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by

I. Montvay
Deutsches Elektronen-Synchrotron DESY, Hamburg

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Numerical Calculation of Hadron Masses in Lattice Quantum Chromodynamics+

I. Montvay

Deutsches Elektronen-Synchrotron DESY, Hamburg

Abstract

Recent numerical Monte Carlo simulations of the hadron spectrum are reviewed. After a general introduction, different ways of calculating the hadron masses in the "quenched approximation" (i.e. neglecting virtual quark loops) are described and the latest results are summarized. The pseudofermion method and the iterative hopping expansion method for the introduction of dynamical quarks is discussed, and the first results about the hadron spectrum including the effect of virtual quark loops are reviewed. A separate section is devoted to the discussion of the questions related to scaling with dynamical quarks. This review is based on the lecture given at the 1984 Aspen Center for Physics as a part of a lecture series about hadron spectrum calculations. Some topics covered by the other lecturers are left out completely. For instance, the glueball mass calculations and the microcanonical way of introducing dynamical quarks are not included. For these questions see, respectively, the lectures of Bernd Berg and Janos Polonyi. These lecture notes were completed in December 1984, later developments are not included.

+ Lecture given at the Aspen Center for Physics, August 1984

Contents

I. Introduction

The numerical calculation of the hadronic mass spectrum is one of the great challenges in lattice quantum chromodynamics. As a results of many years of experimentation with strongly interacting particles, the masses of many hadrons are known to good precision. The confrontation of this important and extensive body of empirical knowledge with the quantum chromodynamics (QCD) theory has two important aspects: first, it can provide the upto now missing irrefutable evidence for QCD as the correct theory of strong interactions; second, in case of a sucessful reproduction of the known hadron masses we would have a marvellous demonstration of the capabilities of a new approach in theoretical particle physics, namely large scale computation. In fact, this would be the first time ever that masses of elementary particles would be theoretically calculated. The number of free parameters for the description of hundreds of hadron masses (and other static hadron properties) are remarkably small: besides the Λ-parameter for the colour gauge coupling there are only the quark masses for different flavours (six for the moment). It is quite sure, that the experience gained in lattice QCD will be extremely useful also beyond the theory of hadrons, namely in other relativistic quantum field theories. Large scale numerical computation could help in the future to extend our theoretical understanding to areas where detailed phenomenological study is not (or not yet) feasible.

At present, we are obviously only at the beginning of this, almost revolutionary, development. The first efforts to develop the numerical methods needed for the hadron mass calculations started only about 3 years ago. The first investigations necessarily had an exploratory character. Still, the progress in the field seems to be rather fast, and as it is usual

in such cases, a great number of papers (good and bad) were produced. This makes a review somewhat difficult and certainly incomplete. Nevertheless, I tried to give a coherent introduction and a detailed list of references to at least some of the interesting topics in the field.

II. Calculation of hadron masses

The quark fields in euclidean lattice QCD are described by anticommuting (Grassmann-) variables defined on lattice sites $x = (x_1,x_2,x_3,x_4)$; $1 \leq x_\mu \leq N_\mu$ (the lattice size is $N_1N_2N_3N_4$). It is convenient to use dimensionless fields, therefore the connection between continuum-fields and lattice-fields is given by (a = lattice spacing):

$$\sqrt{\frac{a^3}{2K}}\,\psi_{cont}(ax) \to \psi_x;\quad \sqrt{\frac{a^3}{2K}}\,\tilde{\psi}_{cont}(ax) \to \tilde{\psi}_x. \tag{2.1}$$

Here K is, in general, an appropriately chosen normalization factor. For Wilson lattice fermions K is the "hopping parameter", which is related in the free fermion case to the bare mass m by

$$K = (8r + 2am)^{-1} \tag{2.2}$$

r is the "Wilson-parameter" satisfying $0 < r \leq 1$. The quark part of QCD lattice action with Wilson-fermions (Wilson, 1974, 1977) is then

$$S_f = \sum_{xy} \tilde{\psi}_y Q_{yx} \psi_x,$$

$$Q_{yx} = \delta_{yx} - K \sum_\mu (r + \gamma_\mu)\, U(x,\mu)\, \delta_{y,x+\hat{\mu}}. \tag{2.3}$$

The SU(3) link variables are denoted here by $U(x,\mu)$. They satisfy $U(x,\mu)^+ = U(x+\hat{\mu},-\mu)$ ($\hat{\mu}$ is a unit vector in the direction μ). The euclidean Dirac-matrices are defined according to $\gamma_\mu = \gamma_\mu^+ = -\gamma_{-\mu}$, and $\sum_\mu$ means a summation over both positive and negative directions: $\mu = \pm 1, \pm 2, \pm 3, \pm 4$.

The "quark-matrix" Q in Eq. (2.3) is neither hermitian nor anti-

hermitian, but is obeys

$$Q_{yx} = \gamma_5\, Q^+_{xy} \gamma_5 \tag{2.4}$$

with $\gamma_5 = \gamma_1\,\gamma_2\,\gamma_3\,\gamma_4$, as usual. In the free case ($U(x,\mu) = 1$) the quark-propagator Q^{-1} can be easily obtained by Fourier-transformation. On a finite lattice with periodic boundary conditions the μ-th component of the momentum $k_\mu = ap_\mu = 2\pi\,\nu_\mu/N_\mu$ has values in the Brillouin-zone (B_μ - arbitrary interger):

$$B_\mu + 1 \leq \nu_\mu \leq B_\mu + N_\mu. \tag{2.5}$$

Introducing the notation

$$(k,x) \equiv 2\pi \left(\frac{\nu_1 x_1}{N_1} + \ldots + \frac{\nu_4 x_4}{N_4}\right), \tag{2.6}$$

the free quark-propagator can be written with $N = N_1N_2N_3N_4$ as

$$G_{yx} \equiv Q^{-1}(U=1)_{yx} = N^{-1} \sum_k e^{-i(k,x-y)}\, \tilde{G}_k\,. \tag{2.7}$$

The momentum-space free propagator $\tilde{G}_k$ is

$$\tilde{G}_k = \{1 - 2K \sum_{\mu>0} (r\cos k_\mu - i\gamma_\mu \sin k_\mu)\}^{-1} =$$

$$= \{1 - 2K \sum_{\mu>0} (r\cos k_\mu + i\gamma_\mu \sin k_\mu)\}\,\cdot$$

$$\cdot \left\{\left[1 - 2K \sum_{\mu>0} r\cos k_\mu\right]^2 + 4K^2 \sum_{\mu>0} \sin^2 k_\mu\right\}^{-1}. \tag{2.8}$$

This form shows, how the fermion doubling problem is solved for Wilson-fermions. In the continuum limit $k_\mu \equiv ap_\mu \to 0$ the denominator is proportional to

$$(\frac{1-8rK}{2K})^2 + k_\mu k_\mu \to a^2(m^2 + p_\mu p_\mu). \qquad (2.9)$$

At the other corners of the Brillouin-zone, however, the mass in the denominator is $m_w = (m + 2wra^{-1})$, if the number of momentum components with $k_\mu - \pi \equiv ap_\mu$ is $w = 1,2,3,4$. Therefore, the mass m_w tends to infinity for $a \to 0$, and the unwanted extra fermions decouple from the physical fermion with mass m.

The global symmetry properties of the Wilson lattice fermion action can be immediately seen in Eq. (2.3). For N_f flavours the quark matrix is block diagonal in flavour. In the individual blocks the hopping parameter has the value of K_f belonging to the bare quark mass m_f of the flavour in question ($f = u,d,s,c,b,t,\ldots$). For non-degenerate flavours the action has an exact $U(1)^{\otimes N_f}$ symmetry corresponding to the conservation of the quark number in each flavour. For N_f flavours with degenerate mass the global symmetry is $U(N_f) = U(1) \otimes SU(N_f)$. The axial part of the global chiral $U(1) \otimes SU(N_f) \otimes SU(N_f)$ symmetry is, however, explicitly broken by the Wilson-term proportional to r, even in the case of zero bare mass $m_f = 0$. The expected situation in QCD is, that the axial-vector symmetry is spontaneously broken by the vacuum expectation value of $\tilde{\psi}\psi$, therefore in the Wilson fermion formulation one has to assume that for vanishing lattice spacing the explicit breaking goes over into a spontaneous breaking.

The anticommuting Grassmann-variables are not well suited for numerical calculations. Therefore, it is convenient to perform the fermion integration by using the bilinearity of the action in fermion fields. In general, the expectation value of a quantity, $F(U, \psi, \tilde{\psi})$ is defined as

$$\langle F \rangle = \frac{\int \prod_x \left(d\psi_x d\tilde{\psi}_x \prod_{\mu>0} dU(x,\mu)\right) e^{-S_g(u) - S_f(U,\psi,\tilde{\psi})} F(U,\psi,\tilde{\psi})}{\int \prod_x \left(d\psi_x d\tilde{\psi}_x \prod_{\mu>0} dU(x,\mu)\right) e^{-S_g(U) - S_f(U,\psi,\tilde{\psi})}} \qquad (2.10)$$

For a purely gluonic quantity depending only on the gauge field U, this is equivalent to

$$\langle F \rangle = \frac{\int dU \exp(-S_{eff}(U))\, F(U)}{\int dU \exp(-S_{eff}(U))},$$

$$dU \equiv \prod_x \prod_{\mu>0} dU(x,\mu). \qquad (2.11)$$

The effective action S_{eff} in the gluonic sector is the sum of the pure gauge action S_g and the negative logarithm of the Matthews-Salam determinant encountered at the integration over the fermionic degrees of freedom:

$$S_{eff}(U) = S_g(U) + S^q_{eff}(U),$$

$$S^q_{eff}(U) = -\ln\det Q(U). \qquad (2.12)$$

The quantities explicitly depending on the quark fields can also be evaluated from the effective action in the gluonic sector. For the product of purely fermionic variables we have, for instance,

$$\langle \psi_{r_1} \tilde{\psi}_{s_1} \psi_{r_2} \tilde{\psi}_{s_2} \cdots \psi_{r_n} \tilde{\psi}_{s_n} \rangle = \frac{\int dU \exp(-S_{eff}(U)) \det (r_1 \cdots r_n, s_1 \cdots s_n) \{Q^{-1}(U)\}}{\int dU \exp (-S_{eff}(U))} . \qquad (2.13)$$

Here the determinant in the nominator is built from the matrix elements $Q^{-1}(U)_{rs}$ of the quark-propagator in the background gauge field U. Note, that the indices (r,s) are abbreviations for all sorts of indices of the quark field, namely, colour-, spin- and flavour-indices.

The hadron masses can be calculated from the expectation values of correlation functions of composite operators carrying different quantum numbers. The choice of the composite operators is to a large extent arbitrary. In fact, for given values of the coupling constants one has to find the optimal operator, which has a strong enough coupling to the hadron in question and, at the same time, its numerical evaluation is not too difficult. In practice this means that in most cases the simplest local multiquark composite operators are taken. Let us now restrict ourselves to the ground state mesons and baryons (in the sense of SU(6)) containing u-, d- and s-quarks. The spin dependence of the operators is dictated in this case by the relativistic generalization of SU(6) symmetry (for a review and references see [Pais, 1966]). The $J^{PC} = 0^{-+}$ pseudoscalar mesons are described by bilinear composite operators like, for instance,

$$\phi_x^{(\pi^+)} = \tilde{d}_{x\alpha a} \gamma_{5,\alpha\beta} u_x^{\beta a},$$

$$\phi_x^{(K^+)} = \tilde{s}_{x\alpha a} \gamma_{5,\alpha\beta} u_x^{\beta a}. \qquad (2.14)$$

Here $u_x^{\alpha a}$, $d_x^{\alpha a}$ and $s_x^{\alpha a}$ stand for the u-, d- and s-flavour components of the quark field $\psi_x^{\alpha a}$, respectively. The indices $\alpha,\beta,\ldots$ denote Dirac spin-indices, whereas a, b, ... are the SU(3) colour-indices. The corresponding 1^{--} vector meson fields are (k = 1,2,3):

$$\phi_{xk}^{(\rho^+)} = \tilde{d}_{x\alpha a} \gamma_{k,\alpha\beta} u_x^{\beta a},$$

$$\phi_{xk}^{(K^{*+})} = \tilde{s}_{x\alpha a} \gamma_{k,\alpha\beta} u_x^{\beta a}. \qquad (2.15)$$

For the baryons the trilinear composite operators can be chosen in different ways (see. e.g. [Joffe, 1981]). In the spin $-\frac{1}{2}$ octet one can use, for instance

$$\phi_{x\alpha}^{(p)} = \varepsilon_{abc} (C\gamma_5)_{\beta\gamma} u_x^{\alpha a} (u_x^{\beta b} d_x^{\gamma c} - d_x^{\beta b} u_x^{\gamma c}),$$

$$\phi_{x\alpha}^{(\Sigma+)} = \varepsilon_{abc} (C\gamma_5)_{\beta\gamma} u_x^{\alpha a} (u_x^{\beta b} s_x^{\gamma c} - s_x^{\beta b} u_x^{\gamma c}),$$

$$\phi_{x\alpha}^{(\Lambda)} = \varepsilon_{abc} (C\gamma_5)_{\beta\gamma} [u_x^{\alpha a} (d_x^{\beta b} s_x^{\gamma c} - s_x^{\beta b} d_x^{\gamma c}) +$$

$$+ d_x^{\alpha a} (s_x^{\beta b} u_x^{\gamma c} - u_x^{\beta b} s_x^{\gamma c}) - 2 s_x^{\alpha a} (u_x^{\beta b} d_x^{\gamma c} - d_x^{\beta b} u_x^{\gamma c})],$$

$$\phi_{x\alpha}^{(\Xi^0)} = \varepsilon_{abc} (C\gamma_5)_{\beta\gamma} s_x^{\alpha a} (s_x^{\beta b} u_x^{\gamma c} - u_x^{\beta b} s_x^{\gamma c}). \qquad (2.16)$$

For the spin $-\frac{3}{2}$ decuplet one can consider:

$$\phi^{(\Delta^{++})}_{xk\alpha} = \varepsilon_{abc}(C\gamma_k)_{\beta\gamma} u_x^{\alpha a} u_x^{\beta b} u_x^{\gamma c},$$

$$\phi^{(\Sigma^{**})}_{xk\alpha} = \varepsilon_{abc}(C\gamma_k)_{\beta\gamma}(u_x^{\alpha a} u_x^{\beta b} s_x^{\gamma c} + u_x^{\alpha a} s_x^{\beta b} u_x^{\gamma c} + s_x^{\alpha a} u_x^{\beta b} u_x^{\gamma c}),$$

$$\phi^{(\Xi^{*0})}_{xk\alpha} = \varepsilon_{abc}(C\gamma_k)_{\beta\gamma}(s_x^{\alpha a} s_x^{\beta b} u_x^{\gamma c} + s_x^{\alpha a} u_x^{\beta b} s_x^{\gamma c} + u_x^{\alpha a} s_x^{\beta b} s_x^{\gamma c}),$$

$$\phi^{(\Omega^{-})}_{xk\alpha} = \varepsilon_{abc}(C\gamma_k)_{\beta\gamma} s_x^{\alpha a} s_x^{\beta b} s_x^{\gamma c}. \qquad (2.17)$$

ε_{abc} denotes, as usual, the totally antisymmetric SU(3) unit tensor, and C is the Dirac-matrix for charge conjugation. In the numerical calculations it is customary to use the following representation of euclidean Dirac-matrices:

$$\gamma_1 = \begin{vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{vmatrix} \qquad \gamma_2 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$\gamma_3 = \begin{vmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{vmatrix} \qquad \gamma_4 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$\gamma_5 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = \gamma_1\gamma_2\gamma_3\gamma_4 \qquad C = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = \gamma_1\gamma_3\gamma_5 \qquad (2.18)$$

The expectation value of the product of two hadron operators can be expressed, using Eq. (2.13), by the products of the quark propagators in some background gauge field configuration U. For the quark flavours u, d, and s one has to take in the propagator the hopping parameter values K_u, K_d and K_s, respectively. (The small mass difference between u- and d-quarks is, however, usually neglected: $K_u = K_d$). Writing out indices explicitly, let us introduce

$$U_{x\alpha a, y\beta b} \equiv Q^{-1}(U, K = K_u)_{x\alpha a, y\beta b},$$

$$D_{x\alpha a, y\beta b} \equiv Q^{-1}(U, K = K_d)_{x\alpha a, y\beta b},$$

$$S_{x\alpha a, y\beta b} \equiv Q^{-1}(U, K = K_s)_{x\alpha a, y\beta b}. \qquad (2.19)$$

Then, for instance for $\langle \phi_x^{(\pi^+)} \phi_y^{(\pi^-)} \rangle$ and $\langle \phi_{xk}^{(\rho^+)} \phi_{y\ell}^{(\rho^-)} \rangle$, one has to calculate, respectively,

$$Tr_{sc}\{\gamma_5 U_{xy} \gamma_5 D_{yx}\},$$

$$Tr_{sc}\{\gamma_k U_{xy} \gamma_\ell D_{yx}\}, \qquad (2.20)$$

where Tr_{sc} stands for a trace over spin and colour indices. Formulae like Eq. (2.20) apply to all flavour non-diagonal mesons. For mesons like $\eta, \eta', \omega, \phi, \ldots$ some combination of flavour-diagonal operators like e.g.

$$\phi_x^{(\bar{u}u)} = \tilde{u}_{x\alpha a} \Gamma_{\alpha\beta} u_x^{\beta a} \qquad (2.21)$$

is needed (Γ is some Dirac-matrix). For the expectation value $\langle \phi_x^{(\bar{u}u)} \phi_y^{(\bar{u}u)} \rangle$

the general expression (2.13) involves the combination

$$Tr_{sc}\{\Gamma\, U_{xy}\Gamma U_{yx}\} - Tr_{sc}\{\Gamma\, U_{xx}\}\, Tr_{sc}\{\Gamma U_{yy}\}. \qquad (2.22)$$

In order to obtain baryon masses the necessary combinations of quark propagators are: for the proton (and similarly for Ξ^0 and Σ^+):

$$\varepsilon_{abc}\varepsilon_{def}(C\gamma_5)_{\beta\gamma}\,(C\gamma_5)_{\varepsilon\phi}\cdot$$

$$\left[U_{x\alpha a,y\delta d}\,U_{x\beta b,y\varepsilon e}\,D_{x\gamma c,y\phi f} + U_{x\alpha a,y\varepsilon d}U_{x\beta b,y\delta e}D_{x\gamma c,y\phi f}\right], \qquad (2.23a)$$

for the Λ-baryons:

$$\varepsilon_{abc}\varepsilon_{def}(C\gamma_5)_{\beta\gamma}(C\gamma_5)_{\varepsilon\phi}\Big[U_{x\alpha a,y\delta d}D_{x\beta b,y\varepsilon e}S_{x\gamma c,y\phi f} +$$

$$+ D_{x\alpha a,y\delta d}U_{x\beta b,y\varepsilon e}S_{x\gamma c,y\phi f} + 4S_{x\alpha a,y\delta d}U_{x\beta b,y\varepsilon e}D_{x\gamma c,y\phi f} -$$

$$- U_{x\alpha a,y\varepsilon d}D_{x\beta b,y\delta e}S_{x\gamma c,y\phi f} - D_{x\alpha a,y\varepsilon d}U_{x\beta b,y\delta e}S_{x\gamma c,y\phi f} -$$

$$-2U_{x\alpha a,y\varepsilon d}D_{x\beta b,y\phi e}S_{x\gamma c,y\delta f} - 2D_{x\alpha a,y\varepsilon d}U_{x\beta b,y\phi e}S_{x\gamma c,y\delta f} -$$

$$-2\,S_{x\alpha a,y\varepsilon d}D_{x\beta b,y\phi e}U_{x\gamma c,y\delta f} - 2S_{x\alpha a,y\varepsilon d}U_{x\beta b,y\phi e}D_{x\gamma c,y\delta f}\Big], \qquad (2.23b)$$

for the Δ^{++}-baryon (and similarly for Ω^-):

$$\varepsilon_{abc}\varepsilon_{def}(C\gamma_k)_{\beta\gamma}(C\gamma_k)_{\varepsilon\phi}\cdot$$

$$\left[U_{x\alpha a,y\delta d}U_{x\beta b,y\varepsilon e}U_{x\gamma c,y\phi f} + 2U_{x\alpha a,y\varepsilon d}U_{x\beta b,y\delta e}U_{x\gamma c,y\phi f}\right], \qquad (2.23c)$$

and for the Σ^{*+}-baryon (and similarly for Ξ^{*0}):

$$\varepsilon_{abc}\varepsilon_{def}(C\gamma_k)_{\beta\gamma}\,(C\gamma_k)_{\varepsilon\phi}\cdot$$

$$\left[U_{x\alpha a,y\delta d}\,U_{x\beta b,y\varepsilon e}\,S_{x\gamma c,y\phi f} + 2U_{x\alpha a,y\varepsilon d}\,U_{x\beta b,y\delta e}\,S_{x\gamma c,y\phi f}\right]. \qquad (2.23d)$$

The numerical calculation of the hadron masses is based on the Källman-Lehmann representation of two point functions. In the euclidean region for a spinless field $\phi(x)$ (for simplicity) we have

$$\langle o|T\{\phi(x)\phi(y)\}|o\rangle = \int_{m_o^2}^{\infty} dm^2\rho(m^2)\,\Delta_E(x-y;m^2), \qquad (2.24)$$

with a positive spectral weight function $\rho(m^2)$ and the euclidean propagator

$$\Delta_E(x;m^2) = \int \frac{d^4k}{(2\pi)^4}\,\frac{e^{ik_\mu x_\mu}}{m^2 + k_\nu k_\nu}\,. \qquad (2.25)$$

Projecting out the zero three-momentum intermediate states by an integration over three-space, one obtains for this "time-slice":

$$\int d^3x \, \langle o|T\{\phi(x)\phi(y)\}|o\rangle = \int_{m_o}^{\infty} dm \, \rho(m^2) \, e^{-m|x_4 - y_4|} . \qquad (2.26)$$

Stable single particle states contribute by a δ-function term in ρ, whereas multiparticle intermediate states give a continuum contribution. For large euclidean time separations the lowest mass m_o dominates, and we have

$$m_o = - \lim_{|x_4 - y_4| \to \infty} \frac{1}{|x_4 - y_4|} \ln \int dx^3 \, \langle o|T\{\phi(x)\phi(y)\}|o\rangle . \qquad (2.27)$$

Another possibility is to do a Fourier-transformation

$$\int dx_4 \, e^{-ip_4 x_4} \int d^3x \, \langle o|T\{\phi(x)\phi(o)\}|o\rangle =$$

$$= \int_{m_o^2}^{\infty} dm^2 \, \frac{\rho(m^2)}{m^2 + p_4^2} . \qquad (2.28)$$

This shows the particle-poles in the (real) energy variable $E = -ip_4$.

Both Eq. (2.26) and Eq. (2.28) can, in principle, be used to extract the lowest masses from the expectation values of products of multiquark operators like in Eq. (2.14- 2.17). One has to keep, however, in mind that the formulae are exact only in the continuum limit. On a finite lattice there are O(a) corrections due to the finite lattice spacing a, and also finite size effects due to the finite physical extension of the lattice. (For some exact results about the spectrum of finite lattice pure gauge theory in the strong coupling region see, for instance, (Schor, 1983, 1984; O'Caroll, 1984a, 1984b). An important modification to the formula (2.26) is due to the periodic boundary conditions, which are introduced in most calculations in order to minimize finite size effects. Due to the periodicity, a signal can propagate between

two points in different ways. Neglecting propagations with more windings, the simple exponential behaviour (2.26) is replaced in the case of mesons by

$$e^{-x_4 am} + e^{-(N_4 - x_4) am} . \qquad (2.29)$$

Here the time difference x_4 is given in lattice units and N_4 is the lattice size in the euclidean time direction. For baryons the corresponding formula is more complicated because TCP-invariance implies the propagation of the opposite parity (charge conjugate) state in opposite time direction, therefore Eq. (2.29) is replaced by

$$(1 + \gamma_4) \{c_+ e^{-x_4 am_+} + c_- e^{-(N_4 - x_4) am_-}\} +$$

$$+ (1 - \gamma_4) \{c_+ e^{-(N_4 - x_4) am_+} + c_- e^{-x_4 am_-}\} . \qquad (2.30)$$

Here c_+ and c_- give the coupling strengths of the two opposite parity states with masses m_+ and m_-, respectively. Besides the effects of finite time extension, the spectrum is, of course, also influenced by the finite spatial extension L of the lattice. A dimensionless measure of the finite size is $\xi \equiv Lm_o$ (with $m_o = m_o(L)$ the lowest mass in the given channel). For large L the mass on the finite lattice tends to the physical mass $M_o = \lim_{L \to \infty} m_o(L)$ and the deviation behaves like (Lüscher, 1984):

$$\delta_o \equiv \frac{M_o - m_o}{m_o} \simeq \frac{c_1}{\xi} e^{-c_2 \xi} . \qquad (2.31)$$

The constants $c_{1,2}$ depend on the quantum numbers. A more detailed formula (Luscher, 1984) relates δ_o to some elastic scattering amplitude and hence c_1

turns out to be proportional to some coupling constant squared. The other constant c_2 is of order 1, therefore finite size effects should go away fast, once some critical size is reached.

III. Monte Carlo Calculations in the Quenched Approximation

A. Generalities

The numerical evaluation of the fermion part of the effective action S^q_{eff} in Eq. (2.12) is rather time consuming, because the "quark determinant" detQ is essentially non-local. The non-locality is due to the fact that S^q_{eff} describes the effect of closed virtual quark loops, and light virtual quarks can propagate to large distances. (For the elaboration on the quark determinant see the next Section.) In the "quenched" or "valence" approximation (Hamber et al., 1981; Marinari et al. 1981a; Weingarten, 1982) virtual quark loops are omitted by neglecting the dependence of $S^q_{eff}(U)$ on the gauge configuration U. In this case in Eqs. (2.11) and (2.13) S^q_{eff} cancels out and the effective gauge field action $S_{eff}(U)$ can be replaced by the pure gauge action $S_g(U)$. The quenched approximation is expected to give a reasonable (say, within, 10%) description of the hadron spectrum at least in the flavour non-singlet channels. This expectation is based on the phenomenological Okubo-Zweig-Iizuka rule (Okubo, 1963; Zweig, 1964; Iizuka, 1966) and on some theoretical results obtained in $1/N_c$ (N_c = number of colours) expansion (t'Hooft, 1974).

Since the gauge field configurations in the quenched approximation are distributed according to the pure gauge action, the scaling of the hadron masses m have to follow the renormalization group equation (RGE) without quarks:

$$\left\{-a \frac{\partial}{\partial a} + \bar{\beta}(g) \frac{\partial}{\partial g}\right\} m = o(a). \qquad (3.1)$$

Here $\bar{\beta}(g)$ is the Callan-Symanzik β-function of the pure gluon theory on the lattice and the right hand side is due to the scale-breaking lattice

artifacts. (For $a \to 0$ it goes to zero by some power of a). The solution of Eq. (3.1) is

$$m = a^{-1} \exp \{-\int_{g_o}^{g} \frac{dx}{\bar{\beta}(x)}\}. \tag{3.2}$$

If the integration constant g_o is replaced by an overall factor c_m, one can write (in analogy with Eq. (5.4):

$$m = c_m a^{-1} (\beta_o g^2)^{-\beta_1/2\beta_o^2} \exp \{-\frac{1}{2\beta_o g^2} - \int_o^g dx \{\frac{1}{\bar{\beta}(x)} +$$

$$+ \frac{1}{\beta_o x^3} - \frac{\beta_1}{\beta_o^2 x})\} \equiv c_m \Lambda_{latt}. \tag{3.3}$$

Here Λ_{latt} is the Λ-parameter of pure lattice gauge theory and β_0 and β_1 are the first two (universal) expansion coefficients of $\bar{\beta}(g)$ given by Eq. (5.2) and (5.3) with $N_c = 3$ and $N_f = 0$. The integral piece in the exponent is not universal, it depends, for instance, on the particular form of lattice action chosen for $S_g(U)$. But, compared to the universal g^{-2} term, it becomes small in the continuum limit $g \to 0$.

In order to obtain the two point functions of the hadronic multiquark operators in the quenched approximation one has to calculate the expectation value of the expressions like in Eq. (2.20- 2.23). In the case of flavour non-singlet mesons the required combination of quark propagators can be represented by Fig. 1 (see Eq. (2.20)). For the flavour singlet mesons, like in Eq. (2.22), one needs combinations as given in Fig. 2, whereas for the baryons in Eqs. (2.23a - 2.23d) one has to calculate combinations like in Fig. 3. In case of Fig. 1 and 3 it is enough to consider quark propagators

originating from a single point, say x. This is because the orientation of the propagator lines can be reversed by using Eq. (2.4). This means that for the flavour non-singlet mesons one has to calculate the expectation value of expressions like

$$Tr_{sc}\{Q_{xy}^{-1} \Gamma \gamma_5 Q_{xy}^{-1+} \gamma_5 \Gamma\} \tag{3.4}$$

Here Γ is some Dirac-matrix describing the spin. In this connection let us note the special role of the flavour non-singlet pseudoscalar mesons with $\Gamma \gamma_5 = \gamma_5^2 = 1$. In this case there are no cancellations in the spin-trace in Eq. (3.4) and the decrease of the hadron propagator for large distances is the slowest, which corresponds to the smallest mass. This observation is the starting point of the derivation of several rigorous mass inequalities (see, for instance, Weingarten, 1983b; Nussinov, 1983; Witten, 1983).

The case of the flavour singlet mesons in Fig. 2 is much more difficult than the propagator configurations in Fig. 1 and 3. First of all, quark propagators starting from two different points are needed. This means that one has to evaluate the quark propagators from every starting point of at least several time-slices. The second, potentially even more dangerous, difficulty is that the second term in Eq. (2.22) contains a non-connected piece which has to be subtracted. This requires very high statistics, and even then the measurement of the correlation at large distances is rather questionable. (For a suggestion how to overcome the difficulties see (Hamber et al., 1983a).) The only attempt, up to now, to calculate flavour singlet meson masses and mixing with glueballs was done recently in the quenched approximation on a small ($4^3 \cdot 8$) lattice in SU(2) gauge theory (Fukugita et al., 1984b).

B. Iterative methods for the calculation of quark propagators. The main task in the quenched hadron mass calculations is the inversion of the quark matrix Q, in order to obtain the required matrix elements of the quark propagator Q^{-1}. Several standard numerical matrix inversion methods (Varga, 1965; Householder, 1964; Stoer et al., 1980; Lanczos, 1950) were tested and successfully applied. The most popular are the variants of the Gauss-Seidel method and the conjugate-gradient method.

Let us write the quark matrix Q in Eq. (2.3) like

$$Q = 1-KM,$$

$$M_{x_2x_1} = \sum_{x,\mu} (r + \gamma_\mu)\, U(x,\mu)\, \delta_{x_2,\, x+\hat{\mu}}\, \delta_{x,x_1} \tag{3.5}$$

The simplest iteration for $p \equiv Q^{-1}i$ (i = some initial vector) is the "Jacobi-iteration":

$$p_{n+1} = i + Kmp_n \qquad (n = 0,1,2\ldots),$$

$$p_o = i, \qquad p = \lim_{n\to\infty} p_n. \tag{3.6}$$

Iterating point by point, that is, taking on the right hand side the already calculated elements of p_{n+1} instead of the old p_n, gives the "Gauss-Seidel iteration". This corresponds to the decomposition $M = M_\ell + M_u$, where M_ℓ has non-zero elements only below the main diagonal (and there $M = M_\ell$; $M_u = 0$. The iterative equation now becomes:

$$p_{n+1} = i + K\,(M_\ell p_{n+1} + M_u p_n). \tag{3.7}$$

In order to improve convergence one can also introduce a relaxation parameter λ and put

$$p_{n+1} = (1-\lambda)\, p_n + \lambda[i + K\,(M_\ell p_{n+1} + M_u p_n)]. \tag{3.8}$$

For small quark masses, still better convergence can be achieved by a "second order" method. Returning to the simple expression in Eq. (3.6) even if point by point iteraction is done, the "first-order" iteration in Eq. (3.8) can be written like

$$p_{n+1} = p_n + \lambda\,(i - Qp_n). \tag{3.9}$$

Performing after this step a second step with

$$p_{n+1} = \rho p_n + (1-\rho)\, p_{n+1} + \lambda\rho\,(1 - Qp_{n+1}), \tag{3.10}$$

one obtains the "second order" iteration:

$$p_{n+2} = p_n + \lambda\,(1-\lambda\rho Q)(i - Qp_n). \tag{3.11}$$

By the appropriate choice of the two parameters λ,ρ a good convergence can be achieved even for smaller values of the quark mass.

The other popular and effective method for the inversion of the quark matrix is the "conjugate gradient" method. It begins with a guess p_o for $p = Q^{-1}i$. Then one has to calculate

$$r_o = Q^+\,(i - Qp_o), \tag{3.12}$$

If the length $|r_o| = \sqrt{(r_o^+, r_o)}$ is zero, than p_o is the solution. Otherwise for n = 0, 1, 2, ...

$$r_{n+1} = r_n - \frac{|r_n|^2}{|Qh_n|^2} Q^+Q\, h_n,$$

$$p_{n+1} = p_n + \frac{|r_n|^2}{|Q\, h_n|^2} h_n. \qquad (3.13)$$

If $|r_{n+1}| = 0$, then p_{n+1} is the solution, because

$$Qp_{n+1} = Qp_n + Q^{+-1} r_n = Qp_{n-1} + Q^{+-1} r_{n-1} = \ldots$$

$$= Qp_o + Q^{+-1} r_o = i. \qquad (3.14)$$

For $|r_{n+1}| \neq 0$ one calculates

$$h_{n+1} = r_{n+1} + \frac{|r_{n+1}|^2}{|r_n|^2} h_n, \qquad (3.15)$$

and returns to Eq. (3.13) for the next n. It can be shown, that the solution is always obtained in a finite number of steps.

On large lattices, a problem for the iterative methods is often due to computer memory limitations, because the iterated vectors have many components (and the gauge configuration itself takes a lot of storage space). The usual way of circumventing these difficulties is to partition the inversion of the quark matrix Q, which is possible because of the locality. A simple way to do

this is to organize the iteration according to times-slices. In this way it is enough to keep only a few (usually upto 3) time-slices in the memory (Bowler et al., 1984a).

C. Hopping expansion method. Another way to obtain information about hadronic two-point (or many point) amplitudes avoids the direct numerical inversion of the quark matrix by concentrating on the expansion coefficients in powers of the hopping parameter K. Knowing the hopping parameter expansion coefficients to sufficiently high orders, and assuming the analyticity of the amplitudes at K = 0, one can investigate different features of the amplitudes at the physical values of K. The starting point (Wilson, 1977; Stamatescu, 1982; Hasenfratz et al., 1981, 1982a, 1982b; Lang et al., 1982) is a formal Taylor-expansion like

$$(1 - KM)^{-1} = \sum_{j=o}^{\infty} K^j M^j \tag{3.16}$$

or, for the fermion part of the effective action in Eq. (2.12):

$$S^q_{eff} = - \ln\det (1 - KM) = - \mathrm{Tr} \ln (1 - KM) = \sum_{j=1}^{\infty} \frac{K^j}{j} \mathrm{Tr} (M^j). \tag{3.17}$$

From the expansion coefficients in Eq. (3.16), (3.17) one can construct the expansion coefficients of the hadronic amplitude in question, and then either by direct application of the hopping parameter series (if the series converges), or by some analytical continuation method, one can calculate the amplitude at the desired value of K. Applying the explicit form of the "hopping matrix" M in Eq. (3.5), it is possible to represent the hopping parameter series as a sum over curves on the lattice. For instance, one can write $\mathrm{Tr}(M^n)$ as

$$\mathrm{Tr}\,(M^n) = \sum_{x_1\mu_1,\ldots,x_n\mu_n} \delta_{x_1,x_n+\hat{\mu}_n} \delta_{x_n,\, x_{n-1}+\hat{\mu}_{n-1}} \cdots \delta_{x_2,x_1+\hat{\mu}_1}$$

$$\mathrm{Tr}_c\{U(x_n,\mu_n) \cdots U(x_2,\mu_2)\, U(x_1,\mu_1)\}$$

$$\mathrm{Tr}_s\{(r + \gamma_{\mu_n}) \cdots (r + \gamma_{\mu_2})\, (r + \gamma_{\mu_1})\}. \tag{3.18}$$

Due to the δ-functions, the sum runs here over all closed loops. A similar representation of $(M^n)_{yx}$ is possible in terms of loops running from the point x to y. Although this representation is physically rather suggestive and appealing, it has the practical disadvantage that the number of curves at high orders is very large. For instance, at 12^{th} order there are more than $4 \cdot 10^6$ closed curves going through a given link, and at 16^{th} order already more than $6 \cdot 10^9$ (the number increases exponentially) (Berg et al., 1982). It is quite clear that the evaluation of the traces in Eq. (3.18) requires a prohibitively large number of multiplications already in these orders. This was the reason why the first numerical calculations of the hadron spectrum (Hasenfratz et al., 1982a, 1982b) were restricted to low orders. A sufficiently high order hopping expansion is possible with the numerical iterative method (Hasenfratz et al., 1983, 1984a). If one wants to obtain, for instance, the matrix element $\langle g|M^n|i\rangle$, then one uses

$$\langle g|M^n|i\rangle = \sum_h \langle g|M|h\rangle\langle h|M^{n-1}|i\rangle. \tag{3.19}$$

This shows how $\langle g|M^n|i\rangle$ is built up from the lower order matrix elements $\langle h|M^{n-1}|i\rangle$. Due to the nearest neighbour structure of the hopping matrix M (see Eq. (3.5)), the consecutive steps of the iteration for $\langle f|M^n|i\rangle$ can be visualized by Fig. 4. From some starting point, in a given order, always a finite number of points is reached. During the iteration it is possible to choose the boundary conditions for the quarks independently from the given boundary conditions (usually periodic) of the gauge configuration. In the case of the "periodic box" iteration the quarks also obey periodic boundary conditions. In the "copied gauge field" iteration the quarks propagate without boundaries over the periodic gauge field background. A mixture of both ways is also possible: "periodic box" in the space directions and "copied gauge field" in the time direction. The advantage of the copied gauge field iteration over the periodic box iteration is, that the quark propagators are defined for continuous momenta (not just for the discrete values in Eq. (2.5)). Therefore, it is possible to analytically continue the hadron propagators to real energies by doing a Laplace-transform with real $E = -ip_4$, instead of the Fourier-transformation in Eq. (2.28). This allows to look directly for the particle singularities (for fixed E in the hopping parameter variable) by a Pade-approximant technique. In such a way direct information is obtained on the nature of the singularity: the localization of cuts or multiple poles (e.g. due to radial excitations (Hasenfratz et al., 1984a) becomes possible. This is potentially a very useful possibility also in an unquenched spectrum calculation with light dynamical quarks, where the resonances $(\rho,\Delta,\ldots)$ are "hidden" behind multiparticle cuts. The price of a high-order calculation with copied gauge field iteration is the growing number of points reached in higher orders (and hence the growing length of the arrays in the computer program). For illustration, the number of points with given

distance $\ell = 0, 1, \ldots, 16$ on a four-dimensional hypercubical lattice is given in Table III.1.

The required order in the hopping expansion depends on the gauge coupling $\beta \equiv 2N_c g^{-2}$ and on the quark mass (it is higher for larger β and for smaller quark mass). At $\beta = 5.7$ in SU(3) ($N_c = 3$), for instance, good results can be achieved in 32^{nd} order. In the test case of free Wilson fermions (with $r = 1$) the position of branch point singularities in the multiquark amplitudes are reproduced in 32^{nd} order within 1-2% in case of mesons and 4-5% in case of baryons (Kunszt, 1983). For non-zero coupling ($g \neq 0$, $\beta < \infty$) the situation is most probably even better. The order of the quark propagator calculation is chosen in such a way, that for the required hadronic amplitude some given order can be achieved. For instance, a 32^{nd} order calculation of the mesonic amplitude in Fig. 1 requires, that the quark propagator iteration runs upto the maximum distance $\ell = 16$ from the given initial point (see Fig. 4). It can be easily seen, that with this set of quark propagators the baryonic amplitude in Fig. 3 can be calculated upto 33^{rd} order. It is also clear, that mesonic amplitudes contain only even powers of K, whereas the baryonic ones both even and odd powers.

D. Results for Wilson-fermions Quenched hadron mass calculations with Wilson-quarks were performed by several authors; besides the already mentioned References (Hamber et al., 1981; Weingarten, 1982; Hasenfratz et al. 1982a, 1982b, Hasenfratz et al., 1984a Kunszt, 1983) see also (Fucito et al., 1982a; Martinelli et al., 1982b, Weingarten, 1983a; Hamber et al., 1983b; Martinelli et al., 1983a; Bernard et al., 1983a; Gupta et al., 1983a; Bowler et al., 1983; Bernard et al., 1983b; Lipps et al., 1983; Gupta et al., 1983b; Fukugita et al., 1983, 1984a; Fucito et al., 1983; Patel et al., 1983; Bowler et al., 1984b; Langguth et al., 1984; Billoire et al., 1985; König et al., 1984; Itoh et al., 1984). Qualitatively rather good results for the spectrum were reported already in the first pioneering papers, although the physical lattice since was very small: typically less than 1 fm. (The situation became even worse when more precise string-tension measurements (Gutbrod et al., 1983; Parisi et al., 1983; Hasenfratz et al., 1984c; Barkai et al., 1984; Otto et al., 1984b) suggested an even smaller lattice spacing than it was thought before). The importance of some minimum lattice sice was realized, however, soon (Hasenfratz et al., 1983; Martinelli et al., 1983a; Bernard et al., 1983a; Gupta et al., 1983a; Bowler et al., 1983; Politzer, 1984) some exploratory studies on larger lattices showd (Lipps et al., 1983; Hasenfratz et al., 1984a; Billoire et al., 1985; König et al., 1984) that the most drastic finite size effects go away if the spacial size of the lattice reaches 1.7 - 2.0 fm and the temporal size is roughly twice as much. The larger temporal size is needed for the separation of the lowest state from radial excitations. The elongated lattice for the calculation of quark propagators can, however, be prepared by copying a symmetric lattice periodically twice in time direction. In the hopping expansion method this is done anyway, if the copied gauge field iteration is applied. In this case one has, however, to do

a high enough order calculation to exploit efficiently the distant time-slices. In practice this means that for an effective time-elongation N_t roughly an order $2N_t$ is required in the copied gauge field hopping expansion.

Assuming a string tension $\sqrt{\sigma}$ = 420 MeV, the recent SU(3) string-tension calculations give for the lattice spacing.

$$a(\beta = 5.7) \tilde{=} 0.21 \text{ fm},$$

$$a(\beta = 5.8) \sim 0.16 \text{ fm}, \qquad (3.20)$$

$$a(\beta = 6.0) \tilde{=} 0.12 \text{ fm}.$$

Therefore, the minimum required lattice size is roughly $8^3 \cdot 16$ (at $\beta = 5.7$), $12^3 \cdot 24$ (at $\beta = 5.8$) and $16^3 \cdot 32$ (at $\beta = 6.0$). Note, that the lattice scale between $\beta = 5.7$ and $\beta = 6.0$ changes more rapidly than "asymptotic scaling" with the two-loop perturbative β-function $\bar{\beta}(g) \to -\beta_0 g^3 - \beta_1 g^5$ in Eqs. (3.2-3.3) would require. Of course, hadron masses on the lattice should scale also according to Eq. (3.20), in order to be consistent with a continuum (scaling) behaviour. The present situation is not in contradiction with such a behaviour between $\beta = 5.7$ and $\beta = 6.0$ (see Table III.2). The errors are, however, still somewhat large, and the $\beta = 6.0$ results presumably suffer from somewhat more finite size effects. The results of (König et al. 1984) may be better from this point of view, but it is not clear what is the influence of the (approximate) blocking procedure introduced by (Mütter et al., 1984a, 1984b).

Note that the critical hopping parameter value K_{cr}, where the pion mass vanishes, is substantially larger at $\beta = 5.7$ ($g^{-2} = 0.95$) and $\beta = 6.0$

($g^{-2} = 1.0$) than the one-loop perturbative value (Kawamoto, 1981a; Stehr et al., 1982):

$$K_{cr}^{1\text{-loop}} = 0.125 + 0.0101786 \frac{N_c^2-1}{2N_c} g^2 . \qquad (3.21)$$

This also shows, that in this region important non-perturbative (or higher order) effects are present.

Besides finite size effects, the other limiting feature of the existing calculations is the statistics. Most calculations use less than 20 propagators per K-value. In (Langguth et al., 1984) 80 meson propagators and 40 baryon propagators were collected, whereas (König et al., 1984) has 72 propagators per K-value. The experience with higher statistics shows, that for light quark masses the calculation of a few hundred propagators is probably not an exageration.

Comparing the numbers in Table III.2 to experimental masses, it turns out that there is a rough agreement between the overall scale given by the masses and the scale (3.20) obtained from the string-tension. The mass ratios, however, deviate from the right ones: to ratio m_p/m_ρ comes out around 1.8 and the Δ -p mass-splitting has a tendency to be too small. (Note in this respect, that in (Billoire et al., 1985) non-relativistic baryon-operators were used, instead of the standard relativistic ones Eqs. 2.16-2.17. The standard operators would have given also there higher baryon masses). The disagreement of mass ratios can come from the explicit breaking of chiral symmetry introduced by Wilson lattice fermions, but perhaps even more probably, can also be characteristic to the quenched approximation.

E. Kogut-Susskind fermions and variant actions. There exist also several quenched hadron spectrum calculations (Marinari et al., 1981a; Hamber et al., 1983b; Bowler et al., 1984b; Billoire et al., 1985; Hamber et al., 1982; Marinari et al., 1984; Gilchrist et al., 1984a, 1984b; Billoire et al., 1984a, 1984b, 1984c) with Kogut-Susskind lattice fermions (Banks et al., 1977; Susskind, 1977; Kawamoto et al., 1981b; Gliozzi, 1982; Kluberg-Stern et al., 1983). In this formulation the problem of chiral symmetry and the associated proliferation of fermion degrees of freedom is treated differently. For zero bare quark mass there is an exact $U(1)_{vector} \otimes U(1)_{axial}$ symmetry of the lattice action, and the spontaneous breaking of the $U(1)_{axial}$ part implies the existence of a massless Goldstone-boson also in the strong coupling region (Blairon et al., 1981; Kluberg-Stern et al. 1981; Jolicoeur et al., 1984). Flavour symmetry (like isospin etc.) is, however, explicitly broken, therefore an important question for the numerical calculation is to study the masses of non-Goldstone pseudoscalar mesons. This was done for SU(2) gauge theory in (Billoire et al., 1984a, 1984c) and the result showed near $\beta = 2.3 - 2.4$ strong evidence for various light flavoured pseudoscalar mesons, in accordance with a Nambu-Goldstone realization of full chiral symmetry in the continuum.

The extraction of hadron masses from the hadronic two-point functions is a non-trivial task for Kogut-Susskind fermions, especially in the baryon sector. This is due to the mixing in flavour and spin-parity caused by the explicit symmetry breaking terms. In the case of SU(3) colour upto now only the simplest (local) hadronic operators were considered, therefore the interpretation of the numerical results in terms of the masses may have some systematic uncertainty. For a collection of some recent results see Table III.3. Comparing the values at $\beta = 5.7$ (where finite size effects are presumably smaller) to Table III.2 we see reasonable agreement for the nucleon

mass. In the case of the ρ-mass, however, where the errors are small, there is a definit disagreement: the apparent lattice spacing seems to be about a factor 1.6 smaller for Wilson-fermions. Correspondingly, there is no problem with the m_p/m_ρ ratio for Kogut-Susskind quarks. This is actually not a surprise, since this ratio is essentially correct already in strong coupling (Kluberg-Stern et al., 1981; Jolicoeur et al., 1984). In summary: the quenched hadron mass calculations with Kogut-Susskind fermions are promising, but the difficult problem of mixing (in flavour and in spin-parity) deserves further study, in particular in the case of non-degenerate flavour masses (Golterman et al., 1984a, 1984b; Göckeler, 1984).

Besides changing the fermion part of the action the gauge part can also be changed, for instance, in order to improve the scaling properties in the intermediate coupling range. Some attempts in this direction were already undertaken (Bowler et al., 1984b; Itoh et al., 1984; Marinari et al. 1984), but within present precision there are no substantial deviations from the simple Wilson gauge action.

F. Other static hadron properties. Once the hadron mass calculation is under control, one can start to calculate a large number of different static hadronic matrix-elements, which are of interest in strong and electroweak interactions (Fucito et al., 1982b; Bernard et al., 1982; Martinelli et al., 1982a; Ali et al., 1983; Cabibbo et al., 1984; Brower et al., 1984; Gottlieb et al., 1984; Velikson et al., 1985). Many of these matrix elements involve the electromagnetic or weak currents which are conserved in the continuum. Such conserved vector currents can be defined on the lattice, too (Karsten et al., 1981). For Wilson-fermions the appropriate choice is (for $N_f = 3$ degenerate quarks):

$$V^s_{x,\mu} = K\, a^{-3}\{\tilde{\psi}_{x+\mu}(r+\gamma_\mu)\, U(x,\mu)\, \frac{\lambda_s}{2}\, \psi_x -$$

$$- \tilde{\psi}_x(r-\gamma_\mu)\, U(x+\hat{\mu}, -\mu)\, \frac{\lambda_s}{2}\, \psi_{x+\mu}\} \qquad (3.22)$$

The Gell-Mann matrices λ_s ($s = 0, 1, \ldots, 8$) act here in flavour. It can be shown, that as a consequence of the equations of motions $\delta S_f/\delta\psi_x = \delta S_f/\delta\tilde{\psi}_x = 0$, the current in Eq. (3.22) satisfies the "conservation equation"

$$\sum_{\mu>0} (V^s_{x,\mu} - V^s_{x-\mu,\mu}) = 0. \qquad (3.23)$$

The advantage of the conserved current in Eq. (3.22) over the "naive" local currents like const $\cdot\, \tilde{\psi}_x\gamma_\mu\psi_x$ is, that as a consequence of Eq. (3.23), the strength of $V^s_{x,\mu}$ is not renormalized. Therefore, the matrix elements of the vector current in Eq. (3.22) can be directly compared to measurable quantities. In the case of other (for instance, axialvector) currents the

only way to determine the multiplicative renormalization is, at present, one-loop perturbation theory (Meyer et al., 1983; Martinelli et al., 1983b, 1983c; Martinelli, 1984; Groot et al., 1984), which is unrealiable in the intermediate coupling constant range. (Examples for the failure of low order perturbation theory are given by K_{cr}, see Eq. (3.21), and by the "overshooting" of asymptotic scaling according to Eq. (3.20).)

The calculation of two-point current amplitudes from Eq. (3.22) requires the quark-propagator combinations depicted in Fig. 5. This is somewhat more difficult than the combination in Fig. 1 for the local currents, because the quark propagator has to be evaluated from two neighbouring initial points. (For a first attempt see (Ali et al., 1983).)

Another way to calculate $q^2 = 0$ matrix elements (like nucleon magnetic moments (Bernard et al., 1982; Martinelli et al., 1982a) is to introduce an appropriate external classical field in the fermion part of the action, and compare the results with and without such fields.

An important piece of information can be obtained by calculating matrix elements of the non-leptonic decay Hamiltonian (for K-mesons, D-mesons, F-mesons, etc.). The required quark propagator diagrams (Cabibbo et al., 1984; Brower et al., 1984) are shown in Fig. 6. The last diagram in the figure is, unfortunately, as difficult as the second part of Fig. 2, but the evaluation of the first two quark propagator configurations gives already some useful information.

G. Outlook. In conclusion, the status of quenched lattice calculations can be considered as satisfactory. The quality of the present results corresponds reasonably to the invested effort. It is quite clear, however, that further improvement is both necessary and possible. Since the quenched calculation is, technically speaking, a part of the final task with dynamical quarks, high standard quenched calculations are absolutely necessary. An example of a nice "two-star" quenched calculation of the hadron spectrum would be: to take $12^3 \cdot 24$ lattice at $\beta = 5.7$, $18^3 \cdot 36$ lattice at $\beta = 5.8$, or $24^3 \cdot 48$ at $\beta = 6.0$ with several thousand quark-propagators per quark-mass. The corresponding hopping expansion calculations would be: 48^{th} order on 12^4 gauge-field at $\beta = 5.7$ etc. In this case the optimal iteration for the quark propagators is presumably on a periodic spacial box, with copied gauge field in the time direction. In such calculations finite size effects would be probably very small (in the range of a percent) and the statistics would be enough to have agood accuracy for light quark masses (perhaps 0.03 - 0.05 in lattice units).

IV. Dynamical Quarks

A. General Formulae.

As it was discussed at the beginning of the previous Section, the quark part S^q_{eff} of the effective gauge field action in Eq. (2.12) describes the effect of closed virtual quark loops on the gauge field dynamics. The resulting interaction is inherently non-local even if the original fermion action (before the integration over the anticommuting fermion variables) was local. This non-locality is the reason why it is so difficult to include dynamical quarks in the numerical calculations.

In the updating procedure always the change of the action is needed for a given change of a link variable $U(x,\mu)$. From Eq. (2.12) and (3.5) it follows

$$\Delta S^q_{eff} \equiv S^q_{eff}(U') - S^q_{eff}(U) = - \ln\det \frac{1 - KM(U')}{1 - KM(U)} . \qquad (4.1)$$

Introducing the notations

$$D \equiv \frac{M(U') - M(U)}{1 - KM(U)} = [1 - KM(U)]^{-1} \Delta M,$$

$$\Delta M_{x_2 x_1} = \sum_{x,\mu} (r + \gamma_\mu)\, \Delta U(x,\mu)\, \delta_{x_2, x+\hat{\mu}}\, \delta_{x, x_1},$$

$$\Delta U(x,\mu) \equiv U'(x,\mu) - U(x,\mu), \qquad (4.2)$$

one obtains

$$\Delta S^q_{eff} = - \ln\det (1 - KD) = - \mathrm{Tr} \ln (1 - KD) =$$

$$= \sum_{j=1}^{\infty} \frac{K^j}{j} \mathrm{Tr}\,(D^j) \qquad (4.3)$$

These formulae are valid actually for a single quark flavour with hopping parameter K. In the many-flavour case the Wilson-fermion action is block diagonal in flavour, therefore ΔS^q_{eff} is a sum over flavours with different hopping parameters $K \to K_q$ ($q = u, d, s, \ldots$).

In the expectation values like Eq. (2.11) and (2.13) the quark determinant det(1-KM) can, in principle, also be considered as a part of the gauge field dependent quantity to be evaluated with the pure gauge statistical distribution $dU \exp(-S_g(U))$. Denoting such pure gauge field averages by $\langle \ldots \rangle_o$, the expectation value $\langle F \rangle$ in Eq. (2.11), for instance, can also be written like

$$\langle F \rangle = \frac{\langle F\, e^{-S^q_{eff}} \rangle_o}{\langle e^{-S^q_{eff}} \rangle_o} . \qquad (4.4)$$

According to this formula one has to perform the Monte Carlo updating with the simple gauge field action $S_g(U)$, and then calculate S^q_{eff} on the given gauge configurations. We shall see below, that such a procedure is impractical for small quark masses, but for the theoretical understanding it could still be useful. In particular, if the quark part of the effective action S^q_{eff} can be considered small, one has the expansion (Joos et al., 1983):

$$\langle F\rangle = \langle F\rangle_0 - \left[\langle S^q_{eff} F\rangle_0 - \langle S^q_{eff}\rangle_0 \langle F\rangle_0\right] +$$

$$+ \frac{1}{2}\langle S^q_{eff}{}^2 F\rangle_0 - \langle S^q_{eff}{}^2\rangle_0 \langle F\rangle_0 - 2\langle S^q_{eff}\rangle_0 \langle S^q_{eff} F\rangle_0 +$$

$$+ 2\langle S^q_{eff}\rangle_0^2 \langle F\rangle_0\Big] \pm \ldots \quad . \tag{4.5}$$

Here, on the right hand side, only the fluctuations of S^q_{eff} matter, namely with

$$\delta F \equiv F - \langle F\rangle_0,$$

$$\delta S^q_{eff} \equiv S^q_{eff} - \langle S^q_{eff}\rangle_0, \tag{4.6}$$

one has

$$\langle F\rangle = \langle F\rangle_0 - \langle \delta S^q_{eff}\, \delta F\rangle_0 + \frac{1}{2}\langle(\delta S^q_{eff})^2 \delta F\rangle_0 \pm \ldots \tag{4.7}$$

This shows, how the non-locality of S^q_{eff} reflects physics: although S^q_{eff} extends over the whole lattice, in expectation values only its correlated fluctuations matter, therefore the non-locality of S^q_{eff} is practically restricted to regions in which correlations are actually produced by virtual quark propagation. In Eq. (4.7) $\langle F\rangle_0$ corresponds to the quenched approximation and the further terms on the right hand side represent the corrections to it. The difficulty in the application of Eq. (4.4) or Eq.

(4.7) to the calculation of unquenched averages lies in the fact that in a gauge configuration ensemble produced by the pure gauge action the fluctuations of S^q_{eff} are very large for light quarks. As an example, for SU(2) gauge group on 10^4 lattice with $N_f = 1$ flavours, this is shown in Fig. 7. Therefore, Eqs. (4.4) and (4.7) can be applied in practice only for heavy quarks. In the above example the meson masses could be determined only for dynamical quark masses $am_q \equiv (2K_q)^{-1} - (2K_{cr})^{-1} > 0.2$ (roughly 200 MeV in physical units) (Montvay, 1983).

Another way to represent the difficulty of numerical calculations with dynamical fermions is to recall the expectation, that the bulk part of the quark determinant is needed just to produce the required renormalization of bare parameters. This means, that a relatively small change in physics (i.e. in mass ratios etc.) is accompanied by a rather inconvenient shift in the scale. The shift in bare parameters can be seen already in the lowest order approximations to S^q_{eff}. According to Eq. (3.17) we have

$$S^q_{eff} = \sum_{j=1}^{\infty} \frac{K^j}{j} \mathrm{Tr}\,(M^j) \equiv \sum_{j=1}^{\infty} S^{q\,(j)}_{eff}. \tag{4.8}$$

Let us now consider only the Wilson-parameter value $r = 1$. In this case Eq. (3.18) implies that the first non-vanishing term is $S^{q\,(4)}_{eff}$ (and only even values of the index j contribute). It can be easily shown that the first two non-vanishing terms look like

$$S_{eff}^{q\,(4)} = -\,16\,K^4 N_f \sum_{\square} \mathrm{Re\ Tr}\ U =$$

$$= -\,4K^4\,N_f \sum_x \sum_{\mu>o} \sum_{\nu\neq\pm\mu} \mathrm{Re\ Tr}\ U_{(\mu\nu)},$$

$$S_{eff}^{q\,(6)} = \frac{K^6}{3} N_f \sum_x \sum_{\mu>o} \sum_{\nu_1\ldots\nu_5} \delta_{-\hat{\mu},\,\hat{\nu}_1+\ldots+\hat{\nu}_5} \mathrm{Re\ Tr}\ U_{(\mu\nu_1\ldots\nu_5)}\ T_{(\mu\nu_1\ldots\nu_5)},$$

$$S_{eff}^{q} = S_{eff}^{q\,(4)} + S_{eff}^{q\,(6)} + \ldots \qquad (4.9)$$

Here N_f degenerate flavours are taken and $\square$ is a positively oriented plaquette, also denoted in the second form of $S_{eff}^{q\,(4)}$ by $(\mu\nu)$. In the 6^{th} order term the factor $T_{(\mu\nu_1\ldots\nu_5)}$ is the Dirac-trace given by

$$T_{(\mu\nu_1\ldots\nu_5)} = \begin{cases} -\,32 & \text{for} \quad (\mu\nu_1\ldots\nu_5) = \\ -\,16 & \text{for} \quad (\mu\nu_1\ldots\nu_5) = \\ -\,16 & \text{for} \quad (\mu\nu_1\ldots\nu_5) = \end{cases}$$

(4.10)

As it is shown by Eq. (4.9), the 4^{th} order term corresponds to a shift

$$\Delta\beta = 16 N_c\, N_f\, K^4 \qquad (4.11)$$

in the coefficient $\beta \equiv 2N_c g^{-2}$ of the 1-plaquette gauge action. Since $\Delta\beta$ is positive, the lattice spacing is decreased by $S_{eff}^{q\,(4)}$ (and also by the whole

S_{eff}^{q}). The lowest order terms in Eq. (4.9) show, how, by the application of the formula (3.18) the quark part of the effective action can be decomposed into a sum over closed Wilson-loops, multiplied by some Dirac-trace and combinatorics factors. Since the mean values of more complicated Wilson-loops are correlated to the single-plaquette expectation value, it is not very surprising that the mean value of S_{eff}^{q} can be approximated quite well by (Joos et al., 1983)

$$S_{eff}^{q} \tilde{=} \frac{1}{W_{11}}\ S_{eff}^{(o)}\ (K\sqrt{W_{11}}). \qquad (4.12)$$

Here W_{11} is the single plaquette expectation value and $S_{eff}^{(0)}(K)$ is the free quark effective action (see Eq. (2.8) for the derivation):

$$S_{eff}^{(o)}\ (K) =$$

$$= -\,2\,N_c N_f \sum_k \ln\left\{\left[1 - 2K\,r \sum_{\mu>0} \cos k_\mu\right]^2 + 4K^2 \sum_{\mu>o} \sin^2 k_\mu\right\}. \qquad (4.13)$$

The approximation formula (4.12) works well only for the average, the fluctuations of S_{eff}^{q} are, unfortunately, not properly reproduced.

Pseudofermion method. For the gauge field updating with dynamical fermions, according to Eq. (4.1-4.3), matrix elements of the quark propagator $Q^{-1} \equiv (1-KM)^{-1}$ have to be calculated. In the "pseudofermion method" of Fucito, Marinari, Parisi and Rebbi (Fucito et al., 1981) this is done by introducing a complex scalar "pseudofermion" field ϕ_x, having the same number of components as the anticommuting quark field ψ_x. The quark propagator matrix elements are obtained by running a separate Monte Carlo calculation for the pseudofermion field over a fixed gauge field configuration. The action of the pseudofermions is given by the matrix

$$\Omega \equiv Q^+Q = 1 - KM - KM^+ + K^2M^+M \tag{4.14}$$

This is positive definit, as required for a Monte Carlo calculation and, due to Eq. (2.4), its determinant is related to the quark determinant by

$$\det Q = \sqrt{\det \Omega} \ . \tag{4.15}$$

The matrix elements of the quark propagator can be obtained as

$$Q^{-1}_{ij} = \frac{\int d\phi^+ d\phi \ \phi_i^+ (Q\phi)_j \exp\ (-\phi_k{}^+ \Omega_{kl} \phi_l)}{\int d\phi^+ d\phi \ \exp\ (-\phi_k^+ \Omega_{kl} \phi_l)} \ . \tag{4.16}$$

Since the quark matrix in Eq. (2.3) has only nearest neighbour matrix elements, the pseudofermion action Ω extends upto next-nearest neighbours. In the pseudofermion Monte Carlo it is convenient to introduce the auxiliary field (Hamber et al., 1983a)

$$\chi = Q_{ij} \ \phi_j \ . \tag{4.17}$$

With χ Eq. (4.16) can be written like

$$Q^{-1}_{ij} = \frac{\int d\phi^+ d\phi \ \exp\ (-\chi_k^+ \chi_k) \ \phi_i^+ \ \chi_j}{\int d\phi^+ d\phi \ \exp\ (-\chi_k^+ \chi_k)} \tag{4.18}$$

In principle, the pseudofermion Monte Carlo has to be run after every change of a single link. This would cost, however, an enormous amount of time, therefore the matrix elements Q^{-1}_{ij} are kept usually for a full sweep over the gauge variables and then evaluated again. This implies some violation of the detailed balance condition for the Markov-process of Monte Carlo integration. Another approximation usually introduced in the pseudofermion method is, that in the expansion (4.3) only the lowest order term with j = 1 is taken. Since D is proportional to the change ΔU of the link variable, this is a good approximation for small changes $\Delta U \to 0$.

The pseudofermion method was tested first in the 2-dimensional Schwinger model (Marinari et al., 1981b; Otto et al., 1983; Burkitt, 1983). First studies in QCD were carried out on small (2^4 and 4^4) lattices (Otto, 1984a; Bhanot et al., 1983; Azcoiti et al., 1983). Some results for the plaquette expectation value and $\langle\tilde{\psi}\psi\rangle$ were obtained on 8^4 lattice in (Hamber et al. 1983c) with Kogut-Susskind fermions. We have seen in the previous Section that, at least in the quenched approximation, good results for the hadron spectrum can be achieved on 8^4 lattice copied at least twice in the time direction for the quark propagator calculation. Such a lattice size is not yet achieved in pseudofermion calculations of the hadron spectrum, but some studies on 4^4 (Azcoiti et al., 1984), or four-times copied 4^4 (Hamber, 1985) lattices were already performed with $N_f = 3$ light Wilson-quarks.

C. Iterative hopping expansion method. The matrix elements of the quark propagator required for the gauge field updating with dynamical fermions can also be directly evaluated in hopping parameter expansion. The change in the quark part of the effective action is given in Eq. (4.3) by the matrix D which has the detailed structure

$$D = \begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix} ;$$

$$D_{11} = (1 - KM)^{-1}_{x,x+\hat{\mu}} (r + \gamma_\mu) \Delta U(x,\mu),$$

$$D_{12} = (1 - KM)^{-1}_{x,x} (r - \gamma_\mu) \Delta U(x,\mu)^+,$$

$$D_{21} = (1 - KM)^{-1}_{x+\hat{\mu},\, x+\hat{\mu}} (r + \gamma_\mu) \Delta U(x,\mu),$$

$$D_{22} = (1 - KM)^{-1}_{x+\hat{\mu},x} (r - \gamma_\mu) \Delta U(x,\mu)^+ . \qquad (4.19)$$

In what follows only the case r = 1 will be considered. In this case the non-zero contributions in the hopping expansion look like

$$D_{11} = \sum_{\ell = 3,5,..} K^\ell M(U)^\ell_{x,x+\hat{\mu}} (1 + \gamma_\mu) \Delta U(x,\mu),$$

$$D_{12} = \sum_{\ell=4,6..} K^\ell M(U)^\ell_{x,x} (1 - \gamma_\mu) \Delta U(x,\mu)^+,$$

$$D_{21} = \sum_{\ell = 4,6..} K^\ell M(U)^\ell_{x+\hat{\mu},x+\hat{\mu}} (1 + \gamma_\mu) \Delta U(x,\mu),$$

$$D_{22} = \sum_{\ell = 3,5,..} K^\ell M(U)^\ell_{x+\hat{\mu},x} (1 - \gamma_\mu) \Delta U(x,\mu)^+ . \qquad (4.20)$$

The "periodic box iteration" of the hopping parameter series (Hasenfratz et al., 1983, 1984a) was adopted for the unquenched updating in (Montvay, 1984). To speed up the code for the evaluation of the required matrix elements $M(U)^\ell$ in Eq. (4.20), a useful observation is that it is enough to compute only for half of the initial spin index values. In the Dirac-matrix representation given by Eq. (2.18) this is trivial on links in direction $\mu = 4$, since $(1 \pm \gamma_4)$ is non-zero only for half of the index values. For the other directions one can use, for instance, (k = 1,2,3):

$$(1 + \gamma_k) \frac{1-\gamma_4}{2} = (1 + \gamma_k) \frac{1 + \gamma_4}{2} \gamma_k \frac{1 - \gamma_4}{2} . \qquad (4.21)$$

This shows, how the two lower components of e.g. $M(U)_{x,x+\hat{k}} (1 + \gamma_k)$ can be expressed by the upper two.

The average relative weight of the different orders of the hopping parameter series in Eq. (4.20) is shown in Table IV.1 for some representative cases. It can be seen that the hopping expansion converges, in the average, reasonably well within 16^{th} order. The 16^{th} order calculation would, however, take still too much time. Actually calculated were in (Montvay, 1984; Langguth et al., 1984) (on every link) 8^{th} or 12^{th} orders. This takes still a lot of time: one sweep on the 8^4 lattice (Langguth et al., 1984) took ~ 40 minutes in 8^{th} order and ~ 240 minutes in 12^{th} order on the CYBER 205 at Karlsruhe University. It is very important, that it is possible to correct, at least in the average, for the omitted higher orders. This is due to the fact that the higher order coefficients are strongly correlated to the lower ones. Such a behaviour is already suggested by the approximate validity of the formula (4.12). Using the correlation allows to estimate the result of the $\ell_{max} = 16$ order series from some lower order (e.g. $\ell_{max} = 8$ or $\ell_{max} = 12$)

calculation. For instance, in the 6^4 calculation of (Montvay, 1984) the ℓ_{max} = 16 result could be obtained in the average by multiplying the ℓ_{max} = 8 number by a factor $\lambda \tilde{=} 1.14$. The same factor needed from ℓ_{max} = 12 to ℓ_{max} = 16 was $\lambda \tilde{=} 1.03$. This brings a substantial gain in computer time but, of course, increases the error for the quark determinant. Monitoring the difference from time to time on a few hundred of links, it turned out that the estimate based on the extrapolation from ℓ_{max} = 8 deviates in the average from the exact ℓ_{max} = 16 value by ~ 16%. The corresponding average deviation for ℓ_{max} = 12 extrapolated to ℓ_{max} = 16 was 5%. The error in the determinant ratio caused by this extrapolation is far from being normally distributed. In most cases the deviation is much less than the average, but sometimes (in a few percent of cases) also errors in the order of 100% occur. It seems plausible that the effect of the few cases, where the error due to the extrapolation from the lower order to ℓ_{max} = 16 is large, averages out and does not influence the updating process on the long run. It is also possible to improve the extrapolation to the higher orders by a more elaborate use of the covariance matrix between individual lower and higher order expansion coefficients.

First results on the hadron spectrum using the hopping expansion method in the updating with light dynamical quarks were obtained in (Langguth et al., 1984) on 8^4 lattice. N_f = 3 degenerate quark flavours were considered in two points of the (β, μ_q)-plane ($\mu_q \equiv (2K_q)^{-1}$):

point A: $\beta = 5.4$, $\mu_q = 3.0675..$ $(K_q = 0.163)$,

point B: $\beta = 5.3$, $\mu_q = 2.9762..$ $(K_q = 0.168)$. (4.22)

For comparison, an 8^4 quenched calculation was performed, too, at β = 5.7. Some planar Wilson-loop expectation values are given in Table IV.2 for these three cases. Also planar and off-axis alongated Wilson-loop expectation values were measured in order to determine the static energy E of an external SU(3)-colour charge pair by

$$aE(R) = - \lim_{T\to\infty} \frac{1}{T} \ln W(R,T). \qquad (4.23)$$

Here W(R,T) stands for a Wilson-loop with length T in the time-direction and euclidean distance R between the endpoints in fixed time-slices. On the 8^4 lattice T is, of course, restricted to $T \leq 4$ by the periodic boundary conditions and R has possible values R = 1, $\sqrt{2}$, $\sqrt{3}$, 2 $\sqrt{5}$, $\sqrt{6}$, $\sqrt{8}$, 3 and $\sqrt{10}$ (larger values of R were not considered because of statistics limitations). The obtained static energies are shown in Fig. 8a-8c. The expected screening due to the virtual quark pairs (Joos et al., 1983) cannot be seen. The static energies with dynamical quarks are, in fact, remarkably similar to the quark-anti-quark potential in Fig. 8c. Very probably, the distance between the external colour charges is not large enough ($R = \sqrt{10}$ corresponds roughly to ~ 0.5 fm, as we shall see below).

The π^-, ρ^-, p- and Δ- masses were determined on the 8^4 configurations by 32^{nd} order (for the baryons 33^{rd} order) "copied gauge field" iteration in hopping parameter. (See previous section.) The results are shown in Fig. 9a - 9c as a function of the quark mass $\mu \equiv (2K)^{-1}$ in the quark propagator. The quenched calculation (Fig. 9c) was already discussed in the previous Section (see Table III.2). In the physical points, where the quark mass in the determinant is equal to the quark mass in the propagator ($\mu_q = \mu$), the unquenched result is

Point A: $am_\pi = 0.79 \pm 0.01$ $am_\rho = 0.95 \pm 0.01$

$am_p = 1.62 \pm 0.02$ $am_\Delta = 1.74 \pm 0.02$

Point B: $am_\pi = 0.3^{+0.1}_{-0.2}$ $am_\rho = 0.62 \pm 0.05$ (4.24)

$am_p = 0.85 \pm 0.15$ $am_\Delta = 1.09 \pm 0.15 = am_p + (0.24 \pm 0.09)$

It can be seen, that point B is quite near to the critical line $\mu_{cr}(\beta)$, where the pion mass (and quark mass) vanishes (see Fig. 10). This is the reason of the deterioration of convergence for ΔS^q_{eff}, as shown by the last line of Table IV.1 . This implies an unknown systematic error in point B. Taking $(\mu_q - \mu_{cr}) \tilde{=} 0.06$ in point A and $(\mu_q - \mu_{cr}) \tilde{=} 0.01$ in point B as an estimate of quark mass in lattice units, and using Eqs. (5.15-5.17) one obtaines for the renormalization group invariant quark mass M_q and lattice spacing a:

Point A: $M_q \cong 170$ MeV $a \cong 0.87$ GeV^{-1}

Point B: $M_q \cong 30$ MeV $a \cong 0.76$ GeV^{-1} (4.25)

(These numbers are different in (Langguth et al., 1984), because there Eq. (5.16) with c = 1 was taken. The estimate for c in Eq. (5.17) is probably closer to the reality.) Assuming the validity of asymptotic scaling (with zero quark mass) in point B, the obtained value of the Λ-parameter is: $\Lambda_{lat}(N_f - 3) \cong 1.7$ MeV. This corresponds (Kawai et al., 1981; Weisz, 1981)

to $\Lambda^{mom}_{\alpha = 1} \cong 180$ MeV. Of course, the question of asymptotic scaling (or scaling in general) cannot be decided on the basis of only two points in the (β, μ_q)-plane. Some evidence for both points A and B being within the scaling region comes, however, from the fact that rotation symmetry is well satisfied for the static energies shown by Figs. 8a-8b.

A direct comparison of the results in Eq. (4.24) with the quenched masses in Table III.2 is difficult because of the different quark mass dependence and because of the shift in scale. In spite of this, one can see already in point A with quark mass $M_q \cong 170$ MeV, that the p/ρ mass ratio is decreased if one compares at similar values of $(\mu - \mu_{cr})$ in Fig. 9a and Fig. 9c. The p/ρ ratio in point B with $M_q \cong 30$ MeV is $m_p/m_\rho \cong 1{,}35$, therefore considerably lower than in the quenched case. The error is, however, still somewhat large to draw a definite conclusion.

V. Scaling with Dynamical Fermions

The detailed study of the scaling properties in pure gauge theory has shown (for SU(2) see (Gutbrod et al., 1984; Mackenzie, 1984; Patel et al., 1984); for SU(3) (Hasenfratz et al., 1984b; Bowler et al., 1985; Gupta et al., 1984), that asymptotic scaling (corresponding to the two-loop perturbative β-function) is not yet reached in the intermediate coupling range, where most of the Monte Carlo calculations of physical quantities are performed. There is, however, an important region where dimensionless ratios of physical quantities are, to a good approximation, independent from the bare coupling. In this "scaling region" physics is dictated by the continuum theory and the change of the lattice scale is given by some general non-perturbative β-function. In order to perform an optimal numerical calculation, the precise knowledge of the scaling region and of the β-function is very important.

In the case of a lattice gauge theory with dynamical fermions a similar situation has to be expected. The additional complication in this case is, however, that the scaling properties depend on two parameters: besides on the gauge coupling also on the dynamical quark mass. Before discussing the quark mass dependence of the lattice renormalization scheme, let us first briefly summarize some facts about the renormalization group equation (RGE) with quarks.

In a mass-independent renormalization scheme (for a reivew see (Peterman, 1979) the RGE for a physical quantity $P(\mu, g, m)$ depending on the renormalization point mass parameter μ, the renormalized coupling g and renormalized quark mass m, is

$$\{\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \gamma(g)\, m \frac{\partial}{\partial m}\} P = 0. \qquad (5.1)$$

Here and in what follows it will be assumed, for simplicity, that the different quark flavours are degenerate in mass. The expansion of the RG-functions is

$$\beta(g) = -\beta_0 g^3 - \beta_1 g^5 - \beta_2 g^7 - \ldots \qquad (5.2)$$

$$\gamma(g) = \gamma_0 g^2 + \gamma_1 g^4 + \ldots$$

In QCD with $SU(N_c)$ colour and N_f flavours we have

$$\beta_0 = \frac{1}{(4\pi)^2}\{\frac{11N_c}{3} - \frac{2N_f}{3}\},$$

$$\beta_1 = \frac{1}{(4\pi)^4}\{\frac{34N_c^2}{3} - N_f(\frac{13N_c}{3} - \frac{1}{N_c})\},$$

$$\gamma_0 = \frac{1}{(4\pi)^2}\,\frac{3(N_c^2-1)}{N_c}\,. \qquad (5.3)$$

These are the universal expansion coefficients. All other coefficients depend on the renormalization scheme (lattice action etc.).

The two standard solutions of the RGE (the so-called standard "renormalization group invariants") are the Λ-parameter and the renormalization group invariant (RGI) quark mass M:

$$\Lambda = \mu\,(\beta_0 g^2)^{-\beta_1/2\beta_0^2}\, e^{-\frac{1}{2\beta_0 g^2}}\, \exp\{-\int_0^g dx\,(\frac{1}{\beta(x)} + \frac{1}{\beta_0 x^3} - \frac{\beta_1}{\beta_0^2 x})\},$$

$$M = m\,(2\beta_0 g^2)^{-\gamma_0/2\beta_0} \exp\{\int_0^g dx\,(\frac{\gamma(x)}{\beta(x)} + \frac{\gamma_0}{\beta_0 x})\}\,. \qquad (5.4)$$

(The normalization of M corresponds here to (Gasser et al., 1982). Every physical quantity is a function of Λ and M only, therefore the "curves of constant physics" in the (μ, g, m)-space are parameterized by M, Λ = const. Using the freedom of finite renormalizations, it is possible to introduce new renormalized parameters by

$$g' = g\, Z_1\,(g,\lambda), \qquad \mu' \equiv \mu,$$

$$m' = m\, Z_2\,(g,\lambda); \tag{5.5}$$

where $\lambda = M/\Lambda$ is a dimensionless measure of the RGI quark mass, and the functions Z_j (j = 1, 2) are assumed to have the small-g expansion

$$Z_j(g,\lambda) = 1 + a_j(\lambda)\, g^2 + \ldots \tag{5.6}$$

Introducing the dimensionless variable $\ell = \frac{m'}{\mu}$ for the renormalized quark mass, one obtains from Eq. (5.1) the new RGE

$$\{\mu \frac{\partial}{\partial \mu} + \beta_g(g',\ell) \frac{\partial}{\partial g} + \beta_\ell(g',\ell) \frac{\partial}{\partial \ell}\}\, P = 0. \tag{5.7}$$

The RG-functions are related to the old ones by

$$\beta_g(g',\ell) = \{\beta(g) \lfloor Z_1 + g \frac{\partial Z_1}{\partial g} \rfloor\}_{g = g\,(g',\lambda)},$$

$$\beta_\ell(g',\ell) = -\ell\, \{1 + \gamma(g) - \beta(g) \frac{\partial \ln Z_2}{\partial g}\}_{g=g(g',\lambda)},$$

$$\lambda \equiv \frac{M}{\Lambda} \equiv \frac{m}{\mu}\, \phi(g) = \ell\phi(g). \tag{5.8}$$

It can also be shown, that the new and old RGI's are related by

$$\Lambda' = \Lambda \exp\{\frac{a_1(\lambda)}{\beta_o}\}, \qquad M' = M. \tag{5.9}$$

Using the lattice as a perturbative renormalization scheme, the simplest convention is to keep the mass-independent scheme corresponding to Eq. (5.1). This was, in fact, done in previous works on the RGI quark mass (Gonzalez-Arroyo et al., 1982; Hamber et al., 1983d; Golterman et al., 1984a, 1984b; Gockeler, 1984). In the nonperturbative region, however, where the numerical calculations are done, it is more convenient to define the renormalization scheme by the hadron masses. In the case of Wilson-fermions one has to find first the critical line $\mu_{cr}(\beta)$ in the (β,μ_q)-plane (with $\beta \equiv 2N_c g^{-2}$ for the gauge coupling, as usual, and $\mu_q \equiv (2K_q)^{-1}$, where K_q is the hopping parameter of the dynamical quarks). In perturbation theory this is equivalent to cancel the linear divergences in the quark self-energy. In general, $\mu_{cr}(\beta)$ is the line where the lowest 0^- mass and the (bare, renormalized and RGI) quark mass vanish. Having the value of the critical hopping parameter $K_{cr} = (2\mu_{cr})^{-1}$, one can define the bare quark mass parameter m_q (in lattice units) by

$$am_q \equiv \frac{1}{2K_q} - \frac{1}{2K_{cr}} = \mu_q - \mu_{cr}. \tag{5.10}$$

In the case of Kogut-Susskind fermions the critical (zero) quark mass is not renormalized, therefore this step is not necessary.

The next step, for any fermion formulation, is to find the lines, where the hadron mass ratios are constant. These are the "lines of constant physics" where the RGI quark mass is constant. Of course, scaling for finite lattice spacing is always somewhat broken by lattice artifacts, therefore

"constancy" means always a statement within given errors and within a given class of hadron masses. To fix the renormalization scheme completely, it is reasonable to assume that the value of hadron masses (or equivalently, of a singled out hadron mass) is constant along the "lines of constant physics". Furthermore, by using the freedom of defining the value of, say, the proton mass for quark mass values different from the physical case, one can arrange that the value of the A-parameter be independent from the RGI quark mass. (See Eq. (5.9)!) By this, the value of the lattice spacing a is fixed everywhere in the "scaling region", where the "lines of constant physics" can be defined at all. Since there seems to be no reason, why this convention should define a quark mass independent scheme, the RGE on the lattice has a quark mass dependent form corresponding to Eq. (5.7):

$$\left\{-a\,\frac{\partial}{\partial a} + \beta_g(g,\mu_q)\,\frac{\partial}{\partial g} + \beta_\mu(g,\mu_q)\,\frac{\partial}{\partial \mu_q}\right\} P = O(a). \tag{5.11}$$

Here the right hand side represents the scale-breaking lattice artifacts, which in the continuum limit $a \to 0$ tend to zero at least as fast as (some power of) the lattice spacing.

The "lines of constant physics" $\mu_q = \mu_q(g)$ are determined by the differential equation

$$\frac{d\mu_q(g)}{dg} = \frac{\beta_\mu(g,\mu_q)}{\beta_g(g,\mu_q)}\ . \tag{5.12}$$

The different values of the RGI quark mass belong to the solutions with different initial conditions. Defining the single variable β-function for a given RGI quark mass by:

$$\beta_q(g) \equiv \beta_g(g,\mu_q(g)), \tag{5.13}$$

the single variable RGE for this quark mass is

$$\left\{-a\,\frac{\partial}{\partial a} + \beta_q(g)\,\frac{\partial}{\partial g}\right\} P = O(a). \tag{5.14}$$

The quark mass dependence of the β-function $\beta_q(g)$ is assumed to appear only in the higher-order non-universal expansion coefficients β_2, γ_1, ... (see Eq. (5.2)). Hence for $g \to 0$ ($\beta \to \infty$) the quark mass dependence disappears and the renormalization scheme becomes indistinguishable from the mass independent scheme of lattice perturbation theory. This (perturbative) mass-independent regime is, however, presumably very difficult to reach by numerical hadron mass calculations.

Since the perturbative regime is presently out of range, in the intermediate coupling range some pragmatic definition of the quark mass can be very useful. A possibility (Langguth et al., 1984) is to introduce the RGI quark mass (M_q) by the lowest vector meson mass m_{1^-}:

$$m_{1^-} = 2\,M_q + E(M_q) \tag{5.15}$$

For heavy quarks (like c, b, or t) $E(M_q)$ can be taken, to a good approximation, from the Schrödinger-equation, assuming some quark-antiquark potential. For light quarks (u, d, and s) we can take, as an empirical value $E(M_q) \cong 0.75$ GeV which agrees well with the ρ - and φ - meson mass. In order to fix the lattice scale, besides Eq. (5.15), still another relation is needed. Near the critical line μ_{cr}, where the RGI quark mass is small, one can assume

$$aM_q \tilde{=} c(\beta)\ (\mu_q - \mu_{cr}) = c(\beta)\ am_q \qquad (5.16)$$

with some function $c(\beta)$. Having nothing better, one can take for $c(\beta)$ the perturbative expression (Gonzalez-Arroyo et al., 1982; Hamber et al., 1983d; Golterman et al., 1984a, 1984b; Gockeler, 1984) at some arbitrarily fixed coupling $\bar{\beta}$ $(\bar{g})$:

$$c(\bar{\beta}) \tilde{=} (2\beta_o \bar{g}^{-2})^{-\gamma_o/2\beta_o} \tilde{=} 2.5 \qquad (5.17)$$

The numerical value here corresponds to $N_f = 3$ and $\bar{\beta} \tilde{=} 5.4$ in SU(3).

An important question is, how the scaling region can look like in the available part of the (β, μ_q)-plane. For $\mu_q \to \infty$ the quark mass tends to infinity and the theory reduces to a pure gauge theory. In this case the gluonic quantities (like string tension, glueball masses, gluonic energy density in thermodynamics etc.) show approximate scaling for $\beta \gtrsim 5.7$. In the region of light quark masses the results of a recent calculation (Langguth et al., 1984) indicate, that a tentative shape of the scaling region for light dynamical quarks could look like shown in Fig. 10. Note the difference of the scaling regions of pure gluonic quantities compared to quantities containing heavy quarks explicitly (like heavy quark bound state masses, quark energy density in thermodynamics etc.).

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Table III.1

The number of points N_ℓ with given lattice distance (measured in links) in a four-dimensional hypercubic lattice.

ℓ	0	1	2	3	4	5	6	7	8
N_ℓ	1	8	32	88	192	360	608	952	1408

ℓ	9	10	11	12	13	14	15	16
N_ℓ	1992	2720	3608	4672	5928	7392	9080	11008

Table III.3

Some results of quenched hadron mass calculations with Kogut-Susskind fermions.

	β = 5.7		β = 6.0	
	(Gilchrist et al., 1984a, 1984b) $10^3 \cdot 16$	(Bowler et al., 1984b) $8^3 \cdot 16$ (copied)	(Billoire et al., 1985) $10^3 \cdot 20$	
$m_\rho a$	0.98 ± 0.06	0.88 ± 0.06	~ 0.37	
$m_p a$	1.21 ± 0.14	1.05 ± 0.30	~ 0.5	
$(m_\pi a)^2/(m_q a)$	7.6	7.0	~ 6.5	

Table III.2

The main results of some recent quenched hadron mass calculations with Wilson-fermions. K_{cr} is the critical hopping parameter where the π-mass vanishes (it is nearly equal to K_u for u- and d-quarks). The quark mass $m_q a$ is defined in Eq. (5.10).

	β = 5.7			β = 6.0		
	(Hasenfratz et al., 1984a; Kunszt et al., 1984) 16^4	(Bowler et al., 1984b) $8^3 \cdot 16$ (copied)	(Langguth et al., 1984) $8^3 \cdot 16$ (copied)	(Lipps et al., 1983) $10^3 \cdot 20$	(Billoire et al., 1985) $10^3 \cdot 20$	(Konig et al., 1984) $16^3 \cdot 28$ (blocked)
K_{cr}	0.1690 ± 0.0005	0.1695 ± 0.0007	0.1696 ± 0.0016	0.1567 ± 0.0001	~ 0.157	---
$m_\rho a$	$0.58^{+0.12}_{-0.06}$	0.53 ± 0.03	0.57 ± 0.01	0.37 ± 0.02	~ 0.35	0.37 ± 0.03
$m_p a$	1.1 ± 0.2	1.11 ± 0.10	0.97 ± 0.14	0.57 ± 0.07	~ 0.45	0.69 ± 0.04
$(m_\Delta - m_p)a$	0.13 ± 0.06	0.02 ± 0.01	0.25 ± 0.08	0.10 ± 0.10	~ 0.07	0.10 ± 0.04
$(m_\pi a)^2/(m_q a)$	~ 3.2	2.85 ± 0.15	2.5 ± 0.3	~ 2.4	~ 2.4	---

Table IV.1

The average relative magnitude (in percent) of different orders of hopping expansion in $A\ S^q_{eff}$. The numbers in the table are obtained from the ratio of the absolute value of a given order divided by the sum of the absolute values upto 16^{th} order. The first row was obtained on 6^4 lattice (Montvay, 1984) the last two rows on 8^4 lattice (Langguth et al., 1984). In both cases 10 hits per link were done in the Metropolis updating and $N_f = 3$ degenerate quark flavours were considered.

order	4	6	8	10	12	14	16
$\beta = 5.7$ $K = 0.15$	44.0	27.9	15.1	7.4	3.3	1.5	0.8
$\beta = 5.4$ A: $K = 0.163$	35.2	27.9	17.7	9.6	5.8	2.6	1.2
$\beta = 5.3$ B: $K = 0.168$	24.2	23.3	17.0	13.7	10.1	7.7	4.0

TABLE IV.2

Wilson-loop expectation values $W_{ij} = \frac{1}{3} TrC_{ij}$ in the points A and B (see Eq. (4.22) for parameters). The numbers in paranthesis are the estimated errors in last numerals. In the last line the Wilson-loop expectation values on the configurations used for the quenched calculation at $\beta = 5.7$ are given.

	W_{11}	W_{12}	W_{13}	W_{22}	W_{23}	W_{33}
A	0.5298 (9)	0.2996 (12)	0.1719 (11)	0.1099 (10)	0.0428 (8)	0.0128 (7)
B	0.5428 (10)	0.3205 (12)	0.1912 (13)	0.1295 (8)	0.0546 (9)	0.0175 (9)
$N_f = 0$, $\beta = 5.7$	0.5468 (10)	0.3218 (11)	0.1922 (11)	0.1298 (8)	0.0557 (7)	0.0186 (7)

Figure Caption

Fig. 1: The quark propagator configuration needed for flavour non-singlet mesons.

Fig. 2: The same as Fig. 1 for flavour singlet mesons.

Fig. 3: The same as Fig. 1 for baryons.

Fig. 4: Illustration of the iteration for the calculation of the hopping expansion coefficient $\langle f|M^n|i\rangle$. In every step the open points are calculated from the full ones.

Fig. 5: The quark propagator configuration for a matrix element of flavour non-singlet point-split current in Eq. (3.22).

Fig. 6: Quark propagator configurations needed for non-leptonic decay matrix elements. The points connected by a dotted line are at the same site.

Fig. 7: The dependence of the fermion part of effective action S^q_{eff} (the average subtracted: $\delta S^q_{eff} \equiv S^q_{eff} - \langle S^q_{eff}\rangle_0$) on the hopping parameter K, for 20 different gauge configurations (Joos et al., 1983). The colour group is SU(2) at $\beta = 2.3$ and $N_f = 1$ flavour is taken.

Fig. 8a: The static energy of an external quark-antiquark pair as a function of the lattice distance for $\beta = 5.4$, $K_q = 0.163$ with $N_f = 3$ degenerate flavours (point A in (Langguth et al., 1984).

Fig. 8b: The same as Fig. 8a for $\beta = 5.3$, $K_q = 0.168$ (point B in (Langguth et al., 1984).

Fig. 8c: The same as Fig. 8a in the pure gluon theory at $\beta = 5.7$. The gauge configurations are those in the quenched calculation of (Langguth et al., 1984).

Fig. 9a: Hadron masses as a function of quark mass parameter in the quark propagator $\mu = (2K)^{-1}$ for $\beta = 5.4$, $K_q = 0.163$ with $N_f = 3$ degenerate flavours (point A in (Langguth et al., 1984).

Fig. 9b: The same as Fig. 9a for $\beta = 5.3$, $K_q = 0.168$ (point B in (Langguth et al., 1984).

Fig. 9c: Hadron masses as a function of quark mass parameter $\mu = (2K)^{-1}$ in the quenched approximation for $\beta = 5.7$ (Langguth et al., 1984).

Fig. 10: Tentative shape of the scaling region in the (β,μ_q)-plane with $N_f = 3$ degenerate Wilson-flavours. Scaling could be valid for purely gluonic quantities to the right of the line (SG), for all quantities to the right of the line (SQ). The critical line with zero quark mass is $\mu_{cr}(\beta)$, its perturbative 1-loop approximation is the dashed line. The cruves μ_q belong to constant RGI quark masses ("constant physics"). The position of the two points A, B measured in (Langguth et al., 1984) is also shown.

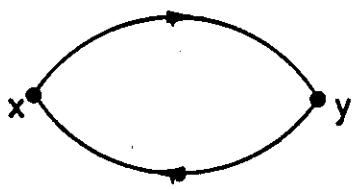

Fig. 1

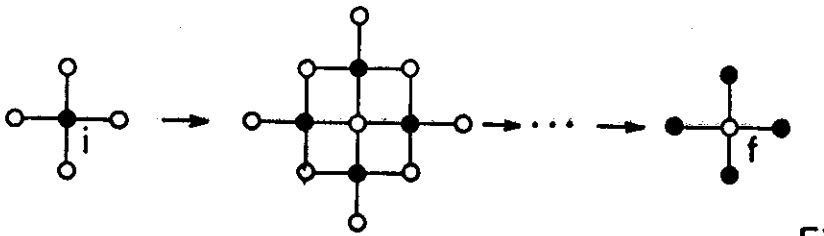

Fig. 4

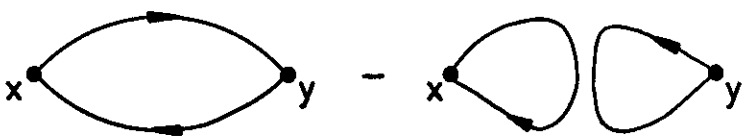

Fig. 2

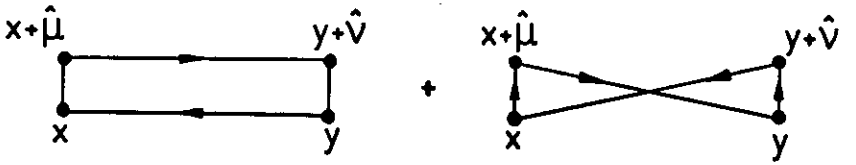

Fig. 5

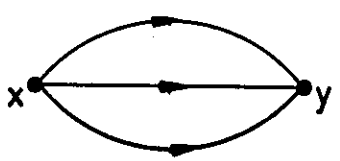

Fig. 3

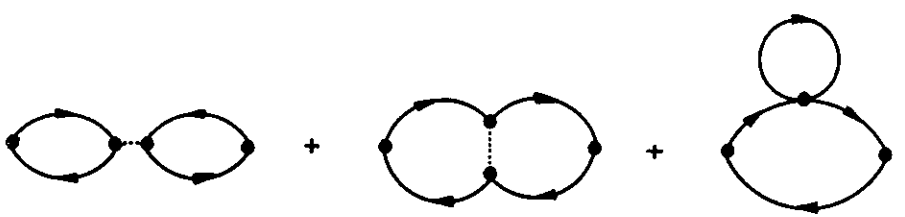

Fig. 6

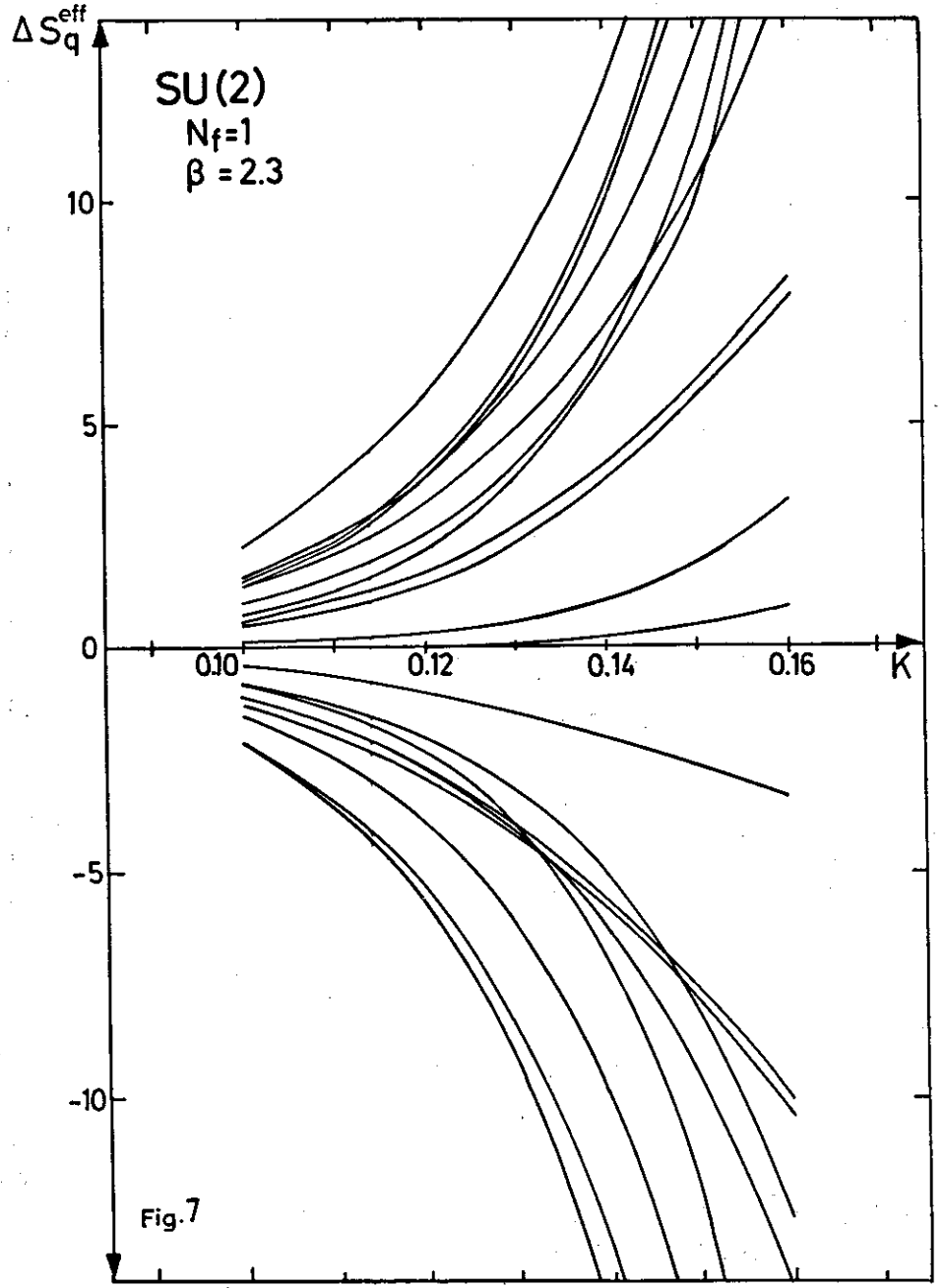
ΔS_q^{eff}
SU(2)
N_f=1
β = 2.3
10
5
0
-5
-10
0.10
0.12
0.14
0.16
K

Fig.7

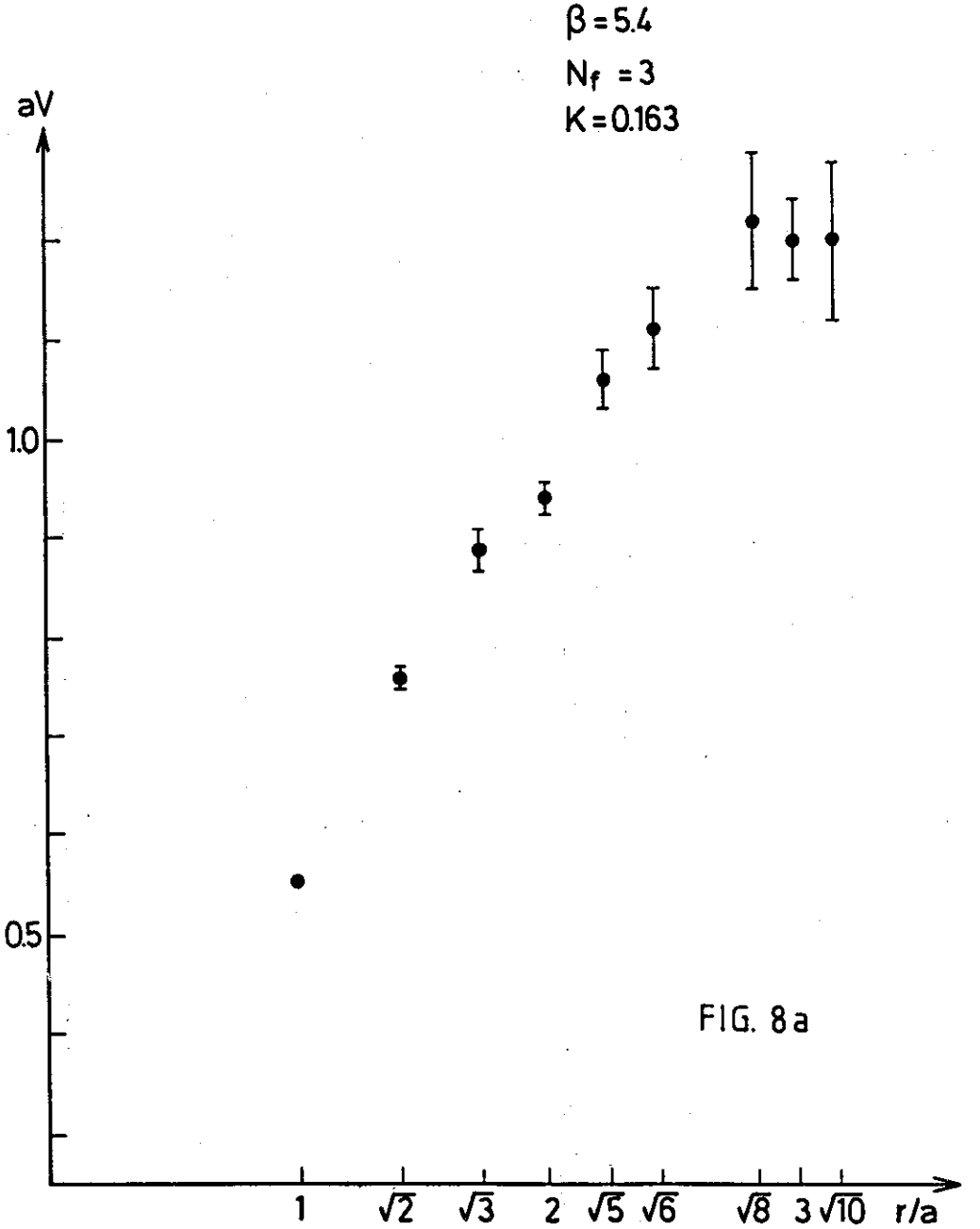
β = 5.4
N_f = 3
K = 0.163
aV
1.0
0.5
1
√2
√3
2
√5
√6
√8
3√10
r/a

FIG. 8a

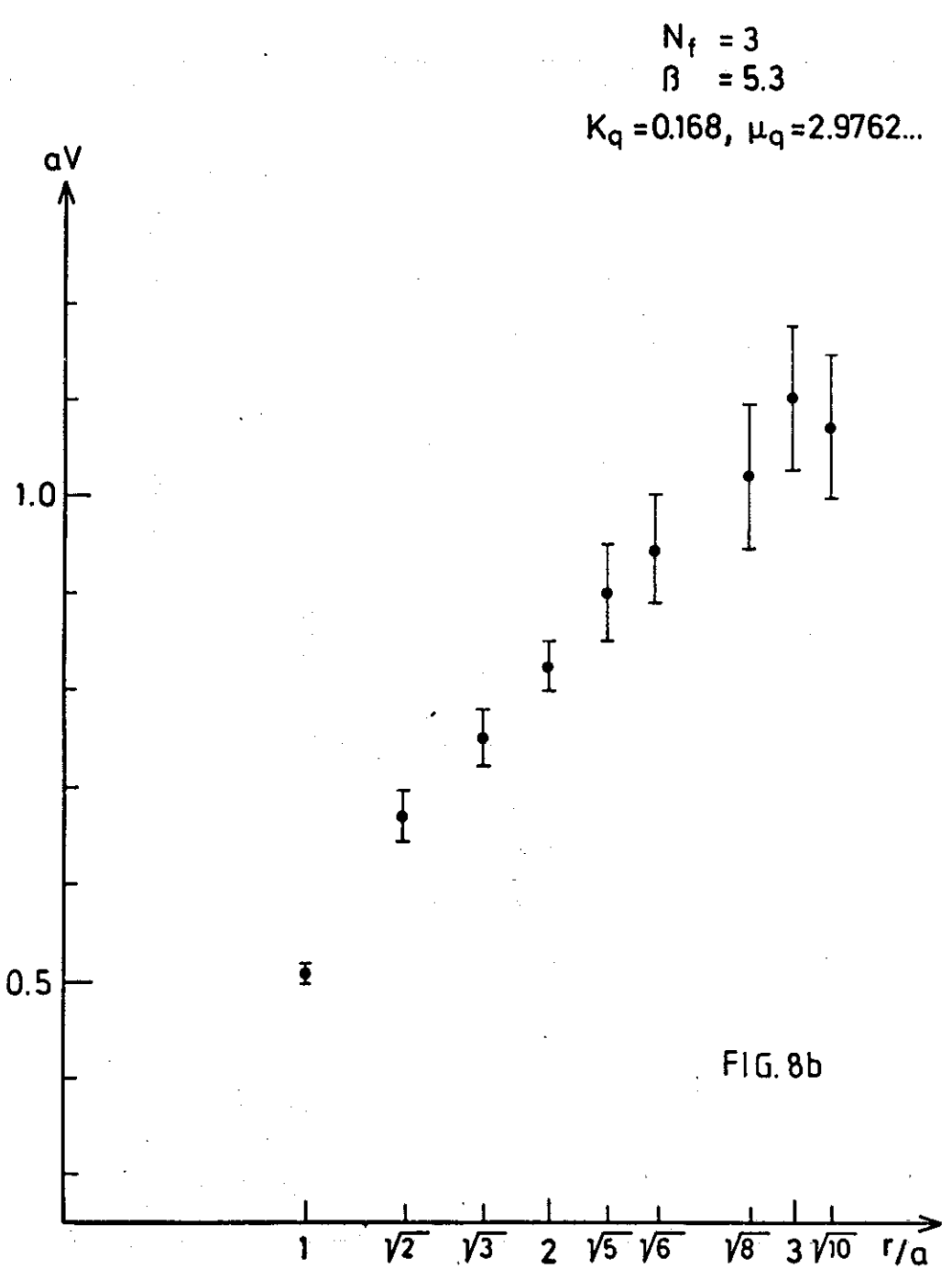
$N_f = 3$
$\beta = 5.3$
$K_q = 0.168$, $\mu_q = 2.9762...$
aV
1.0
0.5
1
$\sqrt{2}$
$\sqrt{3}$
2
$\sqrt{5}$
$\sqrt{6}$
$\sqrt{8}$
3 $\sqrt{10}$
r/a

FIG. 8b

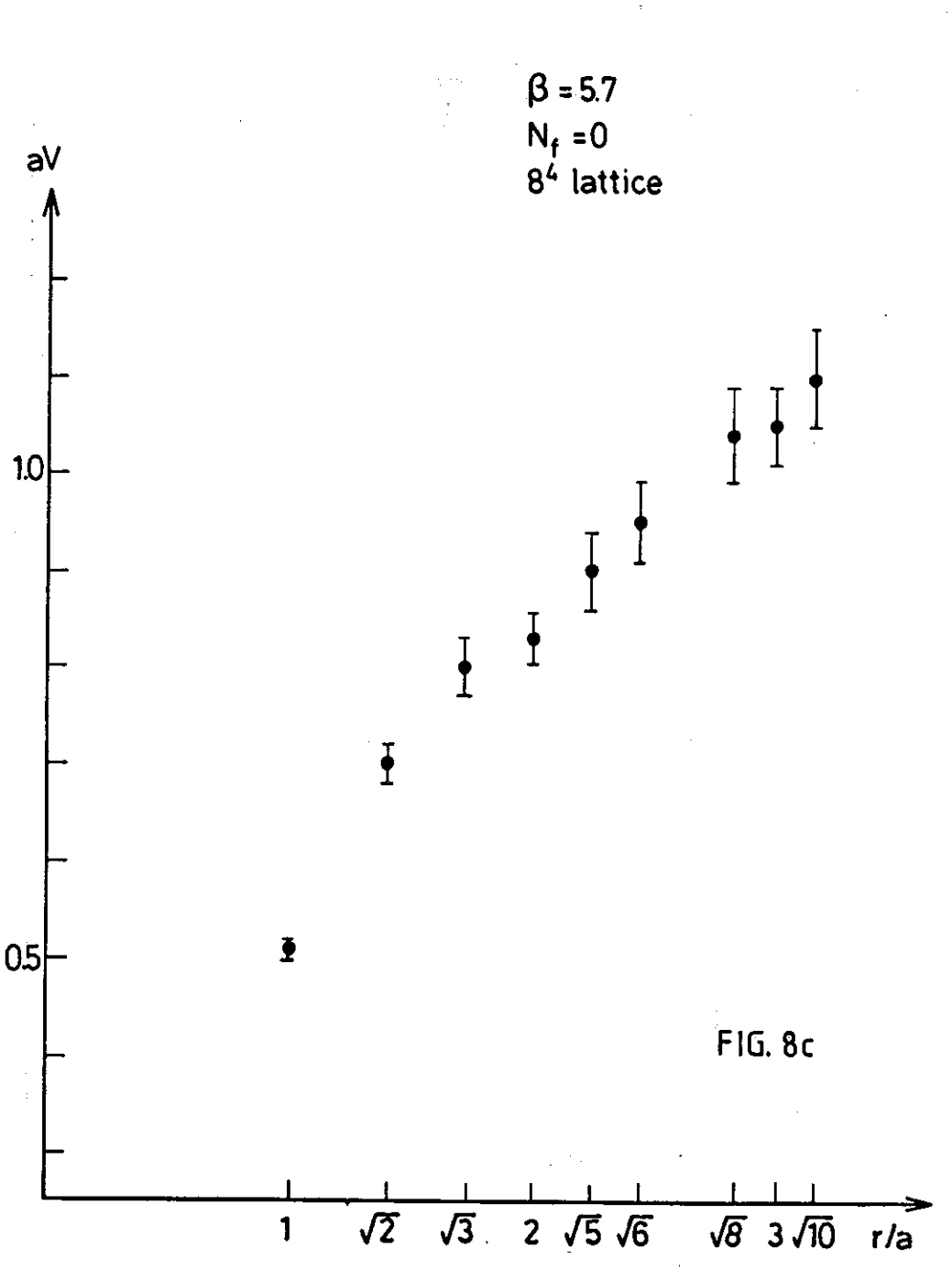
$\beta = 5.7$
$N_f = 0$
8^4 lattice
aV
1.0
0.5
1
$\sqrt{2}$
$\sqrt{3}$
2
$\sqrt{5}$
$\sqrt{6}$
$\sqrt{8}$
3 $\sqrt{10}$
r/a

FIG. 8c

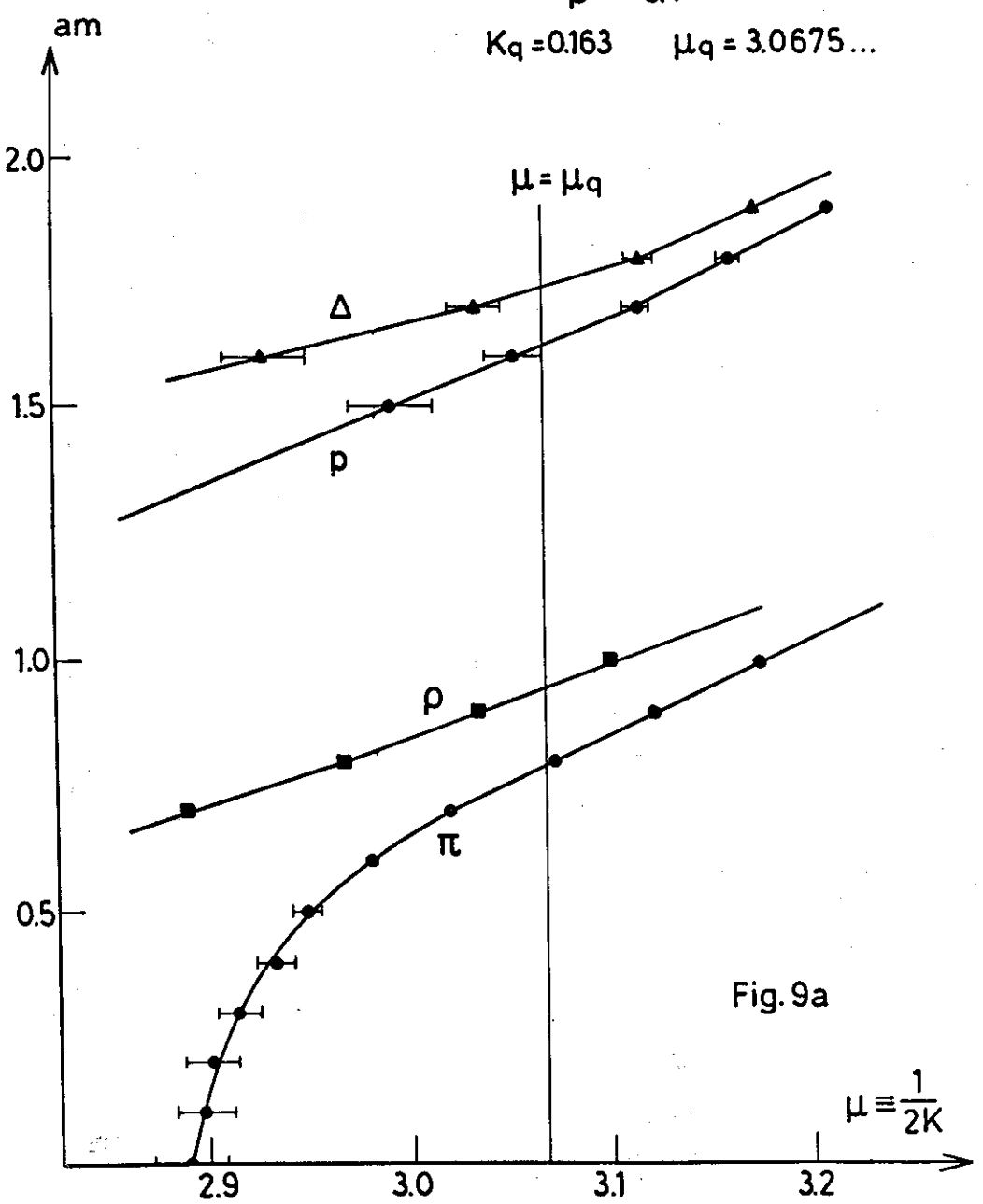
$N_f = 3$
$\beta = 5.4$
$K_q = 0.163$ $\mu_q = 3.0675...$
am
2.0
1.5
1.0
0.5
$\mu = \mu_q$
Δ
p
ρ
π
2.9
3.0
3.1
3.2
$\mu \equiv \frac{1}{2K}$

Fig. 9a

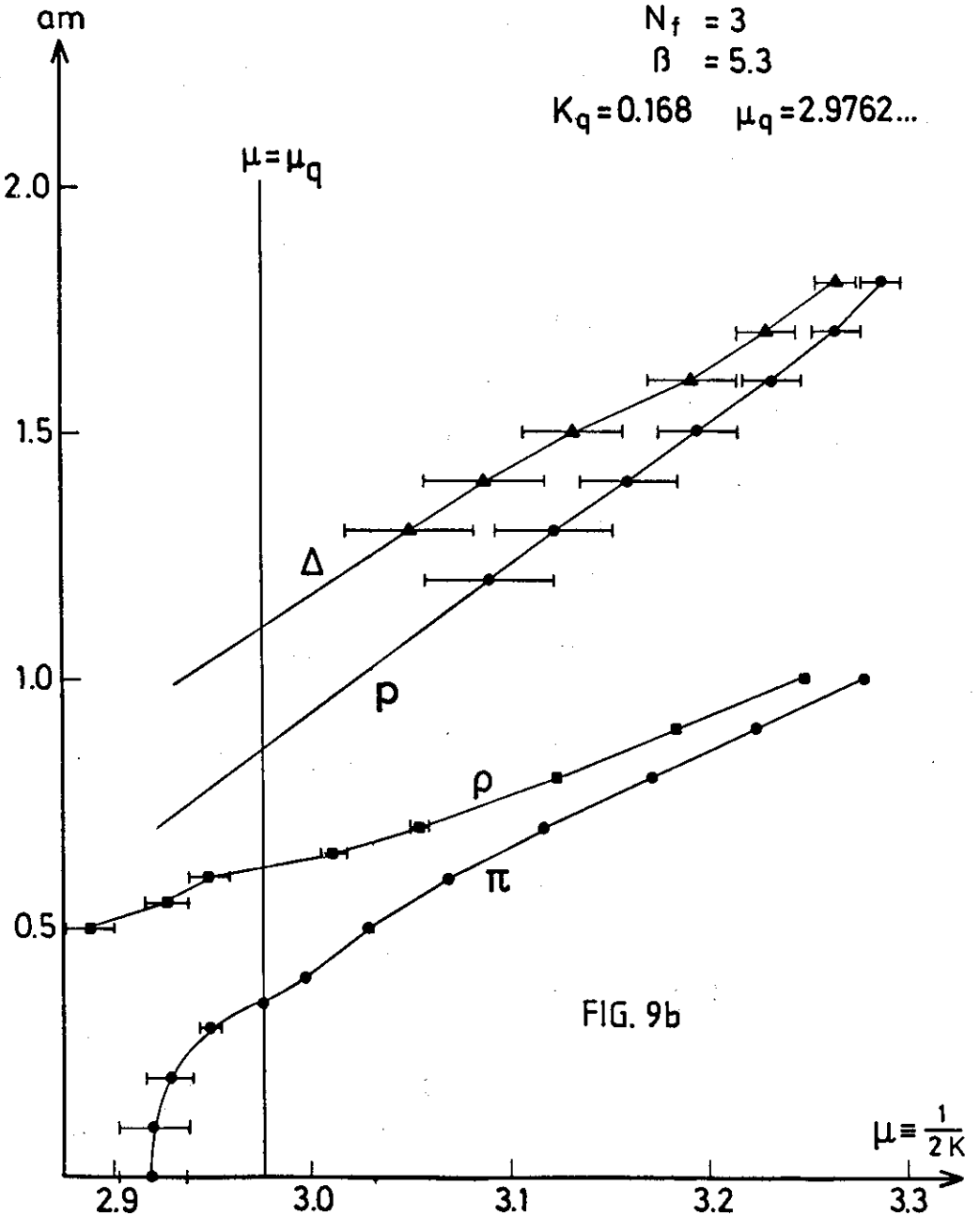
$N_f = 3$
$\beta = 5.3$
$K_q = 0.168$ $\mu_q = 2.9762...$
am
2.0
1.5
1.0
0.5
$\mu = \mu_q$
Δ
p
ρ
π
2.9
3.0
3.1
3.2
3.3
$\mu \equiv \frac{1}{2K}$

FIG. 9b

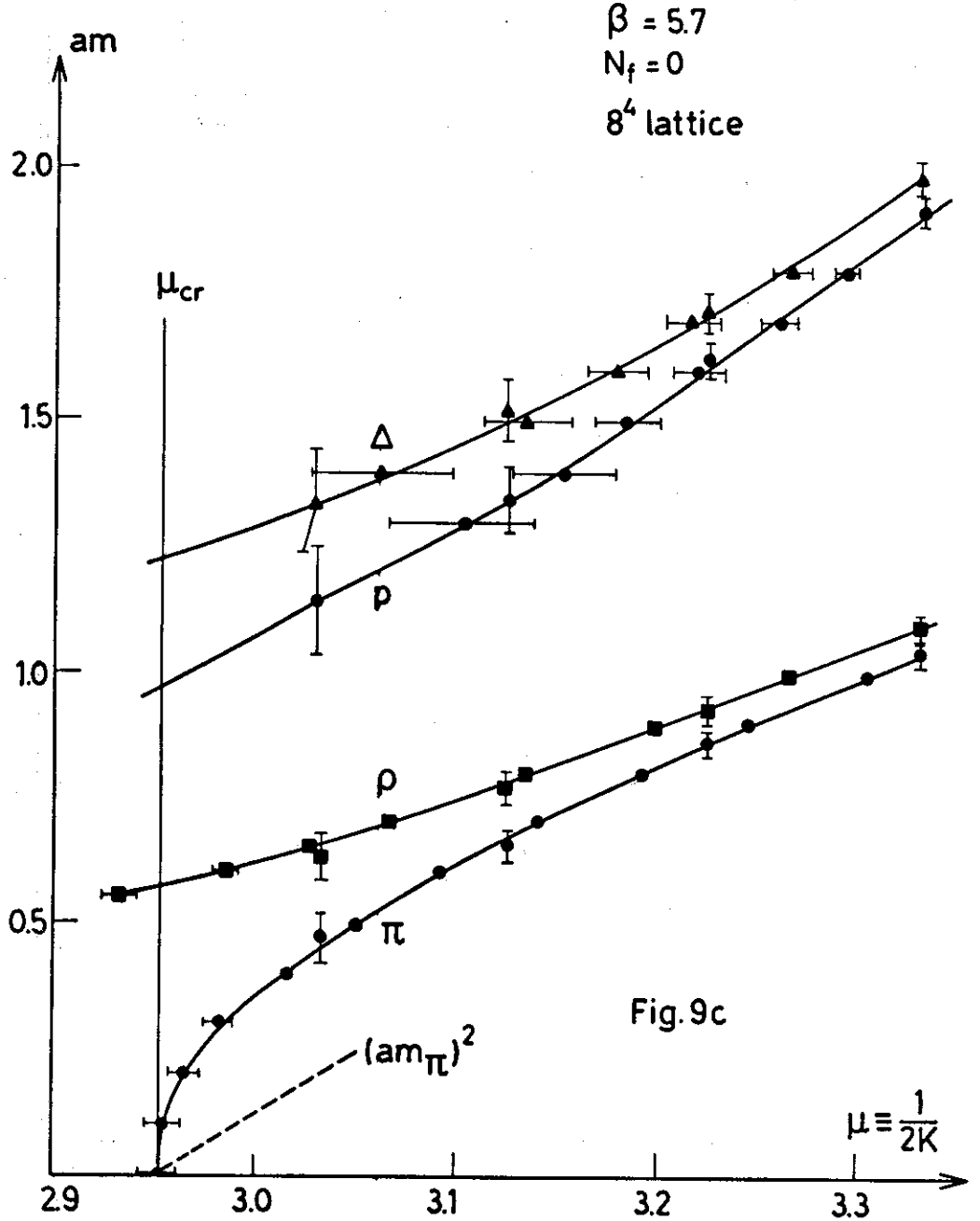
β = 5.7
$N_f = 0$
8^4 lattice
am
2.0
1.5
1.0
0.5
μ_{cr}
Δ
p
ρ
π
$(am_\pi)^2$
$\mu \equiv \frac{1}{2K}$
2.9
3.0
3.1
3.2
3.3

Fig. 9c

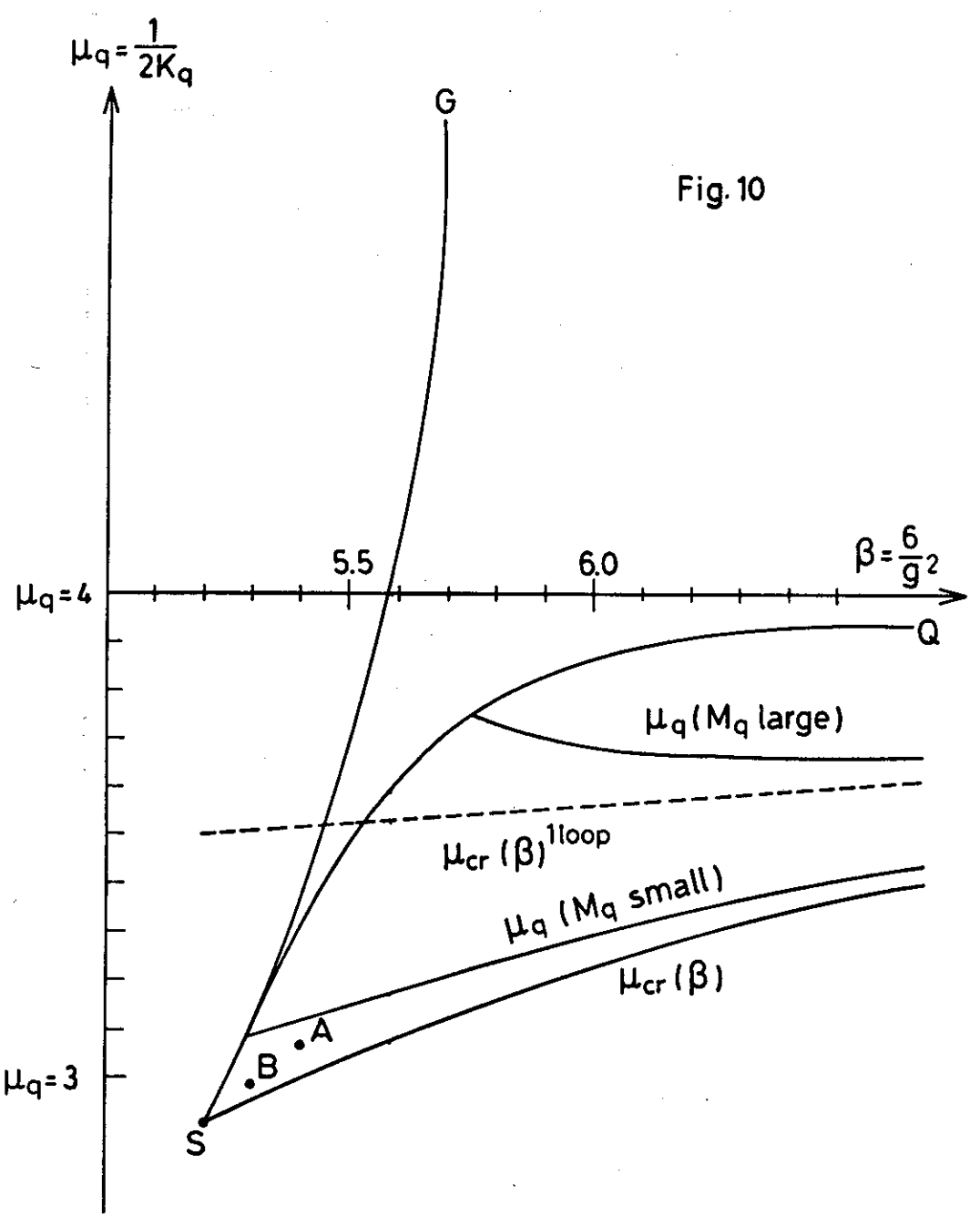
$\mu_q = \frac{1}{2K_q}$
G
5.5
6.0
$\beta = \frac{6}{g^2}$
$\mu_q = 4$
Q
μ_q (M_q large)
$\mu_{cr}(\beta)^{1loop}$
μ_q (M_q small)
$\mu_{cr}(\beta)$
A
B
$\mu_q = 3$
S

Fig. 10