T-duality for transitive Courant algebroids

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Abstract

We develop a theory of T-duality for transitive Courant algebroids. We show that T-duality between transitive Courant algebroids $E \to M$ and $\tilde{E} \to \tilde{M}$ induces a map between the spaces of sections of the corresponding canonical weighted spinor bundles $\mathbb{S}_E$ and $\mathbb{S}_{\tilde{E}}$ intertwining the canonical Dirac generating operators. The map is shown to induce an isomorphism between the spaces of invariant spinors, compatible with an isomorphism between the spaces of invariant sections of the Courant algebroids. The notion of invariance is defined after lifting the vertical parallelisms of the underlying torus bundles $M \to B$ and $\tilde{M} \to B$ to the Courant algebroids and their spinor bundles. We prove a general existence result for T-duals under assumptions generalizing the cohomological integrality conditions for T-duality in the exact case. Specializing our construction, we find that the T-dual of an exact or a heterotic Courant algebroid is again exact or heterotic, respectively.

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1 Introduction

The concept of T-duality appeared first in theoretical physics as a duality between a pair of physical theories related by compactification of a common (possibly hidden) theory along circles of reciprocal radii. Examples include the famous duality between type IIA and type IIB string theories. More generally, it refers to an isomorphism between certain type of structures on a pair of torus bundles over the same manifold.

Precise formulations of T-duality are available in the framework of generalized geometry (in the sense of Hitchin) [7, 2]. Recall that the basic idea of generalized geometry is to replace the tangent bundle TM of a manifold M by a Courant algebroid E. The first examples of Courant algebroids considered in the literature were the exact Courant algebroids. They are obtained from the generalized tangent bundle $\mathbb{T}M := T^*M \oplus TM$ by twisting the canonical Dorfman bracket with a closed 3-form. A more general class of transitive Courant algebroids is the class of heterotic Courant algebroids considered in [2]. These were introduced by Baraglia and Hekmati [2] motivated by T-duality in heterotic string theory. Cavalcanti and Gualtieri [7] developed a theory of T-duality for exact Courant algebroids and Baraglia and Hekmati [2] extended it to heterotic Courant algebroids. The approach of [2] is based on reduction of exact Courant algebroids and uses T-duality for the latter algebroids.

In this article we develop T-duality for transitive Courant algebroids. Let M and $\tilde{M}$ be principal k-torus bundles over a manifold B. We call two transitive Courant algebroids E and $\tilde{E}$ over M and $\tilde{M}$, respectively, T-dual if there exists a certain type of isomorphism between the pullbacks of E and $\tilde{E}$ to the fiber product $N = M \times_B \tilde{M}$ (see Definition 47 for details). We show that T-duality gives rise to a map between the spaces of sections of the corresponding canonical weighted spinor bundles $\mathbb{S}_E$ and $\mathbb{S}_{\tilde{E}}$ intertwining the canonical Dirac generating operators, see Theorem 54. More specifically, we obtain compatible isomorphisms between the spaces of (appropriately defined) invariant sections of E and $\tilde{E}$ as well as between the spaces of invariant sections of $\mathbb{S}_E$ and $\mathbb{S}_{\tilde{E}}$. This implies, in particular, that any invariant geometric structure on the Courant algebroid E gives rise to a corresponding invariant 'T-dual' geometric structure on $\tilde{E}$. A structure solving a system of partial differential equations defined in terms of the Courant algebroid structure on E will be mapped to a solution of the corresponding system on $\tilde{E}$. Examples include integrability equations as considered in [9] and equations of motion of physical theories such as supergravity. We plan to investigate these type of applications in the future.

In Theorem 60 we prove the existence of a T-dual $\tilde{E}$ for a class of transitive Courant algebroids E over a principal torus bundle $M \to B$ under the assumption that certain cohomology classes in $H^2(B, \mathbb{R})$ are integral. The result generalizes a theorem of Bouwknegt, Hannabuss, and Mathai [6] in the exact case, see Section 6.4.1. In the heterotic case we show that the 'T-dual' Courant algebroids obtained from our construction are again heterotic, see Proposition 62.

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2 Preliminary material

To keep the text reasonably self-contained, we recall, following [8, 9], basic facts we need on transitive Courant algebroids and their canonical Dirac generating operator. We assume that the reader is familiar with the definition of Courant algebroids, Dirac generating operators, generalized connections and E-connections. Basic facts on these notions can be found e.g. in [9], the approach and notation of which we preserve along the paper. In this paper we always assume that the Courant algebroids have scalar product of neutral signature. For the definition of densities we keep the conventions from our previous work [9] which coincide with those from [3]. Namely, if V is a vector space of dimension n and $s \in \mathbb{R}$, then the one-dimensional oriented vector space $|\mathrm{det} V^*|^s$ of s-densities on V consists of all maps $\Psi : \Lambda^n V \setminus \{0\} \to \mathbb{R}$ (called s-densities) which satisfy $\omega(\lambda \vec{v}) = |\lambda|^s \omega(\vec{v})$, for any $\vec{v} \in \Lambda^n \setminus \{0\}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Note that, when s is an integer, $|\mathrm{det} V^*|^s$ is canonically isomorphic to $|\mathrm{det} V^*|^{\otimes s}$ and $|\mathrm{det} V^*|^{2s}$ to $(\mathrm{det} V^*)^{2s}$. Any form $\omega \in \Lambda^n V^*$ defines an s-density $|\omega|^s(\vec{v}) = |\omega(v_1, \cdots, v_n)|^s$, where $\vec{v} := v_1 \wedge \cdots \wedge v_n$. If V is oriented then we will identify $\Lambda^n V^*$ and $|\det V^*|$ by the isomorphism which assigns to a positively oriented volume form $\omega \in \Lambda^n V^*$ the density $|\omega|$; the s-density $|\omega|^s$ will be denoted by ω^s when ω is positively oriented and $|\mathrm{det} V^*|^s$ by $(\mathrm{det} V^*)^s$ when V is oriented. The same notation will be used when V is replaced by a vector bundle.

2.1 The canonical Dirac generating operator

Let $(E, \pi, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$ be a regular Courant algebroid over a manifold M, with anchor $\pi : E \to TM$, Dorfmann bracket $[\cdot, \cdot]$ and scalar product (of neutral signature) $\langle \cdot, \cdot \rangle$. Let S be an irreducible $\mathrm{Cl}(E)$-bundle (sometimes called a spinor bundle of E). We denote by $E \ni v \mapsto \gamma_v$ the Clifford action of E on S. We assume that S is $\mathbb{Z}_2$-graded and that the gradation is compatible with the Clifford multiplication (this always holds when E is oriented). Let $|\det S^*|^{1/r}$ be the line bundle of $1/r$-densities on S, where $r := \mathrm{rk}\, S$. An E-connection D on S induces an E-connection on $|\det S^*|^{1/r}$: if $\mathrm{vol}_S \in \Gamma(\Lambda^r S^*)$ is a local volume form on S and $D_e \mathrm{vol}_S = \omega(e) \mathrm{vol}_S$ then $D_e |\mathrm{vol}_S|^{1/r} = \frac{1}{r} \omega(e) |\mathrm{vol}_S|^{1/r}$, for any $e \in E$.

The canonical Dirac generating operator $\not{d}$ of E acts on sections of the canonical weighted spinor bundle of E determined by S. The latter is defined by

$$\mathbb{S} := S \otimes |\det S^*|^{1/r} \otimes |\det T^*M|^{1/2} = \mathcal{S} \otimes L, \tag{1}$$

where $\mathcal{S} := S \otimes |\det S^*|^{1/r}$ is the canonical spinor bundle of S and $L := |\det T^*M|^{1/2}$. The operator $\not{d} : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ is given by

$$\not{d} = \not{D} + \frac{1}{4}\gamma_{T^D}, \tag{2}$$

where $\not{D} := \frac{1}{2}\sum_i \gamma_{\tilde{e}_i} D^{\mathbb{S}}_{e_i}$ is the Dirac operator computed with $D^{\mathbb{S}} := D^{\mathcal{S}} \otimes D^L$, $D^{\mathcal{S}}$ is the E-connection on $\mathcal{S}$ induced by an arbitrary E-connection D^S on S compatible with a given generalized connection D on E, D^L is the E-connection on L defined by D by the rule

$$D^L_v(\mu) = \mathcal{L}_{\pi(v)}\mu - \frac{1}{2}\mathrm{div}_D(v)\mu, \ v \in E, \ \mu \in \Gamma(L), \tag{3}$$

where $\mathrm{div}_D(v) := \mathrm{tr}\,(Dv)$, (e_i) is a frame of E, $(\tilde{e}_i)$ the metrically dual frame (i.e. $\langle e_i, \tilde{e}_j\rangle = \delta_{ij}$) and $T^D \in \Gamma(\Lambda^3 E^*)$ is the torsion of D, viewed as a section of the Clifford bundle $\mathrm{Cl}(E)$ and acting by Clifford multiplication on $\mathbb{S}$. The definition of $\not{d}$ is independent of the choice of generalized connection D and D-compatible E-connection D^S.

2.2 Transitive Courant algebroids

2.2.1 Basic properties

Recall that a vector bundle $\mathcal{G} \to M$ endowed with a tensor field $[\cdot,\cdot] \in \Gamma(\wedge^2 \mathcal{G}^* \otimes \mathcal{G})$ satisfying the Jacobi identity is called a Lie algebra bundle if in a neighborhood of every point $p \in N$ the tensor field has constant coefficients with respect to some local frame. A bundle of quadratic Lie algebras is a Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot])$ endowed with an ad-invariant metric $\langle\cdot,\cdot\rangle_{\mathcal{G}} \in \Gamma(\mathrm{Sym}^2\mathcal{G}^*)$, which we assume of neutral signature.

Let $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ be a bundle of quadratic Lie algebras over a manifold M and $E := T^*M \oplus \mathcal{G} \oplus TM$. Let $\mathrm{pr}_{\mathcal{G}}$ and pr_{TM} be the natural projections from E to $\mathcal{G}$ and TM respectively. As proved in Theorem 2.3 of [8], any Courant algebroid with underlying bundle E, anchor pr_{TM}, scalar product

$$\langle \xi + r_1 + X, \eta + r_2 + Y\rangle = \frac{1}{2}(\eta(Y) + \xi(X)) + \langle r_1, r_2\rangle_{\mathcal{G}}, \ \xi, \eta \in T^*M, \ r_1, r_2 \in \mathcal{G},$$

and whose Dorfman bracket satisfies

$$\mathrm{pr}_{\mathcal{G}}[r_1, r_2] = [r_1, r_2]_{\mathcal{G}}$$

is defined by data (∇, R, H) where ∇ is a connection on the vector bundle $\mathcal{G}$, $R \in \Omega^2(M, \mathcal{G})$ and $H \in \Omega^3(M)$ such that ∇ preserves $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ and $[\cdot, \cdot]_{\mathcal{G}}$, the curvature R^{∇} of ∇ is given by

$$R^{\nabla}(X, Y)r = [R(X, Y), r]_{\mathcal{G}},\ X, Y \in \mathfrak{X}(M),\ r \in \Gamma(\mathcal{G}), \tag{4}$$

and the following relations hold:

$$d^{\nabla} R = 0, \tag{5}$$

$$dH = \langle R \wedge R \rangle_{\mathcal{G}}. \tag{6}$$

We recall that

$$(d^{\nabla} R)(X, Y, Z) := \sum_{\mathfrak{S}(X,Y,Z)} (\nabla_X(R(Y, Z)) - R(\mathcal{L}_X Y, Z)) \quad \text{and}$$

$$\langle R \wedge R \rangle_{\mathcal{G}}(X, Y, Z, W) := 2 \sum_{\mathfrak{S}(X,Y,Z)} \langle R(X, Y), R(Z, W) \rangle_{\mathcal{G}},$$

where $X, Y, Z, W \in \mathfrak{X}(M)$ and $\mathfrak{S}(X, Y, Z)$ denotes cyclic permutations over X, Y, Z. The Dorfman bracket of E is uniquely determined by the relations

$$\begin{aligned} &[X, Y] = \mathcal{L}_X Y + R(X, Y) + i_Y i_X H \\ &[X, r] = \nabla_X r - 2\langle i_X R, r \rangle_{\mathcal{G}} \\ &[r_1, r_2] = [r_1, r_2]_{\mathcal{G}} + 2\langle \nabla r_1, r_2 \rangle_{\mathcal{G}} \\ &[X, \eta] = \mathcal{L}_X \eta,\ [\eta_1, \eta_2] = [r, \eta] = 0, \end{aligned} \tag{7}$$

for any $X, Y \in \mathfrak{X}(M)$, $\eta_1, \eta_2, \eta \in \Omega^1(M)$ and $r, r_1, r_2 \in \Gamma(\mathcal{G})$, together with the condition

$$[u, v] + [v, u] = 2d\langle u, v \rangle, u, v \in \Gamma(E). \tag{8}$$

Such a Courant algebroid is called a **standard Courant algebroid**. As proved in [8], any transitive Courant algebroid (i.e. a Courant algebroid with surjective anchor) is isomorphic to a standard Courant algebroid. A **dissection** of a transitive Courant algebroid E is an isomorphism from E to a standard Courant algebroid. The quadratic Lie algebra bundle $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \langle \cdot, \cdot \rangle_{\mathcal{G}})$ which is a summand of a dissection of E is isomorphic to $\mathrm{Ker}\, \pi / (\mathrm{Ker}\, \pi)^{\perp}$ (with scalar product and Lie bracket induced from E), where $\pi : E \to TM$ is the anchor of E. The following simple lemma holds.

Lemma 1. *Let E be a transitive Courant algebroid with anchor $\pi : E \to TM$. Let $(\mathcal{G}_0, [\cdot, \cdot]_0, \langle \cdot, \cdot \rangle_0)$ be a quadratic Lie algebra bundle, isomorphic to $\mathrm{Ker}\, \pi / (\mathrm{Ker}\, \pi)^{\perp}$. Then E admits a dissection $I_0 : E \to T^*M \oplus \mathcal{G}_0 \oplus TM$.*

Proof. Start with an arbitrary dissection $I : E \to T^*M \oplus \mathcal{G} \oplus TM$, where the target is defined by data (∇, R, H) and a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ The new data

$$\tilde{\nabla}_X := K\nabla_X K^{-1},\ \tilde{R}(X,Y) := KR(X,Y),\ \tilde{H}(X,Y,Z) := H(X,Y,Z),$$

where $K : \mathcal{G} \to \mathcal{G}_0$ is an isomorphism of quadratic Lie algebra bundles, together with $(\mathcal{G}_0, [\cdot,\cdot]_0, \langle\cdot,\cdot\rangle_0)$, define a standard Courant algebroid isomorphic to $T^*M \oplus \mathcal{G} \oplus TM$ (use relations (10) below with $\Phi := 0$ and $\beta := 0$). By composing this isomorphism with I we obtain the required dissection of E. □

Let $E_i := T^*M \oplus \mathcal{G}_i \oplus TM$ $(i = 1, 2)$ be two standard Courant algebroids over a manifold M, defined by quadratic Lie algebra bundles $(\mathcal{G}_i, [\cdot,\cdot]_{\mathcal{G}_i}, \langle\cdot,\cdot\rangle_{\mathcal{G}_i})$ and data $(\nabla^{(i)}, R_i, H_i)$. As proved in Proposition 2.7 of [8], any fiber preserving Courant algebroid isomorphism $I_E : E_1 \to E_2$ is of the form

$$I_E(\eta) = \eta,\ I_E(r) = -2\Phi^* K(r) + K(r),\ I_E(X) = i_X\beta - \Phi^*\Phi(X) + \Phi(X) + X, \tag{9}$$

for any $X \in \mathfrak{X}(M)$, $r \in \Gamma(\mathcal{G}_1)$ and $\eta \in \Omega^1(M)$. Above $\beta \in \Omega^2(M)$, $K \in \mathrm{Isom}(\mathcal{G}_1, \mathcal{G}_2)$ is an isomorphism of quadratic Lie algebra bundles, $\Phi \in \Omega^1(M, \mathcal{G}_2)$,

$$\Phi^*\Phi : TM \to T^*M,\ (\Phi^*\Phi)(X)(Y) := \langle \Phi(X), \Phi(Y)\rangle_{\mathcal{G}_2},$$
$$\Phi^*K : \mathcal{G}_1 \to T^*M,\ (\Phi^*K)(r)(X) := \langle K(r), \Phi(X)\rangle_{\mathcal{G}_2},$$

for any $X, Y \in \mathfrak{X}(M)$ and $r \in \Gamma(\mathcal{G}_1)$, and the next relations are satisfied:

$$\nabla^{(2)}_X r = K\nabla^{(1)}_X (K^{-1}r) + [r, \Phi(X)]_{\mathcal{G}_2},$$
$$KR_1(X,Y) - R_2(X,Y) = (d^{\nabla^{(2)}}\Phi)(X,Y) + [\Phi(X), \Phi(Y)]_{\mathcal{G}_2},$$
$$H_1 - H_2 = d\beta + \langle (KR_1 + R_2) \wedge \Phi\rangle_{\mathcal{G}_2} - c_3, \tag{10}$$

where $c_3(X,Y,Z) := \langle \Phi(X), [\Phi(Y), \Phi(Z)]_{\mathcal{G}_2}\rangle_{\mathcal{G}_2}$, for any $X, Y, Z \in \mathfrak{X}(M)$. The second and third relations (10) are equivalent with relations (46) and (47) of [8] (easy check) but are written in a simpler form. (We decomposed $\mathrm{pr}_{T^*M}(I_E|_{TM})$, which in the notation of [8] is denoted by β, into its symmetric part $-\langle \Phi(\cdot), \Phi(\cdot)\rangle_{\mathcal{G}_2}$ and skew-symmetric part β, see relation (44) of [8]). Under an additional condition on the Courant algebroids E_i relations (10) can be further simplified, as follows.

Lemma 2. *Assume that the adjoint actions $\mathrm{ad}_{\mathcal{G}_i} : \mathcal{G}_i \to \mathrm{Der}(\mathcal{G}_i)$ of the Lie algebra bundles $(\mathcal{G}_i, [\cdot,\cdot]_{\mathcal{G}_i})$ of the standard Courant algebroids E_i are isomorphisms. Then the second relation (10) follows from the first.*

Proof. From the injectivity of $\mathrm{ad}_{\mathcal{G}_2}$, the second relation (10) holds if and only if

$$[KR_1(X,Y) - R_2(X,Y), r]_{\mathcal{G}_2} = [(d^{\nabla^{(2)}}\Phi)(X,Y) + [\Phi(X), \Phi(Y)]_{\mathcal{G}_2}, r]_{\mathcal{G}_2} \quad (11)$$

for any $r \in \Gamma(\mathcal{G}_2)$. Taking the covariant derivative of the first relation (10) we obtain

$$\begin{aligned}[\nabla^{(2)}_Y(\Phi(X)), r]_{\mathcal{G}_2} =& [R_2(X,Y), r]_{\mathcal{G}_2} + \nabla^{(2)}_{\mathcal{L}_X Y} r + \nabla^{(2)}_Y (K\nabla^{(1)}_X (K^{-1}r)) \\ & - K\nabla^{(1)}_X (K^{-1}\nabla^{(2)}_Y r). \end{aligned} \quad (12)$$

Now, a straightforward computation which uses the first relation (10), relation (12), and

$$(d^{\nabla^{(2)}}\Phi)(X,Y) = \nabla^{(2)}_X(\Phi(Y)) - \nabla^{(2)}_Y(\Phi(X)) - \Phi(\mathcal{L}_X Y)$$

shows that

$$\begin{aligned}[(d^{\nabla^{(2)}}\Phi)(X,Y), r]_{\mathcal{G}_2} =& [KR_1(X,Y) - R_2(X,Y), r]_{\mathcal{G}_2} \\ & + \left((\nabla_X K)(\nabla_Y K^{-1}) - (\nabla_Y K)(\nabla_X K^{-1})\right)(r), \end{aligned} \quad (13)$$

where ∇ denotes the connection on $\mathrm{End}(\mathcal{G}_1, \mathcal{G}_2)$ induced by $\nabla^{(1)}$ and $\nabla^{(2)}$. On the other hand, using the Jacobi identity for $[\cdot,\cdot]_{\mathcal{G}_2}$ and the first relation (10), we can compute

$$[[\Phi(X), \Phi(Y)]_{\mathcal{G}_2}, r]_{\mathcal{G}_2} = \left((\nabla_Y K)(\nabla_X K^{-1}) - (\nabla_X K)(\nabla_Y K^{-1})\right)(r). \quad (14)$$

Relations (13) and (14) imply (11). □

We say that two dissections $I_i : E \to T^*M \oplus \mathcal{G}_i \oplus TM$ of a transitive Courant algebroid E are related by (β, K, Φ), where $\beta \in \Omega^2(M)$, $K \in \mathrm{Isom}(\mathcal{G}_1, \mathcal{G}_2)$ and $\Phi :\in \Omega^1(M, \mathcal{G}_2)$, if the isomorphism $I_2 \circ I_1^{-1}$ is given by (9).

The proof of the following proposition is straightforward.

Proposition 3. *If $I_1 : E_1 \to E_2$ and $I_2 : E_2 \to E_3$ are isomorphisms between standard Courant algebroids $E_i = T^*M \oplus \mathcal{G}_i \oplus TM$, defined, according to (9), by (β_1, K_1, Φ_1) and (β_2, K_2, Φ_2) respectively, then $I_2 \circ I_1 : E_1 \to E_3$ is defined by (β_3, K_3, Φ_3) where*

$$K_3 := K_2 K_1, \;\; \Phi_3 := \Phi_2 + K_2 \Phi_1 \quad (15)$$

and, for any $X, Y \in TM$,

$$\beta_3(X,Y) := (\beta_1 + \beta_2)(X,Y) + \langle \Phi_2(X), K_2\Phi_1(Y)\rangle_{\mathcal{G}_2} - \langle \Phi_2(Y), K_2\Phi_1(X)\rangle_{\mathcal{G}_2}. \quad (16)$$

In particular,

$$(\beta_3 - \Phi_3^*\Phi_3)(X,Y) = (\beta_1 - \Phi_1^*\Phi_1)(X,Y) + (\beta_2 - \Phi_2^*\Phi_2)(X,Y) \\ - 2\langle K_2\Phi_1(X), \Phi_2(Y)\rangle_{\mathcal{G}_3}. \tag{17}$$

2.2.2 The canonical Dirac generating operator of a standard Courant algebroid

Let $E = T^*M \oplus \mathcal{G} \oplus TM$ be a standard Courant algebroid as above and $S_{\mathcal{G}}$ an irreducible $\mathrm{Cl}(\mathcal{G})$-bundle, with canonical spinor bundle $\mathcal{S}_{\mathcal{G}}$. Then $S := \Lambda(T^*M)\hat{\otimes}S_{\mathcal{G}}$ is an irreducible spinor bundle of E, with Clifford action

$$\gamma_{\xi+r+X}(\omega\otimes s) = (i_X\omega + \xi\wedge\omega)\otimes s + (-1)^{|\omega|}\omega\otimes(r\cdot s), \tag{18}$$

for any $\xi\in T^*M$, $r\in\mathcal{G}$, $X\in TM$, $\omega\in\Lambda(T^*M)$ and $s\in S_{\mathcal{G}}$. The canonical weighted spinor bundle of E determined by S, as defined in (1), is canonically isomorphic to

$$\mathbb{S} = \Lambda(T^*M)\hat{\otimes}\mathcal{S}_{\mathcal{G}} \tag{19}$$

owing to the canonical isomorphism

$$|\det(\Lambda(TM)\otimes S_{\mathcal{G}}^*|^{\frac{1}{Nr}}\otimes|\det T^*M|^{1/2}\cong|\det S_{\mathcal{G}}^*|^{1/r} \tag{20}$$

given by

$$|(Z_1\otimes s_1^*)\wedge\cdots\wedge(Z_N\otimes s_r^*)|^{\frac{1}{Nr}}\otimes|\alpha_1\wedge\cdots\wedge\alpha_m|^{1/2}\mapsto|s_1^*\wedge\cdots\wedge s_r^*|^{1/r}, \tag{21}$$

where $N := \mathrm{rk}\,\Lambda(TM)$, $r := \mathrm{rk}\,S_{\mathcal{G}}$, (s_i^*) is a local frame of $\mathcal{S}_{\mathcal{G}}^*$, (α_i) is a local frame of T^*M, and (Z_i) is the local frame of $\Lambda(TM)$ determined by the dual frame (X_i) of (α_i).

As shown in Theorem 67 of [9], the canonical Dirac generating operator $\not{d} : \Gamma(\mathbb{S})\to\Gamma(\mathbb{S})$ takes the form

$$\not{d}(\omega\otimes s) = (d\omega - H\wedge\omega)\otimes s + \nabla^{\mathcal{S}_{\mathcal{G}}}(s)\wedge\omega \\ + \frac{1}{4}(-1)^{|\omega|+1}\omega\otimes(C_{\mathcal{G}}s) + (-1)^{|\omega|+1}\bar{R}^E(\omega\otimes s), \tag{22}$$

where $\omega\in\Omega(M)$ and $s\in\Gamma(\mathcal{S}_{\mathcal{G}})$. Above $C_{\mathcal{G}}\in\Gamma(\Lambda^3\mathcal{G}^*)\subset\Gamma(\mathrm{Cl}(\mathcal{G}))$ is the Cartan form $C_{\mathcal{G}}(u,v,w) := \langle[u,v]_{\mathcal{G}},w\rangle_{\mathcal{G}}$ which acts on s by Clifford multiplication, $\nabla^{\mathcal{S}_{\mathcal{G}}}$ is a connection on $\mathcal{S}_{\mathcal{G}}$ induced by (any) connection $\nabla^{S_{\mathcal{G}}}$ on $S_{\mathcal{G}}$ compatible with ∇,

$$\nabla^{\mathcal{S}_{\mathcal{G}}}(s)\wedge\omega = \sum_i \alpha_i\wedge\omega\otimes(\nabla^{\mathcal{S}_{\mathcal{G}}}_{X_i}s)$$

and

$$\bar{R}^E(\omega \otimes s) = \frac{1}{2} \sum_{i,j,k} \langle R(X_i, X_j), r_k \rangle_{\mathcal{G}} (\alpha_i \wedge \alpha_j \wedge \omega) \otimes (\tilde{r}_k s),$$

where (r_k) is a local frame of $\mathcal{G}$, $(\tilde{r}_k)$ the metrically dual frame (i.e. $\langle r_i, \tilde{r}_j \rangle_{\mathcal{G}} = \delta_{ij}$ for any i, j) and $\tilde{r}_k s$ is the Clifford action of $\tilde{r}_k$ on s. Sometimes it will be convenient to write the canonical Dirac generating operator in the form

$$\begin{aligned} \not{d}(\omega \otimes s) &= (d\omega) \otimes s - H \cdot (\omega \otimes s) + \sum_i \alpha_i \cdot (\omega \otimes \nabla^{\mathcal{S}_{\mathcal{G}}}_{X_i} s) \\ &- \frac{1}{4} C_{\mathcal{G}} \cdot (\omega \otimes s) - \frac{1}{2} \sum_{i,j,k} \langle R(X_i, X_j), r_k \rangle_{\mathcal{G}} \tilde{r}_k \cdot \alpha_i \cdot \alpha_j \cdot (\omega \otimes s), \end{aligned} \tag{23}$$

where the dots denote the Clifford action of $\mathrm{Cl}(E) \cong \Lambda E$ on $\mathbb{S}$.

3 The bilinear pairing on spinors

Let $(E, \pi, [\cdot, \cdot], \langle \cdot, \cdot \rangle_E)$ be a rank $2n \geq 2$ regular Courant algebroid over a manifold M, S an irreducible spinor bundle of E of rank r and $\mathcal{S} = S \otimes |\det S^*|^{1/r}$ the canonical spinor bundle of S. Before we state the next proposition we need to define the determinant of a bilinear pairing $\langle \cdot, \cdot \rangle$ on $\mathcal{S}$. For this we consider $\langle \cdot, \cdot \rangle$ as a map $\mathcal{S} \to \mathcal{S}^*$, $v \mapsto \langle v, \cdot \rangle$. Its determinant $\det \mathcal{S} \to \det \mathcal{S}^*$ defines a nowhere vanishing section $\det \langle \cdot, \cdot \rangle \in \Gamma((\det \mathcal{S}^*)^{\otimes 2})$. Since $\mathcal{S} = S \otimes |\det S^*|^{1/r}$, $\det \mathcal{S} = \det S \otimes |\det S^*|$ and $(\det \mathcal{S})^2 \cong (\det S)^2 \otimes |\det S^*|^2 \cong (\det S)^2 \otimes (\det S^*)^2$ is canonically identified with the trivial line bundle, which means that $\det \langle \cdot, \cdot \rangle$ is simply a real-valued function. This function can be computed as follows: let (s_i) be a local frame of S defined on some open set $U \subset M$ and $l := |s_1 \wedge \cdots \wedge s_r|^{-1/r}$. The determinant $\det \langle \cdot, \cdot \rangle$ of $\langle \cdot, \cdot \rangle$ coincides on U with the determinant of the matrix $A = (a_{ij})$ where $a_{ij} := \langle s_i \otimes l, s_j \otimes l \rangle$. Note that $\det \lambda \langle \cdot, \cdot \rangle = \lambda^r \det \langle \cdot, \cdot \rangle$, for any $\lambda \in \mathbb{R}^*$.

Proposition 4. *i) For any $U \subset M$ open and sufficiently small, there is a pairing*

$$\langle \cdot, \cdot \rangle_{\mathcal{S}|_U} : \Gamma(\mathcal{S}|_U) \times \Gamma(\mathcal{S}|_U) \to C^\infty(U) \tag{24}$$

which is $C^\infty(U)$-linear, satisfies

$$\langle u \cdot s, u \cdot \tilde{s} \rangle_{\mathcal{S}|_U} = \langle u, u \rangle_E \langle s, \tilde{s} \rangle_{\mathcal{S}|_U}, \; u \in \Gamma(E|_U), s, \tilde{s} \in \Gamma(\mathcal{S}|_U) \tag{25}$$

and has determinant one if $n > 1$ and -1 if $n = 1$. Any two such pairings differ by multiplication by ± 1.

ii) If n is even then the even and odd parts $\mathcal{S}^0|_U$ and $\mathcal{S}^1|_U$ of $\mathcal{S}|_U$ are orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathcal{S}|_U}$. If n is odd then $\mathcal{S}^0|_U$ and $\mathcal{S}^1|_U$ are isotropic with respect to $\langle\cdot,\cdot\rangle_{\mathcal{S}|_U}$.

iii) The pairing is symmetric if $n \equiv 0,1 \pmod 4$ and skew-symmetric if $n \equiv 2,3 \pmod 4$.

iv) Let D be a generalized connection on E. The pairing $\langle\cdot,\cdot\rangle_{\mathcal{S}|_U}$ is preserved by the E-connection $D^{\mathcal{S}}$ induced by (any) E-connection D^S on S, compatible with D.

The remaining part of this section is devoted to the proof of Proposition 4 and to various corollaries. Let V be an n-dimensional vector space. We begin by considering the irreducible $\mathrm{Cl}(V \oplus V^*)$-module ΛV^* where $V \oplus V^*$ is endowed with its standard metric of neutral signature $\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\xi(Y)+\eta(X))$ and the Clifford action is given by

$$(X+\xi)\omega := i_X\omega + \xi\wedge\omega,\ X\in V,\ \xi\in V^*,\ \omega\in\Lambda V^*.$$

It is well-known that the vector valued bilinear pairing

$$\langle\cdot,\cdot\rangle : \Lambda V^*\otimes\Lambda V^* \to \Lambda^n V^*,\ \langle\omega,\tilde{\omega}\rangle := \left(\omega^t\wedge\tilde{\omega}\right)_{\mathrm{top}}, \tag{26}$$

where ${}^t : \Lambda V^* \to \Lambda V^*$ is defined on decomposable forms by $(\alpha_1\wedge\cdots\wedge\alpha_k)^t := \alpha_k\wedge\cdots\wedge\alpha_1$ and, for a form $\omega\in\Lambda V^*$, $\omega_{\mathrm{top}}\in\Lambda^n V^*$ denotes its component of maximal degree, satisfies (25) (see e.g. [12]). Since the metric of $V\oplus V^*$ has neutral signature, we obtain that (26) is determined (up to multiplication by a non-zero constant), by this property. Note that $\Lambda^{\mathrm{even}}V^*$ and $\Lambda^{\mathrm{odd}}V^*$ are orthogonal with respect to the pairing (26) when n is even and are isotropic when n is odd. Also it is easy to check that the above pairing is non-degenerate, symmetric if $n\equiv 0,1 \pmod 4$ and skew-symmetric if $n\equiv 2,3 \pmod 4$. By choosing a volume form on V, we obtain an $\mathbb{R}$-valued pairing with the same properties. These considerations hold for any irreducible Clifford module in neutral signature and lead to the following lemma.

Lemma 5. *For any $p\in M$, there is a canonical (determined up to multiplication by ±1) $\mathbb{R}$-valued pairing $\langle\cdot,\cdot\rangle_{\mathcal{S}_p}$ on $\mathcal{S}_p$ which satisfies (25) and $\det\langle\cdot,\cdot\rangle_{\mathcal{S}_p}=1$ if $n>1$, respectively $\det\langle\cdot,\cdot\rangle_{\mathcal{S}_p}=-1$ if $n=1$. The pairing is symmetric if $n\equiv 0,1 \pmod 4$ and skew-symmetric if $n\equiv 2,3 \pmod 4$. Moreover, $\mathcal{S}^0_p$ and $\mathcal{S}^1_p$ are orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathcal{S}_p}$ when n is even and are isotropic when n is odd.*

Proof. It remains to prove that we can rescale $\langle\cdot,\cdot\rangle_{\mathcal{S}_p}$ appropriately in order to have $\det\langle\cdot,\cdot\rangle_{\mathcal{S}_p}=1$ or -1. Assume that $n>1$. Using $\det(\lambda\langle\cdot,\cdot\rangle_{\mathcal{S}_p}) =$

$\lambda^r \det(\langle\cdot,\cdot\rangle_{\mathcal{S}_p})$, this reduces to showing that $\det\langle\cdot,\cdot\rangle_{\mathcal{S}_p} > 0$ for any bilinear pairing $\langle\cdot,\cdot\rangle_{S_p}$ which satisfies (25). For this, it is sufficient to compute $\det\langle\cdot,\cdot\rangle_{\mathcal{S}_p}$ using a basis of $\mathcal{S}_p$ of the form $(s_1,\cdots,s_{r/2}, vs_1,\cdots,vs_{r/2})$ where $v\in E$ satisfies $\langle v,v\rangle_{E_p}=1$. We obtain $\det(\langle\cdot,\cdot\rangle_{\mathcal{S}_p}) = (\det A)^2$, where $A=(A_{ij})\in M_{r/2\times r/2}(\mathbb{R})$ with $A_{ij}=\langle s_i\otimes l, s_j\otimes l\rangle_{\mathcal{S}_p}$ when n is even, $A_{ij} = \langle s_i\otimes l, vs_j\otimes l\rangle_{\mathcal{S}_p}$ when $n>1$ is odd and $l := |s_1\wedge\cdots\wedge s_{r/2}\wedge vs_1\wedge\cdots\wedge vs_{r/2}|^{-1/r}$ in both cases. For $n=1$ we obtain instead $\det(\langle\cdot,\cdot\rangle_{\mathcal{S}_p}) = -(\det A)^2$. $\square$

Remark 6. There is the following tautological way to compute $\langle\cdot,\cdot\rangle_{\mathcal{S}_p}$. Consider a bilinear pairing $\langle\cdot,\cdot\rangle_{S_p}$ on S_p, which satisfies (25). Take a local frame (s_i) of S_p and let $l := |s_1\wedge\cdots\wedge s_r|^{-1/r}$. Then

$$\langle s\otimes l, \tilde{s}\otimes l\rangle_{\mathcal{S}} = |\det C|^{-1/r}\langle s,\tilde{s}\rangle_{S_p},\ C := (\langle s_i,s_j\rangle_{S_p})_{i,j}. \tag{27}$$

The next lemma concludes the proof of Proposition 4.

Lemma 7. *Let U be a sufficiently small open subset of M. The section of $(\mathcal{S}^*\otimes\mathcal{S}^*)/\pm 1$ defined by the pairings $\langle\cdot,\cdot\rangle_{\mathcal{S}_p}$ from Lemma 5 lifts to a smooth section $\langle\cdot,\cdot\rangle_{\mathcal{S}|_U}$ of $\mathcal{S}^*|_U\otimes\mathcal{S}^*|_U$, which is preserved by the E-connection $D^{\mathcal{S}}$ on $\mathcal{S}$ induced by any generalized connection D on E.*

Proof. Assume that $E|_U$ admits a local frame (e_i) with $\langle e_i,e_j\rangle_E = \epsilon_i\delta_{ij}$, where $\epsilon_i = 1$ for $i\le n$ and -1 for $i\ge n+1$. On $\mathbb{R}^{2n}$ we consider the standard basis (v_i) and metric $\langle\cdot,\cdot\rangle_{\mathbb{R}^{2n}}$ defined by $\langle v_i,v_j\rangle_{\mathbb{R}^{2n}} = \epsilon_i\delta_{ij}$. Let V be an irreducible $\mathrm{Cl}(\mathbb{R}^{2n})$-module and $\Sigma := U\times V$, which is an irreducible $\mathrm{Cl}(E|_U)$-bundle with Clifford action $\gamma_{e_i}(p,w) := (p, v_i\cdot w)$, for any $(p,w)\in U\times V$. Since E has neutral signature, $S|_U = \Sigma\otimes L$ where L is a line bundle and

$$\mathcal{S}|_U = \Sigma\otimes|\det\Sigma^*|^{1/r}\otimes L\otimes|L^*|,$$

where $r := \dim V$. The bilinear pairing $\langle\cdot,\cdot\rangle_{\mathcal{S}|_U}$ we are looking for is given by

$$\langle s\otimes l,\tilde{s}\otimes l\rangle_{\mathcal{S}} = \langle s,\tilde{s}\rangle_{\Sigma\otimes|\det\Sigma^*|^{1/r}} l^2,\ s,\tilde{s}\in\Sigma\otimes|\det\Sigma^*|^{1/r},\ l\in L\otimes|L^*|,$$

where $\langle s,\tilde{s}\rangle_{\Sigma\otimes|\det\Sigma^*|^{1/r}}$ is the constant pairing on Σ induced by a canonical $\mathbb{R}$-valued pairing on $V\otimes|\det V^*|^{1/r}$ and $l^2\in C^\infty(U)$ under the canonical isomorphism $(L\otimes|L^*|)^2 = U\times\mathbb{R}$. If

$$D_u(e_k) = 2\sum_{j<p}\omega_{pj}(u)(e_p\wedge e_j)(e_k),\ \forall u\in\Gamma(E|_U),$$

where $\omega_{pj}\in\Gamma(E^*|_U)$, then the E-connection D^Σ on Σ defined by

$$D^\Sigma_u(\sigma_\alpha) := \frac{1}{2}\sum_{i<j}\omega_{ji}(u)e_je_i\cdot\sigma_\alpha,\ 1\le\alpha\le r,$$

where (σ_α) is a constant frame of Σ, is compatible with D (see e.g. [9]). Since $\mathrm{trace}(\, e_i e_j \cdot) = 0$, $D^\Sigma(\sigma_1 \wedge \cdots \wedge \sigma_r) = 0$ and the E-connection induced by D^Σ on $\Sigma \otimes |\det \Sigma^*|^{1/r}$, also denoted by D^Σ, satisfies

$$D^\Sigma_u(\sigma_\alpha \otimes l_\Sigma) = \frac{1}{2} \sum_{i<j} \omega_{ji}(u)(e_j e_i \cdot \sigma_\alpha) \otimes l_\Sigma,$$

where $l_\Sigma := |\sigma_1 \wedge \cdots \wedge \sigma_r|^{-1/r}$. Since $\langle \cdot, \cdot \rangle_{\Sigma \otimes |\det \Sigma^*|^{1/r}}$ is constant in the frame $(\sigma_\alpha \otimes l_\Sigma)$ and the Clifford action of $e_i e_j$ is skew-symmetric with respect to $\langle \cdot, \cdot \rangle_{\Sigma \otimes |\det \Sigma^*|^{1/r}}$ (from the property (25) of $\langle \cdot, \cdot \rangle_{\Sigma \otimes |\det \Sigma^*|^{1/r}}$), we obtain that D^Σ preserves $\langle \cdot, \cdot \rangle_{\Sigma \otimes |\det \Sigma^*|^{1/r}}$. Let D^L be an E-connection on L. Then $D^\Sigma \otimes D^L$ is an E-connection on $S|_U$, compatible with D, with the property that the induced connection on $\mathcal{S}|_U$ preserves $\langle \cdot, \cdot \rangle_{\mathcal{S}}$ (easy check). The latter coincides with $D^{\mathcal{S}}|_U$. □

As a consequence of Proposition 4 we obtain, for any $U \subset M$ open and sufficiently small, a canonical (unique modulo ± 1) $C^\infty(U)$-bilinear pairing

$$\langle \cdot, \cdot \rangle_{\mathbb{S}|_U} : \Gamma(\mathbb{S}|_U) \times \Gamma(\mathbb{S}|_U) \to |\det T^*U|, \;\; \langle s \otimes l, \tilde{s} \otimes l \rangle_{\mathbb{S}|_U} := \langle s, \tilde{s} \rangle_{\mathcal{S}|_U} l^2, \quad (28)$$

where $s, \tilde{s} \in \Gamma(\mathcal{S}|_U)$ and $l \in |\det T^*U|^{1/2}$. It satisfies

$$\langle u \cdot (s \otimes l), u \cdot (\tilde{s} \otimes \tilde{l}) \rangle_{\mathbb{S}|_U} = \langle u, u \rangle_E \langle s \otimes l, \tilde{s} \otimes \tilde{l} \rangle_{\mathbb{S}|_U}, \; u \in \Gamma(E|_U), \; s \otimes l, \tilde{s} \otimes \tilde{l} \in \Gamma(\mathbb{S}|_U). \quad (29)$$

When M is oriented, $\langle \cdot, \cdot \rangle_{\mathbb{S}|_U}$ takes values in the bundle $\det T^*U$ of forms of top degree on U. A pairing with similar properties (but with values in $(\det T^*U) \otimes \mathbb{C}$) was constructed in Proposition 3.14 of [13].

Given a standard Courant algebroid $T^*M \oplus \mathcal{G} \oplus TM$, we will often consider, as in the next lemma, an irreducible $\mathrm{Cl}(\mathcal{G})$-bundle $S_{\mathcal{G}}$ of $\mathcal{G}$. This will always be assumed to be $\mathbb{Z}_2$-graded, with gradation compatible with the Clifford action. In the next lemma by a canonical bilinear pairing of the weighted spinor bundle $\mathcal{S}_{\mathcal{G}}|_U$ of $S_{\mathcal{G}}|_U$ we mean a smooth $C^\infty(U)$-bilinear pairing

$$\langle \cdot, \cdot \rangle_{\mathcal{S}_{\mathcal{G}}|_U} : \Gamma(\mathcal{S}_{\mathcal{G}}|_U) \times \Gamma(\mathcal{S}_{\mathcal{G}}|_U) \to C^\infty(U) \quad (30)$$

of normalized determinant, which satisfies

$$\langle u \cdot s, u \cdot \tilde{s} \rangle_{\mathcal{S}_{\mathcal{G}}|_U} = \langle u, u \rangle_{\mathcal{G}} \langle s, \tilde{s} \rangle_{\mathcal{S}_{\mathcal{G}}|_U}, \quad (31)$$

for any $s, \tilde{s} \in \Gamma(\mathcal{S}_{\mathcal{G}}|_U)$ and $u \in \Gamma(\mathcal{G}|_U)$. With the same argument as in Proposition 4, such a pairing exists when $U \subset M$ is a sufficiently small open set and is unique up to multiplication by ± 1. It is preserved by the connection $\nabla^{\mathcal{S}_{\mathcal{G}}}$ induced by any connection $\nabla^{S_{\mathcal{G}}}$ on $S_{\mathcal{G}}$ compatible with ∇.

Lemma 8. *Assume that $E = T^*M \oplus \mathcal{G} \oplus TM$ is a standard Courant algebroid over an oriented manifold M, defined by a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data (∇, R, H). Let $S_{\mathcal{G}}$ be an irreducible $\mathrm{Cl}(\mathcal{G})$-bundle, $\mathcal{S}_{\mathcal{G}} = S_{\mathcal{G}} \otimes |\mathrm{det} S^*|^{1/r}$ the canonical spinor bundle of $S_{\mathcal{G}}$ and $\mathbb{S} = \Lambda(T^*M)\hat{\otimes}\mathcal{S}_{\mathcal{G}}$ the corresponding canonical weighted spinor bundle of E. For any $U \subset M$ open and sufficiently small, the canonical bilinear pairing $\langle\cdot,\cdot\rangle_{\mathbb{S}|_U}$ is given (up to multiplication by ± 1) by*

$$\langle \omega \otimes s, \tilde{\omega} \otimes \tilde{s}\rangle_{\mathbb{S}|_U} = (-1)^{|s|(|\omega|+|\tilde{\omega}|)}(\omega^t \wedge \tilde{\omega})_{\mathrm{top}}\langle s, \tilde{s}\rangle_{\mathcal{S}_{\mathcal{G}}|_U}, \tag{32}$$

where $\langle\cdot,\cdot\rangle_{\mathcal{S}_{\mathcal{G}}|_U}$ is the canonical bilinear pairing of $\mathcal{S}_{\mathcal{G}}|_U$.

Proof. The claim is a consequence of the following general statement: if $(V_i, \langle\cdot,\cdot\rangle_i)$ are Euclidian vector spaces with metrics of neutral signature and S_i are irreducible $\mathrm{Cl}(V_i)$-modules of ranks r_i, with canonical bilinear pairings $\langle\cdot,\cdot\rangle_{\mathcal{S}_i}$ on $\mathcal{S}_i = S_i \otimes |\det S_i^*|^{1/r_i}$, then $S := S_1 \hat{\otimes} S_2$ is an irreducible $\mathrm{Cl}(V_1 \oplus V_2)$-module, with canonical spinor module $\mathcal{S} = \mathcal{S}_1 \hat{\otimes} \mathcal{S}_2$ and the canonical bilinear pairing on $\mathcal{S}$ is given by

$$\langle s_1 \otimes s_2, \tilde{s}_1 \otimes \tilde{s}_2\rangle_{\mathcal{S}} = (-1)^{|s_2|(|s_1|+|\tilde{s}_1|)}\langle s_1, \tilde{s}_1\rangle_{\mathcal{S}_1}\langle s_2, \tilde{s}_2\rangle_{\mathcal{S}_2}. \tag{33}$$

Indeed, the scalar product (33) satisfies (25) (easy check). In order to show that it has normalized determinant, we remark that

$$\det\langle\cdot,\cdot\rangle_{\mathcal{S}} = \det\langle\cdot,\cdot\rangle'_{\mathcal{S}} = (\det\langle\cdot,\cdot\rangle_{\mathcal{S}_1})^{r_2}(\det\langle\cdot,\cdot\rangle_{\mathcal{S}_2})^{r_1} = 1, \tag{34}$$

where

$$\langle s_1 \otimes s_2, \tilde{s}_1 \otimes \tilde{s}_2\rangle'_{\mathcal{S}} := \langle s_1, \tilde{s}_1\rangle_{\mathcal{S}_1}\langle s_2, \tilde{s}_2\rangle_{\mathcal{S}_2}.$$

The first relation in (34) can be checked using that the scalar products $\langle\cdot,\cdot\rangle_{\mathcal{S}}$ and $\langle\cdot,\cdot\rangle'_{\mathcal{S}}$ differ only by a sign (dependent on degrees) when restricted to tensor products of homogeneous elements. (Recall that the even and odd parts of $\mathcal{S}_i$ are orthogonal or isotropic with respect to $\langle\cdot,\cdot\rangle_{\mathcal{S}_i}$.) □

The next corollary will be used to show that the pushforward commutes with the canonical Dirac generating operators (see Section 4.3).

Corollary 9. *In the setting of Lemma 8, let $\nabla^{\mathcal{S}_{\mathcal{G}}}$ be the connection on $\mathcal{S}_{\mathcal{G}}$ induced by an arbitrary connection $\nabla^{S_{\mathcal{G}}}$ on $S_{\mathcal{G}}$, compatible with ∇. Define $\mathcal{E} \in \mathrm{End}\,\Gamma(\mathbb{S}|_U)$ by*

$$\mathcal{E}(\omega \otimes s) := (d\omega) \otimes s + \sum_i (\alpha_i \wedge \omega) \otimes \nabla^{\mathcal{S}_{\mathcal{G}}}_{X_i} s,\ \ \omega \in \Omega(U),\ \ s \in \Gamma(\mathcal{S}_{\mathcal{G}}|_U) \tag{35}$$

where (X_i) *is a local frame of* TU*, with dual frame* (α_i)*. Then, for any* $U \subset M$ *open and sufficiently small and products* $\omega \otimes s, \tilde{\omega} \otimes \tilde{s} \in \Gamma(\mathbb{S}|_U)$ *of homogeneous elements,*

$$\langle \mathcal{E}(\omega \otimes s), \tilde{\omega} \otimes \tilde{s} \rangle_{\mathbb{S}|_U} + \langle \omega \otimes s, \mathcal{E}(\tilde{\omega} \otimes \tilde{s}) \rangle_{\mathbb{S}|_U} = \\ (-1)^{|s|(|\omega|+|\tilde{\omega}|+1)+|\omega|} d\left(\langle s, \tilde{s} \rangle_{\mathcal{S}_{\mathcal{G}}|_U} (\omega^t \wedge \tilde{\omega})_{m-1} \right). \tag{36}$$

Here m *is the dimension of* M *and* ω_{m-1} *denotes the degree* $(m-1)$*-component of a form* $\omega \in \Omega(U)$*.*

Proof. We use the expression (32) of the canonical bilinear pairing $\langle \cdot, \cdot \rangle_{\mathbb{S}|_U}$ of $\mathbb{S}|_U = \Lambda(T^*U)\hat{\otimes}\mathcal{S}_{\mathcal{G}}|_U$. Since $d(\omega^t) = (-1)^{|\omega|}(d\omega)^t$ we obtain

$$\langle (d\omega) \otimes s, \tilde{\omega} \otimes \tilde{s} \rangle_{\mathbb{S}|_U} = \\ (-1)^{|s|(|\omega|+|\tilde{\omega}|+1)+|\omega|} \langle s, \tilde{s} \rangle_{\mathcal{S}_{\mathcal{G}}|_U} \left(d((\omega^t \wedge \tilde{\omega})_{m-1}) + (-1)^{|\omega|+1} (\omega^t \wedge d\tilde{\omega})_{\mathrm{top}} \right). \tag{37}$$

Similarly, since $(\alpha_i \wedge \omega)^t = \omega^t \wedge \alpha_i$ and using that $\nabla^{\mathcal{S}_{\mathcal{G}}}$ preserves $\langle \cdot, \cdot \rangle_{\mathcal{S}_{\mathcal{G}}|_U}$ we obtain

$$\langle (\alpha_i \wedge \omega) \otimes \nabla^{\mathcal{S}_{\mathcal{G}}}_{X_i} s, \tilde{\omega} \otimes \tilde{s} \rangle_{\mathbb{S}|_U} = \\ (-1)^{|s|(|\omega|+|\tilde{\omega}|+1)} (\omega^t \wedge \alpha_i \wedge \tilde{\omega})_{\mathrm{top}} \left(X_i \langle s, \tilde{s} \rangle_{\mathcal{S}_{\mathcal{G}}|_U} - \langle s, \nabla^{\mathcal{S}_{\mathcal{G}}|_U}_{X_i} \tilde{s} \rangle_{\mathcal{S}_{\mathcal{G}}|_U} \right). \tag{38}$$

From (37) and (38) we obtain (36). □

4 Dirac generating operator and operations on spinors

4.1 Behaviour of canonical Dirac generating operators under isomorphisms

The next lemma and proposition are stated for transitive Courant algebroids but the same arguments hold in the larger setting of regular Courant algebroids.

Lemma 10. *Let* $I_E : E_1 \to E_2$ *be an isomorphism of transitive Courant algebroids over a manifold* M *and* S_i *irreducible* $\mathrm{Cl}(E_i)$*-bundles* $(i = 1, 2)$*. Then, for any* $U \subset M$ *open and sufficiently small, there is a unique (up to multiplication by a smooth non-vanishing function) isomorphism* $I_{S|_U} : S_1|_U \to S_2|_U$ *such that*

$$I_{S|_U} \circ \gamma_u = \gamma_{I_E(u)} \circ I_{S|_U}, \ \forall u \in E_1|_U. \tag{39}$$

The map $I_{S|_U}$ is homogeneous (i.e. even or odd). If $\not{d}_1 \in \operatorname{End}\Gamma(S_1|_U)$ is a Dirac generating operator of $E_1|_U$ then $\not{d}_2 := I_{S|_U} \circ \not{d}_1 \circ I_{S|_U}^{-1} \in \operatorname{End}\Gamma(S_2|_U)$ is a Dirac generating operator of $E_2|_U$.

Proof. For any $p \in M$ consider the maps

$$\mathrm{Cl}(E_1)_p \ni u \to \gamma_{I_E(u)} \in \mathrm{End}(S_2)_p,\ \mathrm{Cl}(E_1)_p \ni v \to \gamma_v \in \mathrm{End}(S_1)_p. \quad (40)$$

Since I_E is an isometry, $I_E(uv) = I_E(u)I_E(v)$ in the Clifford algebras and both maps (40) are irreducible representations of $\mathrm{Cl}(E_1)_p$. Since the metric $\langle \cdot, \cdot \rangle_{E_1}$ has neutral signature, the two representations are equivalent, i.e. there is an isomorphism $(I_S)_p : (S_1)_p \to (S_2)_p$ (unique up to a multiplicative factor) such that

$$\gamma_{I_E(u)} = (I_S)_p \circ \gamma_u \circ (I_S)_p^{-1},\ \forall u \in (E_1)_p, \quad (41)$$

which is equivalent to (39). We now show that for any $U \subset M$ open and sufficiently small, the isomorphisms $(I_S)_p$ define an isomorphism $I_{S|_U} : S_1|_U \to S_2|_U$ with the property (39). Assume that $E_1|_U$ admits an orthonormal frame (e_i) and let $(\tilde{e}_i) := (I_E(e_i))$ be the corresponding orthonormal frame of $E_2|_U$. Like in the proof of Lemma 7, $S_i|_U = \Sigma_i \otimes L_i$ where $\Sigma_i := U \times V$ are $\mathrm{Cl}(E_i)$-bundles, constructed using an irreducible $\mathrm{Cl}(\mathbb{R}^{2n})$-module V and the orthonormal frames (e_i) and $(\tilde{e}_i)$ respectively, and L_i are line bundles over U. Restricting U if necessary, we may assume that L_i are isomorphic. Let $I_L : L_1 \to L_2$ be an isomorphism. Then $I_{S|_U} : S_1|_U \to S_2|_U$ defined by $I_{S|_U}(\sigma \otimes l) := \sigma \otimes I_L(l)$ satisfies (39). The even and odd parts of S_1 are given by $S_1^0 = \frac{1}{2}(1+\epsilon\gamma_\omega)S$ and $S_1^1 = \frac{1}{2}(1-\epsilon\gamma_\omega)S$, where $\epsilon \in \{\pm 1\}$ and $\omega = e_1 \cdots e_{2n}$, and similarly for the even and odd parts of S_2 (using $\tilde{\omega} = \tilde{e}_1 \cdots \tilde{e}_n$). Therefore the statement that $I_{S|_U}$ is homogeneous follows from (39), which implies that $I_{S|_U} \circ \gamma_\omega = \gamma_{\tilde{\omega}} \circ I_{S|_U}$. Since $I_{S|_U}$ is homogeneous and $\not{d}_2$ is odd, we obtain that also $\not{d}_1$ is odd. The statement that $\not{d}_2$ satisfies the remaing conditions from the definition of a Dirac generating operator can be checked using (41), which implies

$$[\not{d}_2, \gamma_{I_E(u)}] = I_{S|_U} \circ [\not{d}_1, \gamma_u] \circ I_{S|_U}^{-1}.$$

□

Remark 11. In general, the isomorphisms $I_{S|_U}$ do not glue together to give an isomorphism $I_S : S_1 \to S_2$ compatible with I_E. However, assume that $E_1 = E_2 = E$ and let $S_1 = S_2 = S$ be an irreducible spinor bundle over $\mathrm{Cl}(E)$. If $I_E \in \mathrm{Aut}(E)$ is of the form $I_E(u) = \alpha \cdot u \cdot \alpha^{-1}$, where $\alpha \in \Gamma(\mathrm{Pin}(E))$, then $I_S \in \mathrm{Aut}(S)$ defined by $I_S(s) := \alpha \cdot s$, $s \in \Gamma(S)$, satisfies (39).

The isomorphism $I_{S|_U} : S_1|_U \to S_2|_U$ from Lemma 10 induces an isomorphism $I_{\mathcal{S}|_U} : \mathcal{S}_1|_U \to \mathcal{S}_2|_U$ between the canonical spinor bundles of S_1 and S_2, given by

$$I_{\mathcal{S}}(s \otimes |s_1 \wedge \cdots \wedge s_r|^{-1/r}) := (I_S s) \otimes |I_S s_1 \wedge \cdots \wedge I_S s_r|^{-1/r}, \quad (42)$$

where $s_1 \wedge \cdots \wedge s_r \in \Gamma(\Lambda^r(S_1|_U))$ is non-vanishing. Since $I_{S|_U}$ is unique up to a multiplicative factor, $I_{\mathcal{S}|_U}$ is independent of the choice of $I_{S|_U}$, modulo multiplication by ± 1 (see also Remark 55 of [9]).

Lemma 12. *For any $U \subset M$ open and sufficiently small, the isomorphism $I_{\mathcal{S}|_U}$ preserves the canonical bilinear pairings $\langle\cdot,\cdot\rangle_{\mathcal{S}_i|_U}$ of $\mathcal{S}_i|_U$, i.e. $\langle I_{\mathcal{S}} s, I_{\mathcal{S}}\tilde{s}\rangle_{\mathcal{S}_2|_U} = \epsilon\langle s,\tilde{s}\rangle_{\mathcal{S}_1|_U}$, for all $s,\tilde{s} \in \Gamma(\mathcal{S}_1|_U)$, where $\epsilon \in \{\pm 1\}$ is independent of $s,\tilde{s}$.*

Proof. From relation (39) and the fact that I_E is an isometry, we obtain that bilinear pairing $\langle s,\tilde{s}\rangle'_{\mathcal{S}_1|_U} := \langle I_{\mathcal{S}}(s), I_{\mathcal{S}}(\tilde{s})\rangle_{\mathcal{S}_2|_U}$ on $\mathcal{S}_1$ satisfies (25). Also, $\det\langle\cdot,\cdot\rangle'_{\mathcal{S}_1|_U} = \det\langle\cdot,\cdot\rangle_{\mathcal{S}_2|_U} = 1$, if $\operatorname{rk} E_1 > 2$ (and $= -1$ if $\operatorname{rk} E_1 = 2$). □

From the above lemma, $I_{\mathbb{S}|_U} := I_{\mathcal{S}|_U} \otimes \mathrm{Id}_{|\det T^*U|^{1/2}} : \mathbb{S}_1|_U \to \mathbb{S}_2|_U$ satisfies

$$\langle I_{\mathbb{S}|_U}(s), I_{\mathbb{S}|_U}(\tilde{s})\rangle_{\mathbb{S}_2|_U} = \epsilon\langle s,\tilde{s}\rangle_{\mathbb{S}_1|_U},\quad s,\tilde{s} \in \Gamma(\mathbb{S}_1|_U) \tag{43}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{S}_i|_U}$ are the canonical $|\det T^*U|$-valued bilinear pairings of the canonical weighted spinor bundles $\mathbb{S}_i|_U$ of $E_i|_U$ determined by $S_i|_U$ and $\epsilon \in \{\pm 1\}$ is independent on s and $\tilde{s}$.

Notation 13. The isomorphisms $I_{\mathcal{S}|_U}$ and $I_{\mathbb{S}|_U}$ are determined only up to multiplication by ± 1. In our computations we will often choose (without repeating it each time) one $I_{\mathcal{S}|_U}$, $I_{S|_U}$ or $I_{\mathbb{S}|_U}$, and refer to it as the isomorphism induced by I (or the isomorphism compatible with I) on the spinor bundle, canonical spinor bundle and canonical weighted spinor bundle on U, respectively. The arguments will be independent on this choice. A similar convention will be used for the various canonical bilinear pairings like $\langle\cdot,\cdot\rangle_{\mathbb{S}|_U}$ or $\langle\cdot,\cdot\rangle_{\mathcal{S}_{\mathcal{G}}|_U}$ and for the pullback and pushforward on spinors (which will also be uniquely defined only up to multiplication by ± 1, see the next sections).

Remark 14. In the setting of Lemma 10, assume that $E_i = T^*M \oplus \mathcal{G}_i \oplus TM$ $(i = 1,2)$ are standard Courant algebroids and that $I_S : S_1 \to S_2$ is defined globally. Let $S_{\mathcal{G}_i}$ be irreducible $\mathrm{Cl}(\mathcal{G}_i)$-bundles of rank r and $S_i := \Lambda(T^*M)\hat{\otimes} S_{\mathcal{G}_i}$. Using (19) and (20), one can show that the isomorphism

$$I_{\mathbb{S}} : \mathbb{S}_1 = \Lambda\,(T^*M)\hat{\otimes}\mathcal{S}_{\mathcal{G}_1} \to \mathbb{S}_2 = \Lambda\,(T^*M)\hat{\otimes}\mathcal{S}_{\mathcal{G}_2}$$

induced by $I : E_1 \to E_2$ on the canonical weighted spinor bundles determined by S_i is given in terms of $I_S : S_1 \to S_2$ by

$$I_{\mathbb{S}}(\omega\otimes s\otimes|s_1^*\wedge\cdots\wedge s_r^*|^{\frac{1}{r}}) = |\det(I_S)|^{-\frac{1}{Nr}}\, I_S(\omega\otimes s)\otimes|\tilde{s}_1^*\wedge\cdots\wedge\tilde{s}_r^*|^{\frac{1}{r}}, \tag{44}$$

where $\omega\otimes s \in \Gamma(S_1)$, (s_i^*) and $(\tilde{s}_i^*)$ are local frames of $S^*_{\mathcal{G}_1}$ and $S^*_{\mathcal{G}_2}$ respectively, $N := \operatorname{rk}\Lambda(TM)$, $r := \operatorname{rk} S_{\mathcal{G}_i}$, and $\det(I_S)$ is the determinant of the representation matrix of I_S in the local frames $(\Omega_i \otimes s_j)$ and $(\Omega_i \otimes \tilde{s}_j)$ respectively, where (Ω_i) is the local frame of $\Lambda(T^*M)$ induced by a local frame of TM and (s_i), $(\tilde{s}_i)$ are the frames dual to (s_i^*) and $(\tilde{s}_i^*)$ respectively. (We shall refer to $\det(I_S)$ as the determinant of I_S with respect to the local frames (s_i) and $(\tilde{s}_i)$).

Proposition 15. *In the setting of Lemma 10, if $\not{d}_1$ is the canonical Dirac generating operator of $E_1|_U$ then*

$$\not{d}_2 = I_{\mathbb{S}|_U} \circ \not{d}_1 \circ I_{\mathbb{S}|_U}^{-1} \tag{45}$$

is the canonical Dirac generating operator of $E_2|_U$.

Proof. Let $\nabla^{(1)}$ be a metric connection on $E_1|_U$ and ∇^{S_1} a compatible connection on $S_1|_U$. Let $D^{(1)}$ and D^{S_1} be the generalized connection on $E_1|_U$ and the $E_1|_U$-connection on $S_1|_U$ defined by $\nabla^{(1)}$ and ∇^{S_1} respectively. Let $\nabla^{(2)} := I_E \circ \nabla^{(1)} \circ I_E^{-1}$ and $\nabla^{S_2} := I_{S|_U} \circ \nabla^{S_1} \circ (I_{S|_U})^{-1}$. Then $\nabla^{(2)}$ is a metric connection on $E_2|_U$ and ∇^{S_2} is compatible with $\nabla^{(2)}$. Let $D^{(2)}$ and D^{S_2} be the generalized connection on $E_2|_U$ and the (compatible) $E_2|_U$-connection on $S_2|_U$, defined by $\nabla^{(2)}$ and ∇^{S_2}. As formula (2) for the canonical Dirac generating operator is independent of the choice of generalized connection (and compatible E-connection), we can (and will) choose to compute $\not{d}_1$ and $\not{d}_2$ using $(D^{(1)}, D^{S_1})$ and $(D^{(2)}, D^{S_2})$ respectively.

A straightforward computation using (3) shows that

$$(D^{(2)})^L_{I_E(u)}\mu = (D^{(1)})^L_u \mu, \ \forall u \in E_1|_U, \ \mu \in \Gamma(L|_U) \tag{46}$$

and

$$(D^{\mathcal{S}_2} \otimes (D^{(2)})^L)_u = I_{\mathbb{S}|_U} \circ (D^{\mathcal{S}_1} \otimes (D^{(1)})^L)_{I_E^{-1}(u)} \circ (I_{\mathbb{S}|_U})^{-1}. \tag{47}$$

Relation (47) implies that the Dirac operators $\not{D}^{(2)}$ on $\mathbb{S}_2|_U$ and $\not{D}^{(1)}$ on $\mathbb{S}_1|_U$ computed with $D^{\mathcal{S}_2} \otimes (D^{(2)})^L$ and $D^{\mathcal{S}_1} \otimes (D^{(1)})^L$ respectively, are related by

$$\not{D}^{(2)} = I_{\mathbb{S}|_U} \circ \not{D}^{(1)} \circ (I_{\mathbb{S}|_U})^{-1}. \tag{48}$$

On the other hand, it is easy to see that

$$T^{D^{(2)}}(u, v, w) = T^{D^{(1)}}(I_E^{-1}u, I_E^{-1}v, I_E^{-1}w), \ \forall u, v, w \in E_2|_U$$

which implies that

$$T^{D^{(2)}} = I_E(T^{D^{(1)}}) \tag{49}$$

where $T^{D^{(i)}} \in \Gamma(\Lambda^3 E_i|_U) \subset \Gamma \operatorname{Cl}(E_i|_U)$ and I_E denotes the action induced by the isometry I_E on Clifford algebras. Relations (39) and (49) imply that $\gamma_{T^{D^{(2)}}} = I_{\mathbb{S}|_U} \circ \gamma_{T^{D^{(1)}}} \circ (I_{\mathbb{S}|_U})^{-1}$ which, together with (2 and (48), implies our claim. □

4.2 Pullback of spinors

Let $f : M \to N$ be a submersion and E a transitive Courant algebroid over N. Following [14], we recall the definition of the pullback Courant algebroid $f^!E$. Let $\mathbb{T}M := T^*M \oplus TM$ be the generalized tangent bundle with its standard Courant algebroid structure, given by the Dorfmann bracket

$$[\xi + X, \eta + Y] := \mathcal{L}_X(Y + \eta) - i_Y d\xi, \tag{50}$$

for any $X, Y \in \mathfrak{X}(M)$, $\xi, \eta \in \Omega^1(M)$, scalar product $\langle \xi+X, \eta+Y\rangle := \frac{1}{2}(\xi(Y)+\eta(X))$ and anchor the natural projection from $\mathbb{T}M$ to TM. Consider the direct product Courant algebroid $E \times \mathbb{T}M$ and let $a : E \times \mathbb{T}M \to T(N \times M)$ be its anchor. Then $C := a^{-1}(TM_f)$ is a coisotropic subbundle of $E \times \mathbb{T}M$ over the graph $M_f \subset N \times M$ of f, which we identify with M. Its fiber over $p \in M$ is given by

$$C_p := \{(u, \eta + X) \in E_{f(p)} \times \mathbb{T}_pM \mid \pi(u) = (d_pf)(X)\} \tag{51}$$

and

$$C_p^{\perp} := \{(\frac{1}{2}\pi^*\gamma, -(d_pf)^*\gamma) \mid \gamma \in T^*_{f(p)}N\} \subset C_p, \tag{52}$$

where $\pi^* : T^*N \to E$ is the dual of the anchor $\pi : E \to TN$ composed with the natural identification $E^* \overset{\sim}{\to} E$ induced by the scalar product $\langle \cdot, \cdot \rangle$ of E. The quotient $C/C^{\perp}$ is a Courant algebroid over $M \cong M_f$ with anchor, scalar product and Courant bracket induced from $E \times \mathbb{T}M$. The Courant algebroid $C/C^{\perp}$ was called in [14] the pullback of E by the map f.

Lemma 16. *i) Let $E = T^*N \oplus \mathcal{G} \oplus TN$ be a standard Courant algebroid, defined by a bundle of quadratic Lie algebras $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle \cdot, \cdot \rangle_{\mathcal{G}})$ and data (∇, R, H). Then $f^!E$ is isomorphic to the standard Courant algebroid defined by the bundle of quadratic Lie algebras*

$$(f^*\mathcal{G},\ [\cdot,\cdot]_{f^*\mathcal{G}} := f^*[\cdot,\cdot]_{\mathcal{G}},\ \langle \cdot,\cdot\rangle_{f^*\mathcal{G}} := f^*\langle\cdot,\cdot\rangle_{\mathcal{G}})$$

together with $(f^*\nabla, f^*R, f^*H)$.

ii) Let $I : E_1 \to E_2$ be an isomorphism between two transitive Courant algebroids over N and $a_i : E_i \times \mathbb{T}M \to T(N \times M)$ the anchors of the direct product Courant algebroids $E_i \times \mathbb{T}M$ ($i = 1, 2$). Then I induces a Courant algebroid isomorphism $I^f : f^!E_1 \to f^!E_2$ defined by

$$I^f[(u, \eta + X)] := [(I(u), \eta + X)],\ \forall (u, \eta + X) \in (C_1)_p, \tag{53}$$

where $C_i = (a_i)^{-1}(TM_f)$ and $[(I(u), \eta + X)]$ denotes the class of $(I(u), \eta + X) \in (C_2)_p$ modulo $(C_2)_p^{\perp}$.

iii) Let E be a transitive Courant algebroid over N. Any dissection of E induces a dissection of $f^! E$. Moreover, if $I_i : E \to T^*N \oplus \mathcal{G}_i \oplus TN$ are two dissections of E, related by (β, K, Φ), then the induced dissections of $f^! E$ are related by $(f^*\beta, f^*K, f^*\Phi)$.

Proof. i) We claim that the quadratic Lie algebra bundle $(f^*\mathcal{G}, [\cdot,\cdot]_{f^*\mathcal{G}}, \langle\cdot,\cdot\rangle_{f^*\mathcal{G}})$ together with $(f^*\nabla, f^*R, f^*H)$ define a standard Courant algebroid. The proof reduces to the verification of the conditions stated in Section 2.2.1. The form f^*R is defined by $(f^*R)(X,Y) = R(dfX, dfY) \in \mathcal{G}_{f(p)} = (f^*\mathcal{G})_p$, for any $X, Y \in T_pM$, $p \in M$. To show, for instance, that

$$[d^{f^*\nabla}(f^*R)](X,Y,Z) = 0 \quad \text{for all} \quad X,Y,Z \in \mathfrak{X}(M), \tag{54}$$

cf. equation (5), we notice that it holds for any projectable vector fields $X, Y, Z \in \mathfrak{X}(M)$, since

$$(f^*\nabla)_X[(f^*R)(Y,Z)] = f^*[\nabla_{f_*X} R(f_*Y, f_*Z)],$$
$$(f^*R)(\mathcal{L}_X Y, Z) = f^*[R(\mathcal{L}_{f_*X} f_*Y, f_*Z)]$$

and that it is $C^\infty(M)$-linear in all arguments X, Y, Z. Here f_*X denotes the vector field on N obtained by projection of a projectable vector field $X \in \mathfrak{X}(M)$. Recall that for projectable vector fields we have $dfX = (f_*X)\circ f$. Relation (54) follows. In a similar way we prove that $(f^*\mathcal{G}, [\cdot,\cdot]_{f^*\mathcal{G}}, \langle\cdot,\cdot\rangle_{f^*\mathcal{G}})$ together with $(f^*\nabla, f^*R, f^*H)$ satisfy the remaining conditions for standard Courant algebroids.

One can show that the map

$$F : T^*M \oplus f^*\mathcal{G} \oplus TM \to f^! E,\ F(\eta + r + X) := [(r + df(X), \eta + X)] \tag{55}$$

where $\eta \in T^*_pM$, $r \in \mathcal{G}_{f(p)}$, $X \in T_pM$ and $p \in M$ is arbitrary, is a Courant algebroid isomorphism between the standard Courant algebroid defined by the quadratic Lie algebra bundle $(f^*\mathcal{G}, [\cdot,\cdot]_{f^*\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ together with $(f^*\nabla, f^*R, f^*H)$, and $f^! E$.

ii), iii) Claim ii) is an easy check and claim iii) follows by combining claims ii) and iii) and using (9). □

Our next aim is to define a pullback from spinors of E to spinors of $f^! E$. At first, we assume that E is a standard Courant algebroid.

Remark 17. i) Assume that $E = T^*N \oplus \mathcal{G} \oplus TN$ is a standard Courant algebroid, defined by a bundle of quadratic Lie algebras $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data (∇, R, H). Using the isomorphism (55), we often identify (without repeating it each time) $f^! E$ with the standard Courant algebroid $T^*M \oplus$

$f^*\mathcal{G}\oplus TM$ defined by the quadratic Lie algebra bundle $(f^*\mathcal{G}, f^*[\cdot,\cdot]_{\mathcal{G}}, f^*\langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data $(f^*\nabla, f^*R, f^*H)$. We fix an irreducible $\mathrm{Cl}(\mathcal{G})$-bundle $S_{\mathcal{G}}$. Then $S_{f^*\mathcal{G}} := f^*S_{\mathcal{G}}$ is an irreducible $\mathrm{Cl}(f^*\mathcal{G})$-bundle and $f^*\mathcal{S}_{\mathcal{G}} = \mathcal{S}_{f^*\mathcal{G}}$. The natural map

$$\begin{aligned} f^* : \Gamma(\mathbb{S}_N) =&\Omega(N, \mathcal{S}_{\mathcal{G}}) \to \Gamma(\mathbb{S}_M) = \Omega(M, f^*\mathcal{S}_{\mathcal{G}}),\\ &\omega\otimes s \to f^*(\omega)\otimes f^*(s) \end{aligned} \tag{56}$$

preserves the $\mathbb{Z}_2$-degrees of $\mathbb{S}_N$ and $\mathbb{S}_M$. It is called the pullback on spinors.

ii) Assume in addition that $f : M \to N$ is endowed with a horizontal distribution. For any $X \in \mathfrak{X}(N)$, we denote by $\widehat{X} \in \mathfrak{X}(M)$ the horizontal lift of X and we define a map

$$f^* : \Gamma(E) \to \Gamma(f^!E),\ f^*(\xi + r + X) := f^*(\xi + r) + \widehat{X}. \tag{57}$$

Let $\langle\cdot,\cdot\rangle_E$ and $\langle\cdot,\cdot\rangle_{f^!E}$ be the scalar products of E and $f^!E$. As

$$\langle f^*(u), f^*(v)\rangle_{f^!E} = \langle u, v\rangle_E \circ f,\ u, v \in \Gamma(E),$$

we obtain an induced map $f^* : \Gamma\,\mathrm{Cl}(E) \to \Gamma\,\mathrm{Cl}(f^!E)$, which satisfies

$$f^*(u\cdot v) = f^*(u)\cdot f^*(v),\ \forall u, v \in \Gamma\,\mathrm{Cl}(E) \tag{58}$$

and

$$f^*(u\cdot s) = f^*(u)\cdot f^*(s),\ u \in \Gamma\mathrm{Cl}(E),\ s \in \Gamma(\mathbb{S}_N). \tag{59}$$

Assume now that E is a transitive, but not necessarily standard, Courant algebroid and let $\mathbb{S}_E$ and $\mathbb{S}_{f^!E}$ be canonical weighted spinor bundles of E and $f^!E$, determined by irreducible spinor bundles S_E and $S_{f^!E}$ respectively. In order to be able to construct a pullback map from $\Gamma(\mathbb{S}_E)$ to $\Gamma(\mathbb{S}_{f^!E})$, we assume that the following condition is satisfied: there is a dissection $I : E \to E_N = T^*N \oplus \mathcal{G} \oplus TN$ of E and an irreducible $\mathrm{Cl}(\mathcal{G})$-bundle $S_{\mathcal{G}}$, such that I and the dissection $I^f : f^!E \to E_M = T^*M \oplus f^*\mathcal{G} \oplus TM$ of $f^!E$ induce global isomorphisms $I_S : S_E \to \Lambda(T^*N)\hat{\otimes}S_{\mathcal{G}}$ and $I^f_S : S_{f^!E} \to \Lambda(T^*M)\hat{\otimes}f^*S_{\mathcal{G}}$ between spinor bundles. We shall often refer to $(I, S_{\mathcal{G}})$ as an admissible pair for $\mathbb{S}_E$ and $\mathbb{S}_{f^!E}$. Let

$$\begin{aligned} I_{\mathbb{S}} &: \mathbb{S}_E \to \mathbb{S}_N = \Lambda(T^*N)\hat{\otimes}\mathcal{S}_{\mathcal{G}}\\ I^f_{\mathbb{S}} &: \mathbb{S}_{f^!E} \to \mathbb{S}_M = \Lambda(T^*M)\hat{\otimes}f^*\mathcal{S}_{\mathcal{G}} \end{aligned} \tag{60}$$

be the induced (global) isomorphisms between the canonical weighted spinor bundles determined by S_E, $\Lambda(T^*N)\hat{\otimes}S_{\mathcal{G}}$, $S_{f^!E}$ and $\Lambda(T^*M)\hat{\otimes}f^*S_{\mathcal{G}}$.

Lemma 18. *The map*

$$f^! : \Gamma(\mathbb{S}_E) \to \Gamma(\mathbb{S}_{f^! E}),\ f^! := (I^f_{\mathbb{S}})^{-1} \circ f^* \circ I_{\mathbb{S}} \tag{61}$$

is well defined, up to multiplication by ± 1. *It is called the* pullback on spinors.

Proof. Let

$$I : E_1 = T^*N \oplus \mathcal{G}_1 \oplus TN \to E_2 = T^*N \oplus \mathcal{G}_2 \oplus TN \tag{62}$$

be an isomorphism between standard Courant algebroids and

$$I^f : f^! E_1 = T^*M \oplus f^*\mathcal{G}_1 \oplus TM \to f^! E_2 = T^*M \oplus f^*\mathcal{G}_2 \oplus T^*M$$

the induced isomorphism between their pullbacks. Let $S_{\mathcal{G}_i}$ be irreducible $\mathrm{Cl}(\mathcal{G}_i)$-bundles, such that I and I^f induce global isomorphisms

$$I_S : \Lambda(TN)\hat{\otimes} S_{\mathcal{G}_1} \to \Lambda(TN)\hat{\otimes} S_{\mathcal{G}_2},\ I^f_S : \Lambda(TM)\hat{\otimes} f^* S_{\mathcal{G}_1} \to \Lambda(TM)\hat{\otimes} f^* S_{\mathcal{G}_2}.$$

By considering two admissible pairs for $\mathbb{S}_E$ and $\mathbb{S}_{f^! E}$ the claim reduces to showing that

$$I^f_{\mathbb{S}_{\mathrm{M}}} \circ f^* = \epsilon f^* \circ I_{\mathbb{S}_N} \tag{63}$$

where $\epsilon \in \{\pm 1\}$,

$$I_{\mathbb{S}_N} : \mathbb{S}^1_N \to \mathbb{S}^2_N,\ I^f_{\mathbb{S}_M} : \mathbb{S}^1_M \to \mathbb{S}^2_M \tag{64}$$

are the isomorphisms induced by I and I^f on the canonical weighted spinor bundles

$$\mathbb{S}^i_N = \Lambda(T^*N)\hat{\otimes}\mathcal{S}_{\mathcal{G}_i},\ \mathbb{S}^i_M = \Lambda(T^*M)\hat{\otimes} f^*\mathcal{S}_{\mathcal{G}_i}$$

determined by the spinor bundles

$$S^i_N := \Lambda(T^*N)\hat{\otimes} S_{\mathcal{G}_i},\ S^i_M := \Lambda(T^*M)\hat{\otimes} f^* S_{\mathcal{G}_i}$$

where $\mathcal{S}_{\mathcal{G}_i} = S_{\mathcal{G}_i} \otimes |\det S^*_{\mathcal{G}_i}|^{1/r}$, and $f^* : \mathbb{S}^i_N \to \mathbb{S}^i_M$ are defined by (56). In order to prove (63) we fix a distribution $\mathcal{D} \subset TM$ complementary to $\mathrm{Ker}\, df$ and we decompose orthogonally $f^! E_i = V_i^+ \oplus V^-$, where V_i^+ and V_- are given by

$$(V_i^+)_p = \mathcal{D}^*_p \oplus (\mathcal{G}_i)_{f(p)} \oplus \mathcal{D}_p$$
$$(V^-)_p = (\mathrm{Ker}\, d_p f)^* \oplus \mathrm{Ker}\, d_p f,$$

for any $p \in M$. Assume that I is defined by (β, K, Φ) as in Section 2.2.1. Then, from Lemma 16 iii), I^f is defined by $(f^*\beta, f^*K, f^*\Phi)$ and acts as the identity on V^- while its restiction $I^{f+} := I^f|_{V_1^+} : V_1^+ \to V_2^+$ satisfies

$$(I^{f+})_p(f^*u) = f^*(I_{f(p)}(u)),\ \forall u \in (E_1)_{f(p)},\ p \in N, \tag{65}$$

where $f^* : (E_i)_{f(p)} \to (V_i^+)_p$ are given by (57), constructed using the distribution $\mathcal{D}$. Consider the spinor bundles

$$S_i^+ := \Lambda\, \mathcal{D}^* \hat{\otimes} f^* S_{\mathcal{G}_i},\ S^- = \Lambda\, (\mathrm{Ker}\, df)^* \tag{66}$$

of V_i^+ and V^-. Then $\bar{S}_M^i := S_i^+ \hat{\otimes} S^-$ is a spinor bundle of $f^! E_i$, isomorphic to the spinor bundle S_M^i via the $\mathrm{Cl}(f^! E_i)$- bundle isomorphism

$$T_i : \bar{S}_M^i \to S_M^i,\ T((\omega \otimes s) \otimes \eta) = (-1)^{|s||\eta|} (\omega \wedge \eta) \otimes s \tag{67}$$

where $\omega \in \Lambda \mathcal{D}^*$ and $s \in f^* S_{\mathcal{G}_i}$, $\eta \in S^-$ are homogeneous. Let

$$I_{S^+}^{f_+} := f^* \circ I_{S_N} \circ (f^*)^{-1} : S_1^+ \to S_2^+, \tag{68}$$

where $f^* : S_N^2 \to S_2^+$ and $(f^*)^{-1} : S_1^+ \to S_N^1$ are induced by the pullback (and its inverse). From (65), we obtain that $I_{S^+}^{f_+}$ is compatible with I^{f_+}. Since $I^f|_{V^-} = \mathrm{Id}_{V^-}$,

$$I_{\bar{S}_M}^f (s \hat{\otimes} \eta) := (-1)^{|\eta||I_{S^+}^{f_+}|} I_{S^+}^{f_+}(s) \otimes \eta, \tag{69}$$

for any $s \in S_1^+$ and $\eta \in S^-$ homogeneous, where $|\eta|$ and $|I_{S^+}^{f_+}|$ denote the degrees of η and $I_{S^+}^{f_+}$, is compatible with I^f. The isomorphism $I_{\bar{S}_M}^f$ induces, via the isomorphisms (67), an isomorphism $I_{S_M}^f : S_M^1 \to S_M^2$ compatible with I^f, which maps $S_1^+ \subset S_M^1$ onto $S_2^+ \subset S_M^2$ and whose restriction to S_1^+ coincides with $I_{S^+}^{f_+}$. As we already know, any isomorphism compatible with I^f and acting between S_M^i is uniquely determined up to a multiplicative factor and the isomorphism it induces on the canonical weighted spinor bundles $\mathbb{S}_M^i$ is independent of this factor, up to multiplication by ± 1. It remains to show that the isomorphisms $I_{\mathbb{S}_M}^f : \mathbb{S}_M^1 \to \mathbb{S}_M^2$ and $I_{\mathbb{S}_N} : \mathbb{S}_N^1 \to \mathbb{S}_N^2$ induced by $I_{S_M}^f$ (defined as above) and I_{S_N} are related by (63). For this, we use Remark 14. Let (s_i), $(\tilde{s}_i)$ be local frames of $\mathcal{S}_{\mathcal{G}_1}$, $\mathcal{S}_{\mathcal{G}_2}$ and (s_i^*), $(\tilde{s}_i^*)$ the dual frames. From Remark (14),

$$\begin{aligned} &I_{\mathbb{S}_M}^f ((\omega \otimes s) \otimes |f^* s_1^* \wedge \cdots \wedge f^* s_r^*|^{1/2}) \\ &= I_{S_M}^f (\omega \otimes s) \otimes |f^* \tilde{s}_1^* \wedge \cdots \wedge f^* \tilde{s}_r^*|^{1/2} |\det(I_{S_M}^f)|^{-\frac{1}{r N_h N_v}} \end{aligned} \tag{70}$$

where $\omega \in \Lambda(T^*M)$, $s \in f^* S_{\mathcal{G}_i}$, $N_h := \mathrm{rk}\,(\Lambda \mathcal{D})$, $N_v := \mathrm{rk}\,(\Lambda\, \mathrm{Ker} df)$, $r := \mathrm{rk}\, S_{\mathcal{G}_i}$ and $\det(I_{S_M}^f)$ denotes the determinant of $I_{S_M}^f$ with respect to the local frames $(f^* s_i)$ and $(f^* \tilde{s}_i)$. Similarly,

$$I_{\mathbb{S}_N}((\omega \otimes s) \otimes |s_1^* \wedge \cdots \wedge s_r^*|^{1/2}) = I_{S_N}(\omega \otimes s) \otimes |\tilde{s}_1^* \wedge \cdots \wedge \tilde{s}_r^*|^{1/2} |\det(I_{S_N})|^{-\frac{1}{r N_h}} \tag{71}$$

where $\omega \in \Lambda(T^*N)$, $s \in S_{\mathcal{G}_1}$ and $\det(I_{S_N})$ is the determinant of I_{S_N} with respect to the local frames (s_i) and $(\tilde{s}_i)$. Using (70), (71) together with

$$\det(I^f_{S_M}) = \det(I^{f_+}_{S_+})^{N_v} = \det(I_{S_N})^{N_v} \circ f \tag{72}$$

we obtain (63). (In the first relation (72) we used the definition of $I^f_{S_M}$ and (69) while in the second relation (72) we used the definition (68) of I_{S_N}). □

Proposition 19. *Let $f : M \to N$ be a submersion, E a transitive Courant algebroid over N and $\mathbb{S}_E$, $\mathbb{S}_{f^!E}$ canonical weighted spinor bundles of E and $f^!E$ such that the pullback $f^! : \Gamma(\mathbb{S}_E) \to \Gamma(\mathbb{S}_{f^!E})$ is defined. Let $\not{d}_E \in \mathrm{End}\,\Gamma(\mathbb{S}_E)$ and $\not{d}_{f^!E} \in \mathrm{End}\,\Gamma(\mathbb{S}_{f^!E})$ be the canonical Dirac generating operators of E and $f^!E$. Then*

$$f^! \circ \not{d}_E = \not{d}_{f^!E} \circ f^!. \tag{73}$$

Proof. From the invariance of the canonical Dirac generating operators under isomorphisms (see Proposition 15) and the definition of $f^!$ (see 61)) it is sufficient to prove (73) when $E = T^*N \oplus \mathcal{G} \oplus TN$ is a standard Courant algebroid, as in Remark 17. With the notation of that remark, we need to show that

$$f^* \circ \not{d}_N = \not{d}_M \circ f^* : \Gamma(\mathbb{S}_N) \to \Gamma(\mathbb{S}_M), \tag{74}$$

where $\not{d}_N$ and $\not{d}_M$ are the canonical Dirac generating operators of E and $f^!E = T^*M \oplus f^*\mathcal{G} \oplus TM$, which can be computed using (22). Let m and n be the dimensions of M and N respectively. Let $(X_i)_{1\le i\le m}$ be a local frame of TM such that $(X_i)_{1\le i\le n}$ are projectable and their projections $(f_*X_i)_{i\le n}$ form a local frame of TN and $(X_i)_{n+1\le i\le m}$ are vertical. Let $(\alpha_i)_{1\le i\le n}$ be the dual frame of $(f_*X_i)_{1\le i\le n}$. Then, using $f_*X_i = 0$ for any $i \ge n+1$,

$$\begin{aligned}
&\bar{R}^{f^!E}(f^*(\omega \otimes s))\\
&= \frac{1}{2}\sum_{i,j\le n} \langle f^*(R(f_*X_i, f_*X_j)), f^*r_k\rangle_{f^*\mathcal{G}} f^*(\alpha_i \wedge \alpha_j \wedge \omega) \otimes (f^*(r_k) f^*(s))
\end{aligned}$$

that is,

$$(\bar{R}^{f^!E} \circ f^*)(\omega \otimes s) = (f^* \circ \bar{R}^E)(\omega \otimes s). \tag{75}$$

On the other hand, if $\nabla^{S_\mathcal{G}}$ is compatible with ∇ then $\nabla^{S_{f^*\mathcal{G}}} := f^*\nabla^{S_\mathcal{G}}$ is compatible with $f^*\nabla$ and

$$\nabla^{\mathcal{S}_{f^*\mathcal{G}}} = f^*\nabla^{\mathcal{S}_\mathcal{G}}. \tag{76}$$

Relations (75), (76), $C_{f^*\mathcal{G}} = f^*C_\mathcal{G}$ and the expression of the canonical Dirac generating operator (22) imply (74). □

4.3 Pushforward on spinors

Let $f : M \to N$ be a fiber bundle with compact fibers and M, N oriented. Let E a transitive Courant algebroid over N. In this section we define a pushforward from spinors of $f^! E$ to spinors of E. As for the pullback, we assume first that $E = T^*N \oplus \mathcal{G} \oplus TN$ is a standard Courant algebroid, as in Remark 17. We choose an irreducible $\mathrm{Cl}(\mathcal{G})$-bundle $S_{\mathcal{G}}$, with canonical spinor bundle $\mathcal{S}_{\mathcal{G}}$. Consider an open cover $\mathcal{U} = \{U_i\}$ of N and, for any $U_i \in \mathcal{U}$, a canonical bilinear pairing $\langle \cdot, \cdot \rangle_{\mathcal{S}_{\mathcal{G}}|_{U_i}}$ on $\Gamma(\mathcal{S}_{\mathcal{G}}|_{U_i})$. We define $\langle \cdot, \cdot \rangle_{f^*\mathcal{S}_{\mathcal{G}}|_{f^{-1}(U_i)}} := f^* \langle \cdot, \cdot \rangle_{\mathcal{S}_{\mathcal{G}}|_{U_i}}$, which is a canonical bilinear pairing on $\Gamma(\mathcal{S}_{f^*\mathcal{G}}|_{f^{-1}(U_i)})$, where $\mathcal{S}_{f^*\mathcal{G}} = f^*\mathcal{S}_{\mathcal{G}}$ is the canonical spinor bundle of $f^* S_{\mathcal{G}}$. We denote by $\langle \cdot, \cdot \rangle_{\mathbb{S}_{U_i}}$ and $\langle \cdot, \cdot \rangle_{\mathbb{S}_{f^{-1}(U_i)}}$ the corresponding $\det (T^* U_i)$ and $\det (T^* f^{-1}(U_i))$-valued canonical bilinear pairings on $\Gamma(\mathbb{S}_N|_{U_i})$ and $\Gamma(\mathbb{S}_M|_{f^{-1}(U_i)})$, where $\mathbb{S}_N = \Lambda(T^*N) \hat{\otimes} \mathcal{S}_{\mathcal{G}}$ and $\mathbb{S}_M = \Lambda(T^*M) \hat{\otimes} f^*\mathcal{S}_{\mathcal{G}}$, see relation (32). For any $U_i \in \mathcal{U}$, let

$$f_*^{U_i} : \Gamma(\mathbb{S}_M|_{f^{-1}(U_i)}) = \Omega(f^{-1}(U_i), f^*\mathcal{S}_{\mathcal{G}}) \to \Gamma(\mathbb{S}_N|_{U_i}) = \Omega(U_i, \mathcal{S}_{\mathcal{G}}) \tag{77}$$

be defined by

$$\int_{U_i} \langle f_*^{U_i} s_1, s_2 \rangle_{\mathbb{S}_{U_i}} = \int_{f^{-1}(U_i)} \langle s_1, f^* s_2 \rangle_{\mathbb{S}_{f^{-1}(U_i)}}, \tag{78}$$

for all $s_1 \in \Gamma(\mathbb{S}_M|_{f^{-1}(U_i)})$ and $s_2 \in \Gamma_c(\mathbb{S}_N|_{U_i})$, where $\Gamma_c(V)$ denotes the space of compactly supported sections of a vector bundle V. Using the maps $f_*^{U_i}$ and a partition of unity $\{\lambda_i\}$ of $\mathcal{U}$ we obtain a map

$$f_* : \Gamma(\mathbb{S}_M) = \Omega(M, f^*\mathcal{S}_{\mathcal{G}}) \to \Gamma(\mathbb{S}_N) = \Omega(N, \mathcal{S}_{\mathcal{G}}) \tag{79}$$

defined by

$$(f_* s) = \sum_i \lambda_i f_*^{U_i}(s|_{f^{-1}(U_i)}), \forall\ s \in \Gamma(\mathbb{S}_M). \tag{80}$$

The map (79) is called the pushforward on spinors.

Remark 20. Recall that the pushforward on forms $f_* : \Omega(M) \to \Omega(N)$ has the properties

$$f_* \circ d = d \circ f_*,\ \ f_*((f^*\alpha) \wedge \beta) = \alpha \wedge f_*\beta,\ \ \int_M (f^*\alpha) \wedge \beta = \int_N \alpha \wedge f_*\beta. \tag{81}$$

Let U be a local chart over which the fiber bundle $f : M \to N$ is trivial. Then we can identify $f^{-1}(U)$ with $U \times F$, where F is the compact fiber. The decomposition $U \times F$ induces a bigrading on $\Lambda T^*_p M = \Lambda T^*_x U \otimes \Lambda T^*_t F = \bigoplus_{k,\ell} \Lambda^k T^*_x U \otimes \Lambda^\ell T^*_t F$ for all $p = (x, t) \in U \times F$. Then $f_*\omega = 0$ for every differential form ω on $U \times F$ of type (k,ℓ), $\ell \neq r = \dim F$. Choosing a positively oriented volume form vol_F on the fiber F, we can write every differential form of type (k,r) as $\omega = h\omega_U \wedge \mathrm{vol}_F$, where h is a function on $U \times F$ and ω_U is a differential form on U. Then

$$f_*\omega = \omega_U \int_F h(x,t)\mathrm{vol}_F(t). \tag{82}$$

So f_* is simply integration over the fibers.

The next lemma provides a concrete formulation for the pushforward on spinors in terms of the pushforward on forms.

Lemma 21. *For any $\omega \otimes f^*s \in \Gamma(\mathbb{S}_M)$ such that s is homogenous,*

$$f_*(\omega \otimes f^*s) = (-1)^{r|s|+nr+\frac{r(r-1)}{2}}(f_*\omega) \otimes s, \tag{83}$$

where n and r are the dimensions of N and the fibers of f, respectively. In particular, the pushforward is well-defined (i.e. independent on the choice of $\mathcal{U}$, partition of unity $\{\lambda_i\}$ and canonical bilinear pairings $\langle\cdot,\cdot\rangle_{\mathcal{S}_{\mathcal{G}}|_{U_i}}$).

Proof. We show that for any $\omega \otimes f^*s \in \Gamma(\mathbb{S}_M|_{f^{-1}(U_i)})$ with s homogeneous and $\tilde{\omega} \otimes \tilde{s} \in \Gamma_c(\mathbb{S}_N|_{U_i})$,

$$\int_{U_i} \langle (f_*\omega)\otimes s, \tilde{\omega}\otimes\tilde{s}\rangle_{\mathbb{S}_{U_i}} = (-1)^{r|s|+nr+\frac{r(r-1)}{2}} \int_{f^{-1}(U_i)} \langle \omega\otimes f^*s, f^*(\tilde{\omega}\otimes\tilde{s})\rangle_{\mathbb{S}_{f^{-1}(U_i)}}. \tag{84}$$

In order to prove (84), we assume, without loss of generality, that ω, $\tilde{\omega}$ and $\tilde{s}$ are also homogeneous. If $|\omega| + |\tilde{\omega}| \neq m$ (where $m := n + r$) both terms in (84) vanish. Assume now that $|\omega| + |\tilde{\omega}| = m$. Then

$$\begin{aligned}
\int_{f^{-1}(U_i)} \langle \omega \otimes f^*s, f^*(\tilde{\omega} \otimes \tilde{s})\rangle_{\mathbb{S}_{f^{-1}(U_i)}} &= (-1)^{|s|m+|\omega||\tilde{\omega}|} \int_{f^{-1}(U_i)} f^*(\langle s,\tilde{s}\rangle_{\mathcal{S}_{\mathcal{G}}}\tilde{\omega}) \wedge \omega^t \\
&= (-1)^{|s|m+|\omega||\tilde{\omega}|} \int_{U_i} \langle s,\tilde{s}\rangle_{\mathcal{S}_{\mathcal{G}}}\tilde{\omega} \wedge f_*(\omega^t) \\
&= (-1)^{r(m-\frac{r+1}{2}-|s|)} \int_{U_i} \langle (f_*\omega) \otimes s, \tilde{\omega} \otimes \tilde{s}\rangle_{\mathbb{S}_{U_i}},
\end{aligned}$$

where we used $f_*(\omega^t) = (f_*\omega)^t(-1)^{\frac{r(r-1)}{2}+r(|\omega|-r)}$, which can be checked using (82) and the third property (81) of the pushforward on forms. Relation (84) is proved and implies (83). □

Remark 22. In the above setting, assume that f is endowed with an horizontal distribution, like in Remark 17 ii). Then

$$f_*(f^*(u)\cdot s) = u\cdot f_* s,\ \forall u\in\Gamma\mathrm{Cl}(E),\ s\in\Gamma(\mathbb{S}_{f^!E}). \tag{85}$$

where $f^*:\Gamma\mathrm{Cl}(E)\to\Gamma\mathrm{Cl}(f^!E)$ is the map (57).

Assume now that E is a transitive, but not necessarily standard, Courant algebroid. Then we can define the pushforward $f_!:\Gamma(\mathbb{S}_{f^!E})\to\Gamma(\mathbb{S}_E)$ for any canonical weighted spinor bundles $\mathbb{S}_E$ and $\mathbb{S}_{f^!E}$, for which the pullback $f^!:\Gamma(\mathbb{S}_E)\to\Gamma(\mathbb{S}_{f^!E})$ is defined. Namely, we consider an admissible pair $(I:E\to T^*N\oplus\mathcal{G}\oplus TN, S_{\mathcal{G}})$ for $\mathbb{S}_E$ and $\mathbb{S}_{f^!E}$ and we define the pushforward on spinors

$$f_!:\Gamma(\mathbb{S}_{f^!E})\to\Gamma(\mathbb{S}_E),\ f_!:=(I_{\mathbb{S}})^{-1}\circ f_*\circ I^f_{\mathbb{S}} \tag{86}$$

where $f_*:\Gamma(\mathbb{S}_M)\to\Gamma(\mathbb{S}_N)$ is the map (78). Remark that if $\langle\cdot,\cdot\rangle_{\mathbb{S}_E|_{U_i}}:=(I_{\mathbb{S}})^*\langle\cdot,\cdot\rangle_{\mathbb{S}_{U_i}}$ and $\langle\cdot,\cdot\rangle_{\mathbb{S}_{f^!E}|_{f^{-1}(U_i)}}:=(I^f_{\mathbb{S}})^*\langle\cdot,\cdot\rangle_{\mathbb{S}_{f^{-1}(U_i)}}$, then

$$\int_{U_i}\langle f_! s_1, s_2\rangle_{\mathbb{S}_E|_{U_i}} = \int_{f^{-1}(U_i)}\langle s_1, f^! s_2\rangle_{\mathbb{S}_{f^!E}|_{f^{-1}(U_i)}}, \tag{87}$$

for any $s_1\in\Gamma(\mathbb{S}_{f^!E}|_{f^{-1}(U_i)})$ and $s_2\in\Gamma_c(\mathbb{S}_E|_{U_i})$, where $f^! = (I^f_{\mathbb{S}})^{-1}\circ f^*\circ I_{\mathbb{S}}$, cf. Lemma 18. In particular, (86) is well defined, up to multiplication by ± 1.

Proposition 23. *The pushforward $f_!:\Gamma(f^!E)\to\Gamma(E)$ commutes with the canonical Dirac generating operators, i.e. $f_!\circ \not{d}_{f^!E} = \not{d}_E\circ f_!$.*

Proof. Like in the proof of Proposition 19, it is sufficient to show that

$$f_*\circ\not{d}_M = \not{d}_N\circ f_*, \tag{88}$$

where we preserve the notation from the proof of that proposition. From (85) we know

$$f_*(f^*(u)\cdot s) = u\cdot f_* s,\ \forall u\in\Gamma(T^*N\oplus\mathcal{G}),\ s\in\Gamma(\mathbb{S}_M). \tag{89}$$

Iterating relation (89) and using (58), we see that (89) holds for any $u\in\Gamma\Lambda(T^*N\oplus\mathcal{G})$. Using the expression (23) of the canonical Dirac generating operator and property (89) of f_* (with $u:=H$, α_i, $C_{\mathcal{G}}$), we obtain that

$$\not{d}_N f_*(\tilde{\omega}\otimes f^*s) = f_*\not{d}_M(\tilde{\omega}\otimes f^*s),\ \forall\tilde{\omega}\in\Omega(M),\ s\in\Gamma(\mathcal{S}_{\mathcal{G}}) \tag{90}$$

reduces to

$$f_*\mathcal{E}_M(\tilde{\omega}\otimes f^*s) = \mathcal{E}_N f_*(\tilde{\omega}\otimes s), \tag{91}$$

where

$$\mathcal{E}_N(\omega\otimes s) := (d\omega)\otimes s + \sum_i (\alpha_i\wedge\omega)\otimes\nabla^{\mathcal{S}_\mathcal{G}}_{X_i} s$$

$$\mathcal{E}_M(\tilde{\omega}\otimes f^*s) := (d\tilde{\omega})\otimes f^*s + \sum_i ((f^*\alpha_i)\wedge\tilde{\omega})\otimes f^*(\nabla^{\mathcal{S}_\mathcal{G}}_{X_i} s),$$

for any $\omega\in\Omega(N)$, $\tilde{\omega}\in\Omega(M)$ and $s\in\Gamma(\mathcal{S}_\mathcal{G})$. In order to show (91) it is sufficient to show that for any $U\subset N$ open and sufficiently small and $\beta\in\Gamma_c(\mathbb{S}_N|_U)$,

$$\int_U \langle \mathcal{E}_N f_*(\tilde{\omega}\otimes f^*s),\beta\rangle_{\mathbb{S}_U} = \int_{f^{-1}(U)} \langle \mathcal{E}_M(\tilde{\omega}\otimes f^*s), f^*\beta\rangle_{\mathbb{S}_{f^{-1}(U)}}. \tag{92}$$

From Corollary 9 and $f^*\mathcal{E}_N = \mathcal{E}_M f^*$ we have

$$\begin{aligned}
&\int_U \langle \mathcal{E}_N f_*(\tilde{\omega}\otimes f^*s),\beta\rangle_{\mathbb{S}_U} = -\int_U \langle f_*(\tilde{\omega}\otimes f^*s),\mathcal{E}_N\beta\rangle_{\mathbb{S}_U}\\
&= -\int_{f^{-1}(U)} \langle \tilde{\omega}\otimes f^*s, f^*\mathcal{E}_N\beta\rangle_{\mathbb{S}_{f^{-1}(U)}} = -\int_{f^{-1}(U)} \langle \tilde{\omega}\otimes f^*s, \mathcal{E}_M f^*\beta\rangle_{\mathbb{S}_{f^{-1}(U)}}\\
&= \int_{f-1(U)} \langle \mathcal{E}_M(\tilde{\omega}\otimes f^*s), f^*\beta\rangle_{\mathbb{S}_{f^{-1}(U)}},
\end{aligned}$$

which proves (92). □

5 Actions on transitive Courant algebroids

5.1 Basic properties

In this section we consider a class of actions on a transitive Courant algebroid which generalizes torus actions on exact and, more generally, on heterotic Courant algebroids. For the latter types of Courant algebroids, a notion of T-duality has been developed in [7] and [2] respectively.

Let E be a transitive Courant algebroid over a manifold M, with anchor $\pi : E\to TM$, Dorfmann bracket $[\cdot,\cdot]$ and scalar product $\langle\cdot,\cdot\rangle$. Recall that the automorphism group $\mathrm{Aut}(E)$ of E is the group of orthogonal automorphisms $F : E\to E$ which cover a diffeomorphism $f : M\to M$, such that

$$\pi(F(u)) = (d_p f)\pi(u),\ \forall u\in E_p,\ p\in M$$

and the natural map induced by F on the space of sections of E preserves the Dorfmann bracket. Its Lie algebra is the Lie algebra $\mathrm{Der}(E)$ of derivations of E. This is the subalgebra of $\mathrm{End}\,\Gamma(E)$ of first order linear differential operators $D : \Gamma(E) \to \Gamma(E)$ which satisfy, for any $s, s_1, s_2 \in \Gamma(E)$,

$$\begin{aligned} D[s_1, s_2] &= [Ds_1, s_2] + [s_1, Ds_2] \\ X\langle s_1, s_2\rangle &= \langle Ds_1, s_2\rangle + \langle s_1, Ds_2\rangle \\ \pi \circ D(s) &= \mathcal{L}_X \pi(s), \end{aligned} \tag{93}$$

where $X \in \mathfrak{X}(M)$ is a vector field on M, uniquely determined by D (from the second relation (93)) and usually denoted by $\pi(D)$.

Let $\mathfrak{g}$ be a Lie algebra acting on M by an infinitesimal action

$$\psi : \mathfrak{g} \to \mathfrak{X}(M),\ a \mapsto \psi(a) = X_a.$$

We will always assume (without repeating it each time) that all the infinitesimal actions considered are free, which means that the fundamental vector fields X_a are non-vanishing, for all $a \in \mathfrak{g} \setminus \{0\}$.

Definition 24. *i) An* (infinitesimal) action *of $\mathfrak{g}$ on E which lifts ψ is an algebra homomorphism $\Psi : \mathfrak{g} \to \mathrm{Der}(E)$ which satisfies $\pi\Psi(a) = X_a$ for any $a \in \mathfrak{g}$.*

ii) Let $\Psi : \mathfrak{g} \to \mathrm{Der}(E)$ be an action of E which lifts ψ. An invariant dissection *of E is a dissection $I : E \to T^*M \oplus \mathcal{G} \oplus TM$ for which the action*

$$\mathfrak{g} \ni a \to I \circ \Psi(a) \circ I^{-1} \in \mathrm{Der}(T^*M \oplus \mathcal{G} \oplus TM)$$

preserves the summands T^*M, $\mathcal{G}$ and TM.

We will only consider (without repeating it each time) infinitesimal actions on Courant algebroids for which there is an invariant dissection. The next proposition shows that this is automatically the case if the infinitesimal action is induced from an action of a compact group.

Proposition 25. *Let $\Psi : G \to \mathrm{Aut}(E)$ be an action of a compact group G by automorphisms of a Courant algebroid E over M, hence covering a group action $\psi : G \to \mathrm{Diff}(M)$. Then E admits a dissection invariant under Ψ.*

Proof. By compactness of G there exists a G-invariant positive definite metric h in E. Using the auxiliary metric h we can define a G-invariant splitting $\sigma_0 : TM \to E$ of the anchor map $\pi : E \to TM$, where $\sigma_0(TM)$ is h-orthogonal complement of $\mathrm{Ker}\,\pi$. The section σ_0 of π can be canonically modified to a G-invariant totally isotropic section σ defined by

$$\langle \sigma(X), v\rangle = \langle \sigma_0(X), v - \frac{1}{2}\sigma_0(\pi(v))\rangle$$

for all $X \in T_pM$, $v \in E_p$, $p \in M$. If we define $\mathcal{G}$ as the $\langle \cdot, \cdot\rangle$-orthogonal complement of $\pi^*T^*M \oplus \sigma(TM)$, then $E = \pi^*T^*M \oplus \mathcal{G} \oplus \sigma(TM)$ is a G-invariant dissection. $\square$

In the remaining part of this section we assume that

$$E = T^*M \oplus \mathcal{G} \oplus TM \tag{94}$$

is a standard Courant algebroid, defined by a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data (∇, R, H) as in Section 2.2.1 and we consider in detail the class of actions $\Psi : \mathfrak{g} \to \mathrm{Der}(E)$ which lift ψ and preserve the factors T^*M, $\mathcal{G}$ and TM of E. From the third condition (93), the restriction of Ψ to TM is given by

$$\Psi(a)(X) = \mathcal{L}_{X_a} X,\ \forall a \in \mathfrak{g},\ X \in \mathfrak{X}(M). \tag{95}$$

Since X_a (with $a \in \mathfrak{g} \setminus \{0\}$) are nowhere vanishing we can define

$$\nabla^{\Psi}_{X_a(p)} r := (\Psi(a)(r))\,(p),\ \forall a \in \mathfrak{g},\ r \in \Gamma(\mathcal{G}),\ p \in M,$$

which is a partial connection on $\mathcal{G}$.

Lemma 26. *There is a one to one correspondence between actions $\Psi : \mathfrak{g} \to \mathrm{Der}(E)$ which lift ψ and preserve the factors T^*M, $\mathcal{G}$, TM of E and partial connections ∇^{Ψ} on $\mathcal{G}$ such that the following conditions are satisfied:*

i) ∇^{Ψ} is flat and preserves $[\cdot,\cdot]_{\mathcal{G}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{G}}$;

ii) H and R are invariant, i.e. for any $a \in \mathfrak{g}$,

$$\mathcal{L}_{X_a} H = 0,\ \mathcal{L}_{\Psi(a)} R = 0 \tag{96}$$

where

$$(\mathcal{L}_{\Psi(a)} R)(X,Y) := \nabla^{\Psi}_{X_a}(R(X,Y)) - R(\mathcal{L}_{X_a} X, Y) - R(X, \mathcal{L}_{X_a} Y) \tag{97}$$

for any $X, Y \in \mathfrak{X}(M)$;

iii) for any $a \in \mathfrak{g}$, the endomorphism $A_a := \nabla^{\Psi}_{X_a} - \nabla_{X_a}$ satisfies

$$(\nabla_X A_a)(r) = [R(X_a, X), r]_{\mathcal{G}},\ \forall r \in \Gamma(\mathcal{G}). \tag{98}$$

If i), ii) and iii) are satisfied, then the corresponding action Ψ acts naturally (by Lie derivative) on the subbundle $T^*M \oplus TM$ of E, i.e.

$$\Psi(a)(\xi + X) = \mathcal{L}_{X_a}(\xi + X),\ X \in \mathfrak{X}(M),\ \xi \in \Omega^1(M), \tag{99}$$

and on $\mathcal{G}$ by

$$\Psi(a)(r) = \nabla^{\Psi}_{X_a} r,\ r \in \Gamma(\mathcal{G}). \tag{100}$$

Moreover, for any $a \in \mathfrak{g}$, the endomorphism A_a is a skew-symmetric derivation of $\mathcal{G}$.

Proof. Let Ψ be an action as in the statement of the lemma. From (95),

$$X_a\langle X,\eta\rangle = \langle \mathcal{L}_{X_a}X,\eta\rangle + \langle X, \Psi(a)(\eta)\rangle,\ \forall a\in\mathfrak{g},\ X\in\mathfrak{X}(M),\ \eta\in\Omega^1(M),$$

and from the fact that $\Psi(a)$ preserves $\Omega^1(M)\subset\Gamma(E)$, we obtain that $\Psi(a)(\eta)=\mathcal{L}_{X_a}\eta$. Relation (99) follows. From our comments above, ∇^Ψ defined by (100) is a partial connection on $\mathcal{G}$. Using (7) we obtain that the relations (93) satisfied by Ψ are equivalent to the following conditions: R and H are invariant, ∇^Ψ is flat, preserves $[\cdot,\cdot]_{\mathcal{G}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{G}}$, and

$$\begin{aligned}
&\nabla^\Psi_{X_a}\nabla_X r - \nabla_X\nabla^\Psi_{X_a} r - \nabla_{\mathcal{L}_{X_a}X} r = 0\\
&\mathcal{L}_{X_a}\langle i_X R, r\rangle_{\mathcal{G}} = \langle i_{\mathcal{L}_{X_a}X}R, r\rangle_{\mathcal{G}} + \langle i_X R, \nabla^\Psi_{X_a} r\rangle_{\mathcal{G}}\\
&\mathcal{L}_{X_a}\langle \nabla r,\tilde r\rangle_{\mathcal{G}} = \langle \nabla(\nabla^\Psi_{X_a} r),\tilde r\rangle_{\mathcal{G}} + \langle \nabla r, \nabla^\Psi_{X_a}\tilde r\rangle_{\mathcal{G}},
\end{aligned}\tag{101}$$

for any $a\in\mathfrak{g}$, $X\in\mathfrak{X}(M)$ and $r,\tilde r\in\Gamma(\mathcal{G})$. The first relation (101) is equivalent to (98). Since both ∇ and ∇^Ψ preserve $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ and $[\cdot,\cdot]_{\mathcal{G}}$, the endomorphism A_a is a skew-symmetric derivation. The second relation (101) is equivalent to

$$\langle A_a R(X,Y), r\rangle_{\mathcal{G}} + \langle R(X,Y), A_a r\rangle_{\mathcal{G}} = 0 \tag{102}$$

and follows from the skew-symmetry of the endomorphism A_a. The third relation (101) follows from the fact that ∇ preserves $\langle\cdot,\cdot\rangle_{\mathcal{G}}$, by writing $\nabla^\Psi_{X_a} = \nabla_{X_a}+A_a$ and using relation (98) together with $R^\nabla(X_a,X)(r) = [R(X_a,X),r]_{\mathcal{G}}$. □

Corollary 27. *The skew-symmetric derivations A_a from Lemma 26 satisfy the relation*

$$[R(X_a,X_b),r]_{\mathcal{G}} - [A_a,A_b]r + A_{[a,b]}r = 0,\ \forall a,b\in\mathfrak{g},\ r\in\mathcal{G}, \tag{103}$$

where $[A_a,A_b] := A_aA_b - A_bA_a$ is the commutator of A_a and A_b.

Proof. The claim follows from the flatness of ∇^Ψ together with $\nabla^\Psi_{X_a} = \nabla_{X_a} + A_a$, relations (4), (98) and $[X_a,X_b] = X_{[a,b]}$. □

Remark 28. i) The first relation (101) implies that ∇ is invariant, i.e.

$$(\mathcal{L}_{\Psi(a)}\nabla)_X r := \Psi(a)(\nabla_X r) - \nabla_{\mathcal{L}_{X_a}X} r - \nabla_X(\Psi(a) r) = 0,$$

for any $a\in\mathfrak{g}$, $X\in\mathfrak{X}(M)$ and $r\in\Gamma(\mathcal{G})$.

ii) Like for R, we can define the Lie derivative

$$\begin{aligned}
(\mathcal{L}_{\Psi(a)}\alpha)(X_1,\cdots,X_k) &:= \Psi(a)(\alpha(X_1,\cdots,X_k))\\
&- \alpha(\mathcal{L}_{X_a}X_1,\cdots,X_k) - \cdots - \alpha(X_1,\cdots,\mathcal{L}_{X_a}X_k),
\end{aligned}$$

for any form $\alpha \in \Omega^k(M, \mathcal{G})$. The Lie derivative so defined can be extended in the usual way to forms with values in the tensor bundle $\mathcal{T}(\mathcal{G})$ of $\mathcal{G}$. In particular, for $\alpha \in \Omega^k(M)$ we simply define $\mathcal{L}_{\Psi(a)}\alpha := \mathcal{L}_{X_a}\alpha$. A $\mathcal{T}(\mathcal{G})$-valued form α is called invariant if $\mathcal{L}_{\Psi(a)}\alpha = 0$ for any $a \in \mathfrak{g}$.

iii) The definition of Lie derivative and (98) yield that

$$\mathcal{L}_{\Psi(b)}(A_a)(r) = [R(X_a, X_b), r]_{\mathcal{G}} + [A_b, A_a]r, \ \forall a, b \in \mathfrak{g}. \tag{104}$$

Combining (103) with (104) we obtain that the endomorphisms A_a are invariant when $\mathfrak{g}$ is abelian.

Let E_i $(i = 1, 2)$ be two transititive Courant algebroids over M and $\Psi_i : \mathfrak{g} \to \mathrm{Der}(E_i)$ actions which lift ψ. A fiber preserving Courant algebroid isomorphism $F : E_1 \to E_2$ is called invariant if

$$\Psi_2(a)(F(u)) = F\Psi_1(a)(u), \ \forall a \in \mathfrak{g}, \ u \in \Gamma(E_1). \tag{105}$$

Lemma 29. *Let E_i $(i = 1, 2)$ be standard Courant algebroids over M defined by quadratic Lie algebra bundles $(\mathcal{G}_i, [\cdot, \cdot]_{\mathcal{G}_i}, \langle \cdot, \cdot \rangle_{\mathcal{G}_i})$ and the data $(\nabla^{(i)}, R_i, H_i)$. Assume that E_i are endowed with actions $\Psi_i : \mathfrak{g} \to \mathrm{Der}(E_i)$ which lift $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ and preserve the factors T^*M, $\mathcal{G}_i$ and TM of E_i, and that a fiber preserving Courant algebroid isomorphism $F : E_1 \to E_2$, defined by (β, Φ, K), where $\beta \in \Omega^2(M)$, $\Phi \in \Omega^1(M, \mathcal{G}_2)$ and $K \in \mathrm{Isom}(\mathcal{G}_1, \mathcal{G}_2)$, is given, as in (9). Let $\nabla^{\Psi_i} := \left(\nabla^{(i)}\right)^{\Psi_i}$ $(i = 1, 2)$ be the partial connections associated with $\nabla^{(i)}$ and Ψ_i. Then F is invariant if and only if K maps ∇^{Ψ_1} to ∇^{Ψ_2} and the forms β and Φ are invariant.*

Proof. The proof uses the expression (7) for the Dorfman bracket. □

5.1.1 A class of T^k-actions

Let $(E = T^*M \oplus \mathcal{G} \oplus TM, \langle \cdot, \cdot \rangle, [\cdot, \cdot])$ be a standard Courant algebroid over the total space of a principal T^k-bundle $\pi : M \to B$, where $T^k = \mathbb{R}^k/\mathbb{Z}^k$ denotes the k-dimensional torus. We assume that E is defined by a bundle of quadratic Lie algebras $(\mathcal{G}, \langle \cdot, \cdot \rangle_{\mathcal{G}}, [\cdot, \cdot]_{\mathcal{G}})$ and data (∇, R, H), where ∇ is a connection on the vector bundle $\mathcal{G}$ compatible with the tensor fields $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ and $[\cdot, \cdot]_{\mathcal{G}}$, $R \in \Omega^2(M, \mathcal{G})$ and $H \in \Omega^3(M)$. Recall that these data satisfy the compatibility equations

$$dH = \langle R \wedge R \rangle \ , \quad d^{\nabla} R = 0 \ , \quad R^{\nabla} = \mathrm{ad}_R \ , \tag{106}$$

where $\langle R \wedge R \rangle_{\mathcal{G}}$ is abbreviated as $\langle R \wedge R \rangle$ in harmony with the fact that $\langle \cdot, \cdot \rangle_{\mathcal{G}} = \langle \cdot, \cdot \rangle|_{\mathcal{G} \times \mathcal{G}}$. The Dorfmann bracket, scalar product and anchor of E are then expressed by the usual formulas in terms of the above data.

We assume that the vertical paralellism of π is lifted to an action of $\mathfrak{t}^k$ on E,

$\Psi : \mathfrak{t}^k = \mathbb{R}^k \to \mathrm{Der}(E)$, $\quad a \mapsto \Psi(a) = (\xi + r + X \mapsto \mathcal{L}_{X_a}\xi + \nabla^{\Psi}_{X_a} r + \mathcal{L}_{X_a} X)$,

where X_a is the fundamental vector field of π determined by $a \in \mathfrak{t}^k$ and ∇^{Ψ} is a partial flat connection on $\mathcal{G}$. We recall (see Lemma 26) that ∇^{Ψ} preserves $[\cdot,\cdot]_{\mathcal{G}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ and that

$$\mathcal{L}_{X_a} R = 0,\ \mathcal{L}_{X_a} H = 0,\ (\nabla_X A_a)(r) = [R(X_a, X), r]_{\mathcal{H}}, \tag{107}$$

where $A_a := \nabla^{\Psi}_{X_a} - \nabla_{X_a} \in \mathrm{End}(\mathcal{G})$ is a skew-symmetric derivation, which is invariant since $\mathfrak{t}^k$ is abelian (see Remark 28). Recall also that $\mathcal{L}_{X_a}\nabla = 0$.

We consider

$$\Omega^s_b(M, \mathcal{G}) := \{\alpha \in \Omega^s(M, \mathcal{G}) \mid \mathcal{L}_{\Psi(a)}\alpha = 0\ , i_{X_a}\alpha = 0,\ \forall a \in \mathfrak{t}^k\},$$

the space of basic $\mathcal{G}$-valued s-forms on M. The space $\Omega^s_b(M)$ of basic scalar valued s-forms on M can be defined similarly and coincides with $\pi^*\Omega^s(B) \cong \Omega^s(B)$. The analogous fact for $\Omega^s_b(M, \mathcal{G})$ is stated in the next proposition.

Proposition 30. *$\Omega^s_b(M, \mathcal{G}) \cong \pi^*\Omega^s(B) \otimes \Gamma_{t^k}(\mathcal{G})$, where $\Gamma_{t^k}(\mathcal{G})$ denotes the space of $\mathfrak{t}^k$-invariant (i.e. ∇^{Ψ}-parallel) sections.*

Proof. Let $U \subset B$ be an open set such that $\Lambda^s T^*B|_U$ is trivial. Then any horizontal form $\alpha \in \Omega^s(\pi^{-1}(U), \mathcal{G})$ (i.e. $i_{X_a}\alpha = 0$ for any $a \in \mathfrak{t}^k$) can be written as $\alpha = \sum_i (\pi^*\beta_i) \otimes s_i$ where (β_i) is a basis of $\Lambda^s T^*B|_U$ and $s_i \in \Gamma(\mathcal{G}|_{\pi^{-1}(U)})$. Then $\mathcal{L}_{\Psi(a)}\alpha = \sum_i (\pi^*\beta_i) \otimes \mathcal{L}_{\Psi(a)} s_i$ from where we deduce that $\Omega^s_b(\pi^{-1}(U), \mathcal{G}) = \pi^*\Omega^s(U) \otimes \Gamma_{t^k}(\mathcal{G}|_{\pi^{-1}(U)})$. Using a partition of unity in B one can deduce that the same holds globally for $U = B$. $\square$

In the following we will always identify $\Lambda^s T^*M \otimes \mathcal{G}$ with $\mathcal{G} \otimes \Lambda^s T^*M$, which allows to freely write decomposable elements as $\omega \otimes r$ or as $r \otimes \omega$. Let (e_i) be a basis of $\mathfrak{t}^k$, $X_i := X_{e_i}$ the associated fundamental vector fields and $A_i := A_{e_i} = \nabla^{\Psi}_{X_i} - \nabla_{X_i} \in \mathrm{End}(\mathcal{G})$. We choose a connection $\mathcal{H}$ on the principal bundle $\pi : M \to B$, with connection form $\theta = \sum_{i=1}^k \theta_i e_i$. We introduce the connection

$$\nabla^{\theta} := \nabla + \sum_{i=1}^{k} \theta_i \otimes A_i \tag{108}$$

on the vector bundle $\mathcal{G}$. Since ∇ preserves $[\cdot,\cdot]_{\mathcal{G}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ and A_i are skew-symmetric derivations, we obtain that ∇^{θ} preserves $[\cdot,\cdot]_{\mathcal{G}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{G}}$. The curvature R^{∇} of ∇ and R^{θ} of ∇^{θ} are related by

$$R^{\nabla} = R^{\theta} - \sum_{i=1}^{k} (d\theta_i) \otimes A_i + \sum_{i=1}^{k} \theta_i \wedge \mathrm{ad}_{R(X_i,\cdot)} - \frac{1}{2}\sum_{i,j} (\theta_i \wedge \theta_j) \otimes [A_i, A_j], \tag{109}$$

where for any form $\omega \in \Omega^s(M, \mathcal{G})$ (in particular $\omega := R(X_i, \cdot)$) we define $\mathrm{ad}_\omega \in \Omega^s(M, \mathrm{End}\,\mathcal{G})$ by

$$(\mathrm{ad}_\omega)(Y_1, \cdots, Y_s)(r) := [\omega(Y_1, \cdots, Y_s), r]_{\mathcal{G}}, \ \forall Y_i \in \mathfrak{X}(M).$$

Lemma 31. *The exterior covariant derivative $d^\theta := d^{\nabla^\theta}$ associated with the connection ∇^θ maps basic forms to basic forms:*

$$d^\theta : \Omega^s_b(M, \mathcal{G}) \to \Omega^{s+1}_b(M, \mathcal{G}).$$

Proof. We have to show that $d^\theta \alpha$ is basic for all $\alpha \in \Omega^s_b(M, \mathcal{G})$. Thanks to Proposition 30 it suffices to consider $\alpha = \omega \otimes r = r \otimes \omega$, $\omega \in \Omega_b(M)$, $r \in \Gamma_{\mathfrak{t}^k}(\mathcal{G})$. Using the fact that

$$d^\theta(r \otimes \omega) = \nabla^\theta r \wedge \omega + r \otimes d\omega$$

and that $d\omega$ is basic we can reduce the statement to the case $s = 0$. The $\mathcal{G}$-valued 1-form $\nabla^\theta r$ is horizontal, since

$$\nabla^\theta_{X_i} r = \nabla^\Psi_{X_i} r = \mathcal{L}_{\Psi(e_i)} r = 0, \ \forall 1 \leq i \leq k.$$

On the other hand, as ∇, θ and A_i are $\mathfrak{t}^k$- invariant, so is ∇^θ and

$$\mathcal{L}_{\Psi(a)}(\nabla^\theta r) = \mathcal{L}_{\Psi(a)}(\nabla^\theta) r + \nabla^\theta \mathcal{L}_{\Psi(a)} r = 0, \tag{110}$$

which implies that $\nabla^\theta r$ is $\mathfrak{t}^k$-invariant. ☐

Remark 32. i) When the partial connection ∇^Ψ has trivial holonomy, we can define a bundle $\mathcal{G}_B \to B$ by defining its fiber over $p \in B$ as

$$\mathcal{G}_B|_p := \Gamma_{\mathfrak{t}^k}(\mathcal{G}|_{\pi^{-1}(p)}),$$

the vector space of ∇^Ψ-parallel sections of $\mathcal{G}$ over the torus $\pi^{-1}(p)$. Since ∇^Ψ preserves $[\cdot, \cdot]_{\mathcal{G}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{G}}$, the bundle $\mathcal{G}_B$ inherits a bracket $[\cdot, \cdot]_{\mathcal{G}_B}$ and a scalar product $\langle \cdot, \cdot \rangle_{\mathcal{G}_B}$ which make $(\mathcal{G}_B, [\cdot, \cdot]_{\mathcal{G}_B}, \langle \cdot, \cdot \rangle_{\mathcal{G}_B})$ a bundle of quadratic Lie algebras. Furthermore, $\mathcal{G} = \pi^* \mathcal{G}_B$, $\Gamma_{\mathfrak{t}^k}(\mathcal{G}) = \pi^* \Gamma(\mathcal{G}_B)$ and we can identify by pullback

$$\Omega^s_b(M, \mathcal{G}) = \pi^* \Omega^s(B) \otimes \pi^* \Gamma(\mathcal{G}_B) \cong \Omega^s(B, \mathcal{G}_B).$$

For a basic form $\alpha \in \Omega^s_b(M, \mathcal{G})$ we shall denote by $\alpha^B \in \Omega^s(B, \mathcal{G}_B)$ the corresponding form in the above identification. As the endomorphism A_a of $\mathcal{G}$ is invariant, it is the pullback of an endomorphism A^B_a of $\mathcal{G}_B$. From Lemma 31, ∇^θ induces a connection $\nabla^{\theta, B}$ on $\mathcal{G}_B$. Since ∇^θ preserves $[\cdot, \cdot]_{\mathcal{G}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{G}}$,

$\nabla^{\theta,B}$ preserves $[\cdot,\cdot]_{\mathcal{G}_B}$ and $\langle\cdot,\cdot\rangle_{\mathcal{G}_B}$. The curvature R^θ of ∇^θ is the pullback of the curvature $R^{\theta,B}$ of $\nabla^{\theta,B}$, i.e. $(R^\theta)^B = R^{\theta,B}$. We denote by $d^{\theta,B}$ the exterior covariant derivative defined by $\nabla^{\theta,B}$.

ii) Note that by working locally in a flow box for the vertical foliation of $M\to B$, we can always assume that ∇^Ψ has trivial holonomy. (We recall that a flow box is a domain $V\subset M$ such that for all $p\in\pi(V)\subset B$ the manifolds $\pi^{-1}(p)\cap V$ are diffeomorphic to $\mathbb{R}^k$. From now on we make this assumption.

In order to describe the Courant algebroid E together with the action $\Psi:\mathfrak{t}^k\to\Gamma(E)$ in terms of structures on the base manifold B of the torus bundle, we need to interpret equations (106) and (107) on B. As H and R are invariant, they are of the form

$$\begin{aligned} H &= H_{(3)} + \theta_i\wedge H^i_{(2)} + \theta_i\wedge\theta_j\wedge H^{ij}_{(1)} + H^{ijs}_{(0)}\theta_i\wedge\theta_j\wedge\theta_s\\ R &= R_{(2)} + \theta_i\wedge R^i_{(1)} + R^{ij}_{(0)}\theta_i\wedge\theta_j \end{aligned}\tag{111}$$

where $H_{(3)}$, $H^i_{(2)}$, $H^{ij}_{(1)}$, $H^{ijk}_{(0)}$, $R_{(2)}$, $R^i_{(1)}$, $R^{ij}_{(0)}$ are basic and for simplicity of notation we omit the summation signs.

Lemma 33. *i) The compatibility equations listed in (106) are satisfied if and only if the following conditions hold:*

$$dH^B_{(3)} + H^{i,B}_{(2)}\wedge(d\theta_i)^B = \langle R^B_{(2)}\wedge R^B_{(2)}\rangle_{\mathcal{G}_B}\tag{112}$$
$$dH^{p,B}_{(2)} + 2H^{pi,B}_{(1)}\wedge(d\theta_i)^B = -2\langle R^B_{(2)}\wedge R^{p,B}_{(1)}\rangle_{\mathcal{G}_B}\tag{113}$$
$$dH^{pq,B}_{(1)} + 3H^{ipq,B}_{(0)}(d\theta_i)^B = 2\langle R^{pq,B}_{(0)}, R^B_{(2)}\rangle_{\mathcal{G}_B} - \langle R^{p,B}_{(1)}, R^{q,B}_{(1)}\rangle_{\mathcal{G}_B}\tag{114}$$
$$3dH^{pqs}_{(0)} + 2(\langle R^{pq,B}_{(0)}, R^{s,B}_{(1)}\rangle_{\mathcal{G}_B} + \langle R^{sp,B}_{(0)}, R^{q,B}_{(1)}\rangle_{\mathcal{G}_B} + \langle R^{qs,B}_{(0)}, R^{p,B}_{(1)}\rangle_{\mathcal{G}_B}) = 0\tag{115}$$
$$\langle R^{ij,B}_{(0)}, R^{pq,B}_{(0)}\rangle\theta_i\wedge\theta_j\wedge\theta_p\wedge\theta_q = 0\tag{116}$$
$$d^{\theta,B}R^B_{(2)} + R^{i,B}_{(1)}\wedge(d\theta_i)^B = 0\tag{117}$$
$$d^{\theta,B}R^{p,B}_{(1)} + A^B_p\wedge R^B_{(2)} + 2R^{pi,B}_{(0)}(d\theta_i)^B = 0\tag{118}$$
$$A^B_p\wedge R^{q,B}_{(1)} - A^B_q\wedge R^{p,B}_{(1)} = 2\nabla^{\theta,B}R^{pq,B}_{(0)}\tag{119}$$
$$A^B_s R^{pq,B}_{(0)} + A^B_q R^{sp,B}_{(0)} + A^B_p R^{qs,B}_{(0)} = 0\tag{120}$$
$$R^{\theta,B} = (d\theta_i)^B\otimes A^B_i + \mathrm{ad}_{R^B_{(2)}}\tag{121}$$
$$\mathrm{ad}_{R^{ij,B}_{(0)}} = \frac{1}{2}[A^B_i, A^B_j],\tag{122}$$

where

$$\mathrm{ad}:\mathcal{G}_B\to\mathrm{Der}(\mathcal{G}_B),\ \ \mathrm{ad}_u(v) = [u,v]_{\mathcal{G}_B}\tag{123}$$

is the adjoint representation in the Lie algebra bundle $(\mathcal{G}_B, [\cdot,\cdot]_{\mathcal{G}_B})$ *and* $1 \leq p, q, s \leq k$ *are arbitrary.*

ii) If the compatibility relations (106) are satisfied, then the third relation (107) is satisfied as well if and only if

$$\nabla^{\theta,B}_X A_i^B = [R^{i,B}_{(1)}(X), r]_{\mathcal{G}_B},\ \forall X \in \mathfrak{X}(M). \tag{124}$$

Proof. i) The equations (112)-(116) are obtained from the first relation (106), by comparing

$$\begin{aligned} dH &= dH_{(3)} + (d\theta_i) \wedge H^i_{(2)} - \theta_i \wedge dH^i_{(2)} + 2(d\theta_i) \wedge \theta_j \wedge H^{ij}_{(1)} \\ &+ \theta_i \wedge \theta_j \wedge dH^{ij}_{(1)} + 3H^{ijs}_{(0)}(d\theta_i) \wedge \theta_j \wedge \theta_s + (dH^{ijs}_{(0)}) \wedge \theta_i \wedge \theta_j \wedge \theta_s \end{aligned}$$

with

$$\begin{aligned} \langle R \wedge R \rangle &= \langle R_{(2)} \wedge R_{(2)} \rangle + 2\theta_i \wedge \langle R_{(2)} \wedge R^i_{(1)} \rangle + 2\theta_i \wedge \theta_j \wedge \langle R_{(2)}, R^{ij}_{(0)} \rangle \\ &- \theta_i \wedge \theta_j \wedge \langle R^i_{(1)} \wedge R^j_{(1)} \rangle + 2\theta_i \wedge \theta_j \wedge \theta_s \wedge \langle R^{ij}_{(0)}, R^s_{(1)} \rangle \\ &+ \langle R^{ij}_{(0)}, R^{pq}_{(0)} \rangle \theta_i \wedge \theta_j \wedge \theta_p \wedge \theta_q, \end{aligned}$$

using $d\theta_i \in \Omega^2(B)$, that the exterior derivative maps basic forms to basic forms, that the operation $(\alpha, \beta) \mapsto \langle \alpha \wedge \beta \rangle$ maps a pair of $\mathcal{G}$-valued basic forms to a basic scalar valued form and then interpreting the resulting relations on B. The equations (117)-(120) are obtained from the second relation (106), by computing

$$\begin{aligned} 0 &= d^\nabla R \\ &= d^\nabla R_{(2)} + (d\theta_i) \wedge R^i_{(1)} - \theta_i \wedge d^\nabla R^i_{(1)} + (\nabla R^{ij}_{(0)}) \wedge \theta_i \wedge \theta_j + 2R^{ij}_{(0)} \otimes (d\theta_i) \wedge \theta_j \\ &= d^\theta R_{(2)} - (\theta_i \otimes \mathrm{A}_i) \wedge R_{(2)} + R^i_{(1)} \wedge d\theta_i - \theta_j \wedge \left(d^\theta R^j_{(1)} - (\theta_i \otimes \mathrm{A}_i) \wedge R^j_{(1)} \right) \\ &+ (\nabla^\theta R^{ij}_{(0)}) \wedge \theta_i \wedge \theta_j - A_i(R^{js}_{(0)})\theta_i \wedge \theta_j \wedge \theta_s + 2R^{ij}_{(0)} \otimes (d\theta_i) \wedge \theta_j, \end{aligned} \tag{125}$$

identifying the horizontal and vertical parts in the last expression of (125) and interpreting the result on B. The remaining equations (121) and (122) are obtained by writing R^∇ in terms of R^θ as in (109) and identifying the horizontal and vertical parts in $R^\nabla = \mathrm{ad}_R$.

ii) The third relation (107) is equivalent to relation (124), together with relation (122). □

Since $\nabla^{\theta,B}$ preserves $[\cdot,\cdot]_{\mathcal{G}_B}$, the endomorphism $R^{\theta,B}(X,Y)$ of $\mathcal{G}_B$ is a derivation, for any $X, Y \in \mathfrak{X}(B)$. Recall that $A_i^B \in \mathrm{End}(\mathcal{G}_B)$ is also a derivation. The conditions from Lemma 33 simplify considerably when the adjoint representation (123) of the Lie algebra bundle $(\mathcal{G}_B, [\cdot,\cdot]_{\mathcal{G}_B})$ is an isomorphism. Then

$$A_i^B = \mathrm{ad}_{r_i^B},\ R^{\theta,B}(X,Y) = \mathrm{ad}_{\mathfrak{r}^{\theta,B}(X,Y)} \tag{126}$$

for $r_i^B \in \Gamma(\mathcal{G}_B)$ and $\mathfrak{r}^{\theta,B} \in \Omega^2(B,\mathcal{G}_B)$. From the Bianchi identity we obtain that $\mathfrak{r}^{\theta,B}$ is d^θ-closed.

Corollary 34. *Let $\pi : M \to B$ be a principal T^k-bundle and $\mathcal{H}$ a principal connection on π, with connection form $\theta = \sum_{i=1}^k \theta_i e_i \in \Omega^1(M,\mathfrak{t}^k)$, where (e_i) is a basis of $\mathfrak{t}^k$. There is a one to one correspondence between*

1. *standard Courant algebroids $E = T^*M \oplus \mathcal{G} \oplus TM$ for which the adjoint action of the Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}})$ is an isomorphism, together with an action $\Psi : \mathfrak{t}^k \to \mathrm{Der}(E)$ which lifts the vertical parallelism of π, preserves the factors T^*M, $\mathcal{G}$ and TM of E, and for which the flat partial connection ∇^Ψ has trivial holonomy*

   and

2. *quadratic Lie algebra bundles $(\mathcal{G}_B, \langle\cdot,\cdot\rangle_{\mathcal{G}_B}, [\cdot,\cdot]_{\mathcal{G}_B})$ over B, whose adjoint action is an isomorphism, together with a connection ∇^B on the vector bundle $\mathcal{G}_B$ which preserves $\langle\cdot,\cdot\rangle_{\mathcal{G}_B}$ and $[\cdot,\cdot]_{\mathcal{G}_B}$, sections $r_i^B \in \Gamma(\mathcal{G}_B)$ $(1 \le i \le k)$, a 3-form $H_{(3)}^B \in \Omega^3(B)$, 2-forms $H_{(2)}^{i,B} \in \Omega^2(B)$, 1-forms $H_{(1)}^{ij,B} \in \Omega^1(B)$ and constants $c_{ijp} \in \mathbb{R}$ $(1 \le i,j,p \le k)$ such that*

$$\begin{aligned} dH_{(3)}^B &= \langle \mathfrak{r}^B \wedge \mathfrak{r}^B\rangle_{\mathcal{G}_B} \\ &- (H_{(2)}^{i,B} + 2\langle \mathfrak{r}^B, r_i^B\rangle_{\mathcal{G}_B} - \langle r_i^B, r_j^B\rangle_{\mathcal{G}_B}(d\theta_j)^B) \wedge (d\theta_i)^B, \\ dH_{(2)}^{p,B} &= 2(\langle \nabla^B r_p^B, r_i^B\rangle_{\mathcal{G}_B} - H_{(1)}^{pi,B}) \wedge (d\theta_i)^B - 2\langle \mathfrak{r}^B \wedge \nabla^B r_p^B\rangle_{\mathcal{G}_B} \\ dH_{(1)}^{pq,B} &= -3c_{ipq}(d\theta_i)^B + \langle \mathfrak{r}^B, [r_p^B, r_q^B]_{\mathcal{G}_B}\rangle_{\mathcal{G}_B} - \langle \nabla^B r_p^B \wedge \nabla^B r_q^B\rangle_{\mathcal{G}_B}, \end{aligned} \tag{127}$$

   where $\mathfrak{r}^B \in \Omega^2(B,\mathcal{G}_B)$ is related to the curvature R^B of the connection ∇^B by $R^B(X,Y) = \mathrm{ad}_{\mathfrak{r}^B(X,Y)}$ for any $X, Y \in \mathfrak{X}(B)$.

Proof. The claim follows from Lemma 33, by letting $\nabla^B := \nabla^{\theta,B}$ and simplifying the relations from this lemma, using in an essential way that the adjoint representation of the Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}})$ is an isomorphism. More precisely, relations (122), (124) and (121) determine $R_{(0)}^{ij,B}$, $R_{(1)}^{i,B}$ and $R_{(2)}^B$ respectively by

$$R_{(0)}^{ij,B} = \frac{1}{2}[r_i^B, r_j^B]_{\mathcal{G}_B},\ R_{(1)}^{i,B} = \nabla^B r_i^B,\ R_{(2)}^B = \mathfrak{r}^B - (d\theta_i)^B \otimes r_i^B. \tag{128}$$

Relation (115) with $R^{ij,B}_{(0)}$ and $R^{i,B}_{(1)}$ given by (128) implies that

$$H^{pqs,B}_{(0)} = -\frac{1}{3}\langle [r^B_p, r^B_q]_{\mathcal{G}_B}, r^B_s\rangle_{\mathcal{G}_B} + c_{pqs} \tag{129}$$

for some constants c_{pqs}. Written in terms of $\mathfrak{r}^B$ rather than $R^B_{(2)}$, relations (112), (113), (114) become relations (127). The remaining relations from Lemma 33, with $R^{ij,B}_{(0)}, R^{i,B}_{(1)}, R^B_{(2)}$ and $H^{pqs,B}_{(0)}$ as above and $A_i = \mathrm{ad}_{r^B_i}$ are satisfied. □

Example 35. Under the assumptions of Corollary 34, let $(\mathcal{G}_B, \langle\cdot,\cdot\rangle_{\mathcal{G}_B}, [\cdot,\cdot]_{\mathcal{G}_B})$ be a quadratic Lie algebra bundle over B, whose adjoint action is an isomorphism, together with a connection ∇^B on the vector bundle $\mathcal{G}_B$ which preserves $\langle\cdot,\cdot\rangle_{\mathcal{G}_B}$ and $[\cdot,\cdot]_{\mathcal{G}_B}$. Choose arbitrary sections $r^B_i \in \Gamma(\mathcal{G}_B)$ $(1 \leq i \leq k)$ and define, for any i, j, s, $c_{ijp} := 0$,

$$H^{ijs,B}_{(0)} := -\frac{1}{3}\langle [r^B_i, r^B_j]_{\mathcal{G}_B}, r^B_s\rangle_{\mathcal{G}_B} \tag{130}$$

and

$$H^{ij,B}_{(1)} := \frac{1}{2}\left(\langle \nabla^B r^B_i, r^B_j\rangle_{\mathcal{G}_B} - \langle \nabla^B r^B_j, r^B_i\rangle_{\mathcal{G}_B}\right). \tag{131}$$

With these choices, the third relation (127) is satisfied. For any forms $H^B_{(3)}$ and $H^{i,B}_{(2)}$, such that

$$\mathcal{K}_i := H^{i,B}_{(2)} + 2\langle \mathfrak{r}^B, r^B_i\rangle_{\mathcal{G}_B} - \langle r^B_i, r^B_j\rangle_{\mathcal{G}_B}(d\theta_j)^B \tag{132}$$

is closed and

$$dH^B_{(3)} = \langle \mathfrak{r}^B \wedge \mathfrak{r}^B\rangle_{\mathcal{G}_B} - \mathcal{K}_i \wedge (d\theta_i)^B \tag{133}$$

the relations (127) are satisfied and we thus obtain a standard Courant algebroid together with an action $\Psi : \mathfrak{t}^k \to \mathrm{Der}(E)$ lifting the vertical parallelism of the principal torus bundle $\pi : M \to B$. Note that 2-forms $H^{i,B}_{(2)}$ as required in the above construction do always exist and are unique up to addition of closed forms whereas $H^B_{(3)}$ exists if and only if the closed form $\langle \mathfrak{r}^B \wedge \mathfrak{r}^B\rangle_{\mathcal{G}_B} - \mathcal{K}_i \wedge (d\theta_i)^B$ is exact. It is also unique up to addition of a closed form.

5.2 Invariant spinors

Let E be a transitive Courant algebroid over an oriented manifold M and $\Psi : \mathfrak{g} \to \mathrm{Der}(E)$ an action on E, which lifts an action $\psi : \mathfrak{g} \mapsto \mathfrak{X}(M)$, $a \mapsto X_a$ of $\mathfrak{g}$ on M. Let $\mathbb{S}$ be a canonical weighted spinor bundle of E. Our aim in this section is to define an action of $\mathfrak{g}$ on $\Gamma(\mathbb{S})$. In order to find a proper definition we assume that Ψ integrates to a Lie group action

$$G \to \mathrm{Aut}(E),\ g \mapsto I_E^g : E \to E$$

such that I_E^g induces a globally defined isomorphism $I_{\mathbb{S}}^g : \mathbb{S} \to \mathbb{S}$, for any $g \in \mathfrak{g}$. Recall that

$$I_{\mathbb{S}}^g \circ \gamma_u = \gamma_{I_E^g(u)} \circ I_{\mathbb{S}}^g,\ \forall g \in G,\ u \in E.$$

Consider a curve $g = g(t)$ of G with $g(0) = e$ and $\dot{g}(0) = a$. We choose $I_{\mathbb{S}}^{g(t)}$ depending smoothly on t and such that $I_{\mathbb{S}}^{g(0)} = \mathrm{Id}_{\mathbb{S}}$. Replacing in the above relation g by $g(t)$ and taking the derivative at $t = 0$ we obtain that $\Psi^{\mathbb{S}}(a) := \frac{d}{dt}\big|_{t=0} I_{\mathbb{S}}^{g(t)} \in \mathrm{End}\,\Gamma(\mathbb{S})$ satisfies

$$\Psi^{\mathbb{S}}(a)\gamma_u s = \gamma_{\Psi(a)(u)} s + \gamma_u \Psi^{\mathbb{S}}(a)s,\ \forall u \in \Gamma(E),\ s \in \Gamma(\mathbb{S}),\ a \in \mathfrak{g}. \tag{134}$$

In the following we do not assume that Ψ integrates to an action of G.

Proposition 36. *i) There is a unique linear map*

$$\Psi^{\mathbb{S}} : \mathfrak{g} \to \mathrm{End}\,\Gamma(\mathbb{S})$$

which satisfies relation (134), the Leibniz rule

$$\Psi^{\mathbb{S}}(a)(fs) = f\Psi^{\mathbb{S}}(a)(s) + X_a(f)s,\ \forall f \in C^\infty(M),\ s \in \Gamma(\mathbb{S}),\ a \in \mathfrak{g}, \tag{135}$$

and, for any $U \subset M$ open and sufficiently small, preserves the canonical $\det(T^*U)$*-valued bilinear pairing $\langle \cdot, \cdot \rangle_{\mathbb{S}|_U}$ of $\Gamma(\mathbb{S}|_U)$, i.e.*

$$\mathcal{L}_{X_a}\langle s, \tilde{s}\rangle_{\mathbb{S}|_U} = \langle \Psi^{\mathbb{S}}(a)s, \tilde{s}\rangle_{\mathbb{S}|_U} + \langle s, \Psi^{\mathbb{S}}(a)\tilde{s}\rangle_{\mathbb{S}|_U},\ \forall s, \tilde{s} \in \Gamma(\mathbb{S}|_U),\ a \in \mathfrak{g}. \tag{136}$$

ii) The map $\Psi^{\mathbb{S}} : \mathfrak{g} \to \mathrm{End}\,\Gamma(\mathbb{S})$ satisfies

$$[\Psi^{\mathbb{S}}(a), \Psi^{\mathbb{S}}(b)] = \Psi^{\mathbb{S}}[a, b],\ \forall a, b \in \mathfrak{g}. \tag{137}$$

It is called the action on spinors *induced by Ψ.*

The remaining part of this section is devoted to the proof of Proposition 36. For uniqueness, let $\Psi^{\mathbb{S}}$ and $\tilde{\Psi}^{\mathbb{S}}$ be two maps which satisfy the required conditions. Then $F(a) := \Psi^{\mathbb{S}}(a) - \tilde{\Psi}^{\mathbb{S}}(a)$ is $C^\infty(M)$-linear and commutes with the Clifford action. Hence $F(a) = \lambda(a)\mathrm{Id}_{\mathbb{S}}$, for $\lambda(a) \in C^\infty(M)$. Since $F(a)$ is skew-symmetric with respect to $\langle \cdot, \cdot \rangle_{\mathbb{S}|_U}$, we obtain $\lambda(a) = 0$. The uniqueness follows. For existence we need the following two lemmas.

Lemma 37. *Let E_i $(i = 1, 2)$ be two transitive Courant algebroids over M with actions $\Psi_i : \mathfrak{g} \to \mathrm{Der}(E_i)$, which lift ψ. Let $I_E : E_1 \to E_2$ be an invariant isomorphism and, for any $U \subset M$ open and sufficiently small, $I_{\mathbb{S}|_U} : \mathbb{S}_1|_U \to \mathbb{S}_2|_U$ the induced isomorphism between canonical weighted spinor bundles of $E_i|_U$. Let $\Psi^{\mathbb{S}_i} : \mathfrak{g} \to \mathrm{End}\,\Gamma(\mathbb{S}_i|_U)$ $(i = 1, 2)$ be two maps related by*

$$\Psi^{\mathbb{S}_2}(a) = I_{\mathbb{S}|_U} \circ \Psi^{\mathbb{S}_1}(a) \circ (I_{\mathbb{S}|_U})^{-1}, \ \forall a \in \mathfrak{g}. \tag{138}$$

Then $\Psi^{\mathbb{S}_1}$ satisfies the conditions from Proposition 36 if and only if $\Psi^{\mathbb{S}_2}$ does.

Proof. Let $\Psi^{\mathbb{S}_1} : \mathfrak{g} \to \mathrm{End}\,\Gamma(\mathbb{S}_1|_U)$ be a map which satisfies the conditions from Proposition 36 and $\Psi^{\mathbb{S}_2} : \mathfrak{g} \to \mathrm{End}\,(\mathbb{S}_2|_U)$ be defined by (138). The map $\Psi^{\mathbb{S}_2}$ obviously satisfies (135) and (137) and, from (43), it satisfies (136) as well. Using $I_{\mathbb{S}|_U} \circ \gamma_u = \gamma_{I_E(u)} \circ I_{\mathbb{S}|_U}$ and relation (134) satisfied by $\Psi^{\mathbb{S}_1}$, we obtain

$$\Psi^{\mathbb{S}_2}(a)\gamma_u(s) = \gamma_{I_E \Psi_1(a) I_E^{-1}(u)} s + \gamma_u \Psi^{\mathbb{S}_2}(a)s, \ \forall a \in \mathfrak{g}, \ u \in \Gamma(E_1|_U), \ s \in \Gamma(\mathbb{S}_2|_U). \tag{139}$$

Since I_E is invariant, $\Psi_2(a) = I_E \circ \Psi_1(a) \circ I_E^{-1}$ and we obtain that $\Psi^{\mathbb{S}_2}$ satisfies (134). □

Let $E_M = T^*M \oplus \mathcal{G} \oplus TM$ be a standard Courant algebroid defined by a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data (∇, R, H), with action

$$\Psi : \mathfrak{g} \mapsto \mathrm{Der}(E_M), \ \Psi(a)(\xi + r + X) := \mathcal{L}_{X_a}\xi + \nabla^{\Psi}_{X_a} r + \mathcal{L}_{X_a} X \tag{140}$$

which lifts an action

$$\psi : \mathfrak{g} \mapsto \mathfrak{X}(M), \ a \mapsto X_a$$

of $\mathfrak{g}$ on M. Let $S_{\mathcal{G}}$ be an irreducible $\mathrm{Cl}(\mathcal{G})$-bundle, $\mathcal{S}_{\mathcal{G}} = S_{\mathcal{G}} \otimes |\det S^*|^{1/r}$ the canonical spinor bundle and $\mathbb{S}_M := \Lambda(T^*M)\hat{\otimes}\mathcal{S}_{\mathcal{G}}$ the canonical weighted spinor bundle of E determined by $S_{\mathcal{G}}$.

Lemma 38. *The map*

$$\Psi^{\mathbb{S}_M} : \mathfrak{g} \to \mathrm{End}\,\Gamma(\mathbb{S}_M), \ \Psi^{\mathbb{S}_M}(a)(\omega \otimes s) := (\mathcal{L}_{X_a}\omega) \otimes s + \omega \otimes \nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}_{X_a} s, \tag{141}$$

for any $a \in \mathfrak{g}$, $\omega \in \Omega(M)$ and $s \in \Gamma(\mathcal{S}_{\mathcal{G}})$, satisfies the conditions from Proposition 36. Above $\nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}$ is the partial connection on $\mathcal{S}_{\mathcal{G}}$ induced by any partial connection $\nabla^{\Psi, S_{\mathcal{G}}}$ on $S_{\mathcal{G}}$, compatible with the partial connection ∇^{Ψ}.

Proof. Relation (135) is obviously satisfied. To prove relation (136) we recall that $\langle \cdot, \cdot \rangle_{\mathbb{S}|_U}$ is given by (32), where $\langle \cdot, \cdot \rangle_{\mathcal{S}_{\mathcal{G}}|_U}$ is a canonical bilinear pairing of $\Gamma(\mathcal{S}_{\mathcal{G}}|_U)$. Relation (136 follows from a computation which uses (32),

$$X_a \langle s, \tilde{s} \rangle_{\mathcal{S}_{\mathcal{G}}|_U} = \langle \nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}_{X_a} s, \tilde{s} \rangle_{\mathcal{S}_{\mathcal{G}}|_U} + \langle s, \nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}_{X_a} \tilde{s} \rangle_{\mathcal{S}_{\mathcal{G}}|_U} \tag{142}$$

and the fact that $\nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}$ preserves the degree of sections of $\mathcal{S}_{\mathcal{G}}$. (Relation (142) follows from the fact that ∇^{Ψ} preserves $\langle \cdot, \cdot \rangle_{\mathcal{G}}$, which is of neutral signature, and $\nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}$ is induced by any partial connection $\nabla^{\Psi, S_{\mathcal{G}}}$ on $S_{\mathcal{G}}$ compatible with ∇^{Ψ}. The argument is similar to the one used in the proof of Lemma 7). In order to prove (134), decompose $u = \xi + r + X$. Then

$$\Psi(a)(u) = \mathcal{L}_{X_a}(\xi + X) + \nabla^{\Psi}_{X_a} r,$$

from where we deduce that

$$\begin{aligned} \gamma_{\Psi(a)(u)}(\omega \otimes s) &= \gamma_{\mathcal{L}_{X_a}(\xi + X)}(\omega \otimes s) + \gamma_{\nabla^{\Psi}_{X_a} r}(\omega \otimes s) \\ &= (i_{\mathcal{L}_{X_a} X}\omega + (\mathcal{L}_{X_a}\xi) \wedge \omega) \otimes s + (-1)^{|\omega|} \omega \otimes (\nabla^{\Psi}_{X_a} r) s. \end{aligned} \tag{143}$$

Similar computations show that

$$\begin{aligned} \Psi^{\mathbb{S}_M}(a)\gamma_u(\omega \otimes s) &= \mathcal{L}_{X_a}(i_X\omega + \xi \wedge \omega) \otimes s + (i_X\omega + \xi \wedge \omega) \otimes \nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}_{X_a} s \\ &\quad + (-1)^{|\omega|}(\mathcal{L}_{X_a}\omega \otimes (rs) + \omega \otimes \nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}_{X_a}(rs)) \end{aligned} \tag{144}$$

$$\begin{aligned} \gamma_u \Psi^{\mathbb{S}_M}(a)(\omega \otimes s) &= (i_X \mathcal{L}_{X_a}\omega + \xi \wedge \mathcal{L}_{X_a}\omega) \otimes s + (-1)^{|\omega|}(\mathcal{L}_{X_a}\omega) \otimes (rs) \\ &\quad + (i_X\omega + \xi \wedge \omega) \otimes \nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}_{X_a} s + (-1)^{|\omega|} \omega \otimes (r \nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}_{X_a} s). \end{aligned} \tag{145}$$

Combining relations (143), (144) and (145) and using that $\nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}$ is compatible with ∇^{Ψ} we obtain (134). Relation (137) follows from the definition of the map $\Psi^{\mathbb{S}_M}$ and the flatness of $\nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}$ (which is a consequence of the flatness of ∇^{Ψ}). □

We conclude the proof of Proposition 36 by choosing an invariant dissection $I : E \to E_M$ and isomorphisms $I_{\mathbb{S}|_{U_i}} : \mathbb{S}|_{U_i} \to \mathbb{S}_M|_{U_i}$ compatible with $I|_{U_i}$, where $\mathcal{U} = \{U_i\}$ is a cover of M with sufficiently small open subsets. Using Lemmas 37 and 38, we obtain that the map $\Psi^{\mathbb{S}} : \mathfrak{g} \to \mathrm{End}\, \Gamma(\mathbb{S})$ defined by

$$\Psi^{\mathbb{S}}(a)(s)|_{U_i} := (I_{\mathbb{S}|_{U_i}})^{-1} \circ \Psi^{\mathbb{S}_M}(a) \circ I_{\mathbb{S}|_{U_i}}(s|_{U_i}), \ \forall s \in \Gamma(\mathbb{S})$$

satisfies the conditions from Proposition 36.

Definition 39. *A section of the canonical weighted spinor bundle $\mathbb{S}$ is an invariant spinor if it is annihilated by the operators $\Psi^{\mathbb{S}}(a)$, for all $a \in \mathfrak{g}$.*

Notation 40. Given an action $\Psi : \mathfrak{g} \to \mathrm{Der}(E)$ on a transitive Courant algebroid E, we shall denote by $\Gamma_{\mathfrak{g}}(\mathbb{S})$ the vector space of invariant spinors. Similarly, $\Gamma_{\mathfrak{g}}(E)$ will denote the vector space of invariant sections of E.

Lemma 41. *In the setting of Proposition 36,*

$$\not{d} \circ \Psi^{\mathbb{S}}(a) = \Psi^{\mathbb{S}}(a) \circ \not{d}, \ \forall a \in \mathfrak{g}, \tag{146}$$

where $\not{d}$ is the canonical Dirac generating operator of E.

Proof. From Proposition 15 and Lemma 37, it is sufficient to prove the statement for the Courant algebroid E_M considered in Lemma 38 with action $\Psi^{\mathbb{S}_M}$ defined by (141). We need to show that for any $a \in \mathfrak{g}$, $\omega \in \Omega(M)$ and $s \in \Gamma(\mathcal{S}_{\mathcal{G}})$

$$\not{d}_M \Psi^{\mathbb{S}_M}(a)(\omega \otimes s) = \Psi^{\mathbb{S}_M}(a)\not{d}_M(\omega \otimes s) \tag{147}$$

where $\not{d}_M \in \mathrm{End}\,\Gamma(\mathbb{S}_M)$ is the Dirac generating operator of E_M. We consider an invariant local frame (X_i) of TM. Since ∇^{Ψ} is flat we may (and will) take the local frame (r_k) of $\mathcal{G}$ to be ∇^{Ψ}-parallel. Since ∇^{Ψ} preserves the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{G}}$, the $\langle \cdot, \cdot \rangle_{\mathcal{G}}$- dual frame $(\tilde{r}_k)$ is also ∇^{Ψ}-parallel. Since ∇^{Ψ} preserves the Lie bracket $[\cdot, \cdot]_{\mathcal{G}}$, the Cartan form $C_{\mathcal{G}}$ is ∇^{Ψ}-parallel.

Since R, X_i and r_k are invariant,

$$\mathcal{L}_{X_a} \langle R(X_i, X_j), r_k \rangle_{\mathcal{G}} = 0, \ \forall a \in \mathfrak{g}. \tag{148}$$

From (148), $\nabla^{\Psi} C_{\mathcal{G}} = 0$, the fact that $\nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}$ is compatible with ∇^{Ψ} and the expressions (22), (141) for $\not{d}$ and $\Psi^{\mathbb{S}}$, we see that relation (146) reduces to

$$\nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}_{X_a} \nabla^{\mathcal{S}_{\mathcal{G}}}_{X_i} s = \nabla^{\mathcal{S}_{\mathcal{G}}}_{X_i} \nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}_{X_a} s, \ \forall a \in \mathfrak{g}, \ s \in \Gamma(\mathcal{S}_{\mathcal{G}}), \tag{149}$$

where, we recall, $\nabla^{\mathcal{S}_{\mathcal{G}}}$ is the connection on $\mathcal{S}_{\mathcal{G}}$ induced by any connection on $S_{\mathcal{G}}$ compatible with ∇ and similarly for the partial connections $\nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}$ and ∇^{Ψ}. For any $a \in \mathfrak{g}$, let $A_a := \nabla^{\Psi}_{X_a} - \nabla_{X_a}$. Then

$$\nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}_{X_a} s = \nabla^{\mathcal{S}_{\mathcal{G}}}_{X_a} s - \frac{1}{2} A_a \cdot s \tag{150}$$

where $A_a \cdot s$ denotes the Clifford action of $A_a \in \Gamma(\Lambda^2 \mathcal{G}) \subset \Gamma\,\mathrm{Cl}(\mathcal{G})$ on $s \in \Gamma(\mathcal{S}_{\mathcal{G}})$ (see e.g. Proposition 53 of [9] for more details). From (150), (98) and $\mathcal{L}_{X_a} X_i = 0$ we deduce that (149) is equivalent to

$$R^{\nabla^{\mathcal{S}_{\mathcal{G}}}}(X_a, X_i)s + \frac{1}{2}(\mathrm{ad}_{R(X_a, X_i)})s = 0, \tag{151}$$

where $(\mathrm{ad}_{R(X_a, X_i)})s$ means the Clifford action of $\mathrm{ad}_{R(X_a, X_i)} := [\mathrm{ad}_{R(X_a, X_i)}, \cdot]_{\mathcal{G}} \in \Gamma(\Lambda^2 \mathcal{G}) \subset \Gamma\,\mathrm{Cl}(\mathcal{G})$ on s. In order to prove (151) we remark first that both endomorphisms $R^{\nabla^{\mathcal{S}_{\mathcal{G}}}}(X_a, X_i)$ and $(\mathrm{ad}_{R(X_a,X_i)})$ of $\mathcal{S}_{\mathcal{G}}$ are trace free (the statement for $R^{\nabla^{\mathcal{S}_{\mathcal{G}}}}(X_a, X_i)$ is a consequence of the fact that $\nabla^{\mathcal{S}_{\mathcal{G}}}$ is induced by a connection $\nabla^{S_{\mathcal{G}}}$ on $S_{\mathcal{G}}$). On the other hand, since $\nabla^{\mathcal{S}_{\mathcal{G}}}$ is compatible with ∇, we obtain that $T := R^{\nabla^{\mathcal{S}_{\mathcal{G}}}}(X_a, X_i) \in \mathrm{End}(\mathcal{S}_{\mathcal{G}})$ satisfies

$$T(rs) = (R^{\nabla}(X_a, X_i)r)s + rT(s), \forall r \in \mathcal{G},\ s \in \mathcal{S}_{\mathcal{G}}. \tag{152}$$

The same relation is satisfied by $T := -\frac{1}{2}\mathrm{ad}_{R(X_a,X_i)}$ acting by Clifford multiplication (here we use that $R^{\nabla}(X_a, X_i)(r) = \mathrm{ad}_{R(X_a,X_i)}(r)$ and relation $\omega(r) = -\frac{1}{2}[\omega, r]_{\mathrm{Cl}}$, for any $\omega \in \Lambda^2\mathcal{G} \subset \mathrm{Cl}(\mathcal{G})$, where $[\omega, r]_{\mathrm{Cl}} = \omega r - r\omega$ denotes the commutator of ω and r in the Clifford algebra and $\omega(r)$ the action of $\omega \in \Lambda^2\mathcal{G} \cong \mathfrak{so}(\mathcal{G})$ on $r \in \mathcal{G}$).

To summarize: both $R^{\nabla^{\mathcal{S}_{\mathcal{G}}}}(X_a, X_i)$ and $-\frac{1}{2}(\mathrm{ad}_{R(X_a,X_i)})$ are trace-free and satisfy (152). Since $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ has neutral signature, they coincide. □

5.3 Pullback actions and spinors

Let $f : M \to N$ be a submersion and

$$\psi^M : \mathfrak{g} \to \mathfrak{X}(M),\ a \mapsto X_a^M$$
$$\psi^N : \mathfrak{g} \to \mathfrak{X}(N),\ a \to X_a^N$$

be f-related infinitesimal actions, i.e. $X_a^N \circ f = df X_a^M$ for all $a \in \mathfrak{g}$. Let E be a transitive Courant algebroid over N with anchor $\pi : E \to TN$ and

$$\mathfrak{g} \ni a \mapsto \Psi(a) \in \mathrm{Der}(E)$$

be an action on E which lifts ψ^N. Recall that the pullback Courant algebroid $f^!E$ is the quotient bundle $C/C^{\perp}$ over M (identified with the graph M_f of f), where, for any $p \in M$,

$$C_p := \{(u, \mu + X) \in E_{f(p)} \times \mathbb{T}_p M :\ \pi(u) = (d_p f)(X)\}$$
$$C_p^{\perp} := \{(\frac{1}{2}\pi^*(\gamma), -f^*\gamma),\ \gamma \in T_{f(p)}N\} \subset C_p$$

with the Courant algebroid structure defined at the beginning of Section 4.2. For $U \subset N$ open, a section of $C|_{f^{-1}(U)}$ (and the induced section of $(f^!E)|_{f^{-1}(U)}$) of the form $(f^*u, \mu + X)$ where $u \in \Gamma(E|_U)$, $X \in \mathfrak{X}(f^{-1}(U))$ is f-projectable with $f_*X = \pi(u)$ and $\mu \in \Omega^1(f^{-1}(U))$, will be called distinguished. Let $\mathcal{U} = \{U_i\}$ be an open cover of N, with sufficiently small sets U_i. Any section of $C|_{f^{-1}(U_i)}$ is a $C^{\infty}(f^{-1}(U_i))$-linear combination of distinguished sections. For each U_i we define

$$\widehat{\Psi}^{U_i} : \mathfrak{g} \to \mathrm{End}\,\Gamma(C|_{f^{-1}(U_i)}), \tag{153}$$

such that it satisfies the Leibniz rule

$$\widehat{\Psi}^{U_i}(a)(fs) = X_a^M(f)s + f\widehat{\Psi}^{U_i}(a)(s), \tag{154}$$

for any $a \in \mathfrak{g}$, $f \in C^\infty(f^{-1}(U_i))$, $s \in \Gamma(C|_{f^{-1}(U_i)})$, and on distinguished sections is given by

$$\widehat{\Psi}^{U_i}(a)(f^*u, \mu + X) := (f^*(\Psi(a)u), \mathcal{L}_{X_a^M}(\mu + X)). \tag{155}$$

Lemma 42. *The map $\Psi : \mathfrak{g} \to \operatorname{End}\Gamma(C)$ given by*

$$\widehat{\Psi}(a)(s)|_{f^{-1}(U_i)} = \widehat{\Psi}^{U_i}(a)(s|_{f^{-1}(U_i)}) \tag{156}$$

is well defined, preserves $\Gamma(C^\perp)$, and induces an action

$$f^!\Psi : \mathfrak{g} \to \operatorname{Der}(f^!E) \tag{157}$$

which lifts ψ^M. It is called the pullback action of Ψ.

Proof. The statement that $\widehat{\Psi}$ is well defined reduces to showing that for any $U_k, U_p \in \mathcal{U}$, if

$$\sum_i \lambda_i(f^*u_i, \mu_i + X_i) = 0$$

where $\lambda_i \in C^\infty(f^{-1}(U_k \cap U_p))$ and $(f^*u_i, \mu_i + X_i)$ are distinguished sections on $f^{-1}(U_k \cap U_p)$, then

$$\sum_i \left(X_a^M(\lambda_i)f^*u_i + \lambda_i f^*\Psi(a)(u_i)\right) = 0. \tag{158}$$

This follows by writing $u_i \in \Gamma(E|_{U_k \cap U_p})$ in terms of a frame of $E|_{U_k \cap U_p}$ and using the Leibniz rule for $\Psi(a)$ and that X_a^M projects to X_a^N. The map $\widehat{\Psi}(a)$ takes values in $\Gamma(C)$ since for any distinguished section $(f^*u, \mu + X)$, we have

$$\pi\Psi(a)(u) = \mathcal{L}_{X_a^N}\pi(u) = f_*\mathcal{L}_{X_a^M}X.$$

It preserves $\Gamma(C^\perp)$ since

$$\Psi(a)\pi^*(\gamma) = \pi^*(\mathcal{L}_{X_a^N}\gamma),\ \forall \gamma \in \Omega^1(N).$$

Since Ψ satisfies the relations (93), also $f^!\Psi$ does (easy check). □

When $E = E_N := T^*N \oplus \mathcal{G} \oplus TN$ is a standard Courant algebroid and $\Psi = \Psi^N : \mathfrak{g} \to \operatorname{Der}(E_N)$ preserves the factors T^*N, $\mathcal{G}$ and TN of E_N, the pullback action $f^!\Psi^N$ has a concrete formulation, as follows. Assume that

E_N is defined by a bundle of quadratic Lie algebras $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data (∇, R, H). Recall that we identify $f^! E$ with the standard Courant algebroid $E_M := T^*M \oplus f^*\mathcal{G} \oplus TM$ defined by the bundle of quadratic Lie algebras $(f^*\mathcal{G}, f^*[\cdot,\cdot]_{\mathcal{G}}, f^*\langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data $(f^*\nabla, f^*R, f^*H)$, using the canonical isomorphism F defined by (55). Using this identification, we obtain an action

$$\Psi^M : \mathfrak{g} \to \mathrm{Der}(E_M),\ \ \Psi^M(a) := F^{-1} \circ (f^!\Psi^N)(a) \circ F$$

of $\mathfrak{g}$ on E_M.

Lemma 43. *In the above setting, assume that*

$$\Psi^N(a)(\xi + r + X) := \mathcal{L}_{X_a^N}\xi + \nabla^{\Psi}_{X_a^N} r + \mathcal{L}_{X_a^N} X, \tag{159}$$

where $\xi \in \Omega^1(N)$, $r \in \Gamma(\mathcal{G})$ and $X \in \mathfrak{X}(N)$. Then

$$\Psi^M(\xi + r + X) := \mathcal{L}_{X_a^M}\xi + (f^*\nabla^{\Psi})_{X_a^M} r + \mathcal{L}_{X_a^M} X, \tag{160}$$

where $\xi \in \Omega^1(M)$, $r \in \Gamma(f^*\mathcal{G})$ and $X \in \mathfrak{X}(M)$.

Proof. The isomorphism F given by (55) induces an isomorphism $F : \Gamma(E_M) \to \Gamma(f^! E_N)$ which satisfies

$$F(\xi + f^* r + X) = [(f^*(r + f_* X), \xi + X)] \tag{161}$$

where $r \in \Gamma(\mathcal{G})$, $X \in \mathfrak{X}(M)$ is f-projectable and $\xi \in \Omega^1(M)$. (In the right hand side of (161) $r + f_* X \in \Gamma(\mathcal{G} \oplus TN) \subset \Gamma(E_N)$). Then

$$(f^!\Psi^N)(a) \circ F(\xi + f^* r + X) = [(f^*(\nabla^{\Psi}_{X_a^N} r + \mathcal{L}_{X_a^N} f_* X), \mathcal{L}_{X_a^N}(\xi + X))],$$

and, applying F^{-1}, we obtain (160). □

The next proposition states several compatibilities between pullback actions, isomorphisms, pullback and pushforward on spinors.

Proposition 44. *i) Let (E_i, Ψ_i) $(i = 1, 2)$ be transitive Courant algebroids over N with actions $\Psi_i : \mathfrak{g} \to \mathrm{Der}(E_i)$ which lift ψ^N. If $I : (E_1, \Psi_1) \to (E_2, \Psi_2)$ is invariant with respect to Ψ_i, then $I^f : (f^! E_1, f^!\Psi_1) \to (f^! E_2, f^!\Psi_2)$ is invariant with respect to $f^!\Psi_i$.*

ii) In the setting of Lemma 42, assume that M and N are oriented and let $\Psi^{\mathbb{S}} : \mathfrak{g} \to \mathrm{End}\,\Gamma(\mathbb{S}_E)$ and $(f^!\Psi)^{\mathbb{S}} : \mathfrak{g} \to \mathrm{End}\,\Gamma(\mathbb{S}_{f^! E})$ be the actions on canonical weighted spinor bundles, induced by the actions $\Psi : \mathfrak{g} \to \mathrm{Der}(E)$ and $f^!\Psi : \mathfrak{g} \to \mathrm{Der}(f^! E)$. Assume that the pullback $f^! : \Gamma(\mathbb{S}_E) \to \Gamma(\mathbb{S}_{f^! E})$ is *defined and there is an admissible pair* $(I : E \to T^*N \oplus \mathcal{G} \oplus TN, S_{\mathcal{G}})$ *for* $\mathbb{S}_E$ *and* $\mathbb{S}_{f^!E}$ *such that* I *is invariant, cf. Section 4.2. Then*

$$f^! \circ \Psi^{\mathbb{S}}(a) = (f^!\Psi)^{\mathbb{S}}(a) \circ f^!, \ \forall a \in \mathfrak{g}. \tag{162}$$

If, in addition, $f : M \to N$ *has compact fibers then also the pushforward* $f_! : \Gamma(\mathbb{S}_{f^!E}) \to \Gamma(\mathbb{S}_E)$ *is defined and*

$$f_! \circ (f^!\Psi)^{\mathbb{S}}(a) = (-1)^{r|s|+nr+\frac{r(r-1)}{2}} \Psi^{\mathbb{S}}(a) \circ f_!, \ \forall a \in \mathfrak{g}, \tag{163}$$

where m*,* n *and* r *are the dimension of* M*,* N *and the fibers of* f*.*

Proof. i) We need to check that

$$I^f \circ f^!\Psi_1(a)[(f^*u, \eta + X)] = f^!\Psi_2(a) \circ I^f[(f^*u, \eta + X)]$$

for any distinguished section $[(f^*u, \eta + X)]$ of $f^!E_1$, which follows by applying the definitions of I^f and $f^!\Psi_i$ and using that I is invariant.

ii) Using the admissible pair $(I, S_{\mathcal{G}})$ and Lemma 37, we can assume, without loss of generality, that

$$E = E_N = T^*N \oplus \mathcal{G} \oplus TN, \ f^!E = E_M = T^*M \oplus f^*\mathcal{G} \oplus TM$$

and $\Psi = \Psi^N$, $f^!\Psi = \Psi^M$ are given by (159) and (160) respectively. Let $\mathbb{S}_N := \Lambda(T^*N)\hat{\otimes}\mathcal{S}_{\mathcal{G}}$ and $\mathbb{S}_M := \Lambda(T^*M)\hat{\otimes} f^*\mathcal{S}_{\mathcal{G}}$ be the canonical weighted spinor bundles of E_N and E_M, determined by $S_{\mathcal{G}}$ and its pullback $S_{f^*\mathcal{G}} := f^*S_{\mathcal{G}}$ respectively. From the definition of $f^!$, we need to show that

$$f^*\Psi^{\mathbb{S}_N}(a)(\omega \otimes s) = \Psi^{\mathbb{S}_M}(a)(f^*\omega \otimes f^*s) \tag{164}$$

for any $\omega \otimes s \in \Gamma(\mathbb{S}_N)$, where $\Psi^{\mathbb{S}_N} : \mathfrak{g} \to \mathrm{End}\,\Gamma(\mathbb{S}_N)$ and $\Psi^{\mathbb{S}_M} : \mathfrak{g} \to \mathrm{End}\,\Gamma(\mathbb{S}_M)$ are the induced actions on spinors (given by Lemma 38) and f^* is the map (56). Relation (164) follows from (141), $f^*\mathcal{L}_{X_a^N}\omega = \mathcal{L}_{X_a^M}(f^*\omega)$ and $f^*(\nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}_{X_a^N} s) = (\nabla^{f^!\Psi, \mathcal{S}_{f^*\mathcal{G}}})_{X_a^M}(f^*s)$ for any $s \in \Gamma(\mathcal{S}_{\mathcal{G}})$ (the latter being a consequence of $\nabla^{f^!\Psi} = f^*\nabla^{\Psi}$). The statement for the pushforward can be proved by a similar argument, which uses that $f_*\mathcal{L}_{X_a^M}\omega = \mathcal{L}_{X_a^N} f_*\omega$ for any form ω on M and Lemma (21). □

6 T-duality

6.1 Definition of T-duality

Let $\pi : M \to B$ and $\tilde{\pi} : \tilde{M} \to B$ be principal bundles over the same manifold B with structure group the k-dimensional torus T^k. For notational convenience, we will denote the structure group of $\tilde{\pi}$ by $\tilde{T}^k$ and its Lie algebra by $\tilde{t}^k$. We assume that M, $\tilde{M}$ and B are oriented. Let

$$\mathrm{Lie}\,(T^k) = \mathfrak{t}^k \ni a \mapsto \psi^M(a) := X_a^M,\ \tilde{\mathfrak{t}}^k \ni a \mapsto \psi^{\tilde{M}}(a) := X_a^{\tilde{M}},$$

be the vertical paralellism of π and $\tilde{\pi}$. We denote by

$$N := M \times_B \tilde{M} := \{(m, \tilde{m}) \in M \times \tilde{M} \mid \pi(m) = \tilde{\pi}(\tilde{m})\}$$

the fiber product of M and $\tilde{M}$ and by $\pi_N : N \to M$ and $\tilde{\pi}_N : N \to \tilde{M}$ the natural projections. The actions of T^k on M and $\tilde{T}^k$ on $\tilde{M}$ induce naturally an action of $T^{2k} = T^k \times \tilde{T}^k$ on N, with infinitesimal action

$$\mathfrak{t}^{2k} \ni a \to \psi^N(a) = X_a^N,$$

where, for any $a \in \mathfrak{t}^k := \mathfrak{t}^k \oplus 0 \subset \mathfrak{t}^{2k}$,

$$(\pi_N)_* X_a^N = X_a^M, \quad (\tilde{\pi}_N)_* X_a^N = 0,$$

and for any $a \in \tilde{\mathfrak{t}}^k := 0 \oplus \mathfrak{t}^k \subset \mathfrak{t}^{2k}$,

$$(\pi_N)_* X_a^N = 0, \quad (\tilde{\pi}_N)_* X_a^N = X_a^{\tilde{M}}.$$

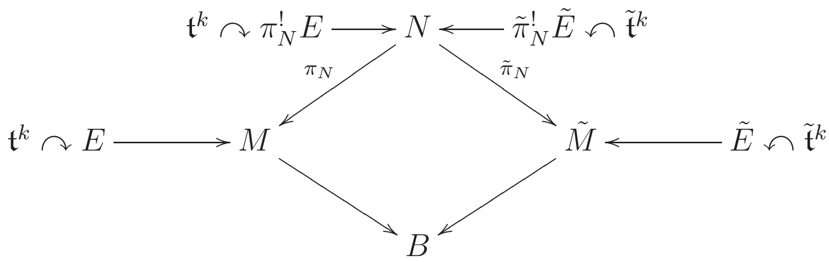
Let E and $\tilde{E}$ be transitive Courant algebroids over M and $\tilde{M}$, and assume they come with actions

$$\Psi : \mathfrak{t}^k \to \mathrm{Der}(E),\ \tilde{\Psi} : \tilde{\mathfrak{t}}^k \to \mathrm{Der}(\tilde{E}),$$

which lift ψ^M and $\psi^{\tilde{M}}$, such that there are invariant dissections $I : E \to T^*M \oplus \mathcal{G} \oplus TM$ and $\tilde{I} : \tilde{E} \to T^*\tilde{M} \oplus \tilde{\mathcal{G}} \oplus T\tilde{M}$ with the property that the partial connections ∇^{Ψ} and $\nabla^{\tilde{\Psi}}$ on $\mathcal{G}$ and $\tilde{\mathcal{G}}$, induced by the actions, have trivial holonomy (it is easy to see that this condition is independent of the choice of invariant dissections). The pullback Courant algebroids $\pi_N^! E$ and $\tilde{\pi}_N^! \tilde{E}$ inherit the pullback actions (see Lemma 42)

$$\pi_N^! \Psi : \mathfrak{t}^k \to \mathrm{Der}(\pi_N^! E),\ \tilde{\pi}_N^! \Psi : \tilde{\mathfrak{t}}^k \to \mathrm{Der}(\tilde{\pi}_N^! \tilde{E}) \tag{165}$$

which lift the infinitesimal actions $\mathfrak{t}^k \ni a \to X_a^N$ and $\tilde{\mathfrak{t}}^k \ni a \to X_a^N$ respectively. The situation is summarized in the following commutative diagram, in which the arrows pointing down are quotient maps with respect to principal T^k-actions: $B = M/T^k = \tilde{M}/\tilde{T}^k = N/T^{2k}$, $M = N/\tilde{T}^k$, $\tilde{M} = N/T^k$ ($T^{2k} = T^k \times \tilde{T}^k$).



The next lemma extends the action $\pi_N^! \Psi$ to a $\mathfrak{t}^{2k}$-action and states some of the properties of this $\mathfrak{t}^{2k}$-action.

Lemma 45. *i) The map*

$$\Psi^{\pi_N^! E} : \mathfrak{t}^{2k} \to \mathrm{Der}(\pi_N^! E)$$

which on $\mathfrak{t}^k$ coincides with $\pi_N^! \Psi$ and the evaluation of which on any $b \in \tilde{\mathfrak{t}}^k$ satisfies the Leibniz rule

$$\Psi^{\pi_N^! E}(b)(fs) = X_b^N(f)s + f\Psi^{\pi_N^! E}(b)(s), \ \forall f \in C^\infty(N), \ s \in \Gamma(\pi_N^! E)$$

and on distinguished sections $[(\pi_N^*(u), \xi + X)]$ of $\pi_N^! E$ is given by

$$\Psi^{\pi_N^! E}(b)[(\pi_N^*(u), \xi + X)] = [(0, \mathcal{L}_{X_b^N}(\xi + X))] \tag{166}$$

is a well defined action on $\pi_N^! E$.

ii) Let (E_1, Ψ_1) be another transitive Courant algebroid over M with an action $\Psi_1 : \mathfrak{t}^k \to \mathrm{Der}(E_1)$ which lifts ψ^M. If $I : E \to E_1$ is an isomorphism invariant with respect to Ψ and Ψ_1, then the pullback isomorphism I^{π_N} : $\pi_N^! E \to \pi_N^! E_1$ is invariant with respect to $\Psi^{\pi_N^! E}$ and $\Psi^{\pi_N^! E_1}$ (the latter defined as $\Psi^{\pi_N^! E}$, using Ψ_1 instead of Ψ).

Proof. Claim i) follows from arguments similar to the proof of Lemma 42. To prove claim ii), we need to show that

$$I^{\pi_N} \circ \Psi^{\pi_N^! E}(a)(s) = \Psi_1^{\pi_N^! E_1}(a) \circ I^{\pi_N}(s), \ \forall a \in \mathfrak{t}^{2k}, \ s \in \Gamma(\pi_N^! E). \tag{167}$$

Relation (167) with $a \in \mathfrak{t}^k$, follows from Proposition 44 i). Relation (167) with $a \in \tilde{\mathfrak{t}}^k$ follows by assuming that s is a distinguished section and using (166) together with the definition of I^{π_N}, cf. equation (53). □

Lemma 46. *i) If $E = T^*M \oplus \mathcal{G} \oplus TM$ is a standard Courant algebroid and Ψ preserves the summands T^*M, $\mathcal{G}$, TM of E, then $\Psi^{\pi_N^! E}$ preserves the summands T^*N, $\pi_N^* \mathcal{G}$, TN of $\pi_N^! E = T^*N \oplus \pi_N^* \mathcal{G} \oplus TN$. The partial connection $\nabla^{\Psi^{\pi_N^! E}}$ on $\pi_N^* \mathcal{G}$ associated to $\Psi^{\pi_N^! E}$ is the pullback of the partial connection ∇^Ψ on $\mathcal{G}$ associated to Ψ:*

$$\nabla^{\Psi^{\pi_N^! E}}_{X_a^N}(\pi_N^* r) = \pi_N^* \nabla^\Psi_{X_a^M} r, \ \nabla^{\Psi^{\pi_N^! E}}_{X_b^N}(\pi_N^* r) = 0, \ \forall r \in \Gamma(\mathcal{G}), \ a \in \mathfrak{t}^k, \ b \in \tilde{\mathfrak{t}}^k. \tag{168}$$

ii) A section of $\pi_N^* \mathcal{G}$ is $\nabla^{\Psi^{\pi_N^! E}}$-parallel if and only if it is the pullback by π_N of a ∇^Ψ-parallel section of $\mathcal{G}$ (or the pullback by $\Pi := \pi \circ \pi_N$ of a section of $\mathcal{G}_B$).

Proof. Claim i) follows from an argument similar to the proof of Lemma 43. Claim ii) follows immediately from claim i) (recall the definition of the bundle $\mathcal{G}_B \to B$ from Section 5.1.1). $\square$

When E is a standard Courant algebroid like in Lemma 45 ii), the partial connection $\nabla^{\Psi^{\pi_N^! E}}$ will be denoted by $\nabla^{\Psi, \pi_N^! E}$. Let $S_{\mathcal{G}}$ be an irreducible $\mathrm{Cl}(\mathcal{G})$-bundle, with canonical spinor bundle $\mathcal{S}_{\mathcal{G}}$. Then $\mathcal{S}_{\pi_N^* \mathcal{G}} = \pi_N^* \mathcal{S}_{\mathcal{G}}$ is the canonical spinor bundle of the irreducible $\mathrm{Cl}(\pi_N^* \mathcal{G})$-bundle $S_{\pi_N^* \mathcal{G}} := \pi_N^* S_{\mathcal{G}}$ and the partial connection $\nabla^{\Psi^{\pi_N^! E}, \mathcal{S}_{\pi_N^* \mathcal{G}}}$ on $\mathcal{S}_{\pi_N^* \mathcal{G}}$ induced by any partial connection on $S_{\pi_N^* \mathcal{G}}$ compatible with $\nabla^{\Psi, \pi_N^! E}$ is the pullback of the partial connection $\nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}$ on $\mathcal{S}_{\mathcal{G}}$ induced by any partial connection on $S_{\mathcal{G}}$ compatible with ∇^{Ψ}, that is,

$$\nabla^{\Psi, \pi_N^* \mathcal{S}_{\mathcal{G}}}_{X_a^N} = (\pi_N^* \nabla^{\Psi, \mathcal{S}_{\mathcal{G}}})_{X_a^N}, \ \forall a \in \mathfrak{t}^{2k}. \tag{169}$$

In a similar way, we construct an action $\tilde{\Psi}^{\tilde{\pi}_N^! \tilde{E}} : \mathfrak{t}^{2k} \to \mathrm{Der}(\tilde{\pi}_N^! \tilde{E})$ which extends $\tilde{\pi}_N^! \tilde{\Psi}$. When $\tilde{E} = T^* \tilde{M} \oplus \tilde{\mathcal{G}} \oplus T\tilde{M}$ is a standard Courant algebroid and $\tilde{\Psi}$ preserves the factors $T^* \tilde{M}$, $\tilde{\mathcal{G}}$ and $T\tilde{M}$ of $\tilde{E}$, we use the notation $\nabla^{\tilde{\Psi}, \tilde{\pi}_N^! \tilde{E}}$ for the partial connection $\nabla^{\tilde{\Psi}^{\tilde{\pi}_N^! \tilde{E}}}$ on $\tilde{\pi}_N^* \tilde{\mathcal{G}}$. It is related to $\nabla^{\tilde{\Psi}}$ by relations analogous to (169).

From now on the Courant algebroids $\pi_N^! E$ and $\tilde{\pi}_N^! \tilde{E}$ will be considered with the $\mathfrak{t}^{2k}$-actions $\Psi^{\pi_N^! E}$ and $\tilde{\Psi}^{\tilde{\pi}_N^! \tilde{E}}$.

Definition 47. *The Courant algebroids E and $\tilde{E}$ are called T-dual if there is an invariant fiber preserving Courant algebroid isomorphism*

$$F : \pi_N^! E \to \tilde{\pi}_N^! \tilde{E}$$

such that the following non-degeneracy condition is satisfied. Let

$$I : E \to T^* M \oplus \mathcal{G} \oplus TM, \ \tilde{I} : \tilde{E} \to T^* \tilde{M} \oplus \tilde{\mathcal{G}} \oplus T\tilde{M},$$

be dissections of E and $\tilde{E}$, and

$$I^{\pi_N} : \pi_N^! E \to T^* N \oplus \pi_N^* \mathcal{G} \oplus TN, \ \tilde{I}^{\tilde{\pi}_N} : \tilde{\pi}_N^! \tilde{E} \to T^* N \oplus \tilde{\pi}_N^* \tilde{\mathcal{G}} \oplus TN$$

the induced dissections of $\pi_N^! E$ and $\tilde{\pi}_N^! \tilde{E}$. Let (β, Φ, K), where $\beta \in \Omega^2(N)$, $\Phi \in \Omega^1(N, \tilde{\pi}_N^* \tilde{\mathcal{G}})$ and $K \in \mathrm{Isom}(\pi_N^* \mathcal{G}, \tilde{\pi}_N^* \tilde{\mathcal{G}})$, be the data which defines the isomorphism

$$\tilde{I}^{\tilde{\pi}_N} \circ F \circ (I^{\pi_N})^{-1} : T^* N \oplus \pi_N^* \mathcal{G} \oplus TN \to T^* N \oplus \tilde{\pi}_N^* \tilde{\mathcal{G}} \oplus TN$$

(according to relation (9) from Section 2.2.1). Then

$$\beta - \Phi^* \Phi : \mathrm{Ker}\ (d\pi_N) \times \mathrm{Ker}\ (d\tilde{\pi}_N) \to \mathbb{R} \tag{170}$$

is non-degenerate.

Definition 48. *The above definition is independent of the choice of dissections.*

Proof. Let $I_i : E \to T^*M \oplus \mathcal{G}_i \oplus TM$ $(i = 1, 2)$ be two dissections of E. Then

$$\hat{F}_i := \tilde{I}^{\tilde{\pi}_N} \circ F \circ (I_i^{\pi_N})^{-1} : T^*N \oplus \pi_N^* \mathcal{G}_i \oplus TN \to T^*N \oplus \tilde{\pi}_N^* \tilde{\mathcal{G}} \oplus TN$$

satisfy

$$\hat{F}_2 = \hat{F}_1 \circ (I_1 \circ I_2^{-1})^{\pi_N}. \tag{171}$$

Assume that the dissections I_1 and I_2 are related by (β, K, Φ). Then from Lemma 16 iii) the induced dissections of $\pi_N^! E$ are related by $(\pi_N^* \beta, \pi_N^* K, \pi_N^* \Phi)$, i.e. the Courant algebroid isomorphism

$$(I_1 \circ I_2^{-1})^{\pi_N} : T^*N \oplus \pi_N^* \mathcal{G}_2 \oplus TN \to T^*N \oplus \pi_N^* \mathcal{G}_1 \oplus TN$$

is given by (9), with (β, K, Φ) replaced by $(\pi_N^* \beta, \pi_N^* K, \pi_N^* \Phi)$. The independence of the non-degeneracy condition (170) on the dissection of E follows from (17). The independence on the dissection of $\tilde{E}$ can be proved similarly. □

Remark 49. Unlike the T-duality for exact Courant algebroids, the definition of T-dual transitive Courant algebroids E and $\tilde{E}$ is not symmetric with respect to E and $\tilde{E}$, in general. This follows from the lack of symmetry in the non-degeneracy condition from Definition 47

Lemma 50. *Let (E_1, Ψ_1) and $(\tilde{E}_1, \tilde{\Psi}_1)$ be transitive Courant algebroids over M and $\tilde{M}$, together with actions*

$$\Psi_1 : \mathfrak{t}^k \to \mathrm{Der}(E_1),\ \ \tilde{\Psi}_1 : \tilde{\mathfrak{t}}^k \to \mathrm{Der}(\tilde{E}_1)$$

which lift ψ^M and $\psi^{\tilde{M}}$ respectively. Assume that

$$G : E_1 \to E,\ \ \tilde{G} : \tilde{E}_1 \to \tilde{E} \tag{172}$$

are invariant, fiber preserving Courant algebroid isomorphisms. If E and $\tilde{E}$ are T-dual, then also E_1 and $\tilde{E}_1$ are T-dual.

Proof. Let $F : \pi_N^! E \to \tilde{\pi}_N^! \tilde{E}$ be an isomorphism which satisfies the conditions from Definition 47. Then the isomorphism

$$F_1 := (\tilde{G}^{\tilde{\pi}_N})^{-1} \circ F \circ G^{\pi_N} : \pi_N^! E_1 \to \tilde{\pi}_N^! \tilde{E}_1. \tag{173}$$

satisfies the same conditions. (For the invariance of $\tilde{G}^{\tilde{\pi}_N}$ and G^{π_N} we use Lemma 45 ii)). □

The next lemma states the conditions that two standard Courant algebroids are T-dual. Let $E = T^*M \oplus \mathcal{G} \oplus TM$ and $\tilde{E} = T^*\tilde{M} \oplus \tilde{\mathcal{G}} \oplus T\tilde{M}$ be standard Courant algebroids over M and $\tilde{M}$, defined by a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data (∇^E, R, H) and, respectively, a quadratic Lie algebra bundle $(\tilde{\mathcal{G}}, [\cdot,\cdot]_{\tilde{\mathcal{G}}}, \langle\cdot,\cdot\rangle_{\tilde{\mathcal{G}}})$ and data $(\nabla^{\tilde{E}}, \tilde{R}, \tilde{H})$ (as there are various connections involved, we choose to use the notation ∇^E rather than ∇ for the connection which is part of the data which defines E; a similar convention is used for $\tilde{E}$). Let $\Psi : \mathfrak{t}^k \to \mathrm{Der}\,(E)$ and $\tilde{\Psi} : \tilde{\mathfrak{t}}^k \to \mathrm{Der}\,(\tilde{E})$ be actions which lift ψ^M and $\psi^{\tilde{M}}$ and preserve the factors of E and $\tilde{E}$.

Lemma 51. *The standard Courant algebroids E and $\tilde{E}$ are T-dual if and only if there are invariant forms $\beta \in \Omega^2(N)$ and $\Phi \in \Omega^1(N, \tilde{\pi}_N^* \tilde{\mathcal{G}})$ and a quadratic Lie algebra bundle isomorphism $K \in \mathrm{Isom}(\pi_N^* \mathcal{G}, \tilde{\pi}_N^* \tilde{\mathcal{G}})$ which maps $\nabla^{\Psi, \pi_N^! E}$ to $\nabla^{\tilde{\Psi}, \tilde{\pi}_N^! \tilde{E}}$ such that the non-degeneracy condition (170) and the following relations hold:*

$$(\tilde{\pi}_N^* \nabla^{\tilde{E}})_X r = K(\pi_N^* \nabla^E)_X (K^{-1} r) + [r, \Phi(X)]_{\tilde{\pi}_N^* \tilde{\mathcal{G}}} \tag{174}$$

$$K \pi_N^* R - \tilde{\pi}_N^* \tilde{R} = d^{\tilde{\pi}_N^* \nabla^{\tilde{E}}} \Phi + c_2 \tag{175}$$

$$\pi_N^* H - \tilde{\pi}_N^* \tilde{H} = d\beta + \langle (K\pi_N^* R + \tilde{\pi}_N^* \tilde{R}) \wedge \Phi \rangle_{\tilde{\pi}_N^* \tilde{\mathcal{G}}} - c_3 \tag{176}$$

where $c_2 \in \Omega^2(N, \tilde{\pi}_N^* \tilde{\mathcal{G}})$ and $c_3 \in \Omega^3(N)$ are defined by

$$c_2(X, Y) := [\Phi(X), \Phi(Y)]_{\tilde{\pi}_N^* \tilde{\mathcal{G}}},$$
$$c_3(X, Y, Z) := \langle \Phi(X), [\Phi(Y), \Phi(Z)]_{\tilde{\pi}_N^* \tilde{\mathcal{G}}} \rangle_{\tilde{\pi}_N^* \tilde{\mathcal{G}}}$$

for any $X, Y, Z \in \mathfrak{X}(N)$.

Proof. Since E and $\tilde{E}$ are standard Courant algebroids, so are $\pi_N^! E$ and $\tilde{\pi}_N^! \tilde{E}$ and an invariant isomorphism $F : \pi_N^! E \to \tilde{\pi}_N^! \tilde{E}$ is defined by invariant data (β, K, Φ), where $\beta \in \Omega^2(N)$, $K \in \mathrm{Isom}(\pi_N^* \mathcal{G}, \tilde{\pi}_N^* \tilde{G})$ and $\Phi \in \Omega^1(N, \tilde{\pi}_N^* \tilde{\mathcal{G}})$. The above relations coincide with relations (10), applied to F and $X, Y, Z \in \mathfrak{X}(N)$. ∎

We end this section with a simple lemma on the existence of preferred dissections of T-dual transitive Courant algebroids.

Lemma 52. *Let E and $\tilde{E}$ be T-dual transitive Courant algebroids (not necessarily in the standard form). Then E and $\tilde{E}$ admit invariant dissections of the form*

$$I : E \to T^*M \oplus \pi^* \mathcal{G}_B \oplus TM, \; \tilde{I} : \tilde{E} \to T^*\tilde{M} \oplus \tilde{\pi}^* \mathcal{G}_B \oplus T\tilde{M}, \tag{177}$$

where $(\mathcal{G}_B, \langle\cdot,\cdot\rangle_{\mathcal{G}_B}, [\cdot,\cdot]_{\mathcal{G}_B})$ is a quadratic Lie algebra bundle on B.

Proof. Let $I : E \to E_M = T^*M \oplus \mathcal{G} \oplus TM$ and $\tilde{I} : \tilde{E} \to E_{\tilde{M}} = T^*\tilde{M} \oplus \tilde{\mathcal{G}} \oplus T\tilde{M}$ be invariant dissections of E and $\tilde{E}$. From Lemma 50, E_M and $E_{\tilde{M}}$ are T-dual. Let $F : \pi_N^! E_M \to \pi_N^! E_{\tilde{M}}$ be an invariant isomorphism, and assume it is defined by data (β, K, Φ). Since F is invariant, the quadratic Lie algebra bundle isomorphism $K : \pi_N^* \mathcal{G} \to \tilde{\pi}_N^* \tilde{\mathcal{G}}$ maps $\nabla^{\Psi, \pi_N^! E}$ to $\nabla^{\tilde{\Psi}, \tilde{\pi}_N^! \tilde{E}}$. From Lemma 46 ii), $K = \Pi^* K_B$ where $K_B : \mathcal{G}_B \to \tilde{\mathcal{G}}_B$ is a quadratic Lie algebra bundle isomorphism and $\mathcal{G} = \pi^* \mathcal{G}_B$, $\tilde{\mathcal{G}} = \tilde{\pi}^* \tilde{\mathcal{G}}_B$ (see Section 5.1.1). Using Lemma 1, we change the dissection $\tilde{I}$ to obtain a new invariant dissection of $\tilde{E}$ with $\tilde{\mathcal{G}} = \tilde{\pi}^* \tilde{\mathcal{G}}_B \cong \tilde{\pi}^* \mathcal{G}_B$ replaced by $\tilde{\pi}^* \mathcal{G}_B$. □

6.2 T-duality and spinors

Assume that E and $\tilde{E}$ are T-dual transitive Courant algebroids and let

$$F : \pi_N^! E \to \tilde{\pi}_N^! \tilde{E}$$

be an invariant isomorphism as in Definition 47. Let $\mathbb{S}_E$, $\mathbb{S}_{\tilde{E}}$, $\mathbb{S}_{\pi_N^! E}$ and $\mathbb{S}_{\tilde{\pi}_N^! \tilde{E}}$ be canonical weighted spinor bundles of E, $\tilde{E}$, $\pi_N^! E$ and $\tilde{\pi}_N^! \tilde{E}$ respectively, such that the pullbacks $\pi_N^! : \Gamma(\mathbb{S}_E) \to \Gamma(\mathbb{S}_{\pi_N^! E})$ and $\tilde{\pi}_N^! : \Gamma(\mathbb{S}_{\tilde{E}}) \to \Gamma(\mathbb{S}_{\tilde{\pi}_N^! \tilde{E}})$ are defined. We consider an admissible pair $(I, S_{\mathcal{G}})$ for $\mathbb{S}_E$ and $\mathbb{S}_{\pi_N^! E}$, and an admissible pair $(\tilde{I}, S_{\tilde{\mathcal{G}}})$ for $\mathbb{S}_{\tilde{E}}$ and $\mathbb{S}_{\tilde{\pi}_N^! \tilde{E}}$, with invariant dissections

$$I : E \to E_M = T^*M \oplus \pi^* \mathcal{G}_B \oplus TM, \;\; \tilde{I} : \tilde{E} \to \tilde{E}_{\tilde{M}} = T^*\tilde{M} \oplus \tilde{\pi}^* \tilde{\mathcal{G}}_B \oplus T\tilde{M}$$

such that $S_{\mathcal{G}} = \pi^* S_B$ and $S_{\tilde{\mathcal{G}}} = \tilde{\pi}^* \tilde{S}_B$, where S_B is an irreducible $\mathrm{Cl}(\mathcal{G}_B)$-bundle and $\tilde{S}_B$ is an irreducible $\mathrm{Cl}(\tilde{\mathcal{G}}_B)$-bundle. We assume that the isomorphism $F_{\mathbb{S}} : \mathbb{S}_{\pi_N^! E} \to \mathbb{S}_{\tilde{\pi}_N^! \tilde{E}}$ induced by F is globally defined. This is equivalent to assuming that the isomorphism $(F_1)_{\mathbb{S}} : \mathbb{S}_N \to \tilde{\mathbb{S}}_N$ compatible with $F_1 := \tilde{I}^{\tilde{\pi}_N} \circ F \circ (I^{\pi_N})^{-1} : \pi_N^! E_M \to \tilde{\pi}_N^! \tilde{E}_{\tilde{M}}$ is globally defined, where $\mathbb{S}_N = \Lambda(T^*N) \hat{\otimes} \Pi^* \mathcal{S}_B$ and $\tilde{\mathbb{S}}_N = \Lambda(T^*N) \hat{\otimes} \Pi^* \tilde{\mathcal{S}}_B$ are spinor bundles of $\pi_N^! E_M = T^*N \oplus \Pi^* \mathcal{G}_B \oplus TN$ and $\tilde{\pi}_N^! \tilde{E}_{\tilde{M}} = T^*N \oplus \Pi^* \tilde{\mathcal{G}}_B \oplus TN$ respectively (and $\Pi = \pi \circ \pi_N = \tilde{\pi} \circ \tilde{\pi}_N$).

Remark 53. When the dissections are chosen such that $\mathcal{G}_B = \tilde{\mathcal{G}}_B$ as quadratic Lie algebra bundles (see Lemma 52) and $S_B = \tilde{S}_B$ as $\mathrm{Cl}(\mathcal{G}_B)$-bundles, F_1 is an automorphism of the vector bundle

$$\pi_N^! E_M = \tilde{\pi}_N^! \tilde{E}_{\tilde{M}} = T^*N \oplus \Pi^* \mathcal{G}_B \oplus TN$$

with scalar product

$$\langle \xi + \Pi^*(r_1) + X, \eta + \Pi^*(r_2) + Y \rangle = \frac{1}{2}(\xi(Y) + \eta(X)) + \Pi^* \langle r_1, r_2 \rangle_{\mathcal{G}_B} \quad (178)$$

and $(F_1)_{\mathbb{S}}$ is an automorphism of the spinor bundle $\mathbb{S}_N = \Lambda(T^*N)\hat{\otimes}\Pi^*\mathcal{S}_B$ of $T^*N \oplus \Pi^*\mathcal{G}_B \oplus TN$. If F_1 belongs to the image of the exponential map $\exp : \mathfrak{so}(T^*N \oplus \Pi^*\mathcal{G}_B \oplus TN) \to \mathrm{SO}_0(T^*N \oplus \Pi^*\mathcal{G}_B \oplus TN)$, then $(F_1)_{\mathbb{S}}$ is automatically globally defined (cf. Remark 11). In fact, in that case F_1 can be lifted to $(F_1)_{\mathbb{S}}$ using the exponential map for $\mathfrak{spin}(T^*N \oplus \Pi^*\mathcal{G}_B \oplus TN)$.

Theorem 54. *i) The map*

$$\tau := (\tilde{\pi}_N)_! \circ F_{\mathbb{S}} \circ \pi_N^! : \Gamma(\mathbb{S}_E) \to \Gamma(\mathbb{S}_{\tilde{E}}) \tag{179}$$

intertwines the canonical Dirac generating operators of E and $\tilde{E}$ and maps invariant spinors to invariant spinors. In particular, there is the following commutative diagram

$$\begin{array}{ccc} \Gamma_{\mathfrak{t}^{2k}}(\mathbb{S}_{\pi_N^! E}) & \xrightarrow{F_{\mathbb{S}}} & \Gamma_{\mathfrak{t}^{2k}}(\mathbb{S}_{\tilde{\pi}_N^! \tilde{E}}) \\ \pi_N^! \uparrow & & \downarrow (\tilde{\pi}_N)_! \\ \Gamma_{\mathfrak{t}^k}(\mathbb{S}_E) & \xrightarrow{\tau} & \Gamma_{\tilde{\mathfrak{t}}^k}(\mathbb{S}_{\tilde{E}}) \end{array}$$

ii) There is an isomorphism $\rho : \Gamma_{\mathfrak{t}^k}(E) \to \Gamma_{\tilde{\mathfrak{t}}^k}(\tilde{E})$ of $C^\infty(B)$-modules which preserves Courant brackets, scalar products and is compatible with τ, i.e.

$$\tau(\gamma_u s) = \gamma_{\rho(u)}\tau(s),\ [\rho(u), \rho(v)]_{\tilde{E}} = \rho[u, v]_E,\ \langle \rho(u), \rho(v)\rangle_{\tilde{E}} = \langle u, v\rangle_E, \tag{180}$$

for any $u, v \in \Gamma_{\mathfrak{t}^k}(E)$ and $s \in \Gamma_{\mathfrak{t}^k}(\mathbb{S}_E)$.

The claim from Theorem 54 i) concerning the canonical Dirac generating operators follows from Propositions 15, 19 and 23. The remaining claims from Theorem 54 will be proved in the next two lemmas.

Lemma 55. *In the above setting, E_M and $\tilde{E}_{\tilde{M}}$ are T-dual. The assertions of Theorem 54 hold for the pair $(E, \tilde{E})$ and canonical weighted spinor bundles $\mathbb{S}_E$ and $\mathbb{S}_{\tilde{E}}$ if and only if they hold for the pair $(E_M, \tilde{E}_{\tilde{M}})$ and canonical weighted spinor bundles $\mathbb{S}_M := \Lambda(T^*M)\hat{\otimes}\pi^*\mathcal{S}_B$ and $\mathbb{S}_{\tilde{M}} := \Lambda(T^*\tilde{M})\hat{\otimes}\tilde{\pi}^*\tilde{\mathcal{S}}_B$ of E_M and $\tilde{E}_{\tilde{M}}$ respectively.*

Proof. The fact that E_M and $\tilde{E}_{\tilde{M}}$ are T-dual follows from Lemma 50. We will assume that the assertions of Theorem 54 hold for $(E, \tilde{E})$ and show that they hold for $(E_M, \tilde{E}_{\tilde{M}})$ as well. The same arguments prove also the converse statement. Using relation (173), we obtain that the map τ_1 defined for the pair $(E_M, \tilde{E}_{\tilde{M}})$ as is defined τ for the pair $(E, \tilde{E})$, that is,

$$\tau_1 := (\tilde{\pi}_N)_* \circ (F_1)_{\mathbb{S}} \circ \pi_N^* : \Gamma(\mathbb{S}_M) \to \Gamma(\mathbb{S}_{\tilde{M}}), \tag{181}$$

where

$$\pi_N^* : \Gamma(\mathbb{S}_M) \to \Gamma(\mathbb{S}_N) = \Omega(N, \Pi^* \mathcal{S}_B)$$
$$(\tilde{\pi}_N)_* : \Gamma(\tilde{\mathbb{S}}_N) = \Omega(N, \Pi^* \tilde{\mathcal{S}}_B) \to \Gamma(\mathbb{S}_{\tilde{M}})$$

are the pullback and pushforward maps (56) and (79), is related to τ by $\tau_1 = \epsilon \tilde{I}_{\mathbb{S}} \circ \tau \circ (I_{\mathbb{S}})^{-1}$. Here $\epsilon \in \{\pm 1\}$ and $I_{\mathbb{S}} : \mathbb{S}_E \to \mathbb{S}_M$, $\tilde{I}_{\mathbb{S}} : \mathbb{S}_{\tilde{E}} \to \mathbb{S}_{\tilde{M}}$ are induced by I, $\tilde{I}$. As I and $\tilde{I}$ are invariant, $I_{\mathbb{S}}$ and $\tilde{I}_{\mathbb{S}}$ map invariant spinors to invariant spinors. This is true also for τ. We obtain that τ_1 maps invariant spinors to invariant spinors. Define

$$\rho_1 : \Gamma_{\mathfrak{k}^k}(E_M) \to \Gamma_{\mathfrak{k}^k}(\tilde{E}_{\tilde{M}}),\ \rho_1 := \tilde{I} \circ \rho \circ I^{-1}.$$

Since (τ, ρ) satisfy (180), so do (τ_1, ρ_1). □

Lemma 56. *The statements from Theorem 54 hold for the pair $(E_M, \tilde{E}_{\tilde{M}})$ and canonical weighted spinor bundles $\mathbb{S}_M$ and $\mathbb{S}_{\tilde{M}}$.*

Proof. i) We prove that the map τ_1 defined by (181) maps invariant spinors to invariant spinors. We denote by ∇^{Ψ} and $\nabla^{\tilde{\Psi}}$ the partial connections defined by the actions on the standard Courant algebroids E_M and $\tilde{E}_{\tilde{M}}$. For the various other partial connections (like $\nabla^{\Psi, \pi_N^! E_M}$, $\nabla^{\Psi, \pi^* \mathcal{S}_B}$, $\nabla^{\Psi, \Pi^* \mathcal{S}_B}$ etc) we use the definition explained after Lemma 45 (keeping in mind that $S_{\mathcal{G}} = \pi^* S_B$ and $\pi_N^* S_{\mathcal{G}} = \Pi^* S_B$). We prove that if $\omega \otimes s \in \Gamma(\mathbb{S}_M)$ is $\mathfrak{k}^k$-invariant then $\pi_N^*(\omega \otimes s) \in \Gamma(\mathbb{S}_N)$ is $\mathfrak{k}^{2k}$-invariant. The $\mathfrak{k}^k$-invariance of $\pi_N^*(\omega \otimes s)$ follows from Proposition 44 ii). In order to prove that $\pi_N^*(\omega \otimes s)$ is $\tilde{\mathfrak{k}}^k$-invariant, we apply the formula

$$\Psi^{\mathbb{S}_N}(a) \pi_N^*(\omega \otimes s) = \mathcal{L}_{X_a^N}(\pi_N^* \omega) \otimes \pi_N^* \omega + (\pi_N^* \omega) \otimes \nabla^{\Psi, \Pi^* \mathcal{S}_B}(\pi_N^* s). \tag{182}$$

If $a \in \tilde{\mathfrak{k}}^k$ then

$$\mathcal{L}_{X_a^N}(\pi_N^* \omega) = \pi_N^* \mathcal{L}_{(\pi_N)_* X_a^N} \omega = 0, \tag{183}$$

since $(\pi_N)_* X_a^N = 0$. On the other hand, from (169), we deduce that

$$\nabla^{\Psi, \Pi^* \mathcal{S}_B}_{X_a^N}(\pi_N^* s) = \pi_N^* (\nabla^{\Psi, \pi^* \mathcal{S}_B}_{(\pi_N)_* X_a^N} r) = 0$$

by using again $(\pi_N)_* X_a^N = 0$. It follows that $\pi_N^*(\omega \otimes s)$ is $\tilde{\mathfrak{k}}^k$-invariant. We have proven that $\pi_N^*(\omega \otimes s)$ is $\mathfrak{k}^{2k}$-invariant. From Lemma 37, $(F_1)_{\mathbb{S}} \pi_N^*(\omega \otimes s)$ is $\mathfrak{k}^{2k}$-invariant (in particular, $\tilde{\mathfrak{k}}^k$-invariant) and from Proposition 44 ii) we obtain that $\tau_1(\omega \otimes s)$ is $\tilde{\mathfrak{k}}^k$-invariant, as needed.

ii) Let $u = \xi + r + X \in \Gamma_{\mathfrak{k}^k}(E_M)$. Then X and ξ are invariant with respect to the standard (by Lie derivatives) action of $\mathfrak{k}^k$ on M and r is ∇^{Ψ}-parallel. We obtain

$$\mathcal{L}_{X_a^N}(\pi_N^* \xi) = 0,\ \forall a \in \mathfrak{k}^{2k},\ \nabla^{\Psi, \pi_N^! E_M}(\pi_N^* r) = 0, \tag{184}$$

where in the second relation (184) we used (168). We claim that there is a unique $\mathfrak{t}^{2k}$-invariant lift $\widehat{X}_0 \in \mathfrak{X}_{\mathfrak{t}^{2k}}(N)$ of X with the property that

$$\mathrm{pr}_{T^*N} F_1(\pi_N^*(\xi + r) + \widehat{X}_0) = \tilde{\pi}_N^*(\tilde{\xi}) \tag{185}$$

for an (invariant) 1-form $\tilde{\xi} \in \Omega^1(\tilde{M})$. To prove the claim we assume that the isomorphism F_1 is defined by data (β, K, Φ) as in Section 2.2.1, where $\beta \in \Omega^2(N)$, $K \in \mathrm{Isom}(\Pi^*\mathcal{G}_B, \Pi^*\tilde{\mathcal{G}}_B)$ and $\Phi \in \Omega^1(N, \Pi^*\tilde{\mathcal{G}}_B)$. Let $\widehat{X}$ be an arbitrary $\mathfrak{t}^{2k}$-invariant lift of X. Then

$$\begin{aligned}
&\mathrm{pr}_{TN} F_1(\pi_N^*(\xi + r) + \widehat{X}) = \widehat{X},\ \mathrm{pr}_{\tilde{\pi}_N^*\tilde{\mathcal{G}}} F_1(\pi_N^*(\xi + r) + \widehat{X}) = K(\pi_N^* r) + \Phi(\widehat{X})\\
&\mathrm{pr}_{T^*N} F_1(\pi_N^*(\xi + r) + \widehat{X}) = \pi_N^*(\xi) - 2\Phi^* K(\pi_N^* r) + i_{\widehat{X}}\beta - \Phi^*\Phi(\widehat{X}).
\end{aligned} \tag{186}$$

From (184) $\pi_N^*(\xi + r)$ is $\mathfrak{t}^{2k}$-invariant and we obtain that $F_1(\pi_N^*(\xi + r) + \widehat{X})$ is also $\mathfrak{t}^{2k}$-invariant. The non-degeneracy of (170) together with the $\mathfrak{t}^{2k}$-invariance of $\mathrm{pr}_{T^*N} F_1(\pi_N^*(\xi + r) + \widehat{X})$ and the last formula (186) imply that there is a unique $\mathfrak{t}^{2k}$- invariant lift $\widehat{X}_0$ of X such that $\mathrm{pr}_{T^*N} F_1(\pi_N^*(\xi+r)+\widehat{X}_0)$ is horizontal with respect to $\tilde{\pi}_N$ and invariant, hence basic. For this lift relation (185) holds.

On the other hand, since $F_1(\pi_N^*(\xi+r)+\widehat{X}_0)$ is $\mathfrak{t}^{2k}$-invariant, its projection to $\Pi^*\tilde{\mathcal{G}}_B$ is $\nabla^{\tilde{\Psi}, \tilde{\pi}_N^! \tilde{E}_{\tilde{M}}}$-parallel and is therefore the pullback of a $\nabla^{\tilde{\Psi}}$-parallel section $\tilde{r}$ of $\tilde{\pi}^*\mathcal{G}_B$. To summarize,

$$F_1(\pi_N^*(\xi + r) + \widehat{X}_0) = \tilde{\pi}_N^*(\tilde{\xi} + \tilde{r}) + \widehat{X}_0 \tag{187}$$

where $\tilde{\xi} \in \Omega^1(\tilde{M})$ is $\tilde{\mathfrak{t}}^k$-invariant and $\nabla^{\tilde{\Psi}}(\tilde{r}) = 0$. We define

$$\rho_1(u) := \tilde{\xi} + \tilde{r} + (\tilde{\pi}_N)_* \widehat{X}_0. \tag{188}$$

Obviously, $\rho_1(u)$ is $\tilde{\mathfrak{t}}^k$-invariant and the resulting map $\rho_1 : \Gamma_{\mathfrak{t}^k}(E_M) \to \Gamma_{\tilde{\mathfrak{t}}^k}(\tilde{E}_{\tilde{M}})$ is $C^\infty(B)$-linear. It remains to prove that (ρ_1, τ_1) satisfy relations (180). In order to prove the first relation (180), let $u := \xi + r + X \in \Gamma_{\mathfrak{t}^k}(E_M)$, $\rho_1(u) = \tilde{\xi} + \tilde{r} + (\tilde{\pi}_N)_* \widehat{X}_0$ constructed as above, $s \in \Gamma_{\mathfrak{t}^k}(\mathbb{S}_M)$ and $\sigma \in \Gamma(\mathbb{S}_{\tilde{N}})$. Using (59) and Remark 22,

$$\pi_N^* \gamma_u(s) = \gamma_{\pi_N^*(\xi+r)+\widehat{X}_0} \pi_N^* s,\ (\tilde{\pi}_N)_* \gamma_{\tilde{\pi}_N^*(\tilde{\xi}+\tilde{r})+\widehat{X}_0} \sigma = \gamma_{\rho_1(u)} (\tilde{\pi}_N)_* \sigma, \tag{189}$$

and we write

$$\begin{aligned}
\tau_1 \gamma_u(s) &= (\tilde{\pi}_N)_* (F_1)_{\mathbb{S}} (\pi_N)^* \gamma_u(s) = (\tilde{\pi}_N)_* (F_1)_{\mathbb{S}} \gamma_{\pi_N^*(\xi+r)+\widehat{X}_0} \pi_N^* s\\
&= (\tilde{\pi}_N)_* \gamma_{\tilde{\pi}_N^*(\tilde{\xi}+\tilde{r})+\widehat{X}_0} (F_1)_{\mathbb{S}} \pi_N^*(s) = \gamma_{\rho_1(u)} (\tilde{\pi}_N)_* (F_1)_{\mathbb{S}} (\pi_N)^*(s)\\
&= \gamma_{\rho_1(u)} \tau_1(s),
\end{aligned}$$

where in the third equality we used Lemma 10 and relation (187). The first relation of (180) is proved. The second relation of (180) follows from the next computation, which uses the first relation of (180) together with $\tau_1 \circ \not{d}_M = \not{d}_{\tilde{M}} \circ \tau_1$ proved in part i) of Theorem 54 (where $\not{d}_M$ and $\not{d}_{\tilde{M}}$ are the Dirac generating operators of E_M and $E_{\tilde{M}}$ acting on $\Gamma(\mathbb{S}_M)$ and $\Gamma(\mathbb{S}_{\tilde{M}})$ respectively). For any $u, v \in \Gamma_{\mathfrak{t}^k}(E_M)$ and $s \in \Gamma_{\mathfrak{t}^k}(\mathbb{S}_M)$, we have

$$\begin{aligned}\gamma_{\rho_1[u,v]_{E_M}} \tau_1(s) &= \tau_1 \gamma_{[u,v]_{E_M}}(s) = \tau_1[[\not{d}_M, \gamma_u], \gamma_v](s) = [[\not{d}_{\tilde{M}}, \gamma_{\rho_1(u)}], \gamma_{\rho_1(v)}]\tau_1(s) \\ &= \gamma_{[\rho_1(u), \rho_1(v)]_{\tilde{E}_{\tilde{M}}}} \tau_1(s).\end{aligned}$$

In order to prove the third relation of (180), we remark that for any $u \in \Gamma_{\mathfrak{t}^k}(E_M)$, $\langle u, u \rangle_{E_M}$ is $\mathfrak{t}^k$-invariant and hence is the pullback of a function $g \in C^\infty(B)$. Similarly, $\langle \rho_1(u), \rho_1(u) \rangle_{\tilde{E}_{\tilde{M}}}$ is the pullback of a function $\tilde{g} \in C^\infty(B)$. We need to show that $g = \tilde{g}$. This follows from the next computation which uses the first relation of (180):

$$\begin{aligned}\tilde{\pi}^*(g)\tau_1(s) &= \tau_1(\pi^*(g)s) = \tau_1(\langle u, u \rangle_{E_M} s) = \tau_1 \gamma_u^2(s) = \gamma_{\rho_1(u)}^2 \tau_1(s) \\ &= \langle \rho_1(u), \rho_1(u) \rangle_{\tilde{E}_{\tilde{M}}} \tau_1(s) = \tilde{\pi}^*(\tilde{g})\tau_1(s).\end{aligned}$$

From the third relation of (180) we obtain that ρ_1 is an isomorphism (of vector spaces and even of $C^\infty(B)$-modules). The proof of the theorem is completed. □

Corollary 57. *The map*

$$\tau := (\tilde{\pi}_N)_! \circ F_{\mathbb{S}} \circ \pi_N^! : \Gamma_{\mathfrak{t}^k}(\mathbb{S}_E) \to \Gamma_{\tilde{\mathfrak{t}}^k}(\mathbb{S}_{\tilde{E}}) \tag{190}$$

is an isomorphism of $C^\infty(B)$-modules.

Proof. This follows from the irreducibility of the spinor bundles together with the fact that τ is $C^\infty(B)$-linear, is not the zero map and intertwines the Clifford multiplications in the commutative diagram

$$\begin{array}{ccc} \Gamma_{\mathfrak{t}^k}(E) \times \Gamma_{\mathfrak{t}^k}(\mathbb{S}_E) & \longrightarrow & \Gamma_{\mathfrak{t}^k}(\mathbb{S}_{\tilde{E}}) \\ \downarrow{\scriptstyle \rho\times\tau} & & \downarrow{\scriptstyle \tau} \\ \Gamma_{\tilde{\mathfrak{t}}^k}(\tilde{E}) \times \Gamma_{\tilde{\mathfrak{t}}^k}(\mathbb{S}_{\tilde{E}}) & \longrightarrow & \Gamma_{\tilde{\mathfrak{t}}^k}(\mathbb{S}_{\tilde{E}}). \end{array}$$

□

Remark 58. As in the T-duality for exact or heterotic Courant algebroids, the map ρ constructed in Theorem 54 can be interpreted as an isomorphism between Courant algebroids over B (see [2] and [7]).

In the next remark we discuss Theorem 54 without the assumption that $F_{\mathbb{S}}$ is globally defined.

Remark 59. i) We claim that the isomorphism $(F_1)_{\mathbb{S}}$ introduced before Remark 53 (hence also $F_{\mathbb{S}}$) is always defined on subsets of N of the form $\Pi^{-1}(V)$, where $V \subset B$ is open and sufficiently small. Indeed, $(F_1)_{\mathbb{S}|_U} \in \mathrm{Isom}(\mathbb{S}_N|_U, \tilde{\mathbb{S}}_N|_U)$ is defined, whenever $U \subset N$ is open and sufficiently small (see Lemma 10). Letting $V := \Pi(U)$, we can find (reducing V is necessary) invariant frames (s_i) and $(\tilde{s}_i)$ of $\mathbb{S}_N$ and $\mathbb{S}_{\tilde{N}}$ on $\Pi^{-1}(V)$. With respect to these frames, $(F_1)_{\mathbb{S}|_U}$ is given by

$$(F_1)_{\mathbb{S}|_U}(s_i) = \sum_j C_{ji}\tilde{s}_j \tag{191}$$

for some functions $C_{ji} \in C^\infty(U)$. From the invariance of $(F_1)_{\mathbb{S}|_U}$, we deduce that $\mathcal{L}_{X_a^N} C_{ji} = 0$ for any $a \in \mathfrak{t}^{2k}$, i.e. $C_{ji} = \Pi_0^*(c_{ji})$ where $c_{ji} \in C^\infty(V)$ and $\Pi_0 : U \to V$ is the restriction of Π. Using that all $\pi_N^! E_M$, $\tilde{\pi}_N^! \tilde{E}_{\tilde{M}}$, $\mathbb{S}_N$ and $\mathbb{S}_{\tilde{N}}$ admit invariant frames on $\Pi^{-1}(V)$, we deduce from the compatibility of $(F_1)_{\mathbb{S}|_U}$ with $F_1|_U$ that

$$(F_1)_{\mathbb{S}|_{\Pi^{-1}(V)}}(s_i) := \sum_j \Pi^*(c_{ji})\tilde{s}_j, \tag{192}$$

defined on $\Pi^{-1}(V)$, is compatible with $F_1|_{\Pi^{-1}(V)}$.

ii) From the above, the map

$$\tau_V := (\tilde{\pi}_N)_! \circ F_{\mathbb{S}} \circ \pi_N^! : \Gamma(\mathbb{S}_E|_{\pi^{-1}(V)}) \to \Gamma(\mathbb{S}_{\tilde{E}}|_{\tilde{\pi}^{-1}(V)})$$

is defined. Theorem 54 still holds, the only difference being that τ is replaced by the locally defined maps τ_V, for any $V \subset B$ open and sufficiently small. (The isomorphism ρ remains defined globally.)

6.3 Existence of a T-dual

Let $\pi : M \to B$ be a principal T^k-bundle and $\mathcal{H}$ a principal connection on π, with connection form $\theta = \sum_{i=1}^k \theta_i e_i \in \Omega^1(M, \mathfrak{t}^k)$, where (e_i) is a basis of $\mathfrak{t}^k$. Let (E, Ψ) be a standard Courant algebroid with an action $\Psi : \mathfrak{t}^k \to \mathrm{Der}(E)$ which lifts the vertical paralellism of π, defined by a quadratic Lie algebra bundle $(\mathcal{G}_B, [\cdot,\cdot]_{\mathcal{G}_B}, \langle\cdot,\cdot\rangle_{\mathcal{G}_B})$ whose adjoint representation is an isomorphism, a connection ∇^B on $\mathcal{G}_B$ which preserves $[\cdot,\cdot]_{\mathcal{G}_B}$ and $\langle\cdot,\cdot\rangle_{\mathcal{G}_B}$, a 3-form $H^B_{(3)}$, 2-forms $H^{i,B}_{(2)}$ and sections $r_i^B \in \Gamma(\mathcal{G}_B)$ $(1 \leq i \leq k)$ as in Example 35 (see also Corollary 34). We denote by (e^i) the dual basis of (e_i).

Theorem 60. *Assume that the closed forms $\mathcal{K}_i$ defined by (132) represent integral cohomology classes in $H^2(B,\mathbb{R})$ and let $\tilde{\pi}: \tilde{M} \to B$ be a principal $\tilde{T}^k$-bundle with connection form $\tilde{\theta} = \sum_{i=1}^k \tilde{\theta}_i e^i$, such that $(d\tilde{\theta}_i)^B = \mathcal{K}_i$ for any i. Then E admits a T-dual $\tilde{E}$, defined on $\tilde{M}$ and*

$$\left[\sum_{i=1}^k (d\theta_i)^B \wedge (d\tilde{\theta}_i)^B \right] = [\langle \mathfrak{r}^B \wedge \mathfrak{r}^B \rangle_{\mathcal{G}_B}] \in H^4(B,\mathbb{R}). \tag{193}$$

Proof. From the expression (132) of $\mathcal{K}_i = (d\tilde{\theta}_i)^B$, we have

$$(d\tilde{\theta}_i)^B = H^{i,B}_{(2)} + 2\langle \mathfrak{r}^B, r_i^B \rangle_{\mathcal{G}_B} - \langle r_i^B, r_j^B \rangle_{\mathcal{G}_B} (d\theta_j)^B. \tag{194}$$

We consider the data formed by the quadratic Lie algebra bundle

$$(\tilde{\mathcal{G}}_B, [\cdot,\cdot]\tilde{}_{\mathcal{G}_B}, \langle\cdot,\cdot\rangle\tilde{}_{\mathcal{G}_B}) := (\mathcal{G}_B, [\cdot,\cdot]_{\mathcal{G}_B}, \langle\cdot,\cdot\rangle_{\mathcal{G}_B}),$$

connection $\tilde{\nabla}^B := \nabla^B$, sections $\tilde{r}_i^B \in \Gamma(\mathcal{G}_B)$ (arbitrarily chosen), 3-form $\tilde{H}^B_{(3)} := H^B_{(3)}$ and 2-forms

$$\tilde{H}^{i,B}_{(2)} := (d\theta_i)^B - 2\langle \mathfrak{r}^B, \tilde{r}_i^B \rangle_{\mathcal{G}_B} + \langle \tilde{r}_i^B, \tilde{r}_j^B \rangle_{\mathcal{G}_B} (d\tilde{\theta}_j)^B. \tag{195}$$

From (195), the 2-form

$$\tilde{\mathcal{K}}_i := \tilde{H}^{i,B}_{(2)} + 2\langle \mathfrak{r}^B, \tilde{r}_i^B \rangle_{\mathcal{G}_B} - \langle \tilde{r}_i^B, \tilde{r}_j^B \rangle_{\mathcal{G}_B} (d\tilde{\theta}_j)^B = (d\theta_i)^B$$

is closed. Since

$$d\tilde{H}_{(3)} = dH_{(3)} = \langle \mathfrak{r}^B \wedge \mathfrak{r}^B \rangle_{\mathcal{G}_B} - \mathcal{K}_i \wedge (d\theta_i)^B = \langle \mathfrak{r}^B \wedge \mathfrak{r}^B \rangle_{\mathcal{G}_B} - \tilde{\mathcal{K}}_i \wedge (d\tilde{\theta}_i)^B,$$

we obtain from Example 35 a standard Courant algebroid $(\tilde{E}, \tilde{\Psi})$ together with an action which lifts the vertical paralellism of $\tilde{\pi}$, such that

$$\begin{aligned} \tilde{H}^{pqs,B}_{(0)} &:= -\frac{1}{3} \langle [\tilde{r}_p^B, \tilde{r}_q^B]_{\mathcal{G}_B}, \tilde{r}_s^B \rangle_{\mathcal{G}_B} \\ \tilde{H}^{ij,B}_{(1)} &:= \frac{1}{2} \left(\langle \nabla^B \tilde{r}_i^B, \tilde{r}_j^B \rangle_{\mathcal{G}_B} - \langle \nabla^B \tilde{r}_j^B, \tilde{r}_i^B \rangle_{\mathcal{G}_B} \right) \end{aligned}$$

and the $\mathcal{G}_B$-valued forms $\tilde{R}^{ij,B}_{(0)}$, $\tilde{R}^{i,B}_{(1)}$ and $\tilde{R}^B_{(2)}$ given by the tilde analogue of (128). The quadratic Lie algebra bundles of E and $\tilde{E}$ are the pullbacks of $(\mathcal{G}_B, [\cdot,\cdot]_{\mathcal{G}_B}, \langle\cdot,\cdot\rangle_{\mathcal{G}_B})$ and, as vector bundles with scalar products,

$$\pi_N^! E = \tilde{\pi}_N^! \tilde{E} = T^*N \oplus \Pi^* \mathcal{G}_B \oplus TN,$$

where the scalar products are given by (178) and $\Pi = \pi \circ \pi_N = \tilde{\pi} \circ \tilde{\pi}_N$.

We claim that E and $\tilde{E}$ are T-dual, i.e. not only that $\pi_N^! E$ and $\tilde{\pi}_N^! \tilde{E}$ are isomorphic as Courant algebroids but that one can choose the isomorphism to be invariant and such that the non-degeneracy condition (170) is satisfied. Such an isomorphism $F : \pi_N^! E \to \tilde{\pi}_N^! \tilde{E}$ (if it exists) is given by a triple (β, K, Φ), where $\beta \in \Omega^2(N)$, $\Phi \in \Omega^1(N, \Pi^* \mathcal{G}_B)$ and $K = \Pi^* K_B$ where $K_B \in \mathrm{Aut}(\mathcal{G}_B)$ is a quadratic Lie algebra bundle automorphism (see the proof of Lemma 52). Let

$$\nabla^{\theta} := \nabla^E + \theta_i \otimes \mathrm{ad}_{r_i} = \pi^* \nabla^B$$
$$\nabla^{\tilde{\theta}} := \nabla^{\tilde{E}} + \tilde{\theta}_i \otimes \mathrm{ad}_{\tilde{r}_i} = \tilde{\pi}^* \nabla^B$$

be the connections on E and $\tilde{E}$ defined before Lemma 31, where $r_i := \pi^*(r_i^B)$, $\tilde{r}_i := \tilde{\pi}^*(\tilde{r}_i^B)$ and to simplify notation we continue to omit the summation sign and we denote by the same symbol 'ad' the adjoint action in the Lie algebra bundles $\mathcal{G}_B$, $\mathcal{G}$, $\tilde{\mathcal{G}}$ or their pullbacks to N. Then

$$\tilde{\pi}_N^* \nabla^{\tilde{E}} = \Pi^* \nabla^B - (\tilde{\pi}_N^* \tilde{\theta}_i) \otimes \Pi^*(\mathrm{ad}_{\tilde{r}_i^B})$$
$$\pi_N^* \nabla^E = \Pi^* \nabla^B - (\pi_N^* \theta_i) \otimes \Pi^*(\mathrm{ad}_{r_i^B}). \tag{196}$$

With these preliminary remarks, we now consider separately the relations from Lemma 51 and we look for $(\beta, K = \Pi^* K_B, \Phi)$ such that these relations are satisfied. Relation (174) can be written in the equivalent way

$$\Pi^* \left(K_B(\nabla^B) K_B^{-1} - \nabla^B \right) = (\pi_N^* \theta_i) \otimes \Pi^*(\mathrm{ad}_{K_B(r_i^B)}) - (\tilde{\pi}_N^* \tilde{\theta}_i) \otimes (\Pi^* \mathrm{ad}_{\tilde{r}_i^B}) + \mathrm{ad} \circ \Phi. \tag{197}$$

Letting

$$K_B := \mathrm{Id}_{\mathcal{G}_B}, \; \Phi := (\tilde{\pi}_N^* \tilde{\theta}_i) \otimes \Pi^*(\tilde{r}_i^B) - (\pi_N^* \theta_i) \otimes \Pi^*(r_i^B), \tag{198}$$

relation (197) is obviously satisfied. Relation (175) is automatically satisfied from Lemma 2 and our hypothesis that the adjoint representation of $(\mathcal{G}_B, [\cdot,\cdot]_{\mathcal{G}_B})$ is an isomorphism. It remains to find an invariant 2-form $\beta \in \Omega^2(N)$ such that relation (176) is satisfied. Now, a straightforward computation which uses the definition of Φ shows that the 3-form

$$c_3(X, Y, Z) := \langle \Phi(X), [\Phi(Y), \Phi(Z)]_{\Pi^* \mathcal{G}_B} \rangle_{\Pi^* \mathcal{G}_B}, \; \forall X, Y, Z \in \mathfrak{X}(N)$$

is given by

$$c_3 = \frac{1}{6} \left(\langle [\tilde{r}_s^B, \tilde{r}_i^B]_{\mathcal{G}_B}, \tilde{r}_j^B \rangle \tilde{\theta}_s \wedge \tilde{\theta}_i \wedge \tilde{\theta}_j - \langle [r_s^B, r_i^B]_{\mathcal{G}_B}, r_j^B \rangle \theta_s \wedge \theta_i \wedge \theta_j \right)$$
$$+ \frac{1}{2} \left(\langle [r_i^B, r_j^B]_{\mathcal{G}_B}, \tilde{r}_s^B \rangle \tilde{\theta}_s \wedge \theta_i \wedge \theta_j - \langle [\tilde{r}_j^B, \tilde{r}_s^B]_{\mathcal{G}_B}, r_i^B \rangle \theta_i \wedge \tilde{\theta}_j \wedge \tilde{\theta}_s \right), \tag{199}$$

where we identify forms on M, $\tilde{M}$ or B with their pullback to N (we omit the pullback signs) and we denote $\langle\cdot,\cdot\rangle_{\mathcal{G}_B}$ by $\langle\cdot,\cdot\rangle$ for simplicity. On the other hand,

$$\begin{aligned}\pi_N^* R &= R^B_{(2)} + \theta_i \wedge R^{i,B}_{(1)} + R^{ij,B}_{(0)} \otimes (\theta_i \wedge \theta_j),\\ \tilde{\pi}_N^* \tilde{R} &= \tilde{R}^B_{(2)} + \tilde{\theta}_i \wedge \tilde{R}^{i,B}_{(1)} + \tilde{R}^{ij,B}_{(0)} \otimes (\tilde{\theta}_i \wedge \tilde{\theta}_j),\end{aligned} \tag{200}$$

where we recall that

$$R^{ij,B}_{(0)} = \frac{1}{2}[r_i^B, r_j^B]_{\mathcal{G}_B},\ R^{i,B}_{(1)} = \nabla^B r_i^B,\ R^B_{(2)} = \mathfrak{r}^B - d\theta_i \otimes r_i^B \tag{201}$$

and similarly for $\tilde{R}^{ij,B}_{(0)}$, $\tilde{R}^{i,B}_{(1)}$ and $\tilde{R}^B_{(2)}$, with r_i^B replaced by $\tilde{r}_i^B$ and θ_i replaced by $\tilde{\theta}_i$. From (200) and (201) we obtain that

$$\begin{aligned}&\langle(\pi_N^* R + \tilde{\pi}_N^* \tilde{R}) \wedge \Phi\rangle_{\Pi^*\mathcal{G}_B} = -\theta_i \wedge \tilde{\theta}_j \wedge d\langle r_i^B, \tilde{r}_j^B\rangle\\ &+ \langle(2\mathfrak{r}^B - (d\theta_i) \otimes r_i^B - (d\tilde{\theta}_i) \otimes \tilde{r}_i^B) \wedge \tilde{r}_j\rangle \wedge \tilde{\theta}_j\\ &- \langle(2\mathfrak{r}^B - (d\theta_i) \otimes r_i^B - (d\tilde{\theta}_i) \otimes \tilde{r}_i^B) \wedge r_j\rangle \wedge \theta_j\\ &+ \frac{1}{2}\left(\langle[r_i^B, r_j^B], \tilde{r}_p^B\rangle \theta_i \wedge \theta_j \wedge \tilde{\theta}_p - \langle[\tilde{r}_i^B, \tilde{r}_j^B], r_p^B\rangle \tilde{\theta}_i \wedge \tilde{\theta}_j \wedge \theta_p\right)\\ &+ \frac{1}{2}\left(\langle[\tilde{r}_i^B, \tilde{r}_j^B], \tilde{r}_p^B\rangle \tilde{\theta}_i \wedge \tilde{\theta}_j \wedge \tilde{\theta}_p - \langle[r_i^B, r_j^B], r_p^B\rangle \theta_i \wedge \theta_j \wedge \theta_p\right)\\ &+ \langle\nabla^B r_i^B, r_j^B\rangle \wedge \theta_i \wedge \theta_j - \langle\nabla^B \tilde{r}_i^B, \tilde{r}_j^B\rangle \wedge \tilde{\theta}_i \wedge \tilde{\theta}_j.\end{aligned} \tag{202}$$

We write the 2-form β as

$$\beta = \beta_{(2)} + \theta_i \wedge \beta^i_{(1)} + \tilde{\theta}_i \wedge \tilde{\beta}^i_{(1)} + f_{ij}\theta_i \wedge \tilde{\theta}_j$$

where $\beta_{(2)}$, $\beta^i_{(1)}$, $\tilde{\beta}^i_{(1)}$ and f_{ij} are defined on B, so that

$$\begin{aligned}d\beta &= d\beta_{(2)} + d\theta_i \wedge \beta^i_{(1)} + d\tilde{\theta}_i \wedge \tilde{\beta}^i_{(1)}\\ &- \theta_i \wedge (d\beta^i_{(1)} + f_{ij} d\tilde{\theta}_j) + \tilde{\theta}_i \wedge (-d\tilde{\beta}^i_{(1)} + f_{ji} d\theta_j)\\ &- df_{ij} \wedge \tilde{\theta}_j \wedge \theta_i.\end{aligned} \tag{203}$$

Finally, we write, as in Section 5.1.1,

$$\begin{aligned}\pi_N^* H &= H^B_{(3)} + \theta_i \wedge H^{i,B}_{(2)} + \theta_i \wedge \theta_j \wedge H^{ij,B}_{(1)} + H^{ijs,B}_{(0)} \theta_i \wedge \theta_j \wedge \theta_s\\ \tilde{\pi}_N^* \tilde{H} &= \tilde{H}^B_{(3)} + \tilde{\theta}_i \wedge \tilde{H}^i_{(2)} + \tilde{\theta}_i \wedge \tilde{\theta}_j \wedge \tilde{H}^{ij,B}_{(1)} + \tilde{H}^{ijs,B}_{(0)} \tilde{\theta}_i \wedge \tilde{\theta}_j \wedge \tilde{\theta}_s.\end{aligned} \tag{204}$$

Using the expressions of $H^{ijs,B}_{(0)}$ and $\tilde{H}^{ijs,B}_{(0)}$, and (199), (202), (203) and (204), we obtain that relation (176) reduces to the following relations:

$$\begin{aligned}
&H^B_{(3)} - \tilde{H}^B_{(3)} - d\beta_{(2)} - (d\theta_i)^B \wedge \beta^i_{(1)} - (d\tilde{\theta}_i)^B \wedge \tilde{\beta}^i_{(1)} = 0\\
&d\beta^i_{(1)} + f_{ij}(d\tilde{\theta}_j)^B = -H^{i,B}_{(2)} + \langle r^B_i, \tilde{r}^B_j\rangle (d\tilde{\theta}_j)^B - 2\langle \mathfrak{r}^B, r^B_i\rangle + \langle r^B_i, r^B_j\rangle (d\theta_j)^B\\
&d\tilde{\beta}^i_{(1)} - f_{ji}(d\theta_j)^B = \tilde{H}^{i,B}_{(2)} - \langle \tilde{r}^B_i, \tilde{r}^B_j\rangle (d\tilde{\theta}_j)^B + 2\langle \mathfrak{r}^B, \tilde{r}^B_i\rangle - \langle \tilde{r}^B_i, r^B_j\rangle (d\theta_j)^B\\
&df_{ij} = d\langle \tilde{r}^B_j, r^B_i\rangle. \qquad (205)
\end{aligned}$$

Recall now that $H^B_{(3)} = \tilde{H}^B_{(3)}$ and

$$\begin{aligned}
(d\tilde{\theta}_i)^B &= H^{i,B}_{(2)} + 2\langle \mathfrak{r}^B, r^B_i\rangle - \langle r^B_i, r^B_j\rangle (d\theta_j)^B\\
(d\theta_i)^B &= \tilde{H}^{i,B}_{(2)} + 2\langle \mathfrak{r}^B, \tilde{r}^B_i\rangle - \langle \tilde{r}^B_i, \tilde{r}^B_j\rangle (d\tilde{\theta}_j)^B.
\end{aligned}$$

It follows that $\beta_{(2)} := 0$, $\beta^i_{(1)} := 0$, $\tilde{\beta}^i_{(1)} := 0$ and $f_{ij} := \langle r^B_i, \tilde{r}^B_j\rangle - \delta_{ij}$ satisfy relations (205), We obtain that the 2-form

$$\beta := (\langle r^B_i, \tilde{r}^B_j\rangle - \delta_{ij})\theta_i \wedge \tilde{\theta}_j$$

satisfies (176). The existence of F is proved. It is clear that it is invariant. The non-degeneracy condition (170) is satisfied, since

$$\beta(X_{\tilde{a}}, X_b) = -\langle \tilde{r}^B_a, r^B_b\rangle + \delta_{ab},\ \ (\Phi^*\Phi)(X_{\tilde{a}}, X_b) = -\langle \tilde{r}_a, r_b\rangle. \qquad \square$$

6.4 Examples of T-duality

In this section we apply Theorem 60 to various classes of transitive Courant algebroids. In particular, we recover, in our setting, the T-duality for exact Courant algebroids [7] and the T-duality for heterotic Courant algebroids [2].

6.4.1 T-duality for exact Courant algebroids

Let $E = T^*M \oplus TM$ be an exact Courant algebroid over the total space of a principal T^k-bundle $\pi : M \to B$, with Dorfmann bracket $[\cdot,\cdot]_H$ twisted by an invariant, closed, 3-form $H \in \Omega^3(M)$, that is,

$$[\xi + X, \eta + Y]_H := \mathcal{L}_X(Y + \eta) - i_Y d\xi + i_Y i_X H, \qquad (206)$$

for any $X, Y \in \mathfrak{X}(M)$, $\xi, \eta \in \Omega^1(M)$, scalar product $\langle \xi + X, \eta + Y\rangle := \frac{1}{2}(\xi(Y) + \eta(X))$ and anchor the natural projection from E to TM. The action of T^k on M lifts naturally to an action on E. Assuming that $H|_{\Lambda^2(\mathrm{Ker}\,\pi)} = 0$, we obtain a Courant algebroid of the type described in Example 35. Choose a connection $\mathcal{H}$ on π, with connection form $\theta = \sum_{i=1}^{k} \theta_i e_i$ (where (e_i) is a basis of $\mathfrak{t}^k$) and write

$$H = H_{(3)} + \sum_{i=1}^{k} \theta_i \wedge H^i_{(2)},$$

where $H_{(3)}$ and $H^i_{(2)}$ are basic. If $[H] \in H^3(M, \mathbb{R})$ is an integral cohomology class, then so is $[H^{i,B}_{(2)}] \in H^2(B, \mathbb{R})$ (for any i) and Theorem 60 can be applied. We recover the existence of a T-dual for exact Courant algebroids, which was proved in [6] (see also Proposition 2.1 of [7]).

6.4.2 Heterotic T-duality

Let G be a compact semisimple Lie group, with a fixed Ad-invariant scalar product of neutral signature $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\sigma : P \to M$ be a principal G-bundle and $\mathcal{H}$ a connection on σ. By definition, the heterotic Courant algebroid defined by the principal G-bundle $\sigma : P \to M$, connection $\mathcal{H}$ and a 3-form $H \in \Omega^3(M)$ is the standard Courant algebroid $E = T^*M \oplus \mathcal{G} \oplus TM$ with the following properties:

1. $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}}, \langle \cdot, \cdot \rangle_{\mathcal{G}})$, as a quadratic Lie algebra bundle, is given by the adjoint bundle $\mathfrak{g}_P := P \times_{\mathrm{Ad}} \mathfrak{g}$. Recall that sections $r \in \Gamma(\mathfrak{g}_P)$ are invariant vertical vector fields on P and can be identified with functions $f : P \to \mathfrak{g}$ which satisfy the equivariance condition $f(pg) = \mathrm{Ad}_{g^{-1}} f(p)$ for any $p \in P$ and $g \in G$. We shall use the notation $r \equiv f$ to denote this identification. Since the Lie bracket $[\cdot, \cdot]_{\mathfrak{g}}$ and scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ of $\mathfrak{g}$ are Ad-invariant, they induce a Lie bracket and a scalar product on $\mathfrak{g}_P$, which make $\mathfrak{g}_P$ a quadratic Lie algebra bundle. The Lie bracket of $\mathfrak{g}_P$ so defined coincides with the usual Lie bracket of invariant, vertical vector fields on P.

2. The connection ∇ which is part of the data (∇, R, H) which defines the standard Courant algebroid E is induced by $\mathcal{H}$, $R = R^{\mathcal{H}} \in \Omega^2(M, \mathfrak{g}_P)$ is the curvature of $\mathcal{H}$ and $dH = \langle R^{\mathcal{H}} \wedge R^{\mathcal{H}} \rangle_{\mathfrak{g}}$.

Assume that M is the total space of a principal T^k-bundle $\pi : M \to B$ and that $\sigma : P \to M$ is the pullback of a principal G-bundle $\sigma_0 : P_0 \to B$. Then, for any $m \in M$ and $g \in T^k$ there is a natural identification between the fibers $P_m := \sigma^{-1}(m)$, $P_{mg} := \sigma^{-1}(mg)$ and $(P_0)_{\pi(m)} := \sigma_0^{-1}(\pi(m))$, and the T^k-action on M lifts naturally to an action on P (such that g acts as the identity map between P_m and P_{mg} in the above identification). We deduce that on any heterotic Courant algebroid $E = T^*M \oplus \mathfrak{g}_P \oplus TM$ there is an induced action

$$\Psi : \mathfrak{t}^k \to \mathrm{Der}(E), \Psi(a)(\xi + r + X) := \mathcal{L}_{X_a^M}\xi + \mathcal{L}_{X_a^P} r + \mathcal{L}_{X_a^M} X \tag{207}$$

where X_a^M and X_a^P denote the fundamental vector fields of the T^k-action on M and P defined by $a \in \mathfrak{t}^k$ and $r \in \Gamma(\mathfrak{g}_P)$ is viewed as an invariant vertical vector field on P. If $r \equiv f$, then $\mathcal{L}_{X_a^P} r \equiv X_a^P(f)$.

Following [2], we shall be interested in heterotic Courant algebroids defined by σ and a particular class of connections $\mathcal{H} := \mathcal{H}^\sigma$ on σ. More precisely, we consider a connection $\mathcal{H}^\pi$ on the principal T^k-bundle $\pi : M \to B$, with connection form $\theta = \sum_{i=1}^k \theta_i e_i \in \Omega^1(M, \mathfrak{t}^k)$ (where (e_i) is a basis of $\mathfrak{t}^k$), a connection $\mathcal{H}^{\sigma_0}$ on the principal G-bundle $\sigma_0 : P_0 \to B$, with connection form $A_0 \in \Omega^1(P_0, \mathfrak{g})$ and a $G \times T^k$-equivariant function $\hat{v} : P \to (\mathfrak{t}^k)^* \otimes \mathfrak{g}$. They define a connection $\mathcal{H}^\sigma$ on σ, with connection form

$$A := \pi_0^* A_0 - \langle \sigma^*\theta, \hat{v}\rangle = \pi_0^* A_0 - \sum_{i=1}^k \sigma^*\theta_i \otimes \hat{v}_i, \tag{208}$$

where $\pi_0 : P \to P_0$ is the natural projection, $\langle \cdot, \cdot \rangle$ denotes the natural contraction between $\mathfrak{t}^k$ and $(\mathfrak{t}^k)^*$, and $\hat{v}_i = \langle \hat{v}, e_i \rangle : P \to \mathfrak{g}$. From the equivariance of $\hat{v}$, the functions $\hat{v}_i$ define sections of $\mathfrak{g}_P = \pi^*\mathfrak{g}_{P_0}$ which are pullback of sections of $\mathfrak{g}_{P_0}$, i.e. $\hat{v}_i = \pi^* \hat{v}_i^B$ for $\hat{v}_i^B \in \Gamma(\mathfrak{g}_{P_0})$ (we use the same notation for the functions $\hat{v}_i$, $\hat{v}_i^B$ and the corresponding sections of $\mathfrak{g}_P$ and $\mathfrak{g}_{P_0}$ respectively). We shall denote by $\widetilde{X}^{A_0}$, $\widetilde{Y}^{\pi_0^* A_0}$, $\widetilde{Y}^A$, the horizontal lifts of $X \in \mathfrak{X}(B)$ and $Y \in \mathfrak{X}(M)$ with respect to $\mathcal{H}^{\sigma_0}$, $\pi_0^*\mathcal{H}^{\sigma_0}$ and $\mathcal{H}^\sigma$ respectively. Here $\pi_0^*\mathcal{H}^{\sigma_0} \subset TP$ denotes the G-invariant horizontal distribution in $\sigma : P \to M$ defined by $(\pi_0^*\mathcal{H}^{\sigma_0})_p = (d_p\pi_0)^{-1}\mathcal{H}^{\sigma_0}_{\pi_0(p)}$, $p \in P$. It coincides with the kernel of the connection form $\pi_0^* A_0$.

Lemma 61. *Let $(E = T^*M \oplus \mathfrak{g}_P \oplus TM, \Psi)$ be the heterotic Courant algebroid defined by $\sigma : P \to M$, the connection $\mathcal{H}^\sigma$ with connection form (208) and a 3-form $H \in \Omega^3(M)$ such that $dH = \langle R^{\mathcal{H}^\sigma} \wedge R^{\mathcal{H}^\sigma} \rangle_{\mathfrak{g}}$, together with the $\mathfrak{t}^k$-action (207). Then the connection ∇^θ on $\mathfrak{g}_P = \pi^*(\mathfrak{g}_{P_0})$ defined in (108) is the pullback of the connection ∇^{A_0} on $\mathfrak{g}_{P_0}$ induced by $\mathcal{H}^{\sigma_0}$ (in the notation of Corollary 34, $\mathcal{G}_B = \mathfrak{g}_{P_0}$ and $\nabla^{\theta, B} = \nabla^{A_0}$).*

Proof. We claim that the horizontal lift $\widetilde{X_a^M}^{\pi_0^* A_0} \in \mathfrak{X}(P)$ of the π-vertical vector field X_a^M determined by $a \in \mathfrak{t}^k$ coincides with the fundamental vector field X_a^P of the T^k-action on P, i.e.

$$\widetilde{X_a^M}^{\pi_0^* A_0} = X_a^P, \ \forall a \in \mathfrak{t}^k. \tag{209}$$

In order to prove (209), let $U \subset B$ be open and sufficiently small such that, over U, σ_0 is the trivial G-bundle and

$$\pi_0 : P|_{\pi^{-1}(U)} = \pi^{-1}(U) \times G \to P_0|_U = U \times G,\ \pi_0(p,g) = (\pi(p), g).$$

For any $X \in \mathfrak{X}(\pi^{-1}(U))$,

$$\widetilde{X}^{\pi_0^* A_0} = X - \langle (\pi_0^* A_0)(X), f_i^* \rangle X^P_{f_i}, \tag{210}$$

where (f_i) is a basis of $\mathfrak{g}$ with dual basis (f_i^*) and for $f \in \mathfrak{g}$, X^P_f is the left invariant vector field on G determined by f (viewed as a vector field on $\pi^{-1}(U) \times G$). On the other hand, $X_a^M \in \mathfrak{X}(\pi^{-1}(U))$, viewed as a vector field on $P = \pi^{-1}(U) \times G$, satisfies $(\pi_0)_* X^M_a = 0$, since $\pi_* X^M_a = 0$ and $\pi_0 = \pi \times \mathrm{Id}$ in our trivializations. Applying (210) to $X := X^M_a$ and using $(\pi_0)_* X^M_a = 0$ we obtain $\widetilde{X^M_a}^{\pi_0^* A_0} = X^M_a$. On the other hand, the action of T^k on $P = \pi^{-1}(U) \times G$ is given by $R_g(m, \tilde{g}) = (mg, \tilde{g})$ which implies that $X^P_a = X^M_a$. Relation (209) follows.

Let ∇ be the connection on $\mathfrak{g}_P$ induced by $\mathcal{H}^\sigma$. Its covariant derivative is given by

$$\nabla_X r \equiv \widetilde{X}^A(f) = \widetilde{X}^{\pi_0^* A_0}(f) - \theta_i(X) \mathrm{ad}_{\hat{v}_i} \circ f,\ X \in \mathfrak{X}(M), \tag{211}$$

where $r \in \Gamma(\mathfrak{g}_P)$ and $r \equiv f$. Here we have used that

$$\widetilde{X}^A = \widetilde{X}^{\pi_0^* A_0} + \theta_i(X) X^P_{\hat{v}_i}$$

and the G-equivariance of f, which implies $X^P_v(f) = -\mathrm{ad}_v \circ f$ for all $v \in \mathfrak{g}$. Applying relation (211) to $X := X^M_a$ and using (209) we obtain

$$\nabla_{X^M_a} r \equiv X^P_a(f) - \langle a, e_i^* \rangle \mathrm{ad}_{\hat{v}_i} \circ f,\ \forall a \in \mathfrak{t}^k, \tag{212}$$

which implies that the skew-symmetric derivation A_a of $\mathfrak{g}_P$, from Lemma 26, is given by

$$A_a(r) = (\mathcal{L}_{X^P_a} - \nabla_{X^M_a}) r \equiv \mathrm{ad}_{\langle \hat{v}, a \rangle} \circ f,\ \forall a \in \mathfrak{t}^k. \tag{213}$$

From its definition (108) and relations (211), (213), ∇^θ is given by

$$\nabla^\theta_X r = \nabla_X r + \sum_{i=1}^k \theta_i(X) A_i(r) \equiv \widetilde{X}^{\pi_0^* A_0}(f), \tag{214}$$

which implies that $\nabla^\theta = \pi^* \nabla^{A_0}$ as needed. ∎

Since ∇^{A_0} preserves the Lie bracket and scalar product of $\mathfrak{g}_{P_0}$, its curvature takes values in the bundle of skew-symmetric derivations of $\mathfrak{g}_{P_0}$ and is of the form $\mathrm{ad}_{\mathfrak{r}^{A_0}}$ where $\mathfrak{r}^{A_0} \in \Omega^2(B, \mathfrak{g}_{P_0})$ (since $\mathfrak{g}$ is semisimple). Like in (111), we decompose $H \in \Omega^3(M)$ using the connection θ.

Proposition 62. *In the above setting, assume that*

$$
\begin{aligned}
H^{ijs,B}_{(0)} &= -\frac{1}{3}\langle [\hat{v}^B_i, \hat{v}^B_j]_{\mathfrak{g}}, \hat{v}^B_s\rangle_{\mathfrak{g}} \\
H^{ij,B}_{(1)} &:= \frac{1}{2}\left(\langle \nabla^{A_0}\hat{v}^B_i, \hat{v}^B_j\rangle_{\mathfrak{g}} - \langle \nabla^{A_0}\hat{v}^B_j, \hat{v}^B_i\rangle_{\mathfrak{g}}\right).
\end{aligned}
\tag{215}
$$

and that the (closed) forms

$$
H^{i,B}_{(2)} + 2\langle \mathfrak{r}^{A_0}, \hat{v}^B_i\rangle_{\mathfrak{g}} - \langle \hat{v}^B_i, \hat{v}^B_j\rangle_{\mathfrak{g}}(d\theta_j)^B. \tag{216}
$$

represent integral cohomology classes. Then (E, Ψ) admits a T-dual which is a heterotic Courant algebroid.

Proof. The conditions (215) mean that (E, Ψ) belongs to the class of standard Courant algebroids with $\mathfrak{t}^k$-action described in Example 35 (in the notation of that example, $r_i = \hat{v}_i$ and $r^B_i = \hat{v}^B_i$). Let $(\tilde{E}, \tilde{\Psi})$ be a T-dual of (E, Ψ), provided by Theorem 60. Then $(\tilde{E}, \tilde{\Psi})$ is defined on the total space of a principal T^k-bundle $\tilde{\pi} : \tilde{M} \to B$, with connection form $\tilde{\theta} = \sum_{i=1}^k \tilde{\theta}_i e^i$, in terms of arbitrarily chosen sections $\tilde{r}^B_i \in \Gamma(\mathfrak{g}_{P_0})$. We define the pullback bundle $\tilde{\sigma} : \tilde{P} \to \tilde{M}$ of $\sigma_0 : P_0 \to B$ by the map $\tilde{\pi}$. The arguments from Theorem 60 and the above lemma show that $\tilde{E}$ is the heterotic Courant algebroid defined by the principal G-bundle $\tilde{\sigma}$, connection $\mathcal{H}^{\tilde{\sigma}}$ with connection form

$$
\tilde{A} = \tilde{\pi}_0^* A_0 - \sum_{i=1}^k \tilde{\sigma}^* \tilde{\theta}_i \otimes \tilde{r}_i \tag{217}
$$

where $\tilde{\pi}_0 : \tilde{P} \to P_0$ is the natural projection, $\tilde{r}_i = \tilde{\pi}^*(\tilde{r}^B_i) \in \Gamma(\mathfrak{g}_{\tilde{P}})$ and 3-form $\tilde{H}$ is constructed as in Theorem 60 (in particular, $\tilde{H}^{ijs,B}_{(0)}$ and $\tilde{H}^{ij,B}_{(1)}$ are given by (215) with $\hat{v}^B_i$ replaced by $\tilde{r}^B_i$). ☐

Remark 63. Our approach from this section provides an alternative viewpoint for the heterotic T-duality developed in [2]. Heterotic Courant algebroids can be obtained from exact Courant algebroids by a reduction procedure described in [2] and the heterotic T-duality from [2] was obtained as a reduction of the T-duality for exact Courant algebroids [7]. Our approach is more direct and makes no reference to exact Courant algebroids.

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