THE LOVÁSZ-CHERKASSKIY THEOREM IN COUNTABLE GRAPHS

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Abstract. Lovász and Cherkasskiy discovered in the 1970s independently that if G is a finite graph with a given set T of terminal vertices such that G is inner Eulerien, then the maximal number of edge-disjoint paths connecting distinct vertices in T is $\sum_{t\in T} \lambda(t, T-t)$ where λ is the local edge-connectivity function.

The maximal size of a system of edge-disjoint T-paths in the Lovász-Cherkasskiy theorem can be characterized by the existence of certain cuts applying Menger's theorem. The infinite generalization of Menger's theorem by Aharoni and Berger (earlier known as the Erdős-Menger conjecture) together with the characterization of infinite Eulerian graphs due to Nash-Williams make possible to generalize the theorem for infinite graphs in a structural way. The aim of the paper is to prove this generalization when T is countable.

1. Introduction

There are several deep results and conjectures in infinite combinatorics whose restriction to finite structures is a well-known classical theorem. For example the results [3], [2] and [1] by Aharoni, Nash-Williams and Shelah are known as Hall's and Kőnig's theorem when only finite graphs are considered. The finite case of the Aharoni-Berger theorem [5] (earlier known as the Erdős-Menger Conjecture) is known as Menger's theorem and the Matroid Intersection Conjecture [6] by Nash-Williams extends the Matroid Intersection Theorem [9] of Edmonds.

There are several common aspects of the problems above. For example assuming the finiteness of the involved structures simplifies the proof significantly. Induction on size or finitely many application of an "augmenting path" type of argument is applicable and sufficient. These tools are unavailable and insufficient respectively in the general case where more complex techniques and structural arguments are required. All these problems express that a "primal-dual complementarity slackness" type of condition holds between suitable primal and dual objects: a matching M in $G=(A,B,E)$ and vertex-cover C consisting of a single vertex from each $e \in M$; disjoint path-system $\mathcal{P}$ between A and B in $G=(V,E)$ with $A,B \subseteq V$ and AB-separation $S \subseteq V$ consisting of choosing a single vertex from each $P \in \mathcal{P}$; common independent set I of matroids M_0 and M_1 and a bipartition $S = S_0 \cup S_1$ of their common ground set such that $S_i \cap I$ spans S_i in M_i for $i \in \{0,1\}$. Alternative characterizations of "primal optimality" can be given through the concept of strong maximality. Let us call an element X of set family $\mathcal{X}$ strongly maximal in $\mathcal{X}$ if $|Y \setminus X| \leq |X \setminus Y|$ for every $Y \in \mathcal{X}$. Note that if $\mathcal{X}$ has only finite

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elements, then strongly maximal means maximum size but it is stronger than that in general. It is known that in the three problems we mentioned that the strong maximality of a matching/disjoint path system/common independent set is equivalent with the existence of a vertex-cover/separation/bipartition such that the corresponding complementarity slackness conditions are satisfied.

Our aim is to prove a theorem in this manner extending the following result obtained by Lovász and Cherkasskiy independently in the 1970s:

Theorem 1.1 (Lovász-Cherkasskiy theorem, [16]). *Let G be a finite graph and let $T \subseteq V(G)$ such that G is inner Eulerian (i.e. $d_G(v)$ is even for every $v \in V(G) \setminus T$). Then the maximal number of pairwise edge-disjoint T-paths*[1] *is*

$$\frac{1}{2}\sum_{t \in T} \lambda_G(t, T-t),$$

where $\lambda_G(t, T-t)$ stands for the maximal number of pairwise edge-disjoint paths between t and $T-t$.

The literal extension of Theorem 1.1 to infinite graphs fails. Indeed, consider the star $K_{1,3}$ and attach a one-way infinite path to its central vertex. Let T consists of the vertices of degree one. Then we have only even degrees out of T and the maximal number of edge-disjoint T-paths is 1 although $\frac{1}{2}\sum_{t\in T}\lambda(t, T-t) = \frac{3}{2}$.

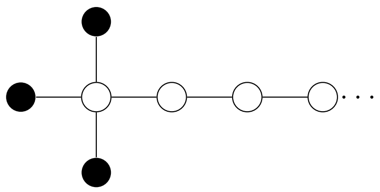

FIGURE 1. The failure of the literal infinite generalization the Lovász-Cherkasskiy theorem. Elements of T are black.

The reason of this discrepancy is that after allowing G to be infinite the condition "G is Eulerian" (i.e. $E(G)$ can be partitioned into edge-disjoint cycles) is no longer equivalent with the property that G has only even degrees. Indeed, in the two-way infinite path each degree is 2 but it is obviously not Eulerian. On the other hand, graphs with infinite degrees can be easily Eulerian. The characterization of infinite Eulerian graphs due to Nash-Williams is one of the fundamental theorems in infinite graph theory:

Theorem 1.2 (Nash-Williams, [19] (p. 235 Theorem 3)). *A (possibly infinite) graph is Eulerian if and only if it does not contain odd cut.*[2]

Simpler proofs for Theorem 1.2 was given by L. Soukup (Theorem 5.1 of [22]) and Thomassen [23] while its analogue for directed graphs (conjectured by Thomassen) was settled affirmatively in [14]. Theorem 1.2 indicates that the condition "for every $v \in V \setminus T$: $d(v)$ is even" should be replaced by "for every $X \subseteq V \setminus T$: $d(X)$ is not odd" in order to

[1] paths connecting distinct vertices in T without having internal vertex in T.

[2] infinite cardinals considered neither odd nor even

allow infinite graphs. The latter means by Theorem 1.2 that the contraction of T results in an Eulerian graph while the former meant the same but restricted to finite graphs.

The literal adaptation of the formula $\frac{1}{2}\sum_{t\in T}\lambda(t, T-t)$ is also not really fruitful in the presence of infinite quantities. Consider for example the graph $(\{u,v\}, E)$ with $T=\{u,v\}$ where E consists of $\aleph_0$ parallel edges between u and v. Then any infinite $\mathcal{P}\subseteq E$, considered as a set of paths of length one, has the same size $\aleph_0$. It demonstrates that cardinality is an overly rough measure in the presence of infinite quantities and urges us to focus on combinatorial instead of quantitative properties of an optimal path-system in Theorem 1.1. In a finite graph a system $\mathcal{P}$ of edge-disjoint T-paths has $\frac{1}{2}\sum_{t\in T}\lambda(t, T-t)$ elements if and only if $\mathcal{P}$ contains $\lambda(t, T-t)$ paths between t and $T-t$ for each $t\in T$. By Menger's theorem it is equivalent with the fact that for every $t\in T$ one can choose exactly one edge from each $P\in\mathcal{P}$ having t as an end-vertex such that the resulting edge set C is a cut separating t from $T-t$. Now we are ready to state our main results:

Theorem 1.3. *Let G be a graph and let $T\subseteq V(G)$ be countable such that there is no $X\subseteq V(G)\setminus T$ where $d_G(X)$ is an odd natural number. Then there exists a system $\mathcal{P}$ of edge-disjoint T-paths such that for every $t\in T$: one can choose exactly one edge from each $P\in\mathcal{P}$ having t as an end-vertex in such a way that the resulting edge set C is a cut separating t and $T-t$.*

We also prove the following closely related theorem.

Theorem 1.4. *Let G be a graph and let $T\subseteq V(G)$ be countable such that there is no $X\subseteq V(G)\setminus T$ where $d_G(X)$ is an odd natural number. Assume that for each $t\in T$ there is a system $\mathcal{P}_t$ of edge-disjoint T-paths covering all the edges incident with t. Then there exists a system $\mathcal{P}$ of edge-disjoint T-paths covering all the edges incident with any $t\in T$.*

We conjecture that the countability of T can be omitted in the theorems above. However, based on the experience with the similar problems mentioned earlier, we suspect that the proof is significantly harder.

Mader generalized Theorem 1.1 for arbitrary finite graphs in [17]. The structural and algorithmic aspects of the problem have been ever since a subject of interest (see for example [21], [15], [7] and [12]) as well the analogous theorems considering vertex-disjoint [11] and internally vertex-disjoint [17] paths.

Conjecture 1.5. *Let G be a graph and let $T\subseteq V(G)$. Then there exist a strongly maximal system $\mathcal{P}$ of edge-disjoint/vertex-disjoint/internally vertex-disjoint T-paths in G.*

We also conjecture that the path-system $\mathcal{P}$ in Conjecture 1.5 can be characterized in the way that it extends the corresponding minimax theorem to infinite graphs based on complementarity slackness conditions. We discuss the details in Section 5. Before we turn to the proof of our main result in Section 4, we need to introduce some notation and recall few results we are going to use in the proof. These are done in Sections 2 and 3 respectively.

2. Notation

In graphs we allow parallel edges but not loops. Technically we represent a graph as a triple $G=(V,E,I)$ where the *incidence function* $\boldsymbol{I}: E \rightarrow [V]^2$ defines the end-vertices of the edges. For $X \subseteq V$ let $\boldsymbol{\delta_G(X)} := \{e \in E : |I(e) \cap X| = 1\}$ and we write $\boldsymbol{d_G(X)}$ for $|\delta_G(X)|$. If graph G is obvious from the context, then we omit the subscript, furthermore, for a singleton $\{x\}$ we write simply $\delta(x)$ and $d(x)$. All the paths in the paper are finite. We refer sometimes the first vertex or last edge of a path. The context will always indicate according which direction we mean this. An *AB-path* for $A, B \subseteq V$ is a path with first vertex in A last vertex in B and no internal vertices in $A \cup B$. A $C \subseteq E$ is a *cut* if $C = \delta(X)$ for some $X \subseteq V$. If G is connected then X is determined by C up to taking complement and the *v-side* of the cut C is the unique X with $C = \delta(X)$ and $v \in X$. We call $\delta(X)$ an AB-cut if $A \subseteq X$ and $B \cap X = \varnothing$ or the other way around. In a connected graph G, cut $\delta(X)$ is $\subseteq$-minimal if and only if the induced subgraphs $G[X]$ and $G[V \setminus X]$ are connected. We extend the definitions above for disconnected graphs G and cuts C living in a single connected component M by considering C as a cut in $G[M]$. For a $U \subseteq V$ and a family $\mathcal{F} = \{X_u : u \in U\}$ of pairwise disjoint subsets of V with $X_u \cap U = \{u\}$ we define the graph $\boldsymbol{G/\mathcal{F}}$ obtained from G by *contracting* X_u to u for $u \in U$ and deleting the resulting loops. More formally $V(G/\mathcal{F}) := (V \setminus \bigcup \mathcal{F}) \cup U$, $E(G/\mathcal{F}) := E \setminus \{e \in E : (\exists u \in U) I(e) \subseteq X_u\}$ and $I(G/\mathcal{F})(e) := \{i_{\mathcal{F}}(u), i_{\mathcal{F}}(v)\}$ where $I(e) = \{u, v\}$ and

$$i_{\mathcal{F}}(v) = \begin{cases} v & \text{if } v \notin \bigcup \mathcal{F} \\ u & \text{if } u \in X_u. \end{cases}$$

3. Preliminaries

Menger's theorem and the other connectivity-related results that we recall in this section have four versions depending on if the graph is directed and if we consider vertex-disjoint or edge-disjoint paths. In all of these theorems the two directed variants are equivalent as well as the two undirected variants which can be shown by simple techniques like splitting edges by a new vertex and blowing up vertices to a highly connected vertex sets. Furthermore, through replacing undirected edges by back and forth directed ones the undirected vertex-disjoint version can be reduced to the directed one.

In this paper we deal only with undirected graphs and edge-disjoint paths so let us always formulate immediately that variant even if historically other version was proved first.

Let a connected graph G and distinct $s, t \in V(G)$ be fixed. For st-cuts C and D we write $C \preceq D$ if the s-side of cut C is a subset of the s-side of D. Note that the st-cuts with $\preceq$ form a complete lattice. For a finite G the optimal (minimal-sized) st-cuts form a distributive sublattice (see [10]) of it. In general graphs the size of the cut is an overly rough measure for optimality. A structural infinite generalization of the class of "optimal" st-cuts is provided by the Aharoni-Berger theorem:

Theorem 3.1 (Aharoni and Berger, [5])**.** *Let G be a (possibly infinite) graph and let $s, t \in V(G)$ be distinct. Then there is a system $\mathcal{P}$ of edge-disjoint st-paths and an st-cut*

C which is orthogonal *to $\mathcal{P}$, i.e. C consists of choosing exactly one edge from each path in $\mathcal{P}$.*

We say that the st-cut C in Theorem 3.1 is an *Erdős-Menger st-cut* and we let $\boldsymbol{\mathfrak{C}(s,t)}$ be the set of such cuts.

Theorem 3.2 (J. [13])**.** $(\mathfrak{C}(s,t), \preceq)$ *is a complete lattice, although usually not a sublattice of all the st-cuts.*

Finally we introduce two more classes $\mathfrak{C}^-(s,t)$ and $\mathfrak{C}^+(s,t)$ of st-cuts with $\mathfrak{C}^-(s,t) \cap \mathfrak{C}^+(s,t) = \mathfrak{C}(s,t)$ and $\mathfrak{C}^+(s,t) := \mathfrak{C}^-(t,s)$. Let $\boldsymbol{\mathfrak{C}^-(s,t)}$ consists of those st-cuts C for which there is a system $\mathcal{W}$ of pairwise edge-disjoint paths starting at s and having C as the set of last edges (considering the paths directed away from s). Such a $\mathcal{W}$ is called an *st-wave* and played an important role in the proof of Theorem 3.1. The cut defined as the last edges of the paths in $\mathcal{W}$ is denoted by $\boldsymbol{C_{\mathcal{W}}}$. The $\preceq$-smallest st-cut $\delta(s)$ is also a wave (considering the edges as paths of length one) that we call the *trivial st-wave*.

Lemma 3.3. $(\mathfrak{C}^-(s,t), \preceq)$ *is a complete lattice and a sup-sublattice of all the st-cuts. After the contraction of the s-side of its largest element to s, there is no non-trivial st-wave in the resulting system.*

We call an st-wave $\mathcal{W}$ *large* if $C_{\mathcal{W}}$ is the largest element of $\mathfrak{C}^-(s,t)$. Note that if the trivial st-wave is the only one, then $\delta(s)$ must be an Erdős-Menger st-cut because $\mathfrak{C} \subseteq \mathfrak{C}^- = \{\delta(s)\}$ and the left side is nonempty by Theorem 3.1. This leads to the following conclusion:

Corollary 3.4. *If there is no non-trivial st-wave, then there is a system $\mathcal{P}$ of edge-disjoint st-paths covering $\delta(s)$.*

Theorem 3.5 (Pym's theorem, [20])**.** *Assume that G is a (possibly infinite) graph, $s,t \in V(G)$ are distinct, moreover, $\mathcal{P}$ and $\mathcal{Q}$ are systems of edge-disjoint st-paths. Then there exists a system $\mathcal{R}$ of edge-disjoint st-paths such that $\delta_{\mathcal{R}}(s) \supseteq \delta_{\mathcal{P}}(s)$ and $\delta_{\mathcal{R}}(t) \supseteq \delta_{\mathcal{Q}}(t)$.*

Let $\mathcal{P}$ be a system of edge-disjoint st-paths and let $\mathcal{W}'$ be a large st-wave. By contracting the t-side of $C_{\mathcal{W}'}$ to t and applying Theorem 3.5 with the st-paths obtained from $\mathcal{W}'$ and from the initial segments of the paths in $\mathcal{P}$ we conclude:

Corollary 3.6. *Let $\mathcal{P}$ be a system of edge-disjoint st-paths. Then there is a large st-wave $\mathcal{W}$ with $\delta_{\mathcal{W}}(s) \supseteq \delta_{\mathcal{P}}(s)$.*

Finally, we will make use of the following classical lemma (see Lemma 3.3.2 and 3.3.3 in [8]):

Lemma 3.7 (Augmenting path lemma)**.** *Assume that G is a (possibly infinite) graph, $s,t \in V(G)$ are distinct and $\mathcal{P}$ is a system of edge-disjoint st-paths in G. Then either there exists an st-cut C orthogonal to $\mathcal{P}$ or there is another system $\mathcal{Q}$ of edge-disjoint st-paths for which $\delta_{\mathcal{Q}}(s) \supset \delta_{\mathcal{P}}(s)$ with $|\delta_{\mathcal{Q}}(s) \setminus \delta_{\mathcal{P}}(s)| = 1$ and $\delta_{\mathcal{Q}}(t) \supset \delta_{\mathcal{P}}(t)$ with $|\delta_{\mathcal{Q}}(t) \setminus \delta_{\mathcal{P}}(t)| = 1$.*

All the definitions and results in the section remain valid (but might sound less natural) if s and t are not vertices but disjoint vertex sets.

4. The proof of the main result

We start by giving a short outline of the proof. In the first two subsections we apply relatively simple techniques in order to reduce Theorem 1.3 to Theorem 1.4 and cut the latter problem into countable sub-problems. The third subsection is devoted to the proof of the reduced problem, namely the countable case of Theorem 1.4. The core of that proof is to show that for every given $e \in \bigcup_{t \in T} \delta(t)$ there is a path P through e such that $G - E(P)$ maintains the premisses of Theorem 1.4.

Proof of Theorem 1.3. We will use only that $\{t \in T : d(t) > 1\}$ is countable instead of the countability of the whole T. As a first step we reduce Theorem 1.3 to the following theorem.

Theorem 4.1. *Let G be a graph and let $T \subseteq V(G)$ such that $d(t) \leq 1$ for all but countably many $t \in T$ and there is no $X \subseteq V(G) \setminus T$ where $d(X)$ is an odd natural number. Assume that for each $t \in T$ there is a system $\mathcal{P}_t$ of edge-disjoint T-paths covering $\delta(t)$. Then there exists a system $\mathcal{P}$ of edge-disjoint T-paths covering $\bigcup_{t \in T} \delta(t)$.*

For $s \in T$ we will write shortly s-wave instead of $s(T - s)$-wave. Recall, it is a system $\mathcal{W}$ of pairwise edge-disjoint paths starting at s such that the set $C_{\mathcal{W}}$ of the last edges of the paths is a cut separating s and $T - s$.

4.1. **Elimination of waves.** We will call shortly the condition about the existence of path-system $\mathcal{P}_t$ in Theorem 4.1 the *linkability condition for t* (w.r.t. G and T) and we refer to the conjunction of these for $t \in T$ as *linkability condition.* First we define a process that we call *wave elimination.* We may assume that G is connected otherwise we define the elimination process component-wise. Let $T' \subseteq T$ be given where $T' = \{t_\xi : \xi < \kappa\}$ and we define by transfinite recursion G_ξ for $\xi \leq \kappa$. Let $G_0 := G$. If G_ξ is already defined then let $\mathcal{W}_\xi$ be a large t_ξ-wave with respect to G_ξ and T (exists by Lemma 3.3). We obtain $G_{\xi+1}$ by contracting the t_ξ-side of the cut $C_{\mathcal{W}_\xi}$ in G_ξ to t_ξ (see Figure 2). If ξ is a limit ordinal then we obtain G_ξ by doing all the previous contractions simultaneously. The recursion is done.

The cardinal $d_{G_\kappa}(X)$ for $X \subseteq V(G_\kappa) \setminus T$ cannot be an odd natural number because $d_{G_\kappa}(X) = d_G(X)$ and G was inner Eulerian w.r.t. T. Furthermore, Corollary 3.4 ensures that for $\xi < \kappa$ there is no non-trivial t_ξ-wave in $G_{\xi+1}$. Since any t_ξ-wave in G_κ is also a t_ξ-wave in $G_{\xi+1}$, it follows that for each $t \in T'$ there is only the trivial t-wave in G_κ. By taking $T' := T$, this is (more than) enough to guarantee the linkability condition at Theorem 4.1 (see Corollary 3.4). Therefore G_κ satisfies the premisses of Theorem 4.1 and hence assuming Theorem 4.1 we may conclude that there is a system $\mathcal{P}$ of T-paths in G_κ covering $\bigcup_{t \in T} \delta_{G_\kappa}(t)$. By using the waves $\mathcal{W}_\xi$, the system $\mathcal{P}$ can be extended to a system $\mathcal{Q}$ of T-paths in G where the $t_\xi(T - t_\xi)$-cut $C_{\mathcal{W}_\xi}$ is orthogonal to $\mathcal{Q}_{t_\xi} := \{Q \in \mathcal{Q} : t_\xi \in V(Q)\}$. Therefore $\mathcal{Q}$ satisfies the requirements of Theorem 1.3.

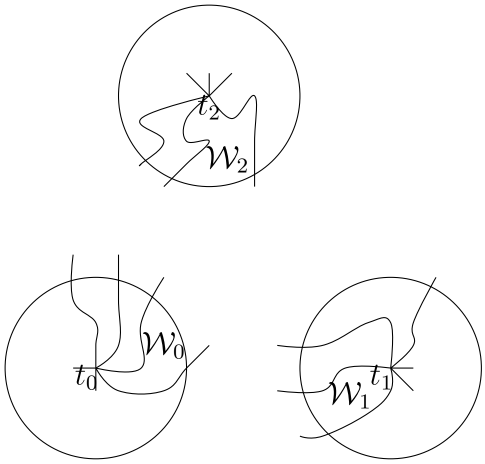

FIGURE 2. The contracted vertex sets during the wave elimination process

4.2. **Reduction to countable graphs.** In the next reduction we show that it is enough to restrict our attention to countable graphs in the proof of Theorem 4.1. First of all, we may assume without loss of generality that T does not span any edges. Indeed, otherwise we delete those edges and a set of T-paths for the remaining system demanded by 4.1 extended by the deleted edges as T-paths of length one suffices 4.1 for the original system.

By applying some basic elementary submodel-type of arguments we cut E into countable pieces each of them satisfying both the inner Eulerian and the linkability condition w.r.t. T. The contraction of T to some t results in an Eulerian graph G/T by Theorem 1.2 thus we can take a partition $\mathcal{O}$ of $E(G/T) = E$ into (edge sets of) G/T-cycles. These are cycles and T-paths in G . Let $T' := \{t \in T : d(t) > 1\}$. For $t \in T'$ let $\mathcal{P}_t$ be a system of T-paths witnessing the linkability condition for t and let $\mathcal{E} := \{E(P) : (\exists t \in T')P \in \mathcal{P}_t\}$. We define a closure operation c on 2^E in the following way. Intuitively we want to close a set $F_0 \subseteq E$ under the property that if it shares an edge with some O or $E(P)$, then it contains it completely. Formally let $c(F_0) := \bigcup_{n\in\mathbb{N}} F_n$ where

$$F_{n+1} := F_n \cup \bigcup\{O \in \mathcal{O} : F_n \cap O \neq \emptyset\} \cup \bigcup\{E(P) \in \mathcal{E} : F_n \cap E(P) \neq \emptyset\}.$$

We call an F c-closed if $c(F) = F$. We claim that c satisfies the following properties:

(1) The family of c-closed sets forms a complete Boolean algebra with respect to the usual $\cup$ and $\cap$;
(2) If F is countable then so is $c(F)$;
(3) If F is c-closed, then graph (V, F, I) and T satisfy the premisses of Theorem 4.1.

Indeed, property (1) follows directly from the construction and (2) holds because of the assumption $|T'| \leq \aleph_0$. The inner Eulerian and linkability for $t \in T'$ parts of condition (3) ensured by F not subdividing any O and $E(P)$ respectively. Recall that $d(t) \leq 1$ for $t \in T \setminus T'$ by assumption. Preservation of the linkability for these t is “automatic”:

Lemma 4.2. *If H is an inner Eulerian graph w.r.t. $T \subseteq V(H)$, then the linkability condition holds for all $t \in T$ with $d(t) \leq 1$.*

Proof. $E(H)$ can be partitioned into the edge sets of cycles and T-paths. If $d(t) = 1$, then the unique edge incident with t cannot be in a cycle so must be in a T-path. □

In order to reduce Theorem 4.1 to countable graphs, it is enough to partition E into countable c-closed sets F_ξ. Indeed, then $G_\xi := (V, F_\xi, I)$ is countable (apart from isolated vertices) and satisfies the premisses of Theorem 4.1 with T by property (3). Hence by applying the countable case of Theorem 4.1, we can take a system $\mathcal{P}_\xi$ of T-paths in G_ξ covering the edges $\bigcup_{t\in T} \delta_{G_\xi}(t)$. Finally, $\bigcup_\xi \mathcal{P}_\xi$ is as desired.

Suppose that the pairwise disjoint countable c-closed sets $\{F_\xi : \xi < \alpha\}$ are already defined for some ordinal α. Then $E \setminus \bigcup_{\xi<\alpha} F_\xi$ is c-closed by property (1). If it is empty then we are done. Otherwise let $F_\alpha := c(\{e\})$ for an arbitrary $e \in E \setminus \bigcup_{\xi<\alpha} F_\xi$, which is countable by property (2). The recursion is done.

4.3. **The proof of Theorem 4.1.** We will make use of the following simple observation.

Observation 4.3. *The deletion of the edges of a T-path preserves the condition that there is no $X \subseteq V \setminus T$ with $d(X)$ odd.*

The core of our proof is the repeated application of the following claim:

Claim 4.4. *For every $t \in T$ and $e \in \delta(t)$ there exists a T-path P through e such that $G - E(P)$ satisfies the linkability condition.*

Indeed, we only need to prove Theorem 4.1 for countable G as discussed in the previous subsection. Assuming Claim 4.4, a system of T-paths covering $\bigcup_{t\in T} \delta(t)$ can be constructed by a straightforward recursion.

Proof of Claim 4.4. First we give a proof in the special case where there is some $s \in T$ such that $d(t) \leq 1$ for all $t \in T - s$. Let us fix a system $\mathcal{P}_s$ of edge-disjoint paths between s and $T - s$ covering $\delta(s)$.

For $e \in \delta(s)$, we simply take the unique $P \in \mathcal{P}_s$ through e. By Observation 4.3 graph $G - E(P)$ is still Eulerian w.r.t. T. By Lemma 4.2 it is enough to check that the linkability condition is preserved for s but it is obviously true witnessed by $\mathcal{P}_s \setminus \{P\}$.

Suppose now that $e \in \delta(t)$ for a $t \in T - s$. If t is an end-vertex of some $P \in \mathcal{P}_s$, then we take P and argue as in the previous paragraph. If it is not the case, then either we replace $\mathcal{P}_s$ by another $\mathcal{P}'_s$ where t is an end-vertex of some $P \in \mathcal{P}'_s$ or chose P to be edge-disjoint from $\mathcal{P}_s$. To do so, let Q be an arbitrary path between t and $T - t$. If $E(Q) \cap E(\mathcal{P}_s) = \emptyset$, then we take $P := Q$ and the linkability condition holds for s since $\mathcal{P}$ lives in $G - E(P)$. If $E(Q) \cap E(\mathcal{P}) \neq \emptyset$, then let $v \in V(Q) \cap V(\mathcal{P})$ be the first common vertex while going along Q from t. Let $P' \in \mathcal{P}_s$ such that $v \in V(P')$. We get $\mathcal{P}'_s$ by replacing P' in $\mathcal{P}_s$ with the path P we obtain by uniting the initial segment of P' from s to v with the initial segment of Q from t to v.

Applying this iteratively together with the technique discussed in Subsection 4.2 we conclude:

Corollary 4.5. *Theorem 4.1 holds whenever there is an $s \in T$ such that $d(t) \leq 1$ for $t \in T - s$.*

Proposition 4.6. *Assume that $G = (V, E, I)$ is an inner Eulerian graph w.r.t. $T \subseteq V$ such that there is no non-trivial s-wave for some $s \in T$. Then for every $f, h \in E$, the linkability condition holds for s in $G - f - h$.*

Proof. We may assume without loss of generality that G is connected, since only the component containing s is relevant. Since deletion of edges in $\delta(s)$ make the linkability for s a weaker requirement, we can also assume that $f, h \in E \setminus \delta(s)$. If G is finite and $X \subseteq V$ with $X \cap T = \{s\}$, then $d(s)$ and $d(X)$ must have the same parity because $d(v)$ is even for $v \in X - s$. This observation of Lovász led immediately to the justification of Proposition 4.6 for finite graphs. Indeed, on the one hand, $d(s) < d(X)$ if $\{s\} \subsetneq X \subseteq V \setminus (T - s)$, since $\delta(s)$ is the only Erdős-Menger $s(T - s)$-cut by assumption. On the other hand, the same parity of $d(s)$ and $d(X)$ ensures $d(s) + 2 \leq d(X)$. The proof of Proposition 4.6 for infinite graphs is more involved and we need some preparation.

For a graph H and distinct $s, t \in V(H)$, we call an Erdős-Menger st-cut C *s-tight* if there is system $\mathcal{P}$ of edge-disjoint paths in H between s and t covering $\delta_H(s)$ but every such a path-system is orthogonal to C.

Lemma 4.7. *Assume that H is a graph, $s, t \in V(H)$ are distinct and there is a system $\mathcal{P}$ of edge-disjoint paths in H between s and t covering $\delta_H(s)$ but there is an $e \in E(H) \setminus \delta_H(s)$ such that $e \in E(\mathcal{P})$ for every such a path-system. Then there exists an s-tight Erdős-Menger st-cut C containing e.*

Proof. We may assume that H is connected, since otherwise we consider the component containing s and t. Let $\mathcal{P}$ and e as in the lemma. Then there is a unique $P_e \in \mathcal{P}$ through e. If $H - e$ is disconnected, then $\mathcal{P} = \{P_e\}$ and $C := \{e\}$ is as desired. Suppose that $H - e$ is connected. Let D be the $\preceq$-smallest Erdős-Menger st-cut in $H - e$ (see Lemma 3.2). We are going to prove that $C := D + e$ is as desired. To do so, it is enough to show that $\mathcal{Q} := \mathcal{P} \setminus \{P_e\}$ is orthogonal to D. Indeed, if it is done, then e must connect the two parts of cut D in $H - e$ and therefore $D + e$ is an st-cut in H and $\mathcal{P}$ is orthogonal to it.

Suppose for a contradiction that $\mathcal{Q}$ is not orthogonal to D.

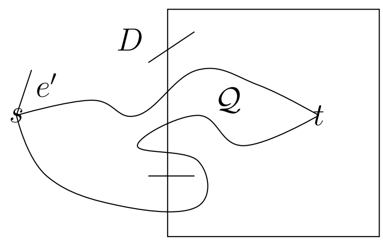

FIGURE 3. Graph $H - e$ where D is not orthogonal to $\mathcal{Q}$. The first edge of P_e is e'.

Let H' what we get by contracting the t-side of D to t in H. Then $D = \delta_{H'-e}(t)$ and it is the only element of $\mathfrak{C}_{H'-e}(s, t)$ since it is the smallest but also the largest one. We apply the Augmenting path lemma 3.7 in $H' - e$ with s, t and the set $\mathcal{Q}'$ of st-paths in $H' - e$ given by the initial segments of the paths in $\mathcal{Q}$. The augmentation must be successful, since otherwise it would give a $D' \in \mathfrak{C}_{H'-e}(s, t)$ with $D' \neq D$. Indeed, $D \setminus E(\mathcal{Q}') \neq \varnothing$ by the indirect assumption but $D' \subseteq E(\mathcal{Q}')$ according to Lemma 3.7. The successful augmentation provides a system $\mathcal{Q}''$ of edge-disjoint st-paths in $H' - e$ covering $\delta_{H'-e}(s)$. Indeed, there is a unique $e' \in \delta_{H'-e}(s)$ which is uncovered by $\mathcal{Q}'$, namely the first edge of P_e, but Augmenting path lemma 3.7 ensures $\delta_{\mathcal{Q}'}(s) \subset \delta_{\mathcal{Q}''}(s)$. Since $D \in \mathfrak{C}_{H-e}(s,t)$, the paths in $\mathcal{Q}''$ can be forward extended in H to obtain a system of edge-disjoint st-paths in $H - e$ covering $\delta_H(s)$ contradicting the obligatory usage of e in the assumption of the lemma. $\square$

Since the only Erdős-Menger $s(T-s)$-cut is $\delta(s)$ (because there is no non-trivial s-wave) and $f \notin \delta(s)$, Lemma 4.7 ensures that there is a system $\boldsymbol{\mathcal{P}_s}$ of edge-disjoint paths in $G - f$ between s and $T - s$ covering $\delta(s)$. Suppose for a contradiction that such a path-system cannot be found in $G - f - h$. By applying Lemma 4.7 again this time with $G - f$ and h, we obtain an s-tight Erdős-Menger $s(T-s)$-cut C in $G - f$ containing h. Let S be the s-side of cut C. Then $\delta_{G-f}(S) = C$ and we must have $f \in \delta_G(S)$ since otherwise the initial segments of the paths in $\mathcal{P}_s$ up to their unique edge in C would form a non-trivial s-wave with respect to G and T. Since Erdős-Menger cuts are $\subseteq$-minimal cuts, $G[S]$ is connected. We define G' by extending $G[S]$ with new vertices $\{t_e : e \in \delta_G(S)\}$ and with the edges $\delta(S)$ where an $e \in \delta(S)$ keeps its original end-vertex in S and gets t_e as the other end-vertex. We define $T' := \{s\} \cup \{t_e : e \in \delta_G(S)\}$.

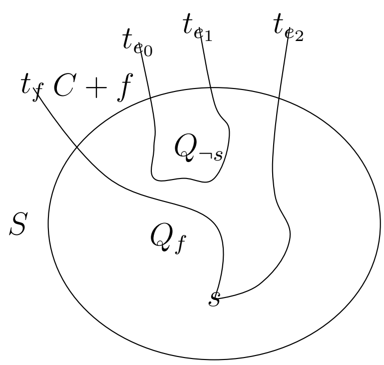

Figure 4. Graph G' and path-system $\mathcal{Q}$.

For $X \subseteq V(G') \setminus T'$, the cardinal $d_{G'}(X)$ cannot be an odd number because $d_{G'}(X) = d_G(X)$ by construction. Moreover, the linkability condition with respect to G' and T' holds, since for s it is witnessed by the initial segments of the paths in $\mathcal{P}_s$ while the connectivity of G' guarantees it for the vertices in $T' - s$. Thus the premisses of Theorem 4.1 are satisfied, furthermore, all the vertices in T' except possibly s has degree 1. By applying Corollary 4.5 to G' and T', we can take a system $\boldsymbol{\mathcal{Q}}$ of T'-paths in G' covering all the edges incident with some vertex in T'.

It cannot happen that all $Q \in \mathcal{Q}$ has s as an end-vertex because then $\mathcal{Q}$ would provide a non-trivial s-wave with respect to G and T (where $f \notin \delta(s)$ is used to ensure 'non-trivial'). Let $Q_{\neg s} \in \mathcal{Q}$ be a path with $s \notin V(Q_{\neg s})$ and let us denote the end-vertices of $Q_{\neg s}$ by $\boldsymbol{t_{e_0}}$ and $\boldsymbol{t_{e_1}}$.

Lemma 4.8. *Every system $\mathcal{R}'$ of edge-disjoint T'-paths in G' covering $\delta(s)$ and avoiding t_f must use all the vertices $\{t_e : e \in C\}$.*

Proof. Suppose for a contradiction that $\mathcal{R}'$ is a counterexample and let $\mathcal{R}$ be the path-system in $G-f$ corresponding to $\mathcal{R}'$. Then $\mathcal{R}$ is a system of edge-disjoint paths starting at s and having exactly their last edges in C such that $C \setminus E(\mathcal{R}) \neq \varnothing$. Since C is an Erdős-Menger $s(T-s)$-cut in $G-f$, the paths in $\mathcal{R}$ can be forward extended to obtain a system $\mathcal{R}^+$ of edge-disjoint $s(T-s)$-paths with $C \cap E(\mathcal{R}^+) = C \cap E(\mathcal{R})$. Then $\mathcal{R}^+$ also covers $\delta(s)$ and $C \setminus E(\mathcal{R}^+) \neq \varnothing$ contradicting the s-tightness of C in $G-f$. $\square$

Let $Q_f \in \mathcal{Q}$ the unique path with $t_f \in V(Q_f)$. We claim that the other end-vertex of Q_f must be s, thus in particular $Q_f \neq Q_{\neg s}$ and hence $f \notin \{e_0, e_1\}$. Indeed, since otherwise the system $\mathcal{Q}_s := \{Q \in \mathcal{Q} : s \in V(Q)\}$ of edge-disjoint T'-paths in G' covers $\delta(s)$ using neither t_f nor the other end-vertex of Q_f which contradicts Lemma 4.8. Now we consider the path-system $\mathcal{Q}_s \setminus \{Q_f\}$. It covers all but one edges in $\delta(s)$ and avoids e_0 and e_1. We apply the Augmenting path lemma 3.7 in G' with $\mathcal{Q}_s \setminus \{Q_f\}, s$ and $\{t_e : e \in C\}$. If the augmentation is successful, then the resulting path-system covers $\delta(s)$ and at least one of e_0 and e_1 is still unused contradicting Lemma 4.8. Thus the Augmenting path lemma ensures that we can pick a single edge from each path in $\mathcal{Q}_s \setminus \{Q_f\}$ such that the resulting edge set C' separates s and $\{t_e : e \in C\}$ in G'. We take the initial segments of the paths in $\mathcal{P}_s$ until the first meeting with C' and continue them forward using the terminal segments of the corresponding paths from $\mathcal{Q}_s \setminus \{Q_f\}$ to obtain a set of T'-paths in G' covering $\delta(s)$ without using t_{e_0}, t_{e_1} and t_f, which contradicts Lemma 4.8. $\square$

Now we can finish the proof of Claim 4.4. Suppose for a contradiction that $G, T, s \in T$ and $e_0 \in \delta(s)$ form a counterexample and $\mathcal{P}_s = \{P_e : e \in \delta(s)\}$ is a system of edge-disjoint T-paths with $e \in E(P_e)$. We may assume that G, T, s, e_0 and $\mathcal{P}_s$ have been chosen to minimize $|E(P_{e_0})|$ among the possible options. We know that $|E(P_{e_0})| \geq 2$ because if P_{e_0} were consists of the single edge e_0, then $P := P_{e_0}$ would satisfy Claim 4.4. We proceed by applying wave elimination with $T' := T - s$ (see Subsection 4.1) and denote the resulting graph by G'. The linkability condition for a $t \in T'$ w.r.t. G and T combined with Corollary 3.6 allows us to choose wave $\mathcal{W}_t$ during the elimination in such a way that $\delta_{\mathcal{W}_t}(t) = \delta_G(t)$. Let us define $\mathcal{W}_s$ to be the trivial s-wave in G. For any T-path Q in G' with end-vertices t and t' there is a unique $W_t \in \mathcal{W}_t$ and $W_{t'} \in \mathcal{W}_{t'}$ containing an extreme edge of Q'. Uniting these paths with Q results in a T-path $\ell(Q)$ between t and t' in G. Furthermore, the images of pairwise edge-disjoint paths under this lifting operation ℓ are pairwise edge-disjoint.

We claim that G', T, s and e_0 must be also a counterexample for Claim 4.4. Suppose for a contradiction that P' witnesses that it is not. We will conclude that then $P := \ell(P')$ shows that G, T, s and e_0 is also not a counterexample. Indeed, we need to check that the linkability condition holds w.r.t. $G - E(P)$ and T. For $t \in T$, we take a path-system $\mathcal{Q}_t$ witnessing the linkability condition in $G' - E(P')$ for t. Since $\mathcal{W}_t$ was chosen in the way to cover $\delta_G(t)$, the point-wise image $\ell[\mathcal{Q}_t]$ witnessing the linkability for t in $G - E(P)$. Thus G', T, s and e_0 form indeed a counterexample.Moreover, the set $\mathcal{P}'_s = \{P'_e : e \in \delta(s)\}$ of the suitable initial segments of the paths in $\mathcal{P}_s$ guarantee by $\left|P'_{e_0}\right| \leq |P_{e_0}|$ that this new counterexample also minimizing.

We may assume (just to simplify the notation) that our original counterexample has chosen in such a way that there is no non-trivial t-wave for $t \in T - s$. Let $f_0 \in E(P_{e_0})$ be the edge right after e_0 in P_{e_0}. We replace e_0 and f_0 by a single new edge h_0 connecting s and the end-vertex of f_0 that is not shared with e_0 (splitting technique by Lovász from [16]). Let P_{h_0} be the path in the resulting graph G' with $E(P_{h_0}) = E(P_{e_0}) - e_0 - f_0 + h_0$ and let us define $\mathcal{P}'_s := \mathcal{P}_s - P_{e_0} + P_{h_0}$. For $X \subseteq V \setminus T$ the quantities $d_G(X)$ and $d_{G'}(X)$ are either both infinite or they have the same parity, thus G' is also inner Eulerian w.r.t. T. The linkability condition for s holds in G' witnessed by $\mathcal{P}'_s$. Let $t \in T - s$ be arbitrary. The linkability condition for t holds in $G - e_0 - f_0$ by Proposition 4.6, moreover, if $h_0 \in \delta_{G'}(t)$, then h_0 is an edge between s and t and hence a T-path itself. Note that G', T, s and h_0 cannot be a counterexample for Claim 4.4 because $|E(P_{h_0})| = |E(P_{e_0})| - 1$. Therefore we can pick some T-path P' in G' through h_0 such that the linkability condition holds in $G' - E(P')$. Let us take then a T-path P in G through e_0 with $E(P) \subseteq E(P') - h_0 + e_0 + f_0$. Since $G' - E(P')$ is a subgraph of $G - E(P)$ and $\delta_{G-E(P)}(t) = \delta_{G'-E(P')}(t)$ holds for $t \in T$, the linkability condition in $G' - E(P')$ implies the linkability in $G - E(P)$. This contradicts the assumption that G, T, s and e_0 form a counterexample.

□

□

5. Outlook

5.1. **Edge-disjoint T-paths in non-Eulerian graphs.** Let G be a graph and let $T \subseteq V(G)$. A *T-subpartition* is a family $\mathcal{A} = \{X_t : t \in T\}$ of pairwise disjoint subsets of $V(G)$ such that $X_t \cap T = \{t\}$. If G is finite, then we call a component Y of $G - \bigcup \mathcal{A}$ *obstructive* if $d(Y)$ is odd. Let $\mathsf{o}(G, \mathcal{A})$ be the number of the obstructive components.

Theorem 5.1 (Mader, [17])**.** *Let G be a finite graph and let $T \subseteq V(G)$. Then the maximal number of pairwise edge-disjoint T-paths is*

$$\min\left\{\frac{1}{2}\left(\sum_{t\in T} d(X_t) - \mathsf{o}(G,\mathcal{A})\right) : \mathcal{A} \text{ is a } T\text{-subpartition}\right\}.$$

Let us define $E(\mathcal{A}) := \bigcup_{t\in T} \delta(X_t)$. In Theorem 5.1, for a system $\mathcal{P}$ of edge-disjoint T-paths and a T-partition $\mathcal{A}$ we have equality if and only if the following conditions hold:

Condition 5.2 (complementarity slackness)**.**

(1) Each $P \in \mathcal{P}$ uses either only a single edge from $E(\mathcal{A})$ (which must connect two vertex sets in $\mathcal{A}$) or two edges incident with a component of $G - \bigcup \mathcal{A}$.

(2) For each component Y of $G - \bigcup \mathcal{A}$, path-system $\mathcal{P}$ uses all but at most one edge from $\delta(Y)$.

Conjecture 5.3. *Let G be a (possibly infinite) graph and let $T \subseteq V(G)$. Then there exists a system $\mathcal{P}$ of edge-disjoint T-paths and a T-subpartition $\mathcal{A}$ satisfying Condition 5.2.*

Although the Lovász-Cherkasskiy theorem 1.1 is a special case of Mader's edge-disjoint T-path theorem 5.1, our Conjecture 5.3 does not seem to imply our main result Theorem 1.3. It motivates to formulate a stronger conjecture based on the extension of the concept of obstructive components.

For a possibly infinite graph G, we define a component Y of $G-\bigcup\mathcal{A}$ to be *obstructive* if after the contraction of $V(G)\setminus Y$ to some vertex v the resulting graph H does not contain a set of pairwise edge-disjoint cycles covering $\delta_H(v)$. This extends our previous definition of obstructive. Indeed, on the one hand, if $d_G(Y)$ is odd, then $d_H(v)$ is the same odd number and hence $\delta_H(v)$ cannot be covered by edge-disjoint cycles. On the other hand, if $d(Y)$ is even, then finding the desired cycles is equivalent to finding a J-join in the connected graph $G[Y]$ where J consists of those $u\in Y$ for which there are odd number of edges between u and v in H.

Condition 5.4.

(1) Each $P\in\mathcal{P}$ uses either only a single edge from $E(\mathcal{A})$ (which must connect two vertex sets in $\mathcal{A}$) or two edges incident with a component of $G-\bigcup\mathcal{A}$.

(2) The path-system $\mathcal{P}$ uses all the edges $E(\mathcal{A})$ except one from $\delta(Y)$ for each obstructive component Y.

Note that if G is inner Eulerian, then cannot be any obstructive components (regardless of the choice of $\mathcal{A}$) and therefore by replacing Condition 5.2 with Condition 5.4 in Conjecture 5.3 it will imply Theorem 1.4 (even without the restriction $|T|\leq\aleph_0$). We also point out that for finite graphs Conditions 5.2 and 5.4 are equivalent because if $d(Y)$ is even, then $\mathcal{P}$ cannot misses exactly one edge from $\delta(Y)$.

Recall a system $\mathcal{P}$ of edge-disjoint/vertex-disjoint/internally vertex-disjoint T-paths is *strongly maximal* if $|\mathcal{Q}\setminus\mathcal{P}|\leq|\mathcal{P}\setminus\mathcal{Q}|$ for every edge-disjoint/vertex-disjoint/internally vertex-disjoint system $\mathcal{Q}$ of T-paths.

Conjecture 5.5. *Let G be a (possibly infinite) graph and let $T\subseteq V(G)$. Then for a system $\mathcal{P}$ of edge-disjoint T-paths the following statements are equivalent:*

(i) $\mathcal{P}$ is a strongly maximal system of edge-disjoint T-paths.

(ii) There exists a T-subpartition $\mathcal{A}$ satisfying Condition 5.2 with $\mathcal{P}$.

(iii) There exists a T-subpartition $\mathcal{A}$ satisfying Condition 5.4 with $\mathcal{P}$.

Note that only the implication $(i)\implies(iii)$ is an open question. Indeed, the implication $(iii)\implies(ii)$ is trivial. Assuming (ii), $\mathcal{P}$ must be an inclusion-wise maximal system of edge-disjoint T-paths. If $|\mathcal{P}\setminus\mathcal{Q}|=\kappa\geq\aleph_0$, then $|E(\mathcal{P}\setminus\mathcal{Q})|=\kappa$ and since each $P\in\mathcal{Q}\setminus\mathcal{P}$ must contain an edge from $E(\mathcal{P}\setminus\mathcal{Q})$, we obtain $|\mathcal{Q}\setminus\mathcal{P}|\leq\kappa$. If $|\mathcal{P}\setminus\mathcal{Q}|=k\in\mathbb{N}$, then let $G':=G-E(\mathcal{P}\cap\mathcal{Q})$. Then $d_{G'}(Y)$ is finite for every component of $G'-\bigcup\mathcal{A}$ and for all of but finitely may Y it is 0, moreover,

$$\frac{1}{2}\left(\sum_{t\in T}d_{G'}(X_t)-\mathsf{o}(G',\mathcal{A})\right)=k,$$

from which $|\mathcal{Q}\setminus\mathcal{P}|\leq k$ follows. Thus $\mathcal{P}$ is strongly maximal.

5.2. **Vertex-disjoint T-paths.** We conjectured already in the Introduction (Conjecture 1.5) the existence strongly maximal systems of T-paths with different concepts of disjointness. We believe that strong maximality can be characterized by the existence of a certain dual object reflecting the corresponding classical theorems of Gallai [11] and Mader [18]. If $T=V(G)$, then a vertex-disjoint system of T-paths is a matching. Infinite matching theory was intensively investigated and is well-understood (see the survey [4]). The existence of a strongly maximal matching in a graph was proven by Aharoni (see Theorem 5.3 of [4]) together with the following theorem:

Theorem 5.6 (Aharoni, Theorem 5.2 of [4])**.** *In every (possibly infinite) graph G there is a matching $M \subseteq E(G)$ and an $X \subseteq V(G)$ such that:*

(1) For each component Y of $G - X$, the edges in M spanned by Y cover all but at most one vertex of Y.
(2) The vertices in X are covered by M in such a way that X does not span any edge in M.
(3) $G[Y]$ is factor-critical[3] whenever Y is a component of $G - X$ for which M does not contain a perfect matching of $G[Y]$.
(4) Let $\Pi(G, X)$ be the bipartite graph whose vertex classes are X and the set $\mathcal{Y}$ of the factor-critical components of $G - X$, furthermore, an xY is an edge if x has a neighbour in Y in G. Then for every $Y \in \mathcal{Y}$ there is a matching in $\Pi(G, X)$ covering X while avoiding vertex Y.

Remark 5.7.

- Properties (1) and (2) at Theorem 5.6 are already sufficient to ensure the strong maximality of the matching M.
- For every strongly maximal matching M there is an X satisfying (1)-(4).
- Property (4) was originally not mentioned by Aharoni but it can be obtained easily by applying for example Lemma 3.6 of [3].
- If there is a matching M for which $V(M)$ is $\subseteq$-maximal (which is always the case in countable graphs), then the set X in Theorem 5.6 is unique.

By omitting the assumption of $T = V(G)$ we leave matchings theory and formulate an infinite generalization of Gallai's theorem [11]:

Conjecture 5.8. *Let G be a (possibly infinite) graph and let $T \subseteq V(G)$. Then there exists a system $\mathcal{P}$ of vertex-disjoint T-paths and an $X \subseteq V(G)$ such that:*

(1) For each component Y of $G - X$, the paths $\{P \in \mathcal{P} : V(P) \subseteq Y\}$ cover all but at most one vertex of $T \cap Y$.
(2) $X \subseteq V(\mathcal{P})$ where $|V(P) \cap X| \leq 1$ for every $P \in \mathcal{P}$.

Assume now that there is a partition $\mathcal{S}$ of T and let us call $\mathcal{S}$-path a T-path whose end-vertices are in different members of $\mathcal{S}$. A minimax formula for the maximal number of $\mathcal{S}$-paths was given by Mader in [18]. We expect the following generalization based on the complementarity slackness conditions to be true:

Conjecture 5.9. *Assume that G is a (possibly infinite) graph, $T \subseteq V(G)$ and $\mathcal{S}$ is a partition of T. Then there exists a system $\mathcal{P}$ of vertex-disjoint $\mathcal{S}$-paths, an $X \subseteq V(G)$ and a partition $\mathcal{Y}$ of $V(G) \setminus X$ such that:*

(0) After the deletion of the vertex set X and the edges of the subgraphs $G[Y]$ for $Y \in \mathcal{Y}$ the resulting graph does not contain any $\mathcal{S}$-path.

[3]does not have a perfect matching but after deleting any vertex the resulting graph has

For $B_Y := (T \cap Y) \cup \{v \in Y : v$ *has a neighbour in some* $Y' \in \mathcal{Y} \setminus \{Y\}\}$:

(1) The paths in $\mathcal{P}$ *cover* X *and all but at most one vertex of* B_Y *for every* $Y \in \mathcal{Y}$.

(2) For every $P \in \mathcal{P}$ *either* $|V(P) \cap X| = 1$ *and* $|V(P) \cap B_Y| \leq 1$ *for every* $Y \in \mathcal{Y}$ *or* $|V(P) \cap X| = 0$ *and there is a unique* $Y_P \in \mathcal{Y}$ *with* $|V(P) \cap B_{Y_P}| = 2$ *while* $|V(P) \cap B_Y| \leq 1$ *for* $Y \in \mathcal{Y} \setminus \{Y_P\}$.

(3) For every $Y \in \mathcal{Y}$ *there is at most one* $P \in \mathcal{P}$ *with* $|V(P) \cap Y| = 1$.

References

[1] R Aharoni, C Nash-Williams, and S Shelah, *Marriage in infinite societies*, Progress in Graph Theory Academic Press, Toronto (1984), 71–79.

[2] R Aharoni, C S. J. Nash-Williams, and S Shelah, *Another form of a criterion for the existence of transversals*, Journal of the London Mathematical Society **2** (1984), no. 2, 193–203.

[3] R Aharoni, C. Nash-Williams, and S Shelah, *A general criterion for the existence of transversals*, Proceedings of the London Mathematical Society **3** (1983), no. 1, 43–68.

[4] R. Aharoni, *Infinite matching theory*, Discrete mathematics **95** (1991), no. 1, 5–22.

[5] R. Aharoni and E. Berger, *Menger's theorem for infinite graphs*, Inventiones mathematicae **176** (2009), no. 1, 1–62.

[6] R. Aharoni and R. Ziv, *The intersection of two infinite matroids*, Journal of the London Mathematical Society **58** (1998), no. 03, 513–525.

[7] M. Babenko and S. Artamonov, *Faster algorithms for half-integral t-path packing*, 28th international symposium on algorithms and computation (isaac 2017), 2017.

[8] R. Diestel, *Graph theory*, Fifth, Graduate Texts in Mathematics, vol. 173, Springer, Berlin, 2018. Paperback edition of [MR3644391]. MR3822066

[9] J. Edmonds, *Submodular functions, matroids, and certain polyhedra*, Combinatorial optimization—eureka, you shrink!, 2003, pp. 11–26.

[10] F Escalante and T Gallai, *Note über kantenschnittverbände in graphen*, Acta Mathematica Hungarica **25** (1974), no. 1-2, 93–98.

[11] T. Gallai, *Maximum-minimum sätze und verallgemeinerte faktoren von graphen*, Acta Mathematica Academiae Scientiarum Hungarica **12** (1964), no. 1-2, 131–173.

[12] S. Iwata and Y. Yokoi, *Combinatorial algorithms for edge-disjoint t-paths and integer free multiflow*, arXiv preprint arXiv:1909.07919 (2019).

[13] A. Joó, *The complete lattice of erdős-menger separations* (2019), available at 1904.06244.

[14] ———, *On partitioning the edges of an infinite digraph into directed cycles*, Advances in Combinatorics **2** (2021), no. 8. https://doi.org/10.19086/aic.18702.

[15] J. C. M. Keijsper, R. A. Pendavingh, and L. Stougie, *A linear programming formulation of mader's edge-disjoint paths problem*, Journal of Combinatorial Theory, Series B **96** (2006), no. 1, 159–163.

[16] L. Lovász, *On some connectivity properties of eulerian graphs*, Acta Mathematica Academiae Scientiarum Hungarica **28** (1976), no. 1-2, 129–138.

[17] W. Mader, *Über die maximalzahl kantendisjunktera-wege*, Archiv der Mathematik **30** (1978), no. 1, 325–336.

[18] ———, *Über die maximalzahl kreuzungsfreierh-wege*, Archiv der Mathematik **31** (1978), no. 1, 387–402.

[19] C S. J. Nash-Williams, *Decomposition of graphs into closed and endless chains*, Proceedings of the London Mathematical Society **3** (1960), no. 1, 221–238.

[20] J. Pym, *The linking of sets in graphs*, Journal of the London Mathematical Society **1** (1969), no. 1, 542–550.

[21] A. Sebő and L. Szegő, *The path-packing structure of graphs*, International conference on integer programming and combinatorial optimization, 2004, pp. 256–270.

[22] L. Soukup, *Elementary submodels in infinite combinatorics*, Discrete Mathematics **311** (2011), no. 15, 1585–1598.

[23] C. Thomassen, *Nash-williams' cycle-decomposition theorem*, Combinatorica **37** (2017), no. 5, 1027–1037.

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