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NON-LINEAR CANONICAL EQUATIONS OF COUPLED SYNCHRO-BETATRON MOTION AND THEIR SOLUTION WITHIN THE FRAMEWORK OF A NON-LINEAR 6-DIMENSIONAL (SYMPLECTIC) TRACKING PROGRAM FOR ULTRA-RELATIVISTIC PROTONS

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Abstract

Starting from the Lagrangian of a charged particle in an electromagnetic field, the Hamiltonian for non-linear coupled synchro-betatron oscillations of ultra-relativistic charged particles (protons) is derived. The canonical variables are x, p_x, z, p_z, σ, η which are well-known from the six dimensional linear theory (SLIM). Keeping only terms up to second order in the canonical momenta p_x, p_z, the equations of motion are then solved for various kinds of magnets (quadrupole, skew quadrupole, bending magnet, synchrotron-magnet, solenoid, sextupole, octupole, dipole kicker) and for cavities, taking into account the effect of energy deviation on the focusing strength. The equations so derived can serve to develope a non-linear, six dimensional (symplectic) tracking program for ultra-relativistic protons.

Contents

1. Introduction and Motivation

In the following we derive a system of non-linear equations for coupled synchro-betatron oscillations based on a Hamiltonian which is written in terms of the variables x, p_x, z, p_z, σ and η which are commonly used in the six-dimensional linear formalism (A. Chao)[1,2,3] and which, as shall be shown, are also canonical in this non-linear formalism. If one only keeps terms up to second order in the canonical momenta (so that the effect of energy deviation on the focusing strengths is automatically accounted for), these equations, which are symplectic, may be used to obtain solutions for the 6-dimensional motion for specific magnet types (quadrupole, skew-quadrupole, combined function dipole, solenoid, sextupole, octupole and kicker) as well as for linear and non-linear rf-cavities (σ is a canonical variable). Since they are written in canonical form, these equations can provide the basis of a non-linear, 6-dimensional tracking program.
Among the several applications of such a program the most interesting is the study of chaotic behaviour:

Because the equations of motion are in Hamiltonian form, the mappings representing the motion are symplectic. Thus, it is possible to study chaotic behaviour[4] in the 6-dimensional case taking into account non-linear fields and energy dependence of the focusing.

Other applications are the studies of:

Non-linear resonances: Because of the non-linearity of the equations of motion, not only the linear but also the non-linear resonances of synchro-betatron motion can be investigated. In particular, the tracking program proposed above would enable the position and width of both linear and non-linear satellite stopbands to be estimated with reasonable precision since the non-linear coupling between the synchrotron and betatron motion is specified in exact canonical form.

Resonance crossing: Although we are dealing here with a 6x6 formalism, the basic equations are organised so that by giving a constant energy deviation:

$$\eta = \frac{E - E_0}{E_0}$$

the required equations for simple betatron motion are available.

Thus, if we consider only the transverse part of the motion, the horizontal and vertical tune shifts

$$\delta Q_x = Q_x(\eta) - Q_x(0) \ ;$$

$$\delta Q_z = Q_z(\eta) - Q_z(0)$$

resulting from the energy dependence of the quadrupole focusing strength can be calculated for fixed values of η. Owing to the synchrotron oscillations, these δQ_x, δQ_z oscillate around zero and it may happen that they oscillate across resonances located nearby. With the program mentioned it should be possible to study the resulting blow up of the transverse amplitudes due to resonance crossing and to ensure that the motion is at the same time fully symplectic.

Chromaticity effects: As is well-known the chromaticity can be corrected with the help of sextupoles so that for off energy particles the Q-shifts δQ_x and δQ_z are eliminated[5)]. In this case, the resonance crossing no longer occurs. Unfortunately, these sextupoles generate additional non-linear resonances which can also present stability problems. Thus, it is necessary to choose an arrangement of sextupoles which minimizes this effect. With the proposed tracking program the efficacy of such a sextupole arrangement could be checked (symplectically) using particles executing energy oscillations and not just with fixed energy particles as is usually the case. This will also automatically take into account the effects of linear and non-linear satellite stopbands.

2. Derivation of the equations of motion

2.1. The Lagrangian for a charged particle

As a starting point we consider the relativistic Lagrangian of a charged particle of charge e and mass m_0 in an electromagnetic field:

$$\mathcal{L} = - m_0 \, c^2 \cdot \sqrt{1 - \frac{v^2}{c^2}} + \frac{e}{c} (\dot{\vec{r}} \cdot \vec{A}) - e \cdot \varphi \quad ; \tag{2.1}$$

$$(v = |\dot{\vec{r}}|)$$

where $\vec{r}$ is the position vector and $\vec{A}$ and $\vec{\varphi}$ are the vector and scalar potentials from which the electric field $\vec{\varepsilon}$ and the magnetic field $\vec{B}$ are given by

$$\vec{\varepsilon} = - \operatorname{grad} \varphi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \quad ; \tag{2.2a}$$

$$\vec{B} = \operatorname{rot} \vec{A} \quad . \tag{2.2b}$$

As usual, the equations of motion are derived from the Euler-Lagrange equations and in cartesian coordinates we have

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} + \frac{\partial \mathcal{L}}{\partial \vec{r}} = 0 \quad . \tag{2.3}$$

2.2. Introduction of the natural coordinates x, z, s

The position vector $\vec{r}$ in Eq. (2.1) refers to a fixed coordinate system. However, in accelerator physics, it is useful to introduce the natural coordinates x, z, s [3]. With this in mind we assume that an ideal closed orbit (design orbit) exists describing the path of a particle of constant energy E_0 (neglecting of course energy variations due to radiation loss and assuming that there are no field errors or correction magnets). We also assume that the closed orbit comprises piecewise flat curves which lie either in the horizontal or vertical plane so that it has no torsion. The design orbit which will be used as the reference system will in the following be described by the vector $\vec{r}_0(s)$ where s is the length along the design orbit. An arbitrary particle orbit $\vec{r}(s)$ is then described by the deviation $\delta\vec{r}$ of the particle orbit $\vec{r}(s)$ from the design orbit $\vec{r}_0(s)$:

$$\vec{r}(s) = \vec{r}_0(s) + \delta\vec{r}(s) \quad .$$

The vector $\delta\vec{r}$ can as usual be described using an orthogonal coordinate system ("dreibein") accompanying the particles and comprising

$$\begin{aligned} &\text{a unit normal vector} && \vec{\nu}(s), \\ &\text{a unit tangent vector} && \vec{\tau}(s), \\ \text{and } &\text{a unit binormal vector} && \vec{\beta}(s) = \vec{\tau}(s) \times \vec{\nu}(s). \end{aligned}$$

We require that the vector $\vec{\nu}(s)$ is directed outwards if the motion takes place in the horizontal plane and upwards if the motion takes place in the vertical plane [3].

Choosing the direction of $\vec{\nu}(s)$ in this way, implies that the curvature $K(s)$ appearing in the Fresnet formulae:

$$\vec{\tau}(s) = \frac{d}{ds}\vec{r}_0(s) \equiv \vec{r}_0{}'(s) \quad ; \tag{2.4}$$

$$\begin{aligned} \frac{d\vec{\tau}}{ds} &= -K(s)\cdot\vec{\nu}(s) \quad ; \\ \frac{d\vec{\nu}}{ds} &= K(s)\cdot\vec{\tau}(s) \quad ; \\ \frac{d\vec{\beta}}{ds} &= 0 \end{aligned} \tag{2.5}$$

is always positive in the horizontal plane and negative in the vertical plane if and only if the centre of curvature lies above the reference trajectory.

In the natural coordinate system we can represent $\delta\vec{r}(s)$ as:

$$\delta\vec{r}(s) = (\delta\vec{r}\cdot\vec{\nu})\cdot\vec{\nu} + (\delta\vec{r}\cdot\vec{\beta})\cdot\vec{\beta}$$

(since the "dreibein" accompanies the particle the $\vec{\tau}$-component of $\delta\vec{r}$ is always zero by definition).

However this representation has the disadvantage that the direction of the normal vector $\vec{\nu}(s)$ changes discontinuously if the particle trajectory is going over from the vertical plane to the horizontal plane and vice versa. Therefore, it is advantageous to introduce new unit vectors $\vec{\tau}$, $\vec{e}_x$ and $\vec{e}_z$ which change their directions continuously.

This is achieved by putting

$$\vec{e}_x(s) = \begin{cases} \vec{\nu}(s), & \text{if the orbit lies in the horizontal plane;} \\ -\vec{\beta}(s), & \text{if the orbit lies in the vertical plane;} \end{cases}$$

$$\vec{e}_z(s) = \begin{cases} \vec{\beta}(s), & \text{if the orbit lies in the horizontal plane;} \\ \vec{\nu}(s), & \text{if the orbit lies in the vertical plane.} \end{cases}$$

Thus, the orbit-vector $\vec{r}(s)$ can be written in the form

$$\vec{r}(s,x,z) = \vec{r}_0(s) + x(s)\cdot\vec{e}_x(s) + z(s)\cdot\vec{e}_z(s) \tag{2.6}$$

and the Fresnet formulae (2.5) now read

$$\begin{aligned} \frac{d}{ds}\vec{e}_x(s) &= K_x(s)\cdot\vec{\tau}(s) \quad ; \\ \frac{d}{ds}\vec{e}_z(s) &= K_z(s)\cdot\vec{\tau}(s) \quad ; \\ \frac{d}{ds}\vec{\tau}(s) &= -K_x(s)\cdot\vec{e}_x(s) - K_z(s)\cdot\vec{e}_z(s) \end{aligned} \tag{2.7}$$

with

$$K_x(s)\cdot K_z(s) = 0 \tag{2.8}$$

where $K_x(s)$, $K_z(s)$ designate the curvatures in the x-direction and the z-direction respectively.

For later considerations we mention that the connection between the curvatures K_x, K_z and the guide fields $B_z^{(o)}$, $B_x^{(o)}$ is given by[2,3)]

$$\begin{cases} K_x = \dfrac{e}{E_0}\cdot B_z^{(o)} \quad ; \\ K_z = -\dfrac{e}{E_0}\cdot B_x^{(o)} \quad . \end{cases} \tag{2.9}$$

From Eqs. (2.4), (2.6) and (2.7) one then has

$$\dot{\vec{r}} = \dot{s} \cdot \left[\frac{d\vec{r}_0}{ds} + x \cdot \frac{d\vec{e}_x}{ds} + z \cdot \frac{d\vec{e}_z}{ds}\right] + \dot{x} \cdot \vec{e}_x + \dot{z} \cdot \vec{e}_z$$

$$= \vec{\tau} \cdot \dot{s}(1 + x \cdot K_x + z \cdot K_z) + \dot{x} \cdot \vec{e}_x + \dot{z} \cdot \vec{e}_z$$

so that for the expressions

$$\sqrt{1 - \frac{v^2}{c^2}} \quad \text{and} \quad (\dot{\vec{r}} \cdot \vec{A})$$

in Eq. (2.1) we have

$$\sqrt{1 - \frac{v^2}{c^2}} = \left\{1 - \frac{1}{c^2}\left[\dot{x}^2 + \dot{z}^2 + (1 + K_x \cdot x + K_z \cdot z)^2 \cdot \dot{s}^2\right]\right\}^{1/2} \quad ;$$

$$(\dot{\vec{r}} \cdot \vec{A}) = \dot{x} \cdot A_x + \dot{z} \cdot A_z + \dot{s}(1 + K_x \cdot x + K_z \cdot z) \cdot A_s \quad . ^{*)}$$

In the new coordinate system x,z,s, the Lagrangian in Eq. (2.1) then becomes

$$\mathcal{L}(x,z,s,\dot{x},\dot{z},\dot{s},t) = -m_0 c^2 \cdot \left\{1 - \frac{1}{c^2}\left[\dot{x}^2 + \dot{z}^2 + (1 + K_x \cdot x + K_z \cdot z)^2 \cdot \dot{s}^2\right]\right\}^{1/2} +$$

$$+ \frac{e}{c} \cdot \left\{\dot{x} \cdot A_x + \dot{z} \cdot A_z + (1 + K_x \cdot x + K_z \cdot z) \cdot \dot{s} \cdot A_s\right\} - e\varphi \qquad (2.10)$$

and the equations of motion take the form

$$\left\{\begin{array}{l} \dfrac{d}{dt} \dfrac{\partial \mathcal{L}}{\partial \dot{x}} = \dfrac{\partial \mathcal{L}}{\partial x} \quad ; \\[2ex] \dfrac{d}{dt} \dfrac{\partial \mathcal{L}}{\partial \dot{z}} = \dfrac{\partial \mathcal{L}}{\partial z} \quad ; \\[2ex] \dfrac{d}{dt} \dfrac{\partial \mathcal{L}}{\partial \dot{s}} = \dfrac{\partial \mathcal{L}}{\partial s} \quad . \end{array}\right. \qquad (2.11)$$

*) The components of a vector $\vec{a}$ with respect to the $(\vec{e}_x, \vec{e}_z, \vec{\tau})$ coordinate system are defined by the equation

$$\vec{a} = a_x \cdot \vec{e}_x + a_z \cdot \vec{e}_z + a_s \cdot \vec{\tau} \quad .$$

2.3. The Hamiltonian in the natural coordinates x, z, s

In order to obtain the equations of motion in canonical form we now use the Lagrangian (Eq. (2.10)) to construct the corresponding Hamiltonian:

$$\mathcal{H} = p_x \cdot \dot{x} + p_z \cdot \dot{z} + p_s \cdot \dot{s} - \mathcal{L} \tag{2.12}$$

where p_x, p_z, p_s are the canonical momenta

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} \; ; \quad p_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} \; ; \quad p_s = \frac{\partial \mathcal{L}}{\partial \dot{s}} \; .$$

Using Eq. (2.10) these are given by

$$\left\{ \begin{aligned} p_x &= \frac{m_0 \dot{x}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c} A_x \; ; \\ p_z &= \frac{m_0 \dot{z}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c} A_z \; ; \\ p_s &= \frac{m_0 \dot{s}}{\sqrt{1 - \frac{v^2}{c^2}}} \cdot (1 + K_x \cdot x + K_z \cdot z)^2 + \frac{e}{c} (1 + K_x \cdot x + K_z \cdot z) \cdot A_s \; . \end{aligned} \right. \tag{2.13}$$

Putting now Eqs. (2.13) and (2.10) into (2.12) we have

$$\mathcal{H} = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} + e \cdot \varphi \tag{2.14}$$

(thus, as is well known, $\mathcal{H}$ here is the sum of the mechanical and field energy).

In this equation, the momenta are still written in terms of the velocity. However, using the relation

$$(p_x - \frac{e}{c} A_x)^2 + (p_z - \frac{e}{c} A_z)^2 + \left[\frac{p_s}{(1 + K_x \cdot x + K_z \cdot z)} - \frac{e}{c} A_s\right]^2 + $$

$$+ m_0^2 c^2 = \frac{m_0^2 c^2}{1 - \frac{v^2}{c^2}} \qquad (2.15)$$

we may finally write the relativistic Hamiltonian for the motion of a particle of charge e and mass m_0 in an electromagnetic field given by the potentials $\vec{A}$ and φ as

$$\mathcal{H}(x,z,s,p_x,p_z,p_s,t) = c \cdot \{m_0^2 c^2 + (p_x - \frac{e}{c} A_x)^2 + (p_z - \frac{e}{c} A_z)^2 + $$

$$+ \left[\frac{p_s}{1 + K_x \cdot x + K_z \cdot z} - \frac{e}{c} A_s\right]^2 \}^{1/2} + e \cdot \varphi \, . \qquad (2.16)$$

The equations of motion are then derived from the canonical equations

$$\begin{cases} \dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} \quad ; \quad \dot{p}_x = - \frac{\partial \mathcal{H}}{\partial x} \quad ; \\ \dot{z} = \frac{\partial \mathcal{H}}{\partial p_z} \quad ; \quad \dot{p}_z = - \frac{\partial \mathcal{H}}{\partial z} \quad ; \\ \dot{s} = \frac{\partial \mathcal{H}}{\partial p_s} \quad ; \quad \dot{p}_s = - \frac{\partial \mathcal{H}}{\partial s} \quad . \end{cases} \qquad (2.17)$$

2.4. The arc length as independent variable

In Eq. (2.17) the time t appeared as independent variable. In order, as is usual in accelerator physics, to introduce the arc length s of the design orbit as independent variable we recall that Eq. (2.17) is equivalent to a version of Hamilton's principle

$$\delta \int_{t_1}^{t_2} dt \cdot \{\dot{x} \cdot p_x + \dot{z} \cdot p_z + \dot{s} \cdot p_s - \mathcal{H}(x,z,s,p_x,p_z,p_s,t)\} = 0 \; ; \qquad (2.18)$$

$$\begin{cases} \delta x(t_1) = \delta z(t_1) = \delta s(t_1) = 0 \; ; \;\; \delta p_x(t_1) = \delta p_z(t_1) = \delta p_s(t_1) = 0 \; ; \\ \delta x(t_2) = \delta z(t_2) = \delta s(t_2) = 0 \; ; \;\; \delta p_x(t_2) = \delta p_z(t_2) = \delta p_s(t_2) = 0 \; ; \\ \delta t(t_1) = \delta t(t_2) = 0 \; , \end{cases}$$

where the variables x, z, s, p_x, p_z, p_s, t are varied independently of each other and are held constant at the end points. (For the usual derivation of the Hamilton equations (2.17) from the variational principle (2.18) the variation of time t is actually not needed[6]. However, in order to be able to carry out the derivation of Eq. (2.10) it is useful, nevertheless, to allow t to vary; see Appendix I).

Eq. (2.18) can now be rewritten using $dt = \frac{dt}{ds}\, ds$ as[7]:

$$\delta \int_{t_1}^{t_2} ds \cdot \{x' \cdot p_x + z' \cdot p_z + t' \cdot (-\mathcal{H}) + p_s(x,z,t,p_x,p_z,-\mathcal{H},s)\} = 0 \; ;$$

$$\begin{cases} \delta x(s_1) = \delta z(s_1) = \delta t(s_1) = 0 \; ; \;\; \delta p_x(s_1) = \delta p_z(s_1) = \delta \mathcal{H}(s_1) = 0 \; ; \\ \delta x(s_2) = \delta z(s_2) = \delta t(s_2) = 0 \; ; \;\; \delta p_x(s_2) = \delta p_z(s_2) = \delta \mathcal{H}(s_2) = 0 \; ; \\ \delta s(s_1) = \delta s(s_2) = 0 \end{cases} \qquad (2.19)$$

with

$$y' \equiv \frac{dy}{ds} \quad (y \equiv x,\, z,\, t)$$

(where we make independent variations of the variables x, z, t, p_x, p_z, $(-\mathcal{H})$, s; and s is the independent variable).

The required equations with s as independent variable are then obtained from the Euler equations of the variational problem (2.19):

$$x' = \frac{\partial K}{\partial p_x} \quad ; \quad p_x' = -\frac{\partial K}{\partial x} \quad ;$$

$$z' = \frac{\partial K}{\partial p_z} \quad ; \quad p_z' = -\frac{\partial K}{\partial z} \quad ; \qquad (2.20)$$

$$t' = \frac{\partial K}{\partial(-\mathcal{H})} \quad ; \quad (-\mathcal{H}') = -\frac{\partial K}{\partial t}$$

with

$$K = -p_s$$

$$= -(1 + K_x \cdot x + K_z \cdot z) \cdot \left\{\frac{(\mathcal{H} - e\varphi)^2}{c^2} - m_0^2 c^2 - (p_x - \frac{e}{c} A_x)^2 - (p_z - \frac{e}{c} A_z)^2\right\}^{1/2}$$

$$- (1 + K_x \cdot x + K_z \cdot z) \cdot \frac{e}{c} A_s \quad . \qquad (2.21)$$

Thus, we once again have a set of equations with canonical structure but this time the Hamiltonian is

$$K = K(x,\ z,\ t,\ p_x,\ p_z,\ -\mathcal{H},\ s)$$

and the canonical variables[7,8,9,10] are

$$(x,\ p_x) \quad ; \quad (z,\ p_z) \quad ; \quad (t,\ -\mathcal{H}) \quad .$$

In the following we choose a gauge in which

$$\varphi = 0 \quad . \qquad (2.22)$$

Then from Eq. (2.14)

$$\mathcal{H} = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv E \quad , \qquad (2.23)$$

(the energy of the particle)

and if we now use the variables (- ct) and η

$$\eta = \frac{E - E_0}{E_0}$$

instead of t and $\mathcal{H}$, Eq. (2.20) gives

$$\left\{\begin{array}{llll} x' = \dfrac{\partial \overline{K}}{\partial \hat{p}_x} & ; & \hat{p}_x' = -\dfrac{\partial \overline{K}}{\partial x} & ; \\[2ex] z' = \dfrac{\partial \overline{K}}{\partial \hat{p}_z} & ; & \hat{p}_z' = -\dfrac{\partial \overline{K}}{\partial z} & ; \\[2ex] (-ct)' = \dfrac{\partial \overline{K}}{\partial \eta} & ; & \eta' = -\dfrac{\partial \overline{K}}{\partial(-ct)} & \end{array}\right. \tag{2.24}$$

with

$$\begin{aligned} \overline{K} &= \frac{c}{E_o} \cdot K \\ &= -(1+K_x \cdot x + K_z \cdot z) \cdot \left\{(1+\eta)^2 - \left(\frac{m_o c^2}{E_o}\right)^2 - (\hat{p}_x - \frac{e}{E_o} A_x)^2 - (\hat{p}_z - \frac{e}{E_o} A_z)^2\right\}^{1/2} \\ &\quad -(1+K_x \cdot x + K_z \cdot z) \cdot \frac{e}{E_o} A_s \quad ; \end{aligned} \tag{2.25}$$

$$\begin{aligned} \hat{p}_x &= \frac{c}{E_o} p_x = \frac{c}{E_o} m\, \upsilon_x + \frac{e}{E_o} A_x \quad ; \\ \hat{p}_z &= \frac{c}{E_o} p_z = \frac{c}{E_o} m\, \upsilon_z + \frac{e}{E_o} A_z \quad . \end{aligned} \tag{2.26}$$

Since the variable t(s) increases without limit, it is more useful to introduce the variable

$$\sigma = s - c \cdot t(s) \tag{2.27}$$

which describes the delay in arrival time at position s of a particle travelling at the speed of light c.

This further change of variables can be achieved using the generating function

$$F_3(p, \overline{q}, s) \equiv F_3(p_x, p_z, \eta, \overline{x}, \overline{z}, \sigma, s) = \underbrace{-p_x \cdot \overline{x} - p_z \cdot \overline{z}}_{\text{identity transformation}} - \sigma \cdot \eta + s \cdot \eta + f(s)$$

The corresponding transformation equations:

$$(-ct) = -\frac{\partial F_3}{\partial \eta} \quad \text{(which leads to} \quad -ct = \sigma - s \quad ; \quad \sigma = s - ct \quad ;)$$

$$\bar{\eta} = -\frac{\partial F_3}{\partial \sigma} \quad \text{(which leads to} \quad \bar{\eta} = \eta)$$

then immediately give (with $f(s) = s$) Hamiltonian equations of the form

$$\left\{ \begin{aligned} x' &= \frac{\partial \hat{K}}{\partial \hat{p}_x} \quad ; \quad \hat{p}_x' = -\frac{\partial \hat{K}}{\partial x} \quad ; \\ z' &= \frac{\partial \hat{K}}{\partial \hat{p}_z} \quad ; \quad \hat{p}_z' = -\frac{\partial \hat{K}}{\partial z} \quad ; \\ \sigma' &= \frac{\partial \hat{K}}{\partial \eta} \quad ; \quad \eta' = -\frac{\partial \hat{K}}{\partial \sigma} \end{aligned} \right. \tag{2.28}$$

with the Hamiltonian

$$\hat{K} = (1+\eta)\cdot\left\{1 - (1 + K_x \cdot x + K_z \cdot z)\cdot\left[1 - \frac{(\hat{p}_x - \frac{e}{E_0} A_x)^2}{(1+\eta)^2} - \frac{(\hat{p}_z - \frac{e}{E_0} A_z)^2}{(1+\eta)^2}\right]^{1/2}\right\}$$
$$- (1 + K_x \cdot x + K_z \cdot z)\cdot\frac{e}{E_0} A_s \quad , \tag{2.29}$$

where the term $\left(\frac{m_0 c^2}{E_0}\right)^2$ in Eq. (2.25) has been dropped since we assume that

$$\left(\frac{m_0 c^2}{E_0}\right)^2 \ll 1$$

and can be neglected.

The canonical equations (2.28) together with the Hamiltonian (2.29) give the defining equations for non-linear coupled synchro-betatron motion and they will serve as the starting point for the developments to follow.

Finally we point out that since $|(\hat{p}_y - \frac{e}{E_o} A_y)| = \left|\frac{c}{E_o} m\upsilon_y\right| \ll 1$ $(y \equiv x,z)$ the square root

$$[1 - \frac{(\hat{p}_x - \frac{e}{E_o} A_x)^2}{(1 + \eta)^2} - \frac{(\hat{p}_z - \frac{e}{E_o} A_z)^2}{(1 + \eta)^2}]^{1/2}$$

in Eq. (2.29) can be expanded in a series:

$$[1 - \frac{(\hat{p}_x - \frac{e}{E_o} A_x)^2}{(1 + \eta)^2} - \frac{(\hat{p}_z - \frac{e}{E_o} A_z)^2}{(1 + \eta)^2}]^{1/2} =$$

$$= 1 - \frac{1}{2} \frac{(\hat{p}_x - \frac{e}{E_o} A_x)^2}{(1 + \eta)^2} - \frac{1}{2} \frac{(\hat{p}_z - \frac{e}{E_o} A_z)^2}{(1 + \eta)^2} + \ldots \tag{2.30}$$

so that in practice the particle motion can be conveniently calculated to various orders of approximation.

Remarks

1) From Eq. (2.28) and (2.29) we can obtain the differential equation for σ:

$$\sigma' = \frac{\partial\hat{K}}{\partial\eta} = 1 - (1 + K_x \cdot x + K_z \cdot z) \cdot \frac{1}{\left[1 - \dfrac{(\hat{p}_x - \frac{e}{E_0} A_x)^2}{(1+\eta)^2} - \dfrac{(\hat{p}_z - \frac{e}{E_0} A_z)^2}{(1+\eta)^2}\right]^{1/2}}$$

which with

$$x' = \frac{\partial\hat{K}}{\partial\hat{p}_x} = + (1 + K_x \cdot x + K_z \cdot z) \cdot \frac{\dfrac{(\hat{p}_x - \frac{e}{E_0} A_x)}{(1+\eta)}}{\left[1 - \dfrac{(\hat{p}_x - \frac{e}{E_0} A_x)^2}{(1+\eta)^2} - \dfrac{(\hat{p}_z - \frac{e}{E_0} A_z)^2}{(1+\eta)^2}\right]^{1/2}} ;$$

$$z' = \frac{\partial\hat{K}}{\partial\hat{p}_z} = + (1 + K_x \cdot x + K_z \cdot z) \cdot \frac{\dfrac{(\hat{p}_z - \frac{e}{E_0} A_z)}{(1+\eta)}}{\left[1 - \dfrac{(\hat{p}_x - \frac{e}{E_0} A_x)^2}{(1+\eta)^2} - \dfrac{(\hat{p}_z - \frac{e}{E_0} A_z)^2}{(1+\eta)^2}\right]^{1/2}}$$

can also be written as

$$\sigma' = 1 - [(1 + K_x \cdot x + K_z \cdot z)^2 + (x')^2 + (z')^2]^{1/2} .$$

This result could also have been obtained directly from Eq. (2.27) (together with Eqs. (2.4, 6, 7)

$$d\sigma = ds - c \cdot dt = ds - |d\vec{r}| ; \quad \Longrightarrow \quad \sigma' = 1 - \left|\frac{d\vec{r}}{ds}\right| .$$

2) To derive the Hamiltonian K in Eq. (2.21) we began from a Lagrangian. It was then simple to derive the generalised momenta p_x, p_z, p_s (Eq. (2.13)) conjugate to the natural coordinates x, z, s without using a canonical transformation. The function K of Eq. (2.21) agrees with that given by C.J.A. Corsten[10] but differs from that given by Courant and Snyder[8]. For a derivation of the Courant and Snyder version see Ref. 9).

2.5. Vector potentials for various magnet types

In order to utilize the Hamiltonian of (2.29), the vector potential,

$$\vec{A} = \vec{A}(x, z, s, t) \;, \tag{2.31}$$

for the commonly occuring types of accelerator magnet must be given. Once $\vec{A}$ is known the fields $\vec{\varepsilon}$ and $\vec{B}$ can be found using Eq. (2.2). In the variables x, z, s, σ these become [11)]

$$\vec{\varepsilon} = \frac{\partial}{\partial\sigma}\vec{A} \quad ; \tag{2.32a}$$

$$\begin{cases} B_x = \dfrac{1}{h}\cdot\{\dfrac{\partial}{\partial z}(h\cdot A_s) - \dfrac{\partial}{\partial s}A_z\} \quad ; \\ B_z = \dfrac{1}{h}\cdot\{\dfrac{\partial}{\partial s}A_x - \dfrac{\partial}{\partial x}(h\cdot A_s)\} \quad ; \\ B_s = \{\dfrac{\partial}{\partial x}A_z - \dfrac{\partial}{\partial z}A_x\} \end{cases} \tag{2.32b}$$

with

$$h = 1 + K_x\cdot x + K_z\cdot z \;. \tag{2.33}$$

Using the freedom of gauge, we can choose any vector potential which leads to the correct form of the fields. Suitable vector potentials are as follows and have been chosen for their simplicity.

2.5.1. Cavity

For a longitudinal electric field

$$\begin{aligned} \varepsilon_x &= 0 \\ \varepsilon_z &= 0 \\ \varepsilon_s &= \varepsilon(s,\sigma) \end{aligned} \tag{2.34}$$

we write

$$\begin{cases} A_x = 0 \quad ; \\ A_z = 0 \quad ; \\ A_s = \displaystyle\int_{\sigma_0}^{\sigma} d\tilde{\sigma}\cdot\varepsilon(s,\tilde{\sigma}) \quad , \end{cases} \tag{2.35}$$

which by (2.31) immediately gives ε_s.

Using a thin lens representation we may write

$$\varepsilon(s,\sigma) = \hat{V} \cdot \sin[k \cdot \frac{2\pi}{L} \cdot \sigma + \Phi] \cdot \delta(s - s_0) \qquad (2.36a)$$

and we obtain using (2.35)

$$A_s = - \frac{L}{2\pi \cdot k} \cdot \hat{V} \cdot \cos[k \cdot \frac{2\pi}{L} \cdot \sigma + \Phi] \cdot \delta(s - s_0) \ , \qquad (2.36b)$$

in which the phase Φ is defined so that the average energy radiated away in the bending magnets is replaced by the cavities and k is the harmonic number. For protons (for which there is no energy loss) one has $\Phi = 0$. The influence of averaged radiation loss on the motion can be taken into account by including in the Hamiltonian $\hat{K}$ (Eq. 2.29)) an additional term

$$\sigma \cdot C_1 \cdot (K_x^2 + K_z^2) \qquad \text{with } C_1 = \frac{2}{3} e^2 \frac{\gamma_0^4}{E_0}$$

This causes a shift of the closed orbit (see the term $\vec{c}_0$ in Ref. 3), Eq. (4.2b)). Thus, in this approach, energy loss effects can be treated canonically. For protons, this term can be neglected.

2.5.2. Transverse magnetic fields

2.5.2.1. Transverse magnetic fields in a straight section

$$A_x = 0 \ ; \ A_z = 0 \ ;$$

$$K_x = K_z = 0 \ ;$$

$$\hat{K} = (1 + \eta) \cdot \{1 - \left[1 - \frac{\hat{p}_x^2}{(1 + \eta)^2} - \frac{\hat{p}_z^2}{(1 + \eta)^2}\right]^{1/2} \} - \frac{e}{E_0} A_s \ ; \qquad (2.37)$$

$$B_x = \partial_z \cdot A_s \ ;$$

$$B_z = - \partial_x \cdot A_s \ ;$$

$$B_s = 0 \ .$$

2.5.2.1.1. Quadrupole

The quadrupole fields are

$$B_x = z \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \ ;$$

$$B_z = x \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \ ;$$

so that we may use the vector potential

$$A_s = \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \cdot \frac{1}{2}(z^2 - x^2) \quad .$$

In the following we rewrite the term $\frac{e}{E_o} A_s$ in (2.37) as

$$\frac{e}{E_o} A_s = \frac{1}{2} g_o \cdot (z^2 - x^2) \, ; \tag{2.38a}$$

$$g_o = \frac{e}{E_o} \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \tag{2.38b}$$

2.5.2.1.2. Skew quadrupole

The fields are

$$\begin{cases} B_x = \frac{1}{2}\left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z}\right)_{x=z=0} \cdot x \quad ; \\ B_z = -\frac{1}{2}\left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z}\right)_{x=z=0} \cdot z \end{cases}$$

so that we may use

$$A_s = \frac{1}{2}\left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z}\right)_{x=z=0} \cdot xz$$

or

$$\frac{e}{E_o} A_s = N_o \cdot xz \tag{2.39a}$$

where

$$N_o = \frac{1}{2} \frac{e}{E_o} \left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z}\right)_{x=z=0} \quad . \tag{2.39b}$$

2.5.2.1.3. Sextupole

$$B_x = \left(\frac{\partial^2 B_z}{\partial x^2}\right)_{x=z=0} \cdot xz \quad ;$$

$$B_z = \left(\frac{\partial^2 B_z}{\partial x^2}\right)_{x=z=0} \cdot \frac{1}{2}(x^2 - z^2)$$

so that

$$\frac{e}{E_o} A_s = -\lambda_o \cdot \frac{1}{6}(x^3 - 3xz^2) \tag{2.40a}$$

with

$$\lambda_o = \frac{e}{E_o} \left(\frac{\partial^2 B_z}{\partial x^2}\right)_{x=z=0} \quad . \tag{2.40b}$$

2.5.2.1.4. <u>Octupole</u>

$$\begin{cases} B_x = \dfrac{1}{6}\left(\dfrac{\partial^3 B_x}{\partial z^3}\right)_{x=z=0} \cdot (z^3 - 3x^2 z) \quad ; \\ B_z = \dfrac{1}{6}\left(\dfrac{\partial^3 B_x}{\partial z^3}\right)_{x=z=0} \cdot (3xz^2 - x^3) \end{cases}$$

so that

$$\frac{e}{E_0} A_s = \mu_0 \cdot \frac{1}{24}(z^4 - 6x^2z^2 + x^4) \tag{2.41a}$$

with

$$\mu_0 = \frac{e}{E_0} \cdot \left(\frac{\partial^3 B_x}{\partial z^3}\right)_{x=z=0} \quad . \tag{2.41b}$$

2.5.2.1.5. <u>Dipole</u>

$$\begin{cases} B_x = \Delta\hat{B}_x \cdot \delta(s - s_0) \\ B_z = \Delta\hat{B}_z \cdot \delta(s - s_0) \end{cases} \tag{2.42a}$$

so that

$$\frac{e}{E_0} A_s = \frac{e}{E_0} \cdot \delta(s - s_0) \cdot [\Delta\hat{B}_x \cdot z - \Delta\hat{B}_z \cdot x] \tag{2.42b}$$

2.5.2.2. <u>Synchrotron-Magnet</u>

$$A_x = A_z = 0 \quad ;$$

$$(K_x, K_z) \neq (0,0) \quad ; \quad K_x \cdot K_z = 0 \quad ;$$

$$\hat{K} = (1 + \eta) \cdot \{1 - (1 + K_x \cdot x + K_z \cdot z) \cdot [1 - \frac{\hat{p}_x^2}{(1+\eta)^2} - \frac{\hat{p}_z^2}{(1+\eta)^2}]^{1/2} \} -$$

$$- (1 + K_x \cdot x + K_z \cdot z) \cdot \frac{e}{E_0} A_s \quad ; \tag{2.43}$$

$$\begin{cases} B_x = \dfrac{1}{1 + K_x \cdot x + K_z \cdot z} \cdot \dfrac{\partial}{\partial z}[(1 + K_x \cdot x + K_z \cdot z) \cdot A_s] \quad ; \\ B_z = -\dfrac{1}{1 + K_x \cdot x + K_z \cdot z} \cdot \dfrac{\partial}{\partial x}[(1 + K_x \cdot x + K_z \cdot z) \cdot A_s] \quad ; \\ B_s = 0 \quad . \end{cases}$$

The field of a synchrotron magnet (combined function magnet) is

$$B_x = B_x^{(o)} + z \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \quad ; \qquad (2.44)$$
$$B_z = B_z^{(o)} + x \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0}$$

so that for the vector potential we may write[8,9,10])

$$\frac{e}{E_o} A_s = -\frac{1}{2} [1 + \frac{e}{E_o} \cdot B_z^{(o)} \cdot x - \frac{e}{E_o} \cdot B_x^{(o)} \cdot z] +$$
$$+ \frac{1}{2} \cdot \frac{e}{E_o} \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \cdot (z^2 - x^2) + \ldots$$

or using Eq. (2.9)[12]):

$$\frac{e}{E_o} A_s = -\frac{1}{2} \cdot (1 + K_x \cdot x + K_z \cdot z) + \frac{1}{2} g_o \cdot (z^2 - x^2) + \ldots \qquad (2.45)$$

with

$$g_o = \frac{e}{E_o} \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \quad .$$

2.5.3. Solenoid fields

$$A_s = 0 \quad ;$$
$$K_x = K_z = 0 \quad ;$$

$$\hat{K} = (1 + \eta) \cdot \{1 - \left[1 - \frac{(\hat{p}_x - \frac{e}{E_o} A_x)^2}{(1 + \eta)^2} - \frac{(\hat{p}_z - \frac{e}{E_o} A_z)^2}{(1 + \eta)^2}\right]^{1/2}\} \quad ; \qquad (2.46)$$

$$\begin{cases} B_x = -\frac{\partial}{\partial s} A_z & ; \\ B_z = +\frac{\partial}{\partial s} A_x & ; \\ B_s = \frac{\partial}{\partial x} A_z - \frac{\partial}{\partial z} A_x & ; \end{cases}$$

$$B_r = \sqrt{B_x^2 + B_z^2} \quad \text{(radial field)}.$$

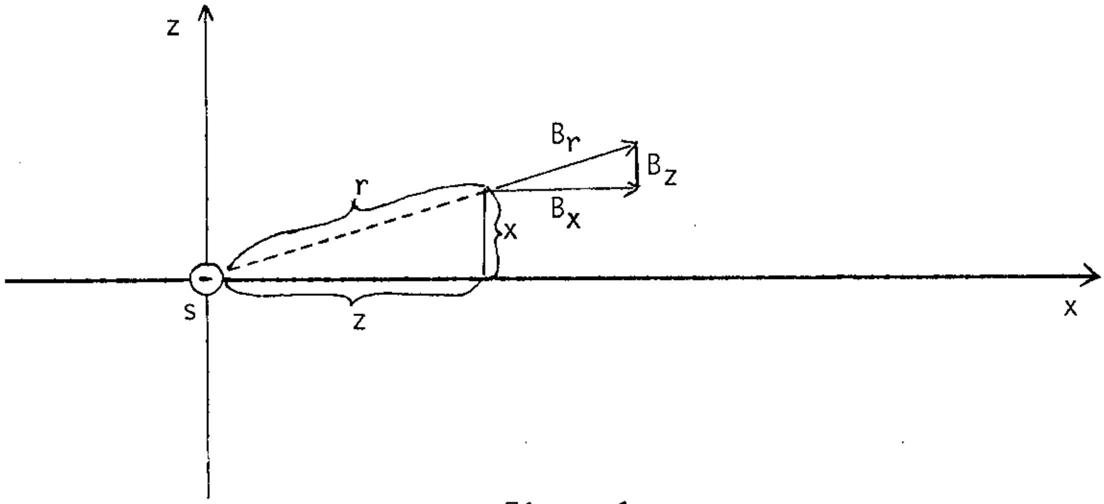

Figure 1

In the current free region (Fig. 1) the radial field B_r and the longitudinal field B_s can be written as power series [13]

$$B_r(x,z,s) = \sum_{\nu=0}^{\infty} b_{(2\nu+1)}(s) \cdot r^{2\nu+1} \quad ; \tag{2.47a}$$

$$B_s(x,z,s) = \sum_{\nu=0}^{\infty} b_{2\nu}(s) \cdot r^{2\nu} \quad . \tag{2.47b}$$

Putting (2.47a,b) into the Maxwell equations

$$\mathrm{div}\,\vec{B} = 0 \quad \Longrightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}(r \cdot B_r) = -\frac{\partial}{\partial s} B_s \quad ;$$

$$\mathrm{rot}\,\vec{B} = 0 \quad \Longrightarrow \quad \frac{\partial}{\partial s} B_r = \frac{\partial}{\partial r} B_s$$

one obtains

$$\sum_{\nu=0}^{\infty} b_{(2\nu+1)}(s) \cdot (2\nu+2) \cdot r^{2\nu} = -\sum_{\nu=0}^{\infty} r^{2\nu} \cdot \frac{d}{ds} b_{2\nu}(s) \quad ;$$

$$\sum_{\nu=0}^{\infty} r^{2\nu+1} \cdot \frac{d}{ds} b_{(2\nu+1)}(s) = \sum_{\nu=0}^{\infty} b_{(2\nu+2)}(s) \cdot (2\nu+2) \cdot r^{2\nu+1} \quad .$$

By equating coefficients of each power one then obtains

$$\begin{aligned} b_{(2\nu+1)}(s) &= -\frac{1}{(2\nu+2)} \cdot b'_{2\nu}(s) \quad ; \\ b_{(2\nu+2)}(s) &= +\frac{1}{(2\nu+2)} \cdot b'_{(2\nu+1)}(s) \quad ; \end{aligned} \tag{2.48}$$

$$(\nu = 0, 1, 2, \ldots)$$

Thus, if the longitudinal field on the s-axis

$$B_s(0,0,s) = b_0(s) \tag{2.49}$$

is known, the coefficients

$$b_1,\ b_2,\ b_3,\ \ldots$$

can be calculated. The field components in the field free region are then given by

$$\left\{\begin{aligned} B_x(x,z,s) &= \frac{x}{r}\cdot B_r = x\cdot\sum_{\nu=0}^{\infty} b_{(2\nu+1)}(s)\cdot(x^2+z^2)^{\nu} \\ B_z(x,z,s) &= \frac{z}{r}\cdot B_r = z\cdot\sum_{\nu=0}^{\infty} b_{(2\nu+1)}(s)\cdot(x^2+z^2)^{\nu} \\ B_s(x,z,s) &= \sum_{\nu=0}^{\infty} b_{2\nu}(s)\cdot(x^2+z^2)^{2\nu}\ . \end{aligned}\right. \tag{2.50}$$

The vector potential leading to the solenoid field of Eq. (2.50) is then:

$$\left\{\begin{aligned} A_x &= -\sum_{\nu}\frac{1}{(2\nu+2)}\cdot b_{(2\nu)}(s)\cdot r^{2\nu}\cdot z\ ; \\ A_z &= +\sum_{\nu}\frac{1}{(2\nu+2)}\cdot b_{(2\nu)}(s)\cdot r^{2\nu}\cdot x\ ; \\ A_s &= 0\ . \end{aligned}\right. \tag{2.51}$$

3. Solution of the equations of motion

Now that the potential A(x,z,s,t) for each magnet type is known it is possible to derive the equations of motion for the various magnets. For this purpose, we truncate the series expansion of the Hamiltonian at second order in the canonical momenta:

$$\left[1 - \frac{(\hat{p}_x - \frac{e}{E_0}A_x)^2}{(1+\eta)^2} - \frac{(\hat{p}_z - \frac{e}{E_0}A_z)^2}{(1+\eta)^2}\right]^{1/2} =$$

$$= 1 - \frac{1}{2}\,\frac{(\hat{p}_x - \frac{e}{E_0}A_x)^2}{(1+\eta)^2} - \frac{1}{2}\,\frac{(\hat{p}_z - \frac{e}{E_0}A_z)^2}{(1+\eta)^2} \quad , \tag{3.1}$$

so that it is possible to solve the equations of motion exactly as shown in the following.

3.1. Cavity

From Eq. (2.29) and (2.36) and using (3.1) (with $K_x = 0$, $K_z = 0$) one obtains the approximate Hamiltonian $\hat{K}$

$$\hat{K} = \frac{1}{2}\,\frac{\hat{p}_x{}^2}{(1+\eta)} + \frac{1}{2}\,\frac{\hat{p}_z{}^2}{(1+\eta)} + \frac{L}{k\cdot 2\pi}\,\frac{e\hat{V}}{E_0}\cos\left[k\cdot\frac{2\pi}{L}\cdot\sigma + \Phi\right]\cdot\delta(s - s_0) \tag{3.2}$$

and the corresponding non-linear canonical equations according to (2.28) are

$$x' = \frac{\hat{p}_x}{(1+\eta)} \quad ; \tag{3.3a}$$

$$\hat{p}_x' = 0 \quad ; \tag{3.3b}$$

$$z' = \frac{\hat{p}_z}{(1+\eta)} \quad ; \tag{3.3c}$$

$$\hat{p}_z' = 0 \quad ; \tag{3.3d}$$

$$\sigma' = -\frac{1}{2}\left\{\frac{\hat{p}_x^2}{(1+\eta)^2} + \frac{\hat{p}_z^2}{(1+\eta)^2}\right\} = -\frac{1}{2}[(x')^2 + (z')^2] \; ; \tag{3.3e}$$

$$\eta' = \frac{e\hat{V}}{E_0}\cdot\sin\left[k\cdot\frac{2\pi}{L}\cdot\sigma + \Phi\right]\cdot\delta(s - s_0) \tag{3.3f}$$

From (3.3c) and (3.3f):

$$\sigma(s_0 + 0) = \sigma(s_0 - 0) \quad ; \tag{3.4a}$$

$$\eta(s_0 + 0) = \eta(s_0 - 0) + \frac{e\hat{V}}{E_0} \cdot \sin\left[k \cdot \frac{2\pi}{L} \cdot \sigma(s_0 - 0) + \Phi\right] \tag{3.4b}$$

Also, by (3.3a) and (3.3b)

$$\frac{d}{ds}\left[(1 + \eta) \cdot x'\right] = 0 \quad \Longrightarrow \quad [1 + \eta(s)] \cdot x'(s) = \text{const};$$

$$\Longrightarrow \quad [1 + \eta(s_0 + 0)] \cdot x'(s_0 + 0) = [1 + \eta(s_0 - 0)] \cdot x'(s_0 - 0) \; ;$$

$$x'(s_0 + 0) = \frac{1 + \eta(s_0 - 0)}{1 + \eta(s_0 + 0)} \cdot x'(s_0 - 0) \quad . \tag{3.4c}$$

Correspondingly from (3.3c) and (3.3d)

$$z'(s_0 + 0) = \frac{1 + \eta(s_0 - 0)}{1 + \eta(s_0 + 0)} \cdot z'(s_0 - 0) \tag{3.4d}$$

and finally

$$x(s_0 + 0) = x(s_0 - 0) \quad ; \tag{3.4e}$$

$$z(s_0 + 0) = z(s_0 - 0) \quad . \tag{3.4f}$$

Eq. (3.4a-f) provide a complete solution to the non-linear canonical equations for cavity fields.

Remarks

1) From Eqs. (3.4c) and (3.4d) it follows that the terms

$$x'(s) \cdot \gamma(s) \qquad \text{and} \qquad z'(s) \cdot \gamma(s)$$

with $\gamma = \dfrac{E}{m_0 c^2}$

are invariants of motion in the longitudinal cavity fields.

2) In the case

$$\left|k \cdot \frac{2\pi}{L} \cdot \sigma\right| \ll 1$$

one can, in place of (3.4b,c,d), use the approximation

$$\eta(s_0 + 0) = \eta(s_0 - 0) + \frac{e\hat{V}}{E_0} \cdot \sin\Phi + \frac{e\hat{V}}{E_0} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\Phi \cdot \sigma(s_0 - 0) \ ;$$

$$x'(s_0 + 0) \approx [1 + \eta(s_0 - 0) - \eta(s_0 + 0)] \cdot x'(s_0 - 0)$$

$$\approx x'(s_0 - 0) - \frac{e\hat{V}}{E_0} \cdot \sin\Phi \cdot x'(s_0 - 0) \ ;$$

$$z'(s_0 + 0) \approx z'(s_0 - 0) - \frac{e\hat{V}}{E_0} \cdot \sin\Phi \cdot z'(s_0 - 0) \ .$$

Alternatively, in matrix terms, also using (3.4a,e,f) we get:

$$\vec{y}(s_0 + 0) = \{\underline{M}(s_0+0,\ s_0-0) + \delta\underline{M}(s_0+0,\ s_0-0)\}\ \vec{y}(s_0-0) + \vec{c} \tag{3.5}$$

where

$$\vec{y}^{\,T} = (x,\ x',\ z,\ z',\ 0,\ \eta) \ ;$$

$$\vec{c} = \left(0,\ 0,\ 0,\ 0,\ 0,\ \frac{e\hat{V}}{E_0}\sin\Phi\right) \ ; \tag{3.5a}$$

$$\begin{cases} M_{\mu\mu} = 1(\mu = 1,\ 2,\ \ldots,\ 6) \ ; \\ M_{65} = \dfrac{e\hat{V}}{E_0} \cdot k \cdot \dfrac{2\pi}{L} \cdot \cos\Phi \ ; \\ M_{\mu\nu} = 0 \quad \text{otherwise} \end{cases} \tag{3.5b}$$

and

$$\delta M_{22} = \delta M_{44} = -\frac{e\hat{V}}{E_0} \cdot \sin\Phi \ ;$$

$$\delta M_{\mu\nu} = 0 \quad \text{otherwise.} \tag{3.5c}$$

This is the form for the solution of the cavity equations given in Ref. (1,2,3) where the coupled synchro-betatron motion was studied in strictly linear terms. In that case, the $\delta \underline{M}$ matrix is non-symplectic and leads to damping of the transverse phase space. This is of course consistent with Eq. (3.4c) where, if the cavity phase Φ is not zero (or π) ($\delta \underline{M} \neq \underline{0}$) so that the particle is accelerated (decelerated), the transverse phase space in terms of variables x, z, p_x, p_z must be conserved since the equations are canonical, but the phase space in terms of the variables x, x', z, z' changes with energy (see Remark 1) above).
(For a discussion of transverse electric dipole fields, see Appendix II.)

3.2. Transverse magnetic fields

3.2.1. Transverse fields in straight sections

3.2.1.1. Quadrupole

From Eq. (2.37), (2.38) and (3.1), the Hamiltonian for a quadrupole is given by

$$\hat{K} = \frac{1}{2} \; \frac{\hat{p}_x^2}{(1+\eta)} + \frac{1}{2} \; \frac{\hat{p}_z^2}{(1+\eta)} + \frac{1}{2} \, g_0 \cdot (x^2 - z^2). \tag{3.6}$$

The corresponding canonical equations are then (see Eq. (2.28))

$$x' = \frac{p_x}{(1+\eta)} \quad ; \tag{3.7a}$$

$$\hat{p}_x' = - g_0 \cdot x \quad ; \tag{3.7b}$$

$$z' = \frac{p_z}{(1+\eta)} \quad ; \tag{3.7c}$$

$$\hat{p}_z' = + g_0 \cdot z \quad ; \tag{3.7d}$$

$$\sigma' = - \frac{1}{2} \left\{ \frac{\hat{p}_x^2}{(1+\eta)^2} + \frac{\hat{p}_z^2}{(1+\eta)^2} \right\} \equiv - \frac{1}{2} \left[(x')^2 + (z')^2 \right] ; \tag{3.7e}$$

$$\eta' = 0 \quad . \tag{3.7f}$$

By eliminating $\hat{p}_x$ (and $\hat{p}_z$) in Eq. (3.7a,b) (and Eq. (3.7c,d)) one has

$$x'' = - g \cdot x \quad ; \tag{3.8a}$$

$$z'' = + g \cdot z \quad . \tag{3.8b}$$

where (see Eq. (2.38b))

$$g = \frac{g_0}{(1 + \eta)} = \frac{e}{E} \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \quad . \tag{3.9}$$

Writing now the solution of Eq. (3.8) in the form

$$\vec{y}(s) = \underline{M}(s,0)\, \vec{y}(0) \tag{3.10a}$$

with $$\vec{y}^{\,T} = (x, \tilde{p}_x, z, \tilde{p}_z) \quad ; \tag{3.10b}$$

$$\begin{cases} \tilde{p}_x \equiv x' \quad ; \\ \tilde{p}_z \equiv z' \quad ; \end{cases} \tag{3.11}$$

we obtain for the 4-dimensional transfer matrix $\underline{M}(s,0)$

a) $g > 0$:

$$M_{11}(s,0) = \cos(\sqrt{g} \cdot s) \quad ;$$

$$M_{12}(s,0) = \frac{1}{\sqrt{g}} \sin(\sqrt{g} \cdot s) \quad ;$$

$$M_{21}(s,0) = -\sqrt{g}\ \sin(\sqrt{g} \cdot s) \quad ;$$

$$M_{22}(s,0) = M_{11}(s,0) \quad ;$$

$$M_{33}(s,0) = \cosh(\sqrt{g} \cdot s) \quad ;$$

$$M_{34}(s,0) = \frac{1}{\sqrt{g}} \sinh(\sqrt{g} \cdot s) \quad ;$$

$$M_{43}(s,0) = \sqrt{g}\ \sinh(\sqrt{g} \cdot s) \quad ;$$

$$M_{44}(s,0) = M_{33}(s,0) \quad ;$$

$$M_{ik}(s,0) = 0 \quad \text{otherwise} \quad . \tag{3.12a}$$

b) $g < 0$:

$$M_{11}(s,0) = \cosh(\sqrt{|g|}\cdot s) \quad ;$$

$$M_{12}(s,0) = \frac{1}{\sqrt{|g|}} \sinh(\sqrt{|g|}\cdot s) \quad ;$$

$$M_{21}(s,0) = \sqrt{|g|} \sinh(\sqrt{|g|}\cdot s) \quad ;$$

$$M_{22}(s,0) = M_{11}(s,0) \quad ;$$

$$M_{33}(s,0) = \cos(\sqrt{|g|}\cdot s) \quad ;$$

$$M_{34}(s,0) = \frac{1}{\sqrt{|g|}} \sin(\sqrt{|g|}\cdot s) \quad ;$$

$$M_{43}(s,0) = -\sqrt{|g|} \sin(\sqrt{|g|}\cdot s) \quad ;$$

$$M_{44}(s,0) = M_{33}(s,0) \quad ;$$

$$M_{ik}(s,0) = 0 \quad \text{otherwise} . \tag{3.12b}$$

From Eq. (3.7f)

$$\eta(s) = \eta(0) \tag{3.13}$$

and finally from Eq. (3.7e)

$$\begin{aligned}
\sigma(s) &= \sigma(0) - \frac{1}{2} \int_0^s d\tilde{s} \cdot \{[x'(\tilde{s})]^2 + [z'(\tilde{s})]^2\} \\
&= \sigma(0) - \frac{g}{4} \{x^2(0) \cdot [s - M_{11}(s,0) \cdot M_{12}(s,0)] - \\
&\qquad - z^2(0) \cdot [s - M_{33}(s,0) \cdot M_{34}(s,0)]\} - \\
&\qquad - \frac{1}{4} \{x'^2(0) \cdot [s + M_{11}(s,0) \cdot M_{12}(s,0)] + \\
&\qquad + z'^2(0) \cdot [s + M_{33}(s,0) \cdot M_{43}(s,0)]\} - \\
&\qquad - \frac{1}{2} \cdot x(0) \cdot x'(0) \cdot M_{12}(s,0) \cdot M_{21}(s,0) - \\
&\qquad - \frac{1}{2} \cdot z(0) \cdot z'(0) \cdot M_{34}(s,0) \cdot M_{43}(s,0) \quad .
\end{aligned} \tag{3.14}$$

3.2.1.2. Skew quadrupole

From Eqs. (2.37) and (2.39) the Hamiltonian for a skew quadrupole is given by

$$\hat{K} = \frac{1}{2}\ \frac{\hat{p}_x^{\,2}}{(1+\eta)} + \frac{1}{2}\ \frac{\hat{p}_z^{\,2}}{(1+\eta)} - N_0 \cdot x\,z \tag{3.15}$$

and the corresponding canonical equations of motion are

$$x' = \frac{\hat{p}_x}{(1+\eta)}\ ; \tag{3.16a}$$

$$\hat{p}_x' = N_0 \cdot z\ ; \tag{3.16b}$$

$$z' = \frac{\hat{p}_z}{(1+\eta)}\ ; \tag{3.16c}$$

$$\hat{p}_z' = -N_0 \cdot x\ ; \tag{3.16d}$$

$$\sigma' = -\frac{1}{2}\,[(x')^2 + (z')^2]\ ; \tag{3.16e}$$

$$\eta' = 0\ . \tag{3.16f}$$

From Eq. (3.16a,b) and (3.16c,d) we obtain

$$x'' = N \cdot z\ ; \tag{3.17a}$$

$$z'' = N \cdot x\ ; \tag{3.17b}$$

where (see Eq. (2.39b))

$$N = \frac{N_0}{(1+\eta)} = \frac{1}{2}\ \frac{e}{E} \cdot \left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z}\right)_{x=z=0}\ . \tag{3.18}$$

Thus the transfer matrix $\underline{M}(s,0)$ (see Eq. (3.10)) can be written as

$$M_{11}(s,0) = \frac{1}{2}\{\cosh(\sqrt{N}\cdot s) + \cos(\sqrt{N}\cdot s)\} \quad ;$$

$$M_{12}(s,0) = \frac{1}{2\sqrt{N}}\{\sinh(\sqrt{N}\cdot s) + \sin(\sqrt{N}\cdot s)\} \quad ;$$

$$M_{13}(s,0) = \frac{1}{2}\{\cosh(\sqrt{N}\cdot s) - \cos(\sqrt{N}\cdot s)\} \quad ;$$

$$M_{14}(s,0) = \frac{1}{2\sqrt{N}}\{\sinh(\sqrt{N}\cdot s) - \sin(\sqrt{N}\cdot s)\} \quad ;$$

$$M_{21}(s,0) = \frac{\sqrt{N}}{2}\{\sinh(\sqrt{N}\cdot s) - \sin(\sqrt{N}\cdot s)\} \quad ;$$

$$M_{22}(s,0) = M_{11}(s,0) \quad ;$$

$$M_{23}(s,0) = \frac{\sqrt{N}}{2}\{\sinh(\sqrt{N}\cdot s) + \sin(\sqrt{N}\cdot s)\} \quad ;$$

$$M_{24}(s,0) = M_{13}(s,0) \quad ;$$

$$M_{31}(s,0) = M_{13}(s,0) \quad ;$$

$$M_{32}(s,0) = M_{14}(s,0) \quad ;$$

$$M_{33}(s,0) = M_{11}(s,0) \quad ;$$

$$M_{34}(s,0) = M_{12}(s,0) \quad ;$$

$$M_{41}(s,0) = M_{23}(s,0) \quad ;$$

$$M_{42}(s,0) = M_{24}(s,0) \quad ;$$

$$M_{43}(s,0) = M_{21}(s,0) \quad ;$$

$$M_{44}(s,0) = M_{11}(s,0) \quad . \qquad (3.19)$$

Eq. (3.16c), on integration, gives:

$$\sigma(s) = \sigma(0) - \frac{1}{2} \int_0^s d\tilde{s} \cdot \{[x'(\tilde{s})]^2 + [z'(\tilde{s})]^2\}$$

$$= \sigma(0) - \frac{1}{8} \cdot [x^2(0) + z^2(0)] \cdot \{M_{11}(s,0) \cdot M_{21}(s,0) + M_{31}(s,0) \cdot M_{41}(s,0)\} -$$

$$- \frac{1}{8} \cdot [x'^2(0) + z'^2(0)] \cdot \{s + M_{11}(s,0) \cdot M_{12}(s,0) + M_{13}(s,0) \cdot M_{14}(s,0)\} +$$

$$+ \frac{1}{2} N \cdot x(0) \cdot z(0) \cdot \{s - M_{11}(s,0) \cdot M_{12}(s,0) - M_{13}(s,0) \cdot M_{14}(s,0)\} -$$

$$- \frac{1}{2} x'(0) \cdot z'(0) \cdot \{M_{11}(s,0) \cdot M_{14}(s,0) + M_{31}(s,0) \cdot M_{34}(s,0)\} -$$

$$- [x(0) \cdot x'(0) + z(0) \cdot z'(0)] \cdot M_{12}(s,0) \cdot M_{21}(s,0) \quad -$$

$$- [x(0) \cdot z'(0) + x'(0) \cdot z(0)] \cdot \frac{1}{2} \{M_{12}(s,0) \cdot M_{23}(s,0) + M_{32}(s,0) \cdot M_{43}(s,0)\} \tag{3.20}$$

and finally from Eq. (3.20)

$$\eta(s) = \eta(0) \quad . \tag{3.21}$$

3.2.1.3. Sextupole

From Eq. (2.37) together with Eq. (2.42) the Hamiltonian for a sextupole is given by

$$\hat{K} = \frac{1}{2}\,\frac{\hat{p}_x^2}{(1+\eta)} + \frac{1}{2}\,\frac{\hat{p}_z^2}{(1+\eta)} + \frac{\lambda_o}{6}\cdot(x^3 - 3xz^2) \tag{3.22}$$

and the canonical equations of motion are

$$x' = \frac{\hat{p}_x}{(1+\eta)} \quad ; \tag{3.23a}$$

$$\hat{p}_x' = -\frac{1}{2}\lambda_o\cdot(x^2 - z^2) \quad ; \tag{3.23b}$$

$$z' = \frac{\hat{p}_z}{(1+\eta)} \quad ; \tag{3.23c}$$

$$\hat{p}_z' = +\lambda_o\cdot xz \quad ; \tag{3.23d}$$

$$\sigma' = -\frac{1}{2}\left[(x')^2 + (z')^2\right] \quad ; \tag{3.23e}$$

$$\eta' = 0 \quad . \tag{3.23f}$$

From Eq. (3.23a,b) and (3.23c,d) one has

$$x'' = -\frac{1}{2}\lambda\cdot(x^2 - z^2) \quad ; \tag{3.24a}$$

$$z'' = \lambda\cdot xz \tag{3.24b}$$

where (see Eq. (2.40b))

$$\lambda = \frac{\lambda_o}{(1+\eta)} = \frac{e}{E}\left(\frac{\partial^2 B_z}{\partial x^2}\right)_{x=z=0} \quad . \tag{3.25}$$

In the case where the sextupole is a thin lens with

$$\lambda = \hat{\lambda} \cdot \delta(s - s_0) \tag{3.26}$$

Eq. (3.23) may be easily integrated, and from Eq. (3.23e,f)

$$\sigma(s_0 + 0) = \sigma(s_0 - 0) \quad ; \tag{3.26a}$$

$$\eta(s_0 + 0) = \eta(s_0 - 0) \quad . \tag{3.26b}$$

Then from Eq. (3.24a,b) one obtains

$$x(s_0 + 0) = x(s_0 - 0) \quad ; \tag{3.26c}$$

$$x'(s_0 + 0) = x'(s_0 - 0) - \frac{\hat{\lambda}}{2}\,[x^2(s_0 - 0) + z^2(s_0 - 0)] \; ; \tag{3.26d}$$

$$z(s_0 + 0) = z(s_0 - 0) \quad ; \tag{3.26e}$$

$$z'(s_0 + 0) = z'(s_0 - 0) + \hat{\lambda} \cdot x(s_0 - 0) \cdot z(s_0 - 0) \; . \tag{3.26f}$$

3.2.1.4. Octupole

The Hamiltonian for an octupole can be written in the form (see Eqs. (2.37) and (2.41):

$$\hat{K} = \frac{1}{2}\,\frac{\hat{p}_x{}^2}{(1+\eta)} + \frac{1}{2}\,\frac{\hat{p}_z{}^2}{(1+\eta)} - \frac{1}{24}\,\mu_0 \cdot (x^4 - 6x^2z^2 + z^4\,) \tag{3.27}$$

and the corresponding canonical equations read

$$x' = \frac{\hat{p}_x}{(1+\eta)} \quad ; \tag{3.27a}$$

$$\hat{p}_x' = \frac{1}{6}\,\mu_0 \cdot (x^3 - 3xz^2) \quad ; \tag{3.27b}$$

$$z' = \frac{\hat{p}_z}{(1+\eta)} \quad ; \tag{3.27c}$$

$$\hat{p}_z' = \frac{1}{6}\,\mu_0 \cdot (z^3 - 3x^2z) \quad ; \tag{3.27d}$$

$$\sigma' = -\frac{1}{2}\,[(x')^2 + (z')^2] \quad ; \tag{3.27e}$$

$$\eta' = 0 \quad . \tag{3.27f}$$

From Eq. (3.27a,b) and (3.27c,d) one gets

$$x'' = \frac{1}{6}\,\mu \cdot (x^3 - 3xz^2) \quad ;$$

$$z'' = \frac{1}{6}\,\mu \cdot (z^3 - 3x^2z)$$

with (see Eq. (2.41b))

$$\mu = \frac{\mu_0}{(1+\eta)} = \frac{e}{E}\,\left(\frac{\partial^3 B_x}{\partial z^3}\right)_{x=z=0} \quad .$$

Considering the octupole as a thin lens with

$$\mu = \hat{\mu} \cdot \delta(s - s_0) \tag{3.28}$$

one obtains by integration

$$\sigma(s_0 + 0) = \sigma(s_0) \quad ; \tag{3.28a}$$

$$\eta(s_0 + 0) = \eta(s_0) \quad ; \tag{3.28b}$$

$$x(s_0 + 0) = x(s_0 - 0) \quad ; \tag{3.28c}$$

$$x'(s_0 + 0) = x'(s_0 - 0) + \frac{\hat{\mu}}{6} \cdot [x^3(s_0 - 0) - 3 \cdot x(s_0 - 0) \cdot z^2(s_0 - 0)]; \tag{3.28d}$$

$$z(s_0 + 0) = z(s_0 - 0) \quad ; \tag{3.28e}$$

$$z'(s_0 + 0) = z'(s_0 - 0) + \frac{\hat{\mu}}{6} \cdot [z^3(s_0 - 0) - 3 \cdot x^2(s_0 - 0) \cdot z(s_0 - 0)]. \tag{3.28f}$$

3.2.1.5. Dipole

The Hamiltonian for a thin lens dipole ($K_x = K_z = 0$) (see Eqs. (2.37) and (2.42)) is:

$$\hat{K} = \frac{1}{2} \frac{\hat{p}_x^{\,2}}{(1 + \eta)} + \frac{1}{2} \frac{\hat{p}_z^{\,2}}{(1 + \eta)} - \frac{e}{E_0} \cdot \delta(s - s_0) \cdot [\Delta\hat{B}_x \cdot z - \Delta\hat{B}_z \cdot x] \tag{3.29}$$

and the canonical equations of motion are

$$x' = \frac{\hat{p}_x}{(1 + \eta)} \quad ; \tag{3.29a}$$

$$\hat{p}_x' = -\frac{e}{E_0} \cdot \delta(s - s_0) \cdot \Delta\hat{B}_z \quad ; \tag{3.29b}$$

$$z' = \frac{\hat{p}_z}{(1 + \eta)} \quad ; \tag{3.29c}$$

$$\hat{p}_z' = +\frac{e}{E_0} \cdot \delta(s - s_0) \cdot \Delta\hat{B}_x \quad ; \tag{3.29d}$$

$$\sigma' = -\frac{1}{2} [(x')^2 + (z')^2] \quad ; \tag{3.29e}$$

$$\eta' = 0 \quad . \tag{3.29f}$$

By eliminating $\hat{p}_x$ and $\hat{p}_z$ one has

$$x'' = -\frac{e}{E}\cdot\delta(s-s_0)\cdot\Delta\hat{B}_z \quad ; \tag{3.30a}$$

$$z'' = +\frac{e}{E}\cdot\delta(s-s_0)\cdot\Delta\hat{B}_x \tag{3.30b}$$

and by integrating Eq. (3.29e,f) and (3.30a,b) one obtains

$$x(s_0+0) = x(s_0-0) \quad ; \tag{3.31a}$$

$$x'(s_0+0) = x'(s_0-0) - \frac{e}{E}\cdot\Delta\hat{B}_z \quad ; \tag{3.31b}$$

$$z(s_0+0) = z(s_0-0) \quad ; \tag{3.31c}$$

$$z'(s_0+0) = z'(s_0-0) + \frac{e}{E}\cdot\Delta\hat{B}_x \quad ; \tag{3.31d}$$

$$\sigma(s_0+0) = \sigma(s_0-0) \quad ; \tag{3.31e}$$

$$\eta(s_0+0) = \eta(s_0-0) \quad . \tag{3.31f}$$

3.2.2. Synchrotron magnet

Using Eqs. (2.43) and (2.45) together with (3.1) the Hamiltonian for a synchrotron magnet is

$$\hat{K} = \frac{1}{2}\,\frac{\hat{p}_x^2}{(1+\eta)} + \frac{1}{2}\,\frac{\hat{p}_z^2}{(1+\eta)} - (K_x\cdot x + K_z\cdot z)\cdot\eta +$$

$$+\frac{1}{2}\,G_1^{(o)}\cdot x^2 + \frac{1}{2}\,G_2^{(o)}\cdot z^2 + \frac{1}{2} \tag{3.32}$$

where

$$G_1^{(o)} = K_x^2 + g_0 \quad ; \tag{3.33a}$$

$$G_2^{(o)} = K_z^2 - g_0 \tag{3.33b}$$

and the canonical equations of motion are

$$x' = \frac{\hat{p}_x}{(1+\eta)} \quad ; \tag{3.34a}$$

$$\hat{p}_x' = -G_1^{(o)} \cdot x + K_x \cdot \eta \quad ; \tag{3.34b}$$

$$z' = \frac{\hat{p}_z}{(1+\eta)} \quad ; \tag{3.34c}$$

$$\hat{p}_z' = -G_2^{(o)} \cdot z + K_z \cdot \eta \quad ; \tag{3.34d}$$

$$\sigma' = -(K_x \cdot x + K_z \cdot z) - \frac{1}{2}[(x')^2 + (z')^2] \quad ; \tag{3.34e}$$

$$\eta' = 0 \quad , \tag{3.34f}$$

whereby the four equations (3.34a-d) can be replaced by second order equations by eliminating $\hat{p}_x$ and $\hat{p}_z$:

$$\begin{cases} x'' = -G_1 \cdot x + K_x \cdot \dfrac{\eta}{1+\eta} \quad ; & (3.35a) \\ z'' = -G_2 \cdot z + K_z \cdot \dfrac{\eta}{1+\eta} & (3.35b) \end{cases}$$

with

$$\begin{cases} G_1 = \dfrac{1}{(1+\eta)} \cdot G_1^{(o)} \quad ; & (3.36a) \\ G_2 = \dfrac{1}{(1+\eta)} \cdot G_2^{(o)} \quad . & (3.36b) \end{cases}$$

Writing now Eq. (3.35) in the form

$$\vec{y}(s) = M(s,0)\,\vec{y}(0) + \vec{q} \tag{3.37a}$$

with

$$\vec{y}^T = (x, \tilde{p}_x, z, \tilde{p}_z): \tag{3.37b}$$

$$\vec{q}^T = (q_1, q_2, q_3, q_4) \quad ; \tag{3.37c}$$

$$\begin{cases} \tilde{p}_x \equiv x' \quad ; \\ \tilde{p}_z \equiv z' \end{cases} \tag{3.37d}$$

(compare (3.10)-(3.11) then

a) for $G_1 > 0$, $G_2 < 0$:

$$M_{11}(s,0) = \cos(\sqrt{G_1}\cdot s) \quad ;$$

$$M_{12}(s,0) = \frac{1}{\sqrt{G_1}}\cdot\sin(\sqrt{G_1}\cdot s) \quad ;$$

$$M_{21}(s,0) = -\sqrt{G_1}\cdot\sin(\sqrt{G_1}\cdot s) \quad ;$$

$$M_{22}(s,0) = M_{11}(s,0) \quad ;$$

$$M_{33}(s,0) = \cosh(\sqrt{|G_2|}\cdot s) \quad ;$$

$$M_{34}(s,0) = \frac{1}{\sqrt{|G_2|}}\cdot\sinh(\sqrt{|G_2|}\cdot s) \quad ;$$

$$M_{43}(s,0) = \sqrt{|G_2|}\cdot\sinh(\sqrt{|G_2|}\cdot s) \quad ;$$

$$M_{44}(s,0) = M_{33}(s,0) \quad ;$$

$$q_1(s,0) = +\frac{K_x}{G_1}\cdot\frac{\eta}{(1+\eta)}\cdot[1-\cos(\sqrt{G_1}\cdot s)] \quad ;$$

$$q_2(s,0) = +\frac{K_x}{\sqrt{G_1}}\cdot\frac{\eta}{(1+\eta)}\cdot\sin(\sqrt{G_1}\cdot s) \quad ;$$

$$q_3(s,0) = +\frac{K_z}{G_2}\cdot\frac{\eta}{(1+\eta)}\cdot[1-\cosh(\sqrt{|G_2|}\cdot s)] \quad ;$$

$$q_4(s,0) = -\frac{K_z}{\sqrt{|G_2|}}\cdot\frac{\eta}{(1+\eta)}\cdot\sinh(\sqrt{|G_2|}\cdot s)] \quad ; \qquad (3.38a)$$

b) for $G_1 < 0$, $G_2 > 0$:

$$M_{11}(s,0) = \cosh(\sqrt{|G_1|}\cdot s) \quad ;$$

$$M_{12}(s,0) = \frac{1}{\sqrt{|G_1|}} \cdot \sinh(\sqrt{|G_1|}\cdot s) \quad ;$$

$$M_{21}(s,0) = \sqrt{|G_1|} \cdot \sinh(\sqrt{|G_1|}\cdot s) \quad ;$$

$$M_{22}(s,0) = M_{11}(s,0) \quad ;$$

$$M_{33}(s,0) = \cos(\sqrt{G_2}\cdot s) \quad ;$$

$$M_{34}(s,0) = \frac{1}{\sqrt{G_2}} \cdot \sin(\sqrt{G_2}\cdot s) \quad ;$$

$$M_{43}(s,0) = -\sqrt{G_2} \cdot \sin(\sqrt{G_2}\cdot s) \quad ;$$

$$M_{44}(s,0) = M_{33}(s,0) \quad ;$$

$$q_1(s,0) = + \frac{K_X}{G_1} \cdot \frac{\eta}{(1+\eta)} \cdot [1 - \cosh(\sqrt{|G_1|}\cdot s)] \quad ;$$

$$q_2(s,0) = - \frac{K_X}{\sqrt{|G_1|}} \cdot \frac{\eta}{(1+\eta)} \cdot \sinh(\sqrt{|G_2|}\cdot s) \quad ;$$

$$q_3(s,0) = + \frac{K_Z}{G_2} \cdot \frac{\eta}{(1+\eta)} \cdot [1 - \cos(\sqrt{G_2}\cdot s)] \quad ;$$

$$q_4(s,0) = + \frac{K_Z}{\sqrt{G_2}} \cdot \frac{\eta}{(1+\eta)} \cdot \sin(\sqrt{G_2}\cdot s) \quad . \tag{3.38b}$$

Furthermore, from Eq. (3.34e):

$$\sigma(s) = \sigma(0) - \int_0^s d\tilde{s} \cdot [K_x \cdot x + K_z \cdot z] - \frac{1}{2} \int_0^s d\tilde{s} \cdot [(x')^2 + (z')^2]$$

$$= -\frac{1}{4} \cdot \{s - M_{11}(s,0) \cdot M_{12}(s,0)\} \cdot [x^2(0) \cdot G_1 + \frac{K_x^2}{G_1} \cdot \left(\frac{\eta}{1+\eta}\right)^2 - 2 \cdot x(0) \cdot K_x \cdot \frac{\eta}{1+\eta}] -$$

$$-\frac{1}{4} \cdot \{s + M_{11}(s,0) \cdot M_{12}(s,0)\} \cdot x'^2(0) -$$

$$-\frac{1}{2} \cdot M_{12}(s,0) \cdot M_{21}(s,0) \cdot [x(0) \cdot x'(0) - x'(o) \cdot \frac{K_x}{G_1} \cdot \frac{\eta}{1+\eta}] -$$

$$- K_x \cdot x(0) \cdot M_{12}(s,0) - K_x \cdot \frac{x'(0)}{G_1} \cdot [1 - M_{11}(s,0)] - \frac{K_x^2}{G_1} \cdot \frac{\eta}{1+\eta} \cdot [s - M_{12}(s,0)] -$$

$$-\frac{1}{4} \cdot \{s - M_{33}(s,0) \cdot M_{34}(s,0)\} \cdot [z^2(0) \cdot G_2 + \frac{K_z^2}{G_2} \cdot \left(\frac{\eta}{1+\eta}\right)^2 - 2 \cdot z(0) \cdot K_z \cdot \frac{\eta}{1+\eta}] -$$

$$-\frac{1}{4} \cdot \{s + M_{33}(s,0) \cdot M_{34}(s,0)\} \cdot z'^2(0) -$$

$$-\frac{1}{2} \cdot M_{34}(s,0) \cdot M_{43}(s,0) \cdot [z(0) \cdot z'(0) - z'(0) \cdot \frac{K_z}{G_2} \cdot \frac{\eta}{1+\eta}] -$$

$$- K_z \cdot z(0) \cdot M_{34}(s,0) - K_z \cdot \frac{z'(0)}{G_2} \cdot [1 - M_{33}(s,0)] - \frac{K_z^2}{G_2} \cdot \frac{\eta}{1+\eta} \cdot [s - M_{34}(s,0)] \ . \tag{3.39}$$

Finally, from Eq. (3.34f)

$$\eta(s) = \eta(0) \ . \tag{3.40}$$

Remarks

If we take into account field errors ΔB_x, ΔB_z in the guide field, Eq. (2.45) must be replaced by

$$\frac{e}{E_0} A_s = -\frac{1}{2}\cdot[1 + \frac{\Delta B_x}{B_x^{(o)}} + \frac{\Delta B_z}{B_z^{(o)}}]\cdot(1 + K_x\cdot x + K_z\cdot z) + \frac{1}{2} g_0\cdot(z^2 - x^2).$$

In this case the (approximate) Hamiltonian takes the form

$$\hat{K} = \frac{1}{2}\frac{\hat{p}_x^2}{(1+\eta)} + \frac{1}{2}\frac{\hat{p}_z^2}{(1+\eta)} - (K_x\cdot x + K_z\cdot z)\cdot\eta + $$

$$+ \frac{1}{2} G_1^{(o)}\cdot x^2 + \frac{1}{2} G_2^{(o)}\cdot z^2 +$$

$$+ \frac{\Delta B_x}{B_x^{(o)}}\cdot K_z\cdot z + \frac{\Delta B_z}{B_z^{(o)}}\cdot K_x\cdot x + \frac{1}{2}[1 + \frac{\Delta B_x}{B_x^{(o)}} + \frac{\Delta B_z}{B_z^{(o)}}] \tag{3.41}$$

and the corresponding canonical equations of motion are now

$$x' = \frac{\hat{p}_x}{(1+\eta)} \; ; \tag{3.41a}$$

$$\hat{p}_x' = -G_1^{(o)}\cdot x + K_x\cdot\eta - \frac{\Delta B_z}{B_z^{(o)}}\cdot K_x \; ; \tag{3.41b}$$

$$z' = \frac{\hat{p}_z}{(1+\eta)} \; ; \tag{3.41c}$$

$$\hat{p}_z' = -G_2^{(o)}\cdot z + K_z\cdot\eta - \frac{\Delta B_x}{B_x^{(o)}}\cdot K_z \; ; \tag{3.41d}$$

$$\sigma' = -(K_x\cdot x + K_z\cdot z) - \frac{1}{2}[(x')^2 + (z')^2] \; ; \tag{3.41e}$$

$$\eta' = 0 \; . \tag{3.41f}$$

By eliminating $\hat{p}_x$ and $\hat{p}_z$ one obtains from (3.32a,b) and (3.32c,d):

$$\begin{cases} x'' = - G_1 \cdot x + \dfrac{\eta}{(1+\eta)} \cdot K_x - \dfrac{\Delta B_z}{B_z^{(o)}} \cdot K_x \quad ; \\[2ex] z'' = - G_2 \cdot z + \dfrac{\eta}{(1+\eta)} \cdot K_z - \dfrac{\Delta B_x}{B_z^{(o)}} \cdot K_z \quad . \end{cases}$$

Comparing these equations with (3.35a,b) one sees that, due to the field errors, additional inhomogenious terms

$$\frac{\Delta B_z}{B_z^{(o)}} \cdot K_x \quad \text{and} \quad \frac{\Delta B_x}{B_x^{(o)}} \cdot K_z$$

appear which give rise to closed orbit shifts.

3.3. Solenoid fields

The Hamiltonian for a solenoid is obtained using Eqs. (2.46) and (2.51a,b) and by keeping only the first order terms in A_x and A_z:

$$\hat{K} = \frac{1}{2} \frac{(\hat{p}_x + H_o \cdot z)^2}{(1+\eta)} + \frac{1}{2} \frac{(\hat{p}_z - H_o \cdot x)^2}{(1+\eta)} \tag{3.42a}$$

with (see Eq. (2.49))

$$H_o = \frac{1}{2} \frac{e}{E_o} \cdot b_o(s) \equiv \frac{1}{2} \frac{e}{E_o} \cdot B_s(0,0,s) \quad . \tag{3.42b}$$

The corresponding canonical equations of motion are then

$$x' = \frac{1}{(1+\eta)} \cdot (\hat{p}_x + H_o \cdot z) \quad ; \tag{3.43a}$$

$$\hat{p}_x' = \frac{1}{(1+\eta)} \cdot (\hat{p}_z - H_o \cdot x) \cdot H_o \quad ; \tag{3.43b}$$

$$z' = \frac{1}{(1+\eta)} \cdot (\hat{p}_z - H_o \cdot x) \quad ; \tag{3.43c}$$

$$\hat{p}_z' = - \frac{1}{(1+\eta)} \cdot (\hat{p}_x + H_o \cdot z) \cdot H_o \quad ; \tag{3.43d}$$

$$\sigma' = - \frac{1}{2} \left\{ \frac{(\hat{p}_x + H \cdot z)^2}{(1+\eta)^2} + \frac{(\hat{p}_z - H \cdot x)^2}{(1+\eta)^2} \right\} \equiv - \frac{1}{2} [(x')^2 + (z')^2] \quad ; \tag{3.43e}$$

$$\eta' = 0 \quad . \tag{3.43f}$$

In order to integrate Eqs. (3.34a-d) it is useful to use new variables namely:

$$\tilde{p}_x = \frac{\hat{p}_x}{(1+\eta)} \quad ; \tag{3.44a}$$

$$\tilde{p}_z = \frac{\hat{p}_z}{(1+\eta)} \quad ; \tag{3.44b}$$

Then, with

$$H = \frac{1}{(1+\eta)} \cdot H_0 \equiv \frac{e}{E} \cdot B_s(0,0,s) \quad , \tag{3.45}$$

we obtain from Eq. (3.44a-d)

$$\vec{y}\,' = \underline{A} \cdot \vec{y} \tag{3.46}$$

where

$$\vec{y}^{\,T} = (x, \tilde{p}_x, z, \tilde{p}_z) \quad ; \tag{3.46a}$$

$$\underline{A} = \begin{pmatrix} 0 & 1 & H & 0 \\ -H^2 & 0 & 0 & H \\ -H & 0 & 0 & 1 \\ 0 & -H & -H^2 & 0 \end{pmatrix} \quad , \tag{3.46b}$$

so that the transfer matrix $\underline{M}$ defined by

$$\vec{y}(s) = \underline{M}(s,0)\,\vec{y}(0) \tag{3.46c}$$

is, in the case of H = const (sharp edged field)[2,3]:

$$M_{11}(s,0) = \frac{1}{2} \cdot (1 + \cos 2\theta) \quad ;$$

$$M_{12}(s,0) = \frac{1}{2H} \cdot \sin 2\theta \quad ;$$

$$M_{13}(s,0) = \frac{1}{2} \cdot \sin 2\theta \quad ;$$

$$M_{14}(s,0) = \frac{1}{2H} \cdot (1 - \cos 2\theta) \quad ;$$

$$M_{21}(s,0) = - H \cdot \frac{1}{2} \cdot \sin 2\theta \quad ;$$

$$M_{22}(s,0) = M_{11}(s,0) \quad ;$$

$$M_{23}(s,0) = - H \cdot \frac{1}{2} \cdot (1 - \cos 2\theta) \quad ;$$

$$M_{24}(s,0) = M_{13}(s,0) \quad ;$$

$$M_{31}(s,0) = -M_{13}(s,0) \quad ;$$

$$M_{32}(s,0) = -M_{14}(s,0) \quad ;$$

$$M_{33}(s,0) = M_{11}(s,0) \quad ;$$

$$M_{34}(s,0) = M_{12}(s,0) \quad ;$$

$$M_{41}(s,0) = -M_{23}(s,0) \quad ;$$

$$M_{42}(s,0) = - M_{13}(s,0) \quad ;$$

$$M_{43}(s,0) = M_{21}(s,0) \quad ;$$

$$M_{44}(s,0) = M_{11}(s,0) \tag{3.47}$$

with

$$\theta = H \cdot s \quad .$$

Finally from Eq. (3.43) we obtain

$$\sigma(s) = \sigma(0) - \frac{1}{2}\, s \cdot \{H^2 \cdot [x^2(0) + z^2(0)] + + 2H \cdot [\tilde{p}_x(0) \cdot z(0) - \tilde{p}_z(0) \cdot x(0)] + + \tilde{p}_x^2(0) + \tilde{p}_z^2(0)\} \tag{3.48}$$

and from Eq. (3.43f)

$$\eta(s) = \eta(0) \quad . \tag{3.49}$$

4. Summary

Starting from the Lagrangian of a charged particle in an electromagnetic field we have investigated the Hamiltonian formalism of non-linear coupled synchro-betatron oscillations for ultra-relativistic charged particles. The canonical variables are x, p_x, z, p_z, σ, η, which are well-known from the six-dimensional linear theory (SLIM). By expanding the Hamiltonian in a power series in these variables, one may obtain various orders of approximation of the canonical equations. In this work we keep terms up to second order in the canonical momenta p_x, p_z and take into account the effect of energy deviation on the focusing strength. These equations of motion are then solved for various kinds of magnets (quadrupole, skew quadrupole, bending magnets, synchrotron-magnet, solenoid, sextupole, octupole, dipole) and for cavities. The equations so derived can be very conveniently coded for computers: To calculate the betatron-oscillations one has only to multiply four-dimensional transfer matrices together. The variable σ can be expressed in terms of the elements of these transfer matrices and the variable η changes its value only in the cavities.

The general form of the Hamiltonian can be the starting point for a non-linear theory in the framework of the six-dimensional formalism. By only taking into account terms up to second order in all variables one obtains just the Hamiltonian of the linear theory used in Ref. 2) and 3).

Appendix I

Variational principle for the canonical equations

The aim of this appendix is to show that the variational problem of Eq. (2.18):

$$\delta J = 0 \tag{1}$$

with

$$J = \int_{t_1}^{t_2} L \cdot dt \quad ; \tag{2}$$

and

$$L = \sum_i \{\dot{q}_i \cdot p_i - \mathcal{H}(q, p, t)\} \quad ; \tag{3}$$

$$\begin{aligned} (q_1, q_2, q_3) &\equiv (x, z, s) \quad ; \\ (p_1, p_2, p_3) &\equiv (p_x, p_z, p_s) \end{aligned} \tag{4}$$

leads to the canonical equation (2.17) when the coordinates q_i and the momenta p_i as well as the time t are allowed to vary independently of each other but are held fixed at the end points $t = t_1$ and $t = t_2$:

$$\begin{cases} \delta q_i(t_1) = \delta q_i(t_2) = 0 \quad ; \\ \delta p_i(t_1) = \delta p_i(t_2) = 0 \quad ; \\ \delta t(t_1) = \delta t(t_2) = 0 \quad . \end{cases} \tag{5}$$

With this in mind we can write the integrand of δJ:

$$\delta J = \int_{t_1}^{t_2} \delta(L \cdot dt) \equiv \int_{t_1}^{t_2} \{L(q+\delta q, \dot{q}+\delta\dot{q}, p+\delta p, t+\delta t) \cdot \frac{d}{dt}(t+\delta t) - L(q, \dot{q}, p, t)\} \cdot dt$$

$$= \int_{t_1}^{t_2} \{\underbrace{[L(q+\delta q, \dot{q}+\delta\dot{q}, p+\delta p, t+\delta t) - L(q, \dot{q}, p, t)]}_{\delta L} + L(q, \dot{q}, p, t) \cdot \frac{d}{dt}\delta t\} \cdot dt \tag{6}$$

as

$$\delta(L \cdot \delta t) = (\delta L + L \cdot \frac{d}{dt}\delta t) \cdot dt \quad . \tag{7}$$

and using

$$\delta\dot{q}_i = \frac{d}{dt}\delta q_i - \dot{q}_i \cdot \frac{d}{dt}\delta t$$

we then obtain

$$\delta(L \cdot dt) = dt \cdot \frac{d(\delta t)}{dt} \cdot \{\sum_i \dot{q}_i \cdot p_i - \mathcal{H}\} +$$

$$+ dt \cdot \sum_i \{\dot{q}_i \cdot \delta p_i + [\frac{d}{dt}\delta q_i - \dot{q}_i \cdot \frac{d}{dt}\delta t] \cdot p_i -$$

$$- \frac{\partial\mathcal{H}}{\partial q_i} \cdot \delta q_i - \frac{\partial\mathcal{H}}{\partial p_i} \cdot \delta p_i\} - dt \cdot \frac{\partial\mathcal{H}}{\partial t} \cdot \delta t$$

$$= dt \cdot \{- \frac{d}{dt}(\mathcal{H} \cdot \delta t) + [\frac{d\mathcal{H}}{dt} - \frac{\partial\mathcal{H}}{\partial t}] \cdot \delta t\} +$$

$$+ dt \cdot \sum_i \{- \left(\frac{\partial\mathcal{H}}{\partial p_i} - \dot{q}_i\right) \cdot \delta p_i - \left(\frac{\partial\mathcal{H}}{\partial q_i} + \dot{p}_i\right) \cdot \delta q_i +$$

$$+ \frac{d}{dt}[\delta q_i \cdot p_i]\} \quad . \tag{8}$$

Since

$$\frac{d\mathcal{H}}{dt} - \frac{\partial\mathcal{H}}{\partial t} = \sum_i \{\frac{\partial\mathcal{H}}{\partial q_i} \cdot \dot{q}_i + \frac{\partial\mathcal{H}}{\partial p_i} \cdot \dot{p}_i\}$$

$$= \sum_i \{[\frac{\partial\mathcal{H}}{\partial q_i} + \dot{p}_i] \cdot \dot{q}_i + [\frac{\partial\mathcal{H}}{\partial p_i} - \dot{q}_i] \cdot \dot{p}_i\}$$

Eq. (8) can then be rearranged to give

$$\delta(L \cdot \delta t) = dt \cdot \frac{d}{dt}\{\sum_i \delta q_i \cdot p_i - \delta t \cdot \mathcal{H}\} +$$

$$+ dt \cdot \sum_i \{[\frac{\partial\mathcal{H}}{\partial q_i} + \dot{p}_i] \cdot (\dot{q}_i \cdot \delta t - \delta q_i) +$$

$$+ [\frac{\partial\mathcal{H}}{\partial p_i} - \dot{q}_i] \cdot (\dot{p}_i \cdot \delta t - \delta p_i)\} \quad . \tag{9}$$

so that the variation δJ can be rewritten as

$$\delta J = \int_{t_1}^{t_2} dt \cdot \sum_i \{[\frac{\partial \mathcal{H}}{\partial q_i} + \dot{p}_i] \cdot (\dot{q}_i \cdot \delta t - \delta q_i) +$$

$$+ [\frac{\partial \mathcal{H}}{\partial p_i} - \dot{q}_i] \cdot (\dot{p}_i \cdot \delta t - \delta p_i)\} + [\sum_i \delta q_i \cdot p_i - \delta t \cdot \mathcal{H}]_{t=t_1}^{t=t_2} . \qquad (10)$$

Because the variations vanish at the end points (see Eq. (5)) the last term in Eq. (10) is zero. The remaining integral can now only be zero for arbitrary variations δq_i, δp_i, δt if the conditions

$$\frac{\partial \mathcal{H}}{\partial q_i} + \dot{p}_i = 0 \quad ;$$
$$\frac{\partial \mathcal{H}}{\partial p_i} - \dot{q}_i = 0 \qquad (11)$$

are satisfied. However, (see Eq. (4)) these are just the Hamilton's equations (2.17).

Appendix II

Equations of motion for transverse electric dipole fields

The vector potential for a constant transverse electric dipole field

$$\vec{\varepsilon} = (\Delta\varepsilon_x, \Delta\varepsilon_z, 0) \tag{1}$$

can be written according to Eqs. (2.2) and (2.32)

$$\begin{aligned} A_x(x,z,s,t) &= \Delta\varepsilon_x \cdot (-ct) = \Delta\varepsilon_x \cdot (\sigma - s) \quad ; \\ A_z(x,z,s,t) &= \Delta\varepsilon_z \cdot (-ct) = \Delta\varepsilon_z \cdot (\sigma - s) \quad ; \\ A_s(x,z,s,t) &= 0 \quad . \end{aligned} \tag{2}$$

Using the Hamiltonian of Eq. (2.29) and expanding to second order in x, p_x, z, p_z, σ, η we then have:

$$\hat{K} = \frac{1}{2}[\hat{p}_x - \frac{e}{E_0}\Delta\varepsilon_x \cdot (\sigma - s)]^2 + \frac{1}{2}[\hat{p}_z - \frac{e}{E_0}\Delta\varepsilon_z \cdot (\sigma - s)]^2 \tag{3}$$

and the resulting canonical equations are

$$x' = \hat{p}_x - \frac{e}{E_0}\Delta\varepsilon_x \cdot (\sigma - s) \quad ; \tag{4a}$$

$$\hat{p}_x' = 0 \quad ; \tag{4b}$$

$$z' = \hat{p}_z - \frac{e}{E_0}\Delta\varepsilon_z \cdot (\sigma - s) \quad ; \tag{4c}$$

$$\hat{p}_z' = 0 \quad ; \tag{4d}$$

$$\sigma' = 0 \quad ; \tag{4e}$$

$$\eta' = x' \cdot \frac{e}{E_0}\Delta\varepsilon_x + z' \cdot \frac{e}{E_0}\Delta\varepsilon_z \quad . \tag{4f}$$

By eliminating $\hat{p}_x$ and $\hat{p}_z$ one obtains from Eq. (4a,b) and (4c,d) together with Eq. (4e) the equations

$$x'' = \frac{e}{E_0} \Delta\varepsilon_x \quad ; \qquad z'' = \frac{e}{E_0} \Delta\varepsilon_z \tag{5}$$

and their solution

$$x'(s) = x'(0) + \frac{e}{E_0} \Delta\varepsilon_x \cdot s \quad ; \tag{6a}$$

$$x(s) = x(0) + x'(0) \cdot s + \frac{1}{2} \frac{e}{E_0} \Delta\varepsilon_x \cdot s^2 \quad ; \tag{6b}$$

$$z'(s) = z'(0) + \frac{e}{E_0} \Delta\varepsilon_z \cdot s \quad ; \tag{6c}$$

$$z(s) = z(0) + z'(0) \cdot s + \frac{1}{2} \frac{e}{E_0} \Delta\varepsilon_z \cdot s^2 \quad . \tag{6d}$$

Using Eq. (6b) and (6d) together with Eq. (4f) one also has

$$\begin{aligned} \eta(s) &= \eta(0) + \frac{e}{E_0} \Delta\varepsilon_x \cdot [x(s) - x(0)] + \frac{e}{E_0} \Delta\varepsilon_z \cdot [z(s) - z(0)] \\ &= \eta(0) + \frac{e}{E_0} \Delta\varepsilon_x \cdot [x'(0) \cdot s + \frac{1}{2} \frac{e}{E_0} \Delta\varepsilon_x \cdot s^2] \quad + \\ &\qquad + \frac{e}{E_0} \Delta\varepsilon_z \cdot [z'(0) \cdot s + \frac{1}{2} \frac{e}{E_0} \Delta\varepsilon_z \cdot s^2] \quad . \end{aligned} \tag{6e}$$

Finally, from Eq. (4e)

$$\sigma(s) = \sigma(0) \quad . \tag{6f}$$

These equations can of course be extracted by elementary methods. The aim here is to show how they arise by applying our Hamiltonian formalism.

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The bending field $B_x^{(o)}$, $B_z^{(o)}$ alone is exactly described by the vector potential

$$\frac{e}{E_o} A_s^{(o)} = -\frac{1}{2} \cdot (1 + K_x \cdot x + K_z \cdot z) \quad .$$

Another vector potential which gives the same field (gauge transformation) is:

$$\frac{e}{E_o} \tilde{A}_s^{(o)} = \frac{1 - (1 + K_x \cdot x + K_z \cdot z)^2}{2(1 + K_x \cdot x + K_z \cdot z)} = \frac{e}{E_o} A_s^{(o)} + \frac{1}{2(1 + K_x \cdot x + K_z \cdot z)}$$

Use of the second potential causes p_s (and hence the Hamiltonian $\hat{K}$) to shift by a constant equal to 1/2. Thus, as expected, there is no change in the canonical equations.
13) E. Freytag and G. Ripken, DESY E3/R1-73/1 (1973)