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2- AND 3-COCHAINS IN 4-DIMENSIONAL SU(2) GAUGE THEORY

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Abstract: Explicit formulae are derived for the 2- and 3-cochains $\Omega^{(2)}_{\mu\nu\rho}(i,j,k)$ and $\Omega^{(3)}_{\mu\nu\rho\sigma}(i,j,k,l)$ in SU(2) gauge theory in 4 dimensions. It turns out that $\Omega^{(3)}_{\mu\nu\rho\sigma}(i,j,k,l)$ is given by the volume of a spherical tetrahedron spanned by the gauge transformations relating the gauges i, j, k, l.

I. Introduction

The investigation of the topological structure of SU(N) gauge theories is of great interest[1]. In particular, we hope that it will help understanding QCD[2].

A signal for topologically non-trivial properties of the gauge fields is a non-vanishing topological charge, which is the 4-dimensional integral of the Chern-Pontryagin density

$$\mathcal{P} = -\frac{1}{32\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\,[F^i_{\mu\nu}F^i_{\rho\sigma}]\,, \tag{1}$$

where

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + [A^i_\mu, A^i_\nu] \tag{2}$$

and the index i specifies a particular gauge. In this paper we shall restrict ourselves to SU(2) gauge fields.

The Chern-Pontryagin density can be written as a total divergence,

$$\mathcal{P} = \partial_\mu\,\Omega^{(0)}_\mu(i)\,, \tag{3}$$

with

$$\Omega^{(0)}_\mu(i) = -\frac{1}{8\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\,[A^i_\nu(\partial_\rho A^i_\sigma + \tfrac{2}{3}A^i_\rho A^i_\sigma)] \tag{4}$$

being the Chern-Simons density or 0-cochain. The latter is gauge variant. What interests us naturally is its gauge variation which is given by the coboundary operation

$$\begin{aligned}\Delta\,\Omega^{(0)}_\mu(i,j) &= \Omega^{(0)}_\mu(i) - \Omega^{(0)}_\mu(j) \\ &= -\frac{1}{24\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\,[v_{ij}^{-1}\partial_\nu v_{ij}\,v_{ij}^{-1}\partial_\rho v_{ij}\,v_{ij}^{-1}\partial_\sigma v_{ij}] \\ &\quad -\frac{1}{8\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\,\partial_\nu\,\mathrm{Tr}\,[\partial_\rho v_{ij}\,v_{ij}^{-1}A^i_\sigma]\,,\end{aligned} \tag{5}$$

where v_{ij} relates the gauges i and j,

$$A_\mu^j = v_{ij}^{-1}(A_\mu^i + \partial_\mu)v_{ij} . \tag{6}$$

$\Delta\Omega_\mu^{(0)}(i,j)$ can again be written as a divergence[3],

$$\Delta\Omega_\mu^{(0)}(i,j) = \partial_\nu \Omega_{\mu\nu}^{(1)}(i,j) , \tag{7}$$

where $\Omega_{\mu\nu}^{(1)}(i,j)$ is the 1-cochain given by

$$\begin{aligned} \Omega_{\mu\nu}^{(1)}(i,j) = &-\frac{1}{8\pi^2}(\alpha - \sin\alpha\cos\alpha)\,\epsilon_{\mu\nu\rho\sigma}\,\vec{e}_\alpha\cdot(\partial_\rho\vec{e}_\alpha \times \partial_\sigma\vec{e}_\alpha) \\ &-\frac{1}{8\pi^2}\,Tr\,[\partial_\rho v_{ij}\, v_{ij}^{-1} A_\sigma^i] \end{aligned} \tag{8}$$

and

$$v_{ij} = \exp(i\vec{\alpha}\cdot\vec{\tau}) = \cos\alpha + i\sin\alpha\; \vec{e}_\alpha\cdot\vec{\tau} . \tag{9}$$

The expression for the 1-cochain has been extended to any semi-simple and compact Lie group in ref. 4.

II. 2- and 3-cochains

It is known that the descent (from the 0- to the 1-cochain, cf. Fig. 1a-b) continues. In this work we shall derive explicit formulae for the 2- and 3-cochains[5].

The gauge variation of $\Omega_{\mu\nu}^{(1)}(i,j)$ is given by the coboundary operation

$$\begin{aligned} \Delta\Omega_{\mu\nu}^{(1)}(i,j,k) &= \Omega_{\mu\nu}^{(1)}(i,j) - \Omega_{\mu\nu}^{(1)}(i,k) + \Omega_{\mu\nu}^{(1)}(j,k) \\ &= -\frac{1}{8\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\,[\,(\alpha - \sin\alpha\cos\alpha)\,\vec{e}_\alpha\cdot(\partial_\rho\vec{e}_\alpha\times\partial_\sigma\vec{e}_\alpha) \\ &\quad + (\beta - \sin\beta\cos\beta)\,\vec{e}_\beta\cdot(\partial_\rho\vec{e}_\beta\times\partial_\sigma\vec{e}_\beta) \\ &\quad - (\gamma - \sin\gamma\cos\gamma)\,\vec{e}_\gamma\cdot(\partial_\rho\vec{e}_\gamma\times\partial_\sigma\vec{e}_\gamma)\,] \end{aligned} \tag{10}$$

$$\begin{aligned} &-\frac{1}{4\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\,[\,(\partial_\rho\alpha\,\vec{e}_\alpha + \sin\alpha\cos\alpha\,\partial_\rho\vec{e}_\alpha + \sin^2\alpha\,\vec{e}_\alpha\times\partial_\rho\vec{e}_\alpha) \\ &\quad\cdot(\partial_\sigma\beta\,\vec{e}_\beta + \sin\beta\cos\beta\,\partial_\sigma\vec{e}_\beta - \sin^2\beta\,\vec{e}_\beta\times\partial_\sigma\vec{e}_\beta)\,] . \end{aligned}$$

In deriving (10) we have made use of the cocycle condition

$$v_{ij}\,v_{jk} = v_{ik} \tag{11}$$

and written

$$v_{ij} = \exp(i\vec{\alpha}\cdot\vec{\tau}) , \quad v_{jk} = \exp(i\vec{\beta}\cdot\vec{\tau}) , \quad v_{ik} = \exp(i\vec{\gamma}\cdot\vec{\tau}) . \tag{12}$$

The cocycle condition (11) defines a spherical triangle by

$$\begin{aligned} \cos\gamma &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\;\vec{e}_\alpha\cdot\vec{e}_\beta , \\ \sin\gamma\,\vec{e}_\gamma &= \sin\alpha\cos\beta\,\vec{e}_\alpha + \cos\alpha\sin\beta\,\vec{e}_\beta - \sin\alpha\sin\beta\;\vec{e}_\alpha\times\vec{e}_\beta \end{aligned} \tag{13}$$

as indicated in Fig. 1c. $\Delta\Omega_{\mu\nu}^{(1)}(i,j,k)$ is again a total divergence,

$$\Delta\Omega_{\mu\nu}^{(1)}(i,j,k) = \partial_\rho\,\Omega_{\mu\nu\rho}^{(2)}(i,j,k) , \tag{14}$$

where $\Omega_{\mu\nu\rho}^{(2)}(i,j,k)$ is the 2-cochain.

We find the expression

$$\begin{aligned} \Omega_{\mu\nu\rho}^{(2)}(i,j,k) = &-\frac{1}{8\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\,(1 + 2\cos\alpha\cos\beta\cos\gamma - \cos^2\alpha - \cos^2\beta - \cos^2\gamma)^{-1} \\ &\cdot\{(\vec{\alpha}+\vec{\beta}-\vec{\gamma})\cdot(\sin\alpha\,\vec{e}_\alpha)[\partial_\sigma(\sin\beta\,\vec{e}_\beta)\cdot(\sin\gamma\,\vec{e}_\gamma) - \sin\beta\,\vec{e}_\beta\cdot\partial_\sigma(\sin\gamma\,\vec{e}_\gamma)] \\ &+ (\vec{\alpha}+\vec{\beta}-\vec{\gamma})\cdot(\sin\beta\,\vec{e}_\beta)[\partial_\sigma(\sin\gamma\,\vec{e}_\gamma)\cdot(\sin\alpha\,\vec{e}_\alpha) - \sin\gamma\,\vec{e}_\gamma\cdot\partial_\sigma(\sin\alpha\,\vec{e}_\alpha)] \\ &+ (\vec{\alpha}+\vec{\beta}-\vec{\gamma})\cdot(\sin\gamma\,\vec{e}_\gamma)[\partial_\sigma(\sin\alpha\,\vec{e}_\alpha)\cdot(\sin\beta\,\vec{e}_\beta) - \sin\alpha\,\vec{e}_\alpha\cdot\partial_\sigma(\sin\beta\,\vec{e}_\beta)]\} . \end{aligned} \tag{15}$$

The derivation of (15) is quite tedious and relegated to the Appendix. It can be shown that for infinitesimal gauge transformations (15) reduces to the form given in ref. 1.

The gauge variation of $\Omega^{(2)}_{\mu\nu\rho}(i,j,k)$ combines 4 spherical triangles to form a spherical tetrahedron as indicated in Fig. 1d. I. e.

$$\begin{aligned}\Delta\Omega^{(2)}_{\mu\nu\rho}(i,j,k,l) &= \Omega^{(2)}_{\mu\nu\rho}(i,j,k) - \Omega^{(2)}_{\mu\nu\rho}(i,j,l) \\ &\quad + \Omega^{(2)}_{\mu\nu\rho}(i,k,l) - \Omega^{(2)}_{\mu\nu\rho}(j,k,l).\end{aligned} \tag{16}$$

We show in the Appendix that (16) can be written in the form

$$\Delta\Omega^{(2)}_{\mu\nu\rho}(i,j,k,l) = \frac{1}{4\pi^2}\epsilon_{\mu\nu\rho\sigma}\left(\alpha\partial_\sigma A + \beta\partial_\sigma B + \gamma\partial_\sigma\Gamma + \delta\partial_\sigma\Delta + \epsilon\partial_\sigma E + \zeta\partial_\sigma Z\right) \tag{17}$$

where A, B, Γ, Δ, E, Z are the angles between two spherical triangles intersecting along the hinges $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ (for the explicit expressions see the Appendix).

We recognize that the term in brackets on the r.h.s. of equ. (17) is Schläfli's differential form[6] for the volume V(i,j,k,l) of the spherical tetrahedron of Fig. 1d, i. e.

$$\frac{1}{2}\left(\alpha\partial_\sigma A + \beta\partial_\sigma B + \gamma\partial_\sigma\Gamma + \delta\partial_\sigma\Delta + \epsilon\partial_\sigma E + \zeta\partial_\sigma Z\right) = \partial_\sigma V(i,j,k,l). \tag{18}$$

This allows us to give an explicit expression for the 3-cochain $\Omega^{(3)}_{\mu\nu\rho\sigma}(i,j,k,l)$ defined by

$$\Delta\Omega^{(2)}_{\mu\nu\rho}(i,j,k,l) = \partial_\sigma\Omega^{(3)}_{\mu\nu\rho\sigma}(i,j,k,l). \tag{19}$$

That is

$$\Omega^{(3)}_{\mu\nu\rho\sigma}(i,j,k,l) = \frac{1}{2\pi^2}\epsilon_{\mu\nu\rho\sigma}V(i,j,k,l). \tag{20}$$

The volume V(i,j,k,l) can be constructed explicitly from the angles A, B, Γ, Δ, E, Z following ref. 7.

The gauge variation of $\Omega^{(3)}_{\mu\nu\rho\sigma}(i,j,k,l)$ combines 5 spherical tetrahedra (see Fig. 1e),

$$\begin{aligned}\Delta\Omega^{(3)}_{\mu\nu\rho\sigma}(i,j,k,l,m) &= \Omega^{(3)}_{\mu\nu\rho\sigma}(i,j,k,l) - \Omega^{(3)}_{\mu\nu\rho\sigma}(i,j,k,m) \\ &\quad + \Omega^{(3)}_{\mu\nu\rho\sigma}(i,j,l,m) - \Omega^{(3)}_{\mu\nu\rho\sigma}(i,k,l,m) + \Omega^{(3)}_{\mu\nu\rho\sigma}(j,k,l,m) \\ &= \frac{1}{2\pi^2}\epsilon_{\mu\nu\rho\sigma}\left[V(i,j,k,l) - V(i,j,k,m)\right. \\ &\quad \left.+ V(i,j,l,m) - V(i,k,l,m) + V(j,k,l,m)\right],\end{aligned} \tag{21}$$

which wind around S^3, the group space of SU(2). The volume of S^3 is $2\pi^2$, so that we can write

$$\Delta\Omega^{(3)}_{\mu\nu\rho\sigma}(i,j,k,l,m) = \epsilon_{\mu\nu\rho\sigma}\, n, \tag{22}$$

where

$$n \in \mathbb{Z}. \tag{23}$$

The latter is a consequence of the fact that the 5 spherical tetrahedra together are compact and so cover S^3.

III. Discussion

The result, that the 3-cochain is given by the volume of the spherical tetrahedron V(i,j,k,l), is not really surprising. E. g. in 2-dimensional U(1) gauge theory the corresponding 1-cochain is a segment of S^1.

As will be discussed in a subsequent paper[8], equ. (22) allows us to derive a local, fully algebraic expression for the topological charge in SU(2) and SU(3) gauge theory.

Appendix

We shall first derive equ. (15). Noticing that $\vec{\gamma}$ in equ. (10) can be expressed in terms of $\vec{\alpha}, \vec{\beta}$ by using the cocycle condition (13), the most general ansatz for the tensor structure of $\Omega^{(2)}_{\mu\nu\varrho}(i,j,k)$ is

$$\Omega^{(2)}_{\mu\nu\varrho}(i,j,k) = -\frac{1}{8\pi^2}\,\epsilon_{\mu\nu\varrho\sigma}\,[\,f_1\,\partial_\sigma\alpha + f_2\,\partial_\sigma\beta + f_3\,(\partial_\sigma\vec{e}_\alpha\cdot\vec{e}_\beta)$$
$$+ f_4\,(\vec{e}_\alpha\cdot\partial_\sigma\vec{e}_\beta) + f_5\,\partial_\sigma\vec{e}_\alpha\cdot(\vec{e}_\alpha\times\vec{e}_\beta) + f_6\,\partial_\sigma\vec{e}_\beta\cdot(\vec{e}_\alpha\times\vec{e}_\beta)\,] \tag{A.1}$$

with

$$f_i \equiv f_i(\alpha,\beta,\vec{e}_\alpha\cdot\vec{e}_\beta)\,. \tag{A.2}$$

Equation (14) is then equivalent to the following set of coupled partial differential equations:

$$\frac{\partial f_2}{\partial\alpha} - \frac{\partial f_1}{\partial\beta} = 2\vec{e}_\alpha\vec{e}_\beta - 2\sin\alpha\sin\beta\,\frac{\gamma-\sin\gamma\cos\gamma}{\sin^3\gamma}\,[1-(\vec{e}_\alpha\vec{e}_\beta)^2], \tag{A.3}$$

$$\frac{\partial f_4}{\partial\alpha} - \frac{\partial f_1}{\partial(\vec{e}_\alpha\vec{e}_\beta)} = 2\sin\beta\cos\beta\,\frac{1-\gamma\cot\gamma}{\sin^2\gamma} + 2\cos\alpha\sin\beta\,\frac{\gamma-\sin\gamma\cos\gamma}{\sin^3\gamma},$$

$$\frac{\partial f_3}{\partial\beta} - \frac{\partial f_2}{\partial(\vec{e}_\alpha\vec{e}_\beta)} = -2\sin\alpha\cos\alpha\,\frac{1-\gamma\cot\gamma}{\sin^2\gamma} - 2\sin\alpha\cos\beta\,\frac{\gamma-\sin\gamma\cos\gamma}{\sin^3\gamma},$$

$$\frac{\partial f_3}{\partial\alpha} - \frac{\partial f_1}{\partial(\vec{e}_\alpha\vec{e}_\beta)} = 0\,, \quad \frac{\partial f_4}{\partial\beta} - \frac{\partial f_2}{\partial(\vec{e}_\alpha\vec{e}_\beta)} = 0\,,$$

$$\frac{\partial(f_4-f_3)}{\partial(\vec{e}_\alpha\vec{e}_\beta)} = 2\sin^2\alpha\sin^2\beta\,\frac{1-\gamma\cot\gamma}{\sin^2\gamma}\,, \quad f_4-f_3 = 2\sin\alpha\sin\beta\,\frac{\gamma}{\sin\gamma}\,,$$

$$\frac{\partial f_5}{\partial\alpha} = 2\sin\alpha\sin\beta\,\frac{\gamma-\sin\gamma\cos\gamma}{\sin^3\gamma}\,, \quad \frac{\partial f_5}{\partial\beta} = 2\sin^2\alpha\,\frac{1-\gamma\cot\gamma}{\sin^2\gamma}\,,$$

$$\frac{\partial f_6}{\partial\alpha} = -2\sin^2\beta\,\frac{1-\gamma\cot\gamma}{\sin^2\gamma}\,, \quad \frac{\partial f_6}{\partial\beta} = -2\sin\alpha\sin\beta\,\frac{\gamma-\sin\gamma\cos\gamma}{\sin^3\gamma}\,,$$

$$f_5 + (\vec{e}_\alpha\vec{e}_\beta)\,\frac{\partial f_5}{\partial(\vec{e}_\alpha\vec{e}_\beta)} + \frac{\partial f_6}{\partial(\vec{e}_\alpha\vec{e}_\beta)} = 2\sin^2\alpha\sin\beta\cos\beta\,\frac{1-\gamma\cot\gamma}{\sin^2\gamma}\,,$$

$$f_6 + (\vec{e}_\alpha\vec{e}_\beta)\,\frac{\partial f_6}{\partial(\vec{e}_\alpha\vec{e}_\beta)} + \frac{\partial f_5}{\partial(\vec{e}_\alpha\vec{e}_\beta)} = -2\sin\alpha\cos\alpha\sin^2\beta\,\frac{1-\gamma\cot\gamma}{\sin^2\gamma}\,,$$

$$[1-(\vec{e}_\alpha\vec{e}_\beta)^2]\,\frac{\partial f_5}{\partial(\vec{e}_\alpha\vec{e}_\beta)} - 2(\vec{e}_\alpha\vec{e}_\beta)\,f_5 = 2\alpha - 2\sin\alpha\cos\alpha\,\frac{1-\gamma\cot\gamma}{\sin^2\gamma} - 2\sin\alpha\cos\beta\,\frac{\gamma-\sin\gamma\cos\gamma}{\sin^3\gamma},$$

$$[1-(\vec{e}_\alpha\vec{e}_\beta)^2]\,\frac{\partial f_6}{\partial(\vec{e}_\alpha\vec{e}_\beta)} - 2(\vec{e}_\alpha\vec{e}_\beta)\,f_6 = -2\beta + 2\sin\beta\cos\beta\,\frac{1-\gamma\cot\gamma}{\sin^2\gamma} + 2\cos\alpha\sin\beta\,\frac{\gamma-\sin\gamma\cos\gamma}{\sin^3\gamma},$$

which can be solved giving

$$\begin{aligned}
f_1 &= (\vec{\alpha}-\vec{\gamma})\cdot\vec{e}_\alpha\,,\\
f_2 &= -(\vec{\beta}-\vec{\gamma})\cdot\vec{e}_\beta\,,\\
f_3 &= -f_4 = -\gamma\,\frac{\sin\alpha\sin\beta}{\sin\gamma}\,,\\
f_5 &= 2\,\frac{(\vec{\alpha}+\vec{\beta}-\vec{\gamma})\cdot\vec{e}_\beta}{1-(\vec{e}_\alpha\vec{e}_\beta)^2}\,,\\
f_6 &= -2\,\frac{(\vec{\alpha}+\vec{\beta}-\vec{\gamma})\cdot\vec{e}_\alpha}{1-(\vec{e}_\alpha\vec{e}_\beta)^2}\,.
\end{aligned} \tag{A.4}$$

Inserting (A.4) into (A.1) gives after a straightforward calculation equ. (15).

We shall prove now equ. (17). The angle A is given by (cf. Fig. 1d)

$$\tan A = -\,\frac{\vec{e}_\alpha\cdot(\vec{e}_\beta\times\vec{e}_\epsilon)}{(\vec{e}_\alpha\times\vec{e}_\beta)(\vec{e}_\alpha\times\vec{e}_\epsilon)}\,. \tag{A.5}$$

The other angles B, Γ, ... follow from (A.5) by permutation. From (A.5) we derive

$$\partial_\sigma A = [1-(\vec{e}_\alpha\vec{e}_\beta)^2]^{-1}[\,\vec{e}_\alpha\vec{e}_\beta\;\vec{e}_\beta\cdot(\vec{e}_\alpha\times\partial_\sigma\vec{e}_\alpha) + \vec{e}_\alpha\cdot(\vec{e}_\beta\times\partial_\sigma\vec{e}_\beta)\,]$$
$$-\,[1-(\vec{e}_\alpha\vec{e}_\epsilon)^2]^{-1}[\,\vec{e}_\alpha\vec{e}_\epsilon\;\vec{e}_\epsilon\cdot(\vec{e}_\alpha\times\partial_\sigma\vec{e}_\alpha) + \vec{e}_\alpha\cdot(\vec{e}_\epsilon\times\partial_\sigma\vec{e}_\epsilon)\,]\,. \tag{A.6}$$

By summing over all terms on the r.h.s. of equ. (17) we obtain (16) expressed in terms of the (non-symmetric) expression (A.1).

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Figure Caption

Fig. 1 Pictorial view of the cochain reduction from the Chern-Pontryagin density down to the "local winding number" n.

Chain of descent

Geometry of Gauges

$P = \partial_\mu \Omega_\mu^{(o)}(i)$

$\Delta\Omega_\mu^{(o)}(i,j) = \partial_\nu \Omega_{\mu\nu}^{(1)}(i,j)$

$\Delta\Omega_{\mu\nu}^{(1)}(i,j,k) = \partial_\rho \Omega_{\mu\nu\rho}^{(2)}(i,j,k)$

$\Delta\Omega_{\mu\nu\rho}^{(2)}(i,j,k,l) = \partial_\sigma \Omega_{\mu\nu\rho\sigma}^{(3)}(i,j,k,l)$

$\Delta\Omega_{\mu\nu\rho\sigma}^{(3)}(i,j,k,l,m) = \epsilon_{\mu\nu\rho\sigma}\, n \ , \ n \in \mathbb{Z}$

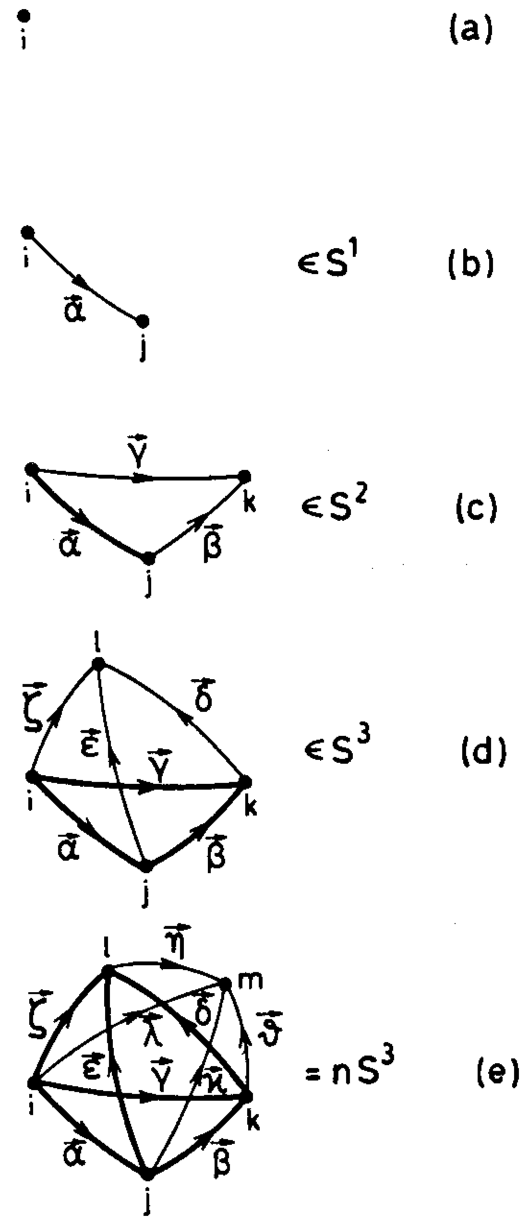

Fig. 1