambda|\psi) = \int d\mu_v(\phi) \prod_{x\in\Lambda} e^{-\lambda(\phi_x+\psi_x)^4}. \tag{3.79}$$

If

$$\lambda/(ma)^4 < c\ , \tag{3.80}$$

where $c = [512e^{\frac{1}{2}}(16e^{\frac{1}{2}}+1)]^{-2}$, then the assertion **B** *is valid for real external fields ψ, which fulfill $\lambda\psi_x^4 \leq \frac{1}{80^2 e}$.*

PROOF: From Lemma 3.2.3. (i) follows

$$b_\kappa \leq e^{8\lambda\kappa^4}. \tag{3.81}$$

Because of (3.78) we choose $\kappa = (\frac{1}{16\lambda})^{\frac{1}{4}}$. Therefore

$$\frac{b_\kappa}{\kappa^2} \leq 4\lambda^{\frac{1}{2}}e^{\frac{1}{2}}, \qquad b_\kappa \leq e^{\frac{1}{2}}. \tag{3.82}$$

From Lemma 3.3.1. with $\gamma = N = 1$ follows that the convergence condition (3.72) is fulfilled, if

$$(16e^{\frac{1}{2}}+1)512\ e^{\frac{1}{2}}\frac{\lambda^{\frac{1}{2}}}{(ma)^2} < 1. \tag{3.83}$$

(3.83) is equivalent to (3.80) and assertion **B** follows from Theorem 2.2.1., Corollary 2.2.2. and 2.2.3. . $\surd$

For the $\lambda\phi^4-$theory with counterterms we obtain

Corollary 3.3.3. *Let the partition function for the $\lambda\phi^4-$theory with counterterms on the lattice $\Lambda \subset (a\mathbb{Z})^\nu$ be*

$$Z(\Lambda|\psi) = \int d\mu_v(\phi) \prod_{x\in\Lambda} e^{-\lambda(\phi_x+\psi_x)^4+\frac{1}{2}\delta m^2(\phi_x+\psi_x)^2+\delta e} \tag{3.84}$$

with $v = (-\Delta + m^2)^{-1}$ and

$$\delta m^2 = \lambda\delta\tilde{m}^2, \qquad \delta\tilde{m}^2 = O(1). \tag{3.85}$$

For coupling constants

$$\lambda < \min(2/(\delta\tilde{m}^2)^2, c(ma)^4) \tag{3.86}$$

with $c = [512e^{1+\delta e}(16e^{1+\delta e}+1]^{-2}$ the assertion **B** *is valid.*

PROOF: We have

$$b_\kappa \leq \max_{\phi_x\in\mathbb{R}} |e^{-\lambda(\phi_x+i\kappa)^4+\frac{1}{2}\delta m^2(\phi_x+i\kappa)^2+\delta e}| = \max_{\phi_x\in\mathbb{R}}\{e^{-\lambda\phi_x^4+6\lambda\kappa^2\phi_x^2-\lambda\kappa^4+\lambda\frac{\delta\tilde{m}^2}{2}\phi_x^2-\lambda\frac{\delta\tilde{m}^2}{2}\kappa^2+\delta e}\}. \tag{3.87}$$

From $-\phi_x^4 + 6\kappa^2\phi_x^2 + \frac{\delta\tilde{m}}{2}\phi_x^2 \leq (3\kappa^2 + \frac{\delta\tilde{m}}{4})^2$ we obtain

$$b_\kappa \leq e^{-\lambda\kappa^4+\lambda(3\kappa^2+\frac{\delta\tilde{m}^2}{4})^2-\lambda\frac{\delta\tilde{m}^2}{2}\kappa^2+\delta e} = e^{8\lambda\kappa^4+\lambda\delta\tilde{m}^2\kappa^2+\lambda\frac{(\delta\tilde{m}^2)^2}{16}+\delta e}. \tag{3.88}$$

Let us choose $\kappa = (\frac{1}{16\lambda})^{\frac{1}{4}}$. From (3.88) follows

$$b_\kappa \leq e^{\frac{1}{2}+\frac{\delta\tilde{m}^2}{4}\lambda^{\frac{1}{2}}+\frac{(\delta\tilde{m}^2)^2}{16}\lambda+\delta e}. \tag{3.89}$$

Because of assumption (3.86) we have

$$\frac{\delta\tilde{m}^2}{4}\lambda^{\frac{1}{2}} + \frac{[\delta\tilde{m}^2]^2}{16}\lambda < \frac{1}{2}. \tag{3.90}$$

From (3.89) and (3.90) follows

$$\frac{b_\kappa}{\kappa^2} < 4\lambda^{\frac{1}{2}}e^{1+\delta e} \quad \text{and} \quad b_\kappa < e^{1+\delta e}. \tag{3.91}$$

with $\kappa = (\frac{1}{16\lambda})^{\frac{1}{4}}$. Because of Lemma 3.3.1. with $\gamma = N = 1$ the convergence condition (3.72) is fulfilled, if

$$(16e^{1+\delta e}+1)512e^{1+\delta e}\frac{\lambda^{\frac{1}{2}}}{(ma)^2} < 1. \tag{3.92}$$

This inequality follows from assumption (3.86). Assertion **B** follows from Theorem 2.2.1., Corollary 2.2.2. and 2.2.3. . $\surd$

The following corollary presents an estimate of the activity for the nonlinear $\sigma-$model with extra mass m :

Corollary 3.3.4. *Let the partition function for the nonlinear σ-model on the lattice $\Lambda \subset (a\mathbb{Z})^\nu$ be*

$$Z(\Lambda|\psi) = \int d\mu_{\frac{N}{f_0}v}(\phi) \prod_{x\in\Lambda} \delta((\phi_x + \psi_x)^2 - 1). \tag{3.93}$$

δ is the Dirac distribution, $(\phi_x)_{x\in\Lambda}$, $(\psi_x)_{x\in\Lambda}$ are fields with N components and the propagator $v = (-\Delta + m^2)^{-1}$ obeys the following inequality

$$|v_{xy}| \leq De^{-\tilde{m}\|x-y\|} \tag{3.94}$$

with $\tilde{m} = m + O(a)$. Then we have for the activity $M(X|\psi)$, $X \subset (a\mathbb{Z})^\nu$, $|X| = n \geq 2$, the following estimate

$$|M(X|\psi)| \leq (n-1)! \frac{e^{-\tilde{m}L(X)}}{8DN^2[4\nu + (am)^2]} \left[\frac{[4\nu + (am)^2]^{\frac{N}{2}+1} De}{\Gamma(\frac{N}{2})(2N)^{\frac{N}{2}-3}}\right]^n f_0^{nN/2} \tag{3.95}$$

for real external fields ψ.

PROOF: With the help of the subtraction trick (Lemma 3.1.1.) follows

$$Z(\Lambda|\psi) = \int d\mu_{\frac{N}{f_0}v - \delta\mathbb{I}}(\phi) \prod_{x\in\Lambda} F_x^{(\delta)}(\phi_x + \psi_x) \tag{3.96a)}$$

with

$$F_x^{(\delta)}(\phi_x + \psi_x) = (2\pi\delta)^{-\frac{N}{2}} \int_{S^{N-1}} d\xi_x e^{-\frac{1}{2\delta}(\xi_x - \phi_x - \psi_x)^2}. \tag{3.96b}$$

Because of Lemma 3.4.1. we choose

$$\delta = \frac{N}{f_0[4\nu + (am)^2]}. \tag{3.97}$$

By Lemma 3.2.3., (iii) we have

$$b_\kappa^X = \min_{c\in\mathbb{R}} \sup_{x\in X} \sup_{\substack{\phi_{x,i}\in\mathbb{C}, i\in\{1,\dots,N\} \\ |Im\ \phi_{x,i}|=\kappa}} |F_x(\phi_{x,1},\dots,\phi_{x,N}) - c| = b_\kappa \leq (2\pi\delta)^{-\frac{N}{2}} O_{N-1} e^{N\kappa^2/2\delta} \tag{3.98}$$

where $O_{N-1} = 2\pi^{\frac{N}{2}}\Gamma(\frac{N}{2})^{-1}$ is the surface of the $(N-1)$-dimensional unit sphere S^{N-1}. By Theorem 3.2.2., Eq. (3.52), with $\gamma = \frac{N}{f_0}$ follows

$$|M(X|\psi)| \leq (n-1)! \frac{f_0\kappa^2 e^{-\tilde{m}L(X)}}{16DN^3} \left[\frac{8DN^3(2\pi\delta)^{-\frac{N}{2}} 2\pi^{\frac{N}{2}}\Gamma(\frac{N}{2})^{-1} e^{N\kappa^2/2\delta}}{f_0\kappa^2}\right]^n. \tag{3.99}$$

We choose

$$\kappa^2 = \frac{2\delta}{N} = \frac{2}{f_0[4\nu + (am)^2]}. \tag{3.100}$$

We insert (3.100) into (3.99) to obtain

$$|M(X|\psi)| \leq (n-1)! \frac{e^{-\tilde{m}L(X)}}{8DN^3[4\nu + (am)^2]} \left[\frac{2^{3-\frac{N}{2}} DN^4 \delta^{-1-\frac{N}{2}} e}{f_0\ \Gamma(\frac{N}{2})}\right]^n. \tag{3.101}$$

With the choice of (3.97) for δ the assertion (3.95) holds. $\surd$

Corollary 3.3.5.. *Let the partition function $Z(\Lambda|\psi)$ for the nonlinear σ-model on the lattice $\Lambda \subset (a\mathbb{Z})^\nu$ be defined by (3.93). Let the following inequality for the coupling constant f_0 be fulfilled*

$$\ln f_0 < \min(\frac{2}{N}\ln\left[\frac{\Gamma(\frac{N}{2})}{256N^3e} \frac{(am)^2}{[4\nu + (am)^2]}\right] - \ln\left[\frac{1}{2N}[4\nu + (am)^2]\right], \frac{2}{N}\ln\left[\frac{\Gamma(\frac{N}{2})}{32e}\right] \ln\left[\frac{1}{2N}[4\nu + (am)^2]\right]). \tag{3.102}$$

*Then assertion **B** is valid for all $\psi_x \in \mathbb{R}$ with $\psi_x^2 \leq \frac{(2N)^{\frac{N}{2}}}{20e}[(4\nu + (am)^2)f_0]^{-\frac{N+2}{2}}$.*

PROOF: By Lemma 3.3.1. we have to show

$$(16b_\kappa + 1)\frac{128N^3 b_\kappa}{f_0(ma)^2\kappa^2} < 1. \tag{3.103}$$

From Lemma 3.2.3. (iii) with $\kappa^2 = \frac{2\delta}{N}$, $\delta = \frac{N}{f_0[4\nu+(am)^2]}$ (cf. Lemma 3.1.3.) follows

$$b_\kappa \leq \frac{2e}{\Gamma(\frac{N}{2})}\left[\frac{(4\nu + (ma)^2)}{2N}f_0\right]^{\frac{N}{2}}. \tag{3.104}$$

From (3.102) follows

$$(16b_\kappa + 1) < 2. \tag{3.105}$$

Because of (3.103) and (3.104) we have to show

$$\frac{256e}{\Gamma(\frac{N}{2})}\frac{[4\nu + (am)^2]N^3}{(am)^2}\left[\frac{4\nu + (am)^2}{2N}f_0\right]^{\frac{N}{2}} < 1. \tag{3.106}$$

(3.106) is valid, because of assumption (3.102). √

In the next corollary an estimate of the activity for the discrete Gaussian model with extra mass is given.

Corollary 3.3.6.. *Let the partition function for the discrete Gaussian model with extra mass m on the lattice $\Lambda \subset (a\mathbb{Z})^\nu$ be*

$$Z(\Lambda|\psi) = \int d\mu_{\beta v}(\phi)\prod_{x\in\Lambda}[\sum_{n\in\mathbb{Z}}\delta(\phi_x - 2\pi n)]. \tag{3.107}$$

δ is the Dirac distribution and the propagator $v = (-\Delta + m^2)^{-1}$ obeys the following inequality

$$|v_{xy}| \leq De^{-\tilde{m}\|x-y\|}. \tag{3.94}$$

Then we have for the activity $M(X|\psi)$, $X \subset (a\mathbb{Z})^\nu$, $|X| = n \geq 2$, the following estimate

$$|M(X|\psi)| \leq (n-1)!\frac{e^{-\tilde{m}L(X)+1}}{2\sqrt{2\pi}}\left[\frac{4eD}{\beta^{3/2}}[4\nu + (am)^2]^{\frac{3}{2}}\right]^{n-1}[4\nu + (am)^2]^{\frac{1}{2}} \tag{3.108}$$

for real external fields ψ.

PROOF: With the help of the subtraction trick (Lemma 3.1.1.) follows

$$Z(\Lambda|\psi) = \int d\mu_{\beta v - \delta \mathbb{1}}(\phi)\prod_{x\in\Lambda}F_x^{(\delta)}(\phi_x + \psi_x) \tag{3.109a}$$

with

$$F_x^{(\delta)}(\phi_x + \psi_x) = (2\pi\delta)^{-\frac{1}{2}}\sum_{n\in(a\mathbb{Z})^\nu}\int d\xi_x e^{-\frac{1}{2}(\xi_x - \phi_x)^2}\delta(\xi_x - 2\pi n). \tag{3.109b}$$

By Lemma 3.1.4. we choose

$$\delta = \frac{\beta}{4\nu + (am)^2}. \tag{3.110}$$

From Lemma 3.2.3. (ii) follows

$$b_\kappa^X = b_\kappa = \min_{c\in\mathbb{R}}\sup_{x\in(a\mathbb{Z})^\nu}\sup_{\substack{\phi_x\in\mathbb{C}\\|Im\ \phi_x|=\kappa}}|F_x^{(\delta)}(\phi_x) - c| \leq \frac{e^{\kappa^2/2\delta}}{\sqrt{2\pi\delta}}. \tag{3.111}$$

By Theorem 3.2.1., Eq.(3.35), we obtain

$$|M(X|\psi)| \leq (n-1)!\ \frac{\kappa^2 e^{-\tilde{m}L(X)}}{16D}\left[\frac{8De^{\kappa^2/2\delta}}{\kappa^2\sqrt{2\pi\delta}}\right]^n. \tag{3.112}$$

The assertion (3.108) is valid, if we choose $\kappa^2 = 2\delta$. √

Corollary 3.3.7. *Let the partition function $Z(\Lambda|\psi)$ for the discrete Gaussian model with extra mass on the lattice $\Lambda \subset (a\mathbb{Z})^\nu$ be defined by (3.107). Suppose that*

$$\beta > \max(\frac{128e^2}{\pi}(4\nu + (am)^2), \; \frac{(128e)^2}{2\pi}\frac{(4\nu + (am)^2)^3}{(ma)^4}). \tag{3.113}$$

Then the assertion **B** *is valid for $\psi_x \in \mathbb{R}$, $\psi_x^2 \leq \frac{(2\pi)^{\frac{1}{2}}}{20e}\{\frac{\beta}{[4\nu+(am)^2]}\}^{\frac{3}{2}}$.*

PROOF: Because of Lemma 3.3.1. with $\gamma = \beta$, $N = 1$ we have to show

$$(16b_\kappa + 1)\frac{128\beta b_\kappa}{(ma)^2\kappa^2} < 1. \tag{1.114}$$

Because of Lemma 3.2.3., (ii) with $\kappa^2 = 2\delta$ we have to show

$$(\frac{16e}{\sqrt{2\pi\delta}} + 1)\frac{64e\beta}{(ma)^2\sqrt{2\pi}\delta^{\frac{3}{2}}} < 1. \tag{3.115}$$

(3.115) follows from assumption (3.113) if $\delta = \frac{\beta}{4\nu+(am)^2}$. $\surd$

3.4. IMPROVED ESTIMATES FOR ACTIVITIES

In the contrast to the supposed boundedness of $F_x(\phi_x)$ in section 3.2. we will suppose here that the derivatives of $F_x(\phi_x)$ increase not faster than $e^{\frac{\epsilon}{4}\phi_x^2}$ (for some ϵ) in this section. For example $F_x(\phi_x)$ may be a polynom. We have the following improved estimate for truncated expectation values:

Theorem 3.4.1. *Let $F_x \in C^\infty$. We consider the truncated expectation value*

$$M(X|\psi) = \langle \prod_{x\in X}[F_x(\phi_x + \psi_x);\ ]\rangle \tag{3.116}$$

for $X = \{x_1, \ldots, x_n\} \subset (a\mathbb{Z})^\nu$, $n \geq 2$. The expectation value is defined by

$$\langle\ .\ \rangle = \int d\mu_{\gamma v}(\phi)[\ .\ ], \qquad v = (-\Delta + m^2)^{-1}. \tag{3.117}$$

Let $\epsilon > 0$ be a constant, which fulfills

$$\epsilon \leq \frac{3}{4}\frac{(ma)^2}{\gamma} \tag{3.118}$$

and

$$c_\epsilon(d) \equiv \sup_{x\in(a\mathbb{Z})^\nu}\sup_{\phi_x\in\mathbb{R}}|e^{-\frac{\epsilon}{4}\phi_x^2}\frac{\partial^d}{\partial\phi_x^d}F_x(\phi_x)| \leq (d-1)!\ C_\epsilon^d h_\epsilon \tag{3.119}$$

where C_ϵ and h_ϵ are $\epsilon-$dependent constants. Then we have the following estimates

(i) *Let D and $\tilde{m}(= m + O(a))$ be constants, such that*

$$|v_{xy}| \leq De^{-\tilde{m}\|x-y\|} \tag{3.94}$$

is fulfilled. Then

$$|M(X|\psi)| \leq (n-1)!\frac{e^{-\tilde{m}L(X)}}{4D\gamma C_\epsilon^2}[8DC_\epsilon^2 h_\epsilon]^n e^{\frac{\epsilon}{2}\sum_{x\in X}|\psi_x|^2}. \tag{3.120}$$

(ii) *We have*

$$a^{-\nu(n-1)}\int_{x_2,\ldots,x_n\in(a\mathbb{Z})^\nu}|M(X|\psi)| \leq (n-1)!\frac{(ma)^2}{4\gamma C_\epsilon^2}[\frac{8\gamma C_\epsilon^2 h_\epsilon e^{\frac{\epsilon}{2}\kappa^2}}{(ma)^2}]^n \tag{3.121}$$

for all $m > 0$, $\psi_x \in \mathbb{C}$ with $|\psi_x| \leq \kappa$, $x \in (a\mathbb{Z})^\nu$.

PROOF: Let us set $a = 1$.

(i) From the tree graph formula (Theorem 2.5.1.) and the inequality (3.94) follows by extracting of a Gaussian factor $e^{-\frac{\epsilon}{2}\phi_x^2}$

$$|M(X|\psi)| \leq (\gamma D)^{n-1} e^{-\tilde{m}L(X)} \sum_{\eta} \sum_{\substack{\bar{x} \\ \bar{x}(1)=x\in X}} \int d\sigma_{n-1} f(\eta|s) \int d\mu_{\gamma v[s]}(\phi)$$
$$e^{\frac{\epsilon}{2}\sum_{x\in X}\phi_x^2} \prod_{l=1}^{n} |e^{-\frac{\epsilon}{2}\phi_{x_l}^2} \frac{\partial^{d_l(\eta)}}{\partial \phi_{x_l}^{d_l(\eta)}} F_{x_l}(\phi_{x_l} + \psi_{x_l})|. \quad (3.122)$$

We will estimate the Gaussian measure by the following method :

$$|\langle F(\phi) G(\phi) \rangle| \leq |\langle G(\phi) \rangle| \sup_{\phi} |F(\phi)|. \quad (3.123)$$

Since the propagator $v[s]$ may be exhibited as a convex combination of partially decoupled interactions (see app. B or [8], Eqs. (3.8)-(3.12)), we get by the assumption (3.118) the inequalities

$$\epsilon\gamma v[s] \leq \epsilon \frac{\gamma}{m^2} 1\!\!1 \leq \frac{3}{4} 1\!\!1. \quad (3.124)$$

We obtain with the $n \times n$—matrix $v_X = (v_{xy})_{x,y\in X}$ and $1\!\!1_X = (\delta_{xy})_{x,y\in X}$

$$|\int d\mu_{\gamma v}(\phi) e^{\frac{\epsilon}{2}\sum_{x\in X}\phi_x^2}| = |\det\Big[\gamma v_X[s][(\gamma v_X[s])^{-1} - \epsilon 1\!\!1_X]\Big]^{-\frac{1}{2}}| =$$
$$= |\det(1\!\!1_X - \epsilon\gamma v_X[s])^{-\frac{1}{2}}| \leq \det(\frac{1}{4} 1\!\!1_X)^{-\frac{1}{2}} = 2^n. \quad (3.125)$$

We insert this on the rhs of inequality (3.122). Thereby

$$|M(X|\psi)| \leq 2^n (\gamma D)^{n-1} e^{-\tilde{m}L(X)} \sum_{\eta} \sum_{\substack{\bar{x} \\ \bar{x}(1)=x\in X}} \int d\sigma_{n-1} f(\eta|s) \prod_{l=1}^{n} \{ \sup_{\phi_{x_l}\in\mathbb{R}} |e^{-\frac{\epsilon}{2}\phi_{x_l}^2}$$
$$\frac{\partial^{d_l(\eta)}}{\partial \phi_{x_l}^{d_l(\eta)}} F_{x_l}(\phi_{x_l} + \psi_{x_l})|\}. \quad (3.126)$$

We have for the bracket $\{\ldots\}$

$$\{\ldots\} \leq \sup_{\phi_{x_l}\in\mathbb{R}} |e^{-\frac{\epsilon}{4}\phi_{x_l} + \epsilon\phi_{x_l}\psi_{x_l} - \frac{\epsilon}{2}\psi_{x_l}^2}| \sup_{\phi_{x_l}\in\mathbb{R}} |e^{-\frac{\epsilon}{4}\phi_{x_l}^2} \frac{\partial^{d_l(\eta)}}{\partial \phi_{x_l}^{d_l(\eta)}} F_{x_l}(\phi_{x_l})|. \quad (3.127)$$

From $-\frac{\epsilon}{4}\phi_{x_l}^2 + \epsilon\phi_{x_l}\psi_{x_l} \leq \epsilon\psi_{x_l}^2$ and (3.119) follows

$$\{\ldots\} \leq |e^{\frac{\epsilon}{2}\psi_{x_l}^2}|(d_l(\eta) - 1)!\, C_\epsilon^{d_l(\eta)} h_\epsilon. \quad (3.128)$$

Therefore

$$|M(X|\psi)| \leq (n-1)!\, (\gamma D)^{n-1} e^{-\tilde{m}L(X)} (2h_\epsilon)^n e^{\frac{\epsilon}{2}\sum_{x\in X}|\psi_x|^2} \sum_{\eta} \int_0^1 d\sigma_{n-1} f(\eta|s) \prod_{l-1}^{n} C_\epsilon^{d_l(\eta)} (d_l(\eta) - 1)!. \quad (3.129)$$

Because of

$$\sum_{l=1}^{n} d_l(\eta) = 2(n-1) \quad (3.130)$$

and the special case (3.30) of Lemma 3.1.5. , we obtain

$$|M(X|\psi)| \leq (n-1)!\, (\gamma D)^{n-1} e^{-\tilde{m}L(X)} (2h_\epsilon)^n e^{\frac{\epsilon}{2}\sum_{x\in X}|\psi_x|^2} C_\epsilon^{2(n-1)} 4^{n-1}. \quad (3.131)$$

This is equivalent to the assertion (3.120).

(ii) Because of $\int_{x\in(a\mathbb{Z})^\nu}|v_{xy}| = \frac{1}{m^2}$, from the tree graph formula (Theorem 2.5.1.) follows

$$\int_{x_2,\ldots,x_n\in(a\mathbb{Z})^\nu}|M(X|\psi)| \le (\frac{\gamma}{m^2})^{n-1}\sum_\eta \sum_{\substack{\bar{x} \\ \bar{x}(1)=x\in X}} \int d\sigma_{n-1}f(\eta|s)\int d\mu_{\gamma v[s]}(\phi)e^{\frac{\epsilon}{2}\sum_{x\in X}\phi_x^2}$$
$$\prod_{l=1}^{n}|e^{-\frac{\epsilon}{2}\phi_{x_l}^2}\frac{\partial^{d_l(\eta)}}{\partial\phi_{x_l}^{d_l(\eta)}}F_{x_l}(\phi_{x_l}+\psi_{x_l})|. \quad (3.132)$$

We get (3.132) from (3.122), if we replace

$$e^{-\tilde{m}L(X)} \longrightarrow 1 \text{ and } D \longrightarrow \frac{1}{m^2}. \quad (3.133)$$

Therefore (3.121) follows from (3.120), if we replace $e^{-\tilde{m}L(X)}$ by 1 and D by $\frac{1}{m^2}$ on the rhs of inequality (3.20). √

In the next Theorem we present a generalization of Theorem 3.4.1. for models with N components.

Theorem 3.4.2. *Let*

$$F_x : \begin{cases} \mathbb{C}^N \longrightarrow \mathbb{C} \\ (\phi_{x,1},\ldots,\phi_{x,N}) \mapsto F_x(\phi_{x,1},\ldots,\phi_{x,N}) \end{cases} \quad (3.134)$$

be functions with N arguments and $F_x \in C^\infty$ for all $x\in(a\mathbb{Z})^\nu$. We consider the following truncated expectation value

$$M(X|\psi) = \langle \prod_{x\in X}[F_x(\phi_x+\psi_x);\ ]\rangle \quad (3.116)$$

for $X=\{x_1,\ldots,x_n\}\subset(a\mathbb{Z})^\nu$, $n\ge 2$. The expectation value is defined by

$$\langle\ .\ \rangle = \int d\mu_{\gamma v}(\phi)[\ .\ ].$$

The kernel of the propagator v is

$$v_{xy,ij} = \delta_{ij}v_{xy}, \qquad v = (-\Delta+m^2)^{-1}. \quad (3.135)$$

Let $\epsilon>0$ be a constant, which fulfills

$$\epsilon \le \frac{3}{4}\frac{(ma)^2}{\gamma} \quad (3.118)$$

and

$$c_\epsilon(m_1,\ldots,m_N) = \sup_{x\in(a\mathbb{Z})^\nu}\ \sup_{\substack{\phi_{x,i}\in\mathbb{R} \\ i\in\{1,\ldots,N\}}} |\prod_{i=1}^N(e^{-\frac{\epsilon}{4}\phi_{x,i}^2}\frac{\partial^{m_i}}{\partial\phi_{x,i}^{m_i}})F_x(\phi_{x,1},\ldots,\phi_{x,N})| \le (\prod_{i=1}^N m_i!)C_\epsilon^{\sum_{j=1}^N m_j}h_\epsilon \quad (3.136)$$

where C_ϵ and h_ϵ are ϵ-dependent constants. Then we have the following estimates

(i) *Let D and $\tilde{m}(=m+O(a))$ be constants, such that*

$$|v_{xy}| \le De^{-\tilde{m}\|x-y\|}. \quad (3.94)$$

Then

$$|M(X|\psi)| \le (n-1)!\ \frac{e^{-\tilde{m}L(X)}}{16D\gamma C_\epsilon^2N^2}[8D\ 2^N\gamma C_\epsilon^2N^2h_\epsilon]^n e^{\frac{\epsilon}{2}\sum_{x\in X}\sum_{i=1}^d|\psi_{x,i}|^2} \quad (3.137)$$

(ii) *We have*

$$a^{-\nu(n-1)}\int_{x_2,\ldots,x_n\in(a\mathbb{Z})^\nu}|M(X|\psi)| \le (n-1)!\ \frac{(ma)^2}{16\gamma C_\epsilon^2N^2}[\frac{8\gamma 2^N C_\epsilon^2h_\epsilon N^2e^{\frac{\epsilon}{2}N\kappa^2}}{(ma)^2}]^n. \quad (3.138)$$

PROOF: Let us set $a = 1$.

(i) With the help of the tree graph formula (Theorem 2.5.1.), inequality (3.94) and the multinominial theorem we obtain by extracting of a Gaussian factor $e^{-\frac{\epsilon}{2}\phi_x^2}$

$$|M(X|\psi)| \leq (\gamma D)^{n-1} e^{-\tilde{m}L(X)} \sum_{\eta} \sum_{\substack{\bar{x} \\ \bar{x}(1)=x\in X}} \int d\sigma_{n-1} f(\eta|s) \int d\mu_{\gamma v[s]}(\phi) e^{\frac{\epsilon}{2}\sum_{x\in X}\sum_{i=1}^{N}\phi_{x,i}^2}$$

$$e^{-\frac{\epsilon}{2}\sum_{x\in X}\sum_{i=1}^{N}\phi_{x,i}^2} |\prod_{l=2}^{n}[\sum_{i=1}^{N} \frac{\partial}{\partial\phi_{x_l,i}} \frac{\partial}{\partial\phi_{x_{\eta(l)},i}}] \prod_{l=1}^{n} F_{x_l}(\phi_{x,l} + \psi_{x,l})| \leq$$

$$\leq (\gamma D)^{n-1} e^{-\tilde{m}L(X)} \sum_{\eta} \sum_{\substack{\bar{x} \\ \bar{x}(1)=x\in X}} \int d\sigma_{n-1} f(\eta|s) \int d\mu_{\gamma v[s]}(\phi) e^{\frac{\epsilon}{2}\sum_{x\in X}\sum_{i=1}^{N}\phi_{x,i}^2}$$

$$\prod_{l=1}^{n}\{ \sum_{\substack{m_1,\ldots,m_N \\ \sum m_j = d_l(\eta)}} \frac{d_l(\eta)!}{m_1!\ldots m_N!} |[\prod_{i=1}^{N} e^{-\frac{\epsilon}{2}\phi_{x_l,i}^2} \frac{\partial^{m_i}}{\partial\phi_{x_l,i}^{m_i}}] F_{x_l}(\phi_{x_l} + \psi_{x_l})|\}. \quad (3.139)$$

Because of $\sum_{\substack{m_1,\ldots,m_N \\ \sum m_j = d_l(\eta)}} 1 \leq N^{d_l(\eta)}$ we obtain from (3.136)

$$\sup_{x\in(a\mathbb{Z})^\nu} \sup_{\substack{\phi_{x,i}\in\mathbb{R} \\ i\in\{1,\ldots,N\}}} \prod_{l=1}^{n}\{ \sum_{\substack{m_1,\ldots,m_N \\ \sum m_j = d_l(\eta)}} \frac{d_l(\eta)}{m_1!\ldots m_N!} |[\prod_{i=1}^{N} e^{-\frac{\epsilon}{2}\phi_{x_l,i}^2} \frac{\partial^{m_i}}{\partial\phi_{x_l,i}^{m_i}}] F_{x_l}(\phi_{x_l} + \psi_{x_l})|\} \leq$$

$$\leq \sup_{x\in(a\mathbb{Z})^\nu} \prod_{l=1}^{n}\{ \sup_{\substack{\phi_{x_l,i}\in\mathbb{R} \\ i\in\{1,\ldots,N\}}} |\prod_{i=1}^{N} e^{-\frac{\epsilon}{4}\phi_{x_l,i}^4 + \epsilon\phi_{x_l,i}\psi_{x_l,i} - \frac{\epsilon}{2}\psi_{x_l,i}^2}|$$

$$\sup_{\substack{\phi_{x_l,i}\in\mathbb{R} \\ i\in\{1,\ldots,N\}}} [\sum_{\substack{m_1,\ldots,m_N \\ \sum m_j = d_l(\eta)}} \frac{d_l(\eta)!}{m_1!\ldots m_N!} |(\prod_{i=1}^{N} e^{-\frac{\epsilon}{4}\phi_{x_l,i}^2} \frac{\partial^{m_i}}{\partial\phi_{x_l,i}^{m_i}} |F_{x_l}(\phi_{x_l} + \psi_{x_l})|]\} \leq$$

$$\leq e^{nN\frac{\epsilon}{2}\kappa^2} (NC_\epsilon)^{2(n-1)} h_\epsilon^n \prod_{l=1}^{n} d_l(\eta)!. \quad (3.140)$$

Estimation of the expectation value in (3.139) as in the proof of Theorem 3.4.1. (see (3.123),(3.125)) yields

$$|M(X|\psi)| \leq (n-1)!\,(\gamma D)^{n-1} e^{-\tilde{m}L(X)} (NC_\epsilon)^{2(n-1)} (2^N h_\epsilon)^n e^{\frac{\epsilon}{2}\sum_{x\in X}\sum_{i=1}^{N}|\psi_{x,i}|^2}$$

$$\sum_{\eta} \int_0^1 d\sigma_{n-1} f(\eta|s) \prod_{l=1}^{n} d_l(\eta)!. \quad (3.141)$$

By the special case (3.29) of Lemma 3.1.5. we obtain

$$|M(X|\psi)| \leq (n-1)!\,\frac{(8\gamma D)^{n-1}}{2} e^{-\tilde{m}L(X)} (NC_\epsilon)^{2(n-1)} (2^N h_\epsilon)^n e^{\frac{\epsilon}{2}\sum_{x\in X}\sum_{i=1}^{N}|\psi_{x,i}|^2}. \quad (3.142)$$

Therefore assertion (3.137) holds.

(ii) The assertion (3.138) follows from (3.137) by the following substitution

$$e^{-\tilde{m}L(X)} \longrightarrow 1 \text{ and } D \longrightarrow \frac{1}{m^2} \quad (3.133)$$

(cf. proof of Theorem 3.4.1., (ii)). ✓

Remark : We get the estimates of Theorem 3.4.1. from the estimates of Theorem 3.2.1., if we substitute :

$$\kappa^2 \longrightarrow \frac{4}{\gamma C_\epsilon^2} \quad \text{and} \quad b_\kappa \longrightarrow 2h_\epsilon e^{\frac{\epsilon}{2}|\psi_x|^2}. \quad (3.143)$$

From this remark follows a lemma, which corresponds to Lemma 3.3.1. :

Lemma 3.4.3. *Consider a model with 1 component, propagator $v = \gamma(-\Delta + m^2)^{-1}$ and external field ψ, which fulfills*

$$\frac{b_\kappa}{\kappa^2}|\psi_x|^2 \le \min(\frac{1}{20},\ b_\kappa) \tag{3.144}$$

for all $x \in (a\mathbb{Z})^\nu$. Suppose (with the notation of Theorem 3.4.1.) that

$$(32h_\epsilon e^{\frac{\epsilon}{2}\kappa^2} + 1)\frac{128h_\epsilon C_\epsilon^2 e^{\frac{\epsilon}{2}\kappa^2}}{(ma)^2}\gamma < 1. \tag{3.145}$$

Then the convergence condition (3.72) is fulfilled.

We want an improved estimate for the activities of the $\lambda\phi^4$–theory without counterterms. For that we will need the following lemma.

Lemma 3.4.4. *We have for $m, n \in \mathbb{N}^* = \{1, 2, \ldots\}$ and positive constants λ, ϵ*

$$|e^{-\frac{\epsilon}{4}x^2}\frac{d^n}{dx^n}(x^m e^{-\lambda x^4}| \le n!(\frac{2}{\epsilon e})^{\frac{m}{2}}(\frac{\epsilon e}{2})^{\frac{n}{2}} m^{\frac{m}{2}} \tag{3.146}$$

if

$$\lambda \le \frac{e}{128}\epsilon^2 \tag{3.147}$$

is fulfilled.

PROOF: Consider the function $f(x) = x^n e^{-\lambda x^4}$, $x \ge 0$. Because of $f(0) = f(\infty) = 0$ and

$$f'(x) = x^{n-1}(n - 4\lambda x^4)e^{-\lambda x^4} = 0 \Longrightarrow x = (\frac{n}{4\lambda})^{1/4} \tag{3.148}$$

the maximum of $f(x)$ is at $x = (\frac{n}{4\lambda})^{\frac{1}{4}}$. Hence

$$|x^n e^{-\lambda x^4}| \le (\frac{n}{4\lambda e})^{\frac{n}{4}}. \tag{3.149}$$

With the help of the Cauchy inequality we obtain

$$|\frac{d^n}{dx^n}e^{-\lambda x^4}| \le \frac{n!}{\kappa^n}\max_{x\in\mathbb{R}}|e^{-\lambda(x\pm i\kappa)^4}| = \frac{n!}{\kappa^n}\max_{x\in\mathbb{R}} e^{-\lambda(x^4 - 6\kappa^2 x^2 + \kappa^4)}. \tag{3.150}$$

From $x^4 - 6\kappa^2 x^2 + 9\kappa^2 \ge 0$ follows

$$|\frac{d^n}{dx^n}e^{-\lambda x^4}| \le \frac{n!}{\kappa^n}\max_{x\in\mathbb{R}} e^{8\lambda\kappa^4}. \tag{3.151}$$

By the special choice $\kappa = (\frac{n}{32\lambda})^{\frac{1}{4}}$ follows

$$|\frac{d^n}{dx^n}e^{-\lambda x^4}| \le n!\,(\frac{32\lambda e}{n})^{\frac{n}{4}}. \tag{3.152}$$

With the help of the Leibniz' formula we have

$$e^{-\frac{\epsilon}{4}x^2}\frac{d^n}{dx^n}(x^m e^{-\lambda x^4}) = e^{-\frac{\epsilon}{4}x^2}\sum_{j=0}^{\min(m,n)}\binom{n}{j}\frac{m!}{(m-j)!}x^{m-j}\frac{d^{n-j}}{dx^{n-j}}e^{-\lambda x^4} =$$

$$= n!\, e^{-\frac{\epsilon}{4}x^2}\sum_{j=0}^{\min(m,n)}\binom{m}{j}x^{m-j}\frac{1}{(n-j)!}\frac{d^{n-j}}{dx^{n-j}}e^{-\lambda x^4}. \tag{3.153}$$

From (3.154) follows

$$|e^{-\frac{\epsilon}{4}x^2}\frac{d^n}{dx^n}(x^m e^{-\lambda x^4})| \le n!\sum_{j=0}^{\min(m,n)}\binom{m}{j}e^{-\frac{\epsilon}{4}x^2}x^{m-j}(32e\lambda)^{\frac{n-j}{4}}. \tag{3.154}$$

Hence with the help of (3.149)

$$|e^{-\frac{\epsilon}{4}x^2}\frac{d^n}{dx^n}(x^m e^{-\lambda x^4})| \le n! \sum_{j=0}^{\min(m,n)} \binom{m}{j} (\frac{m-j}{\epsilon e/2})^{\frac{m-j}{2}} (32e\lambda)^{\frac{n-j}{4}} \le$$

$$\le n!\ (32e\lambda)^{\frac{n}{4}} \sum_{j=0}^{m} \binom{m}{j} [(\frac{m}{\epsilon e/2})^{\frac{1}{2}}]^{m-j}[(32e\lambda)^{-\frac{1}{4}}]^j =$$

$$= n!\ (32e\lambda)^{\frac{n}{4}}[(\frac{m}{\epsilon e/2})^{\frac{1}{2}} + (32e\lambda)^{-\frac{1}{4}}]^m. \quad (3.155)$$

We have to consider two cases :

1. $(32e\lambda)^{-\frac{1}{4}} \le (\frac{m}{\epsilon e/2})^{\frac{1}{2}}$:

$$|\ldots| \le n!\ (32e\lambda)^{\frac{n}{4}} 2^m (\frac{m}{\epsilon e/2})^{\frac{m}{2}} \le n!\ (\sqrt{\frac{2}{\epsilon e}})^m (\sqrt{\frac{\epsilon e}{2}})^n m^{\frac{m}{2}}. \quad (3.156)$$

2. $(32e\lambda)^{-\frac{1}{4}} > (\frac{m}{\epsilon e/2})^{\frac{1}{2}}$:

$$|\ldots| \le n!\ (32e\lambda)^{\frac{n-m}{4}} 2^m \le n!\ (\sqrt{\frac{2}{\epsilon e}})^m (\sqrt{\frac{\epsilon e}{2}})^n. \quad (3.157)$$

The second inequalities of (3.156), (3.157) follows from the assumption $\lambda \le \frac{e}{128}\epsilon^2$. √

Because of

$$|e^{-\frac{\epsilon}{4}\phi_x^2}\frac{\partial^d}{\partial\phi_x^d}e^{-\lambda\phi_x^4}| = 4\lambda|e^{-\frac{\epsilon}{4}\phi_x^2}\frac{\partial^{d-1}}{\partial\phi_x^{d-1}}(\phi_x^3 e^{-\lambda\phi_x^4})| \quad (3.158)$$

we obtain by Lemma 3.4.4.

$$|e^{-\frac{\epsilon}{4}\phi_x^2}\frac{\partial^d}{\partial\phi_x^d}e^{-\lambda\phi_x^4}| \le (d-1)!\ (\frac{\epsilon e}{2})^{\frac{d}{2}} 3^{3/2} 4(\frac{2}{\epsilon e})^2\lambda \quad (3.159)$$

for all $d \in \mathbb{N}^* = \{1, 2, \ldots\}$ if

$$\lambda \le \frac{e}{128}\epsilon^2 \quad (3.147)$$

is fulfilled. Therefore the inequality (3.119) of Theorem 3.4.1. is fulfilled for the $\lambda\phi^4$–theory without counterterms with the constants

$$C_\epsilon = (\frac{\epsilon e}{2})^{\frac{1}{2}}, \qquad h^\epsilon = 3^{3/2}4(\frac{2}{\epsilon e})^2\lambda. \quad (3.160)$$

We obtain by Theorem 3.4.1. with $\gamma = 1$ the following estimate for the activities of the $\lambda\phi^4$–theory without counterterms :

Corollary 3.4.5. *Let the following inequalities for the coupling constant $\lambda > 0$ and the propagator $v = (-\Delta + m^2)^{-1}$ be fulfilled*

$$\lambda \le \frac{9e}{2048}(ma)^4 \quad (3.161)$$

and

$$|v_{xy}| \le De^{-\tilde{m}\|x-y\|}. \quad (3.94)$$

Then we have for the $\lambda\phi^4$–theory without counterterms

$$|M(X|\psi)| \le (n-1)!\ \frac{2e^{-\tilde{m}L(X)}}{3De(ma)^2}(\frac{256\sqrt{3}D}{e})^n e^{\frac{3}{8}(ma)^2\sum_{x\in X}|\psi_x|^2}[\frac{\lambda}{(ma)^2}]^n \quad (3.162)$$

for all $\psi_x \in C,\ x \in X,\ |X| = n \ge 2$.

PROOF: Substitution of (3.160) in (3.120) with $\gamma = 1$ and $\epsilon = \frac{3}{4}(ma)^2$ gives the assertion. √

Substitution of (3.160), $\gamma = 1$ and $\epsilon = \frac{3}{4}(ma)^2$ in Lemma 3.4.3. gives the following corollary :

Corollary 3.4.6. *Suppose*

$$\frac{\lambda}{(ma)^4} < \frac{1}{6747} e^{-\frac{3}{8}(ma\kappa)^2} \tag{3.163}$$

and

$$|\psi_x|^2 \le \min\Big(\frac{1}{80\sqrt{e\lambda}},\ \kappa^2\Big) \tag{3.164}$$

for all $x \in (a\mathbb{Z})^\nu$. *Then the assertion* **B** *is valid for the* $\lambda\phi^4$*-theory without counterterms.*

4. BOREL SUMMABILITY AND ANALYTICITY OF THE ACTIVITIES ON THE LATTICE

The perturbation expansion of the partition function $Z(\Lambda|\psi)$ for the $\lambda\phi^4$-theory is divergent (see introduction). Likewise the perturbation expansion for the activities is divergent. In this chapter we will show, that the perturbation expansion for the $\lambda\phi^4$-theory on the lattice is Borel summable in λ. For the proof we will use the methods of section 3.4. and we will show that the sufficient condition for Borel summability by Nevanlinna-Sokal [7] is valid for small coupling constants λ.

We obtain an analytic expansion for the activities, if we introduce a new coupling constant γ. For that we replace the propagator v by

$$v[\gamma]_{xy} = \gamma v_{xy} + (1-\gamma)\delta_{xy} v_{xy}. \tag{4.1}$$

We have $v[1] = v$. In this manner all lines in the Feynman diagram, which connect *different* points, get a factor γ. In section 2.4. we have shown that the activities consist of point connected Feynman diagrams. Therefore all activities $A(X|\psi)$ with $|X| \ge 2$ vanish if $\gamma = 0$. Hence the polymer system consists only of monomers if $\gamma = 0$. We will show in section 4.2. that the activities are holomorph if γ is in a small complex strip around the imaginary axis.

4.1. Borel summability of the activities for the $\lambda\phi^4$-theory on the lattice

We will use the following Theorem by Nevanlinna-Sokal [7], which presents a sufficient condition for Borel summability :

Theorem 4.1.1. (Nevanlinna-Sokal). *Let f be analytic in $C_R = \{z \in \mathbb{C} \mid Re\ z^{-1} > R^{-1}\}$ and let the*

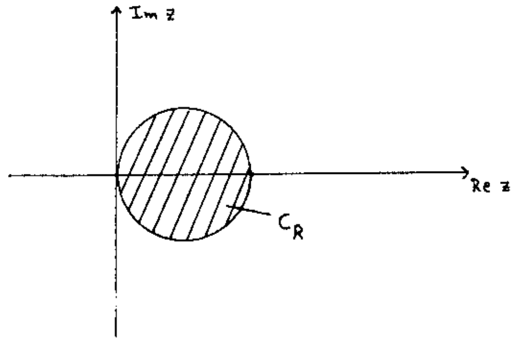


Fig. 4.1 Region of analyticity for f.

following estimate

$$f(z) = \sum_{k=0}^{N-1} a_k z^k + R_N(z), \qquad |R_N(z)| \leq A\sigma^N N!|z|^N \tag{4.2}$$

be fullfilled uniformly in N and $z \in C_R$. Then

(1) $B(t) = \sum_{n=0}^{\infty} a_n t^n/n!$ is convergent for $|t| < 1/\sigma$.

(2) $B(t)$ has an analytic continuation in the complex region

$$S_\sigma = \{t \in C \mid dist(t, \mathbb{R}_+) < 1/\sigma\}, \tag{4.3}$$

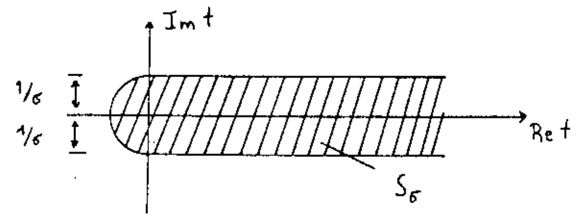


Fig. 4.2 Region of analyticity of the Borel transform

and is satisfying the bound

$$B(t) \leq K \exp(|t|/R) \text{ is uniformly in } S_{\sigma'} \text{ with } \sigma' > \sigma. \tag{4.4}$$

(3) f is represented by the following absolutely convergent integral :

$$f(z) = \frac{1}{z}\int_0^\infty e^{-t/z} B(t)dt \tag{4.5}$$

for all $z \in C_R$.

We consider the activities $A(X|\psi)$ for the partition function

$$Z(\Lambda|\psi) = \int d\mu_v(\phi) \prod_{x\in\Lambda} e^{-\lambda(\phi_x+\psi_x)^4} \tag{4.6}$$

with $v = (-\Delta + m^2)^{-1}$. We need two lemmata for the proof of the Borel summability of $A(X|\psi)$. The first lemma presents the region of analyticity for the activities.

Lemma 4.1.2. *The activities $A(X|\psi)$, $X \subseteq \Lambda$, for the $\lambda\phi^4$–theory are analytic in $\{\lambda \in \mathbb{C} \mid Re\ \lambda > 0\}$.*

PROOF: The integral on the rhs of (4.6) is convergent for $Re\ \lambda > 0$. From the analyticity of $e^{-\lambda(\phi_x+\psi_x)^4}$ follows the analyticity of the partition function $Z(\Lambda|\psi)$ for $Re\ \lambda > 0$. Because of the inversion formula (2.14), the analyticity of $A(X|\psi)$ in $Re\ \lambda > 0$ follows. $\surd$

The next lemma shows, that the assumptions of the Theorem by Nevanlinna-Sokal are fulfilled for the $\lambda\phi^4$–theory :

Lemma 4.1.3. *Let $A(X|\psi)$ be the activity for the $\lambda\phi^4$–theory without counterterms. The partition function $Z(\Lambda|\psi)$ is defined by (4.6). Suppose that positive constants ϵ, c, K exist, such that*

$$0 < \epsilon v \leq c\mathbb{I} < \mathbb{I}, \qquad |v_{xy}| \leq K e^{-m\|x-y\|}. \tag{4.7}$$

Let the asymptotic expansion for $A(X|\psi)$ be

$$A(X|\psi) = \sum_{k=0}^{N-1} a_k \lambda^k + R_N(\lambda). \tag{4.8}$$

Then the following estimates are fulfilled for real ψ_z, $z \in \Lambda$, and $\lambda \in \{z \in \mathbb{C} \mid Re\ z^{-1} > R^{-1}\} = C_R$ with $R = \frac{\epsilon}{128}\epsilon^2$:

(i) *For monomers $X = \{x\}$:*

$$|R_N(\lambda)| \leq \widetilde{A}_1 \sigma_1^N N!\, |\lambda|^N \tag{4.9}$$

with

$$\widetilde{A}_1 = \frac{e^{\frac{\epsilon}{2}\psi_x^2}}{2\pi N\sqrt{1-c}} \tag{4.10}$$

$$\sigma_1 = \frac{64}{\epsilon^2}. \tag{4.11}$$

(ii) *For polymers X with $|X| = n \geq 2$:*

$$|R_N(\lambda)| \leq \widetilde{A}_2 \sigma_2^N N!\, |\lambda|^N \tag{4.12}$$

with

$$\widetilde{A}_2 = \frac{(n-1)!}{4\pi N} \frac{(4Ke\epsilon)^{n-1}}{(1-c)^{n/2}} [\prod_{z\in X} e^{\frac{\epsilon}{2}\psi_z^2}] e^{-mL(X)} \tag{4.13}$$

$$\sigma_2 = \frac{64}{\epsilon^2} \tag{4.14}$$

where $L(X) = \min_\eta \sum_{(xy)\in\eta} \|x-y\|$ (the minimum is over all trees η, which connect all points of X).

PROOF: Let us denote the $\lambda-$ dependent activity $A(X|\psi)$ by A_λ. The function $h(t) = A_{t\lambda}$ possesses for $Re\ \lambda > 0$ derivatives from the right in $t = 0$ of every order. The Taylor expansion is

$$h(t) = \sum_{k=0}^{N-1} \frac{1}{k!} \lim_{t \searrow 0} h^{(k)}(t) t^k + h_N(t) \tag{4.15}$$

with the Lagrangian remainder

$$h_N(t) = \frac{1}{N!} h^{(N)}(s) t^N \tag{4.16}$$

with some $s \in [0, t]$. Therefore we obtain for the power series of the activity

$$A_\lambda = h(1) = \sum_{k=0}^{N-1} a_k \lambda^k + R_N(\lambda). \tag{4.17}$$

The remainder term $R_N(\lambda)$ fulfills

$$|R_N(\lambda)| \leq \max_{t\in[0,1]} |\frac{\partial^N}{\partial t^N} A_{t\lambda}| \, \frac{1}{N!} = \max_{t\in[0,1]} |\frac{\partial^N}{\partial t^N} A_{t|\lambda|}| \, \frac{1}{N!}. \tag{4.18}$$

(i) : For $X = \{x\}$:

$$|R_N(\lambda)| \leq \max_{t\in[0,1]} |\frac{\partial^N}{\partial t^N} \langle e^{-t\lambda(\phi_x+\psi_x)^4}\rangle| \, \frac{1}{N!} \leq |\langle(\phi_x + \psi_x)^{4N}\rangle \, \frac{|\lambda|^N}{N!} \leq$$

$$\leq |\int d\mu_v(\phi) e^{\frac{\epsilon}{2}\phi_x^2}| \, \max_{\phi_x\in\mathbb{R}} |(\phi_x + \psi_x)^{4N} e^{-\frac{\epsilon}{2}\phi_x^2}| \, \frac{|\lambda|^N}{N!}. \tag{4.19}$$

Computation of the Gaussian integral yields

$$\int d\mu_v(\phi) e^{\frac{\epsilon}{2}\phi_x^2} = \det(2\pi v)^{-\frac{1}{2}} \det(2\pi(v^{-1} - \epsilon))^{-\frac{1}{2}} = \det(\mathbb{I} - \epsilon v)^{-\frac{1}{2}}. \tag{4.20}$$

(4.7) implies

$$|\int d\mu_v(\phi) e^{\frac{\epsilon}{2}\phi_x^2}| \leq \frac{1}{\sqrt{1-c}}. \tag{4.21}$$

The maximum in (4.19) is bounded by

$$\max_{\phi_x\in\mathbb{R}} |(\phi_x + \psi_x)^{4N} e^{-\frac{\epsilon}{2}\phi_x^2}| = \max_{\phi_x\in\mathbb{R}} |\phi_x^{4N} e^{-\frac{\epsilon}{2}(\phi_x-\psi_x)^2}| \leq$$

$$\leq \max_{\phi_x\in\mathbb{R}} |e^{-\frac{\epsilon}{4}\phi_x^2 + \epsilon\phi_x\psi_x - \frac{\epsilon}{2}\psi_x^2}| \max_{\phi_x\in\mathbb{R}} |\phi_x^{4N} e^{-\frac{\epsilon}{4}\phi_x^2}|. \tag{4.22}$$

Since $-\frac{\epsilon}{4}\phi_x^2 + \epsilon\phi_x\psi_x = -\epsilon(\frac{\phi_x}{2} - \psi_x)^2 + \epsilon\psi_x^2$, the first maximum on the rhs of inequality (4.22) is bounded by $e^{\frac{\epsilon}{2}\psi_x^2}$. The second maximum is bounded by $(\frac{8N}{\epsilon e})^{2N}$. Hence

$$\max_{\phi_x\in\mathbb{R}} |(\phi_x + \psi_x)^{4N} e^{-\frac{\epsilon}{2}\phi_x^2}| \leq (\frac{8N}{\epsilon e})^{2N} e^{\frac{\epsilon}{2}\psi_x^2}. \tag{4.23}$$

From (4.20), (4.22) and (4.24) follows

$$|R_N(\lambda)| \leq \frac{1}{\sqrt{1-c}} (\frac{8N}{\epsilon e})^{2N} \frac{|\lambda|^N}{N!}. \tag{4.24}$$

From Stirling's formula [20]

$$N^{N+\frac{1}{2}} = \frac{1}{\sqrt{2\pi}} N! \, e^N e^{-\theta/12N}, \qquad 0 < \theta < 1 \tag{4.25}$$

follows

$$N^{2N} \leq \frac{1}{2\pi N} (N!)^2 e^{2N}. \tag{4.26}$$

Hence

$$|R_N(\lambda)| \leq \frac{1}{2\pi N\sqrt{1-c}} (\frac{64}{\epsilon^2})^N |\lambda|^N N! \, . \tag{4.27}$$

This proves assertion (i).

(ii) : For $|X| = n \geq 2$:

With the help of the tree graph formula (Theorem 2.5.1.) and the multinominial Theorem, it follows from (4.18)

$$|R_N(\lambda)| \le \max_{t\in[0,1]} \Big\{ \sum_{\eta} \sum_{\bar{x}(1) \overset{\bar{x}}{=} x\in X} \prod_{(ij)\in\eta} |v_{x_i x_j}| \int d\sigma_{n-1} f(\eta|s) \int d\mu_{v[s]}(\phi) \sum_{\substack{\{n_l\} \\ \sum n_l = N}} \prod_{l=1}^{n} \Big| \frac{1}{n_l!} \frac{\partial^{d_l(\eta)}}{\partial \phi_{x_l}^{d_l(\eta)}} (\phi_{x_l} + \psi_{x_l})^{4n_l} e^{-t|\lambda|(\phi_{x_l}+\psi_{x_l})^4} \Big| \Big\} |\lambda|^N. \quad (4.28)$$

Extraction of a Gaussian factor $e^{-\frac{\epsilon}{2}\phi_x^2}$ yields

$$|R_N(\lambda)| \le \max_{t\in[0,1]} \{ \sum_{\eta} \sum_{\bar{x}(1) \overset{\bar{x}}{=} x\in X} \prod_{(ij)\in\eta} |v_{x_i x_j}| \int d\sigma_{n-1} f(\eta|s) \int d\mu_{v[s]}(\phi) \prod_{z\in X} e^{\frac{\epsilon}{2}\phi_z^2} | \sum_{\substack{\{n_l\} \\ \sum n_l = N}} \prod_{l=1}^{n} \max_{\phi_{x_l}} |e^{-\frac{\epsilon}{2}\phi_{x_l}^2} \frac{1}{n_l!} \frac{\partial^{d_l(\eta)}}{\partial \phi_{x_l}^{d_l(\eta)}} (\phi_{x_l} + \psi_{x_l})^{4n_l} e^{-t|\lambda|(\phi_{x_l}+\psi_{x_l})^4} | \} |\lambda|^N. \quad (4.29)$$

Computation of the Gaussian integral yields

$$\int d\mu_v(\phi) \prod_{z\in X} e^{\frac{\epsilon}{2}\phi_z^2} = \det(2\pi v[s])^{-\frac{1}{2}} \det(2\pi(v[s]^{-1} - \epsilon))^{-\frac{1}{2}} = \\ = \det(v[s](v[s]^{-1} - \epsilon))^{-\frac{1}{2}} = \det(\mathbb{1} - \epsilon v[s])^{-\frac{1}{2}}. \quad (4.30)$$

Since the propagator $v[s]$ is a convex combination of partially decoupled interactions (cf. app. B or [8], Eqs. (3.8)-(3.12)), from assumption (4.7) follows

$$\epsilon v[s] \le \epsilon v \le c\mathbb{1}. \quad (4.31)$$

Hence

$$\int d\mu_v(\phi) \prod_{z\in X} e^{\frac{\epsilon}{2}\phi_z^2} \le (1-c)^{-\frac{n}{2}}. \quad (4.32)$$

Furthermore

$$\max_{\phi_z\in\mathbb{R}} |e^{-\frac{\epsilon}{2}\phi_z^2} \frac{\partial^{d_z(\eta)}}{\partial \phi_z^{d_z(\eta)}} (\phi_z + \psi_z)^{4n_z} e^{-t|\lambda|(\phi_z+\psi_z)^4}| \le \\ \le \max_{\phi_z\in\mathbb{R}} |e^{-\frac{\epsilon}{4}\phi_z^2 + \epsilon\phi_z\psi_z - \frac{\epsilon}{2}\psi_z^2}| \max_{\phi_z\in\mathbb{R}} |e^{-\frac{\epsilon}{4}\phi_z^2} \frac{\partial^{d_z(\eta)}}{\partial \phi_z^{d_z(\eta)}} \phi_z^{4n_z} e^{-t|\lambda|\phi_z^4}|. \quad (4.33)$$

Since $-\frac{\epsilon}{4}\phi_z^4 + \epsilon\phi_z\psi_z = -\epsilon(\frac{\phi_z}{2} - \psi_z)^2 + \epsilon\psi_z^2$, the first maximum on the rhs of inequality (4.33) is bounded by $e^{\frac{\epsilon}{2}\psi_z^2}$. From this and Lemma 3.4.4. we obtain for $\lambda \le \frac{e}{128}\epsilon^2$

$$\max_{\phi_z\in\mathbb{R}} |e^{-\frac{\epsilon}{2}\phi_z^2} \frac{\partial^{d_z(\eta)}}{\partial \phi_z^{d_z(\eta)}} (\phi_z + \psi_z)^{4n_z} e^{-t|\lambda|(\phi_z+\psi_z)^4}| \le e^{\frac{\epsilon}{2}\psi_z^2} d_z(\eta)! \, (\frac{2}{e\epsilon})^{2n_z} (\frac{e\epsilon}{2})^{\frac{d_z(\eta)}{2}} (4n_z)^{2n_z}. \quad (4.34)$$

We insert (4.34) on the rhs of (4.29). This gives

$$|R_N(\lambda)| \le \sum_{\eta} \sum_{\bar{x}(1) \overset{\bar{x}}{=} x\in X} \prod_{(ij)\in\eta} |v_{x_i x_j}| \int d\sigma_{n-1} f(\eta|s)(1-c)^{-\frac{n}{2}} \sum_{\substack{\{n_z\} \\ \sum n_z = N}} \prod_{z\in X} \Big[e^{\frac{\epsilon}{2}\psi_z^2} d_z(\eta)! \, (\frac{2}{e\epsilon})^{2n_z} (\frac{e\epsilon}{2})^{\frac{d_z(\eta)}{2}} \frac{(4n_z)^{2n_z}}{n_z!} \Big] |\lambda|^N. \quad (4.35)$$

Because of $\sum_{z\in X} d_z(\eta) = 2(n-1)$ and $|v_{xy}| \leq Ke^{-m\|x-y\|}$, we obtain

$$|R_N(\lambda)| \leq (n-1)! \, \frac{(K\frac{e\epsilon}{2})^{n-1} e^{-mL(X)}}{(1-c)^{n/2}} [\prod_{z\in X} e^{\frac{\epsilon}{4}\psi_z^2}] (\frac{8}{\epsilon e})^{2N} |\lambda|^N (\sum_{\substack{\{n_z\} \\ \sum n_z = N}} \prod_{z\in X} \frac{n_z^{2n_z}}{n_z!})$$

$$\sum_\eta \int_0^1 d\sigma_{n-1} f(\eta|s) \prod_{z\in X} d_z(\eta)! \, . \quad (4.36)$$

Applying the multinominial Theorem we obtain

$$\sum_{\substack{\{n_z\} \\ \sum n_z = N}} \prod_{z\in X} \frac{(n_z^2)^{n_z}}{n_z!} = \frac{1}{N!} (\sum_{z\in X} n_z^2)^N \leq \frac{N^{2N}}{N!}. \quad (4.37)$$

With the help of Stirling's formula [20]

$$N^{N+\frac{1}{2}} = \frac{1}{\sqrt{2\pi}} N! \, e^N e^{-\theta/12N}, \qquad 0 < \theta < 1 \quad (4.25)$$

follows

$$\sum_{\substack{\{n_z\} \\ \sum n_z = N}} \prod_{z\in X} \frac{n_z^{2n_z}}{n_z!} \leq \frac{e^{2N}}{2\pi N} N! \, . \quad (4.38)$$

We insert (3.29) and (4.38) on the rhs of (4.36) to obtain

$$|R_N(\lambda)| \leq \frac{(n-1)!}{4\pi N} \frac{(4Ke\epsilon)^{n-1} e^{-mL(X)}}{(1-c)^{n/2}} [\prod_{z\in X} e^{\frac{\epsilon}{2}\psi_z^2}] (\frac{8}{\epsilon})^{2N} |\lambda|^N N! \quad (4.39)$$

for $|\lambda| \leq \frac{e}{128}\epsilon^2$. This proves assertion (ii) .√

By Lemma 4.1.2. and 4.1.3. the assumption of the Theorem by Nevanlinna-Sokal are fulfilled. Therefore the activities $A(X|\psi)$ are Borel summable in λ and we have the following Theorem :

Theorem 4.1.4. *Let $A(X|\psi)$, $X \subseteq \Lambda \subset (a\mathbb{Z})^\nu$ be the activity for the partition function*

$$Z(\Lambda|\psi) = \int d\mu_v(\phi) \prod_{x\in\Lambda} e^{-\lambda(\phi_x+\psi_x)^4} \quad (4.6)$$

(i.e. for the $\lambda\phi^4$–theory). Suppose that for positive constants ϵ, c, K the following inequalities

$$0 < \epsilon v \leq c\mathbb{1} < \mathbb{1}, \qquad |v_{xy}| \leq Ke^{-m\|x-y\|}. \quad (4.7)$$

are fulfilled. Then the perturbation expansion

$$A(X|\psi) = \sum_{k=0}^{\infty} a_k \lambda^k \quad (4.43)$$

is Borel summable. More precise, with the notations

$$\sigma = \frac{64}{\epsilon^2}, \qquad R = \frac{e}{128}\epsilon^2 \quad (4.44)$$

the following assertions are valid

(i)

$$B(\lambda) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \lambda^k \text{ converges for } |\lambda| < 1/\sigma. \quad (4.45)$$

(ii) *$B(\lambda)$ has an analytic continuation in the complex region*

$$S_\sigma = \{\lambda \in \mathbb{C} \mid dist(\lambda, \mathbb{R}_+) < 1/\sigma\}, \tag{4.46}$$

and is satisfying the bound

$$|B(\lambda)| \leq const.\ \exp(|\lambda|/R) \ \text{ uniformly in } \ S_{\sigma'},\ \sigma' > \sigma. \tag{4.47}$$

(iii) *$A(X|\psi)$ is represented by the following absolutely convergent integral*

$$A(X|\psi) = \int_0^\infty e^{-t} B(\lambda t) dt \tag{4.48}$$

for all $\lambda \in C_R = \{z \in \mathbb{C} \mid Re\ z^{-1} > R^{-1}\}$.

4.2. Analiticity of the activities

We consider for *finite* sublattices $\Lambda \subset (a\mathbb{Z})^\nu$ the partition function

$$Z(\Lambda|\psi) = \int d\mu_{v[\gamma]}(\phi) \prod_{x\in\Lambda} F_x(\phi_x + \psi_x) \tag{4.49}$$

with

$$v[\gamma]_{xy} = \gamma v_{xy} + (1-\gamma)\delta_{xy} v_{xy}. \tag{4.1}$$

Theorem 4.2.1. *Let $v = (-\Delta + m^2)^{-1}$ be the propagator and $A(X|\psi)$, $X \subseteq \Lambda \subset (a\mathbb{Z})^\nu$, the activities for the partition function $Z(\Lambda|\psi)$, which is defined by (4.49). Let F_x be bounded functions for all $x \in \Lambda$. Then the activities $A(X|\psi)$ are holomorph in γ in the complex strip $-v_{xx}(ma)^2 < Re\ \gamma < v_{xx}(ma)^2$.*

Proof: Because of the inversion formula (2.14), it is sufficient to show the analyticity of $Z(\Lambda|\psi)$ in γ. By Fourier transformation we obtain

$$Z(X|\psi) = \int [\prod_{x\in X} dq_x\ \tilde{F}_x(q_x) e^{iq_x\psi_x} e^{-\frac{1}{2}q_x v_{xx} q_x}] e^{-\frac{1}{2}(q, v[\gamma]q)}. \tag{4.50}$$

From (4.49) follows for positive definite quadratic forms $(q, v[\gamma]q)$

$$|Z(X|\psi)| \leq \prod_{x\in X} \sup_{\phi_x\in\mathbb{R}} |F_x(\phi_x)| < \infty, \tag{4.51}$$

since F_x are supposed to be bounded functions. Since the integral on the rhs of (4.50) is convergent and the e-function $e^{-\frac{1}{2}(q,v[\gamma]q)}$ is analytically in γ, $Z(\Lambda|\psi)$ is an analytic function in γ. By Frobenius' Theorem [19] the following inequality is valid for an eigenvalue $\delta(Re\gamma)$ of the matrix $(v(Re\ \gamma)_{xy})_{x,y\in X}$:

$$\min_{x\in X}(|v_{xx}| - \sum_{y\in X} |(Re\ \gamma)v_{xx}|) \leq |\delta(Re\ \gamma)|. \tag{4.52}$$

The lhs of this inequality is positive for $|Re\ \gamma| < |v_{xx}(ma)^2|$, since $\sum_{y\in X}|v_{xy}| \leq \sum_{y\in(a\mathbb{Z})^\nu}|v_{xy}| = \frac{1}{(ma)^2}$. Hence $|\delta(Re\ \gamma)| > 0$. Because of $\delta(0) > 0$ and the continuous dependence of the eigenvalues from $Re\ \gamma$, we obtain $\delta(Re\ \gamma) > 0$, if $|Re\ \gamma| < |v_{xx}(ma)^2|$. Therefore the quadratic form $(q, v[Re\ \gamma]q)$ is positive-definite for $|Re\ \gamma| < |v_{xx}(ma)^2|$. This proves our assertion. $\surd$

Remark : The activities $A(X|\psi)$ are an analytic continuation of a convergent power series in γ.

5. RENORMALIZATION GROUP AND MAYER EXPANSION

5.1. Renormalization with X-dependent counterterms

We will consider a theory with X-dependent mass and vacuum energy counterterms. The partition function for a finite sublattice $X \subset (a\mathbb{Z})^\nu$ have the following form

$$Z(X|\psi) = \int d\mu_v(\phi)[\prod_{x\in X} e^{-\lambda \mathcal{V}(\phi_x+\psi_x)}]e^{-\delta V_X(\phi+\psi)} \tag{5.1}$$

with X-dependent counterterms

$$\delta V_X(\phi+\psi) = -\sum_{P\subset X}[\delta m^2(P)\sum_{x\in P}(\phi_x+\psi_x)^2 + \delta e(P)]. \tag{5.2}$$

The counterterms can be fixed (for small coupling constants λ and $\langle \frac{\partial^2 \mathcal{V}_x}{\partial\psi^2}\rangle \geq 0$), such that the following renormalization conditions

$$\ln Z(X|\psi)|_{\psi=0} = 0 \tag{5.3a}$$

$$\frac{\partial^2}{\partial\psi^2}\ln Z(X|\psi)|_{\psi=0} = 0 \tag{5.3b}$$

are fulfilled for all finite $X \subset (a\mathbb{Z})^\nu$. In (5.3b) we differentiate with respect to constant fields $\psi_x \equiv \psi$ for all $x \in X$.

PROOF: (i) : We will show that we can find mass counterterms $\delta m^2(P)$, such that (5.3b) is fulfilled. Let us set $\delta e(P) = 0$. Reformulation of the mass counterterms yields

$$\sum_{P\subseteq X}\delta m^2(P)\sum_{x\in P}(\phi_x+\psi_x)^2 = \sum_{x\in X}(\sum_{P\subseteq X}\frac{\delta m^2(P)}{|P|})(\phi_x+\psi_x)^2. \tag{5.4}$$

In the following we will use the notation

$$\delta\tilde{m}^2(P) = \sum_{Y\subseteq P}\frac{\delta m^2(Y)}{|Y|}. \tag{5.5}$$

The renormalization condition (5.3b) is equivalent to

$$\Big\{\sum_{x,y\in X}\langle\langle\lambda^2\frac{\partial\mathcal{V}_x}{\partial\psi}\frac{\partial\mathcal{V}_x}{\partial\psi} - 2\lambda\delta\tilde{m}^2(X)\frac{\partial\mathcal{V}_x}{\partial\psi}\phi_y - 2\lambda\delta\tilde{m}^2(X)\frac{\partial\mathcal{V}_y}{\partial\psi}\phi_x + 4[\delta\tilde{m}^2(X)]^2\phi_x\phi_y\rangle\rangle|_{\psi=0} +$$
$$\sum_{x\in X}\langle\langle-\lambda\frac{\partial^2\mathcal{V}_x}{\partial\psi^2} + 2\delta\tilde{m}^2(X)\rangle\rangle - \langle\langle\sum_{x\in X}(-\lambda\frac{\partial\mathcal{V}_x}{\partial\psi} + 2\delta\tilde{m}^2(X)\phi_x\rangle\rangle^2\Big\}|_{\psi=0} = 0 \quad (5.6)$$

with

$$\langle\langle\ .\ \rangle\rangle = \int d\mu_v(\phi)[\prod_{x\in X} e^{-\lambda \mathcal{V}_x(\phi_x)+\delta\tilde{m}^2(X)\phi_x^2}][\ .\ ]. \tag{5.7}$$

This is equivalent to

$$A[\delta\tilde{m}^2(X)]^2 + B\delta\tilde{m}^2(X) + C \equiv F(\delta\tilde{m}^2(X)) = 0 \tag{5.8}$$

with

$$A = 4\sum_{x,y\in x}\{\langle\langle\phi_x\phi_y\rangle\rangle - \langle\langle\phi_x\rangle\rangle\langle\langle\phi_y\rangle\rangle\} \tag{5.9a}$$

$$B = 2|X|\langle\langle 1\rangle\rangle + O(\lambda) \tag{5.9b}$$

$$C = -\lambda\sum_{x\in X}\langle\langle\frac{\partial^2\mathcal{V}_x}{\partial\psi^2}\rangle\rangle|_{\psi=0}. \tag{5.9c}$$

The function F defined by (5.8) is continuous. For small coupling constants λ and $\langle\frac{\partial^2\mathcal{V}_x}{\partial\psi^2}\rangle \geq 0$ we have $F(\delta\tilde{m}^2(X) = 0) \leq 0$ and $F(\delta\tilde{m}^2(X)) \geq 0$ for large $\delta\tilde{m}^2(X)$. By the mean value theorem exists a positive real solution $\delta\tilde{m}^2(X)$ of Eq. (5.8). By the Möbius inversion formula (Lemma 2.4.2.) we obtain from Eq. (5.5) the mass counterterms $\delta m^2(P),\ P\subseteq X$.

(ii) : We will show that vacuum counterterms $\delta e(P)$ exists , which fulfills (5.3a). Eq. (5.3a) is equivalent to

$$\delta\tilde{e}(X) = \sum_{P\subseteq X}\delta e(P) = -\ln\Big\{\int d\mu_v(\phi)[\prod_{x\in X} e^{-\lambda\mathcal{V}_x(\phi_x)+\sum_{P\subseteq X}\delta m^2(P)\sum_{x\in P}\phi_x^2}]\Big\}. \tag{5.10}$$

The coefficients $\delta e(P)$ are determined with the help of the Möbius inversion formula (Lemma 2.4.2.) . $\surd$

With

$$\mathcal{Z}(X|\psi) = [\prod_{x\in X} e^{-\lambda\mathcal{V}(\phi_x+\psi_x)}]e^{-\delta V_X(\phi_x+\psi_x)} \tag{5.11}$$

the partition function reads

$$Z(X|\psi) = \langle\mathcal{Z}(X|\psi)\rangle \tag{5.12}$$

where $\langle\ .\ \rangle$ denotes the Gaussian expectation value. The polymer representation for $Z(X|\psi)$ is defined by

$$Z(X|\psi) = \sum_{X=\sum Y}\prod_Y B(Y|\psi). \tag{5.13}$$

Polymers of this polymer system are called *molecules* and the activities $B(Y|\psi)$ are called *molecular activities*. The following Theorem gives an expression for $B(Y|\psi)$ and shows $B(Y|\psi) = O(\lambda^{|Y|})$.

Theorem 5.1.1. *The molecular activities of the polymer representation (5.13) are*

$$B(P|\psi) = [\prod_{x\in P} e^{-\lambda\mathcal{V}_x(\phi_x+\psi_x)}]\Big\{\delta_{1,|P|} + \sum_{\substack{\mathcal{P}\\ supp\ \mathcal{P}=P}}\prod_{M\in\mathcal{P}}[e^{-\delta m^2(M)\sum_{x\in M}(\phi_x+\psi_x)^2+\delta e(M)} - 1]\Big\} \tag{5.14}$$

for all $P\subseteq X$. The sum is over all sets $\mathcal{P}$, which consist of sets $P\subseteq X$, such that the graph $\gamma(\mathcal{P})$ is connected (cf. section 2.5.5.) and

$$supp\ \mathcal{P} = \{x\in X \mid \exists M\in\mathcal{P}\ with\ x\in M\} = P. \tag{5.15}$$

Suppose that the renormalization conditions (5.3a,b) are fulfilled. Then

$$B(P|\psi) = O(\lambda^{|P|}). \tag{5.16}$$

PROOF: We split the e-function

$$e^{\delta m^2(P)\sum_{x\in P}(\phi_x+\psi_x)^2+\delta e(P)} = 1 + f_P(\phi+\psi). \tag{5.17}$$

Therefore

$$e^{-\delta V_X(\phi+\psi)} = \prod_{M\subseteq\Lambda} [1 + f_M(\phi+\psi)] = \sum_Q \prod_{M\in Q} f_M(\phi+\psi), \tag{5.18}$$

where the sum is over all sets Q which consist of subsets of X. For Q we define the graph $\gamma(Q)$ (see section 2.2.). Vertices of $\gamma(Q)$ are the elements of Q and two vertices $P_i, P_j \in Q$ are connected by a line, if $P_i \cap P_j \neq \emptyset$. The partition of $\gamma(Q)$ in connected subgraphs $\gamma(\mathcal{P}_i)$ defines the partition

$$Q = \sum_{i=1}^{n} \mathcal{P}_i \tag{5.19}$$

where $\gamma(\mathcal{P}_i)$ is connected. Hence it follows from (5.18)

$$e^{-\delta V_X(\phi_x+\psi_x)} = \sum_{Q=\sum \mathcal{P}} \prod_{\mathcal{P}} \prod_{M\in\mathcal{P}} f_M(\phi+\psi). \tag{5.20}$$

By the distributive law follows

$$e^{-\delta V_X(\phi+\psi)} = \sum_{X\supseteq\sum P} \prod_P \Big[\sum_{\substack{\mathcal{P} \\ supp\, \mathcal{P}=P}} \prod_{M\in\mathcal{P}} f_M(\phi+\psi)\Big]. \tag{5.21}$$

In the same way as in the proof of Lemma 2.3.1. follows

$$e^{-\delta V_X(\phi+\psi)} = \sum_{X=\sum P} \prod_P \Big[\delta_{1,|P|} + \sum_{\substack{\mathcal{P} \\ supp\, \mathcal{P}=P}} \prod_{M\in\mathcal{P}} f_M(\phi+\psi)\Big]. \tag{5.22}$$

This proves assertion (5.14). In the following we will show $\delta m^2(P) = O(\lambda^{|P|})$ and $\delta e(P) = O(\lambda^{|P|})$ by induction. The renormalized activities are defined by

$$Z(X|\psi) = \sum_{X=\sum Y} \prod_Y A^{ren}(Y|\psi) \tag{5.23}$$

and the renormalization conditions (5.3a,b) are equivalent to (see section 1.5.)

$$A^{ren}(X|\psi)|_{\psi=0} = \begin{cases} 1 & \text{if } |X| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.24a}$$

$$\frac{\partial^2}{\partial\psi^2} A^{ren}(X|\psi)|_{\psi=0} = 0. \tag{5.24b}$$

We will show $\delta m^2(P) = O(\lambda^{|P|})$. Since $\delta m^2(P)$ is not dependent from δe, we can suppose $\delta e(X) = 0$ for all $X \subset (a\mathbb{Z})^\nu$. Suppose $|P| = 1$, $P = \{x\}$. We have

$$B(\{x\}|\psi) = e^{-\lambda \mathcal{V}_x(\phi_x+\psi_x)+\delta m^2(x)(\phi_x+\psi_x)^2}. \tag{5.25}$$

Since $A^{ren}(\{x\}|\psi) = \langle B(\{x\}|\psi)\rangle$, it follows from (5.25b)

$$\frac{\partial^2}{\partial\psi^2} A^{ren}(\{x\}|\psi)|_{\psi=0} = \langle -\lambda \frac{\partial^2}{\partial\psi_x^2} \mathcal{V}_x(\phi_x) + \delta m^2(x)\rangle + O(\lambda^2) \doteq 0. \tag{5.26}$$

Therefore $\delta m^2(x) = O(\lambda)$. Let $\delta m^2(X) = O(\lambda^{|X|})$ for all $X \subset (a\mathbb{Z})^\nu$ with $|X| < n$. Consider $P \subset (a\mathbb{Z})^\nu$ with $|P| = n$. Because of Eq. (5.14) and the induction hypothesis, we have

$$B(P|\psi) = \prod_{x\in P} e^{-\lambda\mathcal{V}(\phi_x+\psi_x)}[e^{\delta m^2(P)(\phi_x+\psi_x)^2} - 1] + O(\lambda^{n+1}). \tag{5.27}$$

Hence

$$\frac{\partial^2}{\partial\psi^2}B(P|\psi)|_{\psi=0} = |P|\ \delta m^2(P) + O(\lambda^{n+1}). \tag{5.28}$$

Since

$$A^{ren}(P|\psi) = \sum_{P=\sum Y} \langle\prod_Y [B(Y|\psi);\ ]\rangle \tag{5.29}$$

(see app. A), we obtain from (5.27) and (5.28)

$$\frac{\partial^2}{\partial\psi^2}\langle B(P|\psi)\rangle|_{\psi=0} = |P|\ \delta m^2(P) + O(\lambda^{n+1}) = -\sum_{\substack{P=\sum Y\\ Y\neq P}} \frac{\partial^2}{\partial\psi^2}\langle\prod_Y [B(Y|\psi);\ ]\rangle|_{\psi=0}. \tag{5.30}$$

By induction hypothesis follows $B(Y|\psi) = O(\lambda^{|Y|})$ for all $Y \subset P,\ Y \neq P$. Therefore the rhs of Eq. (5.30) is of order λ^n. This proves $\delta m^2(P) = O(\lambda^n)$.

We will now show $\delta e(P) = O(\lambda^{|P|})$. If $|P| = 1,\ P = \{x\}$, we have

$$B(\{x\}|\psi) = e^{-\lambda\mathcal{V}_x(\phi_x+\psi_x)+\delta m^2(x)(\phi_x+\psi_x)^2+\delta e(x)}. \tag{5.31}$$

From (5.24a) follows

$$A^{ren}(\{x\}|\psi)|_{\psi=0} = \langle B(\{x\}|\psi)\rangle|_{\psi=0} = 1 + \langle -\lambda\mathcal{V}_x(\phi_x) + \delta m^2(x)\phi_x^2 + \delta e(x)\rangle + O(\lambda^2) = 1. \tag{5.32}$$

Therefore $\delta e(x) = O(\lambda)$. Suppose $\delta e(X) = O(\lambda^{|X|})$, if $|X| < n$. Let us consider $P \subset (a\mathbb{Z})^\nu$ with $|P| = n$. From (5.24a) and (5.14) follows

$$B(P|\psi) = \prod_{x\in P} e^{-\lambda\mathcal{V}(\phi_x+\psi_x)}[e^{\delta m^2(P)(\phi_x+\psi_x)^2+\delta e(P)} - 1] + O(\lambda^{n+1}). \tag{5.33}$$

With the help of (5.30) we obtain

$$\langle B(P|\psi)\rangle|_{\psi=0} = \delta e(P)\ |P| + \delta m^2(P)\langle\phi_x^2\rangle + O(\lambda^{n+1}) = -\sum_{\substack{P=\sum Y\\ Y\neq P}} \langle\prod_Y [B(Y|\psi);\ ]\rangle|_{\psi=0}. \tag{5.34}$$

By induction hypothesis the order of the truncated expectation value is λ^n. From $\delta m^2(P) = O(\lambda^n)$ follows $\delta e(P) = O(\lambda^n)$.$\checkmark$

The molecular activities $B(P|\psi)$ are determined by the following recursive equations

$$\langle B(\{x\}|\psi)\rangle|_{\psi=0} = 1 \tag{5.35a}$$

$$\frac{\partial^2}{\partial\psi^2}\langle B(\{x\}|\psi)\rangle|_{\psi=0} = 0 \tag{5.35b}$$

for all $x \in (a\mathbb{Z})^\nu$ and

$$\langle B(P|\psi)\rangle|_{\psi=0} = -\sum_{\substack{P=\sum Y\\ Y\neq P}} \langle\prod_Y [B(Y|\psi);\ ]\rangle|_{\psi=0} \tag{5.35c}$$

$$\frac{\partial^2}{\partial\psi^2}\langle B(P|\psi)\rangle|_{\psi=0} = -\frac{\partial^2}{\partial\psi^2}\sum_{\substack{P=\sum Y\\ Y\neq P}} \langle\langle B(Y|\psi);\ ]\rangle|_{\psi=0}. \tag{5.35d}$$

The renormalized activities are determined by

$$M^{ren}(P|\psi) = A^{ren}(P|\psi) - \delta_{1,|P|} = \sum_{P=\sum Y} \langle\prod_Y [B(Y|\psi) - 1;\ ]\rangle \tag{5.36}$$

with the renormalization conditions

$$M^{ren}(P|\psi)|_{\psi=0} = 0 \tag{5.37a}$$

$$\frac{\partial^2}{\partial\psi^2} M^{ren}(P|\psi)|_{\psi=0} = 0 \tag{5.37b}$$

for all $P \subset (a\mathbb{Z})^\nu$, $|P| < \infty$. The renormalized Mayer amplitude is defined by

$$\mathcal{M}^{ren}(X|\psi) = \frac{a^{-\nu|X|}}{|X|!} M^{ren}(X|\psi) \tag{5.38}$$

and the renormalized augmented Mayer amplitude is defined by (cf. (2.46))

$$\widetilde{\mathcal{M}}^{ren}(X|\psi) = \widetilde{M}^{ren}(X|\psi) \prod_{\substack{x\in X \\ distinct}} n(x)!/(n!\, a^{\nu n}) \tag{5.39}$$

with

$$\widetilde{M}^{ren}(X|\psi) = \sum_{\substack{Q \\ supp\ Q = X}} a(Q) \prod_{P\in Q} M^{ren}(P|\psi) \tag{5.40}$$

(for the definition of $n(x)$, Q, $a(Q)$ see section 2.2.). From the renormalization conditions we obtain a simple relation for the two point renormalized connected free-propagator-amputated Greens functions, if the theory is symmetrical about the transformation $\psi \to -\psi$ on the lattice $\Lambda \subset (a\mathbb{Z})^\nu$:

$$G_c^{ren}(\underline{z}_1, \underline{z}_2) = \frac{\partial^2}{\partial\psi_{z_1}\partial\psi_{z_2}} \sum_{\substack{P \\ z_1, z_2 \in P \subseteq \Lambda}} M^{ren}(P|\psi)|_{\psi=0}. \tag{5.41}$$

In general

$$G_c^{ren}(\underline{z}_1, \ldots, \underline{z}_n) = \sum_{k\geq 1} \sum_{Z=\sum_{k=1}^{k} X_i} \sum_{\substack{\{P_i\} \\ X_i \subseteq \Lambda}} a(Q) \prod_{i=1}^{k} \partial_{X_i} M^{ren}(P_i|\psi)|_{\psi=0} \tag{5.42}$$

with $Z = \{z_1, \ldots, z_k\}$, Cluster $Q := (P_1, \ldots, P_k)$, $\partial_{X_i} = \prod_{x\in X_i} \frac{\partial}{\partial\phi_x}$. Fig. 5.1 shows the both possible forms of the cluster for the 4 point renormalized connected free-propagator-amputated Greens function $G_c^{ren}(\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4)$, if the theory is symmetrical about the transformation $\psi \to -\psi$.

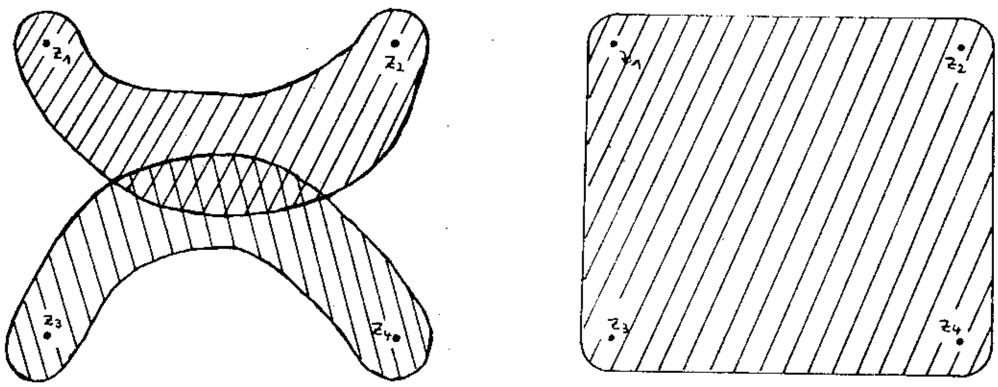


Fig. 5.1 Cluster for the 4 point renormalized connected free-propagator-amputated Greens function $G_c^{ren}(\underline{z}_1, \ldots, \underline{z}_4)$, if the theory is symmetrical about $\psi \to -\psi$.

Because of

$$Z(X|\psi) = \langle \sum_{X=\sum P} \prod_P B(P|\psi) \rangle, \tag{5.43}$$

the renormalized activities $A^{ren}(X|\psi)$ are represented by truncated expectation values (see app. A)

$$A^{ren}(X|\psi) = \sum_{X=\sum P} \langle \prod_P [B(P|\psi);\ ] \rangle. \tag{5.44}$$

The activity $M^{ren}(X|\psi) = -\delta_{1,|X|} + A^{ren}(X|\psi)$ on the doubled lattice is

$$M^{ren}(X|\psi) = \sum_{X=\sum P} \langle \prod_P [B(P|\psi) - 1;\ ] \rangle. \tag{5.45}$$

For these truncated expectation values exist a generalization of the tree graph formula of section 2.5. .

Theorem 5.2.1.. *Let $M_1, \ldots, M_n$ be disjoint subsets of $(a\mathbb{Z})^\nu$ and $B(M_i|\psi)$ molecular activities, which depends only from ϕ_x, ψ_x with $x \in M_i$. Then*

$$\langle [B(M_i|\psi);\ ] \rangle = \sum_\pi \sum_\eta \int_0^1 d\sigma_{n-1} f(\eta|s) \langle \prod_{(ij)\in\eta} [\frac{\partial}{\partial\phi_{M_{\pi(i)}}} v_{M_{\pi(i)}M_{\pi(j)}} \frac{\partial}{\partial\phi_{M_{\pi(j)}}}] \prod_{i=1}^n B(M_i|\psi) \rangle_{v[s]} \tag{5.46}$$

where

$$\frac{1}{2} v[s] = \frac{1}{2} \sum_{i=1}^n v_{M_iM_i} + \sum_{1\le i<j\le n} s_i s_{i+1} \ldots s_{j-1} v_{M_iM_j} \tag{5.47}$$

with $v_{M_iM_j} = \chi_{M_i} v \chi_{M_j}$, χ = characteristic function and

$$\frac{\partial}{\partial\phi_{M_i}} v_{M_iM_j} \frac{\partial}{\partial\phi_{M_j}} = \sum_{\substack{x\in M_i\\ y\in M_j}} \frac{\partial}{\partial\phi_x} v_{xy} \frac{\partial}{\partial\phi_y}. \tag{5.48}$$

The sum is over all permutations

$$\pi\ :\ \begin{cases} \{1,\ldots,n\} \to \{1,\ldots,n\} \\ i \mapsto \pi(i) \end{cases} \tag{5.49}$$

with $\pi(1) = 1$ and over all n-trees η

$$\eta\ :\ \begin{cases} \{2,\ldots,n\} \to \{1,\ldots,n-1\} \\ k \mapsto \eta(k) \end{cases} \tag{5.50}$$

with $\eta(k) < k$.

PROOF: The Fourier transform $\tilde{B}(M|q)$ of $B(M|\psi)$ is defined by

$$B(M|\psi) = \int [\prod_{x\in M} dq_x e^{iq_x(\phi_x+\psi_x)}] \tilde{B}(M|q). \tag{5.51}$$

We insert (5.51) on the rhs of (5.43). This gives

$$Z(X|\psi) = \sum_{X=\sum M} \int \prod_M (\prod_{x\in M} dq_x) \tilde{B}(M|q) e^{-\frac{1}{2}(q,vq)}. \tag{5.52}$$

Hence

$$Z(X|\psi) = \sum_{X=\sum M} \int \Big[\prod_M (\prod_{x\in M} dq_x) e^{-\frac{1}{2}(q,v_{MM}q)} \tilde{B}(M|q)\Big] \prod_{\substack{(RS)\\ R,S\in\{M\},R\neq S}} e^{-\frac{1}{2}(q,v_{RS}q)}. \tag{5.53}$$

We introduce indices for Eq. (5.53). This gives

$$Z(X|\psi) = \sum_{n=1}^{|X|} \sum_{X=\sum_{i=1}^n M_i} \int \Big[\prod_{i=1}^n (\prod_{x\in M_i} dq_x) e^{-\frac{1}{2}(q,v_{M_iM_i}q)} \tilde{B}_i(M|q)\Big] \prod_{\substack{(ij)\\ i\neq j}} e^{-\frac{1}{2}(q,v_{M_iM_j}q)}. \tag{5.54}$$

Furthermore

$$Z(X|\psi) = \sum_{n=1}^{|X|} \sum_{X=\sum_{i=1}^n M_i} \sum_{\mathcal{B}_n} \int \Big[\prod_{i=1}^n (\prod_{x\in M_i} dq_x) e^{-\frac{1}{2}(q,v_{M_iM_i}q)} \tilde{B}_i(M|q)\Big] \prod_{(ij)\in\mathcal{B}_n} [e^{-\frac{1}{2}(q,v_{M_iM_j}q)} - 1]. \tag{5.55}$$

$\mathcal{B}_n$ is the set of all graphs with n vertices. We decompose $\mathcal{B}_n$ into connected Mayer graphs $\mathcal{G}_I$, $I \subseteq \{1,\ldots,n\} \stackrel{\text{def}}{=} \underline{n}$. Hence

$$Z(X|\psi) = \sum_{n=1}^{|X|} \sum_{X=\sum_{i=1}^n M_i} \sum_{\underline{n}=\sum I} \prod_I \sum_{\mathcal{G}_I} \int \Big[\prod_I \prod_{i\in I} dq_x e^{-\frac{1}{2}(q,v_{M_iM_i}q)} \tilde{B}_i(M|q)\Big] \prod_{(ij)\in\mathcal{G}_I} [e^{-\frac{1}{2}(q,v_{M_iM_j}q)} - 1]. \tag{5.56}$$

We omit the indices in (5.56). This gives

$$Z(X|\psi) = \sum_{X=\sum M} \sum_{\{M\}=\sum Q} \prod_Q \tilde{A}(Q,\{M\}) \tag{5.57}$$

with

$$\tilde{A}(Q,\{M\}) = \sum_{\mathcal{G}_Q} \int \Big[\prod_{M\in Q} (\prod_{x\in M} dq_x) e^{-\frac{1}{2}(q,v_{MM}q)} \tilde{B}(M|q)\Big] \prod_{(RS)\in\mathcal{G}_Q} [e^{-\frac{1}{2}(q,v_{M_iM_j}q)} - 1] \tag{5.58}$$

where Q is a subset of the partition $\{M\}$ of X and $\sum Q$ is a partition of $\{M\}$. $\mathcal{G}_X$ labels the set of all Mayer graphs with vertex set Q. By the distributive law follows

$$\sum_{X=\sum M} \sum_{\{M\}=\sum Q} \prod_Q \tilde{A}(Q,\{M\}) = \sum_{X=\sum Q} \prod_Q \Big[\sum_{Q=\sum M} \tilde{A}(Q,\{M\})\Big]. \tag{5.59}$$

From (5.43), (5.57)-(5.59) and the definition of the truncated expectation value follows

$$\sum_{X=\sum M} \langle \prod_M [B(M|\psi);\ ]\rangle = \sum_{X=\sum M} \tilde{A}(X,\{M\}) =$$

$$\sum_{X=\sum M} \sum_{\mathcal{G}_{\{M\}}} \int \Big[\prod_{M\in\{M\}} (\prod_{x\in M} dq_x) e^{-\frac{1}{2}(q,v_{MM}q)} \tilde{B}(M|q)\Big] \prod_{(RS)\in\mathcal{G}_{\{M\}}} [e^{-\frac{1}{2}(q,v_{RS}q)} - 1]. \tag{5.60}$$

Let us set $B(M|\psi) = 0$, if $M \neq M_i$ for all $i = 1,\ldots,n$. Therefore Eq. (5.60) yields

$$\langle \prod_{i=1}^n [B(M_i|\psi);\ ]\rangle = \sum_{G\in\mathcal{G}_n} \int \Big[\prod_{i=1}^n (\prod_{x\in M_i} dq_x) e^{-\frac{1}{2}(q,v_{M_iM_i}q)} \tilde{B}_i(M|q)\Big] \prod_{(ij)\in G} [e^{-\frac{1}{2}(q,v_{M_iM_j}q)} - 1]. \tag{5.61}$$

$\mathcal{G}_n$ labels the set of all Mayer graphs with vertex set $\underline{n} = \{1,\ldots,n\}$. We obtain by the abstract tree graph formula (see app. B, Corollary B.5.)

$$\langle \prod_{i=1}^n [B(M_i|\psi);\ ]\rangle = \sum_\pi \sum_\eta \int_0^1 d\sigma_{n-1} f(\eta|s) \int \Big[\prod_{i=1}^n (\prod_{x\in M_i} dq_x) e^{-\frac{1}{2}(q,v_{M_iM_I}q)} \tilde{B}_i(M|q)\Big]$$

$$\prod_{(ij)\in\eta} [-\sum_{\substack{x\in M_{\pi(i)}\\ y\in M_{\pi(j)}}} q_x v_{xy} q_y] e^{-\frac{1}{2}(q,v[s]q)}. \tag{5.62}$$

We apply the inverse Fourier transformation on the rhs of Eq. (5.62). This proves the assertion. √

In the same way we obtained estimates for the truncated expectation value $\langle \prod_{x\in X} [F_x(\phi_x);\ ]\rangle$ by the tree graph formula, we obtain estimates for the truncated expectation value $\langle \prod_{i=1}^n [B(M_i|\psi);\ ]\rangle$ with the help of the generalized tree graph formula (5.46).

Theorem 5.2.2. *Let $M_1, \ldots, M_n$ be disjoint subsets of $(a\mathbb{Z})^\nu$ and $B(M_i|\psi)$ molecular activities, which depend only on ϕ_x, ψ_x with $x \in M_i$. Suppose that $B(M_i|\psi)$ are holomorph and bounded functions in $S_\kappa^{|M_i|} = \{(\phi_x)_{x\in M_i} \in \mathbb{C}^{|M_i|} \mid |Im\ \phi_x| \le \kappa\}$. We introduce the notations*

$$c_\kappa^{M_i} \stackrel{\text{def}}{=} \min_{c\in\mathbb{R}} \sup_{x\in M_i} \sup_{\substack{\phi_x\in\mathbb{C} \\ |Im\ \phi_x|=\kappa}} |B(M_i|\psi) - c| < \infty \tag{5.63}$$

for all $i \in \{1, \ldots, n\}$ and

$$\mathcal{C}(M_1, \ldots, M_n) = \langle \prod_{i=1}^n [B(M_i|\psi);\ ] \rangle. \tag{5.64}$$

Then we have the following estimates :

(i) *Suppose that for positive constants D, m the following inequality*

$$|v_{xy}| \le De^{-m\|x-y\|} \tag{5.65}$$

is fulfilled. Then

$$\frac{8De^{mL(P)}|\mathcal{C}(M_1, \ldots, M_n)|}{\kappa^2} \le \frac{(n-1)!}{2} \prod_{i=1}^n \left[\frac{8De^{mL(M_i)} c_{\kappa|M_i|}^{M_i}}{\kappa^2} \right] \tag{5.66}$$

with $P = \sum_{i=1}^n M_i$.

(ii) *Let*

$$a^{-\nu} \int_{x\in(a\mathbb{Z})^\nu} |v_{xy}| = \frac{1}{(ma)^2} < \infty. \tag{5.67}$$

Then

$$a^{-\nu(s-1)} \int_{x_2,\ldots,x_s\in(a\mathbb{Z})^\nu} \frac{8|\mathcal{C}(M_1, \ldots, M_n)|}{(ma\kappa)^2} \le \frac{(n-1)!}{2} \prod_{i=1}^n \left[\frac{8c_{\kappa|M_i|}^{M_i}}{(ma\kappa)^2} \right] \tag{5.68}$$

with $M_i = \{x_{n_1+\cdots+n_{i-1}+1}, \ldots, x_{n_1+n_2+\cdots+n_i}\}$, $|M_i| = n_i$, for all $i \in \{1, \ldots, n\}$ and $s = \sum_{i=1}^n n_i$.

PROOF: (i) : By Theorem 5.2.1. and (5.65) follows

$$|\mathcal{C}(M_1, \ldots, M_n)| \le (n-1)! \sum_\eta \int_0^1 d\sigma_{n-1} f(\eta|s) \sup_{\substack{\phi_x\in\mathbb{R} \\ x\in\sum_{i=1}^n M_i}} \left\{ \prod_{i=1}^n |(\sum_{x\in M_i} \frac{\partial}{\partial\phi_x})^{d_i(\eta)} B(M_i|\psi)| \right\} D^{n-1} e^{-mL(M_1,\ldots,M_n)} \tag{5.69}$$

where $d_i(\eta)$ = number of lines in the n-tree η, which emerge from vertex i, and

$$L(M_1, \ldots, M_n) = \min_\eta \sum_{(ij)\in\eta} \min_{\substack{x\in M_i \\ y\in M_j}} \|x-y\| \tag{5.70}$$

i.e. $L(M_1, \ldots, M_n)$ is the length of the shortest polygon, which connects $M_1, \ldots, M_n$. By the multinominial theorem follows

$$|(\sum_{x\in M_i} \frac{\partial}{\partial\phi_x})^{d_i(\eta)} B(M_i|\psi)| \le \sum_{\substack{m_1,\ldots,m_{|M_i|} \\ \sum m_j = d_i(\eta)}} \frac{d_i(\eta)!}{m_1!\ldots m_{|M_i|}!} |(\prod_{j=1}^{|M_i|} \frac{\partial^{m_j}}{\partial\phi_{x_j}^{m_j}}) B(M_i|\psi)|. \tag{5.71}$$

The Cauchy inequality for several variables implies

$$|(\sum_{x\in M_i} \frac{\partial}{\partial\phi_x})^{d_i(\eta)} B(M_i|\psi)| \le \frac{d_i(\eta)!}{\kappa^{d_i(\eta)}} \sum_{\substack{m_1,\ldots,m_{|M_i|} \\ \sum m_j = d_i(\eta)}} \frac{c_{\kappa|M_i|}^{M_i}}{|M_i|^{d_i(\eta)}}. \tag{5.72}$$

Because of

$$\sum_{\substack{m_1,\ldots,m_{|M_i|}\\ \sum m_j=d_i(\eta)}} 1 \leq \sum_{\substack{m_1,\ldots,m_{|M_i|}\\ \sum m_j=d_i(\eta)}} \frac{d_i(\eta)!}{\prod_{j=1}^{|M_i|} m_j!} = \underbrace{(1+\cdots+1)}_{m_i \text{ summands}}{}^{d_i(\eta)} = |M_i|^{d_i(\eta)},$$

we obtain

$$|(\sum_{x\in M_i} \frac{\partial}{\partial\phi_x})^{d_i(\eta)} B(M_i|\psi)| \leq \frac{d_i(\eta)!}{\kappa^{d_i(\eta)}} c^{M_i}_{\kappa|M_i|}. \tag{5.73}$$

We insert this on the rhs of (5.69) and use (3.29). This gives

$$|C(M_1,\ldots,M_n)| \leq (n-1)!\,\frac{D^{n-1}e^{-mL(M_1,\ldots,M_n)}}{2\kappa^{2(n-1)}}8^{n-1}\prod_{i=1}^{n} c^{M_i}_{\kappa|M_i|}. \tag{5.74}$$

From the definition of L follows

$$L(\sum_{i=1}^{n} M_i) \leq L(M_1,\ldots,M_n) + \sum_{i=1}^{n} L(M_i). \tag{5.75}$$

Therefore (5.74) proves the assertion (5.66).

(ii) : The assertion (5.68) follows in the same way as in the proof for (i), if we integrate over $x_2,\ldots,x_{s-1} \in (a\mathbb{Z})^\nu$. $\surd$

5.3. Renormalization group and Mayer expansion

The Fourier transform $\tilde{\phi}$ of the field ϕ on the lattice $(a\mathbb{Z})^\nu$ is defined by

$$\phi_x = \frac{1}{(2\pi)^\nu}\int_{p\in[-\frac{\pi}{a},\frac{\pi}{a}]^\nu} d^\nu p\ \tilde{\phi}_p e^{ipx} \tag{5.76}$$

with $px = \sum_{i=1}^{\nu} p_i x_i$. The field ϕ is called ***high frequent*** if the dominant part of the Fourier integral (5.76) is at large p. The Fourier transform $\tilde{v}$ of the translation invariant propagator v is given by

$$v_{xy} = \frac{1}{(2\pi)^\nu}\int_{p\in[-\frac{\pi}{a},\frac{\pi}{a}]^\nu} d^\nu p\ \tilde{v}_p\ e^{ip(x-y)}. \tag{5.77}$$

The momenta p of the field ϕ on the lattice $(a\mathbb{Z})^\nu$ are bounded by

$$||p|| \leq \frac{\pi}{a}. \tag{5.78}$$

Therefore the UV-cutoff on a lattice with lattice spacing a is $\frac{\pi}{a}$. UV-divergences emerge from the continuum limit $a \to 0$. They are removed by suitable counterterms.

Corresponding to Wilson's renormalization group approach [21] the UV-cutoff decreases, if a renormalization group step is done. For that we split off the propagator v

$$v = v^1 + \cdots + v^N \tag{5.79a}$$

where

$$\tilde{v}^i_p = 0 \ \text{ if } \ ||p|| \notin [K_{i-1}, K_i],\ i = 1,\ldots,N \tag{5.79b}$$

with $0 = K_0 < K_1 < \dots K_{N-1} < K_N = \frac{\pi}{a}$. Let ϕ_i be the process of the covariance v^i. Then we have

$$\phi = \phi^1 + \dots + \phi^N. \tag{5.80}$$

The propagator v^i connects only fields, whose momenta lies in the section $[K_{i-1}, K_i]$. The momenta of the fields ϕ^i increase with increasing index i (in the contrary to the notation of [8]). By the partition formula (Lemma 3.1.2.) we have

$$Z(\Lambda|\psi) = \int d\mu_v(\phi) e^{-V(\phi)} = \int d\mu_{v^1}(\phi^1) \dots d\mu_{v^N}(\phi^N) e^{-V(\phi^1 + \dots + \phi^N)}. \tag{5.81}$$

After integration over the field ϕ^N the propagator v with UV-cutoff $\frac{\pi}{a}$ proceeds to the propagator $v - v^N$ with UV-cutoff $K_{N-1} < \frac{\pi}{a}$. This is called the first renormalization group step. Thereby, the action $V \equiv V^N$ proceeds to a new (in general nonlocal) action V^{N-1}. This new action V^{N-1} is called the ***effective action***. Next, we apply this procedure to the new form of the partition function. After k renormalization group steps we get an UV-cutoff K_{N-k}. The free energy $\ln Z(\Lambda|\psi)$ equals the negative effevtive action after N renormalization group steps. The form of the partition function after $N-k$ renormalization group steps is

$$Z(\Lambda|\psi) = \int d\mu_{v^{[\leq]}}(\phi^{[\leq]}) e^{-V^k(\phi^{[\leq k]} + \psi)} \tag{5.82a}$$

with the field

$$\phi^{[\leq k]} = \sum_{i=1}^{k} \phi^i \tag{5.82b}$$

and the propagator

$$v^{[\leq k]} = \sum_{i=1}^{k} v^i. \tag{5.82c}$$

The effective action V^k is recursive determined by

$$V^N(\phi + \psi) = V(\phi + \psi) \tag{5.83a}$$

$$V^{k-1}(\phi^{[\leq k-1]} + \psi) = -\ln \int d\mu_{v^k}(\phi^k) e^{-V^k(\phi^{[\leq k-1]} + \phi^k + \psi)} \tag{5.83b}$$

with $k \in \{1, \dots, N\}$. Therefore we have after N renormalization group steps

$$V^0(\psi) = -\ln Z(\Lambda|\psi). \tag{5.84}$$

Instead of (5.79a,b) we may split off the propagator $v = (-\Delta + m^2)^{-1}$ in the following manner

$$v = v^1 + \dots + v^N \tag{5.85a}$$

where

$$v^i = \begin{cases} (-\Delta + M^2)^{-1} & \text{if } i = N \\ (-\Delta + M_{i-1}^2)^{-1} - (-\Delta + M_i^2)^{-1} & \text{if } i < N \end{cases} \tag{5.85b}$$

with $m = M_0 < M_1 < \dots < M_{N-1} < M_N = M = O(a^{-1})$. The propagator after k renormalization group steps is

$$v^1 + \dots + v^{N-k} = (-\Delta + m^2)^{-1} - (-\Delta + M_{N+1-k}^2)^{-1}. \tag{5.86}$$

This is a propagator for a theory with Pauli-Villars cutoff M_{N+1-k}. Momenta which are larger than M_{N+1-k} are suppressed in this case. The Pauli-Villars cutoff will decreases after application of a renormalization group step.

Mayer expansion yields a decomposition of the configuration space and the renormalization group approach yields a decomposition of the momentum space. Combination of the Mayer expansion with the renormalization group yields therefore a decomposition of the phase space ('phase space cell expansion'). For estimates of the activity $M(X|\psi)$ we have used the same propagator v for all Polymers P. These estimates are bad for large polymers P and high frequent fields ϕ. Decomposition of the phase space carry to improved estimates for the activities. In this sense the method of iterated Mayer expansions leads to an improved estimate of the activity ([8], [9]). For that the polymers are decomposed in polymers, which are again decomposed in polymers, etc. . Points of the lattice are called *0-vertices*. Ordinary polymers are called *1-vertices*. *k-vertices* are collections of $(k-1)$-vertices. The points of a k-vertex are called *constituents*. Every k-vertex corresponds to an activity. k-vertices interact over the propagator v^{N+1-k}. Since the strength of the propagator v^{N+1-k} increases and the range decreases if the index k decreases, it follows that the elements P' of a k-vertex P interact over a propagator whose range is less and whose strength is larger than the corresponding range and strength of the propagator, which defines the interaction for the k-vertices. The activities for the k-vertex P are expressed by the activities for the $(k-1)$-vertices, which are elements of the k-vertex P. Therefore the estimates for the k-vertices are recursive. For the k-th recursion step an estimate for the propagator v^{N-k+1} is used. This gives better estimates than the estimates obtained by simple Mayer expansion, where we have used bounds for the whole propagator v.

A perturbative formulation for the renormalization group steps was given by Gallavotti and Nicoló [11]. For the proof of the Gallavotti Nicoló tree formula the following partition formula for the truncated expectation value is useful.

Lemma 5.3.1. (Partition formula for truncated expectation values). *Let S be finite set and B_i, $i \in S$, random variables. Then we have for positive propagators v_1, v_2*

$$\langle \prod_{i\in S}[B_i;\ ]\rangle_{v^1+v^2} = \sum_{S=\sum J} \langle \prod_J [\langle \prod_{j\in J}[B_j;\ ]\rangle_{v^2};\ ]\rangle_{v^1}. \tag{5.87}$$

PROOF: By the distributive law follows

$$\sum_{R=\sum I}\prod_I\Big\{\sum_{I=\sum j}\langle\prod_J[\langle\prod_{j\in J}[B_j;\ ]\rangle_{v^2};\ ]\rangle_{v^1}\Big\} = \sum_{R=\sum K}\sum_{\{K\}=\sum I}\prod_I\langle\prod_{J\in I}[\langle\prod_{j\in J}[B_j;\ ]\rangle_{v^2};\ ]\rangle_{v^1}. \tag{5.88}$$

By the definition of the truncated expectation value $\langle\ ;\ \rangle_{v^1}$ we obtain for the rhs of (5.88)

$$\text{rhs} = \sum_{R=\sum K}\langle\prod_K\langle\prod_{j\in K}[B_j;\ ]\rangle_{v^2}\rangle_{v^1} \tag{5.89}$$

and once more by the definition of the truncated expectation value $\langle\ ;\ \rangle_{v^1}$ follows

$$\text{rhs} = \langle\langle\prod_{i\in R}B_i\ \rangle_{v^2}\rangle_{v^1}. \tag{5.90}$$

By the partition formula for expectation values (Lemma 3.2.2.) follows

$$\text{rhs} = \langle\prod_{i\in R}B_i\rangle_{v^1+v^2}. \tag{5.91}$$

Hence

$$\sum_{R=\sum I}\prod_I\Big\{\sum_{I=\sum J}\langle\prod_J[\langle\prod_{j\in J}[B_j;\ ]\rangle_{v^2};\ ]\rangle_{v^1}]\Big\} = \langle\prod_{i\in R}B_i\rangle_{v^1+v^2}. \tag{5.92}$$

The definition of truncated expectation values proves the assertion (5.87). √

The partitions in the partition formula (5.87) are related to tree graphs with depth 2. The maximal vertices of the tree graphs represent the elements $j \in S$ and the vertices with depth 1 represent subsets $J \subseteq S$, such that for the direct successor which represents j, the relation $j \in J$ holds. The generalization of the partition formula for truncated expectation values for the splitting (5.79) of the propagator v leads to the notion of the Gallavotti Nicoló tree. Let us set $\Gamma(k, I)$ for the set of all trees with the depth k and maximal vertices $\in I$. For the splitting (5.79) of the propagator v and the random variables B_i, $i \in I$ we will call a tree $\gamma \in \Gamma(k, I)$, $1 \le k \le N$, *Gallavotti Nicoló tree (GN-tree)*. Two trees γ_1, γ_2 are put together by introducing a new root where the old roots of γ_1 and γ_2 are direct successors of the new root. The new tree is labeled by $\gamma_1 \circ \gamma_2$. In the same way we put together more than two trees. The depth of the new tree $\gamma_1 \circ \gamma_2$ is increased by 1. The truncated expectation value $\mathcal{E}^T(B,\gamma)$, $B = \{B_i,\ i \in I\}$, which corresponds to $\gamma \in \Gamma(k,I)$, is recursively defined by

$$(i) \quad \mathcal{E}^T(I,\gamma) = \langle \prod_{i\in I} [B_i;\ ] \rangle_{v^N} \qquad \text{if } \gamma \in \Gamma(1,I) \tag{5.93a}$$

$$(ii) \quad \mathcal{E}^T(B,\gamma) = \langle \prod_{i=1}^{r} [\mathcal{E}^T(B,\gamma_i);\ ] \rangle_{v^{N-k}} \qquad \text{if } \gamma = \gamma_1 \circ \cdots \circ \gamma_r \in \Gamma(k+1,J),\ \gamma_i \in \Gamma(k,I_i) \tag{5.93b}$$

and $J = \sum_{i=1}^r I_i$. For $\gamma \in \Gamma(k,I)$ the truncated expectation value $\mathcal{E}^T(B,\gamma)$ depends from the field $\phi^{[N-k]} = \sum_{j=1}^{N-k} \phi^j$. The generalization of the partition formula for truncated expectation values is

Corollary 5.3.2. (GN tree formula for truncated expectation values).

$$\langle \prod_{i\in I} [B_i;\ ] \rangle_{v^{N-k+1}+\ldots v^N} = \sum_{\gamma\in\Gamma(k,I)} \mathcal{E}^T(B,\gamma) \tag{5.94}$$

for all $k \in \{1,\ldots,N\}$, $B = \{B_i,\ i \in I\}$.

PROOF (BY INDUCTION): For $k = 1$ follows the assertion by definition (i) of the truncated expectation value $\mathcal{E}^T(B,\gamma)$. Suppose that the assertion is valid for k. By Lemma 5.3.1. we obtain

$$\langle \prod_{i\in I} [B_i;\ ] \rangle_{v^{N-k}+\cdots+v^N} = \sum_{I=\sum J} \langle \prod_J [\langle \prod_{j\in J} [B_j;\ ] \rangle_{v^{N-k+1}+\cdots+v^N};\ ] \rangle_{v^{N-k}}. \tag{5.95}$$

From the induction hypothesis follows

$$\langle \prod_{i\in I} [B_i;\ ] \rangle_{v^{N-k}+\cdots+v^N} = \sum_{I=\sum J} \langle \prod_J [\sum_{\gamma\in\Gamma(k,J)} \mathcal{E}^T(B,\gamma);\ ] \rangle_{v^{N-k}}. \tag{5.96}$$

By the definition (ii) of the truncated expectation value $\mathcal{E}^T(I,\gamma')$ for $\gamma' \in \Gamma(k+1,I)$ follows the assertion for $k+1$. $\surd$

We consider now the special case $B_i \equiv V$ for all i, where V is the action. The GN-trees are characterized by the depth and the number of maximal vertices in this case. Let us denote $\widetilde{\Gamma}(k,n)$ for the set of all GN-trees with depth k and n maximal vertices. We use the notation

$$\gamma^p = \underbrace{\gamma \circ \cdots \circ \gamma}_{p \text{ arguments}}. \tag{5.97}$$

With this notations we have the following corollary for the representation of the effective action.

Corollary 5.3.3. *Let the combinatorial factor* $C(\gamma)$ *for* $\gamma \in \widetilde{\Gamma}(k,n)$, $n \in \mathbb{N}$, $k \in \{1,\ldots,N\}$ *be defined by*

$$(i) \quad C(\gamma) = n! \quad \text{if } \gamma \in \widetilde{\Gamma}(1,n) \tag{5.98a}$$

$$(ii) \quad C(\gamma_1^{p_1} \circ \cdots \circ \gamma_r^{p_r}) = \prod_{i=1}^{r} p_i!\ C(\gamma_i)^{p_i} \quad \text{if } \gamma_i \in \widetilde{\Gamma}(k,n). \tag{5.98b}$$

Then we have with the notations (5.79)-(5.84)

$$V^k(\phi^{[\le k]} + \psi) = \sum_{n\ge 0} \sum_{\gamma\in\widetilde{\Gamma}(N-k,n)} (-1)^{n+1} \frac{\mathcal{E}^T(V,\gamma)}{C(\gamma)} \tag{5.99}$$

for all $k \in \{1,\ldots,N\}$.

PROOF (BY INDUCTION): Let us set $l = N - k$. Suppose $l = 1$. Then

$$V^{N-1}(\phi^{\le[N-1]} + \psi) = -\ln\langle e^{-V(\phi+\psi)} \rangle_{v^N}. \tag{5.100}$$

With the help of the perturbation expansion (2.75) follows

$$V^{N-1}(\phi^{\leq[N-1]}+\psi)=\sum_{n\geq 0}\frac{(-1)^{n+1}}{n!}\langle\underbrace{V;\ldots;V}_{n\text{ arguments}}\rangle_{v^N}. \tag{5.101}$$

By the definition of $\mathcal{E}^T(V,\gamma)$ for $\gamma\in\widetilde{\Gamma}(1,n)$ and $C(\gamma)=n!$ follows the assertion for $l=1$. Suppose that the assertion is valid for l, $1\leq l<N$. By the induction hypothesis and the perturbation expansion (2.75) follows

$$\sum_{n\geq 0}\frac{(-1)^{n+1}}{n!}\langle\underbrace{V;\ldots;V}_{n\text{ arguments}}\rangle_{v^{N-l+1}+\cdots+v^N}=\sum_{n\geq 0}\sum_{\gamma\in\widetilde{\Gamma}(l,n)}(-1)^n\frac{\mathcal{E}^T(V,\gamma)}{C(\gamma)}. \tag{5.102}$$

Let V be of order λ. Comparison of the terms of the order λ^n in (5.102) yields

$$\langle\underbrace{V;\ldots;V}_{n\text{ arguments}}\rangle_{v^{N-l+1}+\cdots+v^N}=n!\sum_{\gamma\in\widetilde{\Gamma}(l,n)}\frac{\mathcal{E}^T(V,\gamma)}{C(\gamma)}. \tag{5.103}$$

Furthermore by the definition of the effective action $V^{N-(l+1)}$ follows

$$V^{N-(l+1)}(\phi^{[\leq N-(l+1)]}+\psi)=-\ln\langle e^{-V(\phi+\psi)}\rangle_{v^{N-l}+\cdots+v^N}=\sum_{n\geq 0}\frac{(-1)^{n+1}}{n!}\langle\underbrace{V;\ldots;V}_{n\text{ arguments}}\rangle_{v^{N-l}+\cdots+v^N}. \tag{5.104}$$

By Lemma 5.3.1. follows

$$V^{N-(l+1)}(\phi^{[\leq N-(l+1)]}+\psi)=\sum_{n\geq 0}\sum_{r\geq 1}\sum_{\substack{\{n_i\}\\ \sum_{i=1}^r=n}}\frac{(-1)^{n+1}}{\prod_{i=1}^r n_i!\prod_{j=1}^n p_j!}\langle\prod_{i=1}^r[\langle\underbrace{V;\ldots;V}_{n\text{ arguments}}\rangle_{v^{N-l}+\cdots+v^N}]\rangle_{v^{N-l}} \tag{5.105}$$

where $p_j=|\{n_i|\ n_i=j\}|$, i.e. p_j equals the number of integers j in the partition $\{n_i\}$. We insert (5.103) on the rhs of (5.105). This gives

$$V^{N-(l+1)}(\phi^{[\leq N-(l+1)]}+\psi)=\sum_{n\geq 0}\sum_{r\geq 1}\sum_{\substack{\{n_i\}\\ \sum_{i=1}^r=n}}\frac{(-1)^{n+1}}{\prod_{i=1}^n p_i!}\langle\prod_{i=1}^r[\sum_{\gamma_i\in\widetilde{\Gamma}(l,n_i)}\frac{\mathcal{E}^T(V,\gamma_i)}{C(\gamma_i)}]\rangle. \tag{5.106}$$

We put together the GN-trees γ_i. This gives

$$V^{N-(l+1)}(\phi^{[\leq N-(l+1)]}+\psi)=\sum_{n\geq 0}\sum_{r\geq 1}\sum_{\gamma_1^{p_1}\circ\cdots\circ\gamma_r^{p_r}\in\widetilde{\Gamma}(l+1,n)}\frac{(-1)^{n+1}}{\prod_{i=1}^n p_i!C(\gamma_i)^{p_i}}\mathcal{E}^T(V,\gamma_1^{p_1}\circ\cdots\circ\gamma_r^{p_r}). \tag{5.107}$$

The assertion for $l+1$ follows by the definition of the combinatorial coefficient $C(\gamma)$. $\surd$

Remark: (i) If the action V is of order λ, then it follows that $\mathcal{E}^T(V,\gamma)$ is of order λ^n and the expansion (5.99) for the effective action V^k is a perturbation expansion.

(ii) Corollary 5.3.3. implies a perturbation expansion for $\ln Z(\Lambda|\psi)$:

$$\ln Z(\Lambda|\psi)=\sum_{n\geq 0}\sum_{\gamma\in\widetilde{\Gamma}(N,n)}(-1)^n\frac{\mathcal{E}^T(V,\gamma)}{C(\gamma)}. \tag{5.108}$$

Fig. 5.2. shows an example for a GN-tree.

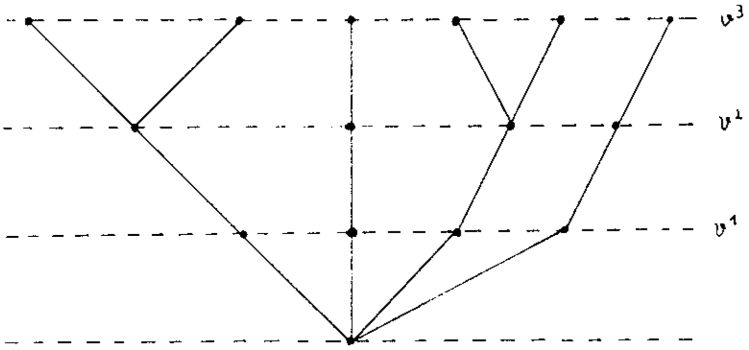


Fig. 5.2 Example for a GN-tree $\gamma \in \Gamma(3,6)$. γ represents the truncated expectation value

$$\langle\langle V;V\rangle_{v^3+v^2};\ \langle V\rangle_{v^3+v^2};\ \langle\langle V;V\rangle_{v^3};\ \langle V\rangle_{v^3}\rangle_{v^2}\rangle_{v^1}.$$

Let $\Lambda \subset (a\mathbb{Z})^\nu$ be a finite sublattice. Let the maximal vertices of a GN-tree represent elements of Λ. The set of all GN-trees γ with depth k and maximal vertices in $X \subseteq \Lambda$ is denoted by $\Gamma(k, X)$. Let $B(X|\psi)$ be random variables which depend only from $\phi_x + \psi_x$, $x \in X$. For the splitting (5.79) of the propagator v we define the truncated expectation value for the GN-trees γ and random variables $B(X|\psi)$:

$$(i) \qquad \mathcal{E}^T(B,\gamma) = B(X|\psi) \ \text{ if } \gamma \in \Gamma(1,X) \tag{5.109a}$$

$$(ii) \qquad \mathcal{E}^T(B,\gamma) = \langle \prod_{i=1}^{r} [\mathcal{E}^T(B,\gamma_i);\ ]\rangle_{v^{N-k+1}}$$

$$\text{if } \gamma = \gamma_1 \circ \cdots \circ \gamma_r \in \Gamma(k+1, X),\ \ \gamma_i \in \Gamma(k, Y_i) \text{ with } \sum_{i=1}^{r} Y_i = X \text{ and } k \in \{1,2,\ldots,N\} \tag{5.109b}$$

$$(iii) \qquad \mathcal{E}^T(B,\gamma) = \prod_{i=1}^{r} \mathcal{E}^T(B,\gamma_i) \ \text{ if } \gamma = \gamma_1 \circ \cdots \circ \gamma_r \in \Gamma(N+2, X),\ \ \gamma_i \in \Gamma(N+1, Y_i)$$

$$\text{with } \sum_{i=1}^{r} Y_i = X. \tag{5.109c}$$

With this notations we obtain for the activities $A(X|\psi)$ after N renormalization group steps :

Corollary 5.3.4. (GN-tree formula for activities). *Let the partition function be defined by*

$$Z(\Lambda|\psi) = \int d\mu_v(\phi) Z(\Lambda|\psi) \tag{5.110a}$$

with

$$Z(\Lambda|\psi) = \sum_{\Lambda = \sum X} \prod_{X} B(X|\psi). \tag{5.110b}$$

Then the activities $A(X|\psi)$ of the polymer representation

$$Z(\Lambda|\psi) = \sum_{\Lambda=\sum X} \prod_X A(X|\psi) \tag{5.111}$$

are

$$A(X|\psi) = \sum_{\gamma\in\Gamma(N+1,X)} \mathcal{E}^T(B,\gamma). \tag{5.112}$$

PROOF: The assertion follows immediately from

$$A(X|\psi) = \sum_{X=\sum Y} \langle \prod_Y [B(Y|\psi);\ ]\rangle_v \tag{5.113}$$

and the GN-tree formula for truncated expectation values (Corollary 5.3.2.). $\surd$

Remarks:

(i) By definition (iii) of the truncated expectation value $\mathcal{E}^T(B,\gamma)$ for $\gamma \in \Gamma(N+2,\Lambda)$ follows

$$Z(\Lambda|\psi) = \sum_{\gamma\in\Gamma(N+2,\Lambda)} \mathcal{E}^T(B,\gamma). \tag{5.114}$$

(ii) With the help of the recursion relation (5.109) for the truncated expectation value $\mathcal{E}^T(B,\gamma)$ and the generalization of the tree graph formula (Theorem 5.2.1.) the activities $A(X|\psi)$ may be expressed by trees, whose vertices consist of trees, whose vertices consist again of trees, etc. . Then the tree estimate for truncated expectation values (Theorem 5.2.2.) leads to a recursive estimate for the activities, *if* there is a suitable estimate for the molecular activities $B(Y|\psi)$ for all $Y \subseteq X$.

(iii) Every GN-tree $\gamma \in \Gamma(k,X)$ corresponds to a k-vertex α with constituent set X. Fig. 5.3 shows an example for a GN-tree γ and the corresponding k-vertex α.

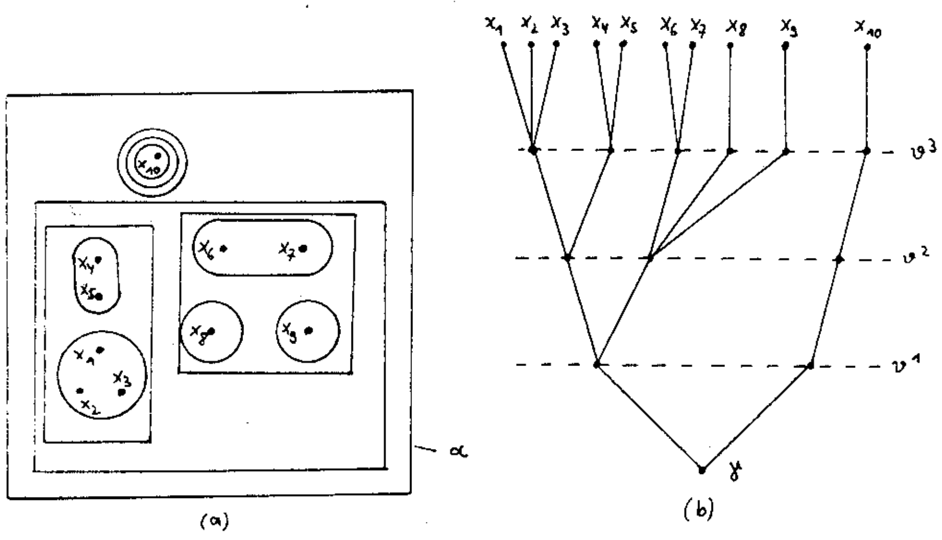


Fig. 5.3 Example for a 4-vertex α with constituent set $X=\{x_1,\dots,x_{10}\}$(a) and the corresponding GN-tree γ with depth 4 (b). The GN-tree γ or 4-vertex α represents

$$\langle\langle\langle B(\{x_1,x_2,x_3\}|\psi);\ B(\{x_4,x_5\}|\psi)\rangle_{v^3};\ \langle B(\{x_6,x_7\}|\psi);\ B(\{x_8\}|\psi);\ B(\{x_9\}|\psi)\rangle_{v^3}\rangle_{v^2};\ \langle B(\{x_{10}\}|\psi)\rangle_{v^3+v^2}\rangle_{v^1}.$$

APPENDIX A. ⋆-CALCULUS, TRUNCATED EXPECTATION VALUES AND COMBINATORIAL COEFFICIENT a(Q)

A.1. ⋆-CALCULUS [17]

Let Γ be an arbitrary finite set. Functions $X : \Gamma \to \mathbb{N}$ are called *multiindices*. For a finite multiindex X we define

$$X! = \prod_{\gamma\in\Gamma} X(\gamma)!, \qquad |X| = \sum_{\gamma\in\Gamma} X(\gamma). \tag{A.1}$$

The addition of multiindices X_1 and X_2 is defined by

$$(X_1 + X_2)(\gamma) = X_1(\gamma) + X_2(\gamma) \tag{A.2}$$

for all $\gamma \in \Gamma$. Let $\mathcal{A}^m$ be the set of all complex valued functions on the set of all multiindices, i.e.

$$\mathcal{A}^m = \{f \,:\, \{X \,:\, \Gamma \to \mathbb{N}\} \to \mathbb{C}\}. \tag{A.3}$$

The ⋆-product of two functions $f_1,\ f_2 \in \mathcal{A}^m$ is defined by

$$(f_1 \star f_2)(X) = \sum_{X = X_1 + X_2} f_1(X_1) f(X_2). \tag{A.4}$$

Addition and multiplication by scalars on $\mathcal{A}^m$ are defined by

$$(f_1 + f_2)(X) = f_1(X) + f_2(X) \tag{A.5}$$

$$(\lambda f)(X) = \lambda f(X) \tag{A.6}$$

for all $f_1, f_2, f \in \mathcal{A}^m$, $\lambda \in \mathbb{C}$. By this definitions $\mathcal{A}^m$ is an algebra with unit element $\mathbb{1}$,

$$\mathbb{1}(X) = \begin{cases} 1 & \text{if } X = 0 \\ o & \text{otherwise.} \end{cases} \tag{A.7}$$

Let us restrict on multiindices X with $X! = 1$. Then X corresponds to a subset of Γ and the familiar ⋆-algebra $(\mathcal{A}, \star)$ is defined by

$$\mathcal{A} = \{f \,:\, \{X \,:\, \Gamma \to \mathbb{N}\} \to \mathbb{N}\} \to \mathbb{C} \mid f(X) = 0 \text{ if } X! \neq 1\}. \tag{A.8}$$

Clearly, $\mathcal{A}$ is a subalgebra of $\mathcal{A}^m$.

To each function $f \in \mathcal{A}^m$ we may associate a formal power series $\tilde{f} \in P[(z_\gamma)_{\gamma\in\Gamma}]$ in the variables $z_\gamma, \gamma \in \Gamma$:

$$f \in \mathcal{A}^m \mapsto \tilde{f}(z) = \sum_{\substack{X \\ X \,:\, \Gamma \to \mathbb{N}}} f(X) z^X, \tag{A.9}$$

where $z^X = \prod_{\gamma\in\Gamma} z_\gamma^{X(\gamma)}$. Conversely, to each formal power series $\tilde{f} \in P[(z_\gamma)_{\gamma\in\Gamma}]$ we associate a function $f \in \mathcal{A}^m$:

$$\tilde{f} \in P[(z_\gamma)_{\gamma\in\Gamma}] \mapsto f \in \mathcal{A}^m \ \text{ with } \ f(X) = \partial_X \tilde{f}(z)|_{z=0}, \tag{A.10}$$

where $\partial_X = \prod_{\gamma\in\Gamma} \frac{\partial^{X(\gamma)}}{\partial z_\gamma^{X(\gamma)}}$. The set $P[(z_\gamma)_{\gamma\in\Gamma}]$ of all formal power series is an algebra and the multiplication

$$(\tilde{f}_1 \cdot \tilde{f}_2)(z) = \tilde{f}_1(z)\tilde{f}_2(z) \tag{A.11}$$

for all $\widetilde{f}_1, \widetilde{f}_2 \in P[(z_\gamma)_{\gamma\in\Gamma}]$ corresponds to the $\star$-product on $\mathcal{A}^m$. Let $f \in \mathcal{A}^m_+ = \{f \in \mathcal{A}^m \mid f(0) = 0\}$ and $\widetilde{f} \in P[z]$ the corresponding formal power series. The *exponential function* $\exp : \mathcal{A}^m_+ \to \mathbb{1} + \mathcal{A}^m_+$ is defined by

$$\exp f = \mathbb{1} + \sum_{n\geq 1} \frac{1}{n!} (\underbrace{f \star \cdots \star f}_{n \text{ factors}}) \tag{A.12}$$

or equivalently

$$\exp f(X) = \sum_{X=\sum Y} \prod_Y f(Y). \tag{A.13}$$

From the algebra-isomorphism of $P[z]$ and $\mathcal{A}^m$ follows

$$(\widetilde{\exp f})(z) = \exp(\widetilde{f}(z)). \tag{A.14}$$

The *logarithm* $\ln : \mathbb{1} + \mathcal{A}^m_+ \to \mathcal{A}^m_+$ is defined by

$$\ln(\mathbb{1} + f) = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} (\underbrace{f \star \cdots \star f}_{n \text{ factors}}) \tag{A.15}$$

or equivalently

$$\ln(\mathbb{1} + f)(X) = \sum_{n\geq 1} \sum_{X=\sum_{i=1}^n Y_i} (-1)^{n+1}(n-1)! \prod_{i=1}^n f(Y_i) \tag{A.16}$$

for all multiindices X with $X \neq 0$. We have

$$[\ln(\widetilde{\mathbb{1} + f})](z) = \ln(1 + \widetilde{f}(z)). \tag{A.17}$$

By Eqs. (A.14) and (A.17) we obtain the following *inversion formulas*

$$\exp \ln(\mathbb{1} + f) = \mathbb{1} + f \tag{A.18}$$

$$\ln \exp f = f \tag{A.19}$$

for all $f \in \mathcal{A}^m_+$. Especially, for $A \in \mathcal{A}_+ \stackrel{\text{def}}{=} \{f \in \mathcal{A} \mid f(0) = 0\}$ and $Z \in \mathbb{1} + \mathcal{A}_+$ follows that

$$Z(X) = (\exp A)(X) = \sum_{X=\sum Y} \prod_Y A(Y) \tag{A.20}$$

is equivalent to

$$A(X) = (\ln Z)(X) = \sum_{n\geq 1} \sum_{X=\sum_{i=1}^n Y_i} (-1)^{n+1}(n-1)! \prod_{i=1}^n Z(Y_i). \tag{A.21}$$

The multiindices may be interpreted here as sets and the sums as disjoint unions.

A.2. Truncated expectation value

Let $\Lambda \subset (a\mathbb{Z})^\nu$ be a finite sublattice and $d\mu_v$ the Gaussian measure with covariance v. Let $\mathcal{A}(\Lambda) = \{f : \wp(\Lambda) \to \mathbb{C}\}$ be the $\star$-algebra for the power set $\wp(\Lambda) = \{X \mid X \subseteq \Lambda\}$. Let $B(X|\phi)$ be a random variable for all

$X \subseteq \Lambda$ with $B(\emptyset|\phi) = 1$ and $B(X|\phi)$ depends only from ϕ_x, $x \in X$. For another random variable $Z(X|\phi)$ the *expectation value* is defined by

$$\mathcal{E}[Z(X|\phi)] = \int d\mu_v(\phi) Z(X|\phi). \tag{A.22}$$

We will use the abbreviation $\mathcal{E}(Z(X|\phi)] = \mathcal{E}(X)$. $\mathcal{E}$ is an element of the $\star$-algebra $\mathcal{A}(\Lambda)$. The *truncated expectation value* is defined by

$$\mathcal{E}^T(X) = (\ln \mathcal{E})(X) \tag{A.23}$$

for all $X \subseteq \Lambda$. Equivalently, the truncated expectation value is completely determined by

$$\mathcal{E}(X) = \sum_{X=\sum Y} \prod_Y \mathcal{E}^T(Y). \tag{A.24}$$

This follows from the inversion formulas (A.20), (A.21). Furthermore we have

$$\mathcal{E}^T(X) = \sum_{n\geq 1} (-1)^{n+1}(n-1)! \sum_{X=\sum_{i=1}^n Y_i} \prod_{i=1}^n \mathcal{E}(Y_i). \tag{A.25}$$

For $Z(X|\phi) = \sum_{X=\sum Y} \prod_Y B(Y|\phi)$ we use the notation

$$\mathcal{E}^T(X) = \sum_{X=\sum Y} \langle \prod_Y [B(Y|\phi);\ ]\rangle. \tag{A.26}$$

By the isomorphism of section A.1. (see Eqs. (A.9), (A.10)) follows

$$\mathcal{E}^T(X) = \partial_X \ln\langle \exp(\sum_{Y\subseteq X} B(Y) z^Y)\rangle|_{z=0}. \tag{A.27}$$

Especially for

$$B(X) = \begin{cases} F_x(\phi_x) & \text{if } |X| = 1, X = \{x\} \\ 0 & \text{if } |X| \geq 2 \end{cases} \tag{A.28}$$

follows

$$\langle \prod_{x\in X} [F_x(\phi_x);\ ]\rangle = \frac{\partial^n}{\partial z^n} \ln\langle \exp(z \sum_{x\in X} F_x(\phi_x))\rangle|_{z=0} \tag{A.29}$$

Let the partition function be defined by

$$Z(\Lambda|\psi) = \langle \prod_{x\in\Lambda} e^{-\lambda \mathcal{V}(x)}\rangle. \tag{A.30}$$

Because of (A.29), the perturbation expansion for $\ln Z(\Lambda|\psi)$ is

$$\ln Z(\Lambda|\psi) = \sum_{n\geq 1} \frac{(-\lambda)^n}{n!} \frac{\partial^n}{\partial s^n} \ln\langle \prod_{x\in\Lambda} e^{s\mathcal{V}(x)}\rangle|_{s=0} = \sum_{n\geq 1} \frac{(-\lambda)^n}{n!} \langle \underbrace{\sum_{x\in\Lambda} \mathcal{V}(x); \ldots; \sum_{x\in\Lambda} \mathcal{V}(x)}_{n \text{ arguments}}\rangle. \tag{A.31}$$

This proves the relation (2.75).

A.3. Combinatorial coefficient $a(Q)$

Theorem A.3.1. *Let Γ be a finite set and*

$$g : \Gamma \times \Gamma \to \{0, 1\} \tag{A.32}$$

with $g(\gamma,\gamma) = -1$ for all $\gamma \in \Gamma$. We associate to a multiindex $X : \Gamma \to \mathbb{N}$ a collection of vertices $\gamma_1, \ldots, \gamma_{|X|}$ and links $(\gamma\gamma')$, if $g(\gamma,\gamma') = -1$. This graph is labeled by $G(X)$. We define a function Φ by

$$\Phi(X) = \prod_{\substack{i<j \\ X(\gamma_i), X(\gamma_j) > 1}} (1 + g(\gamma_i, \gamma_j)). \tag{A.33}$$

Clearly, $\Phi(X) = 0$ holds for $X! > 1$. Let us define

$$Z = \sum_X \Phi(X) z^X. \tag{A.34}$$

Then we have

$$\ln Z = \sum_X \Phi^T(X) z^X \tag{A.35}$$

with

$$\Phi^T(X) = \frac{1}{X!} \sum_{C \subseteq G(X)} (-1)^{l(C)}. \tag{A.36}$$

The sum is over all connected subgraphs C of $G(X)$ with the same set of vertices as for $G(X)$ and $l(C)$ equals the number of links in C.

PROOF [18]: Let $\gamma_1, \ldots, \gamma_n$ be the vertices of $G(X)$ with $n = |X|$. Expansion of the product on the rhs of (A.33) yields

$$\Phi(X) = \sum_{N = \sum I} \prod_I g(I) \tag{A.37}$$

with $N = \{1, \ldots, n\}$ and

$$g(I) = \begin{cases} \sum_{G \in \mathcal{G}_I} \prod_{(ij) \in G} g(\gamma_i, \gamma_j) & \text{if } |I| \geq 2 \\ 1 & \text{otherwise.} \end{cases} \tag{A.38}$$

We insert (A.37) on the rhs of (A.34). This gives

$$Z = \sum_X \sum_{N = \sum I} \prod_I [g(I) \prod_{j \in I} z_{\gamma_j}] \tag{A.39}$$

After resummation we obtain an expansion in the number of vertices

$$Z = \sum_{n \geq 0} \frac{1}{n!} \sum_{\gamma_1 \in \Gamma} \cdots \sum_{\gamma_n \in \Gamma} \left\{ \sum_{\{1,\ldots,n\} = \sum I} \prod_I g(I) \right\} \prod_{i=1}^n z_{\gamma_i}. \tag{A.40}$$

For every partition $\{n_i\}$ with $n = \sum_{i=1}^k n_i$ exists $\frac{n!}{\prod_{i=1}^k n_i! \prod_{r=1}^n p_r!}$ partitions $X = \sum_{i=1}^k Y_i$ with $|X| = n$, $|Y_i| = n_i$ and $p_r = |\{n_l | n_l = r\}|$. Therefore we obtain

$$Z = \sum_{n \geq 0} \frac{1}{n!} \sum_{k \geq 0} \sum_{\substack{\{n_i\} \\ \sum_{i=1}^k = n}} \frac{n!}{\prod_{i=1}^k n_i! \prod_{l=1}^n p_l!} \prod_{i=1}^k (\sum_{\gamma_1, \ldots, \gamma_{n_i} \in \Gamma} g(\gamma_1, \ldots, \gamma_{n_i}) \prod_{j=1}^{n_i} z_{\gamma_j}) \tag{A.41}$$

with the abbreviation

$$g(\gamma_1, \ldots, \gamma_n) = \begin{cases} \sum_{G \in \mathcal{G}_I} \prod_{(ij) \in G} g(\gamma_i, \gamma_j) & \text{if } |I| \geq 2 \\ 1 & \text{otherwise.} \end{cases} \tag{A.42}$$

By the multinominial theorem follows

$$Z = \sum_{k \geq 0} \frac{1}{k!} [\sum_{n \geq 1} \frac{1}{n!} \sum_{\gamma_1, \ldots, \gamma_n \in \Gamma} g(\gamma_1, \ldots, \gamma_n) \prod_{j=1}^{n_i} z_{\gamma_j}]^k. \tag{A.43}$$

Hence

$$\ln Z = \sum_{n \geq 1} \frac{1}{n!} \sum_{\gamma_1, \ldots, \gamma_n \in \Gamma} g(\gamma_1, \ldots, \gamma_n) \prod_{j=1}^{n_i} z_{\gamma_j}. \tag{A.44}$$

Comparison of the coefficients with $\ln Z = \sum_X \Phi^T(X) z^X$ yields

$$\Phi^T(X) = \frac{1}{X!} g(\gamma_1, \ldots, \gamma_{|x|}). \tag{A.45}$$

This proves the assertion. √

By Theorem A.3.1. follows the representation (2.19) of $\ln Z(\Lambda|\psi)$:

Corollary A.3.2. *If*

$$Z(\Lambda|\psi) = \sum_{\Lambda=\sum X} \prod_X A(X|\psi), \tag{A.46}$$

then

$$\ln Z(\Lambda|\psi) = \sum_{x\in\Lambda} \ln A(\{x\}|\psi) + \sum_Q a(Q)\bar{A}(Q|\psi) \tag{A.47}$$

with the notations of section 2.2. .

PROOF: We will use Theorem A.3.1. with $\Gamma = \wp(\Lambda)$(=power set of Λ). By the definition

$$z^X = \bar{A}(X) - \delta_{1,|X|} = \begin{cases} \bar{A}(X) & \text{if } |X| \geq 2 \\ 0 & \text{otherwise} \end{cases} \tag{A.48}$$

follows

$$\sum_{\bar{X}} \Phi(\bar{X}) z^{\bar{X}} = \sum_{\Lambda=\sum X} \prod_X \bar{A}(X), \tag{A.49}$$

if

$$g(X,Y) = \begin{cases} -1 & \text{if } X \cap Y \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

is chosen. $\bar{X}$ denotes the multiindex

$$\bar{X}(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \notin X. \end{cases}$$

By Theorem A.3.1. follows

$$\ln Z(\Lambda|\psi) = \sum_{x\in\Lambda} \ln A(\{x\}|\psi) + \sum_{\bar{X}} \Phi^T(\bar{X}) z^{\bar{X}} = \sum_{x\in\Lambda} \ln A(\{x\}|\psi) + \sum_Q a(Q)\bar{A}(Q|\psi) \tag{A.50}$$

where the sum is over all clusters $Q = (P_1^{n_1}, \ldots, P_k^{n_k})$ with $|P_i| \geq 2$. ✓

With the help of the abstract tree graph formula (Corollary B.4.2.) we obtain an estimate for the combinatorial coefficient $a(Q)$:

Corollary A.3.3. *We have*

$$|a(Q)| \leq \frac{(n-1)!\, e^{n-1}}{\prod_{i=1}^k n_i!} (2\pi)^{n-1} \tag{A.51}$$

for the cluster $Q = (P_1^{n_1}, \ldots, P_k^{n_k})$, $n = \sum_{i=1}^k n_i$.

PROOF: Let G be a Mayer graph with n vertices $1, \ldots, n$. We define

$$w_{ij} = \begin{cases} i\phi_{ij} & \text{if } (ij) \in G \\ 0 & \text{otherwise.} \end{cases} \tag{A.52}$$

Because of

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi_{ij} (e^{w_{ij}} - 1) = \begin{cases} -1 & \text{if } (ij) \in G \\ 0 & \text{otherwise} \end{cases} \tag{A.53}$$

we have

$$\sum_{C\subseteq G} (-1)^{l(C)} = \prod_{\substack{i,j \\ 1\leq i<j\leq n}} [\frac{1}{2\pi} \int d\phi_{ij}] \sum_{M\in\mathcal{G}_n} \prod_{(ij)\in M} [e^{w_{ij}} - 1] \tag{A.54}$$

where $\mathcal{G}_n$ denotes the set of all Mayer graphs with n vertices. By the abstract tree graph formula (Corollary B.4.2.) follows

$$\sum_{C\subseteq G} (-1)^{l(C)} = \sum_\pi \prod_{\substack{i,j \\ 1\leq i<j\leq n}} [\frac{1}{2\pi} \int d\phi_{ij}] \Big\{ \sum_\eta \int_0^1 d\sigma_{n-1} f(\eta|s) \prod_{k=2}^n w_{\pi(k)\eta(\pi(k))} e^{\sum_{1\leq i<j\leq n} w_{\pi(i)\pi(j)}[s]} \Big\}. \tag{A.55}$$

The sum is over all permutations

$$\pi \ : \ \begin{cases} \{1,\ldots,n\} \to \{1,\ldots,n\} \\ i \mapsto \pi(i) \end{cases} \tag{A.56}$$

with $\pi(1)=1$. Since $|w_{ij}| \leq 2\pi$ and $|e^{w_{ij}}| = 1$ we obtain

$$|\sum_{C \subseteq G} (-1)^{l(C)}| \leq (n-1)!\ (2\pi)^{n-1} \sum_{\eta} \int_0^1 d\sigma_{n-1} f(\eta|s). \tag{A.57}$$

The special case

$$\sum_{\eta} \int_0^1 d\sigma_{n-1} f(\eta|s) \leq e^{n-1} \tag{A.58}$$

of the tree estimate (Lemma 3.1.4.) proves the assertion. $\surd$

APPENDIX B. DECOUPLING EXPANSION FOR NONLOCAL INTERACTIONS, TREE GRAPH FORMULA AND ESTIMATES

B.1. Interpolating interaction

For every finite point sets X we define the *multiparticle interaction* by

$$E(X) = \sum_{l} \sum_{x_1,\ldots,x_l \in X} \mathcal{E}^l(x_1,\ldots,x_l). \tag{B.1}$$

$\mathcal{E}^l$ is symmetrically in the arguments $x_1,\ldots,x_l$. The points $x_1,\ldots,x_l$ are not necessarily distinct. The point set occupied by $x_1,\ldots,x_l$ is denoted by $p(x_1,\ldots,x_l)$. A disjoint partition of Y in n subsets Y_i is labeled by $\vec{Y}_n$, i.e.

$$Y = \sum_{i=1}^{n} Y_i \qquad \text{for} \ \ \vec{Y}_n = (Y_1,\ldots,Y_n). \tag{B.2}$$

We will use the notation

$$Y^{(k)} = \sum_{i=1}^{k} Y_i \tag{B.3}$$

for all $k \in \{1,\ldots,n\}$. For $s \in [0,1]$ and a subset $Y \subseteq X$ the l-particle interaction is modified by

$$\mathcal{E}^l(x_1,\ldots,x_l|s\chi_Y) = \prod_{j=1}^{l} s^{\chi_Y(x_j)} \mathcal{E}^l(x_1,\ldots,x_l) \tag{B.4}$$

with the characteristic function χ_Y. Corresponding to (B.4) we define

$$E(X|s\chi_Y) = \sum_{l} \sum_{x_1,\ldots,x_l \in X} \mathcal{E}^l(x_1,\ldots,x_l|s\chi_Y). \tag{B.5}$$

The multiparticle interaction defined by (B.5) is called an ***interpolating interaction***. In the following we define an interpolating interaction.

Definition (of an interpolating interaction)

Let $\tilde{Y}_n$ be a disjoint partition of $Y^{(n)}$. For $s_1, \ldots, s_n \in [0,1]$ the *interpolating interaction* $E(Y^{(n)}|s_1, \ldots, s_{n-1})$ is defined recursively by

$$E_0(Y^{(n)}) = E(Y^{(n)}) \tag{B.6a}$$

$$E_i(Y^{(n)}) = E_{i-1}(Y^{(n)}|s_i\chi_{Y^{(i)}}) + E_{i-1}(Y^{(i)}) - E_{i-1}(Y^{(i)}|s_i\chi_{Y^{(i)}}) \tag{B.6b}$$

for all $i \in \{1, \ldots, n\}$ and

$$E(Y^{(n)}|s_1, \ldots, s_{n-1}) = E_n(Y^{(n)}). \tag{B.6c}$$

We will use also the following notation

$$E_{\tilde{Y}_n}(X|s_1, \ldots, s_{n-1}) = E(Y^{(n)}|s_1, \ldots, s_{n-1}) \tag{B.7}$$

for $X = Y^{(n)}$. The following Lemma shows that $E(Y^{(n)}|s_1, \ldots, s_n)$ is independent from s_n :

Lemma B.1.1.. *An explicit expression for the interpolating interaction is*

$$E_{\tilde{Y}_n}(X|s_1, \ldots, s_{n-1}) = \sum_l \sum_{x_1, \ldots, x_l \in X} \prod_{i=1}^{n-1} s_i^{n_i^l} \mathcal{E}^l(x_1, \ldots, x_l) \tag{B.8}$$

with

$$n_i^l = N_i^l(1 - \delta_{l,N_i^l}), \qquad N_i^l = |\{j \in \{1, \ldots, l\}|\ x_j \in Y^{(i)}\}|. \tag{B.9}$$

With the notation

$$\mathcal{E}^l_{\tilde{Y}_n}(x_1, \ldots, x_l|s_1, \ldots, s_{n-1}) = \prod_{i=1}^{n-1} s_i^{n_i^l} \mathcal{E}^l(x_1, \ldots, x_l) \tag{B.10}$$

the following conditions are fulfilled

(i) **Decoupling**

Suppose that for $x_1, \ldots, x_l \in X$, it exists $j_1, j_2 \in \{1, \ldots, n-1\}$, $j_1 < j_2$, with

$$p(x_1, \ldots, x_l) \cap Y_{j_r} \neq \emptyset, \ r \in \{1, 2\}. \tag{B.11}$$

Then

$$\mathcal{E}^l_{\tilde{Y}_n}(x_1, \ldots, x_l|s_1, \ldots, s_i = 0, \ldots, s_{n-1}) = 0 \tag{B.12}$$

for all $i \in \{j_1, \ldots, j_2 - 1\}$.

(ii) **Reduction**

For

$$Y_n \wedge X \overset{\text{def}}{=} \{Y_1, \ldots, Y_n, X - Y^{(n)}\} \tag{B.13}$$

we have

$$E_{\tilde{Y}_n \wedge X}(X|s_1, \ldots, s_n = 1) = E_{\tilde{Y}_{n-1} \wedge X}(X|s_1, \ldots, s_{n-1}). \tag{B.14}$$

(iii) **Locality**

Suppose that for $x_1, \ldots, x_l \in X$

$$p(x_1, \ldots, x_l) \cap Y_j = \emptyset \ \text{ for all } j > i \text{ or } \ p(x_1, \ldots, x_l) \cap Y_j = \emptyset \ \text{ for all } j \leq i. \tag{B.15}$$

Then

$$\partial_{s_i} \mathcal{E}_{\tilde{Y}_n}(x_1, \ldots, x_l|s_1, \ldots, s_{n-1}) = 0 \tag{B.16}$$

(iv) **Positivity**

$$\text{If } E(X) \geq 0 \text{ then } E_{\tilde{Y}_n}(X|s_1, \ldots, s_{n-1}) \geq 0.. \tag{B.17}$$

PROOF: Without loss of generality we prove the assertion for

$$E(X) = \mathcal{E}^l(x_1, \ldots, x_l) \tag{B.18}$$

where $X \supseteq p(x_1, \ldots, x_l)$. By the recursive definition of the interpolating interaction we have to show

$$E_i(X) = s_i^{n_i^l} E_{i-1}(X) \tag{B.19}$$

with $n_i^l = N_i^l(1 - \delta_{l,N_i^l})$, $N_i^l = |\{j \in \{1, \ldots, l\} | x_j \in Y^{(i)}\}|$. This follows by consideration of two different cases :

1. $p(x_1, \ldots, x_l) \cap (X - Y^{(i)}) \neq \emptyset$:

From Eq. (B.6) follows

$$E_i(X) = E_{i-1}(X | s_i \chi_{Y^{(i)}}). \tag{B.20}$$

With the notation (B.9) we have

$$E_i(X) = \prod_{j=1}^{l} s_i^{\chi_{Y^{(i)}}(x_j)} E_{i-1}(X) = s_i^{N_i^l} E_{i-1}(X). \tag{B.21}$$

This proves (B.19) because of $l = N_i^l$.

2. $x_1, \ldots, x_l \in Y^{(i)}$:

From Eq. (B.6) follows

$$E_i(X) = E_{i-1}(Y^{(i)}) = E_{i-1}(X). \tag{B.22}$$

Because of $l = N_i^l$ we have (B.19). This proves Eq. (B.8). We will now show the conditions (i)-(iv) :

(i) : By assumption (B.11) follows $N_i^l \neq l, 0$ if $i \in \{j_1, \ldots, j_2 - 1\}$. From (B.8) follows

$$\mathcal{E}_{\tilde{Y}_n}^l(x_1, \ldots, x_l | s_1, \ldots, s_{n-1}) = \prod_{i=1}^{n-1} s_i^{N_i^l(1-\delta_{l,N_i^l})} \mathcal{E}^l(x_1, \ldots, x_l). \tag{B.23}$$

This proves Eq. (B.12).

(ii) : If $N_n^l = l$, then $n_n^l = 0$. This gives (B.14).

(iii) : By assumption (B.15) follows $N_i^l = 0$ or $= l$. Therefore $n_i^l = 0$. By Eq. (B.10) the expression $\mathcal{E}_{\tilde{Y}_n}(x_1, \ldots, x_l | s_1, \ldots, s_{n-1})$ is independent from s_i. This proves Eq. (B.16).

(iv) : This follows immediately from the recursive definition of $E_{\tilde{Y}_n}(X | s_1, \ldots, s_{n-1})$ and the fact that convex combinations maintain inequalities. $\surd$

The interpolating interaction may be represented graphically. Every point set Y_i is represented by a horizontal line. Every point $x_j \in Y_i$ of an interaction term $\mathcal{E}^l(x_1, \ldots, x_l)$ is represented by a point on the i-th horizontal line. Because of the symmetry of $\mathcal{E}^l$, the labelling of the points on the lines is unessential. Let us denote the maximal index of lines which have points by max. We associate to every point on the line i the term $s_i s_{i+1} \ldots s_{max-1}$. The product of this terms gives $\prod_{i=1}^{n-1} s_i^{n_i^l}$. This is the s-factor of the interpolating interaction (see Eq. (B.10)). Fig. B.1 shows an example of this construction for a 4-particle interaction.

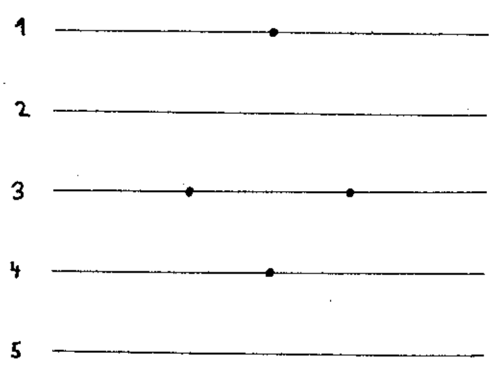


Fig. B.1 Example for the graphical representation of a 4-particle interaction with the s-dependent coefficient $s_1 s_2 s_3^3$.

Let us consider the 2-particle interaction

$$E(X) = \sum_{x,y \in X} \mathcal{E}^2(x,y) \tag{B.24}$$

with $X = \{x_1, \ldots, x_n\}$ and the partition

$$\bar{Y}_n = (Y_1, \ldots, Y_n) \tag{B.25}$$

with $Y_i = \{x_i\}$. Then the interpolating interaction is

$$E_{\bar{Y}_n}(X|s_1, \ldots, s_{n-1}) = 2 \sum_{1 \le i < j \le n} s_i s_{i+1} \ldots s_{j-1} \mathcal{E}^2(x_i, x_j) + \sum_{i=1}^{n} \mathcal{E}^2(x_i, x_i). \tag{B.26}$$

This is the modified propagator $v[s]$ for $\mathcal{E}^2(x,y) = \frac{1}{2} v_{xy}$ (cf. (2.110)). For an l-particle interaction

$$E(\phi) = \sum_{x_1, \ldots, x_l \in X} \rho(x_1, \ldots, x_l) \phi_{x_1} \ldots \phi_{x_l} \tag{B.27}$$

the interpolating interaction may be recursively defined by

$$E_{\bar{Y}_0}(\phi) = E(\phi) \tag{B.28a}$$

$$E_{\bar{Y}_i}(\phi|s_1, \ldots, s_i) = E_{\bar{Y}_{i-1}}(\phi_{X-Y^{(i)}} + s_i \phi_{Y^{(i)}}|s_1, \ldots, s_{i-1}) + (1 - s_i^l) E_{\bar{Y}_{i-1}}(\phi_{Y^{(i)}}|s_1, \ldots, s_{i-1}), \; i \in \{1, \ldots, n\} \tag{B.28b}$$

with the notation

$$(\phi_Y)_x = \chi_Y(x) \phi_x. \tag{B.29}$$

This corresponds to (B.6). The conditions (i)-(iv) of Lemma B.1.1. are also fulfilled.

B.2. Representation of the molecular activities with the help of the interpolating interaction

Lemma B.2.1. *Let the multiparticle interaction be defined by*

$$E(X) = \sum_{l} \sum_{x_1, \ldots, x_l \in X} \mathcal{E}^l(x_1, \ldots, x_l) \tag{B.30}$$

for every finite point set X. Consider the polymer representation

$$e^{E(X)} = \sum_{X = \sum Y} \prod_{Y} B(Y). \tag{B.31}$$

The molecular activities $B(Y)$ are uniquely defined by (B.31)(cf. app. A). We have

$$B(Y) = \sum_{j \le |Y|} \sum_{\substack{\bar{Y}_j \\ Y^{(j)} = Y}} \mathcal{A}(\bar{Y}_j) \tag{B.32}$$

with

$$\mathcal{A}(Y_j) = \int_0^1 ds_1 \ldots ds_{j-1} \prod_{i=1}^{j-1} [\partial_{s_i} E^{(i)}_{\bar{Y}_{i+1}}(Y^{(i+1)}|s_1, \ldots, s_i)] e^{E_{\bar{Y}_j}(Y^{(j)}|s_1, \ldots, s_{j-1})} \tag{B.33}$$

$$E^{(i)}_{\bar{Y}_{i+1}}(Y^{(i+1)}|s_1,\dots,s_i) = \sum_l \sum_{\substack{x_1,\dots,x_l \in Y^{(i+1)} \\ p(x_1,\dots,x_l)-Y^{(i)}=Y_{i+1}}} \mathcal{E}^l(x_1,\dots,x_l|s_1,\dots,s_i) \tag{B.34}$$

where the sum $\sum_{\substack{Y_j \\ Y^{(j)}=Y}}$ *is over all disjoint partitions* $Y = \sum_{i=1}^j Y_i$ *with* $Y_1 = \{x\}$*, for a fixed chosen* $x \in Y$*. Permutations of* $Y_2,\dots,Y_j$ *are considered as different partitions. Suppose that the conditions (i) (decoupling) and (ii) (reduction) of Lemma B.1.1. are fulfilled for the interpolating interaction* $\mathcal{E}^l(x_1,\dots,x_l|s_1,\dots,s_i)$.

PROOF: We will consider the following Kirkwood Salaburg equations with remainder

$$e^{E(X)} = \sum_{\substack{W \\ x\in W\subseteq X}} \{K_n(W)e^{E(X-W)} + R_n(X,W)\} \tag{B.35}$$

for all $n \in \mathbb{N}^* = \{1,2,3,\dots\}$ with

$$K_n(W) = \sum_{j\le n} \sum_{\substack{Y_j \\ Y^{(j)}=W}} A(\bar{Y}_j) \tag{B.36}$$

$$R_n(X,W) = \sum_{\substack{Y_n \\ Y^{(n)}=W}} \int_0^1 ds_1\dots ds_n\, \partial_{s_n}[\{\prod_{i=1}^{n-1} \partial_{s_i} E^{(i)}_{\bar{Y}_{i+1}}(Y^{(i+1)}|s_1,\dots,s_i)\} e^{E_{\bar{Y}_n\wedge W}(X|s_1,\dots,s_n)}]. \tag{B.37}$$

Then $R_n(X,W) = 0$ for $n > |X|$. By the unique solution of the Kirkwood Salsburg equations follows the assertion. Eq. (B.35) is proved by induction. Let $n = 1$. By reduction (Lemma B.1.1., (ii)) follows

$$E_{\bar{Y}_1\wedge X}(X|s_1 = 1) = E(X). \tag{B.38}$$

The mean value theorem yields

$$e^{E(X)} = e^{E_{\bar{Y}_1\wedge X}(X|s_1=0)} + \int_0^1 ds_1 \partial_{s_1} e^{E_{\bar{Y}_1\wedge X}(X|s_1)}. \tag{B.39}$$

By decoupling (Lemma B.1.1., (i)) follows

$$E_{\bar{Y}_1\wedge X}(X|s_1 = 0) = E(Y_1) + E(X - Y_1). \tag{B.40}$$

This proves the assertion for $n = 1$. Let the assertion be valid for n. We apply the derivative in s_n on the rhs of (B.37). Since

$$\partial_{s_n} E_{\bar{Y}_n\wedge X}(X|s_1,\dots,s_n) = \sum_l \sum_{x_1,\dots,x_l\in X} \partial_{s_n} \mathcal{E}^l_{\bar{Y}_n\wedge X}(x_1,\dots,x_l|s_1,\dots,s_n) =$$
$$= \sum_{Y_{n+1}} \sum_l \sum_{\substack{x_1,\dots,x_l\in Y^{(n+1)} \\ p(x_1,\dots,x_l)-Y^{(n)}=Y_{n+1}}} \partial_{s_n} \mathcal{E}^l_{\bar{Y}_n\wedge X}(x_1,\dots,x_l|s_1,\dots,s_n) = \sum_{Y_{n+1}} \partial_{s_n} E^{(n)}_{\bar{Y}_{n+1}}(Y^{(n+1)}|s_1,\dots,s_n) \tag{B.41}$$

it follows

$$R_n(X,W) = \sum_{\substack{Y_{n+1} \\ Y^{(n)}=W}} \int_0^1 ds_1\dots ds_n \prod_{i=1}^n [\partial_{s_i} E^{(i)}_{\bar{Y}_{i+1}}(Y^{(i+1)}|s_1,\dots,s_i)] e^{E_{\bar{Y}_n\wedge X}(X|s_1,\dots s_n)}. \tag{B.42}$$

By reduction (Lemma B.1.1., (ii)) follows

$$E_{\bar{Y}_n}(X|s_1,\dots,s_n) = E_{\bar{Y}_{n+1}\wedge X}(X|s_1,\dots,s_n,s_{n+1} = 1). \tag{B.43}$$

We use the mean value theorem. This gives

$$R_n(X,W) = \sum_{\substack{Y_{n+1} \\ Y^{(n)}=W}} \int_0^1 ds_1\dots ds_n \prod_{i=1}^n [\dots] e^{E_{\bar{Y}_{n+1}\wedge X}(X|s_1,\dots,s_{n+1}=0)} +$$
$$+ \sum_{\substack{Y_{n+1} \\ Y^{(n)}=W}} \int_0^1 ds_1\dots ds_n ds_{n+1}\, \partial_{s_{n+1}} \prod_{i=1}^n [\dots] e^{E_{\bar{Y}_{n+1}\wedge X}(X|s_1,\dots,s_{n+1})} \tag{B.44}$$

By decoupling (Lemma B.1.1., (i))

$$E_{\bar{Y}_{n+1}\wedge X}(X|s_1,\ldots,s_{n+1}=0) = E_{\bar{Y}_{n+1}}(Y^{(n+1)}|s_1,\ldots,s_n) + E(X-Y^{(n+1)}) \tag{B.45}$$

follows

$$\sum_{x\in W\subseteq X}^{W} R_n(X,W) = \sum_{x\in W\subseteq X}^{W}\{\sum_{Y^{(n+1)}=W}^{\bar{Y}_{n+1}} \int_0^1 ds_1\ldots ds_n \prod_{i=1}^{n}[\ldots]e^{E_{\bar{Y}_{n+1}}(Y^{(n+1)}|s_1,\ldots,s_n)}e^{E(X-W)} + R_{n+1}(X,W)\} =$$

$$\sum_{x\in W\subseteq X}^{W}\{\sum_{Y^{(n+1)}=W}^{\bar{Y}_{n+1}} A(\bar{Y}_{n+1})e^{E(X-W)} + R_{n+1}(X,W)\}. \tag{B.46}$$

The assertion (B.35) for $n+1$ follows by induction hypothesis. $\surd$

B.3. Representation of the activities with the help of the interpolating interaction

By Lemma B.2.1. for the representation of the molecular activities follows a representation of the activities for multiparticle interactions :

Theorem B.3.1. *Let the partition function $Z(X)$ for finite point sets be defined by*

$$Z(X) = \int d\mu_v(\phi)e^{E(X)} \tag{B.47}$$

with the multiparticle interaction

$$E(X) = \sum_l \sum_{x_1,\ldots,x_n\in X} \mathcal{E}^l(x_1,\ldots,x_l). \tag{B.48}$$

The activities $A(Y)$, $Y\subseteq X$, are defined uniquely by

$$Z(X) = \sum_{X=\sum Y}\prod_Y A(Y). \tag{B.49}$$

The modified propagator $v[s]$ for parameters $s_1,\ldots,s_{n-1}\in[0,1]$ and partitions $\bar{Y}_n$ of Y is defined by

$$v[s]_{xy} = \sum_{1\le i<j\le n} s_i s_{i+1}\ldots s_{j-1}[\chi_{Y_i}(x)v_{xy}\chi_{Y_j}(y) + \chi_{Y_j}(x)v_{xy}\chi_{Y_i}(y)] + \sum_{i=1}^{n}\chi_{Y_i}(x)v_{xy}\chi_{Y_i}(y). \tag{B.50}$$

Then

$$A(Y) = \sum_{j\le|Y|}\sum_{Y^{(j)}=Y}^{\bar{Y}_j}\int_0^1 ds_1\ldots ds_{j-1}\int d\mu_{v[s]}(\phi)\left\{\prod_{i=1}^{j-1}[\partial_{s_i}E^{(i)}_{\bar{Y}_{i+1}}(Y^{(i+1)}|s_1,\ldots,s_i)]e^{E_{\bar{Y}_j}(Y^{(j)}|s_1,\ldots,s_{j-1})}\right\} \tag{B.51}$$

with

$$E^{(i)}_{\bar{Y}_{i+1}}(Y^{(i+1)}|s_1,\ldots,s_i) = \sum_{\substack{x_1\in Y^{(i)}\\ x_2\in Y_{i+1}}}\left[\frac{1}{2}\frac{\partial}{\partial\phi_{x_1}}v[s]_{x_1x_2}\frac{\partial}{\partial\phi_{x_2}} + \sum_l \sum_{\substack{x_1,\ldots x_l\in Y^{(i+1)}\\ p(x_1,\ldots,x_l)-Y^{(i)}=Y_{i+1}}}\mathcal{E}^l(x_1,\ldots,x_l|s_1,\ldots,s_i)\right] \tag{B.52}$$

where the sum $\sum_{Y(j)=Y}^{Y_j}$ *is over all disjoint partitions* $Y = \sum_{i=1}^{j} Y_i$ *with* $Y_1 = \{x\}$, *for any* $x \in Y$. *Permutations of* $Y_2, \ldots, Y_j$ *are considered as different. The interpolating interaction* $\mathcal{E}^l(x_1, \ldots, x_l|s_1, \ldots s_i)$ *is supposed to fulfill the conditions (i) and (ii) of Lemma B.1.1. . The differential operators in* $E^{(i)}_{\tilde{Y}_{i+1}}(Y^{(i+1)}|s_1, \ldots, s_i)$ *operates only on the e-function on the rhs of (B.51).*

PROOF: Consider the multiparticle interaction

$$\tilde{E}(X) = -i \sum_{x \in X} q_x \phi_x - \frac{1}{2} \sum_{x,y \in X} q_x v_{xy} q_y + \sum_l \sum_{x_1, \ldots, x_l \in X} \mathcal{E}^l(x_1, \ldots, x_l). \tag{B.53}$$

The corresponding interpolating interaction is

$$\tilde{E}_{\tilde{Y}_{i+1}}(Y^{(i+1)}|s_1, \ldots, s_i) = -i \sum_{x \in X} q_x \phi_x - \frac{1}{2} \sum_{x,y \in X} q_x v[s]_{xy} q_y + \sum_l \sum_{x_1, \ldots, x_l \in X} \mathcal{E}^l(x_1, \ldots, x_l|s_1, \ldots, s_i) \tag{B.54}$$

and fulfills the conditions of decoupling and reduction (see Lemma B.1.1.). By Lemma B.2.1. we have

$$e^{-i(q,\phi)_X - \frac{1}{2}(q,vq)_X + \sum_l \sum_{x_1,\ldots,x_l \in X} \mathcal{E}^l(x_1,\ldots,x_l)} =$$

$$= \sum_{X=\sum Y} \prod_Y \Big\{ \sum_{j \le |Y|} \sum_{\substack{Y_j \\ Y(j)=Y}} \int_0^1 ds_1 \ldots ds_{j-1} \prod_{i=1}^{j-1} [\partial_{s_i} (\sum_{\substack{x_1 \in Y^{(i)} \\ \{x_2\} = Y_{i+1}}} -\frac{1}{2} q_{x_1} v[s]_{x_1 x_2} q_{x_2} +$$

$$+ \sum_l \sum_{\substack{x_1, \ldots, x_l \in Y^{(i+1)} \\ p(x_1, \ldots, x_l) - Y^{(i)} = Y_{i+1}}} \mathcal{E}^l(x_1, \ldots, x_l|s_1, \ldots, s_i))]$$

$$e^{-i(q,\phi)_{Y^{(j)}} - \frac{1}{2}(q,v[s]q)_{Y^{(j)}} + \sum_l \sum_{x_1,\ldots,x_l \in Y^{(j)}} \mathcal{E}^l(x_1,\ldots,x_l|s_1,\ldots,s_j)} \Big\}. \tag{B.55}$$

Integration of (B.55) by $\int \prod_{x \in X} \frac{d\phi_x dq_x}{2\pi}$ proves the assertion. √

B.4. EXPLICIT S-DEPENDENCE OF THE DECOUPLING EXPANSION AND PROOF OF THE TREE GRAPH FORMULA

In this section the interpolating interaction defined by Eq. (B.8) will be used. By Theorem B.3.1. we have the following representation of the activities $A(Y)$.

Corollary B.4.1. *Let the partition function be defined by*

$$Z(X) = \int d\mu_v(\phi) e^{E(X)} \tag{B.47}$$

with the interaction

$$E(X) = \sum_l \sum_{x_1, \ldots, x_l \in X} \mathcal{E}^l(x_1, \ldots, x_l) \tag{B.48}$$

for all finite point sets X. Suppose that the interpolating interaction is defined by

$$E_{\tilde{Y}_n}(Y^{(n)}|s_1, \ldots, s_{n-1}) = \sum_l \sum_{x_1, \ldots, x_l \in Y^{(n)}} \prod_{i=1}^{n-1} s_i^{n_i^l} \mathcal{E}^l(x_1, \ldots, x_l) \tag{B.56a}$$

with

$$n_i^l = N_i^l(1-\delta_{l,N_i^l}), \qquad N_i^l = |\{j \in \{1,\ldots,l\}|\ x_j \in Y^{(i)}\}| \tag{B.56b}$$

and the interpolating propagator is defined by

$$v[s]_{xy} = \sum_{1\le i<j\le n} s_i s_{i+1}\ldots s_{j-1}[\chi_{Y_i}(x)v_{xy}\chi_{Y_j}(y) + \chi_{Y_j}(x)v_{xy}\chi_{Y_i}(y)] + \sum_{i=1}^{n}\chi_{Y_i}(x)v_{xy}\chi_{Y_i}(y). \tag{B.50}$$

Then we have for the activities $A(Y)$, $Y \subseteq X$

$$A(Y) = \sum_{t\le|Y|}\sum_{\tilde{l}}\sum_{\substack{Y_t\\ Y^{(t)}=Y}}\sum_{\tilde{N}}\sum_{\tilde{\eta}}\int_0^1 ds_1\ldots ds_{t-1}\int d\mu_{v[s]}(\phi)\Big\{\prod_{a=2}^{t}\partial_{s_{a-1}}\prod_{i=1}^{N_a}[s_{a-1}s_{a-2}\ldots s_{\eta_i(a)}]$$
$$[\frac{l_a!}{\prod_{j=1}^{N_a}p_j^a!\,(l_a-N_a)!}\sum_{\substack{x_j\in Y_{\eta_j(a)}\\ p(x_{N_a+1},\ldots,x_{l_a})=Y_a}}\tilde{\mathcal{E}}^{l_a}(x_1,\ldots,x_l)]e^{E_{Y_t}(Y^{(t)}|s_1,\ldots,s_{t-1})}\Big\} \tag{B.57}$$

with

$$\tilde{\mathcal{E}}^l(x_1,\ldots,x_l) = \begin{cases}\frac{1}{2}\frac{\partial}{\partial\phi_{x_1}}v_{x_1x_2}\frac{\partial}{\partial\phi_{x_2}} + \mathcal{E}^2(x_1,x_2) & \text{if } l=2\\ \mathcal{E}^l(x_1,\ldots,x_l) & \text{otherwise.}\end{cases} \tag{B.58}$$

$\tilde{l} = (l_2,\ldots,l_t)$ *is a* $(t-1)$*-tupel of integers* ≥ 2. $\bar{Y}_t = (Y_1,\ldots,Y_t)$ *is a disjoint partition of* Y *with* $Y_1 = \{x\}$ *for some* $x \in Y$. *Permutations of* $Y_2,\ldots,Y_t$ *define different partitions* $\bar{Y}_t$. *The differential operator in* $\tilde{\mathcal{E}}^l$ *operates only on the e-function on the rhs of Eq. (B.57). We have the condition*

$$1 \le |Y_a| \le l_a. \tag{B.59}$$

$\tilde{N} = (N_2,\ldots,N_t)$ *is a* $(t-1)$*-tupel of positive integers with*

$$1 \le N_a \le l_a - |Y_a|. \tag{B.60}$$

$\tilde{\eta}$ *are the following* $\tilde{l}$ *and* $\tilde{N}$ *dependent functions*

$$\tilde{\eta}: \begin{cases}\{2,\ldots,t\} \to \sum_{k=1}^{\max(N_a)}\{1,\ldots,t-1\}^k\\ a \mapsto (\eta_i(a))_{i=1,\ldots,N_a}\end{cases} \tag{B.61}$$

with $\eta_i(a) < a$ *and the notation*

$$\{1,\ldots,t-1\}^k = \underbrace{\{1,\ldots,t-1\}\times\cdots\times\{1,\ldots,t-1\}}_{k \text{ times}}.$$

p_j^a *is defined by*

$$p_j^a = |\{\eta_i(a)|\ \eta_i(a) = j\}|. \tag{B.62}$$

PROOF: By Theorem B.3.1. follows

$$A(Y) = \sum_{t\le|Y|}\sum_{\tilde{l}}\sum_{\substack{Y_t\\ Y^{(t)}=Y}}\int_0^1 ds_1\ldots ds_{t-1}\int d\mu_{v[s]}(\phi)\Big\{\prod_{a=2}^{t}[\partial_{s_{a-1}}$$
$$\sum_{\substack{x_1,\ldots,x_{l_a}\in Y^{(a)}\\ p(x_1,\ldots,x_{l_a})-Y^{(a-1)}=Y_a}}\tilde{\mathcal{E}}^{l_a}(x_1,\ldots,x_{l_a}|s_1,\ldots,s_{a-1})]e^{E_{Y_t}(Y^{(t)}|s_1,\ldots,s_{t-1})}\Big\}. \tag{B.63}$$

Furthermore

$$\sum_{\substack{x_1,\ldots,x_{l_a}\in Y^{(a)}\\ p(x_1,\ldots,x_{l_a})-Y^{(a-1)}=Y_a}}\tilde{\mathcal{E}}^{l_a}(x_1,\ldots,x_{l_a}|s_1,\ldots,s_{a-1}) =$$
$$= \sum_{N_a=1}^{l_a-|Y_a|}\sum_{\substack{x_j\in Y^{(a-1)} \text{ for } j\le N_a\\ p(x_{N_a+1},\ldots,x_{l_a})=Y_a}}\binom{l_a}{N_a}\tilde{\mathcal{E}}^{l_a}(x_1,\ldots,x_{l_a}|s_1,\ldots,s_{a-1}). \tag{B.64}$$

We insert (B.64) on the rhs of (B.63). This gives

$$A(Y)=\sum_{t\leq|Y|}\sum_{\tilde{l}}\sum_{\substack{Y_t\\ Y^{(t)}=Y}}\sum_{\tilde{N}}\int_0^1 ds_1\dots ds_{t-1}\int d\mu_{v[s]}(\phi)\Big\{\prod_{a=2}^{t}[\partial_{s_{a-1}}$$

$$\sum_{\substack{x_j\in Y^{(a-1)}\text{ for } j\leq N_a\\ p(x_{N_a+1},\dots,x_{l_a})=Y_a}}\binom{l_a}{N_a}\widetilde{\mathcal{E}}^{l_a}(x_1,\dots,x_{l_a}|s_1,\dots,s_{a-1})]e^{E_{\bar{Y}_t}(Y^{(t)}|s_1,\dots,s_{t-1})}\Big\}. \quad \text{(B.65)}$$

With

$$\widetilde{\mathcal{E}}^{l_a}(x_1,\dots,x_{l_a}|s_1,\dots,s_{a-1})=\prod_{i=1}^{a-1}s_i^{N_i^{l_a}}\,\widetilde{\mathcal{E}}^{l_a}(x_1,\dots,x_{l_a}) \quad \text{(B.66a)}$$

$$n_i^{l_a}=N_i^{l_a}(1-\delta_{l,N_i^{l_a}}), \qquad N_i^{l_a}=|\{j\in\{1,\dots,l_a\}|\ x_j\in Y^{(i)}\}| \quad \text{(B.66b)}$$

we obtain

$$\sum_{\substack{x_j\in Y^{(a-1)}\text{ for } j\leq N_a\\ p(x_{N_a+1},\dots,x_{l_a})=Y_a}}\binom{l_a}{N}\widetilde{\mathcal{E}}^{l_a}(x_1,\dots,x_{l_a}|s_1,\dots s_{a-1})=\sum_{\substack{(k_i^a)_{i=1,\dots,N_a}\\ 1\leq k_i^a<a}}\prod_{i=1}^{N_a}(s_{a-1}s_{a-2}\dots s_{k_i^a})$$

$$\frac{l_a!}{\prod_{j=1}^{N_a}p_j^a!(N-N_a)!}\sum_{\substack{x_j\in Y_{k_j^a}\text{ for } j\leq N_a\\ p(x_{N_a+1},\dots,x_{l_a})=Y_a}}\widetilde{\mathcal{E}}^{l_a}(x_1,\dots,x_{l_a}). \quad \text{(B.67)}$$

The sum is over all N_a-tupel $(k_i^a)_{i=1,\dots,N_a}$ with $k_i^a\in\{1,\dots,a-1\}$. p_j^a equals the number of elements x_i in Y_j, i.e.

$$p_j^a=|\{k_j^a|\ k_i^a=j\}|. \quad \text{(B.68)}$$

The tupels (k_i^a) for $a\in\{2,\dots,t\}$ may be replaced by functions $\tilde{\eta}$. We insert (B.67) on the rhs of (B.65) and set

$$\eta_i(a)=k_i^a. \quad \text{(B.69)}$$

This proves Eq. (B.57). √

The decoupling expansion (B.57) is essentially simpler for 2-particle interactions

$$E(X)=\sum_{x\in X}\mathcal{E}^1(x)+\sum_{x,y\in X}\mathcal{E}^2(x,y). \quad \text{(B.70)}$$

For this special case the conditions (B.59) and (B.60) of Corollary B.4.1. are

$$l_a=2,\ |Y_a|=1 \quad \text{(B.71)}$$

for all $a\in\{2,\dots,t\}$. For that the summation over t, $\tilde{l}$, $\bar{Y}_t$, and $\tilde{N}$ on the rhs of (B.57) is a sum over partitions $Y_2,\dots,Y_t$ with

$$t=|Y^{(t)}|, \qquad \bar{Y}_t=(\{x_1\},\{x_2\},\dots,\{x_t\}), \qquad l_a=2, \qquad N_a=1 \quad \text{(B.72)}$$

for all $a\in\{2,\dots,t\}$. The sum over $\tilde{\eta}$ is in this special case a sum over all t-trees η. Thereby the tree graph formula is a special case of the decoupling expansion.

PROOF OF THEOREM 2.5.1. (TREE GRAPH FORMULA): Let us set

$$E(X)=\sum_{x\in X}\ln F_x(\phi_x+\psi_x). \quad \text{(B.73)}$$

By Corollary B.4.1. follows

$$A(X|\psi) = \sum_{\substack{Y_n \\ Y^{(n)}=X}} \sum_{\tilde{\eta}} \int_0^1 d\sigma_{n-1} \prod_{a=2}^{n} [s_{a-2}s_{a-3}\dots s_{\eta_1(a)} \int d\mu_{v[s]}(\phi)$$

$$\Big\{ [\sum_{\substack{x_1 \in Y_{\eta_1(a)} \\ x_2 \in Y_a}} \frac{\partial}{\partial \phi_{x_1}} v_{x_1 x_2} \frac{\partial}{\partial \phi_{x_2}}] \prod_{x \in X} F_x(\phi_x + \psi_x) \Big\}, \quad \text{(B.74)}$$

where

$$\bar{Y}_n = (\{x_1\}, \{x_2\}, \dots, \{x_n\}) \text{ with } x_1 = x \in X \quad \text{(B.75)}$$

is a partition of $X = \{x_1, \dots, x_n\}$. The sum over $\bar{Y}_n$ is a sum over permutations of $x_2, \dots, x_n$. The interpolating propagator $v[s]$ defined by Eq. (B.50) equals the modified propagator defined by (2.110) for the partition $\bar{Y}_n$. With the notations $\eta_1 \equiv \eta$ and

$$f(\eta|s) = \prod_{a=2}^{n} [s_{a-2}s_{a-3}\dots s_{\eta(a)}] \quad \text{(2.107)}$$

follows by Eq. (B.74) the assertion (2.111a). √

The tree graph formula yields a relation of representations by Mayer graphs and by tree graphs. We have for complex w_{ij} with $1 \le i < j \le n,\ i, j, n \in \mathbb{N}$

$$e^{\sum_{1\le i<j\le n} w_{ij}} = \sum_{\{1,\dots,n\}=\sum I} \prod_I \Big\{ \sum_{G \in \mathcal{G}_I} \prod_{(ij)\in G} [e^{w_{ij}} - 1] \Big\}. \quad \text{(B.76)}$$

$\mathcal{G}_I$ is the set of all Mayer graphs with vertex set I. By the tree graph formula we obtain the following corollary :

Corollary B.4.2. (abstract tree graph formula). *We have*

$$\sum_{G\in\mathcal{G}_n} \prod_{(ij)\in G} [e^{w_{ij}} - 1] = \sum_\pi \sum_\eta \int_0^1 d\sigma_{n-1} f(\eta|s) \prod_{k=2}^{n} w_{\pi(k)\eta(\pi(k))} e^{\sum_{1\le i<j\le n} w_{\pi(i)\pi(j)}[s]} \quad \text{(B.77)}$$

with

$$w_{ij}[s] = \begin{cases} s_i s_{i+1} \dots s_{j-1} w_{ij} & \text{if } i < j \\ w_{ii} & \text{if } i = j \end{cases} \quad \text{(B.78)}$$

for all i, j *with* $1 \le i < j \le n$. *The sum is over all permutations*

$$\pi : \begin{cases} \{1, \dots, n\} \to \{1, \dots, n\} \\ i \mapsto \pi(i) \end{cases} \quad \text{(B.79)}$$

with $\pi(1) = 1$ *and all* n*-trees*

$$\eta : \begin{cases} \{2, \dots, n\} \to \{1, \dots, n-1\} \\ k \mapsto \eta(k) \end{cases} \quad \text{(B.80)}$$

with $\eta(k) < k$.

B.5. Estimates for the sum over $\tilde{\eta}$ in the decoupling expansion for nonlocal interactions

This section presents a generalization of the tree estimate (Lemma 3.1.4) and the Lemma by Battle (Lemma 3.1.5.) for the decoupling expansion of nonlocal interactions.

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Lemma B.5.1. (generalization of the tree estimate). *Let $\widetilde{l} = (l_2,\ldots,l_t)$ and $\widetilde{N} = (N_2,\ldots,N_t)$ be $(t-1)$-tuples of positive integers with $N_a < l_a$ for all $a \in \{2,\ldots,t\}$. Let $\mu(a) \geq 0$, $a \in \{1,\ldots,t\}$, be non negative real numbers. With the notation*

$$S_{\widetilde{l},\widetilde{N},t}(\mu) = \sum_{\widetilde{\eta}} \int_0^1 ds_1 \ldots ds_{t-1} \prod_{a=2}^{t} \left\{ \mu(a)^{l_a - N_a} \partial_{s_{a-1}} \prod_{i=1}^{N_a} [s_{a-1} \ldots s_{\eta_i(a)} \mu(\eta_i(a))] \right\} \tag{B.81}$$

we have the inequality

$$S_{\widetilde{l},\widetilde{N},t}(\mu) \leq \prod_{a=2}^{t} \left\{ \frac{N_a}{(\prod_{i=a+1}^{t} N_i)^{N_a}} [\mu(a)]^{l_a - N_a} \exp(\mu(a-1) \prod_{i=a}^{t} N_i) \right\}. \tag{B.82}$$

PROOF: The sum over $\widetilde{\eta}$ may be replaced by the sum over $\eta_i(a) = k_i$ from 1 to $a-1$ for all $a \in \{2,\ldots,t\}$. Therefore

$$S_{\widetilde{l},\widetilde{N},t}(\mu) = \int_0^1 ds_1 \ldots ds_{t-1} \prod_{a=2}^{t} \left\{ \mu(a)^{l_a - N_a} \partial_{s_{a-1}} \prod_{i=1}^{N_a} [\sum_{k_i=1}^{a-1} s_{a-1} s_{a-2} \ldots s_{k_i} \mu(k_i)] \right\} =$$
$$= \int_0^1 ds_1 \ldots ds_{t-1} \prod_{a=2}^{t} \left\{ \mu(a)^{l_a - N_a} \partial_{s_{a-1}} [\sum_{k=1}^{a-1} s_{a-1} s_{a-2} \ldots s_k \mu(k)]^{N_a} \right\}. \tag{B.83}$$

Let us set

$$S_i'(\mu) = \int_0^1 ds_1 \ldots ds_{t-i} \prod_{a=2}^{t+1-i} [\sum_{k=1}^{a-1} s_{a-2} \ldots s_k \mu(k)]^{N_a} \exp(\sum_{k=1}^{t-i} s_{t-i} \ldots s_k \mu(k) \prod_{k=2}^{i} N_{t+2-k}) \tag{B.84}$$

for all $i \in \{1,\ldots,t\}$. We have

$$S_{\widetilde{l},\widetilde{N},t}(\mu) \leq \prod_{a=2}^{t} [N_a \mu(a)^{l_a - N_a}] S_1'(\mu). \tag{B.85}$$

$S_i'(\mu)$ is estimated by recursion. Because of $1 \leq \int_0^1 ds e^{su}$ $(u \geq 0)$, we obtain

$$S_i' \leq \int_0^1 ds_1 \ldots ds_{t-i}^{(1)} ds_{t-i}^{(2)} \ldots ds_{t-i}^{(N_{t-i+1})} \prod_{a=2}^{t-i} [\sum_{k=1}^{a-1} s_{a-2} \ldots s_k \mu(k)]^{N_a}$$
$$[\sum_{k=1}^{t-i} s_{t-i-1} s_{t-i-2} \ldots s_k \mu(k)]^{N_{t-i+1}} \exp(\sum_{l=1}^{N_{t-i+1}} \sum_{k=1}^{t-i} s_{t-i}^{(l)} s_{t-i-1} \ldots s_k \mu(k) \prod_{k=2}^{i} N_{t+2-k}). \tag{B.86}$$

We integrate (B.86) over $s_{t-1}^{(1)},\ldots,s_{t-i}^{(N_{t-i+1})}$. We use the inequality

$$\int_0^1 ds \; u e^{su/a} \leq a e^{u/a}$$

and obtain

$$S_i'(\mu) \leq \int_0^1 ds_1 \ldots ds_{t-i-1} \prod_{a=2}^{t-i} [\sum_{k=1}^{a-1} s_{a-2} \ldots s_k \mu(k)]^{N_a} \frac{1}{(\prod_{k=2}^{i} N_{t+2-k})^{N_{t-i+1}}}$$
$$\exp(\sum_{l=1}^{N_{t-i+1}} \sum_{k=1}^{t-i} s_{t-i-1} s_{t-i-2} \ldots s_k \mu(k) \prod_{k=2}^{i} N_{t+2-k}). \tag{B.87}$$

Thereby

$$S_i' \leq \frac{\exp(\mu(t-i) \prod_{k=2}^{i+1} N_{t+2-k})}{(\prod_{k=2}^{i} N_{t+2-k})^{N_{t-i+1}}} S_{i+1}'(\mu). \tag{B.88}$$

Because of $S'_t(\mu) = 1$ we obtain

$$S_{\widetilde{l},\widetilde{N},t}(\mu) \leq \prod_{a=2}^{t} [N_a \mu(a)^{l_a - N_a}] \frac{e^{\mu(t-1)N_t}}{1} S'_2(\mu) \leq$$

$$\leq \prod_{a=2}^{t} [N_a \mu(a)^{l_a - N_a}] \frac{e^{\mu(t-1)N_t}}{1} \frac{e^{\mu(t-2)N_t N_{t-1}}}{N_t^{N_{t-1}}} S'_3(\mu) \leq \dots$$

$$\dots \leq \prod_{a=2}^{t} \left\{ \frac{N_a}{(\prod_{i=a+1}^{t} N_i)^{N_a}} [\mu(a)]^{l_a - N_a} \exp(\mu(a-1) \prod_{i=a}^{t} N_i) \right\}. \checkmark \quad \text{(B.89)}$$

By the generalization of the tree estimate follows

Lemma B.5.2. (generalization of the Lemma by Battle). *Let $\widetilde{l} = (l_2, \dots, l_t)$ and $\widetilde{N} = (N_2, \dots, N_t)$ be $(t-1)$- tuples of positive integers with $N_a < l_a$ for all $a \in \{2, \dots, t\}$. We use the notations*

$$d_{\widetilde{\eta},\widetilde{N}}(a) = \begin{cases} l_t - N_t & \text{if } a = t \\ c_{\widetilde{\eta}}(a) + l_a - N_a & \text{if } 1 < a < t \\ c_{\widetilde{\eta}}(1) & \text{if } a = 1 \end{cases} \quad \text{(B.90a)}$$

with

$$c_{\widetilde{\eta}}(a) = |\{(b,i)| \; \eta_i(b) = a\}|, \qquad a \in \{1, \dots, t-1\} \quad \text{(B.90b)}$$

$d_{\widetilde{\eta},\widetilde{N}}(a)$ equals the exponent of $\mu(a)$ on the rhs of Eq. (B.81). Then

$$\sum_{\widetilde{\eta}} \int_0^1 ds_1 \dots ds_{t-1} d_{\widetilde{\eta},tN}(1)! \prod_{a=2}^{t} \{ d_{\widetilde{\eta},\widetilde{N}}(a)! \; \partial_{s_{a-1}} \prod_{i=1}^{N_a} [s_{a-1} \dots s_{\eta_i(a)}] \} \leq$$

$$\leq [(\prod_{i=2}^{t} N_i) + 1]^{N_t - l_t} \prod_{a=2}^{t} \left\{ \frac{N_a [(\prod_{i=2}^{t} N_i) + 1]^{l_a}}{(\prod_{i=a+1}^{t} N_i)^{N_a}} (l_a - N_a)! \right\}. \quad \text{(B.91)}$$

PROOF: By Lemma B.5.1. follows for the positive real numbers $t_a, \; a \in \{1, \dots, t\}$:

$$\sum_{\widetilde{\eta}} \int_0^1 ds_1 \dots ds_{t-1} \prod_{a=2}^{t} \left\{ t_a^{d_{\widetilde{\eta},\widetilde{N}}(a)} \partial_{s_{a-1}} \prod_{i=1}^{N_a} [s_{a-1} \dots s_{\eta_i(a)}] \right\} t_1^{d_{\widetilde{\eta},\widetilde{N}}(1)} \leq$$

$$\leq \prod_{a=2}^{t} \left[\frac{N_a}{(\prod_{i=a+1}^{t} N_i)^{N_a}} t_a^{l_a - N_a} \exp(t_{a-1} \prod_{i=a}^{t} N_i) \right]. \quad \text{(B.92)}$$

We multiplicate the factor $\prod_{a=2}^{t} [e^{-t_{a-1}(\prod_{i=a}^{t} N_i + 1)}] e^{-t_t}$. This gives

$$\sum_{\widetilde{\eta}} \int_0^1 ds_1 \dots ds_{t-1} t_1^{d_{\widetilde{\eta},\widetilde{N}}(1)} e^{-t_1(\prod_{i=2}^{t} N_i + 1)} \prod_{a=2}^{t-1} \left\{ t_a^{d_{\widetilde{\eta},tN}(a)} e^{-t_a(\prod_{i=a+1}^{t} N_i + 1)} \right\}$$

$$t_1^{d_{\widetilde{\eta},\widetilde{N}}(t)} e^{-t_t} \prod_{a=2}^{t} \left\{ \partial_{s_{a-1}} \prod_{i=1}^{N_a} [s_{a-1} \dots s_{\eta_i(a)}] \right\} \leq$$

$$\leq e^{-t_1} \prod_{a=2}^{t-1} \left[\frac{N_a}{(\prod_{i=a+1}^{t} N_i)^{N_a}} t_a^{l_a - N_a} e^{-t_a} \right] N_t \; t_t^{l_t - N_t} e^{-t_t}. \quad \text{(B.93)}$$

We integrate (B.93) over t_a from 0 to ∞ and we use

$$\int_0^\infty dt \; t^n e^{-at} = \frac{n!}{a^{n+1}}.$$

This gives

$$\sum_{\widetilde{\eta}} \int_0^1 ds_1 \ldots ds_{t-1}\; d_{\widetilde{\eta},\widetilde{N}}(1)! \prod_{a=2}^{t} \left\{ d_{\widetilde{\eta},\widetilde{N}}(a)! \partial_{s_{a-1}} \prod_{i=1}^{N_a} [s_{a-1} \ldots s_{\eta_i(a)}] \right\}$$

$$\frac{1}{\prod_{a=1}^{t-1} [(\prod_{i=a+1}^{t} N_i) + 1]^{d_{\widetilde{\eta},\widetilde{N}}(a)+1}} \leq \prod_{a=2}^{t} \left[\frac{N_a}{(\prod_{i=a+1}^{t} N_i)^{N_a}} (l_a - N_a)! \right]. \quad \text{(B.94)}$$

By the definition (B.90) of $d_{\widetilde{\eta},\widetilde{N}}(a)$ follows

$$\sum_{a=1}^{t-1} d_{\widetilde{\eta},\widetilde{N}}(a) = \sum_{a=2}^{t-1} l_a + N_t. \quad \text{(B.95)}$$

Therefore the assertion follows by the inequality (B.94). $\surd$

APPENDIX C. GENERATING FUNCTION FOR FREE-PROPAGATOR-AMPUTATED GREENS FUNCTIONS

Lemma C.1.. *The generating function for Greens functions is defined by*

$$T[J] = \frac{1}{\mathcal{N}} \int d\mu_v(\phi) F(\phi) e^{i(J,\phi)}, \quad \text{(C.1)}$$

where $\mathcal{N}$ is fixed by $T[0] = 1$. Then the following relation holds on the lattice $(a\mathbb{Z})^\nu = \Lambda_{tot}$

$$\ln \left[\frac{Z(\Lambda_{tot}|\psi)}{Z(\Lambda_{tot}|\psi = 0)} \right] = \ln T[J] - \frac{1}{2} \int_{x,y \in (a\mathbb{Z})^\nu} J(x) v(x,y) J(y) \quad \text{(C.2)}$$

with

$$Z(\Lambda_{tot}|\psi) = \int d\mu_v(\phi) F(\phi + \psi). \quad \text{(C.3)}$$

The generating function for the free-propagator-amputated Greens functions is

$$\ln \left[\frac{Z(\Lambda_{tot}|\psi)}{Z(\Lambda_{tot}|\psi = 0)} \right].$$

PROOF: Subtraction of the integration variables gives

$$Z(\Lambda_{tot}|\psi) = \int d\mu_v(\phi - \psi) F(\phi) = \det(2\pi v)^{-\frac{1}{2}} \int [\prod_{x \in (a\mathbb{Z})^\nu} d\phi_x] F(\phi)\; e^{-\frac{1}{2}((\phi-\psi), v^{-1}(\phi-\psi))} =$$

$$= \int d\mu_v(\phi) F(\phi) e^{-(\phi, v^{-1}\psi)} e^{-\frac{1}{2}(\psi, v^{-1}\psi)}. \quad \text{(C.4)}$$

With

$$J(x) = \int_{y \in (a\mathbb{Z})^\nu} v^{-1}(x,y) \psi(y) \quad \text{(C.5)}$$

and (C.1) follows (C.2). The derivative with respect to $J(x)$ reads

$$\frac{\delta}{\delta J(x)} = \int_{y \in (a\mathbb{Z})^\nu} v(x.y) \frac{\delta}{\delta \psi(y)}. \quad \text{(C.6)}$$

This yields

$$G_c(x_1,\dots,x_n) = \int_{y_1,\dots y_n\in(a\mathbb{Z})^\nu} v(x_1,y_1)\dots v(x_n,y_n)\frac{\delta^n}{\delta\psi(y_1)\dots\delta\psi(y_n)}\ln\left[\frac{Z(\Lambda_{tot}|\psi)}{Z(\Lambda_{tot}|\psi=0)}\right]|_{\psi=0} \tag{C.7}$$

for $n>2$ and

$$G_c(x_1,x_2) = \int_{y_1,y_2\in(a\mathbb{Z})^\nu} v(x_1,y_1)v(x_2,y_2)\frac{\delta^2}{\delta\psi(y_1)\delta\psi(y_2)}\ln\left[\frac{Z(\Lambda_{tot}|\psi)}{Z(\Lambda_{tot}|\psi=0)}\right]|_{\psi=0} + v(x_1,x_2) \tag{C.8}$$

for $n=2$. The n-point free-propagator-amputated Greens function is therefore

$$G_c(\underline{x}_1,\dots,\underline{x}_n) = \frac{\delta^n}{\delta\psi(x_1)\dots\delta\psi(x_n)}\ln\left[\frac{Z(\Lambda_{tot}|\psi)}{Z(\Lambda_{tot}|\psi=0)}\right]|_{\psi=0}.\ \surd \tag{C.9}$$

APPENDIX D. EQUIVALENCE OF RENORMALIZATION CONDITIONS

Lemma D.1.. *Suppose that the partition function $Z(X|\psi)$ fulfills*

$$Z(X|\psi) = Z(X|-\psi) \tag{D.1}$$

for all finite $X\subset(a\mathbb{Z})^\nu$. Then the renormalization conditions

$$\ln Z(X|\psi)|_{\psi=0} = 0 \tag{D.2a}$$

$$\frac{\partial^2}{\partial\psi^2}\ln Z(X|\psi)|_{\psi=0} = 0 \tag{D.2b}$$

are equivalent to

$$A^{ren}(X|\psi)|_{\psi=0} = \begin{cases} 1 & \text{if } |X|=1 \\ 0 & \text{otherwise}\end{cases} \tag{D.3a}$$

$$\frac{\partial^2}{\partial\psi^2}A^{ren}(X|\psi)|_{\psi=0} = 0 \tag{D.3b}$$

for all finite $X\subset(a\mathbb{Z})^\nu$ with

$$Z(X|\psi) = \sum_{X=\sum Y}\prod_Y A^{ren}(Y|\psi). \tag{D.4}$$

The derivative in (D.2b) and (D.3b) is with respect to constant external fields $\psi_x=\psi$ for all $x\in(a\mathbb{Z})^\nu$.

PROOF: 1.) Suppose that the renormalization conditions (D.2a,b) are fulfilled . We will show the renormalization conditions (D.3a,b). From (D.2a) follows

$$Z(X|\psi=0) = 1. \tag{D.5}$$

By uniqueness of the activities A^{ren} in (D.4) and the renormalization condition (D.2a) follows (D.3a). Because of the symmetry (D.1), follows

$$\frac{\partial}{\partial\psi}Z(X|\psi)|_{\psi=0} = 0. \tag{D.6}$$

By (D.2b), (D.5) and (D.6) we obtain

$$\frac{\partial^2}{\partial \psi^2} Z(X|\psi)|_{\psi=0} = 0. \tag{D.7}$$

With the help of the inversion formula (see app. A, (A.21))

$$A^{ren}(X|\psi) = \sum_{n\geq 1} (-1)^{n-1}(n-1)! \sum_{X=\sum_{k=1}^{n} P_k} \prod_{k=1}^{n} Z(P_k|\psi) \tag{D.8}$$

we obtain the renormalization condition (D.3b).

2.) Suppose that the renormalization conditions (D.3a,b) are fulfilled. By the polymer representation (D.4) and (D.3a) follows (D.2a). Because of the symmetry (D.1), we have

$$\frac{\partial}{\partial \psi} A^{ren}(X|\psi)|_{\psi=0} = 0. \tag{D.9}$$

Therefore (D.3b) and (D.4) prove the renormalization condition (D.2b). √

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