EDGE-CONNECTIVITY AND TREE-STRUCTURE IN FINITE AND INFINITE GRAPHS

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Abstract. We show that every graph admits a canonical tree-like decomposition into its k-edge-connected pieces for all $k \in \mathbb{N} \cup \{\infty\}$ simultaneously.

1. Introduction

Finding a tree-like decomposition of any finite graph into its 'k-vertex-connected pieces', for just one given $k \in \mathbb{N}$ or all $k \in \mathbb{N}$ at once, has been a longstanding quest in graph theory until recently, when it was solved completely by Diestel, Hundertmark and Lemanczyk [16]. One of the complications was that there are many competing notions of what a 'k-vertex-connected piece' of a graph should be. Instead of providing a dozen independent solutions for the dozen different notions of 'k-vertex-connected pieces' that are in use, the ultimate solution deals with all these notions at once. Related results can be found in [2–7, 11–24, 26, 29, 31].

If we consider edge-connectivity instead of vertex-connectivity, however, there does exist a single notion of 'k-edge-connected pieces' that undeniably is the most natural one. Let $k \in \mathbb{N} \cup \{\infty\}$ and let G be any connected graph, possibly infinite. We say that two vertices u, v are *$<k$-inseparable* in G if they cannot be separated in G by fewer than k edges. This defines an equivalence relation on the vertex set of G. Its equivalence classes are the 'k-edge-connected pieces' of G, its *k-edge-blocks*. A set of vertices of G is an *edge-block* if it is a k-edge-block for some k. Note that two edge-blocks are either disjoint or one contains the other. In this paper we find a canonical tree-like decomposition of any connected graph, finite or infinite, into its k-edge-blocks—for all $k \in \mathbb{N} \cup \{\infty\}$ simultaneously. To state our result, we only need a few intuitive definitions.

An edge set $F \subseteq E(G)$ *distinguishes* two edge-blocks of G, not necessarily k-edge-blocks for the same k, if they are included in distinct components of $G - F$. An edge set F distinguishes two edge-blocks *efficiently* if it does so with least possible size. Note that if F distinguishes two edge-blocks efficiently, then F must be a *bond*, a cut with connected sides. A set B of bonds *distinguishes* some set of edge-blocks of G *efficiently* if every two disjoint edge-blocks in this set are distinguished efficiently by a bond in B. Two cuts F_1, F_2 of G are *nested* if F_1 has a side V_1 and F_2 has a side V_2 such that $V_1 \subseteq V_2$. Note that this is symmetric. The fundamental cuts of a spanning tree, for example, are (pairwise) nested. Our main result reads as follows:

Theorem 1. *Every connected graph G has a nested set of bonds that efficiently distinguishes all the edge-blocks of G.*

The nested sets $N = N(G)$ that we construct, one for every G, have two strong additional properties:

- They are canonical in that they are invariant under isomorphisms: if $\phi\colon G \to G'$ is a graph-isomorphism, then $\phi(N(G)) = N(\phi(G))$.
- For every $k \in \mathbb{N}$, the subset $N_k \subseteq N$ formed by the bonds of size less than k is equal to the set of fundamental cuts of a tree-cut decomposition of G that decomposes G into its k-edge-blocks.

Tree-cut decompositions are decompositions of graphs similar to tree-decompositions but based on edge-cuts rather than vertex-separators. They have been introduced by Wollan [32], and they are more general than the 'tree-partitions' introduced by Seese [30] and by Halin [27]; see Section 4.

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The second additional property above is best possible in the sense that N_k cannot be replaced with N: there exists a graph G (see Example 4.4) that has no nested set of cuts which, on the one hand, distinguishes all the edge-blocks of G efficiently, and on the other hand, is the set of fundamental cuts of some tree-cut decomposition. (This is because the 'tree-structure' defined by a nested set of cuts may have limit points, and hence not be representable by a graph-theoretical tree.)

This paper is organised as follows. In Section 2 we introduce the tools and terminology that we need. In Section 3 we prove our main result, Theorem 1, and we show that we obtain a canonical set N. In Section 4 we relate each N_k to a tree-cut decomposition. In Section 5 we remark a fact about ∞-edge-blocks.

2. Tools and terminology

We use the graph-theoretic notation of Diestel's book [9]. Throughout this paper, $G = (V, E)$ denotes any connected graph, finite or infinite. The following lemma is well known [9, Exercise 8.12]; we provide a proof for the reader's convenience.

Lemma 2.1. *Every edge of a graph lies in only finitely many bonds of size k of that graph, for any $k \in \mathbb{N}$.*

Proof. Let e be any edge of a graph G, and suppose for a contradiction that e lies in infinitely many distinct bonds $B_0, B_1, \ldots$ of size k, say. Let F be an inclusionwise maximal set of edges of G such that F is included in B_n for infinitely many n (all n, without loss of generality). Then $|F| < k$ because the bonds are distinct, and any bond $B_n \supsetneq F$ gives rise to a path P in $G - F$ that links the endvertices of e. Now all the infinitely many bonds B_n must contain an edge of the finite path P. But by the choice of F, each edge of P lies in only finitely many B_n, a contradiction. □

Corollary 2.2. *Let G be any connected graph, $k \in \mathbb{N}$, and let $F_0, F_1, \ldots$ be infinitely many distinct bonds of G of size at most k such that each bond F_n has a side A_n with $A_n \subsetneq A_m$ for all $n < m$. Then $\bigcup_{n \in \mathbb{N}} A_n = V$.*

Proof. If the inclusion is proper, then any A_0–$(V \setminus \bigcup_n A_n)$ path in G admits an edge that lies in infinitely many F_n, contradicting Lemma 2.1. □

2.1. **Cuts, bonds and separations.** The *order* of a cut is its size. A *cut-separation* of a graph G is a bipartition $\{A, B\}$ of the vertex set of G, and it *induces* the cut $E(A, B)$. Then the order of the cut $E(A, B)$ is also the *order* of $\{A, B\}$. Recall that in a connected graph, every cut is induced by a unique cut-separation in this way, to which it *corresponds.* A *bond-separation* of G is a cut-separation that induces a bond of G, a cut with connected sides. We say that a cut-separation *distinguishes* two edge-blocks (*efficiently*) if its corresponding cut does, and we call two cut-separations *nested* if their corresponding cuts are nested. Thus, two cut-separations $\{A, B\}$ and $\{C, D\}$ are nested if one of the four inclusions $A \subseteq C$, $A \subseteq D$, $B \subseteq C$ or $B \subseteq D$ holds.

2.2. **Key tool.** The proof of our main result relies on a result from [20]. To state it, we shall need the following definitions. Let $\mathcal{A}$ be some set and $\sim$ a reflexive and symmetric binary relation on $\mathcal{A}$. We say that two elements a and b of $\mathcal{A}$ are *nested* if $a \sim b$ and two elements of $\mathcal{A}$ which are not nested *cross.* A subset of $\mathcal{A}$ is called nested if its elements are pairwise nested. In our setting, $\mathcal{A}$ will be the set of all the bond-separations of a connected graph G that efficiently distinguish some edge-blocks of G, and $\sim$ will encode 'being nested' for bond-separations.

Given $a, b \in \mathcal{A}$, we call $c \in \mathcal{A}$ a *corner* of a and b if every element of $\mathcal{A}$ which is nested with both a and b is also nested with c. When $a = \{A, B\}$ and $b = \{C, D\}$ are two bond-separations, then c will usually be one of the following four possible corners: either $\{A \cap C, B \cup D\}$, $\{A \cap D, B \cup C\}$, $\{B \cap D, A \cup C\}$ or $\{B \cap C, A \cup D\}$. These are the four possibilities of how a new cut-separation can be built from $\{A, B\}$ and $\{C, D\}$ using just '$\cup$' and '$\cap$'. Note that sometimes an intersection may be empty so some of the four possibilities may not be valid cut-separations; and sometimes a possibility is a cut-separation but not an element of $\mathcal{A}$. We will see in Lemma 3.2 that every possibility that happens to lie in $\mathcal{A}$ is already a corner of $\{A, B\}$ and $\{C, D\}$, provided that $\{A, B\}$ and $\{C, D\}$ cross.

Consider a family $(\, \mathcal{A}_i \mid i \in I\,)$ of non-empty subsets of $\mathcal{A}$ and some function $|\cdot|\colon I \to \mathbb{N}$, where I is a possibly infinite index set. We call $|i|$ the *order* of the elements of $\mathcal{A}_i$. We will consider I to be the collection of all the unordered pairs formed by two disjoint edge-blocks of G, and each $\mathcal{A}_i$ will consist of all the bond-separations of G that efficiently distinguish the two edge-blocks forming the pair i. Then every $\mathcal{A}_i$ will be non-empty because the edge-blocks forming i are disjoint. Our choice for $|i|$ will be the unique natural number that is the order of all the bond-separations in $\mathcal{A}_i$. Note that each of the two edge-blocks forming i will be a k-edge-block for some $k > |i|$.

When we wish to prove Theorem 1 without its additional properties, then it suffices to find a subset $N \subseteq \mathcal{A}$ that meets each $\mathcal{A}_i$ and that is nested. One of the main results of [20] states that we can find N if the setup of the sets $\mathcal{A}_i$ and their order function $|\cdot|$ satisfies a number of properties. The result can be applied even when I is infinite, and moreover it ensures that N is 'canonical' for the given setup. To state the properties and the result, we need one more definition.

The *k-crossing number* of a, for an $a \in \mathcal{A}$ and $k \in \mathbb{N}$, is the number of elements of $\mathcal{A}$ that cross a and lie in some $\mathcal{A}_i$ with $|i| = k$. Note that in our case, every bond-separation of order k can only possibly lie in sets $\mathcal{A}_i$ with $|i| = k$. Thus, the k-crossing number of a bond-separation or arbitrary finite order will be the number of efficiently distinguishing bond-separations of order k crossing it.

We say that the family $(\, \mathcal{A}_i \mid i \in I\,)$ *thinly splinters* if it satisfies the following three properties:

(i) For every $i \in I$ all elements of $\mathcal{A}_i$ have finite k-crossing number for all $k \leq |i|$.

(ii) If $a_i \in \mathcal{A}_i$ and $a_j \in \mathcal{A}_j$ cross with $|i| < |j|$, then $\mathcal{A}_j$ contains some corner of a_i and a_j that is nested with a_i.

(iii) If $a_i \in \mathcal{A}_i$ and $a_j \in \mathcal{A}_j$ cross with $|i| = |j| = k \in \mathbb{N}$, then either $\mathcal{A}_i$ contains a corner of a_i and a_j with strictly lower k-crossing number than a_i, or else $\mathcal{A}_j$ contains a corner of a_i and a_j with strictly lower k-crossing number than a_j.

The following theorem from [20] will be the key ingredient for our proof of Theorem 1:

Theorem 2.3 ([20, Theorem 1.2])**.** *If $(\, \mathcal{A}_i \mid i \in I\,)$ thinly splinters with respect to some reflexive symmetric relation $\sim$ on $\mathcal{A} := \bigcup_{i \in I} \mathcal{A}_i$, then there is a set $N = N((\, \mathcal{A}_i \mid i \in I\,)) \subseteq \mathcal{A}$ which meets every $\mathcal{A}_i$ and is nested, i.e., $n_1 \sim n_2$ for all $n_1, n_2 \in N$. Moreover, this set N can be chosen invariant under isomorphisms: if ϕ is an isomorphism between $(\mathcal{A}, \sim)$ and $(\mathcal{A}', \sim')$, then we have $N((\, \phi(\mathcal{A}_i) \mid i \in I\,)) = \phi(N((\, \mathcal{A}_i \mid i \in I\,)))$.*

3. Proof of Theorem 1

Let G be any connected graph, possibly infinite, and consider the set $\mathcal{A}$ with the relation $\sim$ of 'being nested', the family $(\, \mathcal{A}_i \mid i \in I\,)$ and the function $|\cdot|$, all defined with regard to the efficiently distinguishing bond-separations of G like in Section 2.2. Our aim is to employ Theorem 2.3 to deduce Theorem 1. In order to do that, we first have to verify that $(\, \mathcal{A}_i \mid i \in I\,)$ thinly splinters. To this end, we verify all the three properties (i)–(iii) below. The following lemma clearly implies property (i):

Lemma 3.1. *Every finite-order bond-separation of a graph G is crossed by only finitely many bond-separations of G of order at most k, for any given $k \in \mathbb{N}$.*

Proof. Our proof starts with an observation. If two bond-separations $\{A, B\}$ and $\{A', B'\}$ cross, then A' contains a vertex from A and a vertex from B. Let $v \in A' \cap A$ and $w \in A' \cap B$. Since $G[A']$ is connected, there exists a path from v to w in $G[A']$. This path, and thus $G[A']$, must contain an edge from A to B. Similarly, $G[B']$ must contain an edge from A to B.

Now suppose for a contradiction that there are infinitely many bond-separations of order at most a given $k \in \mathbb{N}$, which all cross some finite-order bond-separation $\{A, B\}$. Without loss of generality, all the crossing bond-separations have order k. Using our observation, the pigeon-hole principle and the finite order of $\{A, B\}$, we find two edges $e, f \in E(A, B)$ and infinitely many bond-separations $\{A_0, B_0\}, \{A_1, B_1\}, \ldots$ that all cross $\{A, B\}$ so that $e \in G[A_n]$ and $f \in G[B_n]$ for all $n \in \mathbb{N}$. Let P be a path in G that links an endvertex v of e to an endvertex w of f. Now v is contained in all the A_n and w is contained in all the B_n, thus for every $\{A_n, B_n\}$ there exists an edge of P with one end in A_n and the other in B_n. However, every $\{A_n, B_n\}$ corresponds to a bond of size k of G and, again by the pigeon-hole principle, infinitely many of theses bonds must contain the same edge of P. This contradicts Lemma 2.1. □

Next, to show the second property, we need the following lemma:

Lemma 3.2. *If two cut-separations $\{A_1, B_1\}$ and $\{A_2, B_2\}$ cross, and a third cut-separation $\{X, Y\}$ is nested with both $\{A_1, B_1\}$ and $\{A_2, B_2\}$, then $\{X, Y\}$ is nested with $\{A_1 \cap A_2, B_1 \cup B_2\}$ (provided that this is a cut-separation).*

Proof. As $\{X, Y\}$ is a cut-separation that is nested with $\{A_1, B_1\}$ and $\{A_2, B_2\}$, either X or Y is a subset of B_1 or B_2, in which case it is immediate that $\{X, Y\}$ is nested with $\{A_1 \cap A_2, B_1 \cup B_2\}$ as desired, or, one of X and Y is a subset of A_1 and one of X and Y is a subset of A_2. However, since $A_1 \cup A_2 \neq V(G)$ (as $\{A_1, B_1\}$ and $\{A_2, B_2\}$ cross) it needs to be the case that either $X \subseteq A_1 \cap A_2$ or $Y \subseteq A_1 \cap A_2$, so in either case $\{X, Y\}$ is nested with $\{A_1 \cap A_2, B_1 \cup B_2\}$ as desired. □

Using this lemma, we can now show property (ii):

Lemma 3.3. *If $\{A, B\} \in \mathcal{A}_i$ and $\{C, D\} \in \mathcal{A}_j$ cross with $|i| < |j|$, then $\mathcal{A}_j$ contains some corner of $\{A, B\}$ and $\{C, D\}$ that is nested with $\{A, B\}$.*

Proof. Let us denote the two edge-blocks in j as U and U' so that $U \subseteq C$ and $U' \subseteq D$. Since the order of $\{A, B\}$ is less than $|j|$, we may assume without loss of generality that $U, U' \subseteq A$. We claim that either $\{A \cap C, B \cup D\}$ or $\{A \cap D, B \cup C\}$ is the desired corner in $\mathcal{A}_j$, and we refer to them as *corner candidates.* Both are cut-separations that distinguish U and U', and both are nested with $\{A, B\}$. Furthermore, by Lemma 3.2, every cut-separation that is nested with both $\{A, B\}$ and $\{C, D\}$ is also nested with both corner candidates. It remains to show that at least one of the two corner candidates has order at most $|j|$, because then it lies in $\mathcal{A}_j$ as desired.

Let us assume for a contradiction that both corner candidates have order greater than $|j|$. Then the two inequalities

$$|E(A \cap C, B \cup D)| + |E(B \cap D, A \cup C)| \leq |E(A, B)| + |E(C, D)|$$
$$|E(A \cap D, B \cup C)| + |E(B \cap C, A \cup D)| \leq |E(A, B)| + |E(C, D)|$$

imply

$$|E(B \cap D, A \cup C)| < |i| \quad \text{and} \quad |E(B \cap C, A \cup D)| < |i|.$$

Recall that the edge-blocks forming the pair i are k-edge-blocks for some values k greater than $|i|$. One of the edge-blocks of the pair i is contained in B, and due to the latter two inequalities, this edge-block must be contained entirely either in $B \cap D$ or in $B \cap C$. But then either $\{B \cap D, A \cup C\}$ or $\{B \cap C, A \cup D\}$ is a cut-separation of order less than $|i|$ that distinguishes the two edge-blocks forming the pair i, contradicting the fact that an order of at least $|i|$ is required for that. □

Finally, to show the third property, we need the following lemma:

Lemma 3.4. *Let $\{A_1, B_1\}$ and $\{A_2, B_2\}$ be crossing cut-separations such that both $\{A_1 \cap A_2, B_1 \cup B_2\}$ and $\{A_1 \cup A_2, B_1 \cap B_2\}$ are cut-separations as well. Then every cut-separation that crosses both $\{A_1 \cap A_2, B_1 \cup B_2\}$ and $\{A_1 \cup A_2, B_1 \cap B_2\}$ must also cross both $\{A_1, B_1\}$ and $\{A_2, B_2\}$.*

Proof. Consider any cut-separation $\{X, Y\}$ that crosses both $\{A_1 \cap A_2, B_1 \cup B_2\}$ and $\{A_1 \cup A_2, B_1 \cap B_2\}$. Since $\{X, Y\}$ crosses $\{A_1 \cap A_2, B_1 \cup B_2\}$, both X and Y contain a vertex from $A_1 \cap A_2$. Since $\{X, Y\}$ crosses $\{A_1 \cup A_2, B_1 \cap B_2\}$, both X and Y contain a vertex from $B_1 \cap B_2$. Hence $\{X, Y\}$ crosses both $\{A_1, B_1\}$ and $\{A_2, B_2\}$. □

Let us now show property (iii):

Lemma 3.5. *If $\{A, B\} \in \mathcal{A}_i$ and $\{C, D\} \in \mathcal{A}_j$ cross with $|i| = |j| = k \in \mathbb{N}$, then either $\mathcal{A}_i$ contains a corner of $\{A, B\}$ and $\{C, D\}$ with strictly lower k-crossing number than $\{A, B\}$, or else $\mathcal{A}_j$ contains a corner of $\{A, B\}$ and $\{C, D\}$ with strictly lower k-crossing number than $\{C, D\}$.*

Proof. Let us assume without loss of generality that the k-crossing number of $\{A, B\}$ is less than or equal to the k-crossing number of $\{C, D\}$, and let us denote the edge-blocks in j as U and U' so that $U \subseteq C$ and $U' \subseteq D$. We consider two cases.

In the first case, $\{A, B\}$ distinguishes the two edge-blocks U and U'. Hence $U \subseteq A \cap C$ and $U' \subseteq B \cap D$, say. Then both $\{A \cap C, B \cup D\}$ and $\{B \cap D, A \cup C\}$ distinguish the two edge-blocks U and U' that form the pair j, and so they have order at least $|j| = k$. Furthermore, we have

$$|E(A \cap C, B \cup D)| + |E(B \cap D, A \cup C)| \leq |E(A, B)| + |E(C, D)| = 2k, \tag{1}$$

so both $\{A \cap C, B \cup D\}$ and $\{B \cap D, A \cup C\}$ must have order exactly k. In particular, both are contained in $\mathcal{A}_j$, and they are corners of $\{A, B\}$ and $\{C, D\}$ by Lemma 3.2. Next, we assert that the k-crossing numbers of $\{A \cap C, B \cup D\}$ and $\{B \cap D, A \cup C\}$ in sum are less than the sum of the k-crossing numbers of $\{A, B\}$ and $\{C, D\}$. Indeed, all the k-crossing numbers involved are finite by property (i), and the two cut-separations $\{A, B\}$ and $\{C, D\}$ cross which allows us to deduce the desired inequality between the sums by Lemmas 3.2 and 3.4, as follows:

- by Lemma 3.2, every $\{X, Y\} \in \mathcal{A}$ of order k that crosses at least one of $\{A \cap C, B \cup D\}$ and $\{B \cap D, A \cup C\}$ must cross at least one of $\{A, B\}$ and $\{C, D\}$; and
- by Lemma 3.4, every $\{X, Y\} \in \mathcal{A}$ of order k that crosses both $\{A \cap C, B \cup D\}$ and $\{B \cap D, A \cup C\}$ must cross both $\{A, B\}$ and $\{C, D\}$.

But then the strict inequality between the sums, plus our initial assumption that the k-crossing number of $\{A, B\}$ is less than or equal to that of $\{C, D\}$, implies that one of $\{A \cap C, B \cup D\}$ and $\{B \cap D, A \cup C\}$ must have a k-crossing number less than the one of $\{C, D\}$, as desired.

In the second case, $\{A, B\}$ does not distinguish the two edge-blocks U and U'. Recall that all the edge-blocks in the two pairs i and j are ℓ-edge-blocks for some values $\ell > k$. Hence $U \cup U' \subseteq A$, say. Let us denote by U'' the edge-block in i that is contained in B. Then either $U'' \subseteq B \cap C$ or $U'' \subseteq B \cap D$, say $U'' \subseteq B \cap D$. In total:

$$U \subseteq A \cap C, \;\; U' \subseteq A \cap D \text{ and } U'' \subseteq B \cap D.$$

Therefore, $\{A \cap C, B \cup D\}$ distinguishes the two edge-blocks U and U' forming the pair j which imposes an order of at least k, and $\{B \cap D, A \cup C\}$ distinguishes the two edge-blocks forming the pair i which imposes an order of at least k as well. Combining these lower bounds with (1) we deduce that both $\{A \cap C, B \cup D\}$ and $\{B \cap D, A \cup C\}$ have order exactly k. In particular, they are contained in $\mathcal{A}_j$ and $\mathcal{A}_i$ respectively, and they are corners of $\{A, B\}$ and $\{C, D\}$ by Lemma 3.2. Repeating the final argument of the first case, we deduce from the strict inequality between the sums of the k-crossing numbers that either $\{A \cap C, B \cup D\} \in \mathcal{A}_j$ has strictly lower k-crossing number than $\{C, D\}$, or else $\{B \cap D, A \cup C\} \in \mathcal{A}_i$ has strictly lower k-crossing number than $\{A, B\}$, completing the proof. □

We can now prove our main result:

Proof of Theorem 1. Let G be any connected graph. By Lemma 3.1, Lemma 3.3 and Lemma 3.5 we may apply Theorem 2.3 to the family $(\, \mathcal{A}_i \mid i \in I \,)$ defined at the beginning of the section. This results in the desired nested set $N(G) \subseteq \mathcal{A}$. To see that it is canonical, note that any isomorphism $\phi\colon G \to G'$ induces an isomorphism between $(\mathcal{A}, \sim)$ and $(\mathcal{A}', \sim')$, where the latter is defined like the former but with regard to G'. Thus, by the 'moreover' part of Theorem 2.3, we indeed obtain that $\phi(N(G)) = N(\phi(G))$. □

4. Nested sets of bonds and tree-cut decompositions

Recall that, given a connected graph G, we denote by $N = N(G)$ the canonical set of nested bonds from Theorem 1 that efficiently distinguishes all the edge-blocks of G. Furthermore, recall that the subset $N_k \subseteq N$ is formed by the bonds in N of order less than k. In this section, we show that:

- For every $k \in \mathbb{N}$, the subset $N_k \subseteq N$ is equal to the set of fundamental cuts of a tree-cut decomposition of G that decomposes G into its k-edge-blocks.

To this end, we first introduce the notion of a tree-cut decomposition. Recall that a *near-partition* of a set M is a family of pairwise disjoint subsets $M_\xi \subseteq M$, possibly empty, such that $\bigcup_\xi M_\xi = M$.

Let G be a graph, T a tree, and let $\mathcal{X} = (X_t)_{t \in T}$ be a family of vertex sets $X_t \subseteq V(G)$ indexed by the nodes t of T. The pair $(T, \mathcal{X})$ is called a *tree-cut decomposition* of G if $\mathcal{X}$ is a near-partition of $V(G)$. The vertex sets X_t are the *parts* or *bags* of the tree-cut decomposition $(T, \mathcal{X})$ and we say that $(T, \mathcal{X})$ *decomposes* the graph G into its non-empty parts. In this paper, we require the nodes with non-empty parts to be *dense* in T in that every edge of T lies on a path in T that links up two nodes with non-empty parts.

If $(T, \mathcal{X})$ is a tree-cut decomposition, then every edge t_1t_2 of its *decomposition tree* T induces a cut $E(\bigcup_{t \in T_1} X_t \,, \bigcup_{t \in T_2} X_t\,)$ of G where T_1 and T_2 are the two components of $T - t_1t_2$ with $t_1 \in T_1$ and $t_2 \in T_2$. Here, the nodes with non-empty parts densely lying in T ensures that both unions are non-empty, which is required of the sides of a cut. We call these induced cuts the *fundamental cuts* of the tree-cut decomposition $(T, \mathcal{X})$. Note that unlike the fundamental cuts of a spanning tree, the fundamental cuts of a tree-cut decomposition need not be bonds.

It is important that parts of a tree-cut decomposition are allowed to be empty, as the following example demonstrates.

Example 4.1. Let the graph G arise from the disjoint union of three copies G_1, G_2 and G_3 of K^4 by selecting one vertex $v_i \in G_i$ for all $i \in [3]$ and adding all edges $v_i v_j$ ($i \neq j \in [3]$). Then the 3-edge-blocks of G are the three vertex sets $V(G_1)$, $V(G_2)$ and $V(G_3)$. Since $N(G)$ is canonical, we have $N_3(G) = \{ F_1, F_2, F_3 \}$ where $F_i := \{ v_i v_j \mid j \neq i \}$. However, we cannot find a tree-cut decomposition $(T, \mathcal{X})$ of G such that, on the one hand, T is a tree on three nodes t_1, t_2, t_3 and $X_{t_i} = V(G_i)$ for all $i \in [3]$, and on the other hand, the fundamental cuts of $(T, \mathcal{X})$ are precisely the bonds in $N_3(G)$: the decomposition tree T would then be a path of length two, and hence would induce two fundamental cuts, but $N_3(G)$ consists of three bonds.

To relate N_k to a tree-cut decomposition, we will use a theorem by Gollin and Kneip. In order to state their theorem, we need to introduce separation systems and S-trees first.

4.1. **Separation systems and S-trees.** Separation systems and S-trees are two fundamental tools in graph minor theory. In this section we briefly introduce the definitions from [8–10] that we need.

A *separation of a set* V is an unordered pair $\{A, B\}$ such that $A \cup B = V$. The ordered pairs (A, B) and (B, A) are its *orientations*. Then the *oriented separations* of V are the orientations of its separations. The map that sends every oriented separation (A, B) to its *inverse* (B, A) is an involution that reverses the partial ordering

$$(A, B) \leq (C, D) \;:\Leftrightarrow\; A \subseteq C \text{ and } B \supseteq D$$

since $(A, B) \leq (C, D)$ is equivalent to $(D, C) \leq (B, A)$.

More generally, a *separation system* is a triple $(\vec{S}, \leq, {}^*)$ where $(\vec{S}, \leq)$ is a partially ordered set and ${}^*\colon \vec{S} \to \vec{S}$ is an order-reversing involution. We refer to the elements of $\vec{S}$ as *oriented separations*. If an oriented separation is denoted by $\vec{s}$, then we denote its *inverse* $\vec{s}^*$ as $\overleftarrow{s}$, and vice versa. That * is *order-reversing* means $\vec{r} \leq \vec{s} \Leftrightarrow \overleftarrow{r} \geq \overleftarrow{s}$ for all $\vec{r}, \vec{s} \in \vec{S}$.

A *separation* is an unordered pair of the form $\{\vec{s}, \overleftarrow{s}\}$, and then denoted by s. Its elements $\vec{s}$ and $\overleftarrow{s}$ are the *orientations* of s. The set of all separations $\{\vec{s}, \overleftarrow{s}\} \subseteq \vec{S}$ is denoted by S. When a separation is introduced as s without specifying its elements first, we use $\vec{s}$ and $\overleftarrow{s}$ (arbitrarily) to refer to these elements.

Separations of sets, and their orientations, are an instance of this abstract setup if we identify $\{A, B\}$ with $\{ (A, B), (B, A) \}$. Hence the cut-separations of a graph define a separation system. Here is another example: The set $\vec{E}(T) := \{ (x, y) \mid xy \in E(T) \}$ of all *orientations* (x, y) of the edges $xy = \{x, y\}$ of a tree T forms a separation system with the involution $(x, y) \mapsto (y, x)$ and the natural partial ordering on $\vec{E}(T)$ in which $(x, y) < (u, v)$ if and only if $xy \neq uv$ and the unique $\{x, y\}$–$\{u, v\}$ path in T is $\mathring{x}yTu\mathring{v} = yTu$.

An *S-tree* is a pair (T, α) such that T is a tree and $\alpha\colon \vec{E}(T) \to \vec{S}$ propagates the ordering on $\vec{E}(T)$ and commutes with inversion: that $\alpha(\vec{e}) \leq \alpha(\vec{f})$ if $\vec{e} \leq \vec{f} \in \vec{E}(T)$ and $(\alpha(\overleftarrow{e}))^* = \alpha(\vec{e})$ for all $\vec{e} \in \vec{E}(T)$; see Figure 1. A tree-decomposition $(T, \mathcal{V})$, for example, makes T into an S-tree for the set of separations it induces [9, §12.5]. Similarly, a tree-cut decomposition $(T, \mathcal{X})$ makes T into an S-tree for the set of cut-separations which correspond to its fundamental cuts. For oriented edges $(x, y) \in \vec{E}(T)$ we will write $\alpha(x, y)$ instead of $\alpha((x, y))$.

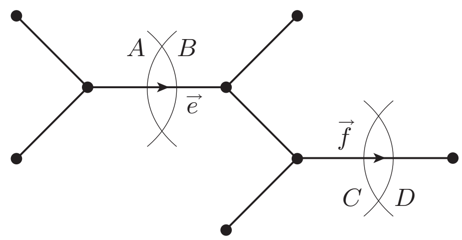

FIGURE 1. An S-tree with $\alpha(\vec{e}) = (A, B) \le (C, D) = \alpha(\vec{f})$ [9]

An *isomorphism* between two separation systems is a bijection between their underlying sets that respects both their partial orderings and their involutions. We need the following fragment of [25, Theorem 1] by Gollin and Kneip:

Theorem 4.2. *Let G be any connected graph, and let $\vec{S}$ be any nested separation system formed by oriented cut-separations of G. Then the following assertions are equivalent:*

(i) *There exists an S-tree (T, α) such that $\alpha\colon \vec{E}(T) \to \vec{S}$ is an isomorphism between separation systems;*
(ii) *$\vec{S}$ contains no chain of order-type $\omega + 1$.*

4.2. **N_k is a set of fundamental cuts.** The following theorem clearly implies that N_k is the set of fundamental cuts of a tree-cut decomposition of G that decomposes G into its k-edge-blocks:

Theorem 4.3. *Let G be any connected graph and $k \in \mathbb{N}$. Every nested set of bonds of G of order less than k is the set of fundamental cuts of some tree-cut decomposition of G.*

Proof. Let G be any connected graph, $k \in \mathbb{N}$, and let B be any nested set of bonds of G of order less than k. We write S for the set of bond-separations which correspond to the bonds in B.

First, we wish to use Theorem 4.2 to find an S-tree (T, α) such that $\alpha\colon \vec{E}(T) \to \vec{S}$ is an isomorphism. For this, it suffices to show that B cannot contain pairwise distinct bonds $F_0, F_1, \dots, F_\omega$ such that each bond F_α has a side A_α with $A_\alpha \subsetneq A_\beta$ for all $\alpha < \beta \le \omega$. This is immediate from Corollary 2.2.

Second, we wish to find a tree-cut decomposition $(T, \mathcal{X})$ whose fundamental cuts are precisely equal to the bonds in B. We define the parts X_t of $(T, \mathcal{X})$ by letting

$$X_t := \bigcap \{\, D \mid (C, D) = \alpha(x, t) \text{ where } xt \in E(T) \,\}.$$

Then clearly the parts X_t are pairwise disjoint. To see that $\bigcup_t X_t$ includes the whole vertex set of G, consider any vertex $v \in V(G)$. We orient each edge $t_1 t_2 \in T$ towards the t_i with $v \in D$ for $(C, D) = \alpha(t_{3-i}, t_i)$. By Corollary 2.2 we may let t be the last node of a maximal directed path in T; then all the edges of T at t are oriented towards t, and $v \in X_t$ follows. Therefore, $\mathcal{X}$ is a near-partition of $V(G)$. It is straightforward to see that B is the set of fundamental cuts of $(T, \mathcal{X})$. □

4.3. **N is not a set of fundamental cuts.** Finally, we show that there exists a graph G that has no nested set of cuts which, on the one hand, distinguishes all the edge-blocks of G efficiently, and on the other hand, is the set of fundamental cuts of some tree-cut decomposition.

Example 4.4. This example is a variation of [20, Example 4.9]. Consider the locally finite graph displayed in Figure 2. This graph G is constructed as follows. For every $n \in \mathbb{N}_{\ge 1}$ we pick a copy of $K^{2^{n+2}}$ together with $n + 2$ additional vertices $w_1^n, \dots, w_{n+2}^n$. Then we select 2^n vertices of the $K^{2^{n+2}}$ and call them

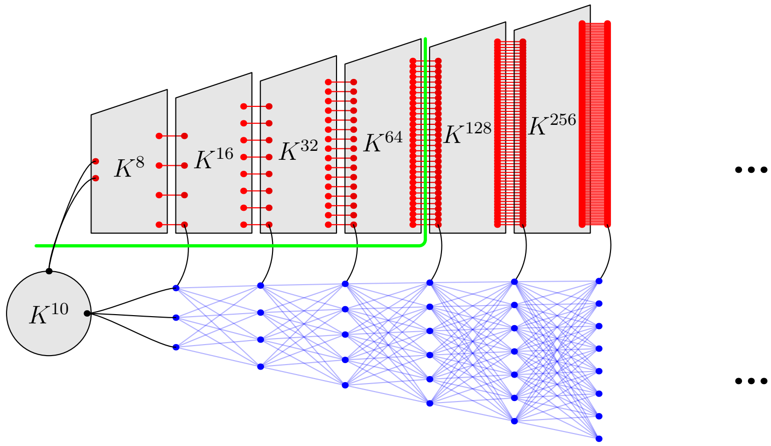

FIGURE 2. The only cut that efficiently distinguishes the two edge-blocks defined by K^{64} and by K^{128} is drawn in green.

$u_1^n, \ldots, u_{2^n}^n$. Furthermore, we select 2^{n+1} vertices of the $K^{2^{n+2}}$, other than the previously chosen u_i^n, and call them $v_1^n, \ldots, v_{2^{n+1}}^n$. Now we add all the red edges $v_i^n u_i^{n+1}$, all the blue edges $w_i^n w_j^{n+1}$, and if $n \geq 2$ we also add the black edge $u_1^n w_1^n$. Finally, we disjointly add one copy of K^{10} and join one vertex v_1^0 of this K^{10} to u_1^1 and u_2^1; and we select another vertex $w_1^0 \in K^{10}$ distinct from v_1^0 and add all edges $w_1^0 w_i^1$. This completes the construction.

Now the vertex sets of the chosen $K^{2^{n+2}}$ are $(2^{n+2} - 1)$-edge-blocks B_n. The only cut-separation that efficiently distinguishes B_n and B_{n+1} is $F_n := \{\bigcup_{k=1}^n B_n\,,\, V \smallsetminus \bigcup_{k=1}^n B_n\}$. Additionally, the vertex set of the K^{10} is a 9-edge-block B_0. The only cut-separation that efficiently distinguishes B_0 and B_1 is $F_0 := \{B_0, V \smallsetminus B_0\}$. Therefore, $N(G)$ must contain all the cuts corresponding to the cut-separations F_n $(n \in \mathbb{N})$. But the cut-separations F_n define an $(\omega + 1)$-chain

$$(B_1, V \smallsetminus B_1) < (B_1 \cup B_2, V \smallsetminus (B_1 \cup B_2)) < \cdots < (V \smallsetminus B_0, B_0),$$

so $N(G)$ cannot be equal to the set of fundamental cuts of a tree cut-decomposition of G by Theorem 4.2.

5. A remark on ∞-edge-blocks

By the second property of our nested set $N(G)$, we find a tree-cut decomposition of any connected graph G into its k-edge-blocks, one for every $k \in \mathbb{N}$. But for $k = \infty$, such a decomposition does not in general exist, e.g., consider Example 4.4 with each K^n of the graph replaced by $K^{\aleph_0}$ (or any other infinitely edge-connected graph). The reason why, however, is not that there are no meaningful tree-cut decompositions of G into its ∞-edge-blocks, but that we considered only those decompositions whose sets of fundamental cuts are equal to $N(G)$. If we drop this requirement, then we find tree-cut decompositions of G into its ∞-edge-blocks, meaningful in the sense that all their fundamental cuts are finite. Let us call a graph *finitely separable* if any two of its vertices can be separated by finitely many edges. And let us call a spanning tree, respectively a tree-cut decomposition, *finitely separating* if all its fundamental cuts are finite. The following theorem has been introduced in [1] as Theorem 3.9, and it is Theorem 5.1 in [28]:

Theorem 5.1 ([1])**.** *Every finitely separable connected graph has a finitely separating spanning tree.*

If G is any connected graph, then the graph $\tilde{G}$ obtained from G by collapsing every ∞-edge-block to a single vertex is finitely separable and connected. Hence $\tilde{G}$ has a finitely separating spanning tree by the theorem, and this tree is easily translated to a finitely separating tree-cut decomposition of G, even with all parts non-empty:

Theorem 5.2 ([28]). *Every connected graph has a finitely separating tree-cut decomposition into its ∞-edge-blocks.* □

This result, phrased in terms of S-trees, is extensively used in [28] to study infinite edge-connectivity.

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