DETECTING AND DESCRIBING RAMIFICATION FOR STRUCTURED RING SPECTRA

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ABSTRACT. Ramification for commutative ring spectra can be detected by relative topological Hochschild homology and by topological André-Quillen homology. In the classical algebraic context it is important to distinguish between tame and wild ramification. Noether's theorem characterizes tame ramification in terms of a normal basis and tame ramification can also be detected via the surjectivity of the trace map. We transfer the latter fact to ring spectra and use the Tate cohomology spectrum to detect wild ramification in the context of commutative ring spectra. We study ramification in examples in the context of topological K-theory and topological modular forms.

1. INTRODUCTION

Classically, ramification is studied in the setting of extensions of rings of integers in number fields. If $K \subset L$ is an extension of number fields and if $\mathcal{O}_K \to \mathcal{O}_L$ is the corresponding extension of rings of integers, then a prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ ramifies in L, if $\mathfrak{p}\mathcal{O}_L = \mathfrak{p}_1^{e_1} \cdot \ldots \cdot \mathfrak{p}_s^{e_s}$ in \mathcal{O}_L and $e_i > 1$ for at least one $1 \leq i \leq s$. The ramification is *tame* when the ramification indices e_i are all relatively prime to the residue characteristic of \mathfrak{p} and it is *wild* otherwise. Auslander and Buchsbaum [AB59] considered ramification in the setting of general noetherian rings. If $K \subset L$ is a finite *G*-Galois extension, then $\mathcal{O}_K \to \mathcal{O}_L$ is unramified, if and only if $\mathcal{O}_K = \mathcal{O}_L^G \to \mathcal{O}_L$ is a Galois extension of commutative rings and this in turn says that $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L \cong \prod_G \mathcal{O}_L$ (see [CHR65, Remark 1.5 (d)], [AB59] or [Rog08a, Example 2.3.3] for more details).

Our main interest is to investigate notions of ramified extensions of ring spectra and to study examples.

Rognes [Rog08a, Definition 4.1.3] introduces G-Galois extensions of ring spectra. A map $A \rightarrow B$ of commutative ring spectra is a G-Galois extension for a finite group G, if certain cofibrancy conditions are satisfied, if G acts on B from the left through commutative A-algebra maps and if the following two conditions are satisfied:

- (1) The map from A to the homotopy fixed points of B with respect to the G-action, $i: A \to B^{hG}$ is a weak equivalence.
- (2) The map

$$(1.1) h: B \wedge_A B \to \prod_G B$$

is a weak equivalence.

Here, h is right adjoint to the composite map

$$B \wedge_A B \wedge G_+ \longrightarrow B \wedge_A B \longrightarrow B,$$

induced by the G-action $B \wedge G_+ \cong G_+ \wedge B \to B$ on B and the multiplication on B.

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Condition (1) is the fixed point condition familiar from ordinary Galois theory. Condition (2) is needed to ensure that the map $A \to B$ is *unramified*. Among other things, it implies for instance that the A-endomorphisms of B correspond to the group elements in the sense that

$$j: B \wedge G_+ \to F_A(B, B)$$

is a weak equivalence, where j is right adjoint to the composite map

$$(B \wedge G_+) \wedge_A B \to B \wedge_A B \to B$$

which is again induced by the G-action and the multiplication on B.

If A is the Eilenberg-MacLane spectrum $H\mathcal{O}_K$ and $B = H\mathcal{O}_L$ for a G-Galois extension $K \subset L$, then $H\mathcal{O}_K \to H\mathcal{O}_L$ is a G-Galois extension of ring spectra if and only if $\mathcal{O}_K \to \mathcal{O}_L$ is a G-Galois extension of commutative rings.

For certain Galois extensions Ausoni and Rognes [AR08] conjecture a version of Galois descent for algebraic K-theory. A descent result that covers many of the conjectured cases is established in [CMNN20]. In some cases, descent can be established even in the presence of ramification. Ausoni [Aus05, Theorem 10.2] shows for instance that the canonical map $K(\ell_p) \to K(ku_p)^{hC_{p-1}}$ is an equivalence after *p*-completion despite the fact that the inclusion of the *p*-completed connective Adams summand, ℓ_p , into *p*-completed topological connective K-theory, ku_p , should be viewed as a tamely ramified extension of commutative ring spectra. In other cases that are not Galois extensions, for instance in cases, that we will identify as wildly ramified, one can consider a modified version of descent [CMNN20, §5.4].

How can we detect ramification? The unramified condition from (1.1) ensures for instance that $A \to B$ is separable [Rog08a, Definition 9.1.1] and this in turn implies that the canonical map from B to the relative topological Hochschild homology, $\mathsf{THH}^A(B)$, is an equivalence and that the spectrum of topological André-Quillen homology $\mathsf{TAQ}^A(B)$ [Bas99] is trivial. So if we know for a map of commutative ring spectra $A \to B$ that $B \to \mathsf{THH}^A(B)$ is not a weak equivalence or that $\pi_*\mathsf{TAQ}^A(B) \neq 0$, then this is an indicator for ramification. We will study examples of non-vanishing TAQ in 2.1 and study relative topological Hochschild homology in examples related to level-2-structures on elliptic curves in 2.2.

An interesting class of examples arises as connective covers of G-Galois extensions. Akhil Mathew shows in [Mat16a, Theorem 6.17] that connective Galois extensions are algebraically étale: the induced map on homotopy groups is étale in a graded sense. So, in particular, connective covers of Galois extensions are rarely Galois extensions, because several known examples of Galois extensions such as $KO \to KU$, $L_p \to KU_p$ and examples of Galois extensions in the context of topological modular forms are far from behaving nicely on the level of homotopy groups.

We use relative topological Hochschild homology and topological André-Quillen homology in order to detect ramification in the cases $ko \to ku$, $\ell \to ku_{(p)}$, $\mathsf{tmf}_0(3)_{(2)} \to \mathsf{tmf}_1(3)_{(2)}$, $\mathsf{tmf}_{(3)} \to \mathsf{tmf}_0(2)_{(3)}$, $\mathsf{Tmf}_{(3)} \to \mathsf{Tmf}_0(2)_{(3)}$ and $\mathsf{tmf}_0(2)_{(3)} \to \mathsf{tmf}(2)_{(3)}$. We also study a version of the discriminant map in the context of structured ring spectra and apply it to the examples $\ell \to ku_{(p)}$ and $ko \to ku$ in 2.3.

For certain finite extensions of discrete valuation rings tame ramification is equivalent to being log-étale (see for instance [Rog09, Example 4.32]). It is known by work of Sagave [Sag14], that $\ell \to ku_{(p)}$ is log-étale if one considers the log structures generated by $v_1 \in \pi_{2p-2}\ell$ and $u \in \pi_2(ku)$. We show that $ko \to ku$ is not log-étale if one considers the log structures generated by the Bott elements $\omega \in \pi_8(ko)$ and $u \in \pi_2(ku)$.

Emmy Noether shows [Noe32, §2] that tame ramification is equivalent to the existence of a normal basis. Tame ramification can also be detected by the surjectivity of the trace map [CF67, Theorem 2, Chapter 1, §5]. This in turn yields a vanishing of Tate cohomology.

Using Tate cohomology as a possible criterion for wild ramification is for instance suggested by Rognes in [Rog14]. Rognes also shows a version of Noether's theorem in [Rog08b, Theorem 5.2.5]: If a spectrum with a *G*-action *X* is in the thick subcategory generated by spectra of the form $G_+ \wedge W$, then $X^{tG} \simeq *$, so in particular, if *B* has a normal basis, $B \simeq G_+ \wedge A$, then $B^{tG} \simeq *$.

We use the Tate spectrum in order to propose a definition of tame and wild ramification of maps of ring spectra and study examples in the context of topological K-theory, topological modular forms and cochains on classifying spaces with coefficients in Morava E-theory aka Lubin-Tate spectra.

Several of our examples use topological modular forms with level structures. The spectrum of topological modular forms, TMF, arises as the global sections of a structure sheaf of E_{∞} -ring spectra on the moduli stack of elliptic curves, \mathcal{M}_{ell} . A variant of it, Tmf, lives on a compactified version, $\overline{\mathcal{M}}_{ell}$. Its connective version is denoted by tmf. There are other variants corresponding to level structures on elliptic curves. Recall that a $\Gamma(n)$ -structure (or level *n*-structure for short) carries the datum of a chosen isomorphism between the *n*-torsion points of an elliptic curve and the group $(\mathbb{Z}/n\mathbb{Z})^2$. A $\Gamma_1(n)$ -structure corresponds to the choice of a point of exact order *n* whereas a $\Gamma_0(n)$ -structure comes from the choice of a subgroup of order *n* of the *n*-torsion points. See [KM85, Chapter 3] for the precise definitions and for background. These level structures give rise to a tower of moduli problems (see [KM85, p. 200])



with corresponding spectra $\mathsf{TMF}(n)$, $\mathsf{TMF}_1(n)$ and $\mathsf{TMF}_0(n)$ and their compactified versions $\mathsf{Tmf}(n)$, $\mathsf{Tmf}_1(n)$ and $\mathsf{Tmf}_0(n)$ [HL16, Theorem 6.1].

In [MM15] Mathew and Meier prove that the maps $\mathsf{Tmf}[\frac{1}{n}] \to \mathsf{Tmf}(n)$ are not Galois extensions but they satisfy Tate vanishing, which might be seen as an indication of tame ramification. In contrast, we will show that $\mathsf{tmf}(n)^{tGL_2(\mathbb{Z}/n\mathbb{Z})}$ is non-trivial if 2 does not divide n or if n is a power of 2.

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2. Detecting ramification

2.1. Topological André-Quillen homology. For a map of connective commutative ring spectra $i: A \to B$ we use the connectivity of the map to determine the bottom homotopy group of $\mathsf{TAQ}^A(B)$ [Bas99]. The non-triviality of $\mathsf{TAQ}^A(B)$ indicates that the map i is ramified.

Algebraic cases. If $\mathcal{O}_K \to \mathcal{O}_L$ is an extension of number rings with corresponding extension of number fields $K \subset L$, then of course we cannot use a connectivity argument for understanding TAQ, but here, the algebraic module of Kähler differentials, $\Omega^1_{\mathcal{O}_L | \mathcal{O}_K}$, is isomorphic to the first

Hochschild homology group $\mathsf{HH}_1^{\mathcal{O}_K}(\mathcal{O}_L)$ which in turn is $\pi_0\mathsf{TAQ}^{H\mathcal{O}_K}(H\mathcal{O}_L)$ [BR04, Theorem 2.4]. The ramification type of $\mathcal{O}_K \to \mathcal{O}_L$ can be read off the *different*, *i.e.*, the annihilator of the module of Kähler differentials.

The connective Adams summand. Let ℓ denote the Adams summand of connective p-localized topological complex K-theory, $ku_{(p)}$. Here, p is an odd prime.

The inclusion $i: \ell \to ku_{(p)}$ induces an isomorphism on π_0 and π_1 . Thus, by the Hurewicz theorem for topological André-Quillen homology [Bas99, Lemma 8.2], [BGR08, Lemma 1.2] we get that $\pi_2 \mathsf{TAQ}^{\ell}(ku_{(p)})$ is the bottom homotopy group and is isomorphic to the second homotopy group of the cone of i, cone(i) and this in turn can be determined by the long exact sequence

$$\ldots \to \pi_2(\ell) = 0 \to \pi_2(ku_{(p)}) \to \pi_2(\operatorname{cone}(i)) \to \pi_1(\ell) = 0 \to \ldots$$

hence $\pi_2 \mathsf{TAQ}^{\ell}(ku_{(p)}) \cong \mathbb{Z}_{(p)}$.

We know from [DLR20] that $\ell \to k u_{(p)}$ shows features of a tamely ramified extension of number rings and Sagave shows [Sag14, Theorem 6.1] that $\ell \to k u_{(p)}$ is log-étale.

Real and complex connective topological K-theory. The complexification map $c: ko \to ku$ induces an isomorphism on π_0 and an epimorphism on π_1 , so it is a 1-equivalence. Hence again $\pi_2 \operatorname{cone}(c) \cong \pi_2(\mathsf{TAQ}^{ko}(ku))$ is the bottom homotopy group, but here we obtain an extension

$$0 \to \pi_2 k u = \mathbb{Z} \to \pi_2 \text{cone}(c) \to \pi_1(ko) = \mathbb{Z}/2\mathbb{Z} \to 0.$$

In order to understand $\pi_2 \operatorname{cone}(c)$ we consider the cofiber sequence

$$\Sigma KO \xrightarrow{\eta} KO \xrightarrow{c} KU \xrightarrow{\delta} \Sigma^2 KO$$

and the commutative diagram on homotopy groups

$$\mathbb{Z}/2\mathbb{Z} = \pi_2 ko \xrightarrow{\pi_2(c)} \pi_2 ku \longrightarrow \pi_2 \text{cone}(c) \longrightarrow \pi_1 ko \xrightarrow{\pi_1(c)} 0 = \pi_1 ku$$
$$\cong \bigvee_{\pi_2(\tau_{ko})} \cong \bigvee_{\pi_2(\tau_{ku})} \bigvee_{g} \cong \bigvee_{\pi_1(\tau_{ko})} \cong \bigvee_{\pi_1(\tau_{ku})} \mathbb{Z}/2\mathbb{Z} = \pi_2 KO \xrightarrow{\pi_2(c)} \pi_2 KU \xrightarrow{\pi_2(\delta)} \pi_2 \Sigma^2 KO \xrightarrow{\pi_2(\Sigma\eta)} \pi_1 KO \xrightarrow{\pi_1(c)} 0 = \pi_1 KU.$$

Here $\tau_e \colon e \to E$ denotes the map from the connective cover e of E to E. The middle vertical map g is the map induced by the cofiber sequences. By the five lemma, g is an isomorphism hence

$$\pi_2 \operatorname{cone}(c) \cong \pi_2 \Sigma^2 KO \cong \mathbb{Z}.$$

So this group is also torsion free. We will later see that $ko \to ku$ is not log-étale and we will see some other indicators for wild ramification, but the bottom homotopy group of $\mathsf{TAQ}^{ko}(ku)$ does not detect that.

Connective topological modular forms with level structure (case n = 3). We consider $\mathsf{tmf}_1(3)$. Its homotopy groups are $\pi_*(\mathsf{tmf}_1(3)) \cong \mathbb{Z}[\frac{1}{3}][a_1, a_3]$ with $|a_i| = 2i$. See [HL16] for some background. There is a C_2 -action on $\mathsf{tmf}_1(3)$ coming from the permutation of elements of exact order three and one denotes by $\mathsf{tmf}_0(3)$ the connective cover of the homotopy fixed points, $\mathsf{tmf}_1(3)^{hC_2}$. There is a homotopy fixed point spectral sequence that was studied in detail in [MR09] for the periodic versions. In [HL16, p. 407] it is explained how to adapt this calculation to the connective variants: The terms in the spectral sequence with $s > t - s \ge 0$ can be ignored. The C_2 -action on the a_i 's is given by the sign-action, so if τ generates C_2 , then $\tau(a_i^n) = (-1)^n a_i^n$.

This implies that only $H^0(C_2; \pi_0(\mathsf{tmf}_1(3))) \cong \mathbb{Z}[\frac{1}{3}]$ survives to $\pi_0(\mathsf{tmf}_0(3))$. For π_1 we get a contribution from $H^1(C_2; \pi_2(\mathsf{tmf}_1(3)))$, giving a $\mathbb{Z}/2\mathbb{Z}$ generated by the class of a_1 (this detects an η). For $\pi_2(\mathsf{tmf}_0(3))$ the class of a_1^2 generates a copy of $\mathbb{Z}/2\mathbb{Z}$.

Hence the map $j: \mathsf{tmf}_0(3)_{(2)} \to \mathsf{tmf}_1(3)_{(2)}$ is 1-connected, so $\pi_2 \mathsf{TAQ}^{\mathsf{tmf}_0(3)_{(2)}}(\mathsf{tmf}_1(3)_{(2)})$ is the bottom homotopy group and is isomorphic to $\pi_2(\mathsf{cone}(j))$ which sits in an extension. We can use the commutative diagram of commutative ring spectra from [HL16, Theorem 63]



in order to determine to $\pi_2(\operatorname{cone}(j))$. By [LN14, Theorem 1.2] there is a cofiber sequence of $\operatorname{tmf}_1(3)_{(2)}$ -modules

$$\Sigma^{6} \operatorname{tmf}_{1}(3)_{(2)} \xrightarrow{v_{2}} \operatorname{tmf}_{1}(3)_{(2)} \xrightarrow{} ku_{(2)}$$

and hence $\pi_2(\mathsf{tmf}_1(3)_{(2)}) \cong \pi_2(ku_{(2)}).$

The diagram

commutes and the 5-lemma implies that $\pi_2(\operatorname{cone}(j)) \cong \mathbb{Z}_{(2)}$.

Connective topological modular forms with level structure (case n = 2, p = 3). Forgetting a $\Gamma_0(2)$ -structure yields a map $f: \mathsf{tmf}_{(3)} \to \mathsf{tmf}_0(2)_{(3)}$ such that f is a 3-equivalence. We will recall more details about these spectra at the beginning of 2.2. Again, we obtain that the bottom non-trivial homotopy group of the spectrum of topological André-Quillen homology is $\pi_4(\mathsf{TAQ}^{\mathsf{tmf}_{(3)}}(\mathsf{tmf}_0(2)_{(3)})) \cong \pi_4(\mathsf{cone}(f))$. There is a short exact sequence

$$0 = \pi_4 \mathsf{tmf}_{(3)} \to \pi_4 \mathsf{tmf}_0(2)_{(3)} = \mathbb{Z}_{(3)} \to \pi_4 \mathsf{cone}(f) \to \pi_3 \mathsf{tmf}_{(3)} \cong \mathbb{Z}/3\mathbb{Z} \to 0$$

so a priori $\pi_4 \operatorname{cone}(f)$ could be isomorphic to $\mathbb{Z}_{(3)}$ or to $\mathbb{Z}_{(3)} \oplus \mathbb{Z}/3\mathbb{Z}$.

There is an equivalence $[Beh06, \S2.4]$

$$\operatorname{tmf}_{(3)} \wedge T \simeq \operatorname{tmf}_0(2)_{(3)}$$

where $T = S^0 \cup_{\alpha_1} e^4 \cup_{\alpha_1} e^8$ with α_1 denoting the generator of π_3^s at 3. Thus T is part of a cofiber sequence

$$S^0 \to T \to \Sigma^4 \operatorname{cone}(\alpha_1)$$

and we obtain a cofiber sequence

$$\mathsf{tmf}_{(3)} = \mathsf{tmf}_{(3)} \land S^0 \to \mathsf{tmf}_{(3)} \land T \simeq \mathsf{tmf}_0(2)_{(3)} \to \mathsf{tmf}_{(3)} \land \Sigma^4 \operatorname{cone}(\alpha_1)$$

and thus

$$\pi_4(\operatorname{cone}(f)) \cong \pi_4(\operatorname{tmf}_{(3)} \wedge \Sigma^4 \operatorname{cone}(\alpha_1)) \cong \pi_0(\operatorname{tmf}_{(3)} \wedge \operatorname{cone}(\alpha_1)).$$

But as we have a short exact sequence

$$0 = \pi_0(\Sigma^3 \mathsf{tmf}_{(3)}) \to \pi_0(\mathsf{tmf}_{(3)}) \cong \mathbb{Z}_{(3)} \to \pi_0(\mathsf{tmf}_{(3)} \land \operatorname{cone}(\alpha_1)) \to 0$$

we obtain

$$\pi_4(\mathsf{TAQ}^{\mathsf{tmf}_{(3)}}(\mathsf{tmf}_0(2)_{(3)})) \cong \mathbb{Z}_{(3)}.$$

2.2. Relative topological Hochschild homology. In [DLR20] (see also [DLR] for a correction) we show that the relative topological Hochschild homology spectra $\mathsf{THH}^{\ell}(ku_{(p)})$ and $\mathsf{THH}^{ko}(ku)$ have highly non-trivial homotopy groups. Here, we extend these results to the relative THH-spectra of $\mathsf{tmf}_{(3)} \to \mathsf{tmf}_0(2)_{(3)}$, $\mathsf{Tmf}_{(3)} \to \mathsf{Tmf}_0(2)_{(3)}$ and $\mathsf{tmf}_0(2)_{(3)} \to \mathsf{tmf}(2)_{(3)}$. For formulas concerning the coefficients of elliptic curves we refer to [Del75].

Recall that we have $\mathsf{tmf}_0(2)_{(3)} \simeq \tau_{\geq 0} \mathsf{tmf}(2)_{(3)}^{hC_2}$. By [Sto12, §7] we know that $\pi_* \mathsf{tmf}(2)_{(3)} \cong \mathbb{Z}_{(3)}[\lambda_1, \lambda_2]$ with $|\lambda_i| = 4$ and with C_2 -action given by $\lambda_1 \mapsto \lambda_2$ and $\lambda_2 \mapsto \lambda_1$ [Sto12, Lemma 7.3]. Since $|C_2|$ is invertible in $\pi_* \mathsf{tmf}(2)_{(3)}$ the E^2 -page of the homotopy fixed point spectral sequence is given by

$$H^*(C_2, \pi_*\mathsf{tmf}(2)_{(3)}) = H^0(C_2, \pi_*\mathsf{tmf}(2)_{(3)}) = \pi_*(\mathsf{tmf}(2)_{(3)})^{C_2}.$$

Thus, we have

$$\pi_* \mathsf{tmf}_0(2)_{(3)} = \mathbb{Z}_{(3)}[\lambda_1 + \lambda_2, \lambda_1 \lambda_2] = \mathbb{Z}_{(3)}[a_2, a_4]$$

with $a_2 = -(\lambda_1 + \lambda_2)$ and $a_4 = \lambda_1 \lambda_2$. Recall the following facts about the homotopy of $\mathsf{tmf}_{(3)}$ (see for instance [DFHH14, p. 192]): We have

$$\pi_* \mathsf{tmf}_{(3)} = \begin{cases} \mathbb{Z}_{(3)}\{1\}, & * = 0; \\ \mathbb{Z}/3\mathbb{Z}\{\alpha_1\} & * = 3; \\ \mathbb{Z}_{(3)}\{c_4\}, & * = 8; \\ \mathbb{Z}_{(3)}\{c_6\}, & * = 12; \\ 0, & * = 4, 5, 6, 7 \end{cases}$$

where α_1 is the image of $\alpha_1 \in \pi_3(S_{(3)})$ under $\pi_3(S_{(3)}) \to \pi_3 \operatorname{tmf}_{(3)}$. By [Sto12, Proof of Proposition 10.3] we have that the map $\pi_* \operatorname{tmf}_{(3)} \to \pi_* \operatorname{tmf}_0(2)_{(3)}$ satisfies $c_4 \mapsto 16a_2^2 - 48a_4$ and $c_6 \mapsto -64a_2^3 + 288a_2a_4$. (There is a discrepancy between our sign for c_6 and that in [Sto12].)

We know from personal communication with Mike Hill that there is a fiber sequence of $\mathsf{tmf}_0(2)_{(3)}$ -modules

$$\mathsf{Tmf}_{0}(2)_{(3)} \longrightarrow \mathsf{tmf}_{0}(2)_{(3)}[a_{2}^{-1}] \times \mathsf{tmf}_{0}(2)_{(3)}[a_{4}^{-1}] \xrightarrow{f} \mathsf{tmf}_{0}(2)_{(3)}[(a_{2}a_{4})^{-1}].$$

See [HM17, Proposition 4.24] for the analogous statement at p = 2. The kernel of $\pi_*(f)$ has $\mathbb{Z}_{(3)}$ -basis

$$\{(a_2^n a_4^m, -a_2^n a_4^m) | n, m \in \mathbb{N}\}$$

and the cokernel has $\mathbb{Z}_{(3)}$ -basis

$$\{\frac{1}{a_2^n a_4^m} \mid n \ge 1, m \ge 1\}.$$

We get that in negative degrees $\pi_*\mathsf{Tmf}_0(2)_{(3)}$ is given by

$$\bigoplus_{n,m\geq 1} \mathbb{Z}_{(3)}\left\{\frac{1}{a_2^n a_4^m}\right\},$$

where $\frac{1}{a_2^n a_4^m}$ has degree -4n - 8m - 1. The $\pi_* \mathsf{tmf}_0(2)_{(3)}$ -action is given by

$$a_2 \cdot \frac{1}{a_2^n a_4^m} = \begin{cases} \frac{1}{a_2^{n-1} a_4^m}, & \text{if } n \ge 2\\ 0, & \text{otherwise} \end{cases}$$

and analogously for a_4 .

By the gap theorem (see for instance [Kon]) we have $\pi_* \mathsf{Tmf}_{(3)} \cong 0$ for -21 < * < 0. Lemma 2.1.

$\pi_*(\mathsf{tmf}_0(2)_{(3)} \wedge_{\mathsf{tmf}_{(3)}} \mathsf{tmf}_0(2)_{(3)}) \cong \mathbb{Z}_{(3)}[a_2, a_4, r]/r^3 + a_2r^2 + a_4r =: \mathbb{Z}_{(3)}[a_2, a_4, r]/I$ where r has degree 4 and is mapped to zero under the multiplication map.

Proof. As above we use that we have an equivalence of $\mathsf{tmf}_{(3)}$ -modules $\mathsf{tmf}_{(3)} \wedge T \simeq \mathsf{tmf}_0(2)_{(3)}$. Here, T is defined by the cofiber sequences

$$S^3_{(3)} \xrightarrow{\alpha_1} S^0_{(3)} \longrightarrow \operatorname{cone}(\alpha_1) \longrightarrow S^4_{(3)}$$

and

$$S^7_{(3)} \xrightarrow{\phi} \operatorname{cone}(\alpha_1) \longrightarrow T \longrightarrow S^8_{(3)}$$

where $S_{(3)}^7 \xrightarrow{\phi} \operatorname{cone}(\alpha_1) \to S_{(3)}^4$ is equal to α_1 . We get an equivalence of left $\operatorname{tmf}_0(2)_{(3)}$ -modules $\operatorname{tmf}_0(2)_{(2)} \wedge \operatorname{tmf}_0(2)_{(2)} \sim \operatorname{tmf}_0(2)_{(2)} \wedge \operatorname{tmf}_0(2)_{(3)} \wedge T$.

$$\operatorname{tmf}_{0}(2)_{(3)} \wedge_{\operatorname{tmf}_{(3)}} \operatorname{tmf}_{0}(2)_{(3)} \simeq \operatorname{tmf}_{0}(2)_{(3)} \wedge_{\operatorname{tmf}_{(3)}} (\operatorname{tmf}_{(3)} \wedge T) \simeq \operatorname{tmf}_{0}(2)_{(3)} \wedge T.$$

Smashing the above cofiber sequences with $\mathsf{tmf}_0(2)_{(3)}$ gives cofiber sequences of $\mathsf{tmf}_0(2)_{(3)}$ -modules

$$\Sigma^{3} \mathsf{tmf}_{0}(2)_{(3)} \xrightarrow{\bar{\alpha}_{1}} \mathsf{tmf}_{0}(2)_{(3)} \longrightarrow \mathsf{tmf}_{0}(2)_{(3)} \wedge \operatorname{cone}(\alpha_{1}) \xrightarrow{\delta} \Sigma^{4} \mathsf{tmf}_{0}(2)_{(3)}$$

and

$$\Sigma^{7} \mathsf{tmf}_{0}(2)_{(3)} \xrightarrow{\phi} \mathsf{tmf}_{0}(2)_{(3)} \wedge \operatorname{cone}(\alpha_{1}) \longrightarrow \mathsf{tmf}_{0}(2)_{(3)} \wedge T \xrightarrow{\Delta} \Sigma^{8} \mathsf{tmf}_{0}(2)_{(3)}.$$

The map $\bar{\alpha}_1$ is zero in the derived category of $\mathsf{tmf}_0(2)_{(3)}$ -modules, because $\pi_*(\mathsf{tmf}_0(2)_{(3)})$ is concentrated in even degrees. We therefore get an equivalence of $\mathsf{tmf}_0(2)_{(3)}$ -modules

$$\mathsf{tmf}_0(2)_{(3)} \wedge \operatorname{cone}(\alpha_1) \simeq \mathsf{tmf}_0(2)_{(3)} \vee \Sigma^4 \mathsf{tmf}_0(2)_{(3)}.$$

This implies that $\mathsf{tmf}_0(2)_{(3)} \wedge \operatorname{cone}(\alpha_1)$ has non-trivial homotopy groups only in even degrees and therefore that $\bar{\phi}$ is zero in the derived category of $\mathsf{tmf}_0(2)_{(3)}$ -modules. We get an equivalence of $\mathsf{tmf}_0(2)_{(3)}$ -modules

$$\mathsf{tmf}_{0}(2)_{(3)} \wedge T \simeq \mathsf{tmf}_{0}(2)_{(3)} \vee \Sigma^{4} \mathsf{tmf}_{0}(2)_{(3)} \vee \Sigma^{8} \mathsf{tmf}_{0}(2)_{(3)}.$$

We can assume that the map $\mathsf{tmf}_{(3)} \to \mathsf{tmf}_0(2)_{(3)}$ factors in the derived category of $\mathsf{tmf}_{(3)}$ -modules as

$$\mathsf{tmf}_{(3)} \longrightarrow \mathsf{tmf}_{(3)} \land \operatorname{cone}(\alpha_1) \longrightarrow \mathsf{tmf}_{(3)} \land T \xrightarrow{\simeq} \mathsf{tmf}_0(2)_{(3)}$$

This implies that the inclusion in the first smash factor

 $\eta_L \colon \mathsf{tmf}_0(2)_{(3)} \to \mathsf{tmf}_0(2)_{(3)} \wedge_{\mathsf{tmf}_{(3)}} \mathsf{tmf}_0(2)_{(3)}$

is given by

$$\mathsf{tmf}_0(2)_{(3)} \longrightarrow \mathsf{tmf}_0(2)_{(3)} \wedge \operatorname{cone}(\alpha_1) \longrightarrow \mathsf{tmf}_0(2)_{(3)} \wedge T \xrightarrow{\simeq} \mathsf{tmf}_0(2)_{(3)} \wedge_{\mathsf{tmf}_{(2)}} \mathsf{tmf}_0(2)_{(3)}$$

We obtain that the map

$$\mathsf{tmf}_0(2)_{(3)} \wedge \operatorname{cone}(\alpha_1) \longrightarrow \mathsf{tmf}_0(2)_{(3)} \wedge T \simeq \mathsf{tmf}_0(2)_{(3)} \wedge_{\mathsf{tmf}_{(3)}} \mathsf{tmf}_0(2)_{(3)} \longrightarrow \mathsf{tmf}_0(2)_{(3)}$$

is a left inverse for $\mathsf{tmf}_0(2)_{(3)} \to \mathsf{tmf}_0(2)_{(3)} \land \operatorname{cone}(\alpha_1)$. It is also clear that the inclusion in the second smash factor $\eta_R \colon \mathsf{tmf}_0(2)_{(3)} \to \mathsf{tmf}_0(2)_{(3)} \land \mathsf{tmf}_{(3)} \mathsf{tmf}_0(2)_{(3)}$ is given by

$$\mathsf{tmf}_0(2)_{(3)} \xrightarrow{\simeq} \mathsf{tmf}_{(3)} \wedge T \longrightarrow \mathsf{tmf}_0(2)_{(3)} \wedge T \xrightarrow{\simeq} \mathsf{tmf}_0(2)_{(3)} \wedge_{\mathsf{tmf}_{(3)}} \mathsf{tmf}_0(2)_{(3)}.$$

We claim that

 $\eta_R(a_2) \in \pi_4(\mathsf{tmf}_0(2)_{(3)} \wedge_{\mathsf{tmf}_{(3)}} \mathsf{tmf}_0(2)_{(3)}) \cong \pi_4(\mathsf{tmf}_0(2)_{(3)} \wedge T) \cong \pi_4(\mathsf{tmf}_0(2)_{(3)} \wedge \operatorname{cone}(\alpha_1))$ maps to 3 times a unit under

$$\delta_4 \colon \pi_4(\mathsf{tmf}_0(2)_{(3)} \wedge \operatorname{cone}(\alpha_1)) \to \pi_4(\Sigma^4 \mathsf{tmf}_0(2)_{(3)}) \cong \mathbb{Z}_{(3)}.$$

By commutativity of the diagram

it suffices to show that $a_2 \in \pi_4(\mathsf{tmf}_{(3)} \wedge T)$ maps to 3 times a unit under the bottom map. This follows by the exact sequence

We define r to be the unique element in $\pi_4(\mathsf{tmf}_0(2)_{(3)} \wedge \operatorname{cone}(\alpha_1))$ that maps to that unit under δ_4 and that is in the kernel of the composition of $\pi_4(\mathsf{tmf}_0(2)_{(3)} \wedge \operatorname{cone}(\alpha_1)) \cong \pi_4(\mathsf{tmf}_0(2)_{(3)} \wedge T)$ and the multiplication map

$$\pi_4(\mathsf{tmf}_0(2)_{(3)} \wedge T) \cong \pi_4(\mathsf{tmf}_0(2)_{(3)} \wedge_{\mathsf{tmf}_{(3)}} \mathsf{tmf}_0(2)_{(3)}) \to \pi_4(\mathsf{tmf}_0(2)_{(3)}).$$

We have that $3r - \eta_R(a_2)$ is in the image of $\pi_4(\mathsf{tmf}_0(2)_{(3)}) \to \pi_4(\mathsf{tmf}_0(2)_{(3)} \land \operatorname{cone}(\alpha_1))$ and thus can be written as $3r - \eta_R(a_2) = n \cdot a_2$ for an $n \in \mathbb{Z}_{(3)}$. Applying the map

 $\pi_4(\mathsf{tmf}_0(2)_{(3)}\wedge \operatorname{cone}(\alpha_1)) \cong \pi_4(\mathsf{tmf}_0(2)_{(3)}\wedge T) \cong \pi_4(\mathsf{tmf}_0(2)_{(3)}\wedge_{\mathsf{tmf}_{(3)}}\mathsf{tmf}_0(2)_{(3)}) \to \pi_4(\mathsf{tmf}_0(2)_{(3)})$ gives n = -1.

We claim that $\eta_R(a_4) \in \pi_8(\mathsf{tmf}_0(2)_{(3)} \wedge T)$ maps to 3 times a unit under

$$\Delta_8: \pi_8(\mathsf{tmf}_0(2)_{(3)} \wedge T) \to \pi_8(\Sigma^8 \mathsf{tmf}_0(2)_{(3)}).$$

As above one sees that it suffices to show that a_4 maps to 3 times a unit under the map $\pi_8(\mathsf{tmf}_{(3)} \wedge T) \to \pi_8(\Sigma^8 \mathsf{tmf}_{(3)})$. For this we consider the exact sequence

Using that $\pi_4(\mathsf{tmf}_{(3)}) = 0 = \pi_5(\mathsf{tmf}_{(3)})$ one gets that $\pi_8(\mathsf{tmf}_{(3)}) \cong \pi_8(\mathsf{tmf}_{(3)} \land \operatorname{cone}(\alpha_1))$, and under this isomorphism the first map in the exact sequence identifies with

$$\pi_8(\mathsf{tmf}_{(3)}) \cong \mathbb{Z}_{(3)}\{c_4\} \to \pi_8(\mathsf{tmf}_0(2)_{(3)}) \cong \mathbb{Z}_{(3)}\{a_2^2\} \oplus \mathbb{Z}_{(3)}\{a_4\}, \quad c_4 \mapsto 16a_2^2 - 48a_4.$$

As $\pi_6(\mathsf{tmf}_{(3)}) = 0 = \pi_7(\mathsf{tmf}_{(3)})$, one gets that $\pi_7(\mathsf{tmf}_{(3)} \wedge \operatorname{cone}(\alpha_1)) \cong \pi_7(\Sigma^4 \mathsf{tmf}_{(3)})$, and under this isomorphism the third map in the exact sequence identifies with

$$\pi_8(\Sigma^8 \mathsf{tmf}_{(3)}) \cong \mathbb{Z}_{(3)} \to \pi_3(\mathsf{tmf}_{(3)}) \cong \mathbb{Z}/3\mathbb{Z}\{\alpha_1\}, \quad 1 \mapsto \alpha_1.$$

One obtains that the second map in the exact sequence maps a_4 to $3 \cdot m$ and a_2^2 to $9 \cdot m$ for a unit $m \in \mathbb{Z}_{(3)}$.

Since the map $\pi_*(\mathsf{tmf}_{(3)}) \to \pi_*(\mathsf{tmf}_0(2)_{(3)})$ maps c_4 to $16a_2^2 - 48a_4$, we have the equation

$$16 \cdot a_2^2 - 48 \cdot a_4 = 16 \cdot \eta_R(a_2)^2 - 48 \cdot \eta_R(a_4)$$

in $\pi_*(\mathsf{tmf}_0(2)_{(3)} \wedge_{\mathsf{tmf}_{(3)}} \mathsf{tmf}_0(2)_{(3)})$. Replacing $\eta_R(a_2)$ by $3r + a_2$ and using torsion-freeness one gets the equation

$$\eta_R(a_4) = a_4 + 3r^2 + 2a_2r.$$

We apply the map $\Delta_8: \pi_8(\mathsf{tmf}_0(2)_{(3)} \wedge T) \to \pi_8(\Sigma^8 \mathsf{tmf}_0(2)_{(3)})$ to this equation and obtain by torsion-freeness of $\pi_*(\mathsf{tmf}_0(2)_{(3)})$

$$\Delta_8(r^2) = m.$$

We thus have an isomorphism of left $\pi_*(\mathsf{tmf}_0(2)_{(3)})$ -modules

$$\pi_*(\mathsf{tmf}_0(2)_{(3)} \wedge_{\mathsf{tmf}_{(3)}} \mathsf{tmf}_0(2)_{(3)}) \cong \pi_*(\mathsf{tmf}_0(2)_{(3)}) \oplus \pi_*(\mathsf{tmf}_0(2)_{(3)}) \{r\} \oplus \pi_*(\mathsf{tmf}_0(2)_{(3)}) \{r^2\} \oplus \pi_*(\mathsf{tm$$

Since the map $\pi_*(\mathsf{tmf}_{(3)}) \to \pi_*(\mathsf{tmf}_0(2)_{(3)})$ maps c_6 to $-64a_2^3 + 288a_2a_4$, we have

$$-64 \cdot a_2^3 + 288 \cdot a_2 \cdot a_4 = -64 \cdot \eta_R(a_2)^3 + 288 \cdot \eta_R(a_2) \cdot \eta_R(a_4)$$

in $\pi_*(\mathsf{tmf}_0(2)_{(3)} \wedge_{\mathsf{tmf}_{(3)}} \mathsf{tmf}_0(2)_{(3)})$. Replacing $\eta_R(a_2)$ by $3r + a_2$ and $\eta_R(a_4)$ by $a_4 + 3r^2 + 2a_2r$ and using torsion-freeness one gets

$$r^3 + a_2 r^2 + a_4 r = 0$$

This implies the lemma.

Theorem 2.2. The relative THH-spectrum $\text{THH}^{\text{tmf}_{(3)}}(\text{tmf}_0(2)_{(3)})$ is not trivial. More precisely,

$$\mathsf{THH}^{\mathsf{tmf}_{(3)}}_{*}(\mathsf{tmf}_{0}(2)_{(3)}) \cong \mathbb{Z}_{(3)}[a_{2}, a_{4}] \oplus \bigoplus_{i \ge 0} \Sigma^{14i+5} \mathbb{Z}_{(3)}[a_{2}]$$
$$\cong \pi_{*}\mathsf{tmf}_{0}(2) \oplus \bigoplus_{i \ge 0} \Sigma^{14i+5} \pi_{*}\mathsf{tmf}_{0}(2)/(a_{4})$$

Proof. We use the Tor spectral sequence

$$E_{*,*}^{2} = \operatorname{Tor}_{*,*}^{\pi_{*}(\operatorname{tmf}_{0}(2)_{(3)} \wedge_{\operatorname{tmf}_{(3)}} \operatorname{tmf}_{0}(2)_{(3)})} (\pi_{*}\operatorname{tmf}_{0}(2)_{(3)}, \pi_{*}\operatorname{tmf}_{0}(2)_{(3)}) \Rightarrow \pi_{*}\operatorname{THH}^{\operatorname{tmf}_{(3)}}(\operatorname{tmf}_{0}(2)_{(3)})$$

in order to calculate relative topological Hochschild homology. For determining

$$\operatorname{For}_{*,*}^{\mathbb{Z}_{(3)}[a_2,a_4,r]/I}(\mathbb{Z}_{(3)}[a_2,a_4],\mathbb{Z}_{(3)}[a_2,a_4])$$

we consider the free resolution of $\mathbb{Z}_{(3)}[a_2, a_4]$ as a $\mathbb{Z}_{(3)}[a_2, a_4, r]/I$ -module

$$\dots \longrightarrow \Sigma^{12} \mathbb{Z}_{(3)}[a_2, a_4, r]/I \xrightarrow{r^2 + a_2 r + a_4} \longrightarrow \Sigma^4 \mathbb{Z}_{(3)}[a_2, a_4, r]/I \xrightarrow{r} \mathbb{Z}_{(3)}[a_2, a_4, r]/I.$$

$$\text{ving} (-) \otimes_{\mathbb{Z}_{+^{1}}} [a_2, a_4, r]/I \xrightarrow{\mathbb{Z}_{(2)}} [a_2, a_4] \text{ yields}$$

Applying $(-) \otimes_{\mathbb{Z}_{(3)}[a_2,a_4,r]/I} \mathbb{Z}_{(3)}[a_2,a_4]$ yields

$$\dots \longrightarrow \Sigma^{12} \mathbb{Z}_{(3)}[a_2, a_4] \xrightarrow{a_4} \Sigma^4 \mathbb{Z}_{(3)}[a_2, a_4] \xrightarrow{0} \mathbb{Z}_{(3)}[a_2, a_4]$$

and hence we get

$$E_{n,*}^{2} = \begin{cases} \pi_{*}(\mathsf{tmf}_{0}(2)_{(3)}), & n = 0; \\ \Sigma^{4+12k} \mathbb{Z}_{(3)}[a_{2}], & n = 2k+1, k \ge 0; \\ 0, & \text{otherwise.} \end{cases}$$

We note that all non-trivial classes in positive filtration degree have an odd total degree. Since the edge morphism $\pi_*(\mathsf{tmf}_0(2)_{(3)}) \to \mathsf{THH}^{\mathsf{tmf}_{(3)}}_*(\mathsf{tmf}_0(2)_{(3)})$ is the unit, the classes in filtration degree zero cannot be hit by a differential and the spectral sequence collapses at the E^2 -page. Since $E_{n,m}^2 = E_{n,m}^\infty$ is a free $\mathbb{Z}_{(3)}$ -module for all n, m, there are no additive extensions.

As for the connective covers we have an equivalence of $\mathsf{Tmf}_{(3)}$ -modules $\mathsf{Tmf}_{(3)} \wedge T \simeq \mathsf{Tmf}_0(2)$ [Mat16b, §4.6] such that the map $\mathsf{Tmf}_{(3)} \to \mathsf{Tmf}_0(2)_{(3)}$ factors in the derived category of $\mathsf{Tmf}_{(3)}$ -modules as

 $\mathsf{Tmf}_{(3)} \to \mathsf{Tmf}_{(3)} \land \operatorname{cone}(\alpha_1) \to \mathsf{Tmf}_{(3)} \land T \simeq \mathsf{Tmf}_0(2)_{(3)}.$

Using the gap theorem, one can argue analogously to the proof of Lemma 2.1 to show that

$$\pi_*(\mathsf{Tmf}_0(2)_{(3)} \wedge_{\mathsf{Tmf}_{(3)}} \mathsf{Tmf}_0(2)_{(3)}) \cong \pi_*\mathsf{Tmf}_0(2)_{(3)}[r]/(r^3 + a_2r^2 + a_4r)$$

Theorem 2.3. There is an additive isomorphism

$$\mathsf{THH}^{\mathsf{Tmf}_{(3)}}(\mathsf{Tmf}_{0}(2)_{(3)}) \cong \pi_{*}\mathsf{Tmf}_{0}(2)_{(3)} \oplus \bigoplus_{i \ge 0} \Sigma^{14i+5} \mathbb{Z}_{(3)}[a_{2}] \oplus \bigoplus_{i \ge 1} \Sigma^{14i} \mathbb{Z}_{(3)}\{\frac{1}{a_{2}^{i}a_{4}}\}.$$

Proof. As above we have the following free resolution of $\pi_*(\mathsf{Tmf}_0(2)_{(3)})$ as a module over

$$C_* = \pi_*(\mathsf{Tmf}_0(2)_{(3)} \wedge_{\mathsf{Tmf}_{(3)}} \mathsf{Tmf}_0(2)_{(3)}):$$

 $\dots \xrightarrow{r} \Sigma^{12}C_* \xrightarrow{r^2 + a_2r + a_4} \Sigma^4 C_* \xrightarrow{r} C_* \longrightarrow \pi_* \mathsf{Tmf}_0(2)_{(3)} \longrightarrow 0.$

We get that the E^2 -page of the Tor spectral sequence

$$E_{*,*}^2 = \mathsf{Tor}_{*,*}^{C_*}(\pi_*\mathsf{Tmf}_0(2)_{(3)}, \pi_*\mathsf{Tmf}_0(2)_{(3)}) \Longrightarrow \pi_*\mathsf{THH}^{\mathsf{Tmf}_{(3)}}(\mathsf{Tmf}_0(2)_{(3)})$$

is given by

$$\begin{split} E_{n,*}^{2} &= \begin{cases} \pi_{*} \operatorname{Tmf}_{0}(2)_{(3)}, & n = 0; \\ \Sigma^{4+12k} \pi_{*} \operatorname{Tmf}_{0}(2)_{(3)} / a_{4}, & n = 2k+1, k \geqslant 0; \\ \ker \left(\Sigma^{12k} \pi_{*} \operatorname{Tmf}_{0}(2)_{(3)} \xrightarrow{\cdot a_{4}} \Sigma^{12(k-1)+4} \pi_{*} \operatorname{Tmf}_{0}(2)_{(3)} \right), & n = 2k, k > 0 \end{cases} \\ &= \begin{cases} \pi_{*} \operatorname{Tmf}_{0}(2)_{(3)}, & n = 0; \\ \Sigma^{4+12k} \mathbb{Z}_{(3)}[a_{2}], & n = 2k+1, k \geqslant 0; \\ \Sigma^{12k} \bigoplus_{n \geqslant 1} \mathbb{Z}_{(3)} \{\frac{1}{a_{n}^{n} a_{4}}\}, & n = 2k, k > 0. \end{cases} \end{split}$$

Since all non-trivial classes in positive filtration have an odd total degree, the spectral sequence collapses at the E^2 -page. There are no additive extensions, because the $E^{\infty} = E^2$ -page is a free $\mathbb{Z}_{(3)}$ -module.

Theorem 2.4. We have an additive isomorphism

$$\pi_*\mathsf{THH}^{\mathsf{tmf}_0(2)_{(3)}}(\mathsf{tmf}(2)_{(3)}) \cong \mathbb{Z}_{(3)}[\lambda_1, \lambda_2] \oplus \bigoplus_{i \ge 0} \Sigma^{10i+5} \mathbb{Z}_{(3)}[\lambda_1].$$

Proof. The map $\pi_* \mathsf{tmf}_0(2)_{(3)} \to \pi_* \mathsf{tmf}(2)_{(3)}$ is given by $\mathbb{Z}_{(3)}[\lambda_1 + \lambda_2, \lambda_1 \lambda_2] \to \mathbb{Z}_{(3)}[\lambda_1, \lambda_2]$. One easily sees that

$$\mathbb{Z}_{(3)}[\lambda_1,\lambda_2] \cong \mathbb{Z}_{(3)}[\lambda_1+\lambda_1,\lambda_1\lambda_2] \oplus \mathbb{Z}_{(3)}[\lambda_1+\lambda_2,\lambda_1\lambda_2]\lambda_1,$$

so $\pi_* \operatorname{tmf}(2)_{(3)}$ is a free $\pi_* \operatorname{tmf}_0(2)_{(3)}$ -module. We get

$$\pi_*(\mathsf{tmf}(2)_{(3)} \wedge_{\mathsf{tmf}_0(2)_{(3)}} \mathsf{tmf}(2)_{(3)}) = \pi_*\mathsf{tmf}(2)_{(3)} \otimes_{\pi_*\mathsf{tmf}_0(2)_{(3)}} \pi_*\mathsf{tmf}(2)_{(3)}$$
$$= \mathbb{Z}_{(3)}[\lambda_1, \lambda_2, a]/a^2 + \lambda_1 a - \lambda_2 a,$$

where $a = \eta_R(\lambda_1) - \lambda_1$. Let $C_* = \mathbb{Z}_{(3)}[\lambda_1, \lambda_2, a]/a^2 + \lambda_1 a - \lambda_2 a$. We have the following free resolution of $\pi_* \operatorname{tmf}(2)_{(3)}$ as a C_* -module:

$$\dots \xrightarrow{\cdot a} \Sigma^8 C_* \xrightarrow{\cdot (a+\lambda_1-\lambda_2)} \Sigma^4 C_* \xrightarrow{\cdot a} C_* \longrightarrow \pi_* \mathsf{tmf}(2)_{(3)} \longrightarrow 0$$

Thus, the E^2 -page of the Tor spectral sequence

$$E_{*,*}^2 = \mathsf{Tor}_{*,*}^{C_*}(\pi_*\mathsf{tmf}(2)_{(3)}, \pi_*\mathsf{tmf}(2)_{(3)}) \Longrightarrow \pi_*\mathsf{THH}^{\mathsf{tmf}_0(2)_{(3)}}(\mathsf{tmf}(2)_{(3)})$$

is given by

$$E_{n,*}^{2} = \begin{cases} \pi_{*} \mathsf{tmf}(2)_{(3)}, & n = 0; \\ \Sigma^{8k+4} \pi_{*} \mathsf{tmf}(2)_{(3)} / (\lambda_{1} - \lambda_{2}), & n = 2k+1, k \ge 0; \\ 0, & \text{otherwise.} \end{cases}$$

Since the non-trivial classes in positive filtration have odd total degree, the spectral sequence collapses at the E^2 -page. There are no additive extensions, because the $E^2 = E^{\infty}$ -page is a free $\mathbb{Z}_{(3)}$ -module.

2.3. The discriminant map. If $A \to B$ is a *G*-Galois extension for a finite group *G*, then the discriminant map $\mathfrak{d}_{B|A} \colon B \to F_A(B, A)$ is a weak equivalence [Rog08a, Proposition 6.4.7]. The map $\mathfrak{d}_{B|A}$ is right adjoint to the *trace pairing*

$$B \wedge_A B \xrightarrow{\mu} B \xrightarrow{tr} A$$

where $(A \to B) \circ tr$ is homotopic to $\sum_{g \in G} g$ and μ is the multiplication map of B. Rognes proposes that the deviation of $\mathfrak{d}_{B|A}$ from being a weak equivalence might be used for measuring ramification. We show in the examples of $\ell_p \to ku_p$ and $ko \to ku$ that \mathfrak{d} does detect ramification, but it does not give any information about the type of ramification.

Proposition 2.5. There is a cofiber sequence

$$ku_p \xrightarrow{\mathfrak{o}_{ku_p|\ell_p}} F_{\ell_p}(ku_p,\ell_p) \longrightarrow \bigvee_{i=1}^{p-2} \Sigma^{-2p+2i+2} H\mathbb{Z}_p$$

Proof. We know that $F_{\ell_p}(ku_p, \ell_p)$ can be decomposed as

$$F_{\ell_p}(ku_p,\ell_p) \simeq F_{\ell_p}(\bigvee_{i=0}^{p-2} \Sigma^{2i}\ell_p,\ell_p) \cong \prod_{i=0}^{p-2} \Sigma^{-2i}\ell_p \simeq \bigvee_{i=0}^{p-2} \Sigma^{-2i}\ell_p$$

and $\mathfrak{d}_{ku_n|\ell_n}$ can be identified with a map

$$\bigvee_{i=0}^{p-2} \Sigma^{2i} \ell_p \to \bigvee_{i=0}^{p-2} \Sigma^{-2i} \ell_p$$

As $\pi_* k u_p$ is a free graded $\pi_* \ell_p$ -module, we can calculate the effect of $\mathfrak{d}_{k u_p | \ell_p}$ algebraically via the trace pairing: The element $\Sigma^{2i} 1 \in \pi_* \Sigma^{2i} \ell_p$ maps an element u^i to $tr(u^{i+j})$ and this is

$$tr(u^{i+j}) = \begin{cases} 0, & (p-1) \nmid i+j, \\ (p-1)u^{i+j}, & (p-1) \mid i+j. \end{cases}$$

Hence on the level of homotopy groups $\mathfrak{d}_{ku_p|\ell_p}$ maps $1 \in \pi_0 \Sigma^0 \ell_p$ to $(p-1) \cdot 1 \in \pi_* \Sigma^0 \ell_p = \pi_* \ell_p$ and it maps $\Sigma^{2i} 1 \in \pi_* \Sigma^{2i} \ell_p$ via multiplication with $(p-1)v_1 = (p-1)u^{p-1}$ to $\pi_* \Sigma^{-2p+2i+2} \ell_p$. On the summands $\Sigma^{2i} \ell_p$ we get the following maps:



As (p-1) is a unit in $\pi_0(\ell_p)$ the cofiber of

$$\Sigma^{2i}\ell_p \xrightarrow{(p-1)v_1} \Sigma^{-2p+2i+2}\ell_p$$

is $\Sigma^{-2p+2i+2}H\mathbb{Z}_p$.

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Note that $ko \simeq \tau_{\geq 0} ku^{hC_2}$, but as the trace map $tr: ku \to ku^{hC_2}$ has the connective spectrum ku as a source, it factors through $\tau_{\geq 0} ku^{hC_2} \simeq ko$ and we obtain a discriminant $\mathfrak{d}_{ku|ko}: ku \to F_{ko}(ku, ko)$. We fix notation for $\pi_* ko$ as

$$\pi_* ko = \mathbb{Z}[\eta, y, \omega] / (2\eta, \eta^3, \eta y, y^2 - 4\omega)$$

with $|\eta| = 1$, |y| = 4 and $|\omega| = 8$.

Proposition 2.6. There is a cofiber sequence $ku \xrightarrow{\mathfrak{d}_{ku|ko}} F_{ko}(ku, ko) \longrightarrow \Sigma^{-2} H\mathbb{Z}$.

Proof. The cofiber sequence $\Sigma ko \xrightarrow{\eta} ko \xrightarrow{c} ku \xrightarrow{\delta} \Sigma^2 ko \xrightarrow{\eta} \Sigma ko$ induces a cofiber sequence

$$F_{ko}(\Sigma ko, ko) \xrightarrow{\eta} F_{ko}(\Sigma^2 ko, ko) \xrightarrow{\delta} F_{ko}(ku, ko) \xrightarrow{c} F_{ko}(ko, ko) \xrightarrow{\eta} F_{ko}(\Sigma ko, ko)$$

which is equivalent to

$$\Sigma^{-1}ko \xrightarrow{\eta} \Sigma^{-2}ko \xrightarrow{\delta} F_{ko}(ku, ko) \xrightarrow{c} ko \xrightarrow{\eta} \Sigma^{-1}ko$$

This is the 2-fold desuspension of the cofiber sequence of ku and hence

$$F_{ko}(ku, ko) \simeq \Sigma^{-2} ku$$

We consider the composition $c_* \circ \mathfrak{d}_{ku|ko} : ku \to F_{ko}(ku, ko) \to F_{ko}(ku, ku)$. As c_* is part of the cofiber sequence

$$F_{ko}(ku, \Sigma ko) \xrightarrow{\eta_*} F_{ko}(ku, ko) \xrightarrow{c_*} F_{ko}(ku, ku)$$

and as η is trivial on ku, we know that c_* induces a monomorphism on the level of homotopy groups.

As $\mathfrak{d}_{ku|ko}$ is adjoint to the trace pairing, the composite

$$\pi_*ku \longrightarrow \pi_*F_{ko}(ku, ko) \longrightarrow \pi_*F_{ko}(ku, ku)$$

can be identified with

$$\pi_*ku \longrightarrow \pi_*F_{ko}(ku, ku) \xrightarrow{(\mathrm{id}+t)_*} F_{ko}(ku, ku)$$

where t denotes the generator of C_2 and the first map is adjoint to the multiplication $ku \wedge_{ko} ku \rightarrow ku$.

The target of c_* is $F_{ko}(ku, ku) \simeq F_{ku}(ku \wedge_{ko} ku, ku)$, and we know by work of the first author, documented in [DLR, Proof of Lemma 0.1] that

$$\pi_* F_{ku}(ku \wedge_{ko} ku, ku) \cong \operatorname{Hom}_{ku_*}(ku_*[s]/(s^2 - su), \Sigma^{-*}ku_*),$$

so we can control the effect of $c_* \circ \mathfrak{d}_{ku|ko}$ on homotopy groups.

Note that t induces a ku-linear map $t_* : ku \to t^*ku$, where t^*ku is the ku-module given by restriction of scalars along t.

As $t^2 = id$, we therefore obtain

$$F_{ko}(ku, ku) \xrightarrow{t_*} F_{ko}(ku, t^*ku)$$

and a commutative diagram

Here, β induces the map on π_* that sends an $f: (ku \wedge_{ko} ku)_* \to \Sigma^{-i} ku_*$ to

$$(ku \wedge_{ko} ku)_* \xrightarrow{(t \wedge \mathrm{id})_*} (ku \wedge_{ko} ku)_* \xrightarrow{f} \Sigma^{-i} ku_* \xrightarrow{t} \Sigma^{-i} ku_*$$

If we denote the right unit $\eta_R \colon ku \to ku \wedge_{ko} ku$ applied to u by u_r , then we have the relation $2s + u_r = u$. As $(t \wedge id)_*(u) = -u$ and $(t \wedge id)_*(u_r) = u_r$, this implies that

$$(t \wedge \mathrm{id})_*(2s) = 2s - 2u$$

Torsionfreeness then yields $(t \wedge id)_*(s) = s - u$.

The adjoint of the multiplication map $\pi_* ku \to \pi_* F_{ko}(ku, ku)$ maps u^i to the map that sends 1 to $\Sigma^{-2i}u^i$ and s to zero. Therefore, the composite $c_* \circ \mathfrak{d}_{ku|ko}$ maps u^i to the map with values $1 \mapsto \Sigma^{-2i}(u^i + (-1)^i u^i)$ and

$$s \mapsto (s-u)u^i \mapsto -t(u^{i+1}) = (-1)^i u^{i+1}$$

In order to understand the effect of $\mathfrak{d}_{ku|ko}$ we consider the diagram

$$\pi_* ku \xrightarrow{(\mathfrak{d}_{ku|ko})_*} \pi_* F_{ko}(ku, ko) \cong \pi_{*+2}(ku) \xrightarrow{c_*} \pi_* F_{ko}(ku, ku) \cong \pi_*(\Sigma^{-2}ku \lor ku)$$

$$c^* \downarrow \qquad c^* \downarrow$$

$$\pi_* F_{ko}(ko, ko) \cong \pi_*(ko) \xrightarrow{c_*} \pi_* F_{ko}(ko, ku) \cong \pi_*(ku)$$

where we can identify $c^* \colon \pi_* F_{ko}(ku, ko) \cong \pi_{*+2}(ku) \to \pi_*(ko)$ with $\pi_* \Sigma^{-2} \delta$.

The application of c^* gives the restriction to the unit $c \colon ko \to ku$. Say $(\mathfrak{d}_{ku|ko})_*(u^2) = x \in \pi_6(ku)$. Then $\pi_*\Sigma^{-2}\delta(x) = \lambda y$, and as $c_*(y) = 2u^2$, we obtain that $c^*(\mathfrak{d}_{ku|ko})_*(u^2) = \Sigma^{-4}y$ and therefore $(\mathfrak{d}_{ku}, y_k)_*(u^2) = u^3$

Similarly
$$c^*(\mathfrak{d}_{ku|ko})_*(u^4) = \Sigma^{-8} 2\omega$$
 and $(\mathfrak{d}_{ku|ko})_*(u^4) = u^5$ and in general
 $(\mathfrak{d}_{ku|ko})_*(u^{2i}) = u^{2i+1}.$

Restriction to the unit of the odd powers of u gives zero.

All the u^i send s to $\pm u^{i+1}$ under $c_* \circ (\mathfrak{d}_{ku|ko})_*$, so also the odd powers of u have to hit a generator under $(\mathfrak{d}_{ku|ko})_*$, so as a map from ku to $\Sigma^{-2}ku$ the map $\mathfrak{d}_{ku|ko}$ has cofiber $\Sigma^{-2}H\mathbb{Z}$.

3. Describing ramification

3.1. Log-étaleness. It is shown in [RSS15] and [Sag14] that $\ell \to ku_{(p)}$ is log-étale with respect to the log structures that are generated by v_1 and by u. We will use the class $u \in \pi_2 ku_{(2)}$ in order to define a pre-log structure for $ko_{(2)} \to ku_{(2)}$ and show that $ko_{(2)} \to ku_{(2)}$ is not log-étale with respect to this pre-log structure. This indicates that the map is not tamely ramified. We use the notation from [Sag14].

Let ω denote the Bott element $\omega \in \pi_8 ko_{(2)}$. The complexification map sends ω to u^4 . By [Sag14, Lemma 6.2] we have an exact sequence

 $\pi_1 \mathsf{TAQ}^C(ku_{(2)}) \searrow$

$$\longrightarrow \pi_0 \Big(k u_{(2)} \wedge \gamma(D(u)) / \gamma(D(w)) \Big) \longrightarrow \pi_0 \mathsf{TAQ}^{(ko_{(2)}, D(w))}(k u_{(2)}, D(u)) \longrightarrow \pi_0 \mathsf{TAQ}^C(k u_{(2)}),$$

where $C = ko_{(2)} \wedge_{S^{\mathcal{J}}D(w)} S^{\mathcal{J}}D(u)$ and D(u), $D(\omega)$ are the pre-log structures for the elements u and ω as in [Sag14, Construction 4.2]. We have that $\gamma(D(w))$ and $\gamma(D(u))$ have the homotopy type of the sphere and that $\gamma(D(w)) \rightarrow \gamma(D(u))$ is multiplication by 4. Therefore we get

$$\pi_0\Big(ku_{(2)}\wedge\gamma(D(u))/\gamma(D(w))\Big)=\mathbb{Z}/4\mathbb{Z}.$$

We want to show that $\pi_1 \mathsf{TAQ}^C(ku_{(2)}) = 0 = \pi_0 \mathsf{TAQ}^C(ku_{(2)})$. By [Bas99, Lemma 8.2] it suffices to show that $C \to ku_{(2)}$ is an 1-equivalence. Since $\pi_1(ku_{(2)}) = 0$, it is enough to show that the map is an isomorphism on π_0 . Since $S^{\mathcal{J}}D(w)$ and $S^{\mathcal{J}}D(u)$ are concentrated in nonnegative \mathcal{J} -space degrees by [RSS15, Example 6.8], they are connective. Thus, it is enough to show that $S^{\mathcal{J}}D(w) \to S^{\mathcal{J}}D(u)$ induces an isomorphism on π_0 . For this, we only have to prove that $H_0(S^{\mathcal{J}}D(w),\mathbb{Z}) \to H_0(S^{\mathcal{J}}D(u),\mathbb{Z})$ is an isomorphism. Since this map is a ring map we only need to know that both sides are \mathbb{Z} . This follows from [RSS18, Proposition 5.2, Corollary 5.3]. Hence we obtain the following result:

Theorem 3.1. The map $(ko_{(2)}, D(\omega)) \rightarrow (ku_{(2)}, D(u))$ is not log-étale.

One could try to distinguish between tame and wild ramification by testing for log-étaleness. In many examples, however, it is less obvious what a suitable log structure would be.

3.2. Wild ramification and Tate cohomology. In the algebraic context of Galois extensions of number fields and corresponding extension of number rings tame ramification yields a normal basis and a surjective trace map. Both facts are actually also sufficient in order to distinguish tame from wild ramification. For structured ring spectra it does not work to impose these properties on the level of homotopy groups, because even for finite faithful Galois extensions these would not hold. Instead we propose a different criterion that uses the Tate construction.

Remark 3.2. Let G be a finite group. Usually one calls a G-module M cohomologically trivial, if $\hat{H}^i(H; M) = 0$, for all $i \in \mathbb{Z}$ and all H < G. If M is a commutative ring S, however, it suffices to require $\hat{H}^i(G; S) = 0$ for all $i \in \mathbb{Z}$: In particular, $\hat{H}^0(G; S) = 0$, and hence the trace map $tr_G: S \to S^G$ is surjective. Thus 1_{S^G} is in the image of the norm, say $N_G[x] = 1_{S^G}$ for $[x] \in S_G$. If H < G then we consider the diagram

$$\begin{array}{c} H^{0}(G;S) = S^{G} \xrightarrow{i^{*}} H^{0}(H;S) = S^{H} \\ & \searrow \\ N_{G} \\ \downarrow \\ H_{0}(G;S) = S_{G} \xrightarrow{tr_{H}^{G}} H_{0}(H;S) = S_{H} \end{array}$$

and therefore we can express can express 1_{S^H} as

$$1_{S^H} = i^*(1_{S^G}) = i^* N_G[x] = N_H tr_H^G[x],$$

so 1_{S^H} is in the image of N_H and $\hat{H}^0(H;S) = 0$. But $\hat{H}^*(H;S)$ is a graded commutative ring with unit $[1_{S^H}] = 0$, and thus $\hat{H}^*(H;S) = 0$.

The same argument shows that the surjectivity of the trace map suffices for being cohomologically trivial.

Even if $A \to B$ is a *G*-Galois extension of ring spectra, it is not true, that this implies that *B* is faithful as an *A*-module. An example, due to Wieland is the *C*₂-Galois extension $F((BC_2)_+, H\mathbb{F}_2) \to F((EC_2)_+, H\mathbb{F}_2) \simeq H\mathbb{F}_2$ which is not faithful: The $F((BC_2)_+, H\mathbb{F}_2)$ module spectrum $(H\mathbb{F}_2)^{tC_2}$ is not trivial, but $H\mathbb{F}_2 \wedge_{F((BC_2)_+, H\mathbb{F}_2)} (H\mathbb{F}_2)^{tC_2} \sim *$.

Note that for a map $A \to B$ between connective commutative ring spectra with a finite group G acting on B via commutative A-algebra maps it makes sense to replace the usual homotopy fixed point condition by the condition that A is weakly equivalent to $\tau_{\geq 0}B^{hG}$. In many examples B^{hG} won't be connective. The map $A \to B$ factors through $A \to B^{hG} \to B$, but as A is connective, we can consider the induced map on connective covers and obtain a map of commutative ring spectra

$$\tau_{\geq 0}A = A \to \tau_{\geq 0}B^{hG} \to \tau_{\geq 0}B = B,$$

that turns $\tau_{\geq 0} B^{hG}$ into a commutative A-algebra spectrum.

For any spectrum X we denote by $\tau_{<0}X$ the cofiber of the map $\tau_{\geq 0}X \to X$.

The following result is probably well-known.

Lemma 3.3. Let G be a finite group and let e be a naive connective G-spectrum. Then

$$\tau_{\geq 0}e^{hG} \to e^{hG} \to \tau_{<0}e^{tG}$$

is a cofiber sequence and $\tau_{<0}e^{tG} \simeq \tau_{<0}e^{hG}$.

Proof. We consider the norm sequence

$$e_{hG} \xrightarrow{N} e^{hG} \longrightarrow e^{tG}$$

As e_{hG} is a connective spectrum, we have that $\pi_{-1}e_{hG} = 0$. Hence, applying $\tau_{\geq 0}$ still gives rise to a cofiber sequence

$$\tau_{\geqslant 0} e_{hG} = e_{hG} \xrightarrow{\tau_{\geqslant 0} N} \tau_{\geqslant 0} e^{hG} \longrightarrow \tau_{\geqslant 0} e^{tG}.$$

We combine the norm cofiber sequences with the defining cofiber sequence of $\tau_{<0}$ and obtain



Thus $\tau_{<0}e^{hG} \simeq \tau_{<0}e^{tG}$ and the cofiber sequence in the second row then yields the claim. \Box

Remark 3.4. In many cases, if $B^{tG} \not\simeq *$, then $\pi_*(B^{tG})$ is actually periodic. As the canonical Künneth map

$$\pi_*(B^{tG}) \otimes_{\pi_*(B^{hG})} \pi_*(B) \to \pi_*(B^{tG} \wedge_{B^{hG}} B)$$

is a map of graded commutative rings and as $\pi_*(B^{tG}) \cong \pi_*(B^{hG})$ in negative degrees, a periodicity generator in a negative degree would map to zero in π_*B for connective B and hence $\pi_*(B^{tG}) \otimes_{\pi_*(B^{hG})} \pi_*(B)$ is the zero ring. But then also $\pi_*(B^{tG} \wedge_{B^{hG}} B) \cong 0$ and

$$B^{tG} \wedge_{B^{hG}} B \simeq *.$$

Therefore B would not be a faithful B^{hG} -module in these cases. This emphasizes the importance of replacing the condition that A be weakly equivalent to B^{hG} by the requirement that $A \simeq \tau_{\geq 0}(B^{hG})$.

From Lemma 3.3 we also know that in order to show that $B^{tG} \not\simeq *$ for connective B it is sufficient to show that $\tau_{\leq 0}B^{hG}$ is not trivial.

In [Rog08a, Proposition 6.3.3] Rognes assumes that $A \simeq B^{hG}$, but that assumption is actually not needed for the following:

Theorem 3.5. Assume that G is a finite group, B is a cofibrant commutative A-algebra on which G acts via maps of commutative A-algebras. If B is dualizable and faithful as an A-module and if

$$h: B \wedge_A B \xrightarrow{\sim} F(G_+, B),$$

then $B^{tG} \simeq *$.

Proof. We consider the following commutative diagram in which the columns arise from the norm cofiber sequence.

As $F(G_+, B)$ is free over G, we have $F(G_+, B)^{tG} \simeq *$ and the norm map is an equivalence between $F(G_+, B)_{hG}$ and $F(G_+, B)^{hG}$. The map h is equivariant and hence it induces weak equivalences on homotopy orbits, homotopy fixed points and the Tate construction. This shows that

$$N: (B \wedge_A B)_{hG} \to (B \wedge_A B)^{hG}$$

is a weak equivalence.

The left horizontal map is a weak equivalence because B is dualizable as an A-module. The map $B \wedge_A (B_{hG}) \to (B \wedge_A B)_{hG}$ is always a weak equivalence. Therefore the map

$$B \wedge_A N \colon B \wedge_A (B_{hG}) \to B \wedge_A (B^{hG})$$

is a weak equivalence and thus $B \wedge_A (B^{tG}) \simeq *$. As B is a faithful A-module, this implies that $B^{tG} \simeq *$.

Remark 3.6. Therefore, if G is a finite group, B is a cofibrant commutative A-algebra on which G acts via maps of commutative A-algebras. If B is dualizable and faithful as an A-module and if $B^{tG} \not\simeq *$, then we know that $h: B \wedge_A B \to F(G_+, B)$ cannot be a weak equivalence, and hence $A \to B$ is ramified.

We propose a definition of tame and wild ramification for commutative ring spectra and justify our definition by investigating several examples.

In the following we denote by $\tau_{\geq 0} X$ the connective cover of a spectrum X.

Definition 3.7. Assume that $A \to B$ is a map of commutative ring spectra such that G acts on B via commutative A-algebra maps and B is faithful and dualizable as an A-module.

If $A \simeq B^{hG}$ (or $A \simeq \tau_{\geq 0} B^{hG}$ if A and B are connective), then we call $A \to B$ tamely ramified if $B^{tG} \simeq *$. Otherwise, $A \to B$ is wildly ramified.

Remark 3.8. To compute the homotopy of B^{tG} we use the Tate spectral sequence

$$E_{n,m}^2 = \hat{H}^{-n}(G; \pi_m(B)) \Longrightarrow \pi_{n+m} B^{tG}$$

which is of standard homological type, multiplicative and conditionally convergent. In particular by [Boa99, Theorem 8.2], it converges strongly if it collapses at a finite stage.

We will now investigate our criterion for wild ramification in examples. First, we establish faithfulness:

Lemma 3.9. The map $\operatorname{tmf}_0(2)_{(3)} \to \operatorname{tmf}(2)_{(3)}$ identifies $\operatorname{tmf}(2)_{(3)}$ as a faithful $\operatorname{tmf}_0(2)_{(3)}$ -module.

Proof. For the map $\mathsf{tmf}_0(2)_{(3)} \to \mathsf{tmf}(2)_{(3)}$ we know that C_2 acts on $\mathsf{tmf}(2)_{(3)}$ via commutative $\mathsf{tmf}_0(2)_{(3)}$ -algebra maps and that $\mathsf{tmf}_0(2)_{(3)} \simeq \tau_{\geq 0}(\mathsf{tmf}(2)_{(3)}^{hC_2})$. The trace map $tr: \mathsf{tmf}(2)_{(3)} \to \mathsf{tmf}(2)_{(3)}$

 $\operatorname{tmf}(2)_{(3)}^{hC_2}$ factor through $\tau_{\geq 0}(\operatorname{tmf}(2)_{(3)}^{hC_2}) \simeq \operatorname{tmf}_0(2)_{(3)}$, because $\operatorname{tmf}(2)_{(3)}$ is connective. As in [Rog08a, Lemma 6.4.3] one can show that the composite

$$\mathsf{tmf}_{0}(2)_{(3)} \simeq \tau_{\geq 0}(\mathsf{tmf}(2)_{(3)}^{hC_{2}}) \longrightarrow \mathsf{tmf}(2)_{(3)} \xrightarrow{tr} \tau_{\geq 0}(\mathsf{tmf}(2)_{(3)}^{hC_{2}}) \simeq \mathsf{tmf}_{0}(2)_{(3)}$$

is homotopic to the map that is the multiplication by $|C_2| = 2$. As 2 is invertible in $\pi_0 \operatorname{tmf}_0(2)_{(3)}$, the trace map $tr: \operatorname{tmf}(2)_{(3)} \to \operatorname{tmf}_0(2)_{(3)}$ is a split surjective map of $\operatorname{tmf}_0(2)_{(3)}$ -modules and hence $\operatorname{tmf}_0(2)_{(3)} \to \operatorname{tmf}(2)_{(3)}$ is faithful.

Lemma 3.10. The spectrum $tmf_0(2)_{(3)}$ is faithful as a $tmf_{(3)}$ -module spectrum.

Proof. We already mentioned Behrens' identification [Beh06, Lemma 2] $\operatorname{tmf}_0(2)_{(3)} \simeq \operatorname{tmf}_{(3)} \wedge T$ where $T = S^0 \cup_{\alpha_1} e^4 \cup_{\alpha_1} e^8$ with $\alpha_1 \in (\pi_3 S)_{(3)}$. Note that α_1 is nilpotent of order 2 because $(\pi_6 S)_{(3)} = 0$.

Assume that M is a $tmf_{(3)}$ -module with

$$* \simeq M \wedge_{\mathsf{tmf}_{(3)}} \mathsf{tmf}_0(2)_{(3)} \simeq M \wedge T.$$

Then the cofiber sequences

$$S^0 \longrightarrow T \longrightarrow \Sigma^4 \operatorname{cone}(\alpha_1) \text{ and } \operatorname{cone}(\alpha_1) \longrightarrow T \longrightarrow S^8$$

imply that $\Sigma^4 \operatorname{cone}(\alpha_1) \wedge M \simeq \Sigma M$ and $\Sigma^8 M \simeq \Sigma \operatorname{cone}(\alpha_1) \wedge M$ and therefore

$$\Sigma^{10}M \simeq M.$$

The equivalence is induced by a class in $\pi_{10}S_{(3)} \cong \mathbb{Z}/3\mathbb{Z}\{\beta_1\}$. As this is nilpotent, we get that $M \simeq *$.

Remark 3.11. It is known that $ko \to ku$ is faithful [Rog08a, Proposition 5.3.1] and dualizable and it is clear that $\ell \to ku_{(p)}$ is faithful and dualizable as the inclusion of a summand. As $\mathsf{tmf}_1(3)_{(2)}$ can be identified with $\mathsf{tmf}_{(2)} \land DA(1)$ as a $\mathsf{tmf}_{(2)}$ -module [Mat16b, Theorem 4.12], where DA(1) is a finite cell complex realizing the double of $A(1) = \langle Sq^1, Sq^2 \rangle$, it is dualizable. An argument as in [Rog08a, Proof of Proposition 5.4.5] shows that $\mathsf{tmf}_{(2)} \to \mathsf{tmf}_1(3)_{(2)}$ is faithful.

At the moment we don't know whether $\mathsf{tmf}_0(3)_{(2)} \to \mathsf{tmf}_1(3)_{(2)}$ is faithful. The diagram



commutes, so if M is a $\mathsf{tmf}_0(3)_{(2)}$ -module spectrum with $M \wedge_{\mathsf{tmf}_0(3)_{(2)}} \mathsf{tmf}_1(3)_{(2)} \simeq *$, then multiplication by 2 is a trivial self-map on M.

Meier shows [Mei] that $\mathsf{tmf}_1(3)$ is not perfect as a $\mathsf{tmf}_0(3)$ -module, hence $\mathsf{tmf}_1(3)$ is not a dualizable $\mathsf{tmf}_0(3)$ -module.

Meier also proves that $\mathsf{tmf}[\frac{1}{n}] \to \mathsf{tmf}(n)$ is dualizable for all n. By combining his result with Lemma 3.9 and Lemma 3.10 we obtain that $\mathsf{tmf}(2)_{(3)}$ is dualizable and faithful as a $\mathsf{tmf}_{(3)}$ -module.

We show that the extensions $\mathsf{tmf}_0(3)_{(2)} \to \mathsf{tmf}_1(3)_{(2)}$ and $\mathsf{tmf}_{(3)} \to \mathsf{tmf}(2)_{(3)}$ have non-trivial Tate spectra. For ku the Tate spectrum with respect to the complex conjugation C_2 -action satisfies

$$ku^{tC_2} \simeq \bigvee_{i \in \mathbb{Z}} \Sigma^{4i} H\mathbb{Z}/2\mathbb{Z}.$$

This result is due to Rognes (compare [Rog08a, §5.3]).

Theorem 3.12. For $tmf_1(3)_{(2)}$ with its C_2 -action we obtain an equivalence of spectra

$$\mathsf{tmf}_1(3)_{(2)}^{tC_2} \simeq \bigvee_{i \in \mathbb{Z}} \Sigma^{8i} H\mathbb{Z}/2\mathbb{Z}.$$

Proof. We use the calculations in [MR09]. They compute the homotopy fixed point spectral sequence

$$E_{n,m}^2 = H^{-n}(C_2; \pi_m \mathsf{TMF}_1(3)_{(2)}) \Longrightarrow \pi_{n+m} \mathsf{TMF}_0(3)_{(2)},$$

where $\pi_* \mathsf{TMF}_1(3)_{(2)} = \mathbb{Z}_{(2)}[a_1, a_3][\Delta^{-1}]$ with $\Delta = a_3^3(a_1^3 - 27a_3)$. From their computations we deduce the following behaviour of the Tate spectral sequence

(3.1)
$$E_{n,m}^2 = \hat{H}^{-n}(C_2; \pi_m \mathsf{TMF}_1(3)_{(2)}) \Longrightarrow \pi_{n+m} \mathsf{TMF}_1(3)_{(2)}^{tC_2}:$$

Let $R_{n,m}$ be the bigraded ring $\mathbb{Z}/2[a_1, a_3][\Delta^{-1}][\zeta^{\pm}]$ with $|\zeta| = (-1, 0)$. If we assign odd weight to a_1, a_3 and ζ , then the E^2 -page of the Tate spectral sequence is the even part of $R_{n,m}$. Alternatively, it is given by

$$E_{*,*}^2 = S_*[\Delta^{-1}][x^{\pm}],$$

where S_* is the subalgebra of $\mathbb{Z}/2\mathbb{Z}[a_1, a_3]$ generated by a_1^2, a_1a_3, a_3^2 , and where $x = \zeta a_3^3 \in E_{-1,18}$. Note that a_3^2 is invertible in this ring with $a_3^{-2} = ((a_1a_3)a_1^2 - 27a_3^2)\Delta^{-1}$. By Mahowald-Rezk's computations the first non-trivial differential is d^3 and we have

$$d^{3}(a_{1}^{2}) = (x(a_{1}a_{3})a_{3}^{-4})^{3}, \qquad d^{3}(a_{1}a_{3}) = 0, \qquad d^{3}(a_{3}^{2}) = x^{3}(a_{1}a_{3})a_{3}^{-8},$$

$$d^{3}(x) = 0, \qquad d^{3}(\Delta^{-1}) = 0.$$

Using the Leibniz rule we get that the class $c_{n,m,k,l,i} = (a_1^2)^n (a_1 a_3)^m (a_3^2)^k \Delta^{-l} x^i$ with $n, m, k, l \in \mathbb{N}$ and $i \in \mathbb{Z}$ has differential

$$d^{3}(c_{n,m,k,l,i}) = (n+k)x^{3}(a_{1}a_{3})a_{3}^{-10}c_{n,m,k,l,i}$$

It follows that ker d^3 is generated as \mathbb{F}_2 -vector space by the classes $c_{n,m,k,l,i}$ with n + k = 0 in \mathbb{F}_2 . We claim that

$$E_{*,*}^4 \cong \mathbb{F}_2[x^\pm, \Delta^\pm].$$

To see this, note the following: If n + k = 0 in \mathbb{F}_2 and m > 0, then $c_{n,m,k,l,i}$ is zero in $E_{*,*}^4$ because

$$d^{3}(c_{n,m-1,k+5,l,i-3}) = c_{n,m,k,l,i}$$

If n + k = 0 in \mathbb{F}_2 and n, k > 0, then we have $c_{n,0,k,l,i} = c_{n-1,2,k-1,l,i}$. This is in the image of d^3 , because n - 1 + k - 1 = 0 in \mathbb{F}_2 and 2 > 0. If n = 0 in \mathbb{F}_2 and n > 0, then

$$c_{n,0,0,l,i} = (a_1^2)^n \Delta^{-l} x^i$$

= $(a_1 a_3)^2 (a_1^2)^{n-1} a_3^{-2} \Delta^{-l} x^i$
= $(a_1 a_3)^2 (a_1^2)^{n-1} ((a_1 a_3) a_1^2 + a_3^2) \Delta^{-1} \Delta^{-l} x^i$
= $c_{n,3,0,l+1,i} + c_{n-1,2,1,l+1,i}$,

and both of these summands are in the image of d^3 . Furthermore, note that in $E_{*,*}^4$ we have

$$\Delta = (a_1 a_3)^3 + a_3^4 = c_{0,3,0,0,0} + a_3^4 = a_3^4$$

This implies that for k = 0 in \mathbb{F}_2 we have

$$c_{0,0,k,l,i} = (a_3^4)^{\frac{k}{2}} \Delta^{-l} x^i \equiv \Delta^{-l + \frac{k}{2}} x^i$$

in $E_{*,*}^4$. We thus get a surjective map $\mathbb{F}_2[x^{\pm}, \Delta^{\pm}] \to E_{*,*}^4$, which is injective, because the classes $\Delta^l x^i$ for $l, i \in \mathbb{Z}$ are not divisible by $(a_1 a_3)$ in $S_*[\Delta^{-1}][x^{\pm}]$.

From Mahowald-Rezk's computations we get that the next non-trivial differential is d^7 and that we have

$$d^{7}(x) = 0$$
 and $d^{7}(\Delta) = x^{7}\Delta^{-4}$.

This gives $E_{*,*}^8 = 0$.

We now want to determine the behaviour of the Tate spectral sequence

(3.2)
$$E_{n,m}^2 = \hat{H}^{-n}(C_2; \pi_m \mathsf{tmf}_1(3)_{(2)}) \Longrightarrow \pi_{n+m} \mathsf{tmf}_1(3)_{(2)}^{tC_2}$$

If we assign again odd weight to a_1 , a_3 and ζ , then the E^2 -page is the even part of

 $\mathbb{Z}/2\mathbb{Z}[a_1,a_3][\zeta^{\pm}],$

and one sees that the map of spectral sequences from (3.2) to (3.1) is injective. We get that d^3 is the first non-trivial differential in (3.2) and that we have

$$d^{3}(a_{1}a_{3}) = 0, \qquad d^{3}(a_{1}^{2}) = (a_{1}\zeta)^{3},$$

$$d^{3}(a_{3}^{2}) = a_{1}a_{3}^{2}\zeta^{3}, \qquad d^{3}(a_{3}\zeta) = (a_{1}a_{3})\zeta^{4},$$

$$d^{3}(a_{1}\zeta) = 0, \qquad d^{3}(\zeta^{2}) = a_{1}\zeta^{5}.$$

Note that an \mathbb{F}_2 -basis of the E^3 -page is given by the classes

$$\begin{aligned} d_{n,m,i} &= (a_1^2)^n (a_3^2)^m (\zeta^2)^i, \\ e_{n,m,i} &= (a_1^2)^n (a_1 a_3) (a_3^2)^m (\zeta^2)^i, \\ f_{n,m,i} &= (a_1^2)^n (a_3^2)^m (a_1 \zeta) (\zeta^2)^i, \\ g_{n,m,i} &= (a_1^2)^n (a_3^2)^m (a_3 \zeta) (\zeta^2)^i, \end{aligned}$$

for $n, m \in \mathbb{N}$ and $i \in \mathbb{Z}$.

The d^3 -differential on these classes is given by

$$d^{3}(d_{n,m,i}) = (n+m+i) \cdot f_{n,m,i+1},$$

$$d^{3}(e_{n,m,i}) = (n+m+i) \cdot g_{n+1,m,i+1},$$

$$d^{3}(f_{n,m,i}) = (n+m+i) \cdot d_{n+1,m,i+2},$$

$$d^{3}(g_{n,m,i}) = (n+m+i+1) \cdot e_{n,m,i+2}.$$

We get

$$E_{*,*}^4 = \bigoplus_{\substack{m \in \mathbb{N}, i \in \mathbb{Z} \\ m+i=0 \text{ in } \mathbb{F}_2}} \mathbb{F}_2\{d_{0,m,i}\} \oplus \bigoplus_{\substack{m \in \mathbb{N}, i \in \mathbb{Z} \\ m+i+1=0 \text{ in } \mathbb{F}_2}} \mathbb{F}_2\{g_{0,m,i}\}.$$

The map of spectral sequences from (3.2) to (3.1) satisfies

$$d_{0,m,i} \mapsto \Delta^{\frac{m-3i}{2}} x^{2i}, \qquad g_{0,m,i} \mapsto \Delta^{\frac{m-3i-1}{2}} x^{2i+1}.$$

In particular, one sees that it is injective on E^4 -pages. We conclude that the next non-trivial differential in spectral sequence (3.2) is d^7 and that we have

$$d^{7}(d_{0,m,i}) = \frac{m-3i}{2}g_{0,m,i+3}, \qquad d^{7}(g_{0,m,i}) = \frac{m-3i-1}{2}d_{0,m+1,i+4}.$$

We obtain that

$$E_{*,*}^8 = \bigoplus_{i \in \mathbb{Z}} \mathbb{F}_2\{d_{0,0,4i}\} = \bigoplus_{i \in \mathbb{Z}} \mathbb{F}_2\{\zeta^{8i}\}.$$

Since the E^8 -page is concentrated in the zeroth row, the spectral sequence collapses at this stage. This gives the answer on the level of homotopy groups. As $\mathsf{tmf}_1(3)^{tC_2}$ is an E_{∞} -ring spectrum [McC96] it is in particular an E_2 -ring spectrum and therefore a result by Hopkins-Mahowald (see [MNN15, Theorem 4.18]) implies that $\mathsf{tmf}_1(3)^{tC_2}$ receives a map from $H\mathbb{F}_2$ and therefore is a generalized Eilenberg-MacLane spectrum of the claimed form.

Theorem 3.13. The Σ_3 -action on tmf(2)₍₃₎ yields

$$\operatorname{tmf}(2)_{(3)}^{t\Sigma_3} \simeq \bigvee_{i \in \mathbb{Z}} \Sigma^{12i} H\mathbb{Z}/3\mathbb{Z}.$$

Proof. We use the calculation of [Sto12]. She proves that $\mathsf{Tmf}(2)_{(3)}^{t\Sigma_3} \simeq *$ via the Tate spectral sequence

$$E_{n.m}^{2} = \hat{H}^{-n} \big(\Sigma_{3}; \pi_{m} (\mathsf{Tmf}(2)_{(3)}) \big) \Longrightarrow \pi_{n+m} (\mathsf{Tmf}(2)_{(3)}^{t\Sigma_{3}}).$$

The E^2 -page is given by

$$\mathbb{Z}/3\mathbb{Z}[\alpha, \beta^{\pm}, \Delta^{\pm}]/\alpha^2$$

with $|\alpha| = (-1, 4)$, $|\beta| = (-2, 12)$ and $|\Delta| = (0, 24)$, and the differentials are determined by $d^5(\Delta) = \alpha \beta^2$ and $d^9(\alpha \Delta^2) = \beta^5$.

Since $tmf(2)_{(3)}$ is the connective cover of $Tmf(2)_{(3)}$ the E^2 -page of the Tate spectral sequence

$$\bar{E}_{n,m}^2 = \hat{H}^{-n} \big(\Sigma_3; \pi_m(\mathsf{tmf}(2)_{(3)}) \big) \Longrightarrow \pi_{n+m}(\mathsf{tmf}(2)_{(3)}^{t\Sigma_3})$$

is the $\mathbb{Z}/3\mathbb{Z}$ -module

$$\bigoplus_{\substack{k,l\in\mathbb{Z}\\k+2l\geqslant 0}} \mathbb{Z}/3\mathbb{Z}\{\beta^k \Delta^l\} \oplus \bigoplus_{\substack{k,l\in\mathbb{Z}\\1+3k+6l\geqslant 0}} \mathbb{Z}/3\mathbb{Z}\{\alpha\beta^k \Delta^l\}.$$

Using the map of Tate spectral sequences $\bar{E}^*_{*,*} \to E^*_{*,*}$ one sees that

$$\bar{E}^6_{*,*} = \bigoplus_{\substack{k,l \in \mathbb{Z} \\ k+6l \ge 0}} \mathbb{Z}/3\mathbb{Z}\{\beta^k(\Delta^3)^l\} \oplus \bigoplus_{\substack{k,l \in \mathbb{Z} \\ 13+3k+18l \ge 0}} \mathbb{Z}/3\mathbb{Z}\{(\alpha\Delta^2)\beta^k(\Delta^3)^l\}.$$

Since $E^6_{*,*} = \mathbb{Z}/3\mathbb{Z}[\alpha\Delta^2, \beta^{\pm}, \Delta^{\pm 3}]/(\alpha\Delta^2)^2$ the map $\bar{E}^6_{*,*} \to E^6_{*,*}$ is injective. Thus, \bar{d}^9 is determined by \bar{d}^9 is determined by \bar{d}^9 . mined by d^9 and one gets

$$\bar{E}^{10}_{*,*} = \bigoplus_{k \in \mathbb{Z}} \mathbb{Z}/3\mathbb{Z}\{(\beta^{-6}\Delta^3)^k\}$$

The class $\beta^{-6}\Delta^3$ has bidegree (12,0), and so $\bar{E}_{*,*}^{10}$ is concentrated in line zero and the spectral sequence collapses at this stage.

So we can view the extensions $ko_{(2)} \rightarrow ku_{(2)}$, and $\mathsf{tmf}_{(3)} \rightarrow \mathsf{tmf}_{(2)}_{(3)}$ as being wildly ramified and $\mathsf{tmf}_0(3)_{(2)} \to \mathsf{tmf}_1(3)_{(2)}$ has a non-trivial Tate construction.

In contrast, $KO \rightarrow KU$ is a faithful C_2 -Galois [Rog08a, §5], $\mathsf{TMF}_0(3) \rightarrow \mathsf{TMF}_1(3)$ and $\mathsf{Tmf}_0(3) \to \mathsf{Tmf}_1(3)$ are both faithful C_2 -Galois extensions [MM15, Theorem 7.12]. In general, $\mathsf{TMF}[1/n] \to \mathsf{TMF}(n)$ is a faithful $GL_2(\mathbb{Z}/n\mathbb{Z})$ -Galois extension [MM15, Theorem 7.6] and the Tate spectrum $\mathsf{Tmf}(n)^{tGL_2(\mathbb{Z}/n\mathbb{Z})}$ is contractible [MM15, Theorem 7.11].

For general n, constructions of $\mathsf{tmf}_1(n)$ and $\mathsf{tmf}_0(n)$ are tricky: For some large n, $\pi_1 \mathsf{Tmf}_1(n)$ is non-trivial. Lennart Meier constructs a connective version of $\mathsf{Tmf}_1(n)$ with trivial π_1 as an E_{∞} -ring spectrum in [Mei], so that there are E_{∞} -models of $\mathsf{tmf}_1(n)$ for all n.

As $\pi_0(\mathsf{tmf}(n)) \cong \mathbb{Z}[\frac{1}{n}, \zeta_n]$ where ζ_n is a primitive *n*th root of unity, the defining cofiber sequence of $\operatorname{tmf}(n)^{tGL_2(\mathbb{Z}/n\mathbb{Z})}$ gives

$$\dots \longrightarrow \pi_0 \mathsf{tmf}(n)_{hGL_2(\mathbb{Z}/n\mathbb{Z})} \xrightarrow{N} \pi_0 \mathsf{tmf}(n)^{hGL_2(\mathbb{Z}/n\mathbb{Z})} \longrightarrow \pi_0 \mathsf{tmf}(n)^{tGL_2(\mathbb{Z}/n\mathbb{Z})} \longrightarrow 0$$

because $\mathsf{tmf}(n)_{hGL_2(\mathbb{Z}/n\mathbb{Z})}$ is connective.

By the homotopy orbit spectral sequence we get that $\pi_0 \operatorname{tmf}(n)_{hGL_2(\mathbb{Z}/n\mathbb{Z})} \cong \mathbb{Z}[\frac{1}{n}, \zeta_n]_{GL_2(\mathbb{Z}/n\mathbb{Z})}$. As $\tau_{\geq 0} \operatorname{tmf}(n)^{hGL_2(\mathbb{Z}/n\mathbb{Z})} \simeq \operatorname{tmf}[\frac{1}{n}]$, we know that $\mathbb{Z}[\frac{1}{n}] \cong \pi_0 \operatorname{tmf}[\frac{1}{n}]$. For every n > 1 we have $|GL_2(\mathbb{Z}/n\mathbb{Z})| = \varphi(n)n^3 \prod_{p|n} (1 - \frac{1}{p^2})$. Here, φ denotes the Euler

 φ -function and p runs over all primes dividing n.

Meier shows [Mei], that tmf(n) is a perfect tmf[1/n]-module spectrum and hence dualizable. In general we do not know whether $\mathsf{tmf}(n)$ is faithful as a $\mathsf{tmf}[1/n]$ -module. For n = 2, $\mathsf{tmf}(2)_{(3)}$ is faithful and dualizable over $\mathsf{tmf}_{(3)}$, as we saw in Remark 3.11.

We cannot determine the homotopy type of the $GL_2(\mathbb{Z}/n\mathbb{Z})$ -Tate construction of $\mathsf{tmf}(n)$ for arbitrary n, but we can identify cases where it is non-trivial:

Theorem 3.14. Assume $n \ge 2$, $2 \nmid n$ or $n = 2^k$ for some $k \ge 1$. Then $\operatorname{tmf}(n)^{tGL_2(\mathbb{Z}/n\mathbb{Z})} \simeq *.$

Proof. By [KM85, p. 282] we know that $GL_2(\mathbb{Z}/n\mathbb{Z})$ acts on ζ_n via the determinant map: For $A \in GL_2(\mathbb{Z}/n\mathbb{Z})$ and ζ_n we get

$$A.\zeta_n^i = \zeta_n^{i \cdot \det(A)}.$$

Therefore, the norm map $N \colon \mathbb{Z}[\frac{1}{n}, \zeta_n]_{GL_2(\mathbb{Z}/n\mathbb{Z})} \to \mathbb{Z}[\frac{1}{n}]$ sends ζ_n^i to

$$\sum_{A \in GL_2(\mathbb{Z}/n\mathbb{Z})} \zeta_n^{i \cdot \det(A)} = |SL_2(\mathbb{Z}/n\mathbb{Z})| \sum_{r \in (\mathbb{Z}/n\mathbb{Z})^{\times}} \zeta_n^{ir},$$

in particular it sends a primitive *n*-th root of unity ζ to $|SL_2(\mathbb{Z}/n\mathbb{Z})|\mu(n)$ with μ denoting the Möbius function.

If n is square-free, then $\mu(n) = \pm 1$. Any power of ζ_n has order d with $d \mid n$, so this d is squarefree as well, so $N(\zeta_n^i)$ is equal to $|SL_2(\mathbb{Z}/n\mathbb{Z})|\mu(n)$ or a multiple of it.

If *n* contains a square of a prime, then $\mu(n) = 0$, so $N(\zeta_n) = 0$ but of course $N(1) = |GL_2(\mathbb{Z}/n\mathbb{Z})|$. If $d \mid n$, and *d* is squarefree, then the corresponding power of ζ_n can give a non-trivial multiple of $|SL_2(\mathbb{Z}/n\mathbb{Z})|$.

If $2 \nmid n$ then $|GL_2(\mathbb{Z}/n\mathbb{Z})|$ and $|SL_2(\mathbb{Z}/n\mathbb{Z})|$ are not units in $\mathbb{Z}[\frac{1}{n}]$: Let p be an odd prime factor of n. Then in $n^3 \prod_{p|n} (1 - \frac{1}{p^2})$ we have a factor of p - 1 and this is even, but 2 is not invertible in $\mathbb{Z}[\frac{1}{n}]$.

If $n = 2^k$ for some $k \ge 1$, we obtain

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$$|SL_2(\mathbb{Z}/n\mathbb{Z})| = 2^{3k}\frac{3}{4}$$

which contains 3 as a non-invertible factor.

Remark 3.15. For many *n* the Tate construction $\mathsf{tmf}(n)^{tGL_2(\mathbb{Z}/n\mathbb{Z})}$ is actually trivial. If $n = 2^k 3^\ell$ with $k, \ell \ge 1$ for instance, the order of $GL_2(\mathbb{Z}/n\mathbb{Z})$ is invertible in $\mathbb{Z}[\frac{1}{n}]$. Similarly, if $n = 2 \cdot 3 \cdot \ldots \cdot p_m$ is the product of the first *m* prime numbers for any $m \ge 2$, then the group order is invertible as well.

We close with a periodic example: Let E_n be the Lubin-Tate spectrum for the Honda formal group law over $W(\mathbb{F}_{p^n})$. For any finite group G, $F(BG_+, E_n) \to E_n$ is faithful in the K_n -local category [BR11, Theorem 4.4]. At the moment we don't know whether E_n is a dualizable $F(BG_+, E_n)$ -module for any finite group G.

In [BR11, Theorem 5.1] it is shown that $F((BC_{p^r})_+, E_n) \to E_n$ is ramified and one can also consider more general groups than C_{p^r} , but the type of ramification was *not* determined. The following result indicates that $F((BC_{p^r})_+, E_n) \to E_n$ is wildly ramified.

Theorem 3.16. For all $r \ge 1$ and $n \ge 1$

$$E_n^{tC_{p^r}} \not\simeq *.$$

Proof. The Tate spectral sequence

$$E_2^{s,t} = \hat{H}^{-s}(C_{p^r}; \pi_t E_n) \Rightarrow \pi_{s+t}(E_n^{tC_{p^r}})$$

has as E^2 -term

$$\hat{H}^{-s}(C_{p^r}; \pi_t E_n) \cong \begin{cases} \pi_t E_n^{C_{p^r}} / p^r = \pi_t E_n / p^r, & \text{for } s \text{ even,} \\ \ker(N) / \operatorname{im}(t-1) = 0, & \text{for } s \text{ odd.} \end{cases}$$

As $\pi_*(E_n)$ is concentrated in even degrees, the whole E_2 -term is concentrated in bidegrees (s, t) where s and t are even. Therefore, all differentials have to be trivial and $E_2 = E_{\infty}$. Thus $\pi_*(E_n^{tC_pr})$ is highly non-trivial.

Proposition 3.17. Assume that G is a finite group with a non-trivial cyclic subgroup $C_{p^k} < G$ for some prime p. Then E_n^{tG} is non-trivial when E_n is the Lubin-Tate spectrum at the prime p.

Proof. The restriction map induces a map on Tate constructions $E_n^{tG} \to E_n^{tC_{p^k}}$. McClure [McC96] shows that the E_{∞} -structure on Tate constructions $E_n^{tG} = t((E_n)_G)^G$ is compatible with inclusions of subgroups and Greenlees-May show [GM95, Proposition 3.7] that for any subgroup H < G the H-spectrum $t((E_n)_G)$ is equivalent to $t((E_n)_H)$. Therefore the inclusion of fixed points $t((E_n)_G)^G \to t((E_n)_G)^H$ is a map of E_{∞} -ring spectra. As we know that $E_n^{tC_{p^k}} = t((E_n)_G)^{C_{p^k}}$ is non-trivial by Theorem 3.16, E_n^{tG} cannot be trivial, either.

References

- [AB59] Maurice Auslander and David A. Buchsbaum, On ramification theory in noetherian rings, Amer. J. Math. 81 (1959), 749–765.
- [Aus10] Christian Ausoni, On the algebraic K-theory of the complex K-theory spectrum, Invent. Math. 180 (2010), no. 3, 611–668.
- [Aus05] Christian Ausoni, Topological Hochschild homology of connective complex K-theory, Amer. J. Math. 127 (2005), no. 6, 1261–1313.
- [AR08] Christian Ausoni and John Rognes, The chromatic red-shift in algebraic K-theory, Enseign. Math. 54 (2008), no. 2, 9–11.
- [BGR08] Andrew Baker, Helen Gilmour, and Philipp Reinhard, Topological André-Quillen homology for cellular commutative S-algebras, Abh. Math. Semin. Univ. Hambg. 78 (2008), no. 1, 27–50.
- [BR11] Andrew Baker and Birgit Richter, Galois theory and Lubin-Tate cochains on classifying spaces, Cent. Eur. J. Math. 9 (2011), no. 5, 1074–1087.
- [BBGS] Tobias Barthel, Agnès Beaudry, Paul Goerss, and Vesna Stojanoska, *Constructing the determinant* sphere using a Tate twist. arXiv:1810.06651.
- [Bas99] Maria Basterra, André-Quillen cohomology of commutative S-algebras, J. Pure Appl. Algebra 144 (1999), no. 2, 111–143.
- [BR04] Maria Basterra and Birgit Richter, (Co-)homology theories for commutative (S-)algebras, Structured ring spectra, 2004, pp. 115–131.
- [Beh06] Mark Behrens, A modular description of the K(2)-local sphere at the prime 3, Topology 45 (2006), no. 2, 343–402.
- [Boa99] J. Michael Boardman, Conditionally convergent spectral sequences, Homotopy invariant algebraic structures (Baltimore, MD, 1998), 1999, pp. 49–84.
- [CF67] John William Scott Cassels and Albrecht Fröhlich (eds.), Algebraic number theory, Proceedings of an instructional conference organized by the London Mathematical Society (a NATO Advanced Study Institute) with the support of the International Mathematical Union, Academic Press, London; Thompson Book Co., Inc., Washington, D.C., 1967.
- [CHR65] Stephen U. Chase, David K. Harrison, and Alex Rosenberg, Galois theory and Galois cohomology of commutative rings, Mem. Amer. Math. Soc. 52 (1965), 15–33.
- [CMNN20] Dustin Clausen, Akhil Mathew, Niko Naumann, and Justin Noel, Descent in algebraic K-theory and a conjecture of Ausoni-Rognes, J. Eur. Math. Soc. (JEMS) 22 (2020), no. 4, 1149–1200.
 - [Del75] Pierre Deligne, Courbes elliptiques: formulaire d'après J. Tate, Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), 1975, pp. 53–73. Lecture Notes in Math., Vol. 476.
- [DFHH14] Christopher L. Douglas, John Francis, André G. Henriques, and Michael Hill A. (eds.), Topological modular forms, Mathematical Surveys and Monographs, vol. 201, American Mathematical Society, Providence, RI, 2014.
 - [DLR20] Bjørn Ian Dundas, Ayelet Lindenstrauss, and Birgit Richter, Towards an understanding of ramified extensions of structured ring spectra, Mathematical Proceedings of the Cambridge Philosophical Society 168 (2020), no. 3, 435–454.
 - [DLR] Bjørn Ian Dundas, Ayelet Lindenstrauss, and Birgit Richter, Erratum for: Towards an understanding of ramified extensions of structured ring spectra, Mathematical Proceedings of the Cambridge Philosophical Society. to appear.

- [GM95] John P. C. Greenlees and J. Peter May, Generalized Tate cohomology, Vol. 113, 1995.
- [HL16] Michael A. Hill and Tyler Lawson, Topological modular forms with level structure, Invent. Math. 203 (2016), no. 2, 359–416.
- [HM17] Michael A. Hill and Lennart Meier, The C₂-spectrum Tmf₁(3) and its invertible modules, Algebr. Geom. Topol. 17 (2017), no. 4, 1953–2011.
- [KM85] Nicholas M. Katz and Barry Mazur, Arithmetic moduli of elliptic curves, Annals of Mathematics Studies, vol. 108, Princeton University Press, Princeton, NJ, 1985.
- [Kon] Johan Konter, The homotopy groups of the spectrum Tmf. arXiv:1212.3656.
- [LN14] Tyler Lawson and Niko Naumann, Strictly commutative realizations of diagrams over the Steenrod algebra and topological modular forms at the prime 2, Int. Math. Res. Not. 10 (2014), 2773–2813.
- [MR09] Mark Mahowald and Charles Rezk, Topological modular forms of level 3, Pure Appl. Math. Q. 5 (2009), no. 2, 853–872. Special Issue: In honor of Friedrich Hirzebruch. Part 1.
- [Mat16a] Akhil Mathew, The Galois group of a stable homotopy theory, Adv. Math. 291 (2016), 403–541.
- [Mat16b] Akhil Mathew, The homology of tmf, Homology Homotopy Appl. 18 (2016), no. 2, 1–29.
- [MM15] Akhil Mathew and Lennart Meier, Affineness and chromatic homotopy theory, J. Topol. 8 (2015), no. 2, 476–528.
- [MNN15] Akhil Mathew, Niko Naumann, and Justin Noel, On a nilpotence conjecture of J. P. May, J. Topol. 8 (2015), no. 4, 917–932.
- [McC96] James E. McClure, E_{∞} -ring structures for Tate spectra, Proc. Amer. Math. Soc. **124** (1996), no. 6, 1917–1922.
 - [Mei] Lennart Meier, Connective models for topological modular forms. in preparation.
- [Noe32] Emmy Noether, Normalbasis bei Körpern ohne höhere Verzweigung, J. Reine Angew. Math. 167 (1932), 147–152.
- [Rog08a] John Rognes, Galois extensions of structured ring spectra., Mem. Amer. Math. Soc. 192 (2008), no. 898, viii–98.
- [Rog08b] John Rognes, Stably dualizable groups, Mem. Amer. Math. Soc. 192 (2008), no. 898, 99–137.
- [Rog09] John Rognes, *Topological logarithmic structures*, New topological contexts for Galois theory and algebraic geometry (BIRS 2008), 2009, pp. 401–544.
- [Rog14] John Rognes, Algebraic K-theory of strict ring spectra, Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II, 2014, pp. 1259–1283.
- [RSS15] John Rognes, Steffen Sagave, and Christian Schlichtkrull, Localization sequences for logarithmic topological Hochschild homology, Math. Ann. 363 (2015), no. 3-4, 1349–1398.
- [RSS18] John Rognes, Steffen Sagave, and Christian Schlichtkrull, Logarithmic topological Hochschild homology of topological K-theory spectra, J. Eur. Math. Soc. (JEMS) 20 (2018), no. 2, 489–527.
- [Sag14] Steffen Sagave, Logarithmic structures on topological K-theory spectra, Geom. Topol. 18 (2014), no. 1, 447–490.
- [Sto12] Vesna Stojanoska, Duality for topological modular forms, Doc. Math. 17 (2012), 271–311.