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DESY 20-096
arXiv:2005.13547
May 2020

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ISSN 0418-9833

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The Superconformal χ ing Equation

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ABSTRACT: Crossing symmetry provides a powerful tool to access the non-perturbative dynamics of conformal and superconformal field theories. Here we develop the mathematical formalism that allows to construct the crossing equations for arbitrary four-point functions in theories with superconformal symmetry of type I, including all superconformal field theories in $d = 4$ dimensions. Our advance relies on a supergroup theoretic construction of tensor structures that generalizes an approach which was put forward in [1] for bosonic theories. When combined with our recent construction of the relevant superblocks, we are able to derive the crossing symmetry constraint in particular for four-point functions of arbitrary long multiplets in all 4-dimensional superconformal field theories.

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1 Introduction

Conformal field theories describe very special points in the space of quantum field theories that seem to provide unique views into non-perturbative dynamics through a variety of rather complementary techniques, such as holography, integrability, localization and the conformal bootstrap. One of the principal analytical tools for conformal field theory are conformal partial wave (or block) expansions

that were proposed early on in [2]. The role they play in the study of models with conformal symmetry is very similar to the role of Fourier analysis in systems with translational symmetry. While conformal blocks are entirely determined by kinematics, they allow to separate very neatly the dynamical meat of a theory from its kinematical bones. For example, an N -point function of local operators in a conformal field theory can be a very complicated object. If expanded in conformal blocks, however, the coefficients factorize into a set of three-point couplings, i.e. most of the complicated dependence on the insertion points resides in the kinematical skeleton of a conformal field theory. This is the reason conformal blocks expansions are so important.

Conformal blocks for four-point functions of local operators in bosonic conformal field theories are relatively well studied by now, see e.g. [3–18] and references therein. On the other hand, while we know of many examples of such theories in $d = 3$ dimensions, most conformal field theories in $d \geq 4$ seem to possess supersymmetry. The enhancement from conformal to superconformal symmetry should lead to simplifications, at least once the kinematical aspects are well under control. This, however, is not yet the case. In fact, while four-point blocks of half-BPS operators or the superprimary components of more general supermultiplets have been constructed and applied, see e.g. [19–38], relatively little is actually known about blocks and block expansions for more generic external multiplets that span long(er) representations of the superconformal algebra. On the other hand it has been shown in [39] that the bootstrap with long multiplets is significantly more constraining on CFT data than the bootstrap with e.g. external BPS operators, see also [40]. This provides strong motivation to investigate blocks and crossing symmetry for long multiplets, which is the main goal of our work.

In order to explain the main results of this paper, let us briefly review a few basic facts about conformal partial wave expansions in bosonic conformal field theories. We start from some four-point correlator $G(x_i)$ with its full dependence on the insertion points x_i of the fields. As is well known, conformal symmetry implies that $G(x_i)$ is fully determined by a some function of the two cross ratios u, v one can form from four points in \mathbb{R}^d . More precisely, it is possible to write the correlation function G as

$$G(x_i) = \Omega(x_i)g(u, v) . \tag{1.1}$$

We stress that such a behavior is not restricted to scalar correlation functions. If the fields carry spin, then G takes values in the space of polarizations of the four fields. The function g , on the other hand takes values in the space of four-point tensor structures whose dimension is smaller than that of the space of polarizations, in general, at least for $d > 3$. Hence, one should think of Ω as a rectangular matrix. We shall refer to such a matrix valued function Ω of the insertion points as four-point *tensor*

factor. In some sense to become clear below it combines all four-point tensor structures into one single object Ω . Many authors have studied tensor structures for spinning four-point functions in conformal field theories, see e.g. [6, 41–46].

The tensor factor $\Omega(x_i)$ is restricted but not determined by conformal symmetry. In fact, there is some obvious ‘gauge’ freedom that is associated with matrix-valued functions $\zeta(u, v)$ one can move back and forth between the tensor factor Ω and the function $g(u, v)$, i.e. the gauge symmetry acts as $(\Omega, g) \rightarrow (\Omega\zeta^{-1}, \zeta g)$. The function g of the cross ratios may be expanded in terms of conformal partial waves which, after the influential work of Dolan and Osborn [3, 4], are characterised as eigenfunctions of the so-called Casimir differential equations. The form of these equations, however, depends on the gauge choice that is made when splitting G into Ω and g . For four-point functions of identical scalar fields of weight Δ_0 , for example, Dolan and Osborn chose $\Omega_s = x_{12}^{-2\Delta_0} x_{34}^{-2\Delta_0}$. Note that this factor $\Omega = \Omega_s$ also depends on a split of the four points into two sets of two, a choice usually referred to as a channel. Here we have displayed the factor Ω for the so-called s -channel. The t -channel is obtained by exchanging the fields inserted at x_2 and x_4 . With their pick of Ω_s , Dolan and Osborn worked out the associated Casimir differential equation for the function g_s and similarly for g_t . Solutions of these Casimir equations provided them with a set of blocks $g_{s/t}^{\Delta, l}(u, v)$ in which one can then expand g_s and g_t

$$G(x_i) = \Omega_s(x_i) \sum p_{\Delta, l} g_s^{\Delta, l}(u, v) = \Omega_t(x_i) \sum p_{\Delta, l} g_t^{\Delta, l}(u, v) . \quad (1.2)$$

The equality between the first and second sum is the famous crossing symmetry equation. An important observation is that writing this equation does actually not require a complete knowledge of the tensor factors. It is sufficient to know the ratio of the s - and t -channel Ω

$$M(u, v) = \Omega_t^{-1}(x_i) \Omega_s(x_i) = \left(\frac{v}{u}\right)^{\Delta_0} ,$$

which is a function of the two cross ratios only. We call this important object M the *crossing factor* M . In the case of spinning fields the crossing factor becomes a matrix. This ratio of s - and t -channel tensor factors is not to be confused with the crossing or fusing matrix of the conformal group. While the crossing factor relates the s - and t -channel tensor factors, the crossing matrix relates the conformal blocks in the two channels by providing the expansion coefficients of s -channel blocks in terms of t -channel ones, see [47–49] for some recent discussion in the context of higher dimensional conformal field theory.

In [50] it was noticed that scalar four-point functions G admit an different gauge choice for the factor Ω such that the associated Casimir equations take the form of an eigenvalue equation for an

integrable 2-particle Hamiltonian of Calogero-Sutherland type. This was later explained in [11, 51] through harmonic analysis on the conformal group and then extended to fields with spin in which case the quantum mechanical potential becomes matrix valued. For spinning fields, the tensor structures of such a Calogero-Sutherland gauge were constructed recently in [1]. The goal of our work is to extend all this to the case of superconformal symmetry. In [52] we have constructed the Casimir equations for superconformal symmetries of type I. The form of these equations allows us to compute superblocks systematically as finite sums of spinning bosonic blocks. What was missing up to now is the construction of the associated tensor structures and in particular the crossing factor M . Below we fill this gap and construct both the tensor structures and the crossing factor for all superconformal algebras of type I. Explicit formulas for the crossing factors in 4-dimensional superconformal algebras will be given in our forthcoming paper [53]. Early work on tensor structures for four-point correlators of superconformal field theories includes [21, 54–58].

Let us now describe the plan of this work in more detail. The next section contains some basic background material on superconformal algebras, where we introduce the notion of superspace and discuss the infinitesimal and global action of the conformal symmetry thereon. Special attention will be paid to the action of the so-called Weyl inversion, which plays an important role in later sections. Section 3 contains the first new result of this work. There we construct a special family $g(x_i)$ of supergroup elements that depend on the insertion points of the fields along with a matrix realization that uniquely encodes the correlation function $G(x_i)$. This generalizes a similar formula for bosonic conformal field theories in [1] to the supersymmetric setup. In section 4 we begin to specialize the discussion to superconformal algebras of type I, i.e. to cases in which the R-symmetry group contains an abelian factor $U(1)$. After introducing super Cartan coordinates through a particular KAK factorization of the superconformal group we can construct the tensor factors Ω for any choice of spins and any channel through an elegant group theoretical construction. This then allows us to build the crossing factor M as a quotient of s - and t -channel tensor factors and prove its conformal invariance explicitly. Our main result, which is stated in eqs. (4.13), (4.42), expresses the crossing factor M through representation matrices of some particular family of elements of the group K that is generated by dilations, rotations and R-symmetry transformations. All constructions in section 2-4 are illustrated at the example of $\mathfrak{g} = \mathfrak{sl}(2|1)$ of the $\mathcal{N} = 2$ superconformal algebra in $d = 1$ dimensions. Let us also note that our discussion includes the purely bosonic case $\mathcal{N} = 0$ for which the crossing factor was not constructed previously beyond a few special spin assignments. As a corollary to our discussion we state the crossing factor for arbitrary spinning four-point functions in 3-dimensional conformal field

theories. For all other higher dimensional examples, bosonic as well as supersymmetric, our results for the crossing factor are stated in the form of a precise easy-to-follow algorithm. In order to obtain ready-to-use formulas one needs to input some classical results from the group theory of rotations $SO(d)$. We will discuss this for certain mixed correlators in $\mathcal{N} = 1$ superconformal theories in an accompanying work [53].

2 Superspace and Superconformal Symmetry

In order to state and prove our main results we need some background on supergroups, superspaces and the action of superconformal symmetry thereon. Here we want to review these concepts and at the same time introduce a mathematical language that is appropriate for our subsequent discussion. In particular, we recall the notion of superspace in the second subsection and explain how one constructs an infinitesimal action of the superconformal algebra thereon. This action is lifted to global transformations in the third subsection, with some special focus on the so-called Weyl inversion, a close relative of the conformal inversion which is guaranteed to exist in any superconformal field theory. For more mathematical minded readers we have incorporated a more abstract and introductory subsection on the concept of supergroups. While this helps to make equations in subsequent subsections mathematically rigorous, readers who feel familiar with supergroups and superspaces are encouraged to skip the first subsection, at least upon first reading.

2.1 Some basics on superalgebras and supergroups

In this subsection we introduce some very basic notions and notations concerning supergroups. Our conventions agree with [59–61]. Let \mathfrak{h} be some Lie superalgebra, i.e. a graded vector space $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ with a graded Lie bracket. We denote the latter by $[\cdot, \cdot]_{\pm}$. The associated *universal enveloping algebra* $U(\mathfrak{h})$ is the graded associative algebra generated by elements $X \in \mathfrak{h}$, with relations such that graded commutators are given by the Lie bracket. In a slight abuse of notations we shall denote the graded commutators in the universal enveloping algebra by $[\cdot, \cdot]_{\pm}$ as well.

The universal enveloping algebra comes equipped with a co-product Δ , i.e. with a homomorphism

$$\Delta : U(\mathfrak{h}) \rightarrow U(\mathfrak{h}) \otimes U(\mathfrak{h}) .$$

Here, the tensor product is to be understood in the graded sense, i.e. elements are multiplied as

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{|a_2||b_1|} a_1 a_2 \otimes b_1 b_2 ,$$

where $|a| = 0$ if a is even and $|a| = 1$ if a is odd, as usual. On the generating elements $X \in \mathfrak{h} \subset U(\mathfrak{h})$, the co-product is given by

$$\Delta(X) = X \otimes 1 + 1 \otimes X . \quad (2.1)$$

From here one can extend Δ uniquely to the entire universal enveloping algebra as a homomorphism of graded algebras. The co-product is the algebraic structure that allows us to build tensor products of any two representations of the Lie superalgebra \mathfrak{h} or its universal envelop $U(\mathfrak{h})$.

Let us now turn to another algebra that we can associate to \mathfrak{h} , namely the so-called *structure algebra* $\mathcal{F}(\mathfrak{h})$. By definition, \mathcal{F} is a graded commutative algebra whose generators x_A are associated to the basis elements X^A of the Lie superalgebra \mathfrak{h} . The elements x_A possess the same degree $|x_A| = |A|$ as the generators X^A , i.e. x_A is an ordinary bosonic variable if X^A is even while x_A is a Grassmann variable in case X^A is odd. From the construction we have sketched here it is evident that \mathcal{F} can be thought of as the *algebra of functions* on the supergroup associated with \mathfrak{h} which is generated here from set of coordinate functions, one for each element of the Lie superalgebra.

The two algebras we have associated to \mathfrak{h} up to now are actually closely related. In the case of bosonic groups, the generators X of the Lie algebra give rise to (right) invariant vector fields that act on functions as some first order differential operators. These differential operators \mathcal{R}_X can be multiplied and added and thereby provide an action of elements a in the universal enveloping algebra $U(\mathfrak{h})$ through differential operators \mathcal{R}_a of higher order. One may combine the application of any such differential operator to a function on the group with the evaluation at the group unit e to obtain a map that assigns as number

$$\mathcal{R}_a(f)(e) = (a, f) = f(a) \in \mathbb{C} \quad (2.2)$$

to a pair of an element $a \in U(\mathfrak{h})$ and a (complex valued) function f on the group. In other words, elements of $U(\mathfrak{h})$ give linear functionals of the algebra of functions or structure algebra $\mathcal{F}(\mathfrak{h})$ and vice versa. In this form, the statement remains true for Lie superalgebras and is often expressed by saying that $\mathcal{F}(\mathfrak{h})$ and $U(\mathfrak{h})$ are dual to each other, see also [62] for a nice discussion of this point.

Equipped with these two algebraic structures, namely the universal enveloping algebra $U(\mathfrak{h})$ and the structure algebra $\mathcal{F}(\mathfrak{h})$, we want to introduce the concept of *supergroup elements* h . Let us first give a formal definition according to which h is an even element of the graded tensor product $U(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})$ that satisfies

$$(\Delta \otimes id)h = \overset{1}{h} \overset{2}{h} . \quad (2.3)$$

Here, the application of the co-product Δ to the first tensor factor of h produces an element in $U(\mathfrak{h}) \otimes U(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})$. The factors on the right hand side are elements in the same threefold tensor product. More concretely, $\overset{2}{h}$ is the element $1 \otimes h$ with trivial entry in the first tensor factor. Similarly $\overset{1}{h}$ denotes the element h with trivial entry in the second tensor factor.

The element h is not uniquely characterized by these properties, but we do not need to be more specific. It might be helpful to think of h as the object $h = \exp(x_A X^A)$. The element $x_A X^A$ in the exponent is even and upon expansion of the exponential provides us with an even element in the graded tensor product $U(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})$. In order to construct this element one moves all the elements x_A of the structure algebra to the right of the superalgebra generators X^B using

$$x_A X^B = (-1)^{|A||B|} X_B x_A ,$$

which implements our convention to consider the graded tensor product of $U(\mathfrak{h})$ and $\mathcal{F}(\mathfrak{h})$ rather than the ordinary one. After the reordering we indeed obtain an infinite sum of products between elements in the universal enveloping algebra $U(\mathfrak{h})$ with elements of the structure algebra $\mathcal{F}(\mathfrak{h})$. If we apply the co-product in the universal enveloping algebra we formally obtain

$$(\Delta \otimes id)h = e^{x_A(X^A \otimes 1 + 1 \otimes X^A)} = e^{x_A(X^A \otimes 1)} e^{x_A(1 \otimes X^A)} = \overset{1}{h} \overset{2}{h} . \quad (2.4)$$

In writing the single exponential as a product of exponentials we used the fact that the exponent is an even object so that $x_A(X^A \otimes 1)$ commutes with $x_A(1 \otimes X^A)$. In conclusion, we have constructed an object h with the properties we demanded in the previous paragraph, at least formally. In physics, it is customary to evaluate h in some representation π of the Lie superalgebra \mathfrak{h} or, equivalently, its universal enveloping algebra. Thereby one obtains a finite dimensional supermatrix $h^\pi = (\pi \otimes id)h$ with entries from the structure algebra \mathcal{F} . In the following we often use the symbol h for such a matrix h rather than the universal element $h \in U(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})$.

What we have explained so far actually suffices as background for most of our discussion below, except for the construction of an infinitesimal action of the conformal superalgebra on superspace in the next subsection. To obtain explicit formulas for the first order derivative operators \mathcal{R}_X that are associated with the elements $X \in \mathfrak{h}$ let us first extend the structure algebra $\mathcal{F}(\mathfrak{h})$ of “functions on the supergroup” to a differentially graded algebra $d\mathcal{F}(\mathfrak{h})$ of “differential forms on the supergroup”. The latter is a bi-graded commutative algebra generated by elements x_A and dx_A , with a second grading associated to the form degree. On the algebra $d\mathcal{F}(\mathfrak{h})$ we can define a differential d that squares to zero $d^2 = 0$ and

satisfies the graded Leibniz rule

$$d(f \wedge g) = df \wedge g + (-1)^{\deg(f)} f \wedge dg .$$

Here $\deg(f)$ denotes the form degree of f . Let us stress that there is no additional sign associated with the \mathbb{Z}_2 grading that distinguishes between even (bosonic) and odd (fermionic) elements. This means that d is treated as an even object. Hence, for a given A , x_A and dx_A possess the same degree, i.e. dx_A is even [odd] in case x_A is even [odd].

Since the structure algebra $\mathcal{F}(\mathfrak{h})$ is contained in the larger differentially graded algebra $d\mathcal{F}(\mathfrak{h})$ we can also think of the supergroup element $h \in U(\mathfrak{h}) \otimes \mathcal{F}(\mathfrak{h})$ as an element of the differential graded algebra $U(\mathfrak{h}) \otimes d\mathcal{F}(\mathfrak{h})$ with the additional rule that $dX^A = X^A d$, i.e. we consider the generators X^A of the Lie superalgebra as constants and the differential d as even. Now it makes sense to consider the Maurer-Cartan form

$$dhh^{-1} \in U(\mathfrak{h}) \otimes d\mathcal{F}(\mathfrak{h}) . \quad (2.5)$$

If we apply the differential to the equation (2.3) that characterizes h we obtain

$$\Delta(dhh^{-1}) = \left(\begin{matrix} 1 & 2 \\ dh & h \end{matrix} + \begin{matrix} 1 & 2 \\ h & dh \end{matrix} \right) \begin{matrix} 2 \\ h^{-1} \end{matrix} \begin{matrix} 1 \\ h^{-1} \end{matrix} = \begin{matrix} 1 & 1 \\ dh & h^{-1} \end{matrix} + \begin{matrix} 2 & 2 \\ dh & h^{-1} \end{matrix} . \quad (2.6)$$

We conclude that the Maurer-Cartan form takes values in the Lie superalgebra $\mathfrak{h} \subset U(\mathfrak{h})$, as it is the case for usual bosonic Lie groups. Consequently, it may be expanded as

$$dhh^{-1} = dx_A C_{AB} X^B \quad \text{where} \quad C_{AB} \in \mathcal{F}(\mathfrak{h}) . \quad (2.7)$$

The matrix elements C_{AB} possess degree $|A| + |B|$, i.e. they are even elements of the structure algebra if $|A| = |B|$ and odd otherwise. We also stress that the elements C_{AB} depend on the choice of the supergroup element h . One of the main uses of the matrix elements C_{AB} is to construct the right-invariant vector fields, i.e. an action of the Lie superalgebra \mathfrak{h} through first order differential operators acting on the structure algebra $\mathcal{F}(\mathfrak{h})$. These vector fields are given by

$$\mathcal{R}_{X^A} = \mathcal{R}_A := \mathcal{C}_{AB}^G \partial_B ,$$

where $\mathcal{C} = C^{-1}$ denotes the inverse of C and ∂_B is the (graded) derivative with respect to the coordinate x_B . Its action on an arbitrary function $f \in \mathcal{F}(\mathfrak{h})$ can be read off from $df = dx_B (\partial_B f)$. In particular, when acting on the individual coordinate functions, x_A obeys $(\partial_B x_A) = \delta_{A,B}$. The action of partial derivatives on products of functions satisfies the graded Leibniz rule which implies that

$$\partial_B x_A = (\partial_B x_A) + (-1)^{|A||B|} x_A \partial_B = \delta_{A,B} + (-1)^{|A||B|} x_A \partial_B . \quad (2.8)$$

Since we have assumed that the differential d acts trivially on the generators X^A of the universal enveloping algebra, i.e. $(dX^A) = 0$ we conclude that $\partial_\beta X^A = 0$, i.e. the generators X^A are constant objects on the supergroup satisfying

$$\partial_B X^A = (-1)^{|A||B|} X^A \partial_B . \quad (2.9)$$

With this list of properties of the partial derivatives we conclude our construction of the right invariant vector fields (2.8) and thereby our short mathematical review of superalgebras and the theory of supergroups. The formulation we have introduced here is well adapted to our needs below and also paves the way for some interesting extensions, see the concluding section.

2.2 Superspace and the infinitesimal action of superconformal symmetry

This subsection serves two purposes. On the one hand we need to introduce the notion of superspace that is one of the crucial ingredients throughout the rest of the paper. In addition we shall also construct an action of the superconformal algebra \mathfrak{g} through “first differential operators” on superspace. This infinitesimal action of the superconformal symmetry on superspace will play only a minor role below since most of our analysis is based on global transformations.

To set up notations let us denote the superconformal algebra by \mathfrak{g} . Its bosonic subalgebra \mathfrak{g}_0 consists of d -dimensional conformal transformations in $\mathfrak{so}(1, d+1)$ as well as R-symmetry transformations in some Lie algebra \mathfrak{u} . To define superspace we pick some decomposition

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{p}$$

of \mathfrak{g} into two Lie subalgebras \mathfrak{p} and \mathfrak{m} . The standard choice would be to define \mathfrak{p} as the span of all elements in \mathfrak{g} that lower the eigenvalue of the dilation generator $D \in \mathfrak{g}_0$, i.e.

$$\mathfrak{p} := \mathfrak{g}_{\leq 0} = \text{span}(X \in \mathfrak{g} \mid [D, X] = \alpha X, \alpha \leq 0) .$$

For this choice, \mathfrak{m} then consists of generators P of translations and the supercharges Q . We shall briefly comment on other choices below. We also choose a basis X^A of elements in \mathfrak{g} that is compatible with the decomposition (2.10). Elements X^A that lie in the subspace \mathfrak{m} will be labeled by lower case Latin indices while those that lie in the complement \mathfrak{p} carry Greek indices.

The decomposition of the Lie superalgebra \mathfrak{g} into \mathfrak{m} and \mathfrak{p} determines a decomposition of the corresponding universal enveloping algebra $U(\mathfrak{g}) = U(\mathfrak{m}) \otimes U(\mathfrak{p})$ as well as of the structure algebra $\mathcal{F}(\mathfrak{g}) = \mathcal{F}(\mathfrak{m}) \otimes \mathcal{F}(\mathfrak{p})$. Recall that the structure algebras $\mathcal{F}(\mathfrak{m})$ and $\mathcal{F}(\mathfrak{p})$ are generated by the coordinates x_a and x_α , respectively, with x_a and x_α being Grassmann variables if the corresponding

elements X^a and X^α are fermionic generators of the Lie superalgebra. The structure algebra $\mathcal{F}(\mathfrak{m})$ is what is referred to a *superspace* $\mathcal{M} = \mathcal{F}(\mathfrak{m})$. Loosely speaking one may think of it as the algebra of “functions on the supergroup M ”, though we have not defined what we mean by a supergroup and do not intend to do so.

Now that we know what superspace is let us construct an infinitesimal action of the superconformal symmetry thereon. Here we shall closely follow the general constructions we outlined in the previous subsection and introduce supergroup elements $m = m(x_a)$ and $p = p(x_\alpha)$. In case of m we work with the following standard choice

$$m(x_a) = e^{x_a X^a}. \quad (2.10)$$

The infinitesimal action of the conformal algebra on the coordinates x_a of our superspace descends from the left-regular action of \mathfrak{g} and thus can be computed from the Maurer-Cartan form,

$$dgg^{-1} = dx_A C_{AB}^G X^B. \quad (2.11)$$

In computing the Maurer-Cartan form for \mathfrak{g} it is usual to relate it to the Maurer-Cartan forms that are associated with \mathfrak{m} and \mathfrak{p}

$$dmm^{-1} = dx_a C_{ab}^M X^b, \quad dpp^{-1} = dx_\alpha C_{\alpha\beta}^P X^\beta.$$

With our choice $g = mp$ of the supergroup element g as a product of the two elements m and p it follows that

$$\begin{aligned} dgg^{-1} &= dx_A \partial_A (mp)(mp)^{-1} = dx_a (\partial_a m)m^{-1} + dx_\alpha m(\partial_\alpha p)p^{-1}m^{-1} = \\ &= dx_a C_{ab}^M X^b + dx_\alpha m C_{\alpha\beta} X^\beta m^{-1} = dx_a C_{ab}^M X^b + dx_\alpha C_{\alpha\beta}^P \left((M_1)_{\beta a} X^a + (M_2)_{\beta\gamma} X^\gamma \right). \end{aligned} \quad (2.12)$$

The last equality defines the two matrices $M_{1,2}$,

$$mX^\beta m^{-1} = (M_1)_{\beta a} X^a + (M_2)_{\beta\gamma} X^\gamma. \quad (2.13)$$

From the equation (2.12) we can read off the coefficients C_{AB}^G of the Maurer-Cartan form for \mathfrak{g} . The inverse \mathcal{C}^G of this matrix is easily seen to take the form

$$\mathcal{C}^G = \begin{pmatrix} \mathcal{C}^M & 0 \\ -M_2^{-1} M_1 \mathcal{C}^M & M_2^{-1} \mathcal{C}^P \end{pmatrix},$$

where the first row/column corresponds to direction in \mathfrak{m} while the second row/column collects all the directions in \mathfrak{p} . As stated before, the matrix \mathcal{C}^G provides us with the right-invariant vector fields (2.8)

on the conformal supergroup. To project these operators to the superspace one simply sets $\partial_\alpha = 0$,

$$\mathcal{R}^{(M)} = \begin{pmatrix} \mathcal{C}^M & 0 \\ -M_2^{-1}M_1\mathcal{C}^M & M_2^{-1}\mathcal{C}^P \end{pmatrix} \begin{pmatrix} \partial \\ 0 \end{pmatrix} = \begin{pmatrix} \mathcal{C}_{ab}^M\partial_b \\ -(M_2^{-1}M_1\mathcal{C}^M)_{\alpha b}\partial_b \end{pmatrix}.$$

This is the main result of this subsection. As mentioned above, the differential operators on superspace depend on \mathcal{C}^M and hence on the choice of the supergroup element m . The choice of the supergroup element p , on the other hand, is irrelevant since the coefficients \mathcal{C}^P of the Maurer-Cartan form dpp^{-1} dropped out in the last step when we set all derivatives ∂_α to zero.

Our result (2.14) applies to all decompositions of \mathfrak{g} into two Lie subalgebras \mathfrak{m} and \mathfrak{p} . As we pointed out in the first paragraph, the standard choice is to take \mathfrak{p} to contain generators that do not increase the conformal weight. In that case, the structure algebra $\mathcal{M} = \mathcal{F}(\mathfrak{m})$ is called the standard superspace. If the superconformal algebra \mathfrak{g} is of type I, however, there exist other natural choices to which the constructions of this subsection apply. In a type I superalgebra the R-symmetry contains a $U(1)$ subalgebra which commutes with all bosonic generators but assigns the fermionic ones a non-trivial R-charge ± 1 . As usual, we can decompose the Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\leq 0} \oplus \mathfrak{g}_{> 0}$ by splitting off those generators in $\mathfrak{g}_{> 0}$ that strictly increase the conformal weight. These consist of supercharges Q and generators of translations. In a type I superalgebra we can now split the space \mathfrak{q} or supercharges Q according to the sign of their $U(1)$ R-charge as $\mathfrak{q} = \mathfrak{q}_+ \oplus \mathfrak{q}_-$. With this in mind we can introduce two new decompositions $\mathfrak{g} = \mathfrak{m}_\pm \oplus \mathfrak{p}_\pm$ of the superconformal algebra where

$$\mathfrak{p}_\pm = \mathfrak{g}_{\leq 0} \oplus \mathfrak{q}_\pm, \quad \mathfrak{m}_\pm = \mathfrak{g}_1 \oplus \mathfrak{q}_\mp = \mathfrak{g}/\mathfrak{p}_\pm.$$

From the properties of type I Lie superalgebras, one may easily show that both \mathfrak{p}_\pm and \mathfrak{m}_\pm are subalgebras of \mathfrak{g} . The associated superspaces $\mathcal{M}_\pm = \mathcal{F}(\mathfrak{m}_\pm)$ are called the chiral and anti-chiral superspace, respectively.

Example: As an example, let us illustrate the construction of superspace and the differential operators in the case of the 1-dimensional $\mathcal{N} = 2$ superconformal algebra $\mathfrak{g} = \mathfrak{sl}(2|1)$. The smallest faithful representation of \mathfrak{g} is 3-dimensional. We may choose the generators as

$$D = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

for the four bosonic generators and

$$Q_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Q_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad S_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

for the fermionic ones. Here we shall consider the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{p}$ with the Lie superalgebra \mathfrak{m} spanned by P, Q_+ and Q_- . The corresponding superspace \mathcal{M} is generated by one bosonic variable u along with two Grassmann variables θ and $\bar{\theta}$. In this case the supergroup element m we introduced above takes the following matrix form

$$m(x) = e^{uP + \theta Q_+ + \bar{\theta} Q_-} = \begin{pmatrix} 1 & X & \theta \\ 0 & 1 & 0 \\ 0 & -\bar{\theta} & 1 \end{pmatrix}, \quad (2.14)$$

where $X = u - \frac{1}{2}\theta\bar{\theta}$ and $x = (u, \theta, \bar{\theta})$ represents the three generators of the structure algebra.

The construction we outlined above provides us with an action of the superconformal algebra \mathfrak{g} on this superspace with differential operators $\mathcal{R}_X = u$ of the form

$$p = \partial_u, \quad k = -u^2\partial_u - u\theta\partial_\theta - u\bar{\theta}\partial_{\bar{\theta}}, \quad (2.15)$$

$$d = u\partial_u + \frac{1}{2}\theta\partial_\theta + \frac{1}{2}\bar{\theta}\partial_{\bar{\theta}}, \quad r = \theta\partial_\theta - \bar{\theta}\partial_{\bar{\theta}}, \quad (2.16)$$

$$q_+ = \partial_\theta - \frac{1}{2}\bar{\theta}\partial_u, \quad q_- = \partial_{\bar{\theta}} - \frac{1}{2}\theta\partial_u, \quad (2.17)$$

$$s_+ = -(u + \frac{1}{2}\theta\bar{\theta})q_+, \quad s_- = (u - \frac{1}{2}\theta\bar{\theta})q_-. \quad (2.18)$$

As we pointed out in our discussion above, the choice of p is not relevant for the final result. We encourage the reader to derive these explicit expressions from our general formula (2.14).

2.3 Global superconformal symmetry and Weyl inversions

Having constructed superspace along with an action of the superconformal algebra thereon, our next task is to construct the action of global conformal transformations. As we shall see in a moment, most of the global transformations act in an obvious way. The only exception are special conformal transformations. For bosonic conformal symmetry, the easiest way to construct these is through the conformal inversion of translations. We follow essentially the same strategy in the supersymmetric context, except we need to replace the conformal inversion by a closely related Weyl inversion. The latter extends nicely to superconformal algebras while conformal inversions may not actually exist, see below.

Defining the action of global conformal transformations on superspace requires a little bit of preparation. We shall think of a global symmetry transformation as being associated to a supergroup element $h = h(s)$. We may consider h as a matrix whose matrix elements are functions on the supergroup, i.e. elements of the structure algebra generated by the coordinates s_a and s_α . The graded commutative

algebra that is generated by these coordinates is just another copy of the algebra that is generated by x_a and x_α . From now on we shall suppress the dependence on s again. The left action of such an element h on the supergroup element $g(x) = m(x_a)p(x_\alpha)$ is simply given by the left multiplication $g(x) \mapsto hg(x)$. In order to obtain the action on superspace, we need to factorize $hg(x)$ as

$$hg(x) = m(y(x, h))p(x, h) = e^{y(x, h)_a X^a} p(x, h) . \quad (2.19)$$

This factorization defines the h transform $h(x)_a$ of the superspace coordinates x_a . Note that $y(x, h)_a$ are elements in the tensor product of two structure algebras, the one generated by coordinates x and the one that is generated by s . It is particularly easy to apply this definition to rotations, dilations and R-symmetries since these form a subgroup K that respects the split of \mathfrak{g} into \mathfrak{m} and \mathfrak{p} . In fact, the Lie algebra \mathfrak{k} is even a subalgebra of \mathfrak{p} . In order to factorize

$$kg(x) = km(x)p(x) = m(y(x, k))p(x, k)$$

for some $k \in K^{-1}$ all we need to do is move k through m . Since the generators X^a transform in some representation κ of K , the effect can be captured by a linear transformation of the coordinates x_a , i.e. $y(x, k)_a = \kappa_{ab}(k)x_b$. Also (super-)translations are easy to discuss. These are associated with elements $h(c) = m(c)$ so that multiplication of h with $g(x)$ only requires to multiply $m(c)m(x) = m(y(x, c))$. Since bosonic translations commute among each other and with the supercharges Q , the only non-trivial terms in the product $m(c)m(x)$ come from the non-vanishing anti-commutators of the supercharges. But these can be evaluated easily in concrete examples and hence the computation of $c(x)$ is straightforward.

It now remains to discuss the action of special (super-)conformal transformations. We will not discuss these directly but instead focus on one particular global superconformal transformation, namely the superconformal extension of the Weyl inversion w . As we shall see, this Weyl inversion relates special super conformal transformations to supertranslations, just as in the bosonic case.

Before we enter the discussion of the Weyl inversion, let us briefly recall how the ordinary inversion of conformal field theories is constructed. By definition, the *conformal group* is a Lie group with $\mathfrak{g} = \mathfrak{so}(d+1, 1)$ as its Lie algebra. Let $O(d+1, 1)$ be the group of pseudo-orthogonal matrices. Its identity component is denoted by $SO^+(d+1, 1)$. This group can be realised as the quotient

$$SO^+(d+1, 1) = Spin(d+1, 1)/\mathbb{Z}_2$$

¹Here we assume that all the matrix elements are constant functions on the supergroup, i.e. they are proportional to the identity element of the structure algebra.

of the universal covering group $Spin(d+1, 1)$ by its centre. Both $SO^+(d+1, 1)$ and $Spin(d+1, 1)$ act on the compactified Euclidean space, but only the first action is faithful. In the case of $Spin(d+1, 1)$, both elements of the centre act trivially. Obviously, both $SO^+(d+1, 1)$ and $Spin(d+1, 1)$ possess the same Lie algebra $\mathfrak{g} = \mathfrak{so}(d+1, 1)$. The conformal inversion

$$Ix^\mu = \frac{x^\mu}{x^2}$$

is an element of $O(d+1, 1)$, but it resides in a component that is not connected to the identity component, i.e. the conformal inversion I is not an element of $SO^+(d+1, 1)$. We can improve on this issue by multiplying the inversion with some spatial reflection. The so-called Weyl inversion $w = s_{e_d} \circ I$ involves the reflection on \mathbb{R}^d that sends x_d to $-x_d$ and it belongs to $SO^+(d+1, 1)$. We can actually construct the Weyl inversion explicitly through the following exponential of conformal generators,

$$w = e^{\pi \frac{K_d - P_d}{2}}. \quad (2.20)$$

There are two elements of $Spin(d+1, 1)$ which project to w . We use the expression (2.20) as our definition of the Weyl inversion for $Spin(d+1, 1)$. One can check that its square is the non-trivial element of the centre, i.e. that $w^2 = -1$.

In passing to the superconformal algebra we use the same formula (2.20) to define the Weyl element and hence the Weyl inversion. The bosonic part \mathfrak{g}_0 of the superconformal algebra \mathfrak{g} is generated by the bosonic conformal algebra \mathfrak{g}_{bos} along with the generators $U \in \mathfrak{u}$ of R-symmetry transformations. The latter commute with all elements of \mathfrak{g}_0 and hence the associated universal enveloping algebras satisfy $U(\mathfrak{g}_0) \cong U(\mathfrak{g}_{bos}) \otimes U(\mathfrak{u})$. By construction w lies in $w \in U(\mathfrak{g}_{bos})$ and it is trivial in the $U(\mathfrak{u})$,

$$w = w \otimes e \in U(\mathfrak{g}_{bos}) \otimes U(\mathfrak{u}) \cong U(\mathfrak{g}_0).$$

While the action of the element w on generators of the R-symmetry transformations is trivial, its action on the fermionic generators is not. Using that conjugation of the generator D of dilations with the Weyl inversion is given by $\text{Ad}_{w_{bos}}(D) = -D$ we obtain

$$\frac{1}{2} \text{Ad}_w(Q) = \text{Ad}_w([D, Q]) = [\text{Ad}_w(D), \text{Ad}_w(Q)] = -[D, \text{Ad}_w(Q)],$$

i.e. when a supercharge Q is acted upon by the Weyl inversion it is sent to a generator whose conformal weight is $-1/2$. Consequently, the Weyl inversion interchanges generators of supertranslations and super special conformal transformations. For superconformal algebras of type I, see the final paragraph of the previous subsection for a definition, one can similarly use that $\text{Ad}_w(R) = wRw^{-1} = R$ to deduce

$$\text{Ad}_w(\mathfrak{q}_\pm) \subset \mathfrak{s}_\pm. \quad (2.21)$$

In conclusion we have seen that the super Weyl inversion exists for all superconformal algebras and we stated some of its most important properties. This is to be contrasted with the fact that a supersymmetric analogue of the ordinary conformal inversion may actually not exist. Assuming that one could choose the superconformal group such that the inversion I belonged to the bosonic conformal subgroup, then the arguments leading to eq. (2.21) with $w \times e$ replaced by $I \times e$ would remain valid. On the other hand, as the example $\mathfrak{g} = \mathfrak{sl}(4|1)$ shows, the fact that I commutes with rotations is inconsistent with eq. (2.21), bearing in mind that \mathfrak{q}_+ and \mathfrak{s}_+ are non-isomorphic modules of the rotation group. Fortunately for us, the existence of the super Weyl inversion will suffice.

Example: Let us briefly discuss super-conformal transformations and in particular the super Weyl inversion for the Lie superalgebra $\mathfrak{sl}(2|1)$. As we discussed at the end of the previous subsection, this Lie superalgebra admits a 3-dimensional representation. All generators have been spelled out in this representations above. Within this representation, the supergroup element $m(x)$ takes the form (2.14). The subgroup K is generated by dilations and $U(1)_R$ symmetry transformations which are generated by D and R , i.e. $k = \exp(\lambda D + \vartheta R)$. Under global transformations with elements $k \in K$ the superspace coordinates $x = (u, \theta, \bar{\theta})$ transform as

$$y(x, k) = (e^\lambda u, e^{\frac{1}{2}\lambda + \vartheta} \theta, e^{\frac{1}{2}\lambda - \vartheta} \bar{\theta}) . \quad (2.22)$$

Here we can either think of λ and ϑ as some real parameters of the transformation or as coordinates on the supergroup, i.e. as two generators of the structure algebra. Supertranslations with an element $m(c) = m(v, \eta, \bar{\eta})$ act as $m(c)m(x) = m(c(x))$ with

$$y(x, c) = c(x) = (u + v + \frac{1}{2}\theta\bar{\eta} + \frac{1}{2}\bar{\theta}\eta, \theta + \eta, \bar{\theta} + \bar{\eta}) . \quad (2.23)$$

The components of $c = (v, \eta, \bar{\eta})$ are generators of the structure algebra. It remains to discuss the Weyl inversion. Within the 3-dimensional representation it is straightforward to compute the Weyl inversion from eq. (2.20),

$$w = e^{\pi \frac{K-P}{2}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Note that $w^2 = \text{diag}(-1, -1, 1)$, i.e. it squares to -1 within the bosonic conformal group and is trivially extended within the R-symmetry group. It is now straightforward to compute the action of the Weyl inversion on superspace by decomposing the matrix $wm(x) = m(w(x))p(x, w)$ with $w(x) = y(x, w)$ given by

$$w(u) = -\frac{1}{u} , \quad w(\theta) = \frac{\theta}{u} , \quad w(\bar{\theta}) = \frac{\bar{\theta}}{u} . \quad (2.24)$$

Note that the action of w on the bosonic coordinate u is the same as in bosonic conformal field theory. This had to be the case, since in the chosen coordinate system on the superspace \mathcal{M} the action of the conformal algebra generators on x is the same as in bosonic theory. Furthermore, we have $w(p, q_+, q_-)w^{-1} = (-k, -s_+, s_-)$, in accordance with the relations $w^{-1}(P, Q_+, Q_-)w = (-K, -S_+, S_-)$ satisfied by 3×3 matrices. Often such conditions are used to derive the action of the inversion. In the approach here, this is not necessary as the action of w can be computed directly. This concludes our discussion of the 1-dimensional $\mathcal{N} = 2$ superspace and the global action of the superconformal symmetry on it.

3 Lifting Correlators to the Supergroup

This section contains the first new result of the present work. We establish an isomorphism between the solutions of superconformal Ward identities that are satisfied by a four-point function of arbitrary spinning fields and certain covariant functions on the superconformal group, to which one may also refer as K -spherical functions. The construction finds roots in ideas from [63], and generalises that of [1] to the superconformal setting. One key ingredient in our formula is a family of supergroup elements $g(x_i)$ that depends on the insertion points of the four fields in superspace. These will play an important role in the following sections as well. In the first subsection we state all this precisely before we illustrate the formulas at the example of $\mathfrak{g} = \mathfrak{sl}(2|1)$ in the second. The third subsection contains the proof of our statement.

3.1 Statement of the result

Let us now consider a four-point function in some superconformal field theory. To each field we associate a copy of our superspace \mathcal{M} . The generators x_{ia} of these spaces carry a label $i = 1, \dots, 4$ in addition to the label a we introduced in the previous section. The corresponding supergroup elements $m_i = m(x_i)$ are given by

$$m(x_i) = e^{x_{ia}X^a}. \quad (3.1)$$

Here the summation over a is understood. Given any pair of labels i, j we define the variables $x_{ij} = (x_{ija}) \in \mathcal{M}_i \otimes \mathcal{M}_j$ through

$$m(x_{ij}) = m(x_j)^{-1}m(x_i). \quad (3.2)$$

Concrete expressions for the components of x_{ij} can be worked out from the anti-commutator relations of the supercharges Q . One may think of $m(x_i)$ as a function on superspace with values in the

universal enveloping algebra or, more concretely, after evaluation in a fundamental representation of the Lie superalgebra \mathfrak{g} , as a matrix valued function on superspace.

In the last section we also introduced the Weyl element w through equation (2.20). Note that w is constructed out of generators of the bosonic conformal group only. In particular it acts trivially within the R-symmetry group U . We can think of w as a grouplike element in the universal enveloping algebra or, after application of a fundamental representation, as a concrete matrix such as in eq. (2.24). With the help of the Weyl inversion, let us define a new family of supergroup elements n through

$$n(x) = w^{-1}m(x)w . \quad (3.3)$$

Since m involves only generators $X^a \in \mathfrak{g}_{>0}$ of the superconformal algebra that raise the conformal weight, i.e. generators P of translations and supercharges Q , the element n is built using generators Y^a from the algebra $\mathfrak{g}_{<0}$ that lower the conformal weight, see our previous discussion of the Weyl inversion. This means that n involves special conformal generators K as well as the fermionic generators S .

In order to proceed, let us introduce another supergroup element $k = k(t)$ using the remaining generators $X \in \mathfrak{g}_0$ that commute with the generator of dilations and therefore neither appear in n nor in m . It means that k is built from the generators of dilations, rotations and R-symmetry transformations, all of which are even (bosonic). Given the three supergroup elements m, n, k we can now decompose $wm(x)$ as

$$wm(x) = m(y(x))n(z(x))k(t(x)) , \quad (3.4)$$

where the components of $y(x) = (y(x)_a)$, $z(x) = (z(x)_a)$ and $t(x) = (t(x)_\theta)$ are certain functions of the superspace coordinates x_i that can be worked out concretely on a case-by-case basis. We shall state concrete formulas in some examples below. Let us stress that it is through this factorization (3.4) that we introduce the action of the Weyl inversion w on superspace, i.e. by definition $y(x) = wx$. We consider the functions y, z and t as given for now and use them to introduce

$$y_{ij} = y(x_{ij}) = wx_{ij} , \quad z_{ij} = z(x_{ij}) , \quad t_{ij} = t(x_{ij}) . \quad (3.5)$$

By definition we have

$$wm(x_{ij}) = m(y_{ij})n(z_{ij})k(t_{ij}) . \quad (3.6)$$

The components of x_{ij}, y_{ij}, z_{ij} and t_{ij} are elements in the four-fold tensor product $\mathcal{M}^4 \cong \mathcal{M}^{\otimes 4}$ of the superspace \mathcal{M} , one copy for each insertion point. This is all we need to know about the superconfiguration space of the four insertion points.

So, let us now consider some four-point correlation function G in a quantum field theory with superconformal symmetry given by \mathfrak{g} . The fields Φ of our theory are organized in supermultiplets. We label these supermultiplets through the quantum numbers of their superprimaries. These consist of a conformal weight Δ , a spin λ and the R-charges q . The collection of these quantum numbers determine a finite dimensional irreducible representation $\rho = \rho_{\Delta,\lambda,q}$ on the Lie algebra $\mathfrak{k} = \mathfrak{g}_0$ that is spanned by dilations, rotations and R-symmetries. We denote the carrier space of this representation by $V = V_\rho$ and shall often refer to it as the space of superpolarizations. Let us stress that elements of V_ρ are associated with polarizations of the superprimary in the supermultiplet Φ . In our four-point function we have four supermultiplets whose superprimary components transform in representations ρ_i , $i = 1, \dots, 4$. The polarizations of these four superprimary fields span the vector spaces V_i . Given these data, we now consider the space $\mathcal{F}(\mathfrak{g}) \otimes V_{1234}$ of “functions F on the supergroup” that take values in the vector space $V_{1234} = V_1 \otimes \dots \otimes V_4$. Among its elements we restrict to those functions F that possess the following covariance property²

$$F(k_l g k_r) = \left(\rho_1(k_l) \otimes \rho_2(w k_l w^{-1}) \otimes \rho_3(k_r^{-1}) \otimes \rho_4(w k_r^{-1} w^{-1}) \right) F(g) , \quad (3.7)$$

for all $k_l, k_r \in K$. In analogy with ordinary Lie theory, such an F will be called a K -spherical function. To digest the mathematical meaning of this formula a bit better, let us pretend for a moment that we are dealing with some ordinary Lie algebra \mathfrak{g} rather than a superalgebra. In that case, g as well as k_l, k_r are elements of the bosonic group G . When we write $F(g)$ we let the group element g act as a global symmetry transformation on the space $\mathcal{F}(\mathfrak{g})$ of functions on the group and evaluate the result at the group unit. Stated more directly we simply evaluate the vector valued function F at the point g of the group manifold. Almost the same is true for superalgebras except that g is a matrix whose matrix elements are taken from some Grassmann algebra and F is a prescription that turns such a matrix into a vector $F(g)$ whose components are elements of that Grassmann algebra. To evaluate $F(k_l g k_r)$ we employ the left-right action of $K \times K$ on the space $\mathcal{F}(\mathfrak{g})$ of functions on the supergroup to transform F into new element of $F^{(k_l, k_r)}$ of the space $\mathcal{F}(\mathfrak{g}) \otimes V_{1234}$. When we apply this transformed $F^{(k_l, k_r)}$ to g we obtain another vector $F^{(k_l, k_r)}(g) = F(k_l g k_r)$ with Grassmann valued components. The covariance condition (3.7) selects those elements F for which the two vectors $F(g)$ and $F(k_l g k_r)$ are related by a specific matrix rotation that is obtained from representation matrices of k_l and k_r in the representations ρ_i . The precise construction of this matrix, which also involves conjugation with

² Mathematically minded readers should think of g as a supergroup element $g \in U(\mathfrak{g}) \otimes \mathcal{F}$ where \mathcal{F} can be any graded commutative algebra, see section 2.1. The object $F(g) \in \mathcal{F} \otimes V_{1234}$ is then obtained using the duality between $\mathcal{F}(\mathfrak{g})$ and $U(\mathfrak{g})$, see eq. (2.2).

the Weyl element w in two of the four tensor factors will become clear in the third subsection.

Let us now come back to our correlation function G_4 . By construction, $G_4(x_i)$ is a function on the four-fold tensor product \mathcal{M}^4 of superspace that takes values in the space V_{1234} of polarizations, i.e. $G_4 \in \mathcal{M}^4 \otimes V_{1234}$. Being the four-point function in some superconformal field theory, G_4 transforms in a very special way under superconformal transformations. This can be expressed in terms of a set of superconformal Ward identities. As a consequence of these covariance properties one may show that, given G_4 , there exists a unique function $F \in \mathcal{F}(\mathfrak{g}) \otimes V_{1234}$ on the supergroup with covariance property (3.7) such that

$$G_4(x_i) = \left(1 \otimes \rho_2(k(t_{21}))^{-1} \otimes 1 \otimes \rho_4(k(t_{43}))^{-1} \right) F(g(x_i)), \quad (3.8)$$

$$\text{where } g(x_i) = n(y_{21})^{-1} m(x_{31}) n(y_{43}) . \quad (3.9)$$

The argument of F is a product of supergroup elements, i.e. an element of $U(\mathfrak{g}) \otimes \mathcal{M}^4$ or some matrix representation thereof. After the application of F we obtain an element of $\mathcal{M}^4 \otimes V_{1234}$. We may think of this as a vector valued function on the four-fold tensor product of superspaces which can be compared to G_4 . The factor in front of F , that relates $F(g(x_i))$ to $G_4(x_i)$ is a certain matrix of functions on \mathcal{M} that acts non-trivially on the two factors V_2 and V_4 . We shall also refer to eq. (3.8) as the supersymmetric *lifting formula*.

Let us remark that there is a quick sanity check of our formula, namely one may verify that both sides of the lifting formula (3.8) satisfy the same Ward identities for infinitesimal transformations generated by elements in $X \in \mathfrak{g}_{\geq 0}$. The latter is spanned by translations, supercharges Q , rotations, dilations and R-symmetry transformations. The key observation is that

$$\sum_{j=1}^4 \mathcal{R}_X^{(j)} g(x_i) = [X \otimes id, g(x_i)] . \quad (3.10)$$

Recall that the argument $g(x_i)$ of F may be considered as a matrix whose entries are functions on the four-fold product of superspace. On these matrix elements we act with the sum of right invariant vector fields \mathcal{R}_X for $X \in \mathfrak{g}_{\geq 0}$, acting on one set of superspace coordinates each. The differential operators \mathcal{R} were constructed in the previous section. Our claim is that the resulting matrix of functions on superspace is the same as for the matrix commutator of the representation matrix for X with the product of supergroup elements. This property holds essentially by construction of the argument of F . This is not a full proof of our formula yet since the argument cannot easily be extended to special (super-)conformal transformations. We give a complete derivation in the third subsection after we have

illustrated the notations and constructions we introduced in this section for the $\mathcal{N} = 2$ superconformal algebra in $d = 1$ dimension.

3.2 Illustration for 1-dimensional superconformal algebra

Let us continue to illustrate our constructions and statements in the example of the $\mathcal{N} = 2$ superconformal algebra in $d = 1$. Recall that the fundamental representation of this algebra is 3-dimensional and hence we realize all our supergroup elements as 3×3 matrices with components in the superspace. The elements $m(x)$ were constructed in eq. (2.14) already. The Weyl inversion w and its action on superspace were worked out in eqs. (2.24) and (2.24), respectively. It is easy to determine the 3×3 matrices $n(x)$ to take the form

$$n(x) = w^{-1}m(x)w = \begin{pmatrix} 1 & 0 & 0 \\ -X & 1 & -\theta \\ -\bar{\theta} & 0 & 1 \end{pmatrix}, \quad (3.11)$$

where $X = u - \frac{1}{2}\theta\bar{\theta}$ is the same even combination of superspace coordinates $(u, \theta, \bar{\theta})$ that appeared in our formula (2.14) for $m(x)$. The central ingredient in our construction above is the factorization formula (3.4) for $wm(x)$. In the case of $\mathfrak{g} = \mathfrak{sl}(2|1)$ this reads

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & X & \theta \\ 0 & -\bar{\theta} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{u} \left(1 + \frac{\theta\bar{\theta}}{2u}\right) & \theta/u \\ 0 & 1 & 0 \\ 0 & -\bar{\theta}/u & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ u + \frac{1}{2}\theta\bar{\theta} & 1 & \theta \\ \bar{\theta} & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{u} \left(1 - \frac{\theta\bar{\theta}}{2u}\right) & 0 & 0 \\ 0 & u \left(1 - \frac{\theta\bar{\theta}}{2u}\right) & 0 \\ 0 & 0 & 1 - \frac{\theta\bar{\theta}}{u} \end{pmatrix}. \quad (3.12)$$

Comparing the first of the three factors with the expression (2.14) for $m(y)$ we deduce

$$y(x) = (Y + \frac{1}{2}\eta\bar{\eta}, \eta, \bar{\eta}) = w(u, \theta, \bar{\theta}) = \left(\frac{-1}{u}, \frac{\theta}{u}, \frac{\bar{\theta}}{u}\right). \quad (3.13)$$

This agrees of course with the result we found in eq. (2.24). Turning to the second matrix factor in the factorization formula and comparing with eq. (3.11) for $n(z)$ we conclude

$$z(x) = (Z + \frac{1}{2}\zeta\bar{\zeta}, \zeta, \bar{\zeta}) = (-u, -\theta, -\bar{\theta}). \quad (3.14)$$

Using the representation matrices for D and R that we spelled out in eq. (2.14), the third factor, finally, can be written as

$$k(t(x)) = e^{-\log u^2 D + \frac{\theta\bar{\theta}}{2u} R}. \quad (3.15)$$

We lift the matrix equation (3.12) to the following factorization identity for supergroup elements

$$wm(x) = we^{x \cdot X} = e^{w(x) \cdot X} e^{-x \cdot X^w} e^{-\log u^2 D + \frac{\theta\bar{\theta}}{2u} R}, \quad (3.16)$$

where $X^w = w^{-1}(P, Q_+, Q_-)w = (-K, -S_+, S_-)$. Given several points x_i in superspace, we can now compute the supercoordinates x_{ij} by evaluating the product $m(x_j)^{-1}m(x_i)$. The result is given by $x_{ij} = (u_{ij}, \theta_{ij}, \bar{\theta}_{ij})$ with

$$u_{ij} = u_i - u_j - \frac{1}{2}\theta_i\bar{\theta}_j - \frac{1}{2}\bar{\theta}_i\theta_j, \quad \theta_{ij} = \theta_i - \theta_j, \quad \bar{\theta}_{ij} = \bar{\theta}_i - \bar{\theta}_j. \quad (3.17)$$

For completeness let us also state how the Weyl inversion acts on x_{ij}

$$w(x_{ij}) = (-u_{ij}^{-1}, u_{ij}^{-1}\theta_{ij}, u_{ij}^{-1}\bar{\theta}_{ij}). \quad (3.18)$$

Of course this coincides with the formula (3.13) applied to the superspace coordinates x_{ij} . At this point we have explained all the ingredients that are needed to construct the supergroup elements $g(x_i)$ that were introduced in eq. (3.9).

Let us now consider a four-point function G_4 of primary fields with conformal weights Δ_i and R-charges r_i for $i = 1, \dots, 4$. Given Δ and r , the corresponding representation ρ_i of the group $K = SO(1, 1) \times U(1)$ reads

$$\rho_{\Delta, r}(e^{\lambda D + \kappa R}) = e^{-\Delta\lambda + r\kappa}. \quad (3.19)$$

Since the group K is abelian, the space V of polarizations is 1-dimensional and so is the tensor product $V_{1234} = V_1 \otimes \dots \otimes V_4$. According to our general result (3.8), there exists a unique functional F with the covariance properties

$$F(e^{\lambda_1 D + \kappa_1 R} g e^{\lambda_r D + \kappa_r R}) = e^{(\Delta_2 - \Delta_1)\lambda_l + (r_1 + r_2)\kappa_l} e^{(\Delta_3 - \Delta_4)\lambda_r - (r_3 + r_4)\kappa_r} F(g), \quad (3.20)$$

such that the lifting formula reads

$$G_4(x_i) = \Omega(x_i) F(e^{-w(x_{21}) \cdot X^w} e^{x_{31} \cdot X} e^{w(x_{43}) \cdot X^w}) \quad (3.21)$$

and the prefactor Ω is given by

$$\Omega = \Omega(x_i) = \frac{e^{r_2 \frac{\theta_{12}\bar{\theta}_{12}}{2u_{12}} + r_4 \frac{\theta_{34}\bar{\theta}_{34}}{2u_{34}}}}{u_{12}^{2\Delta_2} u_{34}^{2\Delta_4}}. \quad (3.22)$$

It is instructive to verify that the commutation relations (3.10) hold for the argument of F and to evaluate G_4 for Weyl-inverted arguments $w(x_i)$, thereby showing that the right hand side of the lifting formula (3.21) indeed satisfy the same conformal Ward identities as the four-point function.

3.3 Proof of the lifting formula

The goal of this subsection is to prove the main result (3.8) for an arbitrary superconformal group. Before doing that, let us give one more definition, an extension of the factorization formula (3.4)

$$hm(x) = m(y(x, h))n(z(x, h))k(t(x, h)), \quad (3.23)$$

from the Weyl inversion $h = w$ to arbitrary elements h of the superconformal group. This formula also extends our analysis in section 2.3 where we studied the action of global conformal transformations on superspace. At the time we only cared about the first factor $m(y(x, h))$ in the product on the right hand side. The new formula (3.23) extends the action of global superconformal transformations to the whole superconformal group. Otherwise all the additional explanations we provided in section 2.3. remain applicable. The extended factorization formula involves three sets of functions $y(x, h) = (y(x, h)_a)$, $z(x, h) = (z(x, h)_a)$ and $t(x, h) = (t(x, h)_\rho)$. For $h = w$ we recover the functions we introduced in the previous section.

A four-point correlation function G_4 satisfies a set of Ward identities. For global superconformal transformations h these may be written in the form

$$G_4(x_i^h) = \left(\bigotimes_{i=1}^4 \rho_i(k(t(x_i, h))) \right) G_4(x_i) . \quad (3.24)$$

Note that correlation functions are essentially invariant under these transformations except some factors depending in the weight, spin and the R-charges. This dependence is encoded in the choice of representations ρ_i , as we explained above. In a first step we want to lift the correlator G_4 to an object $F_4 \in \mathcal{F}_1 \otimes V_1 \otimes \cdots \otimes \mathcal{F}_4 \otimes V_4$, where \mathcal{F}_i are supercommuting copies of the structure algebra $\mathcal{F}(\mathfrak{g})$ of functions on the supergroup. This can be done in a unique way if we require

$$F_4(m(x_i)) = G_4(x_i) , \quad F_4(g_i n_i k_i) = \bigotimes_{i=1}^4 \rho_i(k_i^{-1}) F_4(g_i) . \quad (3.25)$$

Here our notations are the same as in section 3.1, see our extended discussion before equation (3.8). The Ward identities (3.24) satisfied by G_4 imply the following invariance conditions satisfied by F_4 under simultaneous left multiplication of its four arguments by an element h of the superconformal group,

$$F_4(hm(x_i)) = F_4\left(m(x_i^h) n(z(x_i, h)) k(t(x_i, h))\right) \quad (3.26)$$

$$= \left(\bigotimes_{i=1}^4 \rho_i(k(t(x_i, h))^{-1}) \right) G_4(x_i^h) = G_4(x_i) = F(m(x_i)) . \quad (3.27)$$

Other than the Ward identity, we have used the definitions (3.23) and (3.25). Given this element F_4 and the Weyl inversion w we can construct a new object $F \in \mathcal{F}(\mathfrak{g}) \otimes V_{1234}$ through the prescription

$$F(g) := F_4(e, w^{-1}, g, gw^{-1}) . \quad (3.28)$$

While this might look a bit bizarre at first, it is easy to verify that it defines a K -spherical function F , i.e. that F satisfies the covariance law (3.7). Indeed, from the definition (3.28) of F , the left invariance

condition (3.26) and the right covariance law in eq. (3.25) of F_4 we obtain

$$\begin{aligned} F(k_l g k_r) &= F_4(e, w^{-1}, k_l g k_r, k_l g k_r w^{-1}) = F_4(k_l^{-1}, w^{-1} w k_l^{-1} w^{-1}, g k_r, g w^{-1} w k_r w^{-1}) \\ &= \left(\rho_1(k_l) \otimes \rho_2(w k_l w^{-1}) \otimes \rho_3(k_r^{-1}) \otimes \rho_4(w k_r^{-1} w^{-1}) \right) F(g) . \end{aligned}$$

In conclusion we have shown that a correlation function G_4 provides us with a K -spherical function F . It is actually not difficult to invert the map and recover G_4 from F . Suppressing the last two arguments and their corresponding prefactors for simplicity, we have

$$\begin{aligned} F_4(m(x_1), m(x_2)) &= (1 \otimes \rho_2(k(t_{21})^{-1})) F_4(m(x_1)n(y_{21}), m(x_2)k(t_{21})^{-1}n(z_{21})^{-1}) \\ &= (1 \otimes \rho_2(k(t_{21})^{-1})) F_4(m(x_1)n(y_{21}), m(x_1)m(x_2)k(t_{21})^{-1}n(z_{21})^{-1}) \\ &= (1 \otimes \rho_2(k(t_{21})^{-1})) F_4(m(x_1)n(y_{21}), m(x_1)w^{-1}m(y_{21})) \\ &= (1 \otimes \rho_2(k(t_{21})^{-1})) F_4(m(x_1)n(y_{21}), m(x_1)n(y_{21})w^{-1}) . \end{aligned}$$

In the first step we used the covariance property (3.25) of F_4 in the first two arguments to multiply the first argument with $n(y_{21})$ and the second with $k(t_{21})^{-1}n(z_{21})^{-1}$. Since the latter contains a factor k it needed to be compensated by a rotation in the second factor of the space of superpolarizations. Then we inserted the definition of $m(x_{21})$ and used that

$$m(x_{21}) = w^{-1}m(y_{21})n(z_{21})k(t_{21}) .$$

This factorization formula is essentially the definition of y_{21} , z_{21} and t_{21} . Finally we moved the Weyl element w^{-1} through m using that $n = w^{-1}mw$. We can now apply the same steps to the third and fourth argument to obtain

$$F_4(m(x_i)) = (1 \otimes \rho_2(k(t_{21})^{-1}) \otimes 1 \otimes \rho_4(k(t_{43})^{-1})) F_4(g_{12}(x_i), g_{12}(x_i)w^{-1}, g_{34}(x_i), g_{34}(x_i)w^{-1}) , \quad (3.29)$$

where we introduced the elements

$$g_{ij} = m(x_i)n(y_{ji}) .$$

Finally, we can use the invariance property (3.26) of F for $h = g_{12}^{-1}$ to obtain

$$F_4(m(x_i)) = (1 \otimes \rho_2(k(t_{21})^{-1}) \otimes 1 \otimes \rho_4(k(t_{43})^{-1})) F_4(e, w^{-1}, g(x_i), g(x_i)w^{-1}) .$$

Here $g(x_i)$ is the element we introduced in eq. (3.9). Using our definition of the functional F in eq. (3.28) and the relation between F_4 and G_4 we have thereby established the lifting formula (3.8).

From the above derivation, one may deduce the following transformation properties of g_{ij} and $k(t_{ji})$ under superconformal transformations

$$g_{ij}(x^h) = h g_{ij}(x) k(t(x_i, h))^{-1} , \quad (3.30)$$

$$k(t_{ji}^h) = k^w(t(x_i, h)) k(t_{ji}) k(t(x_j, h))^{-1} , \quad (3.31)$$

where $k^w = wkw^{-1}$. Indeed these are necessary for the right hand side of eq. (3.29) to satisfy the same Ward identities as the left hand side. A complete proof of the two transformation laws can be found in appendix A. These two formulas will play a significant role in the computation of the crossing factor to which we turn next.

4 Tensor Structures and Crossing Symmetry Equations

Having lifted the spinning four-point function G_4 from superspace to the superconformal group through eq. (3.8) we can now employ (super)group theoretic constructions to study superconformal correlators. In the first subsection we employ a supersymmetric version of the Cartan or KAK decomposition for superconformal groups of type I to factorize four-point functions into the product of a tensor factor $\Theta = \Theta(x_i)$ and a function Ψ that depends on superconformal cross ratios only.³ This part of our analysis extends constructions in [1] to the superconformal setting. We can perform the factorization for different channels. The supercrossing factor, i.e. the ratio of the corresponding tensor factors Θ_s and Θ_t for the s - and t -channel, is studied at the end of the first subsection. There we establish its superconformal invariance and compute it for bosonic conformal symmetries in any dimension d .

At this stage, all quantities depend on fermionic variables. In particular, the function Ψ still depends on some number of nilpotent invariants. By expanding all quantities in the Grassmann variables we construct the crossing factor in the second subsection. This is then used to write the crossing symmetry constraints in the independent coefficients of the operator product expansions in terms of functions of two bosonic cross ratios only. As shown in [52], the latter may be expanded into wave functions of some Calogero-Sutherland Hamiltonian. By collecting all the material we have put together through our discussion of the example $\mathfrak{g} = \mathfrak{sl}(2|1)$ we can finally calculate the crossing factor for $\mathcal{N} = 2$ superconformal field theories in $d = 1$ dimension, see third subsection.

³As explained in the introduction, our group theoretic factorization $G_4 = \Theta\Psi$ is reminiscent of the factorization $G_4 = \Omega g$ used in most of the CFT literature. The difference between the two factorizations can be quantified through the ratio $\Theta\Omega^{-1}$ which has a non-trivial dependence on cross ratios. As we shall see below, Θ and Ψ are more universal than the factors Ω and g .

4.1 Cartan coordinates, tensor and crossing factors

We will now construct the tensor structures, starting from the lifting formula (3.8). Note that eq. (3.8) treats each of the four insertion points differently and hence it breaks the permutation symmetry of correlators in a Euclidean theory. Different permutations σ of the four points are associated with different channels. We refer to the channel that is associated with the identity permutation $\sigma = \sigma_s = id$ as the s -channel. Another important case for us is the permutation $\sigma = \sigma_t = (24)$ which we call the t -channel. In any case, given the choice of the channel σ , we can extend the lifting formula (3.8) to become

$$G_4(x_i) = \rho_{\sigma(2)}(k(t_{\sigma(2)\sigma(1)})^{-1})\rho_{\sigma(4)}(k(t_{\sigma(4)\sigma(3)})^{-1})F_\sigma(g(x_{\sigma(i)})) . \quad (4.1)$$

Here, the factor $\rho_{\sigma(i)}$ acts on the $\sigma(i)^{th}$ tensor factor in the space of superpolarizations and it acts trivially on all other tensor factors.

In order to proceed we adopt a new coordinate system that we refer to as Cartan coordinates. So far we have decomposed supergroup elements g into factors m , n and k . Now we consider a different decomposition in which supergroup elements g are written as

$$g = k_l \eta_l a \eta_r k_r , \quad (4.2)$$

where k_l and k_r are associated with the subgroup K that is obtained through exponentiation of rotations, dilations and R-symmetry transformations. Similarly, the factors η_l and η_r are associated with fermionic generators. More specifically, each of these factors contains half of the generators in the odd subspace \mathfrak{g} . In the following we consider Lie superalgebra \mathfrak{g} of type I for which the internal symmetry group contains a $U(1)$ -factor which allows us to decompose the fermionic generators according to the sign of the $U(1)$ R-charge. It turns out that half of the supercharges Q possess positive R-charge while the others possess negative R-charge and similarly for the super special conformal transformations S . Let us agree that η_l uses generators of negative charge while η_r is build from generators with positive charge. The central factor $a = a(u_1, u_2)$, finally, depends on two bosonic coordinates u_1, u_2 only and it is assumed to take the form

$$a(u_1, u_2) = e^{\frac{u_1+u_2}{4}(P_1+K_1)-i\frac{u_1-u_2}{4}(P_2-K_2)} . \quad (4.3)$$

Let us note that a factorization of supergroup elements g in the form (4.2) is not unique. In fact, given any such factorization we can produce another factorization of the very same form by the transformation

$$(k_l, \eta_l; k_r, \eta_r) \rightarrow (k_l b, b^{-1} \eta_l b; b^{-1} k_r, b^{-1} \eta_r b) , \quad (4.4)$$

where b are elements associated with the subalgebra $\mathfrak{so}(d-2) \oplus \mathfrak{u}_r \subset \mathfrak{k}$ and therefore commute with $a = a(u_1, u_2)$. At the same time, the elements $b^{-1}\eta_{l,r}b$ can still be written as exponentials of fermionic generators with negative(l) and positive(r) $U(1)$ R-charge, respectively. Hence our gauge transformation (4.4) respects the Cartan decomposition. Elements b form the stabilizer group $B = SO(d-2) \times U_r$ of the Cartan decomposition (4.2). For later use we introduce a projector P by integrating b over the entire stabilizer group B ,

$$P = \frac{1}{\text{Vol } B} \int_B d\mu b = \frac{1}{\text{Vol } B} \int_B d\mu(\beta) b(\beta) , \quad (4.5)$$

where μ is the Haar measure on B . For pedagogical reasons we have introduced some coordinates β on B so that the element b could be written explicitly as a function $b = b(\beta)$ on B with values in $U(\mathfrak{b})$. As we indicated, P can be considered as an element in the universal enveloping algebra $U(\mathfrak{g})$. More concretely, after evaluation in some representation we can also think of P as a matrix. By construction, this matrix has two important properties

$$P^2 = P \quad , \quad bP = P . \quad (4.6)$$

We can verify the second equation very easily using the integral representation of P and the left invariance of the Haar measure. The first property then follows from $bP = b(\beta)P$ by performing an additional integration over B since P is a constant on B .

In our analysis below we will apply the projector P to an a function $f(u_i, \theta, \bar{\theta})$ that takes values in the representation space V_{1234} of K . The latter may be considered a carrier space for a representation of B by restriction from K to its subgroup B . The representation of P on such an object f is denoted by \mathcal{P} , i.e.

$$\mathcal{P}[f(u_i, \theta, \bar{\theta})] = \frac{1}{\text{Vol } B} \int_B d\mu \chi(b) f(u_i, \theta^b, \bar{\theta}^b) , \quad (4.7)$$

where θ^b and $\bar{\theta}^b$ denotes the action of b on the Grassmann coordinates θ and $\bar{\theta}$ and $\chi(b)$ is a shorthand for the action of b on the finite dimensional vector space V_{1234} of superpolarizations,

$$\chi(b) = \rho_1(b) \otimes \rho_2(bw^{-1}) \otimes \rho_3(b) \otimes \rho_4(bw^{-1}) . \quad (4.8)$$

In practical computations it is convenient to make some specific choices for the Cartan factors that remove the gauge freedom (4.4). Such gauge fixing conditions are arbitrary and at the end of every calculation one has to check that the result does not depend on them.

Let us now apply the Cartan factorization to the argument $g(x_{\sigma(i)})$ of the functional F_σ in eq. (4.1),

$$g(x_{\sigma(i)}) = k_{\sigma,l}(x_i) \eta_{\sigma,l}(x_i) a_\sigma(x_i) \eta_{\sigma,r}(x_i) k_{\sigma,r}(x_i) . \quad (4.9)$$

The formula (4.1) and covariance properties of F_σ give

$$\begin{aligned}
G_4(x_i) &= \rho_{\sigma(2)}(k(t_{\sigma(2)\sigma(1)})^{-1})\rho_{\sigma(4)}(k(t_{\sigma(4)\sigma(3)})^{-1})F_\sigma(g(x_{\sigma(i)})) \\
&= \rho_{\sigma(2)}(k(t_{\sigma(2)\sigma(1)}))^{-1}\rho_{\sigma(4)}(k(t_{\sigma(4)\sigma(3)}))^{-1}F_\sigma(k_{\sigma,l}\eta_{\sigma,l}a_\sigma\eta_{\sigma,r}k_{\sigma,r}) \\
&= \rho_{\sigma(1)}(k_{\sigma,l})\rho_{\sigma(2)}(k(t_{\sigma(2)\sigma(1)})^{-1}k_{\sigma,l}^w)\rho_{\sigma(3)}(k_{\sigma,r}^{-1})\rho_{\sigma(4)}(k(t_{\sigma(4)\sigma(3)})^{-1}(k_{\sigma,r}^{-1})^w)F_\sigma(\eta_{\sigma,l}a_\sigma\eta_{\sigma,r}).
\end{aligned} \tag{4.10}$$

For simplicity, we dropped the dependence of Cartan factors on the insertion points, i.e. for example $k_{\sigma,l} = k_{\sigma,l}(x_i) = k_l(x_{\sigma(i)})$. We will discuss the concrete functional dependence of the insertion points a bit later.

Let us spell out the previous formula for the s and t -channel. In the s -channel one obtains

$$G_4(x_i) = \rho_1(k_{s,l})\rho_2(k(t_{21})^{-1}k_{s,l}^w)\rho_3(k_{s,r}^{-1})\rho_4(k(t_{43})^{-1}(k_{s,r}^w)^{-1})\mathcal{P}_s F_s(\eta_{s,l}a_s\eta_{s,r}), \tag{4.11}$$

while the t -channel gives

$$G_4(x_i) = \rho_1(k_{t,l})\rho_4(k(t_{41})^{-1}k_{t,l}^w)\rho_3(k_{t,r}^{-1})\rho_2(k(t_{23})^{-1}(k_{t,r}^w)^{-1})\mathcal{P}_t F_t(\eta_{t,l}a_t\eta_{t,r}). \tag{4.12}$$

Here we introduced projector \mathcal{P} that was defined in eq. (4.7) explicitly to stress that $F(\eta_l a \eta_r)$ takes value in the space of B -invariants. Roughly speaking, the two factors in front of F_s and F_t are the s - and t -channel tensor structures.

The ratio of these s - and t -channel tensor structures is referred to as supercrossing factor and we denote it by \mathcal{M} . As we can read off the the previous two formulas the supercrossing factor takes the form

$$\mathcal{M}_{st}(x_i) = \mathcal{P}_t \bigotimes_{i=1}^4 \rho_i(\kappa_i) \mathcal{P}_s, \tag{4.13}$$

where the four elements κ_i are given by

$$\kappa_1 = k_{t,l}^{-1}k_{s,l}, \quad \kappa_2 = k_{t,r}^w k(t_{23})k(t_{21})^{-1}k_{s,l}^w \tag{4.14}$$

$$\kappa_3 = k_{t,r}k_{s,r}^{-1}, \quad \kappa_4 = (k_{t,l}^w)^{-1}k(t_{41})k(t_{43})^{-1}(k_{s,r}^w)^{-1}. \tag{4.15}$$

It is important to stress the two projectors in eq. (4.13) make the supercrossing factor independent of any gauge fixing conditions for our gauge symmetry (4.4). In fact one can easily check using eq. (4.6) that any gauge transformation with some element b is absorbed by the projectors.

Our main goal is to compute the matrix \mathcal{M} explicitly. Note that it depends on the insertion points x_i in superspace through the dependence of the factors $k(t_{ij}) = k(t(x_{ij}))$, that were defined in eq. (3.6), as well as through the factors $k_{l,r}$ in the Cartan decomposition (4.2) of the supergroup elements

$g_{s,t}(x_i)$. In order to compute the matrix \mathcal{M}_{st} we first show that it is invariant under superconformal transformation, i.e. $\mathcal{M}_{st}(x_i^h) = \mathcal{M}_{st}(x_i)$. This then implies that it is a function of cross ratios only and so it can be computed after moving the insertion points into a special positions.

To see that \mathcal{M}_{st} is a conformal invariant we must study the dependence of the four tensor components one after another. We have already stated the transformation behavior of the factors $k(t_{ij})$ at the end of the previous section, see eq. (3.31). What we need to study now is the transformation behavior of the factors $k_{l,r}$ in the Cartan decomposition (4.2). To this end let us first note that, according to eq. (3.30), the supergroup elements $g_\sigma(x_i)$ transform as

$$g_\sigma(x_i^h) = k(t(x_{\sigma(1)}, h)) g_\sigma(x_i) k(t(x_{\sigma(3)}, h))^{-1} . \quad (4.16)$$

Because of the gauge freedom of the Cartan decomposition which we described in eq. (4.4), knowing the behavior of $g_\sigma(x_i)$ under conformal transformations does not allow us to uniquely determine the transformation law of the factors, but we can conclude that

$$k_{\sigma,l}(x_i^h) = k(t(x_{\sigma(1)}, h)) k_{\sigma,l}(x_i) b_\sigma(x_i, h) \quad , \quad k_{\sigma,r}(x_i^h) = b_\sigma^{-1}(x_i, h) k_{\sigma,r}(x_i) k(t(x_{\sigma(3)}, h))^{-1} \quad (4.17)$$

for some factor b that may depend on the channel, the superspace insertion points x_i and the superconformal transformation h , yet must be the same for the left and right factors k_l and k_r . For the case of s - and t -channels, these become

$$k_{s/t,l}(x_i^h) = k(t(x_1, h)) k_{s/t,l} b_{s/t}(x_i, h) \quad , \quad k_{s/t,r}(x_i^h) = b_{s/t}^{-1}(x_i, h) k_{s/t,r} k(t(x_3, h))^{-1} . \quad (4.18)$$

With these transformation laws it is now easy to verify that all four tensor components κ_i of the crossing factor M are indeed invariant under superconformal transformations, up to gauge transformation, i.e.

$$\kappa_i(x_k^h) = b_t^{-1}(x_k, h) \kappa_i(x_k) b_s(x_k, h) \quad , \quad \kappa_j(x_k^h) = w b_t^{-1}(x_k, h) w^{-1} \kappa_j(x_k) w b_s(x_k, h) w^{-1} , \quad (4.19)$$

where $i = 1, 3$ and $j = 2, 4$. To get the last two relations one employs the formula for $k(t_{ji}^h)$ given in eq. (3.31). Using the definition (4.7) of the projectors $\mathcal{P}_s = \mathcal{P}_t$ and the property (4.6) of $P \in U(\mathfrak{b})$ we see that $\mathcal{M}_{st}(x_i)$ is indeed invariant under conformal transformations.

The analysis we have performed in this section holds for conformal and superconformal symmetries alike. It is actually quite instructive to evaluate the final formula (4.13) for the crossing factor for spinning correlators in bosonic conformal field theories. In this case it is in fact rather easy to obtain \mathcal{M}_{st} since we can effectively reduce the problem to one on the 2-dimensional conformal group. We will deviate from previous notations and use G to denote the bosonic conformal group $SO(d+1, 1)$ and assume $d > 2$.

Since the crossing factor is conformally invariant, in computing $\mathcal{M}(u, v)$ we may assume that x_i are any points that give the correct cross ratios u and v . In particular, all points can be assumed to lie in the 2-dimensional plane P that is spanned by the first two unit vectors e_1, e_2 of the d -dimensional space \mathbb{R}^d . In this case, the element $g_\sigma(x_i)$ is seen to belong to the conformal group of the plane, i.e. $g_\sigma(x_i) \in G_P = SO(3, 1) \subset G$. Within this group $g_\sigma(x_i)$ admits a unique Cartan decomposition, which can also serve as its Cartan decomposition in G , bearing in mind that the torus $A \subset G_P \subset G$ of the Cartan decomposition of G is actually a subgroup of G_P . Put in another way, the Cartan decomposition of G_P defines a particular gauge fixing for Cartan factors of $g(x_i)$. Note that all relevant rotations are generated by the element M_{12} , which commutes with the Weyl inversion w when $d > 2$. Hence we conclude that the factors κ_i that arise in the transition from s - to t -channel must be of the form

$$\kappa_i = e^{\gamma_i D} e^{\varphi_i M_{12}} , \quad (4.20)$$

for some functions γ_i and φ_i that depend on the insertion points x_i of the four fields through their two cross ratios. Having determined the general form of κ_i , we can find the undetermined coefficients by a direct calculation. Since we can perform the calculation in any conformal frame we set for convenience,

$$x_1 = \frac{\cosh^2 \frac{u_1}{2} + \cosh^2 \frac{u_2}{2}}{2 \cosh^2 \frac{u_1}{2} \cosh^2 \frac{u_2}{2}} e_1 - i \frac{\cosh^2 \frac{u_1}{2} - \cosh^2 \frac{u_2}{2}}{2 \cosh^2 \frac{u_1}{2} \cosh^2 \frac{u_2}{2}} e_2 , \quad x_2 = 0 , \quad x_3 = e_1 , \quad x_4 = \infty e_1 . \quad (4.21)$$

Then it follows

$$\kappa_1 = \kappa_3 = e^{\gamma D + \alpha M_{12}} , \quad \kappa_2 = \kappa_4 = e^{\gamma D - \alpha M_{12}} , \quad (4.22)$$

where

$$e^{4\gamma} = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2} , \quad e^{2i\alpha} = \frac{\cosh \frac{u_1}{2}}{\cosh \frac{u_2}{2}} . \quad (4.23)$$

To complete this description let us also quote from [1] that

$$e^{u_i} = 1 - \frac{2}{z_i} (1 + \sqrt{1 - z_i}) , \quad (4.24)$$

$$\text{where } u = z_1 z_2 = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} , \quad v = (1 - z_1)(1 - z_2) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} . \quad (4.25)$$

Let us note that \mathcal{M} was originally defined using representations of $K = SO(1, 1) \times SO(d)$, but is computed using only representation theory of $SO(1, 1) \times SO(2)$.

Example: To make the last point manifest, let us give some more details for conformal theories in $d = 3$ dimensions. Let us decompose the factors $k_l = d_l r_l$ and $k_r = d_r r_r$ into dilations $d_{l/r}$ and rotations $r_{l/r}$. Following [1, 11, 51] we parametrize the elements r of the 3-dimensional rotation group

through Euler angles,

$$r(\phi, \theta, \psi) = e^{-\phi M_{12}} e^{-\theta M_{23}} e^{-\psi M_{12}} . \quad (4.26)$$

With this choice of coordinates, the elements κ_i have $\phi = \pm\alpha$ and $\theta = \psi = 0$. Next let us recall that matrix elements of the spin- j representation of $SU(2)$ read

$$t_{mn}^j(\phi, \theta, \psi) = \langle j, m | g(\phi, \theta, \psi) | j, n \rangle = e^{-i(m\phi+n\psi)} d_{mn}^j(\theta) . \quad (4.27)$$

Here, the function d_{mn}^j is known as Wigner's d -function. It is expressed in terms of Jacobi polynomials $P_n^{(\alpha, \beta)}$ as

$$d_{mn}^j(\theta) = i^{m-n} \sqrt{\frac{(j+m)!(j-m)!}{(j+n)!(j-n)!}} \left(\sin \frac{\theta}{2}\right)^{m-n} \left(\cos \frac{\theta}{2}\right)^{m+n} P_{j-m}^{(m-n, m+n)}(\cos \theta) . \quad (4.28)$$

For $\theta = 0$, the only non-zero matrix elements are those with $m = n$. Furthermore

$$t_{nn}^j(\pm\alpha, 0, 0) = e^{\mp i n \alpha} P_{j-n}^{(0, 2n)}(1) = e^{\mp i n \alpha} = \left(\frac{\cosh \frac{u_1}{2}}{\cosh \frac{u_2}{2}}\right)^{\mp \frac{n}{2}} . \quad (4.29)$$

Since the stabilizer group $B = SO(d-2)$ for a bosonic conformal field theory in $d = 3$ dimensions is trivial, so is the projector P . Putting all this together we conclude that the crossing factor reads

$$(\mathcal{M}_{st})_{pqrs}^{ijkl} = \left(\frac{u}{v}\right)^{-\frac{1}{4} \sum \Delta_i} \left(\frac{\cosh \frac{u_1}{2}}{\cosh \frac{u_2}{2}}\right)^{\frac{1}{2}(i+k-j-l)} \delta_p^i \delta_q^j \delta_r^k \delta_s^l , \quad (4.30)$$

where u, v are the usual s -channel cross ratios and $u_i = u_i(u, v)$ are functions thereof, see eq. (4.24). The first factor in this result for the spinning crossing factor is well known from scalar correlators. The correction it receives for spinning correlators are diagonal in the space of polarizations but depend on the eigenvalues of the generator J_z for rotations around one particular direction e_z .

4.2 Blocks and crossing symmetry equation

In case of bosonic conformal theories, the crossing factor we have just computed along with spinning conformal blocks is all it takes to write down crossing symmetry constraints. For superconformal symmetries of type I, some more work is needed in order to spell out these equations. We describe the additional elements in this subsection before we illustrate the entire formalism at the example for $\mathcal{N} = 2$ superconformal theories in $d = 1$ dimensions in the next. Along the way we also review the construction of conformal blocks from [52]. In order not to clutter the presentation too much, the first part of our discussion focuses on the s -channel. Other channels can be dealt with similarly.

In Subsection 4.1 we have shown that the four-point function of primary fields in an arbitrary representation of a conformal superalgebra of type I can be written as

$$G_4(x_i) = \Theta_s(x_i) \Psi_s(u_i; \theta, \bar{\theta}) , \quad (4.31)$$

where the supertensor factor $\Theta_s(x_i)$ depends on the insertion points of the fields through

$$\Theta_s(x_i) = \omega^{-1/2}(u_1, u_2) \rho_1(k_{s,l}) \rho_2(k(t_{21})^{-1} k_{s,l}^w) \rho_3(k_{s,r}^{-1}) \rho_4(k(t_{43})^{-1} (k_{s,r}^w)^{-1}) P_s , \quad (4.32)$$

and Ψ_s is a function of the cross ratios, including all nilpotent/fermionic superconformal invariants, that is given by

$$\Psi_s(u_i; \theta, \bar{\theta}) = \omega^{1/2}(u_1, u_2) F_s(\eta_{s,l} a_s \eta_{s,r}) . \quad (4.33)$$

In splitting eq. (4.11) into a product of a supertensor factor and a function Ψ_s of the cross ratios we have included a scalar factor

$$\omega(u_1, u_2) = 4(-1)^{2-d} \left(\sinh \frac{u_1}{2} \sinh \frac{u_2}{2} \right)^{2d-2} \coth \frac{u_1}{2} \coth \frac{u_2}{2} \left| \sinh^{-2} \frac{u_1}{2} - \sinh^{-2} \frac{u_2}{2} \right|^{d-2} , \quad (4.34)$$

which depends on the bosonic cross u_1, u_2 ratios only and may in fact be interpreted as the volume of $K \times K$ bosonic orbits on the conformal group, see [51] for details. The factor ω is conventional, but has some advantages that will be pointed out below.

Let us now further analyse the factor F_s in formula (4.33) by expanding it in the fermionic variables. The Grassmann variables θ that multiply the odd generators of negative $U(1)$ R-charge in the exponent of η_l generate an algebra Λ_θ while those variables $\bar{\theta}$ that multiply the positively charged odd generators in the exponent of η_r give rise to a Grassmann algebra $\Lambda_{\bar{\theta}}$. Before the expansion, the wave functions $\Psi_s(u_1, u_2; \theta, \bar{\theta})$ are vector valued, with two copies of the bosonic subgroup K acting in the image of F . The first copy, which we refer to as K_l acts on $V_{(12)} = V_1 \otimes V_2'$. Except for the conjugation with the Weyl inversion in the second tensor component, one may think of $V_{(12)}$ as the space of superpolarizations for the first two fields. Similarly, the second copy K_r acts on $V_{(34)} = V_3 \otimes V_4'$. When we perform the fermionic expansion, the coefficients sit in the representation spaces

$$V_l = V_{(12)} \otimes \Lambda_\theta , \quad V_r = V_{(34)} \otimes \Lambda_{\bar{\theta}} \quad (4.35)$$

of K_l and K_r . Note that the bosonic subgroup K acts on the two Grassmann algebras so that indeed both spaces form a representation of K . We also refer to the spaces V_l and V_r as spaces of polarizations, as opposed to $V_{(12)}$ and $V_{(34)}$ which we have called the spaces of superpolarizations.

As we explained before, the covariance properties of F imply that Ψ takes values in the subspace of B -invariants, i.e. in the space

$$\mathcal{T} = (V_l \otimes V_r)^B ,$$

which we also refer to as the space of tensor structures. One may think of its elements as B -invariant elements in the space of function of the Grassmann variables θ and $\bar{\theta}$ that take values the space of

superpolarizations. Let us fix some basis of elements ω^I in \mathcal{T} and denote the dual basis by $\hat{\omega}_I$. We can collect these elements into two objects

$$v_s(x_i) = (\omega^1(x_i), \dots, \omega^T(x_i)), \quad \hat{v}_s(x_i) = (\hat{\omega}_1(x_i), \dots, \hat{\omega}_T(x_i)), \quad (4.36)$$

where $T = \dim \mathcal{T}$ is the number of tensor structures. One may think of v_s as a rectangular matrix from the space \mathcal{T} of tensor structures to the space $V_{(12)} \otimes V_{(34)}$ with matrix elements in the Grassmann algebra $\Lambda_\theta \otimes \Lambda_{\bar{\theta}} = \Lambda_{\mathfrak{g}_1}$. Through the Cartan decomposition of $g(x_i)$, the Grassmann variables θ and $\bar{\theta}$ are concrete functions on the superspace \mathcal{M}_4^\otimes . We have displayed this dependence on the supercoordinates x_i explicitly.

The coefficients in the fermionic expansion of Ψ are functions ψ_I of the two bosonic cross ratios u_1, u_2 that take values in the space of $V_{(12)} \otimes V_{(34)}$ of superpolarizations. We can write this in the form

$$\Psi(u_1, u_2; \theta, \bar{\theta}) = v_s(x_i) \cdot \psi(u_1, u_2) = v_s(x_i) P_s \psi(u_1, u_2). \quad (4.37)$$

Putting eqs. (4.31) and (4.37) together we can now write the four-point function as

$$G_4(x_i) = \Theta_s(x_i) v_s(x_i) \cdot \psi(u_1, u_2). \quad (4.38)$$

We want to expand G_4 into superblocks, i.e. eigenfunctions of the super Casimir operator. The latter turns out to take a particularly simple form when evaluated on $\psi(u_1, u_2)$. As we have shown in [52], one finds that

$$Cas_s G_4(x_i) = \Theta_s(x_i) v_s(x_i) \cdot (H_0^{V_l, V_r} + A) \psi(u_1, u_2). \quad (4.39)$$

Here H_0 is the spinning Calogero-Sutherland Hamiltonian for bosonic blocks, i.e. H_0 takes the form

$$H_0 = -\frac{\partial_2}{\partial u_1^2} - \frac{\partial_2}{\partial u_2^2} + V(u_1, u_2), \quad (4.40)$$

where $V(u_1, u_2)$ is a potential that takes values in the space of $T \times T$ matrices. The precise form of the potential depends on the pair (V_l, V_r) of representations of K , but it is the same one obtains for the spinning Casimir operator of the bosonic conformal algebra in Calogero-Sutherland gauge. In $d = 3, 4$ dimensions such matrix potentials were worked out explicitly in [11, 51]. The second term A is a matrix valued potential that was shown to be nilpotent and the precise form of these terms is remarkably simple, see [52].

The eigenfunctions of the Hamiltonian $H_0 = H_0^{V_l, V_r}$ we have just described will be denoted by $\psi_0(\lambda_i; u_i) = \psi_0^{V_l, V_r}(\lambda_i, u_i)$. Here λ_i denote the eigenvalues of the (second and higher order) Hamiltonians which are directly related to the (spin and weight) quantum numbers of the intermediate fields

in the conformal field theory. Functions $\psi_0(u_i)$ are well studied, and explicit expressions exist at least in dimension $d \leq 4$, see in particular [10]. Eigenfunctions of the full Hamiltonian \mathcal{H} will be denoted by $\psi(\lambda_i; u_i)$. Nilpotency of A guarantees that quantum mechanical perturbation theory truncates at some order $N - 1 \leq \dim_{\mathfrak{g}_+}$, so that we can obtain exact results by summing just a few orders of the perturbative expansion. It turns out that, at any order of the expansion, the perturbation may be evaluated explicitly with some input from the representation theory of $SO(d+2)$. It results in expressions superconformal blocks as finite linear combinations of spinning bosonic blocks. In this sense our results provide a complete solution of the Casimir equations for type I superconformal symmetry and in particular for 4-dimensional conformal field theories with any number \mathcal{N} of supersymmetries.

Here we have described the Casimir equation and its solution for the s -channel but it is clear that similar discussions apply to all channels. Reinstating the subscripts s and t we end up with blocks $\psi_s(\lambda_i; u_i^s)$ and $\psi_t(\lambda_i; u_i^t)$. The eigenvalues $\lambda_i = \lambda_i(\mathcal{O})$ are related to the quantum numbers (weight, spin, R charges) of the intermediate supermultiplets \mathcal{O} . Let us also stress that these blocks ψ are multi-component objects with T components $\psi^I, I = 1, \dots, T$ labeled by a basis of four-point tensor structures. For each eigenvalue one can actually find T independent solutions which are usually labeled by pairs $(a, b) = ab$ of three-point tensor structures for the relevant operators products. Consequently, the blocks $\psi = (\psi^{I,ab})$ carry two sets of labels, an index I running over four point tensor structures and an index ab that enumerates pairs of three point structures. The arguments $u_i^{s/t}$ of the blocks are functions on superspace that are invariant under superconformal transformations. They are related by an exchange of the labels 2 and 4 and we can express one in terms of the other. Equating the s - and t -channel expansion of the four-point function G_4 one finds that

$$\sum_I \sum_{\mathcal{O}} \lambda_{12\mathcal{O},a} \lambda_{34\mathcal{O},b} M_{st}^{II}(u_i^s) \psi_s^{I,ab}(\lambda_i(\mathcal{O}); u_i^s) = \sum_{\mathcal{O}} \lambda_{14\mathcal{O},a} \lambda_{23\mathcal{O},b} \psi_t^{J,ab}(\lambda_i(\mathcal{O}); u_i^t), \quad (4.41)$$

where the indexes a, b and I, J numerate three and four-point tensor structures, respectively, and the crossing factor $M_{st} = M_{st}(u_i)$ is given by

$$M_{st} = \hat{v}_t(x_i)^t \sqrt{\frac{\omega(u_i^t)}{\omega(u_i^s)}} \mathcal{M}_{st}(u_i, \theta_s, \bar{\theta}_s) v_s(x_i). \quad (4.42)$$

Note that the matrix elements of M_{st} depend on the bosonic cross ratios only. The summation in eq. (4.41) runs over all superprimary fields \mathcal{O} in the theory. Here we have expressed the crossing factor $\mathcal{M}_{st}(x_i)$ which we defined in eq. (4.13) in terms of the s -channel invariants and we think of the t -channel invariants on the right hand side as functions of the s -channel ones. Practically, it is easier to relate the t - and s -channel invariants u_i^t and u_i^s to the usual bosonic cross ratios and expand on both sides in the nilpotent invariants.

The additional factors v_s and v_t express the supercrossing factor \mathcal{M} in terms of its action in the space \mathcal{T} of tensor structures. Let us note that all three factors that appear in eq. (4.42) are well defined and straightforward to compute explicitly, even though the computations can be a bit cumbersome. The only additional information one then needs in order to evaluate the crossing symmetry constraint (4.41) is the relation between the bosonic cross ratios u_i^s and u_i^t in the two different channels. These are not difficult to determine from the Cartan decomposition. Let us stress, however, that the relations between u_i^s and u_i^t involve fermionic invariants so that there is an additional fermionic Taylor expansion to be performed on the right hand side of eq. (4.41) when we express u_i^t in terms of u_i^s . We will illustrate all this now in the case of $\mathfrak{g} = \mathfrak{sl}(2|1)$.

4.3 Illustration for 1-dimensional superconformal algebra

We can now put all the above together and compute the crossing factor between the s - and the t -channel for the $\mathcal{N} = 2$ superconformal algebra in one dimension. To this end, the first step is to find the group elements $g_s(x_i)$ and $g_t(x_i)$ which appear in the argument of the covariant function F . In turn, this requires the supergroup elements $m(x_i)$, see eq. (2.14), and the Weyl element (2.24). Then one computes the products $m(x_j)m(x_i)$ and $wm(x)$ to construct the variables x_{ij} and the action of the Weyl inversion on superspace. For the example at hand this was carried out in subsection 3.2.

These calculations provide all the input that is needed to determine $g_s(x_i)$. The supergroup elements $g_t(x_i)$ for the t -channel are obtained by exchanging the labels 2 and 4. At this point, g_s and g_t depend on four sets of superspace variables, i.e. they are 3×3 matrices whose elements are functions in all the $u_i, \theta_i, \bar{\theta}_i$ for $i = 1, \dots, 4$. Since the crossing factor is a superconformal invariant, we can apply superconformal transformations to gauge fix the coordinates of the four insertion points. The following choice turns out to be convenient

$$x_1 = (x, \theta_1, \bar{\theta}_1), \quad x_2 = (0, 0, 0), \quad x_3 = (1, \theta_3, \bar{\theta}_3), \quad x_4 = (\infty, 0, 0). \quad (4.43)$$

With this gauge choice, the entries of the matrices $g_s(x_i)$ and $g_t(x_i)$ depend on the bosonic coordinate x and the four Grassmann variables $\theta_{1,3}$ and $\bar{\theta}_{1,3}$ only.

In the second step we have to find the Cartan decomposition for both families g_s and g_t . For our 1-dimensional theory, the Cartan coordinates are introduced as

$$g = e^{\kappa R} e^{\lambda_t D} e^{\bar{q} Q_- + \bar{s} S_-} e^{\frac{u}{2}(P+K)} e^{q Q_+ + s S_+} e^{\lambda_r D}. \quad (4.44)$$

This agrees with the general prescription (4.2), except that the torus of elements a is parametrized by a single variable u in this case. Through straightforward manipulations of supermatrices one finds

the following expressions for the Cartan coordinates of g_s and g_t in our gauge (4.43). For the bosonic Cartan coordinates in s -channel one has

$$\cosh^2 \frac{u_s}{2} = \frac{1}{x} \left(1 - \frac{1}{2} \theta_3 \bar{\theta}_3 - \frac{\theta_1 \bar{\theta}_1}{2x} + \frac{\theta_1 \bar{\theta}_3}{x} + \frac{\theta_1 \bar{\theta}_1 \theta_3 \bar{\theta}_3}{4x} \right), \quad e^{-2\kappa_s} = 1 + \frac{\theta_1}{x} (\bar{\theta}_1 - \bar{\theta}_3), \quad (4.45)$$

$$e^{\lambda_{s,l} - \lambda_{s,r}} = \left(1 - x - \frac{1}{2} \theta_1 \bar{\theta}_1 - \frac{1}{2} \theta_3 \bar{\theta}_3 + \theta_1 \bar{\theta}_3 \right) \left(x - \frac{1}{2} \theta_1 \bar{\theta}_1 \right), \quad (4.46)$$

$$e^{\lambda_{s,l} + \lambda_{s,r}} = \left(1 + \frac{1}{2} \theta_3 \bar{\theta}_3 \right) \left(x - \frac{1}{2} \theta_1 \bar{\theta}_1 \right), \quad (4.47)$$

while in the t -channel these coordinates read

$$\cosh^2 \frac{u_t}{2} = x \left(1 + \frac{1}{2} \theta_3 \bar{\theta}_3 + \frac{\theta_1 \bar{\theta}_1}{2x} - \theta_1 \bar{\theta}_3 + \frac{\theta_1 \bar{\theta}_1 \theta_3 \bar{\theta}_3}{4x} \right), \quad e^{-2\kappa_t} = 1 + \bar{\theta}_3 (\theta_3 - \theta_1), \quad (4.48)$$

$$e^{\lambda_{t,l} - \lambda_{t,r}} = - \left(1 - x - \frac{1}{2} \theta_1 \bar{\theta}_1 - \frac{1}{2} \theta_3 \bar{\theta}_3 + \theta_1 \bar{\theta}_3 \right) \left(1 + \frac{1}{2} \theta_3 \bar{\theta}_3 \right), \quad (4.49)$$

$$e^{\lambda_{t,l} + \lambda_{t,r}} = \left(1 - \frac{1}{2} \theta_3 \bar{\theta}_3 \right) \left(x + \frac{1}{2} \theta_1 \bar{\theta}_1 \right). \quad (4.50)$$

In order to extract $\sinh(u_t/2)$ from the first and $\exp \lambda_{t,l}$, $\exp \lambda_{t,r}$ from the last two lines one has to take some square roots. Here we use the following convention

$$e^{\lambda_{t,r}} = i \left(\frac{x}{1-x} \right)^{\frac{1}{2}} - \dots, \quad e^{\lambda_{t,l}} = -i \sqrt{x(1-x)} - \dots, \quad \sinh(u_t/2) = i \sqrt{1-x} - \dots. \quad (4.51)$$

The fermionic Cartan coordinates, on the other hand, are given by the following expressions

$$q_s = e^{\frac{1}{2} \lambda_{s,r}} \left(\theta_3 - \frac{\theta_1}{x} \left(1 - \frac{1}{2} \theta_3 \bar{\theta}_3 \right) \right), \quad s_s = e^{-\frac{1}{2} \lambda_{s,r}} \frac{\theta_1}{x}, \quad (4.52)$$

$$\bar{q}_s = e^{-\frac{1}{2} \lambda_{s,l}} (\bar{\theta}_3 - \bar{\theta}_1), \quad \bar{s}_s = -e^{\frac{1}{2} \lambda_{s,l}} \frac{\bar{\theta}_3}{x}, \quad (4.53)$$

$$q_t = e^{\frac{1}{2} \lambda_{t,r}} (\theta_3 - \theta_1), \quad s_t = -e^{-\frac{1}{2} \lambda_{t,r}} \theta_1 \left(1 - \frac{1}{2} \theta_3 \bar{\theta}_3 \right), \quad (4.54)$$

$$\bar{q}_t = -e^{-\frac{1}{2} \lambda_{t,l}} \left(\bar{\theta}_1 - \bar{\theta}_3 \left(x + \frac{1}{2} \theta_3 \bar{\theta}_1 \right) \right), \quad \bar{s}_t = e^{\frac{1}{2} \lambda_{t,l}} \bar{\theta}_3. \quad (4.55)$$

This concludes the second step of the construction, namely the determination of the Cartan coordinates in the two channels.

As a third step we want to compute the supercrossing factor \mathcal{M}_{st} between the two channels that was defined in eq. (4.13). Note that for our superconformal algebra the group K is generated by dilations D and R-symmetry transformations R only. It is abelian and hence all its irreducible representations are 1-dimensional. Therefore, the supercrossing factor \mathcal{M}_{st} consists just of a single function in the variables $x, \theta_{1,3}$ and $\bar{\theta}_{1,3}$. It depends, of course, on the choice of representations for the external superfields.

We shall pick four such representations (Δ_i, r_i) , corresponding to the conformal weight and the R-charges of the superprimaries, as before. The associated representations ρ of $K = SO(1, 1) \times U(1)$ were introduced in eq. (3.19). Note that in our gauge (4.43) the factors $k(t_{41})$ and $k(t_{43})$ are trivial. Therefore, we have

$$\kappa_1 = e^{(\lambda_{s,l}-\lambda_{t,l})D+(\kappa_s-\kappa_t)R}, \quad \kappa_4 = e^{(\lambda_{t,l}+\lambda_{s,r})D-\kappa_t R}, \quad (4.56)$$

$$\kappa_3 = e^{(\lambda_{t,r}-\lambda_{s,r})D}, \quad \kappa_2 = e^{-(\lambda_{t,r}+\lambda_{s,l}-\log x^2)D+(\kappa_s-\frac{1}{2}\theta_3\bar{\theta}_3+\frac{\theta_1\bar{\theta}_1}{2x})R}. \quad (4.57)$$

The matrix \mathcal{M} can be written in terms of superspace coordinates by inserting our explicit formulas (4.45)-(4.50) for the Cartan coordinates in the s - and t -channel. This gives

$$\begin{aligned} \mathcal{M}_{st} = & e^{\frac{i\pi}{2}(\Delta_2+\Delta_4-\Delta_1-\Delta_3)} x^{-2\Delta_1} \alpha^{\frac{3}{2}\Delta_1-\frac{1}{2}\Delta_2-\frac{1}{2}\Delta_3-\frac{1}{2}\Delta_4} \times \\ & \times \beta^{\frac{1}{2}\Delta_1+\frac{1}{2}\Delta_2-\frac{3}{2}\Delta_3+\frac{1}{2}\Delta_4} e^{r_1(\kappa_s-\kappa_t)+r_2(\kappa_s-\frac{1}{2}\theta_3\bar{\theta}_3+\frac{\theta_1\bar{\theta}_1}{2x})-r_4\kappa_t}, \end{aligned} \quad (4.58)$$

where α and β denote the following superspace elements

$$\alpha = x + \frac{1}{2}\theta_1\bar{\theta}_1, \quad \beta = 1 - \frac{1}{2}\theta_3\bar{\theta}_3. \quad (4.59)$$

In order to compute the crossing factor M_{st} we are now instructed to find the map (4.36) in both s - and t -channel. The general construction of v is easy to implement since all representations are 1-dimensional. One finds

$$v = (1, q\bar{q}, q\bar{s}, s\bar{q}, s\bar{s}, qs\bar{q}\bar{s}). \quad (4.60)$$

Once we insert the expressions (4.45)-(4.55) for Cartan coordinates in the two channels we obtain

$$v_s = \left(1, -\frac{(\bar{\theta}_1-\bar{\theta}_3)(\theta_1-x\theta_3)}{x^{3/2}\sqrt{1-x}}, \frac{(\theta_1-x\theta_3)\bar{\theta}_3+\frac{1}{4}\Omega}{x^{3/2}}, \frac{(\bar{\theta}_1-\bar{\theta}_3)\theta_1+\frac{1}{4}\Omega}{x^{3/2}}, \frac{-\theta_1\bar{\theta}_3\sqrt{1-x}}{x^{3/2}}, \frac{\Omega}{x^2} \right), \quad (4.61)$$

$$v_t = \left(1, i\frac{(\theta_1-\theta_3)(\bar{\theta}_1-x\bar{\theta}_3)}{\sqrt{1-x}}, \frac{x\bar{\theta}_3(\theta_1-\theta_3)+\frac{1}{4}\Omega}{\sqrt{x}}, \frac{\theta_1(\bar{\theta}_1-x\bar{\theta}_3)+\frac{1}{4}\Omega}{\sqrt{x}}, i\theta_1\bar{\theta}_3\sqrt{1-x}, \Omega \right), \quad (4.62)$$

where $\Omega = \theta_1\bar{\theta}_1\theta_3\bar{\theta}_3$. Now we have all the elements that are needed to compute the crossing factor M_{st} which we defined in eq. (4.42) as

$$M_{st} = \hat{v}_t^T \sqrt{\frac{\sinh u_t}{\sinh u_s}} \mathcal{M}_{st} v_s, \quad (4.63)$$

where $\sinh u$ is a special instance of the function $\omega(u)$ that we introduced in eq. (4.33). All factors that enter our expression for \mathcal{M}_{st} belong to the algebra $\mathbb{C}[x, x^{-1}] \otimes \mathcal{A}$ where \mathcal{A} is the 6-dimensional algebra that is spanned by the elements

$$e_1 = 1, \quad e_2 = \theta_1\bar{\theta}_1, \quad e_3 = \theta_1\bar{\theta}_3, \quad e_4 = \theta_3\bar{\theta}_1, \quad e_5 = \theta_3\bar{\theta}_3, \quad e_6 = \Omega. \quad (4.64)$$

If we represent the e_i by the canonical (column) vector, the row vectors $v_{s/t}$ become 6×6 matrices whose entries are functions of x . Similarly we can also turn the factor $\sqrt{\frac{\sinh u_t}{\sinh u_s}} \mathcal{M}_{st}$ into a 6×6 matrix if we replace the elements e_i by their matrix representation in the left regular representation of \mathcal{A} . Multiplying all these matrices the final result is a 6×6 matrix of functions in x which is given by

$$M_{st} = v_t^{-1} \sqrt{\frac{\sinh u_t}{\sinh u_s}} \mathcal{M}_{st} v_s . \quad (4.65)$$

Having computed the crossing factor between s - and t -channel there is only one final step left, namely to relate the s - and t -channel cross ratios. Since the arguments of the functions f in the two channels are related by a change of variables that involves Grassmann coordinates, we need to perform a fermionic Taylor expansion in order to write the crossing equation in terms of functions of the bosonic cross ratio x only, e.g. in the t -channel this expansion of $f_t(\cosh^2 \frac{ut}{2})$ takes the following form

$$f_t = \left(1 + x \left(\frac{1}{2} \theta_3 \bar{\theta}_3 + \frac{\theta_1 \bar{\theta}_1}{2x} - \theta_1 \bar{\theta}_3 + \frac{\theta_1 \bar{\theta}_1 \theta_3 \bar{\theta}_3}{4x} \right) \partial + \frac{1}{4} x \Omega \partial^2 \right) f_t(x) . \quad (4.66)$$

Upon substitution, the crossing factor is a 6×6 matrix of second order differential operators in x . This concludes our construction of the crossing symmetry equations for long multiplets of $\mathcal{N} = 2$ superconformal field theories in $d = 1$ dimension.

5 Conclusions and Outlook

In this work we have laid out a systematic theory that allows to decompose four-point functions of local operators into superconformal blocks. It applies to all superconformal field theories in which the superconformal symmetry is of type I, i.e. for which the R-symmetry contains a $U(1)$ subgroup. This is the case for all superconformal field theories in $d = 4$ dimensions and a few other cases, in particular in $d = 1, 2$ and also 3-dimensional $\mathcal{N} = 2$ theories. In a first step we lifted the four point correlation function of arbitrary (long) operators to a function on the conformal supergroup, see eq. (3.8). A crucial ingredient in this auxiliary step was to assign a special family of supergroup elements $g(x_i)$ to the superspace insertion points x_i of the four fields. Let us stress that this first step is still entirely general in that it applies to all superconformal algebras and not just those of type I. The specialization became necessary for the second step in which we introduced a special supersymmetric version of the Cartan or KAK coordinates on the supergroup. As we had shown in our previous work [52], these coordinates are chosen to bring the Casimir equations into a remarkably simple form that allows to construct all superblocks as finite linear combinations of spinning bosonic blocks. The main purpose of the present work was to determine the associated tensor factors that map functions of two cross

ratios back to the original correlation function $G_4(x_i)$ on superspace. These tensor factors consist of two factors, a map $\Theta(x_i)$ on the space of superpolarizations, see eq. (4.32), and a map $v(x_i)$ from the space of superpolarizations to the space of tensor structures that was defined in eq. (4.36). The full evaluation of these two factors required to perform the Cartan decomposition of $g(x_i)$ explicitly. This is in principle straightforward, but can be a bit cumbersome, in particular for higher dimensions $d > 2$. We have illustrated the explicit computation at the example of the $\mathcal{N} = 2$ superconformal algebra in $d = 1$ dimension in the last subsection. Higher dimensional examples will be treated in forthcoming work.

For some applications and in particular in order to write down crossing symmetry equations, the tensor factors are actually not that important. What is needed, in addition to the conformal blocks, of course, is only the ratio of tensor factors between different channels. This quantity, which we dubbed the *crossing factor* is a superconformal invariant and hence it can be computed in any (super)conformal frame. Here we computed it for the $\mathcal{N} = 2$ superconformal algebra in $d = 1$. Along with our previous results on conformal blocks for this symmetry algebra, see [52], this allows to write down crossing symmetry constraints for long multiplets, recovering results from [39]. In the latter paper it was shown that the numerical (super-)conformal bootstrap involving long multiplets is significantly more constraining than the bootstrap with the short or the superprimary components of long multiplets. Our new derivation of these constraints, however, is now entirely algorithmic and it can be extended without any significant additional difficulty to higher dimensional superconformal algebras of type I. Let us also stress once again that the computation of the crossing factor in higher dimensional theories is significantly simpler than the computation of tensor factors. We have illustrated this at the example of bosonic conformal algebras where the computation of the crossing factor was reduced to computations in the subgroup $SO(1, 3)$ of the d -dimensional conformal group $SO(1, d + 1)$.

Our focus here was on developing the general theory. Concrete applications in particular to 4-dimensional superconformal theories will be addressed in forthcoming work. In particular we will spell out the crossing symmetry constraint between two channels of a four-point function involving two half-BPS and two long operators in a 4-dimensional $\mathcal{N} = 1$ superconformal theory. This requires to combine all the elements of our approach. On the one hand we apply the constructions and results of [52] to spell out the Casimir-equations in Calogero-Sutherland gauge and we use them to construct analytic expressions for the conformal blocks as finite sums of spinning bosonic blocks. When restricted to the superprimary fields at the bottom of the long multiplets, our new blocks coincide with those constructed in [37]. On the other hand, we evaluate our formula (4.42) for the crossing factor

in the example of $\mathfrak{sl}(1|4)$. Combining these two types of input we obtain crossing equations that can be exploited with existing numerical techniques. Since the superblocs are finite linear combinations of spinning bosonic blocks with coefficients whose analytical form is known, the evaluation of the superblocs only requires the numerical evaluation of 4-dimensional spinning bosonic blocks which has been developed in the past, see in particular [46]. Given the experience with the long multiplet bootstrap in $d = 2$ dimensions, see [39], we expect that numerical studies of the extended crossing equations can improve on the constraints obtained from the restricted equations in [37]. This may provide new clues also on the elusive minimal $\mathcal{N} = 1$ minimal superconformal field theory.

Of course it would also be interesting to spell out crossing symmetry constraints for other correlators in superconformal theories such as the multiplets of R-currents or the stress tensor multiplets. In principle our approach applies to such quantities as well, as long as the superconformal algebra is of type I. Of course, applications to various types of shorter multiplets containing conserved operators should provide some simplifications which we did not address in this work. It would be interesting to study these in more detail with a view on possible extensions of recent results in [64]. Similarly, when summing over all operators in the crossing equations, one has to take shorting conditions for the blocks of short intermediate exchange into account. Usually it is done on the case by case basis [28, 65–67]. It would be tempting to adopt the Calogero-Sutherland approach for a systematic analysis, at least in $d = 4$. Let us also mention that the situation is getting even more complicated in the case of non-unitary theories [68].

Another interesting direction concerns correlation functions involving non-local operators such as boundaries, interfaces and (line, surface, ...) defects. Block expansions for a large class of defect two-point functions are known, see e.g. [69–72]. A Calogero-Sutherland theory of such blocks was developed in [73]. It would be interesting to supplement this by a theory of tensor structures and to extend both ingredients, blocks and tensor structures, to the superconformal algebras. We will return to this issue in future work. An example of physically relevant 1-dimensional defects are superconformal light-ray operators which model high-energy scattering in supersymmetric gauge theories. Their two and three-point correlation functions in the BFKL limit was already calculated in [74–76]. These results may be considered as a first step in the realisation of the bootstrap programme for super lightray operators. Block expansions and bootstrap equations for supersymmetric defects have also been studied and applied e.g. in [30, 34, 38, 77, 78].

The restriction to type I superalgebras is certainly a limiting one that we would like to overcome in view of possible applications of the bootstrap e.g. to the 6-dimensional $(2, 0)$ theory [79] or to

many relevant examples of superconformal field theories in $d = 3$, see [80–84] for some recent work and further references. With the exception of the $\mathcal{N} = \infty$ superconformal algebra in $d = 3$, non of the superalgebras in these examples is of type I. While it is possible to treat some special cases with methods similar to those described here, in particular in low dimensions, it is not clear to us whether our approach does admit a systematic extension. This remains an interesting challenge for future research.

Acknowledgements: We thank James Drummond, Aleix Gimenez-Grau, Paul Heslop, Mikhail Isachenkov, Madalena Lemos, Pedro Liendo, Junchen Rong, Philine van Vliet for comments and fruitful discussion. The work of ES was supported by ERC grant 648630 IQFT. VS and IB acknowledge support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy – EXC 2121 „Quantum Universe” – 390833306.

A Proof of covariance laws

In this appendix we prove the transformation properties of g_{ij} and $k(t(y_{ji}))$ that were stated in eqs. (3.30) and (3.31) at the end of section 3. These are heavily used in section 4 in establishing conformal invariance of the crossing factor.

Proposition Under a superconformal transformation h , elements g_{ij} and $k(t_{ji})$ transform as

$$g_{ij}(x^h) = hg_{ij}(x)k(t(x_i, h))^{-1}, \quad k(t_{ji}^h) = wk(t(x_i, h))w^{-1}k(t_{ji})k(t(x_j, h))^{-1}. \quad (\text{A.1})$$

Proof: Consider the system of equations

$$m(x_i)n(y_{ji}) = g_{ij}(x), \quad (\text{A.2})$$

$$m(x_j)k(t_{ji})^{-1}n(z_{ji})^{-1} = g_{ij}(x)w^{-1}. \quad (\text{A.3})$$

The first equation is the definition of $g_{ij}(x)$ and the second was shown in the course of the derivation of eq. (3.8) in section 3.3. Let us apply a transformation h to all x_i -s. Furthermore, we use

$$m(x^h) = hm(x)k(t(x, h))^{-1}n(z(x, h))^{-1} \quad (\text{A.4})$$

which follows at once from definitions of $k(t(x, h))$ and $n(z(x, h))$. Doing these two steps, we get

$$hm(x_i)k(t(x_i, h))^{-1}n(z(x_i, h))^{-1}n(y_{ji}^h) = g_{ij}(x^h), \quad (\text{A.5})$$

$$hm(x_j)k(t(x_j, h))^{-1}n(z(x_j, h))^{-1}k(t_{ji}^h)^{-1}n(z_{ji}^h)^{-1} = g_{ij}(x^h)w^{-1}. \quad (\text{A.6})$$

By comparing the two systems of equations we see that

$$h^{-1}g_{ij}(x^h) = g_{ij}(x)k_{ij}n_{ij}, \quad h^{-1}g_{ij}(x^h)w^{-1} = g_{ij}(x)w^{-1}k'_{ij}n'_{ij}, \quad (\text{A.7})$$

for some $k_{ij}, k'_{ij}, n_{ij}, n'_{ij}$. By substituting the first of these equations into the second it follows

$$g_{ij}(x)k_{ij}n_{ij}w^{-1} = g_{ij}(x)w^{-1}k'_{ij}n'_{ij}, \quad (\text{A.8})$$

and consequently

$$k_{ij}n_{ij} = (w^{-1}k'_{ij}w)(w^{-1}n'_{ij}w). \quad (\text{A.9})$$

However, now the grading with respect to the dilation weight implies $n_{ij} = n'_{ij} = 1$. Further, a comparison of eqs. (A.2) and (A.5) yields $k_{ij} = k(t(x_i, h))^{-1}$. Now the proposition follows from equations (A.7). Namely, the first claim is obtained by substituting the results for k_{ij} and n_{ij} into the first equation. For the second, let us substitute $n'_{ij} = 1$ and $k'_{ij} = wk_{ij}w^{-1}$ into the second equation in eq. (A.7). This gives

$$h^{-1}g_{ij}(x^h) = g_{ij}(x)k_{ij} = g_{ij}(x)k(t(x_i, h))^{-1}. \quad (\text{A.10})$$

Finally, we substitute eq. (A.6) on the left hand side and eq. (A.3) on the right. Cancelling $m(x_j)$ factors, we arrive at

$$k(t(x_j, h))^{-1}n(z(x_j, h))^{-1}k(t_{ji}^h)^{-1}n(z_{ji}^h)^{-1}w = k(t_{ji})^{-1}n(z_{ji})^{-1}wk(t(x_i, h))^{-1}. \quad (\text{A.11})$$

The grading on \mathfrak{g} allows to pick the k -factors from both sides

$$k(t(x_j, h))^{-1}k(t_{ji}^h)^{-1} = k(t_{ji})^{-1}wk(t(x_i, h))^{-1}w^{-1}. \quad (\text{A.12})$$

Rearranging terms now gives the second claim. This completes the proof of the proposition.

B Supermanifolds and Lie supergroups

In this appendix we collect some properties of supermanifolds and Lie supergroups, following [59]. We hope these may be useful to some readers by offering more details on some constructions in sections 2, 3 and 4, which are however self-contained.

Recall that, by definition, a supermanifold M is a topological space X together with a sheaf A of superalgebras, such that around any point $x \in X$ there is an open neighbourhood U with $A(U) \cong C^\infty(U) \otimes \Lambda_n$, where Λ_n is the Grassmann algebra on n generators. The number n is called the odd

dimension of M . For any open set $V \subset X$, $A(V)$ is a commutative superalgebra. It is a non-trivial, but familiar, fact that the supermanifold can be completely recovered from its structure algebra $A(X)$. Some constructions regarding supermanifolds are more easily formulated in terms of a certain coalgebra $A(X)^*$ rather than the structure algebra itself. The $A(X)^*$ is defined as the space of all elements in the full dual $A(X)'$ which vanish on some ideal of finite codimension in $A(X)$. Elements of $A(X)^*$ are referred to as *distributions with finite support*. One observes that $A(X)^*$ is a supercocommutative coalgebra. Namely, let i and Δ be the natural injection and the diagonal map

$$i : A(X)' \otimes A(X)' \rightarrow (A(X) \otimes A(X))', \quad i(v \otimes w)(f \otimes g) = (-1)^{|w||f|}v(f)w(g), \quad (\text{B.1})$$

$$\Delta : A(X)' \rightarrow (A(X) \otimes A(X))', \quad (\Delta v)(f \otimes g) = v(fg), \quad v, w \in A(X)', \quad f, g \in A(X). \quad (\text{B.2})$$

Then one can show $\Delta(A(X)^*) \subset A(X)^* \otimes A(X)^*$, so the diagonal map makes $A(X)^*$ into a coalgebra. One again has that $A(X)^*$ determines the sheaf A . For example, X as a set can be recovered either as the set of all homomorphisms $A(X) \rightarrow \mathbb{R}$, or as the set of all group-like elements in $A(X)^*$. The coalgebra $A(X)^*$ also plays a prominent role in the theory of Lie supergroups and their actions on supermanifolds, as will be outlined presently.

B.1 Lie supergroups

Let \mathfrak{g} be a Lie superalgebra, H a group and $\pi : H \rightarrow \text{Aut}(U(\mathfrak{g}))$ a representation of H by algebra automorphisms. Further, write $F(H)$ for the group algebra of H . The *smash product* $E(H, \mathfrak{g}, \pi)$ is a supercocommutative Hopf algebra constructed as follows:

- 1) As a vector space $E = F(H) \otimes U(\mathfrak{g})$.
- 2) The multiplication in $F(H)$ and $U(\mathfrak{g})$ is defined in the usual way and $h x h^{-1} = \pi(h)x$.
- 3) The comultiplication Δ , counit η and the antipode σ are defined as

$$\Delta(h) = h \otimes h, \quad \Delta(x) = 1 \otimes x + x \otimes 1, \quad \eta(h) = 1, \quad \eta(x) = 0, \quad (\text{B.3})$$

$$\sigma(h) = h^{-1}, \quad \sigma(x) = -x, \quad \sigma(AB) = (-1)^{|A||B|}\sigma(B)\sigma(A). \quad (\text{B.4})$$

In these formulas $h \in H$, $x \in \mathfrak{g}$ and $A, B \in U(\mathfrak{g})$ are arbitrary. The set of group-like elements of E is precisely H and that of primitive elements is \mathfrak{g} . Here \mathfrak{g} is identified with a subspace of $U(\mathfrak{g})$ in the obvious way. Conversely, given a supercocommutative Hopf algebra E with the group of group-like elements H and the Lie superalgebra of primitive elements \mathfrak{g} one can show that a representation π exists such that $E = E(H, \mathfrak{g}, \pi)$. Now assume that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra and G_0 the connected, simply connected Lie group whose Lie algebra is \mathfrak{g}_0 . Then there is a unique representation

π on \mathfrak{g} by Lie superalgebra automorphisms which reduces to the adjoint representation on \mathfrak{g}_0 . The smash product $E(G_0, \mathfrak{g}, \pi)$ is called the simply-connected Lie-Hopf algebra associated with \mathfrak{g} and denoted by $E(\mathfrak{g})$.

A supermanifold (X, A) is said to be a Lie supergroup if the coalgebra $A(X)^*$ is a Hopf algebra. By the above remarks, in this case $A(X)^*$ is a smash product $E(G_0, \mathfrak{g}, \pi)$ with $X = G_0$. In fact, if X is simply connected, it can be shown that $A(X)^* = E(\mathfrak{g})$ for some Lie superalgebra, called the Lie superalgebra of (X, A) .

B.2 Supergroup actions

Assume now that $G = (G_0, A)$ is a Lie supergroup and $M = (Y, B)$ another supermanifold. We will say that G acts on M if there is a map $A(G_0)^* \otimes B(Y)^* \rightarrow B(Y)^*$, $u \otimes w \mapsto u \cdot w$, which satisfies

$$\Delta u = \sum_i u'_i \otimes u''_i, \quad \Delta w = \sum_j w'_j \otimes w''_j \implies \Delta(u \cdot w) = \sum_{i,j} (-1)^{|u''_i||w'_j|} u'_i \cdot w'_j \otimes u''_i \cdot w''_j. \quad (\text{B.5})$$

In this case, the structure algebra $B(Y)$ is a $A(G_0)^*$ -module through

$$\pi : A(G_0)^* \rightarrow \text{End}(B(Y)), \quad \langle w, \pi(u)f \rangle = (-1)^{|u||w|} \langle \sigma(u) \cdot w, f \rangle. \quad (\text{B.6})$$

The later is called the coaction representation of G . The action of G is fully determined by the corresponding coaction representation. Bearing in mind that $A(G_0)^* = E(\mathfrak{g})$, we see that a Lie supergroup action can be thought of as a pair of representations of the underlying group G_0 and of Lie superalgebra \mathfrak{g} on the vector space $B(Y)$, which satisfy a compatibility condition.

Dually, there is a map $\varphi : B(Y) \rightarrow B(Y) \otimes A(G_0)$ that makes $B(Y)$ into a comodule-algebra of $A(G_0)$. This means that φ is a morphism of algebras which is compatible with the Hopf algebra structure of $A(G_0)$. For example, φ satisfies

$$(1 \otimes \Delta) \circ \varphi = (\varphi \otimes 1) \circ \varphi : B(Y) \rightarrow B(Y) \otimes A(G_0) \otimes A(G_0), \quad (\text{B.7})$$

along with a number of other compatibility conditions, see e.g. [85]. Let p be a point in G_0 , considered as a morphism $p : A(G_0) \rightarrow \mathbb{R}$. Then one can form the map $(1 \otimes p) \circ \varphi : B(Y) \rightarrow B(Y)$. For obvious reasons, we refer to such compositions with p as *evaluations*. Running over all points p , we get a representation of the G_0 on $B(Y)$. This agrees with the coaction representation π from above.

In section 2.3 it was shown how the action of a superconformal group on a superspace may be computed explicitly. As was pointed out, the action requires the introduction of the tensor factor $A(G_0) = \mathcal{F}(\mathfrak{g})$ in the image of φ - generators of $A(G_0)$ were denoted collectively by the letter s . In the case of bosonic transformations, s could be evaluated in the sense just explained. For a fixed group element like the

Weyl inversion, only the evaluated action makes sense. The evaluated action of the bosonic group G_0 and the infinitesimal action of the conformal Lie superalgebra described in section 2.2 fit together to form a representation of the Lie-Hopf algebra $A(G_0)^*$ on $B(Y) = \mathcal{M}$.

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