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ON THE $\lambda\phi^4$ TRIVIALITY ISSUE IN A ROBERTSON-WALKER SPACE-TIME

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The existence and interaction problem of $\lambda\phi^4$ in a 4-dimensional homogenuous and isotropic space-time is studied. The approach is based on a variational technique. In this first work, and only for reasons of technical and conceptual simplicity, a static metric is assumed. Within the limits of our approach we find that existence and interaction are basically independent of the constant space-curvature for a static space-time. The strength of the interaction, however, has curvature-dependent bounds. In all cases the only phase which hints at an interacting theory is the one with a negative, but for large values of the UV cutoff logarithmically vanishing, bare coupling constant.

I. INTRODUCTION

Our simplest renormalizable field theory in Minkowski space is $\lambda\phi^4$, and yet surprisingly little is actually known about it. Even the most basic questions about its existence (is its spectrum bounded from below?) and, assuming a positive answer, about triviality and interaction (is its spectrum a free field spectrum?) have not yet been answered in a precise way. The little we know about them is, in a nutshell, the following: For $\lambda > 0$ the theory is very likely either nonexistent or otherwise trivial [1]; for $\lambda < 0$ but vanishing when the regulator is removed the theory has good chances of being bounded from below and interacting [2,3]. It is furthermore asymptotically free [3,4,5]. It is however precarious: only its renormalized version is bounded from below, but not its regularized one [2]. For $\lambda < 0$ but finite when the UV cutoff is removed the theory does not exist.

The setting in which we believe these results to be true is the following: a 4-dimensional Minkowski space, a regulator of the UV divergences, Λ, a Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \lambda\phi^4 \tag{I.1}$$

and a flow of the bare parameters m and λ as functions of Λ such that the renormalized Green's functions are finite when $\Lambda\to\infty$. There is only one ingredient in this framework which is necessarily incompatible with our present understanding of physics: the assumption of a flat Lorentzian space-time. A field theory involves energy densities and these imply, because of General Relativity, a Riemannian space-time structure. Thus the first extension of the standard setting for triviality studies is to consider $\lambda\phi^4$ in a given Riemannian background space-time. Of course this will not be the last word either. The background field will become a dynamic and eventually a quantum field, but this is a long road to go. The first step is to consider a given Riemannian background space-time. This is already such a broad generalization that we will limit ourselves in this first study to a homogeneous, isotropic and static space-time. Although homogeneity and isotropy are generally accepted large-scale features of our universe and a reasonable assumption for the study of $\lambda\phi^4$, the same is not true for time-independence. It is enough to recall that a static space-time requires a nonzero cosmological constant [6] and that this is ruled out by the extremely low experimental value of the vacuum energy density. Thus we will have to consider eventually a time-dependent space-time. In this first study, however, only the static solution will be considered, which, due to the existence of a global time-like Killing vector field, allows a natural definition of positive energy solutions to the free field equations, and thus of the associated vacuum [7].

We will follow a variational technique first introduced into Quantum Field Theory by Schiff [8] and further developed by Barnes and Ghandour [9]. It leads to an approximate effective potential and, under certain circumstances, to an upper bound of the energy density of the ground state. In this last case, and if the upper bound implies that the theory is unbounded from below, the result is rigorous. If not, it will only be approximate. The method leads either to nonexistence or triviality for $\lambda > 0$ and to an interacting precarious theory for $\lambda < 0$ but vanishing when $\lambda \to \infty$ [2]. It has been carefully studied by Stevenson [2], extended to include odd terms [10] and applied at finite temperature [11]. We believe it to be a sensible approach to the study of existence and interaction. It will provide our calculational framework.

The next section reviews free field solutions of a scalar field in a static Robertson-Walker metric. We will need them as trial fields of the variational approach. Section III gives a parameter dependent bound to the regularized energy density. Minimization, zero point energy subtraction and renormalization lead in section IV to the socalled gaussian effective potential, which is then analyzed. The last section is devoted to comments and conclusions.

II. FREE FIELDS

Consider a static Robertson-Walker space-time described by the line element (e. g. ref. 12)

$$ds^2 = dt^2 - a^2\left[d\chi^2 + f^2(\chi)\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)\right], \qquad \text{(II.1)}$$

where $r \propto f(\chi)$ is determined by the constant space-curvature K which is normalized to +1,0 or -1 (closed, flat and open, respectively) according to

$$f(\chi) = \begin{cases} \sin\chi & 0 \leq \chi \leq \pi \\ \chi & 0 \leq \chi < \infty \\ \sinh\chi & 0 \leq \chi < \infty \end{cases} \qquad \text{for } K = \begin{cases} +1 \\ 0 \\ -1 \end{cases} . \qquad \text{(II.2)}$$

Notice that a^2 carries the dimensions of the spatial line element. Introducing $\eta \equiv \frac{t}{a}$, a^2 becomes a conformal scale factor. Using these adimensional coordinates η, χ, θ and φ (0,1,2,3) one obtains immediately the following nonzero metrictensor components

$$g^{00} = a^{-2}, \quad g^{11} = -a^{-2}, \quad g^{22} = -\left(a f(\chi)\right)^{-2}, \quad g^{33} = -\left(a f(\chi) \sin\theta\right)^{-2}, \qquad \text{(II.3)}$$

from which the following scalar curvature is obtained:

$$R = \frac{6K}{a^2} . \qquad \text{(II.4)}$$

The free scalar field Lagrangian density is (e. g. ref. 13)

$$\mathcal{L}_0 = \frac{1}{2}\left[-g\right]^{1/2}\left[g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi) - \left(m^2 + \xi R\right)\phi^2\right], \qquad \text{(II.5)}$$

where $g \equiv \det g_{\mu\nu} = -\left(a^4 f^2(\chi) \sin\theta\right)^2$ and ∇_ν is the covariant derivative. Notice that $\mathcal{L}_0$ is adimensional, as so are our coordinates. The equations of motion are

$$(\Box + m^2 + \xi R)\phi = 0, \tag{II.6}$$

where the covariant D'Alambertian is

$$\Box\phi \equiv [-g]^{-\frac{1}{2}} \partial_\mu \left([-g]^{\frac{1}{2}} g^{\mu\nu} \partial_\nu \phi \right) = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi . \tag{II.7}$$

The free fields, solutions of (II.6), can be written as [13]

$$\phi(x) = \int d\tilde{\mu}(k) \left[a_{\vec{k}}\, u_{\vec{k}}(x) + a^+_{\vec{k}}\, u^*_{\vec{k}}(x) \right], \tag{II.8}$$

where the measure will be specified below and $a_{\vec{k}}$ and $a^+_{\vec{k}}$ are annihilation and creation operators which allow to build a Fock space. The time and space dependence of $u_{\vec{k}}(x)$ can be separated as

$$u_{\vec{k}}(x) = \frac{1}{a}\, \mathcal{Y}_{\vec{k}}(\vec{x})\, \chi_k(\eta), \tag{II.9}$$

where $\vec{x} = \chi, \theta, \varphi$ for $K = \pm 1$ and $a\vec{x} = x, y, z$ for $K = 0$. Furthermore the factor a^{-1} carries the dimensions of $u_{\vec{k}}(x)$ which carries the dimensions of $\phi(x)$, and

$$\chi_k(\eta) = \frac{1}{\sqrt{2\omega_k}}\, e^{-i\omega_k \eta} . \tag{II.10}$$

With

$$\Box\phi \equiv \frac{1}{a^2}\left(\partial_\eta \partial_\eta \phi - \Delta\phi \right)$$
$$\Delta\phi = [h]^{-\frac{1}{2}} \partial_i \left([h]^{\frac{1}{2}} h^{ij} \partial_j \phi \right), \quad h^{ij} = -a^2 g^{ij}, \tag{II.11}$$

one finds $\mathcal{Y}_{\vec{k}}(\vec{x})$ solving

$$\Delta \mathcal{Y}_{\vec{k}}(\vec{x}) = (K - k^2)\, \mathcal{Y}_{\vec{k}}(\vec{x}) = (a^2 m^2 + 6\xi K - \omega_k^2)\, \mathcal{Y}_{\vec{k}}(\vec{x}). \tag{II.12}$$

The solutions of (II.12) are [14]

$$\mathcal{Y}_{\vec{k}}(\vec{x}) = \begin{cases} (2\pi)^{-\frac{3}{2}} e^{i\vec{k}\vec{x}}, & \vec{k} = (k_1, k_2, k_3) \\ \Pi^{(\pm)}_{kJ}(\chi)\, Y^M_J(\theta,\varphi), & \vec{k} = (k, J, M) \end{cases} \quad \text{for } K = \begin{cases} 0 \\ \pm 1, \end{cases} \tag{II.13}$$

where the domains of definition of $\vec{k}$ are

$$\begin{array}{lll} -\infty < k_i < \infty & ; \; k = |\vec{k}| & \\ M = -J, -J+1, \dots, J & \begin{cases} J = 0, 1, \dots, k-1 \; ; \; k = 1, 2, \dots \\ J = 0, 1, \dots \quad ; \; 0 < k < \infty \end{cases} & \text{for } K = \begin{cases} 0 \\ +1 \\ -1 . \end{cases} \end{array} \tag{II.14}$$

The measure is given by

$$\int d\tilde{\mu}(k) = \begin{cases} \int d^3k \\ \sum\limits_{k,J,M} \\ \int dk \sum\limits_{J,M} \end{cases} \quad \text{for} \quad K = \begin{cases} 0 \\ +1 \\ -1 \end{cases} \tag{II.15}$$

and the normalization is fixed through

$$\begin{gathered} \int d^3x\, h^{\frac{1}{2}}\, \mathcal{Y}_{\vec{k}}(\vec{x})\, \mathcal{Y}^*_{\vec{k}'}(\vec{x}) = \delta(\vec{k}, \vec{k}') \\ [a_{\vec{k}}, a^+_{\vec{k}'}] = \delta(\vec{k}, \vec{k}') \\ \int d\tilde{\mu}(k')\, \delta(\vec{k}, \vec{k}')\, f(\vec{k}') = f(\vec{k}) . \end{gathered} \tag{II.16}$$

These equations lead to the following expressions for $\Pi^{(+)}_{kJ}(\chi)$ [15]:

$$\Pi^{(+)}_{kJ}(\chi) = \left[\frac{\pi}{2} k^2 (k^2-1)(k^2-4) \cdots (k^2 - J^2) \right]^{-\frac{1}{2}} \sin^J\chi \left(\frac{d}{d\cos\chi} \right)^{J+1} \cos(k\chi) \tag{II.17}$$

anf for $\Pi^{(-)}_{kJ}(\chi)$ [16]:

$$\Pi^{(-)}_{kJ}(\chi) = \left[\frac{\pi}{2} k^2 (k^2+1)(k^2+4) \cdots (k^2 + J^2) \right]^{-\frac{1}{2}} \sinh^J\chi \left(\frac{d}{d\cosh\chi} \right)^{J+1} \cos(k\chi) . \tag{II.18}$$

III. AN UPPER BOUND ON THE ENERGY DENSITY

Let us now introduce the interaction term

$$\mathcal{L}_{int} = -[-g]^{1/2}\lambda\phi^4 . \tag{III.1}$$

The hamiltonian density is then given by

$$\mathcal{H} = \frac{1}{2}a^2 f^2(\chi)\sin\theta\left[(\partial_\eta\phi)^2 + h^{ij}(\partial_i\phi)(\partial_j\phi) + a^2m^2\phi^2 + 2a^2\lambda\phi^4\right], \tag{III.2}$$

where the term ξR of (II.5) has been absorbed into the bare mass, m. The only modification due to the curvature is in the spatial derivatives of (III.2).

Our trial field will be of the form

$$\phi(x) = \phi_0 + \phi_\Omega(x) , \tag{III.3}$$

where ϕ_0 is a constant background field and $\phi_\Omega(x)$ is a free quantum field of mass Ω. Thus ω_k will be given by (cf. (II.12))

$$\omega_k^2(\Omega) = a^2\Omega^2 - K + k^2 . \tag{III.4}$$

The variational parameters will be ϕ_0 and Ω. The ground state corresponding to $\phi_\Omega(x)$ is $|0_\Omega\rangle$. It satisfies

$$a_{\vec{k}}\,|0_\Omega\rangle = 0 . \tag{III.5}$$

The energy density of the true ground state of the theory will be bounded from above by

$$\mathcal{E}_{true} \leqslant \langle 0_\Omega|\mathcal{H}|0_\Omega\rangle \equiv \mathcal{E}(\phi_0,\Omega) . \tag{III.6}$$

The computation of the rhs of (III.6) is straightforward with the help of (II.16) and (III.5). It gives

$$\begin{aligned}\mathcal{E}(\phi_0,\Omega) = \frac{1}{2} f^2(\chi)\sin\theta\Bigg\{\int d\tilde{\mu}(k)\frac{\omega_k(\Omega)}{2}|\mathcal{Y}_{\vec{k}}(\vec{x})|^2 \\ +\int d\tilde{\mu}(k)\frac{h^{ij}}{2\omega_k(\Omega)}(\partial_i\mathcal{Y}^*_{\vec{k}}(\vec{x}))\partial_j\mathcal{Y}_{\vec{k}}(\vec{x}) + a^4m^2\phi_0^2 + a^4m^2\int d\tilde{\mu}(k)\frac{1}{2\omega_k(\Omega)}|\mathcal{Y}_{\vec{k}}(\vec{x})|^2 \\ +2\lambda a^4\phi_0^4 + 12\lambda a^4\phi_0^2\int d\tilde{\mu}(k)\frac{1}{2\omega_k(\Omega)}|\mathcal{Y}_{\vec{k}}(\vec{x})|^2 + 6\left(\int d\tilde{\mu}(k)\frac{1}{2\omega_k(\Omega)}|\mathcal{Y}_{\vec{k}}(\vec{x})|^2\right)^2\Bigg\}.\end{aligned} \tag{III.7}$$

It will be convenient to redefine m, ϕ_0 and Ω so as to absorb all the a-factors; this makes all the variables dimensionless. Eq. (III.7) can be written in a more compact form by introducing the integrals

$$I_n^{(K)}(\Omega^2) \equiv \int\frac{d\tilde{\mu}(k)}{2\omega_k(\Omega)}\,|\mathcal{Y}_{\vec{k}}(\vec{x})|^2(\omega_k(\Omega))^{2n} . \tag{III.8}$$

The notation is such that it coincides for $K = 0$ with the one of reference 2. We are interested in the cases $K = \pm 1$. For these the sum over M can be performed immediately with the help of

$$\sum_{M=-J}^{J} |Y_J^M(\theta,\varphi)|^2 = \frac{2J+1}{4\pi} \tag{III.9}$$

so that

$$\begin{aligned} I_n^{(+)}(\Omega^2) &= \frac{1}{4\pi}\sum_{k=1}^{\infty}\sum_{J=0}^{k-1}\frac{2J+1}{2\omega_k(\Omega)}\left(\Pi_{kJ}^{(+)}(\chi)\right)^2(\omega_k(\Omega))^{2n} \\ I_n^{(-)}(\Omega^2) &= \frac{1}{4\pi}\int_0^\infty dk\sum_{J=0}^{\infty}\frac{2J+1}{2\omega_k(\Omega)}\left(\Pi_{kJ}^{(-)}(\chi)\right)^2(\omega_k(\Omega))^{2n} .\end{aligned} \tag{III.10}$$

The sum over J can easily be performed recalling that (III.10) cannot depend on χ as our space is homogeneous. Then, by choosing $\chi = 0$ and using (II.17,18) one obtains

$$\begin{aligned} I_n^{(+)}(\Omega^2) &= \frac{1}{4\pi^2}\sum_{k=1}^{\infty}\frac{k^2}{\sqrt{k^2+\Omega^2-1}}(k^2+\Omega^2-1)^n \\ I_n^{(-)}(\Omega^2) &= \frac{1}{4\pi^2}\int_0^\infty dk\,\frac{k^2}{\sqrt{k^2+\Omega^2+1}}(k^2+\Omega^2+1)^n .\end{aligned} \tag{III.11}$$

The other integral which appears in (III.7) is

$$\bar{J}(\Omega^2) \equiv \int \frac{d\tilde{\mu}(k)}{2\omega_k(\Omega)} h^{ij}(\partial_i Y_{\vec{k}}^*(\vec{x}))\partial_j Y_{\vec{k}}(\vec{x}) . \qquad \text{(III.12)}$$

It is again convenient to work it out at $\chi = 0$. This then leads to

$$\bar{J}(\Omega^2) = I_1(\Omega^2) - \Omega^2 I_0(\Omega^2), \qquad \text{(III.13)}$$

where we have not indicated the curvature-superscript as the relation holds for all its values (this will be the meaning of neglecting the superscript from now on).

Eqs. (III.8 and 13) allow one to write (III.7) as

$$\mathcal{E}(\phi_0, \Omega) = I_1(\Omega^2) + \frac{1}{2}(m^2 - \Omega^2)\, \bar{I}_0(\Omega^2) + \frac{1}{2} m^2 \Phi_0^2 + \lambda \Phi_0^4 + 6\lambda \Phi_0^2 I_0(\Omega^2) + 3\bar{I}_0^{\,2}(\Omega^2), \qquad \text{(III.14)}$$

where the factor $f^2(\chi)\sin\theta$ of (III.7) has been dropped. This is formally the same expression for all three values of K, only the functions $I_n(\Omega^2)$ differ:

$$\bar{I}_n^{(0)}(\Omega^2) = \frac{1}{4\pi^2} \int_0^\infty dk\, k^2 (k^2 + \Omega^2)^{n - \frac{1}{2}}$$

$$\bar{I}_n^{(+)}(\Omega^2) = \frac{1}{4\pi^2} \sum_{k=1}^{\infty} k^2 (k^2 + \Omega^2 - 1)^{n - \frac{1}{2}} \qquad \text{(III.15)}$$

$$I_n^{(-)}(\Omega^2) = \frac{1}{4\pi^2} \int_0^\infty dk\, k^2 (k^2 + \Omega^2 + 1)^{n - \frac{1}{2}} .$$

These integrals or sums are UV divergent for $n \geq -1$. We will regularize them with an UV cutoff Λ. Notice also that for $\Omega = 0$ and for $K = 0$ and $+1$ IR divergences appear for $n \leq -1$ and $n < 1/2$, respectively. They will, wherever necessary, be regularized by keeping Ω slightly off zero. The fact that the UV behaviour is universal (i. e. curvature independent) whereas this is not the case for the IR behaviour is of course due to curvature being a conspicuous large-scale feature but only a subtle small-scale feature of space-time. This will explain many of our results.

The functions of (III.15) can be expanded around Ω_0^2 leading to the following expressions

$$\bar{I}_1(\Omega^2) = \bar{I}_1(\Omega_0^2) + \frac{1}{2}(\Omega^2 - \Omega_0^2)\, \bar{I}_0(\Omega_0^2) - \frac{1}{8}(\Omega^2 - \Omega_0^2)^2 \bar{I}_1(\Omega_0^2) + \Delta(\Omega^2, \Omega_0^2)$$

$$I_0(\Omega^2) = I_0(\Omega_0^2) - \frac{1}{2}(\Omega^2 - \Omega_0^2)\, \bar{I}_1(\Omega_0^2) + \Gamma(\Omega^2, \Omega_0^2) \qquad \text{(III.16)}$$

$$\bar{I}_{-1}(\Omega^2) = \bar{I}_{-1}(\Omega_0^2) + \tilde{\Pi}(\Omega^2, \Omega_0^2) ,$$

where $\bar{I}_1(\Omega^2)$ is quartically, $\bar{I}_0(\Omega^2)$ quadratically and $\bar{I}_1(\Omega^2)$ logarithmically divergent. The functions $\Delta(\Omega^2, \Omega_0^2)$, $\Gamma(\Omega^2, \Omega_0^2)$ and $\tilde{\Pi}(\Omega^2, \Omega_0^2)$ are UV finite and related by

$$\Gamma(\Omega^2, \Omega_0^2) = 2 \frac{d\Delta(\Omega^2, \Omega_0^2)}{d\Omega^2}$$

$$\tilde{\Pi}(\Omega^2, \Omega_0^2) = -2 \frac{d\Gamma(\Omega^2, \Omega_0^2)}{d\Omega^2} \qquad \text{(III.17)}$$

$$I_{-2}(\Omega^2) = -\frac{2}{3} \frac{d\tilde{\Pi}(\Omega^2, \Omega_0^2)}{d\Omega^2} .$$

They can easily be computed for $K = 0$ [2], giving

$$\Delta^{(0)}(\Omega^2, \Omega_0^2) = \frac{1}{128\pi^2}\left[2\Omega^4 \ln \frac{\Omega^2}{\Omega_0^2} - 2\Omega_0^2(\Omega^2 - \Omega_0^2) - 3(\Omega^2 - \Omega_0^2)^2\right]; \qquad \text{(III.18)}$$

the other functions can then be obtained from (III.17). For $K = -1$, (III.15) allows to obtain the relation

$$\Delta^{(-)}(\Omega^2, \Omega_0^2) = \Delta^{(0)}(\Omega^2 + 1, \Omega_0^2 + 1) . \qquad \text{(III.19)}$$

We have not been able to compute $\Delta^{(+)}(\Omega^2, \Omega_0^2)$ (see, however, ref. 17), but we do not need an explicit expression for it. It will suffice to know some of its properties, which moreover turn out to be K-independent. Let us just list them,

$$\Delta(\Omega^2, \Omega_0^2) \underset{\Omega^2 \gg 1}{\sim} \frac{1}{64\pi^2} \Omega^4 \ln \Omega^2$$

$$\Delta(\Omega_0^2, \Omega_0^2) = 0 \qquad \text{(III.20)}$$

$$\frac{d\Delta(\Omega^2, \Omega_0^2)}{d\Omega^2} \geq 0$$

and

$$\Gamma(\Omega^2, \Omega_0^2) \underset{\Omega^2 \gg 1}{\sim} \frac{1}{16\pi^2} \Omega^2 \ln \Omega^2$$

$$\Gamma(\Omega_0^2, \Omega_0^2) = 0 \tag{III.21}$$

$$\frac{d\Gamma(\Omega^2, \Omega_0^2)}{d\Omega^2} \begin{array}{l} < 0 \\ > 0 \end{array} \quad \text{for} \quad \begin{array}{l} \Omega^2 < \Omega_0^2 \\ \Omega^2 > \Omega_0^2 \end{array}$$

$$\frac{d^2\Gamma(\Omega^2, \Omega_0^2)}{(d\Omega^2)^2} > 0$$

IV. THE GAUSSIAN EFFECTIVE POTENTIAL

Eq. (III.14) is a highly UV divergent expression, which is regularized by a cutoff Λ. The following steps render it explicitly finite[9]:

i) Minimize $\mathcal{E}(\phi_0, \Omega)$ for fixed ϕ_0: this gives $\Omega(\phi_0)$.

ii) Subtract the zero point energy by, for instance, the following procedure:

$$\mathcal{E}(\phi_0, \Lambda, m(\Lambda), \lambda(\Lambda)) \equiv \mathcal{E}(\phi_0, \Omega(\phi_0)) - \mathcal{E}(0, \Omega_0) \quad , \tag{IV.1}$$

where $\Omega_0 \equiv \Omega(0)$ and in the lhs the explicit Λ-dependence due to $I_n(\Omega^2)$ as well as the implicit of the bare parameters has been specified.

iii) Take, for the adequate large Λ behaviour of $m(\Lambda)$ and $\lambda(\Lambda)$ the limit $\Lambda \to \infty$ which then exists and leads to the socalled gaussian effective potential of the renormalized theory,

$$V_G(\phi_0) \equiv \lim_{\Lambda \to \infty} \mathcal{E}(\phi_0, \Lambda, m(\Lambda), \lambda(\Lambda)) \, . \tag{IV.2}$$

This is a nonperturbative approximation to the true effective potential.

It should be mentioned, however, that only under special circumstances (IV.2) will be an upper bound to the true vacuum energy density*. Indeed, as long as the cutoff Λ is finite and as long as the regularization does not introduce an indefinite metric into the Hilbert space,

$$\mathcal{E}_{true}(\phi_0, \Lambda, m(\Lambda), \lambda(\Lambda)) \leqslant \mathcal{E}(\phi_0, \Lambda, m(\Lambda), \lambda(\Lambda)) \tag{IV.3}$$

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* I owe this insight to M. Lüscher.

as expected from a variational method. Then also

$$\begin{aligned}&[\mathcal{E}_{true}(\phi_0,\Lambda,m(\Lambda),\lambda(\Lambda)) - \mathcal{E}_{true}(0,\Lambda,m(\Lambda),\lambda(\Lambda))]\\ &+[\mathcal{E}_{true}(0,\Lambda,m(\Lambda),\lambda(\Lambda)) - \mathcal{E}(0,\Lambda,m(\Lambda),\lambda(\Lambda))] \leq\\ &\mathcal{E}(\phi_0,\Lambda,m(\Lambda),\lambda(\Lambda)) - \mathcal{E}(0,\Lambda,m(\Lambda),\lambda(\Lambda))\end{aligned} \tag{IV.4}$$

holds. Let us now remove the regulator by taking the $\Lambda \to \infty$ limit. Then, and only if the limits of both square-bracketted quantities exist, does one obtain

$$V_{true}(\phi_0) + c \leq V_G(\phi_0) \tag{IV.5}$$

where c is a nonpositive unknown constant and $V_{true}(0) = V_G(0) = 0$. If $c = 0$, $V_G(\phi_0)$ is an upper bound to $V_{true}(\phi_0)$. If $c \neq 0$ then (IV.5) is still useful as far as existence studies are concerned, as unboundedness from below of $V_G(\phi_0)$ implies the same for $V_{true}(\phi_0)$. Under other circumstances $V_G(\phi_0)$ can only be considered an approximation to $V_{true}(\phi_0)$, but not a bound.

$V_G(\phi_0)$ as given by (IV.2) is explicitly finite when written in terms of the renormalized mass and coupling constant defined as

$$m_R^2 \equiv \left.\frac{d^2 V_G(\phi_0)}{d\phi_0^2}\right|_{\phi_0 = 0} \tag{IV.6}$$

and

$$\lambda_R \equiv \frac{1}{4!}\left.\frac{d^4 V_G(\phi_0)}{d\phi_0^4}\right|_{\phi_0 = 0}, \tag{IV.7}$$

where the renormalization is performed at the origin. It is also useful to introduce

$$m_R^2(\Lambda) \equiv \left.\frac{d^2\mathcal{E}(\phi_0,\Lambda,m(\Lambda),\lambda(\Lambda))}{d\phi_0^2}\right|_{\phi_0 = 0} \tag{IV.8}$$

and

$$\lambda_R(\Lambda) \equiv \frac{1}{4!}\left.\frac{d^4\mathcal{E}(\phi_0,\Lambda,m(\Lambda),\lambda(\Lambda))}{d\phi_0^4}\right|_{\phi_0 = 0} \tag{IV.9}$$

such that

$$\begin{aligned} m_R^2 &= \lim_{\Lambda\to\infty} m_R^2(\Lambda)\\ \lambda_R &= \lim_{\Lambda\to\infty} \lambda_R(\Lambda)\,. \end{aligned} \tag{IV.10}$$

We will in the following skip any details of the computations of $V_G(\phi_0)$ as they parallel similar computations of references 2 and 10. We will, however, give the main formulas and results in order to facilitate the understanding and the analysis. Let us consider separately the cases $m_R^2 > 0$, $m_R^2 < 0$ and $m_R^2 = 0$.

1. $\underline{m_R^2 > 0}$

From (III.14) having substituted Ω by $\Omega(\phi_0)$ one obtains through (IV.8,9)

$$\begin{aligned} m_R^2(\Lambda) &= \Omega_0^2 = m^2(\Lambda) + 12\lambda(\Lambda)\, I_0(m_R^2(\Lambda))\\ \lambda_R(\Lambda) &= \lambda(\Lambda)\,\frac{1 - 12\lambda(\Lambda)\, I_{-1}(m_R^2(\Lambda))}{1 + 6\lambda(\Lambda)\, I_{-1}(m_R^2(\Lambda))} \end{aligned} \tag{IV.11}$$

and

$$\begin{aligned} \mathcal{E}(\phi_0,\Lambda,m(\Lambda),\lambda(\Lambda)) &= \frac{1}{8}(\Omega^2(\phi_0) - m_R^2(\Lambda))^2\, I_{-1}(m_R^2(\Lambda))\\ &+ \Delta(\Omega^2(\phi_0), m_R^2(\Lambda)) - \frac{1}{2}(\Omega^2(\phi_0) - m_R^2(\Lambda))\,\Gamma(\Omega^2(\phi_0), m_R^2(\Lambda))\\ &+ \frac{1}{2} m_R^2(\Lambda)\phi_0^2 - 2\lambda(\Lambda)\phi_0^4 + 3\lambda(\Lambda)\Big[\phi_0^2 - \frac{1}{2}(\Omega^2(\phi_0) - m_R^2(\Lambda))\, I_{-1}(m_R^2(\Lambda))\\ &+ \Gamma(\Omega^2(\phi_0), m_R^2(\Lambda))\Big]^2, \end{aligned} \tag{IV.12}$$

where $m^2(\Lambda)$ has been traded for $m_R^2(\Lambda)$ by using (IV.11). Furthermore (III.16) has been used. It is seen in obtaining these results that a finite theory requires

$$1+6\lambda(\Lambda)\, I_{-1}(m_R^2(\Lambda)) > 0 \,. \tag{IV.13}$$

It can be assumed, without loss of generality, that $m(\Lambda)$ is tuned in such a way that $m_R^2(\Lambda)$ approaches m_R^2 without logarithmic subdominant terms, i. e.

$$m_R^2(\Lambda) \underset{\Lambda\gg 1}{\sim} m_R^2 + O\left(\frac{1}{\Lambda}\right) . \tag{IV.14}$$

As only logarithmic divergences appear in (IV.12,13) all the $m_R^2(\Lambda)$ can thus be substituted by their limit m_R^2.

Eq. (IV.13) allows the following possible large Λ behaviours of $\lambda(\Lambda)$:

a)

$$\lambda(\Lambda) = -\frac{1}{6 I_{-1}(m_R^2)}\left(1+\frac{1}{a\, I_{-1}(m_R^2)}\right) + O\left(\frac{1}{I_{-1}^3(m_R^2)}\right) \tag{IV.15}$$

with $a < 0$ and where $O\left(I_{-1}^{-3}(m_R^2)\right)$ just means subdominant terms. Eq. (IV.11) gives

$$\lambda_R = \frac{a}{2} < 0 \tag{IV.16}$$

and this with (IV.15) gives from (IV.12)

$$V_G(\phi_0) = \Delta\left(\Omega^2(\phi_0), m_R^2\right) - \frac{\left(\Omega^2(\phi_0) - m_R^2\right)^2}{16\lambda_R} + \frac{1}{2}\Omega^2(\phi_0)\phi_0^2 \,, \tag{IV.17}$$

which is explicitly finite. $\Omega(\phi_0)$ is either given by a solution of $\delta V_G(\phi_0) / \delta\Omega(\phi_0) = 0$, i.e.

$$4\lambda_R\left[\phi_0^2 + \Gamma(\Omega^2, m_R^2)\right] = \Omega^2 - m_R^2 \tag{IV.18}$$

or by

$$\Omega = 0 \tag{IV.19}$$

depending on which corresponds to the absolute minimum. Now, because of the properties of $\Gamma(\Omega^2, m_R^2)$ (see (III.21)) it is clear that as $\Omega^2 \geq 0$ eq. (IV.18) does not have a solution beyond a certain $\phi_0 = \phi_c$. Then necessarily for $\phi_0 \geq \phi_c$ (IV.19) applies and (IV.17) reads

$$V_G(\phi_0 \geq \phi_c) = \Delta(0, m_R^2) - \frac{m_R^4}{16\lambda_R} \tag{IV.20}$$

On the contrary in some neighborhood of $\phi_0 = 0$ eq. (IV.18) will be operative, as (IV.19) would lead to a ϕ_0-independent potential and thus to $m_R^2 = 0$, contrary to our assumption. This implies that (IV.20) has to be positive so that the absolute minimum is given by (IV.18) and not (IV.19) in some neighborhood of $\phi_0 = 0$. Thus

$$0 > \lambda_R > \frac{m_R^4}{16\,\Delta(0, m_R^2)} \,. \tag{IV.21}$$

Recall from (III.20) that $\Delta(0, m_R^2) < 0$. Also, where (IV.18) is relevant one finds

$$\frac{d\, V_G(\phi_0)}{d\phi_0} = \Omega^2(\phi_0)\,\phi_0 \,. \tag{IV.22}$$

A study of (IV.18) performed with the help of (III.20,21) shows that the gaussian effective potential is a function which starts with value 0 at $\phi_0 = 0$, increases according to (IV.22) until, at some value $\phi_0 = \phi_1 \leq \phi_c$, it goes over to a constant value given by (IV.20).

The theory is bounded from below and interacting.

b)

$$\lambda(\Lambda) = -\frac{a}{6 I_{-1}(m_R^2)} + O\left(\frac{1}{I_{-1}^2(m_R^2)}\right) \tag{IV.23}$$

with $a<1$ leads, through (IV.12) and minimization, to

$$\Omega^2(\Phi_0) = m_R^2 + O\left(\frac{1}{I_1(m_R^2)}\right) \tag{IV.24}$$

so that

$$V_G(\Phi_0) = \frac{1}{2} m_R^2 \Phi_0^2 \; . \tag{IV.25}$$

Obviously $\lambda_R = 0$ and the theory is trivial.

c)

$$\lambda(\Lambda) = a + O\left(\frac{1}{I_{-1}(m_R^2)}\right) \tag{IV.26}$$

with $a>0$ leads, following the same steps, to

$$\Omega^2(\Phi_0) = m_R^2 + \frac{2\Phi_0^2}{I_1(m_R^2)} + O\left(\frac{1}{I_1^2(m_R^2)}\right) , \tag{IV.27}$$

so that

$$V_G(\Phi_0) = \frac{1}{2} m_R^2 \Phi_0^2 - 2a\Phi_0^4 \tag{IV.28}$$

with $\lambda_R = -2a < 0$. The theory is unbounded from below.

2. $\underline{m_R^2 < 0}$

Now one obtains

$$\begin{aligned} &\Omega_0 = 0 \\ &m_R^2(\Lambda) = m^2(\Lambda) + 12\lambda(\Lambda) I_0(0) \equiv m_R^2 + O\left(\frac{1}{\Lambda}\right) \\ &\lambda_R(\Lambda) = \lambda(\Lambda) \end{aligned} \tag{IV.29}$$

and

$$\begin{aligned} \mathcal{E}(\Phi_0,\Lambda,m(\Lambda),\lambda(\Lambda)) = &-\frac{1}{8}\Omega^4(\Phi_0) I_{-1}(0) + \Delta(\Omega^2(\Phi_0),0) \\ &-\frac{1}{2}\left(\Omega^2(\Phi_0) - m_R^2\right)\left(\Phi_0^2 - \frac{1}{2}\Omega^2(\Phi_0) I_{-1}(0) + \Gamma(\Omega^2(\Phi_0),0)\right) \\ &+\frac{1}{2}\Omega^2(\Phi_0)\Phi_0^2 - 2\lambda(\Lambda)\Phi_0^4 + 3\lambda(\Lambda)\left(\Phi_0^2 - \frac{1}{2}\Omega^2(\Phi_0) I_{-1}(0) + \Gamma(\Omega^2(\Phi_0),0)\right)^2 . \end{aligned} \tag{IV.30}$$

Also, as before

$$1 + 6\lambda(\Lambda) I_{-1}(0) > 0 \; . \tag{IV.31}$$

This allows again three different behaviours.

a)

$$\lambda(\Lambda) = -\frac{1}{6 I_{-1}(0)}\left(1 - \frac{a}{I_1(0)}\right) + O\left(\frac{1}{I_{-1}^3(0)}\right) \tag{IV.32}$$

with $a>0$, which plugged into (IV.30) gives

$$\begin{aligned} V_G(\Phi_0) = \; &\Delta(\Omega^2(\Phi_0),0) + \frac{1}{2} m_R^2\left(\Phi_0^2 + \Gamma(\Omega^2(\Phi_0),0)\right) \\ &+\frac{1}{2}\Omega^2(\Phi_0)\Phi_0^2 - \frac{1}{4}\Omega^2(\Phi_0) m_R^2 I_{-1}(0) + \frac{a}{8}\Omega^4(\Phi_0) , \end{aligned} \tag{IV.33}$$

where, obviously,

$$\Omega(\Phi_0) = 0 \, , \tag{IV.34}$$

This gives

$$V_G(\Phi_0) = \frac{1}{2} m_R^2 \Phi_0^2 \, , \tag{IV.35}$$

which is unbounded from below.

b)

$$\lambda(\Lambda) = -\frac{a}{6\, I_{-1}(0)} + O\left(\frac{1}{I_{-1}^2(0)}\right) \qquad \text{(IV.36)}$$

with $a < 1$ leads to exactly the same conclusion.

c)

$$\lambda(\Lambda) = a + O\left(\frac{1}{I_{-1}(0)}\right) \qquad \text{(IV.37)}$$

with $a > 0$ gives

$$\Omega^2(\phi_0) = \frac{2\Phi_0^2}{I_{-1}(0)} + O\left(\frac{1}{I_{-1}^2(0)}\right) \qquad \text{(IV.38)}$$

so that

$$V_G(\phi_0) = \frac{1}{2} m_R^2 \phi_0^2 - 2a\, \phi_0^4 \qquad \text{(IV.39)}$$

with $\lambda_R = -2a < 0$. The theory is again unbounded.

3. $m_R^2 = 0$

This can be considered a limiting case of either $m_R^2 > 0$ or $m_R^2 < 0$. It leads at best to triviality.

It should be mentioned in ending this section that IR divergences appear in intermediate steps for the cases $m_R^2 \leq 0$ if $K = +1$ or 0. They are, however, harmless, i. e. they can be absorbed into the bare parameters by suitable regularization and renormalization [18], as is clear from the fact that the renormalized expressions are IR finite.

V. CONCLUSIONS

The whole analysis of the previous section has been performed without any reference to the structure of space-time. It is the same for all static, homogeneous and isotropic metrics, whether the space is closed, open or flat. Only for the unique case which leads to an interacting theory (1a of the previous section) is there a curvature dependence in the lower bound of (eq. (IV.21)) coming in through $\Delta(0, m_R^2)$. Indeed, for a flat space, as $\Delta^{(0)}(0, m_R^2) = -\frac{m_R^2}{128\pi^2}$,

$$-8\pi^2 < \lambda_R^{(0)} < 0 \qquad \text{(V.1)}$$

so that it does not depend on m_R^2. This is not surprising, as for $K = 0$ the only energy scale we have at our disposal is m_R (actually m_R/a, undoing the rescaling performed after (III.7)) so that $\lambda_R^{(0)}$ cannot depend on it. This is not so for $K = \pm1$, and indeed then $\Delta^{(K)}(0, m_R^2)$ are more complicated functions of m_R^2. Choosing $K = -1$ for which (III.19) provides an explicit expression,

$$\Delta^{(-1)}(0, m_R^2) = \frac{1}{128\pi^2}\left[-2\ln(1 + m_R^2) + 2m_R^2 - m_R^4\right], \qquad \text{(V.2)}$$

one sees that the lower bound of $\lambda_R^{(-1)}$ depends on m_R^2. This is an expected result, as for $K \neq 0$ a new scale, the curvature, enters into the problem. Thus it is a priori reasonable to expect that the limits on the coupling constant $\lambda_R^{(K\neq0)}$ depend on the ratio of the two scales at our disposal, basically what we call m_R. When the curvature tends towards zero, m_R goes towards infinity and

$$\Delta^{(K\neq0)}(0, m_R^2) \underset{m_R^2 \gg 1}{\sim} \Delta^{(0)}(0, m_R^2), \qquad \text{(V.3)}$$

as expected.

For large curvature, the relevant small m_R behaviour is given by (cf. (III.15,16,19))

$$\Delta^{(-)}(0, m_R^2) \underset{m_R^2 \ll 1}{\sim} - \frac{1}{192\pi^2} m_R^6$$
$$\Delta^{(+)}(0, m_R^2) \underset{m_R^2 \ll 1}{\sim} - \frac{3}{32\pi^2} m_R \qquad \text{(V.4)}$$

which implies that the lower bounds on λ_R tend to $-\infty$ and zero, respectively. At the large curvature limit the renormalized coupling of the open space theory can take any negative value, whereas for a closed space ϕ^4 theory it is zero. This in fact reflects asymptotic freedom.

As far as our approximations are reliable, our conclusion is that the issue of existence of $\lambda\phi^4$ is likely to be basically independent of the curvature of a static Robertson-Walker space-time, as expected from its UV character. The issue of interaction, however, looks more curvature dependent, inasmuch as the bounds on λ_R depend on it. A "smaller" space implies a weaker interaction.

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