DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 85-043
May 1985

SPIN AND STATISTICS OF QUANTUM TOPOLOGICAL CHARGES

by

D. Buchholz

II. Institut f. Theoretische Physik, Universität Hamburg

H. Epstein

Inst. Hautes Etudes Scient., Bures-sur-Yvette, France

ISSN 0418-9833

NOTKESTRASSE 85 · 2 HAMBURG 52

DESY 85-043
May 1985

ISSN 0418-9833

SPIN AND STATISTICS OF QUANTUM TOPOLOGICAL CHARGES

D. Buchholz[1] and H. Epstein[2]

1) II. Institut für Theoretische Physik, Universität Hamburg, D 2000 Hamburg 50

2) IHES, F 91440 Bures-sur Yvette, France

Abstract: It is shown that the familiar connection between spin and statistics holds for massive particles carrying a non-localizable charge.

In Memoriam Prof. Vladimir Jurko Glaser

1. Introduction

Since its first formulation more than forty years ago, the spin-statistics theorem [1] has undergone a number of refinements which have largely clarified its physical foundations, cf. [2] and the references quoted there. A conceptually very satisfactory derivation has been given in [3]. There it was shown that particles carrying a localizable charge, such as baryon number or strangeness, can only be (para-) bosons or fermions. Other statistics are excluded by the principle of locality of observables. Moreover, it was found that there exist local field operators (with normal space-like commutation relations [4]) which generate the particle states from the vacuum. Using the methods developed in [5] it was then possible to establish the familiar relation

$$(-1)^{2s} = \text{sign } \lambda \tag{1.1}$$

between the spin s and the statistics parameter λ of these particles.

It is a remarkable fact that these results can be derived from first principles without any a priori information on non-observable quantities, such as charge-carrying fields. The assumption that the particles carry a localizable charge is, however, crucial. This requirement restricts the range of application of these results to the cases which were already covered by the classical arguments.

It is the aim of the present note to extend the spin-statistics theorem to particles carrying a non-localizable charge (e.g. a global gauge charge or a topological charge). In order to avoid unsettled infrared problems[1)] we will restrict our attention to theories with a completely massive particle spectrum. It is then reasonable to assume that the single particle states are described by vectors in the subspace of some irreducible representation of the Poincaré group or its covering group, respectively, and that the mass hyperboloid of each particle is separated by a mass gap from the rest of the energy-momentum spectrum in its super-selection sector.

Using the latter assumption and the spacelike commutativity of observables it was shown in [7] that single particle states are local or string-like excitations of a vacuum state. A string-like localization is exactly what one expects from a particle carrying a gauge charge or a topological charge; in view of the results in [7] this is, on the other hand, the worst possible dislocalization. We mention as an aside that charged states with such localization properties have recently been constructed in a Z_2 lattice gauge theory with matter fields [8].

The string-like localization of particles established in [7] still allows one to determine the statistics of these particles and

1) Note that the fundamental problem of defining the spin and statistics of infra particles is still open, cf. [6] and the references quoted there.

to construct collision states with the appropriate symmetry properties. As in the case of localizable charges only (para-) Bose- or Fermi statistics can occur [7]; the so called infinite statistics, which was left open as a possibility in [3], has been excluded in [9]. So both the left hand and the right hand side of relation (1.1) are intrinsically defined for massive particles carrying a non-localizable charge. Hence what remains to be done is to establish the equality sign in this relation.

This task is facilitated by recent developments in [4] which make possible a translation of the results in [7] into a field-theoretic setting. For later reference we state the properties which are relevant to our analysis.

States: The physical states of interest are represented by vectors in some Hilbert space H. On H there exists a continuous, unitary representation

$$(a,\Lambda) \rightarrow U(a,\Lambda) \qquad (1.2)$$

of the Poincaré group $P_+^\uparrow$, or its covering group $\overline{P_+^\uparrow}$, and the translations $U(a):=U(a,1)$ fulfil the relativistic spectrum condition (positivity of the energy). The (unique) vacuum state is represented by a vector Ω which is invariant under Poincaré transformations $U(a,\Lambda)$, and to each particle type ι there corresponds a subspace $K_\iota \subset H$ on which the unitaries $U(a,\Lambda)$ act as an irreducible representation of

$\overline{P}_{+}^{\uparrow}$ with mass m_ι and spin s_ι.

Fields: For each particle of type ι there exists a family of linear spaces $F_\iota(S)$ of (bounded) operators on H which are labeled by the spacelike cones[2)] $S \subset \mathbb{R}^4$. The operators $\psi \in F_\iota(S)$, called fields, have the following commutation and covariance properties

i) If S_1 and S_2 are spacelike separated and if $\psi_1 \in F_\iota(S_1)$, $\psi_2 \in F_\iota(S_2)$, then

$$\psi_2^* \cdot \psi_1 = \text{sign}\, \lambda_\iota \; \psi_1 \cdot \psi_2^* , \tag{1.3}$$

where λ_ι is the statistics parameter of the particle in question. Thus sign $\lambda_\iota = 1$ if the particle is a (para-) boson and sign $\lambda_\iota = -1$ if it is a (para) fermion.

ii) If $\psi \in F_\iota(S)$, then

$$U(a,\Lambda)\, \psi\, U(a,\Lambda)^{-1} \in F_\iota(a+\Lambda\cdot S), \tag{1.4}$$

where $a+\Lambda\cdot S$ is the Poincaré-transformed region S. We may assume (if necessary, after a regularization) that the operator-valued func-

---

2) A spacelike cone $S \subset \mathbb{R}^4$ may be visualized as a string which fattens. It can be represented in the form
$$S = a + \bigcup_{\lambda>0} \lambda \cdot \mathcal{O},$$
where $a \in \mathbb{R}^4$ and $\mathcal{O} \subset \mathbb{R}^4$ is some open double cone whose closure lies in the causal complement of the origin.
Notation: if S, S_1, S_2 are subsets of $\mathbb{R}^4$ and $\lambda \in \mathbb{R}$ we set $\lambda \cdot S = \{\lambda \cdot s : s \in S\}$ and $S_1 \pm S_2 = \{s_1 \pm s_2 : s_1 \in S_1, s_2 \in S_2\}$.

tions $(a,\Lambda) \to U(a,\Lambda)\,\psi\, U(a,\Lambda)^{-1}$ are smooth, i.e. C^∞, in the norm topology.

iii) Let $E(\Delta)$, $\Delta \subset \mathbb{R}^4$ be the spectral projections of the mass operator on H. Then we have for each spacelike cone S

$$E(\{m_\iota\})F_\iota(S)\,\Omega \subset K_\iota , \tag{1.5}$$

and there exist non-zero vectors in $E(\{m_\iota\})F_\iota(S)\,\Omega$. Moreover,

$$E(\Delta\setminus\{m_\iota\})\,\psi\,\Omega = 0 \tag{1.6}$$

for some open neighbourhood Δ of m_ι and all $\psi \in F_\iota(S)$.

iv) The vacuum has the Reeh-Schlieder property in the following sense: if S_1, S_2 are arbitrary spacelike cones, then

$$\overline{F_\iota(S_1)\,\Omega} = \overline{F_\iota(S_2)\,\Omega}, \tag{1.7}$$

i.e. the fields in a fixed cone generate all states of a given charge. Similarly, one has for the "charge-conjugate sector"

$$\overline{F_\iota(S_1)^*\Omega} = \overline{F_\iota(S_2)^*\,\Omega}, \tag{1.8}$$

where $F_\iota(S)^* = \{\psi^* : \psi \in F_\iota(S)\}$.

We recall that the existence of the fields ψ follows from the fundamental principle of locality of observables and the assumption that the mass hyperboloid of the states in K_ι is isolated from the energy-momentum spectrum of all other states in the same superselection sector [4,7]. This result may be regarded as the solution of the "one-particle problem" of collision theory [10].

As we shall see, the locality and covariance properties of the fields ψ provide enough "analyticity in momentum space" so as to establish the connection (1.1) between the spin and statistics of particles carrying a non-localizable charge. Thus, as far as the kinematical properties are concerned, there is nothing exotic about these particles.

This raises the question of how these particles can be distinguished from their localizable counterparts in collision processes. What immediately comes to mind is that the analyticity properties of the corresponding S-matrix should differ from the one established for localizable particles [11]. At first sight this possibility does not look very attractive, because it deviates from the view that the analyticity of the S-matrix agrees with the indications of perturbation theory. But on the other hand it is conceivable that a less stringent analytic structure of the S-matrix might allow one to avoid some no-go theorems, without coming into conflict with the principle of locality of observables. (For speculations in this direction see the conclusions in [7]). It would therefore be desirable to gather more information about the structure of the S-matrix of particles carrying a non-localizable charge.

2. The Two Point Function in Momentum Space

We begin with an investigation of the two point functions of the fields ψ. It is our aim to establish certain specific analyticity properties of these functions in momentum space.

In order to simplify the subsequent geometrical discussions we fix a Lorentz frame and consider a special set of spacelike cones. Namely, let $\underline{S} \subset \mathbb{R}^3$ be any open, convex, salient cone in the time $t = 0$ plane of Minkowski space $\mathbb{R}^4$ and let $S = \underline{S}''$ be its causal completion. Then S is a spacelike cone (cf. footnote 2) with apex at the origin of $\mathbb{R}^4$. Since we will only consider spacelike cones S which are obtained in this way we can identify them with their "base" $\underline{S}$. We will also deal with the dual cone $\tilde{S}$ of $\underline{S}$, i.e. the set[3)]

$$\tilde{S} = \{q \in \mathbb{R}^4 : \underline{q}.\underline{x} > 0 \text{ for all } \underline{x} \in \bar{\underline{S}} \setminus \{0\}\} . \tag{2.1}$$

Note that the dual cone of a salient cone is open and non-empty.

Now let $\underline{S}_1, \underline{S}_2$ be cones such that $\underline{S}_{1,2} := \underline{S}_2 - \underline{S}_1$ is salient, and let $\psi_i \in \mathcal{F}(S_i)$ $i = 1,2$ be fields satisfying the conditions (1.5) and (1.6). (We omit the particle index ι in the following). Then we consider the functions on $\mathbb{R}^4$

---

3) For $a,b \in \mathbb{R}^4$ we set $a \cdot b = a_0 b_0 - \underline{a} \cdot \underline{b}$ and $a^2 = a \cdot a$.

$$w^{+}_{1,2}(x) = (\Omega, \psi^{*}_{2}\, U(x)\, \psi_{1}\, \Omega)$$
$$w^{-}_{1,2}(x) = \text{sign } \lambda \cdot (\Omega,\ \psi_{1}\, U(-x)\ \psi^{*}_{2}\ \Omega)\ , \tag{2.2}$$

and the commutator function

$$C_{1,2}(x) = w^{+}_{1,2}(x) - w^{-}_{1,2}(x)\ . \tag{2.3}$$

The Fourier transforms of $w^{\pm}_{1,2}$ are measures which (because of the spectrum condition) have support in the forward and backward lightcones $\bar{V}_{+}$ and $\bar{V}_{-}$, respectively. Bearing in mind the space-like commutation relations (1.3) of the fields and the covariance properties (1.4) it is clear that $C_{1,2}(x)$ vanishes whenever $x \in (\mathcal{S}_2 - \mathcal{S}_1)' = \underline{\mathcal{S}}'_{1,2}$. Thus $C_{1,2}$ has support in the region $\underline{\mathcal{S}}_{1,2} + \bar{V}_{+} \cup \bar{V}_{-}$. Next we decompose $C_{1,2}$ into an advanced and retarded part, setting

$$a_{1,2}(x) = \Theta(-x_0) \cdot C_{1,2}(x)$$
$$r_{1,2}(x) = -\,\Theta(x_0) \cdot C_{1,2}(x)\ . \tag{2.4}$$

Since supp $a_{1,2} \subset \overline{\underline{\mathcal{S}}_{1,2} + V_{-}}$ and supp $r_{1,2} \subset \overline{\underline{\mathcal{S}}_{1,2} + V_{+}}$ it follows that the Fourier transforms $\tilde{a}_{1,2}(p)$ and $\tilde{r}_{1,2}(p)$ are boundary values (in the sense of distributions) of two functions which are analytic in[4] $\{k = p+iq\colon q \in V_{+} \cap \tilde{\mathcal{S}}_{1,2}\}$ and $\{k = p+iq\colon q \in V_{-} \cap \tilde{\mathcal{S}}_{1,2}\}$,

4) We reserve the letters p,q for elements of $\mathbb{R}^4$ and the letter k for elements of $\mathbb{C}^4$.

respectively. The same is true for the amputated functions $\frac{1}{2\pi i}(p^2 - m^2)\tilde{a}_{1,2}(p)$ and $\frac{1}{2\pi i}(p^2 - m^2)\tilde{r}_{1,2}(p)$. Now since $\tilde{a}_{1,2}(p) - \tilde{r}_{1,2}(p) = \tilde{C}_{1,2}(p)$, these amputated functions coincide in the region

$$\{p : p^2 < 0\} \cup \{p : \sqrt{p^2} \in \Delta,\ p_0 > 0\} \tag{2.5}$$

because of the spectrum condition and the fact that the mass hyperboloid of the particle is isolated, cf. relations (1.5) and (1.6). So the corresponding analytic functions are actually branches of a single function $h_{1,2}$ which is analytic in the domain

$$\{k = p + iq\colon q \in \tilde{\mathcal{S}}_{1,2},\ k^2 \notin [\mathbb{R}^{+} \setminus \Delta^2]\}, \tag{2.6}$$

where $\Delta^2 := \{\mu^2 : \mu \in \Delta\}$. This can be proved either by using the Jost-Lehmann-Dyson representation [12] or by a direct geometric method (cf. the remarks below).

Taking into account that the operator-valued functions $(x, \Lambda) \to U(x,\Lambda)\, \psi\, U(x,\Lambda)^{-1}$ are smooth, it can also be shown that $h_{1,2}(p+iq)$ has a C^{∞} boundary value as $q \to 0$. Therefore, if $\sqrt{p^2} \in \Delta$, one obtains for the Fourier transform of the commutator function the representation

$$\tilde{C}_{1,2}(p) = h_{1,2}(p)\ \delta(p^2 - m^2)\ , \tag{2.7}$$

and consequently (because of the spectrum condition)

$$(\psi_2\Omega, U(x)E(\{m\})\psi_1\Omega) = \int d^4p\,\theta(p_o)\delta(p^2-m^2)h_{1,2}(p)e^{ipx} \tag{2.8}$$

$$(\psi_1^*\Omega, U(x)E(\{m\})\psi_2^*\Omega) = \mathrm{sign}\,\lambda\cdot\int d^4p\,\theta(p_o)\delta(p^2-m^2)h_{1,2}(-p)e^{ipx}.$$

We note that the existence of anti particles can be derived from this result [7,9]: namely, if the first function in (2.8) is different from 0, the same is true for the second one because of the analyticity properties of $h_{1,2}$. Hence if the vector $\psi\Omega$ has a nonvanishing component in the single particle space K, then the charge-conjugate vector $\psi^*\Omega$ has a nonvanishing component in some single particle space K_c with the same mass as K. We will see below that the states in K_c have also the same spin as the states in K.

We conclude this discussion of the two point functions with a description of the domain of analyticity of $h_{1,2}$ on the complex mass hyperboloid $k^2 = m^2$. According to (2.6) this domain is given by

$$\Gamma_{1,2} = \{k = p+iq:\ k^2 = m^2,\ q \in \tilde{S}_{1,2}\}. \tag{2.9}$$

We shall prove that $\Gamma_{1,2}$ is an open simply connected subset of the complex hyperboloid.

Choosing proper coordinates, we may assume that the vectors $q = (q^o, q^1, 0, 0)$, $q^1 > 0$ are elements of $\tilde{S}_{1,2}$. Then we consider the Lorentz transformations Λ_θ, $\theta \in \mathbb{R}$ acting on $k = (k^o, k^1, k^2, k^3)$ according to

$$\Lambda_\theta k = (\cos\theta\cdot k^o + i\sin\theta\cdot k^1,\ \cos\theta\cdot k^1 + i\sin\theta\cdot k^o, k^2, k^3). \tag{2.10}$$

If $p \in H_+$, where

$$H_\pm = \{p :\ p^2 = m^2, \pm p_o > 0\} \tag{2.11}$$

are the positive and negative shells of the mass hyperboloid, respectively, it follows immediately from (2.10) that $\Lambda_\theta\cdot p \in \Gamma_{1,2}$ if $0 < \theta < \pi$; moreover, $\Lambda_\pi\cdot p \in H_-$. Conversely, if $k' = p' + iq' \in \Gamma_{1,2}$ and $q' = (q'^o, q'^1, 0, 0)$, $q'^1 > 0$ one finds by a straightforward calculation a unique $p \in H_+$ and θ, $0 < \theta < \pi$ such that $\Lambda_\theta\cdot p = k'$.

Changing the direction of $q = (q^o, q^1, 0, 0)$, $q^1 > 0$ by rotations one obtains in this way a complete description of $\Gamma_{1,2}$. In particular one obtains a contraction of $\Gamma_{1,2}$ onto H_+ (by letting θ decrease to 0) or onto H_- (by letting θ tend to π), thus $\Gamma_{1,2}$ is simply connected. We note that this description of $\Gamma_{1,2}$ allows a simple direct proof of the asserted analyticity of $h_{1,2}$.

3. Single Particle Wave Functions

In the next step of our analysis we examine the wave functions of the single particle states $E(\{m\})\psi\,\Omega$ in momentum space. We begin with some notation.

We denote by $\overline{L_+^\uparrow}$ and $\overline{L_+(\mathbb{C})}$ the covering groups of the connected real and complex Lorentz groups, respectively. As usual, the group $\overline{L_+(\mathbb{C})}$ is identified with $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ and $\overline{L_+^\uparrow}$ with its subgroup consisting of the elements $\Lambda := (A,\bar{A}) \in SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. If $p \in \mathbb{R}^4$ we set

$$\underset{\sim}{p} = p^\circ \cdot \sigma_0 + \underline{p} \cdot \underline{\sigma} \quad \text{and} \quad \tilde{p} = p^\circ \cdot \sigma_0 - \underline{p} \cdot \underline{\sigma}, \tag{3.1}$$

where σ_0 is the 2×2 unit matrix and σ_i $i = 1,2,3$ are the Pauli matrices. For $\underline{\Lambda} := (A,B) \in \overline{L_+(\mathbb{C})}$ we define $\underline{\Lambda} \cdot p$ by

$$\underset{\sim}{(\Lambda p)} := A\, \underset{\sim}{p}\, B^T . \tag{3.2}$$

Since $\tilde{p} \cdot \underset{\sim}{p} = p^2 \sigma_0$ this implies

$$\widetilde{(\Lambda p)} = B^{T-1}\, \tilde{p}\, A^{-1} . \tag{3.3}$$

According to our assumptions the vectors $E(\{m\})\psi\,\Omega$ are elements of the given subspace $K \subset H$ on which the unitaries $U(a,\Lambda)$ act as an irreducible representation of $\overline{P_+^\uparrow}$ with mass m and spin s. We may therefore identify K with the Hilbert space $L^2(H_+,s)$ of

$\mathbb{C}^{2s+1}$-valued functions $\phi = \{\phi_\alpha\}_{\alpha=-s,\dots s}$ on H_+ which are equipped with the scalar product

$$(\phi_1,\phi_2) = \sum_{\alpha,\beta} \int d^4p\,\theta(p_0)\,\delta(p^2-m^2)\,\overline{\phi_{1\alpha}}(p)\,\mathcal{D}^s_{\alpha\beta}(\tilde{p})\,\phi_{2\beta}(p). \tag{3.3}$$

Here $\mathcal{D}^s$ is a well known representation of $GL(2,\mathbb{C})$ which satisfies

$$\mathcal{D}^s(M^*) = \mathcal{D}^s(M)^*, \quad \mathcal{D}^s(M^T) = \mathcal{D}^s(M)^T \tag{3.4}$$

for $M \in GL(2,\mathbb{C})$. Setting

$$\mathcal{D}^s(\underline{\Lambda}) := \mathcal{D}^s(A) \quad \text{if} \quad \underline{\Lambda} = (A,B), \tag{3.5}$$

the irreducible representation U_1 of $\overline{P_+^\uparrow}$ on $L^2(H_+,s)$ is given by

$$(U_1(a,\Lambda)\,\phi)_\alpha(p) = \sum_\beta e^{ipa} \cdot \mathcal{D}^s_{\alpha\beta}(\Lambda)\,\phi_\beta(\Lambda^{-1}p). \tag{3.6}$$

The identification of K with $L^2(H_+,s)$ is established by a unitary V, mapping K onto $L^2(H_+,s)$, such that

$$VU(a,\Lambda) \restriction K = U_1(a,\Lambda)V. \tag{3.7}$$

In order to simplify notation we write

$$(VE(\{m\})\psi\,\Omega)_\alpha(p) =: (\psi\,\Omega)_\alpha(p) \quad \alpha = -s,\dots s . \tag{3.8}$$

In view of the fact that the field operators ψ transform smoothly under Poincaré transformations, it is clear that the wave functions $(\psi\,\Omega)_\alpha(p)$ are smooth on H_+. It is our aim to extend them analytically to certain specific domains of the complex mass hyperboloid.

To this end we choose field operators ψ_β $\beta = -s,\dots s$ such that for all Lorentz transformations Λ in some neighbourhood of the identity $1 \in \overline{L_+^\uparrow}$

$$U(0,\Lambda)\,\psi_\beta\, U(0,\Lambda)^{-1} \in \mathcal{F}(\mathcal{S}). \tag{3.9}$$

Moreover, we assume that the $(2s+1) \times (2s+1)$ matrix

$$\Psi_{\alpha\beta}(p) := (\psi_\beta \Omega)_\alpha(p) \qquad \alpha,\ \beta = -s,\dots s \tag{3.10}$$

is invertible in a neighbourhood of some point $p^{(0)} \in H_+$. (That such field operators exist follows from relation (1.5) and the Reeh-Schlieder property (1.7) of Ω). Similarly, we choose field operators ψ'_β $\beta = -s,\dots s$, localized (in the sense of relation (3.9)) in the opposite cone $-\mathcal{S}$, such that the matrix $\Psi'_{\alpha\beta}$ (p) is also invertible at $p^{(0)}$. Then we consider the matrix-valued function

$$\Psi'(p)^* \mathcal{D}^s(\tilde{p})\Psi(p). \tag{3.11}$$

It follows from the results of the previous section that this function can be analytically continued to the domain

$$\Gamma = \{ k = p+iq \ :\ k^2 = m^2,\ q \in -\tilde{\mathcal{S}} \}, \tag{3.12}$$

and it has C^∞ boundary values on H_+ and H_-, respectively. It is an immediate consequence of this fact that $\Psi(p)$ and $\Psi'(p)$ are invertible at almost all points $p^{(0)}$ of H_+ (with the possible exception of some closed set of measure zero).

The information about the analytic properties of the products (3.11) is sufficient to equip the wave functions Ψ with the structure of a (trivial) analytic vector bundle over Γ, i.e. the wave functions are analytic apart from some possibly non-analytic, but universal factor. Using also the transformation properties (3.6) under Lorentz transformations it will become clear that the wave functions themselves can be analytically continued to Γ. To verify this it is convenient [5] to consider the functions

$$\Phi(\Lambda) = \mathcal{D}^s(\Lambda)\ .\ \Psi(\Lambda^{-1}p^{(0)}), \quad \Lambda \in \overline{L_+^\uparrow} \tag{3.13}$$

and Φ' (constructed analogously from Ψ'), respectively; $p^{(0)} \in H_+$ is kept fixed in the following. Note that

$$\Phi_{\alpha\beta}(\Lambda) = (U(0,\Lambda)\psi_\beta U(0,\Lambda)^{-1}\Omega)_\alpha(p^{(0)}) =: \Psi_{\Lambda\alpha\beta}(p^{(0)}), \tag{3.14}$$

so for Λ sufficiently close to 1 these functions are the "intrinsic wave functions" of operators in $\mathcal{F}(\mathcal{S})$, and similarly for Φ'. Taking into account that for $\Lambda \in \overline{L_+^\uparrow}$

$$\mathcal{D}^s(\Lambda)^* \, \mathcal{D}^s(\tilde{p}) \, \mathcal{D}^s(\Lambda) = \mathcal{D}^s(\widetilde{\Lambda^{-1} p}) \tag{3.15}$$

(cf. relation (3.2)), we obtain the crucial relation

$$\Phi'(\Lambda)^* \mathcal{D}^s(\tilde{p}^{(0)}) \Phi(\Lambda\Lambda_o) = \Psi'(\Lambda^{-1} p^{(0)})^* \mathcal{D}^s(\widetilde{\Lambda^{-1} p}^{(0)}) \Psi_{\Lambda_o}(\Lambda^{-1} p^{(0)}) =: H_{\Lambda_o}(\Lambda). \tag{3.16}$$

This relation tells us that for Λ_o sufficiently close to 1 the left-hand side of (3.16) can be analytically continued in Λ to

$$L(p^{(0)}) = \{\underline{\Lambda} \in \overline{L_+(\mathbb{C})} : \underline{\Lambda}^{-1} p^{(0)} \in \Gamma\} \tag{3.17}$$

with C^∞ boundary values on $\overline{L_+^\uparrow}$ and $\overline{L_+^\uparrow}$, respectively. Moreover, this expression is smooth in Λ_o, and the derivatives have the same analyticity properties in Λ as $H_{\Lambda_o}(\Lambda)$.

With this input we can continue $\Phi(\Lambda)$ analytically to the domain $L(p^{(0)})$: let $t \to \underline{\Lambda}_1(t)$ $0 \leq t \leq 1$ be any smooth path in $\overline{L_+(\mathbb{C})}$ such that $\underline{\Lambda}_1(0) \in \overline{L_+^\uparrow}$ and $\underline{\Lambda}_1(t) \in L(p^{(0)})$ for $t > 0$. We assume temporarily that the determinant of $H_{\Lambda_o}(\Lambda)$ does not vanish along the chosen path if $\Lambda_o \in \overline{L_+^\uparrow}$ is sufficiently close to 1. Then we have for $\Lambda \in \overline{L_+^\uparrow}$ in a small neighbourhood of the initial point $\underline{\Lambda}_1(0)$

$$\Phi(\Lambda\Lambda_o)^{-1} \cdot \Phi(\Lambda) = H_{\Lambda_o}(\Lambda)^{-1} \cdot H_1(\Lambda), \tag{3.18}$$

hence $\Phi(\Lambda\Lambda_o)^{-1} \cdot \Phi(\Lambda)$ can be analytically continued along this path. We can therefore use (3.18) to define a flat connection (parallel transport) on the $GL(2s+1,\mathbb{C})$-valued functions Ξ which are analytic

in a neighbourhood of $\underline{\Lambda}_1(t)$ and have C^∞ boundary values on $\overline{L_+^\uparrow}$. Namely, denoting the elements of the Lie-algebra of $\overline{L_+^\uparrow}$ by a,b we obtain a covariant derivative of these functions by setting, first for $\Lambda \in \overline{L_+^\uparrow}$ close to $\underline{\Lambda}_1(0)$,

$$(\nabla_a \Xi)(\Lambda) = \frac{d}{ds}[\Xi(\Lambda e^{sa}) + \Xi(\Lambda) \cdot \Phi(\Lambda e^{sa})^{-1} \Phi(\Lambda)]\big|_{s=0} . \tag{3.19}$$

It follows from the very definition of this derivative that

$$\nabla_a \nabla_b - \nabla_b \nabla_a - \nabla_{[a,b]} = 0 , \tag{3.20}$$

i.e., the curvature is zero. We can now extend the action of ∇_a to the functions Ξ in a neighbourhood of the chosen path by analytic continuation from (3.19), and it is obvious that the relation (3.20) remains valid. Since the Lie-algebra of $\overline{L_+^\uparrow} \simeq \{(A,\bar{A}) : A \in SL(2,\mathbb{C})\}$ coincides with the Lie-algebra of $\overline{L_+(\mathbb{C})} \simeq SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ (as a complex Lie-group) it follows that ∇_a defines a flat connection on the functions Ξ, as claimed.

Now according to well-known theorems any flat connection on a simply connected domain is completely integrable (i.e. a "pure gauge") [13]. This means that $\Phi(\Lambda)$ can be analytically continued along $\underline{\Lambda}_1(t)$. We recall that this continuation is obtained as the solution of the system of differential equations

$$(\nabla_a \Xi)(\underline{\Lambda}) = 0 \tag{3.21}$$

with the boundary value $\Phi(\Lambda)$ on $\overline{L^{\uparrow}_{+}}$.

Suppose now that the determinant of $H_1(\Lambda)$ has zeroes along the path $\underline{\Lambda}_1(t)$. Then we choose some $\Lambda_0 \in \overline{L^{\uparrow}_{+}}$ in an arbitrarily small neighbourhood of 1 such that the determinant of $H_1(\Lambda \cdot \Lambda_0)$ does not vanish along $\underline{\Lambda}_1(t)$ (this is always possible). The continuation of $\Phi(\Lambda)$ is now achieved by making use of the indentity

$$\Phi(\Lambda) = \Phi(\Lambda \cdot \Lambda_0) \cdot H_1(\Lambda \cdot \Lambda_0)^{-1} \cdot H_{\Lambda_0^{-1}}(\Lambda \cdot \Lambda_0) \tag{3.22}$$

for $\Lambda \in \overline{L^{\uparrow}_{+}}$ close to $\underline{\Lambda}_1(0)$. Each factor on the right hand side of this equation can be continued along $\underline{\Lambda}_1(t)$, (the first one according to the previous discussions). So Φ can be analytically continued along any path in $L(p^{(0)})$ starting at $\overline{L^{\uparrow}_{+}}$. But $L(p^{(0)})$ is a simply connected subset of $\overline{L_{+}(\mathbb{C})}$ since Γ is simply connected, so by applying the monodromy theorem we find that Φ can be analytically continued to $L(p^{(0)})$. It is easy to see that this continuation has smooth boundary values on $\overline{L^{\downarrow}_{+}}$.

We can return now to our original problem, the analytic continuation of the wave functions $\Psi(p)$. For $\Lambda \in \overline{L^{\uparrow}_{+}}$ we have

$$\Psi(\Lambda^{-1} \cdot p^{(0)}) = \mathcal{D}^{s}(\Lambda^{-1}) \cdot \Phi(\Lambda), \tag{3.23}$$

and since $\mathcal{D}^{s}$ is an analytic representation of $\overline{L_{+}(\mathbb{C})}$ the right-hand side of this equation can be continued to $L(p^{(0)})$. Moreover,

this continuation depends only on $\underline{\Lambda}^{-1} \cdot p^{(0)}$, which shows that $\Psi(p)$, $p \in H_{+}$ can be analytically continued to Γ. Taking into account that Φ has smooth boundary values on $\overline{L^{\downarrow}_{+}}$ it also follows that the continuation of Ψ has smooth boundary values on H_{-}.

This concludes our discussion of the analytic properties of the single particle wave functions $(\psi\,\Omega)_{\alpha}(p)$. We remark that these results hold also if the Poincaré transformations act on the single particle space $\mathcal{K}$ like a finite direct sum of irreducible representations.

4. The Spin of Antiparticles

We now interrupt our analysis for a brief digression to the question of how the spins of particles and antiparticles are related. As we already pointed out it follows from the results of Sec. 2 that the space

$$K_c = \overline{E(\{m\})\, F(S)^*\, \Omega} \tag{4.1}$$

is non-trivial. We will show now that the Poincaré transformations $U(a,\Lambda)$ act on K_c as an irreducible representation of $\overline{P_+^\uparrow}$ with mass m and spin s. Thus the spins of particles and antiparticles are equal also in the presence of non-localizable charges.

We begin by noting that the space K_c is stable under the action of the unitaries $U(a,\Lambda)$ as a consequence of the Reeh-Schlieder property (1.8). So K_c can be decomposed into irreducible subspaces of definite spin. In order to see that in this decomposition only the spin s appears we proceed as follows: let P_ν and $M_{\rho\sigma}$ be the generators of $U(a)$ and $U(0,\Lambda)$, respectively, and let

$$W^\mu = \tfrac{1}{2}\, \epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \tag{4.2}$$

be the Pauli-Lubanski operator. As is well-known, the operator $W_\mu \cdot W^\mu$ commutes with all Poincaré transformations, and

$$W_\mu \cdot W^\mu \restriction K = -\, m^2 \cdot s(s+1)\cdot 1 \,. \tag{4.3}$$

Using this fact we will exhibit certain specific field operators creating states from the vacuum which are orthogonal to all states with spin s and mass m: if $\psi \in F(S)$ we define

$$\delta^\mu(\psi) = \tfrac{1}{2}\, \epsilon^{\mu\nu\rho\sigma}[P_\nu, [M_{\rho\sigma}, \psi]] \,. \tag{4.4}$$

Since ψ is smooth with respect to Poincaré transfomations this expression is well defined, and $\delta^\mu(\psi) \in F(S)$. Setting

$$\psi_s = \delta_\mu(\delta^\mu(\psi)) + m^2 \cdot s(s+1)\cdot \psi \tag{4.5}$$

and using relations (1.5), (4.3) as well as the invariance of Ω under Poincaré transformations we obtain

$$E(\{m\})\, \psi_s\, \Omega = 0 \,. \tag{4.6}$$

Now if $\psi' \in F(-S)$ is any field operator it follows from (4.6) and the analyticity properties of the two point function established in Sec. 2 that

$$(\psi_s^*\, \Omega,\; E(\{m\})\psi'^{*}\Omega) = 0. \tag{4.7}$$

Taking also the Reeh-Schlieder property (1.8) into account we thus arrive at

$$0 = E(\{m\})\psi^*_s \Omega = (W_\mu W^\mu + m^2 s(s+1))E(\{m\})\psi^* \Omega . \qquad (4.8)$$

This shows that the vectors $E(\{m\})\psi^* \Omega$, $\psi \in \mathcal{F}(S)$ (and therefore all states in K_c) have spin s.

It remains to verify that in the decomposition of K and K_c with respect to the momentum operator P,

$$K = \int d^4 p\, \theta(p_0)\delta(p^2 - m^2)K(p) \text{ and } K_c = \int d^4 p\, \theta(p_0)\delta(p^2-m^2)K_c(p), \qquad (4.9)$$

the spaces $K(p)$ and $K_c(p')$ have for almost all $p, p' \in H_+$ the same dimension (namely 2s+1). It then follows from the well-known representation theory of $\overline{P_+^\uparrow}$ that $U(a,\Lambda) \upharpoonright K_c$ is an irreducible representation with spin s and mass m.

Let $\psi_\sigma \in F(S)$ $\sigma = 1,\dots,r$ and $\psi'_\tau \in F(-S)$ $\tau = 1,\dots r$ be arbitrary field operators. Then we have according to relation (2.8)

$$(\psi_\sigma \Omega, U(x)E(\{m\})\psi'_\tau \Omega) = \int d^4 p\, \theta(p_0)\delta(p^2-m^2)h_{\sigma\tau}(p)e^{ipx}$$
$$(4.10)$$
$$(\psi'^*_\tau \Omega, U(x)E(\{m\})\psi^*_\sigma \Omega) = \operatorname{sign}\lambda \cdot \int d^4 p\, \theta(p_0)\delta(p^2-m^2)h_{\sigma\tau}(-p)e^{ipx},$$

where $h_{\sigma\tau}$ are analytic functions on the domain Γ (cf. 3.12) with smooth boundary values on H_+ and H_-, respectively.

It is obvious that $h_{\sigma\tau}(p)$ is just the scalar product of the components of $E(\{m\})\psi_\sigma \Omega$ and $E(\{m\})\psi'_\tau \Omega$ in $K(p)$ in the decomposition (4.9) of K; an analogous statement holds for $\operatorname{sign}\lambda \cdot h_{\sigma\tau}(-p')$ and $K_c(p')$. In order to see that $K(p)$ and $K_c(p')$ have the same dimension it is therefore sufficient to show that the determinant of the $r \times r$ matrix $h_{\sigma\tau}(p)$, $\sigma,\tau = 1,\dots r$ is different from 0 for almost all $p \in H_+$, whenever the determinant of $h_{\sigma\tau}(-p')$ is different from 0 for some $-p' \in H_-$, and vice versa. But this is an immediate consequence of the fact that the determinant of $h_{\sigma\tau}$ is an analytic function on Γ with smooth boundary values on H_+ and H_-. So, unless it is identical to 0, this determinant can only vanish on H_+ (respectively H_-) on closed sets of measure 0. Hence the dimensions of $K(p)$ and $K_c(p')$ must be equal.

As in the case of the single particle space K we will identify K_c with the Hilbert space of wave functions $L^2(H_+,s)$. We denote the canonical unitary mapping K_c onto $L^2(H_+,s)$ by V_c and write

$$(V_c E(\{m\})\psi^* \Omega)_\alpha(p) =: (\psi^* \Omega)_\alpha(p) \qquad \alpha = -s,\dots s. \qquad (4.11)$$

If $\psi \in F(S)$ these functions have the same analytic properties as those established for $(\psi\Omega)_\alpha(p)$ in Sec. 3, and again it is convenient to deal with the matrix-valued functions

$$\Psi^c_{\alpha\beta}(p) := (\psi^*_\beta \Omega)_\alpha(p) \qquad \alpha,\beta = -s,\dots s. \qquad (4.12)$$

We will use these matrices in the final step of our argument.

5. The Spin-Statistics Theorem

We are now in a position to establish in the present setting the familiar connection between the spin and statistics of particles. The proof essentially boils down to showing that there is an anti-linear relation (related to charge conjugation) between the wave functions

$$\Psi_{\alpha\beta}(p) = (\psi_\beta \Omega)_\alpha(p) \quad \text{and} \quad \Psi^c_{\alpha\beta}(p) = (\psi^*_\beta \Omega)_\alpha(p) \tag{5.1}$$

of particles and antiparticles, respectively.

More precisely, we will prove that the matrix-valued functions on H_+

$$\Phi(p) = \mathcal{D}^s(\tfrac{1}{m}\,\underset{\sim}{p})\cdot\mathcal{D}^s(\sigma_2)\,.\,\overline{\Psi(-p)}, \tag{5.2}$$

where the bar denotes complex conjugation, coincide with $\Psi^c(p)$ up to some phase factor ω. Note that we have used here the same symbol for the wave functions (defined on H_+) and their analytic continuations to H_-. This will, however, not cause any confusion since p will, in the following, always denote an element of H_+, hence $-p \in H_-$.

We recall that if one replaces in (5.1) the field-operators ψ_β by $U(0,\Lambda)\psi_\beta U(0,\Lambda)^{-1}$ then the corresponding wave functions transform according to

$$\Psi(p) \to \mathcal{D}^s(\Lambda)\,\Psi(\Lambda^{-1}p), \tag{5.3}$$

and similarly for $\Psi^c(p)$. Taking into account that

$$\sigma_2\cdot A^{-1}\cdot\sigma_2 = A^T \quad \text{for} \quad A \in SL(2,\mathbb{C}) \tag{5.4}$$

it follows by analytic continuation from (5.3) that the same law of transformation holds also for $\Phi(p)$ if $\Lambda \in \overline{L^\uparrow_+}$ is sufficiently close to 1.

Now let $\underline{S}_1, \underline{S}_2 \subset \mathbb{R}^3$ be any two open, convex and salient cones such that the cone $\underline{S}_{1,2} = \underline{S}_2 - \underline{S}_1$ is salient, and let $\psi_{i,\beta} \in \mathcal{F}(S_i)$ $\beta = -s,\ldots s$, $i = 1,2$ be field operators localized in these cones. We denote the various (matrix-valued) wave functions associated with these fields by Ψ_i, Ψ^c_i, and Φ_i, respectively. As we have seen in Sec. 2 the function

$$H_{1,2}(p) = \Psi_2(p)^*\,\mathcal{D}^s(\tilde{p})\,\Psi_1(p) \tag{5.5}$$

can be analytically continued to

$$\Gamma_{1,2} = \{k : k^2 = m^2,\ \mathrm{Im}\,k \in \tilde{S}_{1,2}\} \tag{5.6}$$

with smooth boundary values on H_-, and the same is true for the individual wave functions $\Psi_1(p)$ and $\Psi_2(p)^*$ in this expression, cf. Sec. 3. (Clearly the continuation of the latter function is given by

$k \to \Psi_2(\bar{k})^*$ for $k^2 = m^2$ and $\mathrm{Im}\, k \in S_2$).

Now according to relation (2.8) we have

$$H_{1,2}(-p) = \text{sign}\,\lambda\,.[\Psi_1^c(p)^*\,\mathcal{D}^s(\tilde{p})\Psi_2^c(p)]^T \tag{5.7}$$

which, in view of the above remarks, leads to

$$\overline{\Psi_2(-p)^*\,\mathcal{D}^s(-\tilde{p})\,\Psi_1(-p)} = \text{sign}\lambda\cdot\Psi_2^c(p)^*\,\mathcal{D}^s(\tilde{p})\,\Psi_1^c(p). \tag{5.8}$$

Using once more the relation (5.4) and the fact that $\mathcal{D}^s(-\tilde{p}) = (-1)^{2s}\cdot\mathcal{D}^s(\tilde{p})$ we thus arrive at

$$\eta\,.\,\Phi_2(p)^*\,\mathcal{D}^s(\tilde{p})\,\Phi_1(p) = \Psi_2^c(p)^*\,\mathcal{D}^s(\tilde{p})\,\Psi_1^c(p), \tag{5.9}$$

where

$$\eta = \text{sign}\,\lambda\cdot(-1)^{2s}\ . \tag{5.10}$$

If we could choose in this argument $\psi_{1\beta} = \psi_{2\beta}$ it would immediately follow from the positivity of both the factor of η in (5.9) and the right-hand side of this relation that $\eta = 1$. But in the present setting we must argue differently. We consider the expression

$$\omega_1(p) = \Psi_1^c(p)\,\Phi_1(p)^{-1}, \tag{5.11}$$

provided the inverse of $\Phi_1(p)$ exists. Varying the fields $\psi_{1,\beta}$ within $F(S_1)$ and keeping $\psi_{2,\beta}$ fixed it is clear from Eq. (5.9) that $\omega_1(p)$ does not depend on the specific choice of the fields $\psi_{1,\beta}$. So if in particular the fields $\psi_{1,\beta} \in F(S_1)$ are such that the Lorentz transformed fields $U(0,\Lambda)\psi_{1,\beta}U(0,\Lambda)^{-1}$ are still elements of $F(S_1)$ for $\Lambda \in \overline{L_+^\uparrow}$ close to 1, it follows from the transformation properties (5.3) of Ψ^c and Φ that

$$\omega_1(p) = \mathcal{D}^s(\Lambda)\,\omega_1(\Lambda^{-1}p)\,\mathcal{D}^s(\Lambda)^{-1}. \tag{5.12}$$

Choosing Λ in the little group of p and using the irreducibility of $\mathcal{D}^s$ we find that $\omega_1(p)$ is a multiple of the identity so that $\omega_1(\Lambda p) = \omega_1(p)$, i.e. ω_1 is locally constant. But $p \in H_+$ was arbitrary, so ω_1 is also globally constant.

It remains to show that ω_1 does not depend on the cone S_1. To see this notice that ω_1 does not change if one replaces S_1 by a smaller, respectively larger cone, provided these cones are admissible (i.e. open, convex, and salient). So, given any other cone S_0 one must only choose an interpolating sequence of admissible cones $S_1, S_2, \ldots S_n = S_0$ such that either $S_{i+1} \supset S_i$ or $S_{i+1} \subset S_i$. Since ω_1 does not change if one proceeds from S_i to S_{i+1} it is then clear that it does not depend on the localization cone S_1, i.e. $\omega_1 = \omega$. So we have, in particular,

$$\Psi_i^c(p) = \omega\,.\,\Phi_i(p) \qquad i = 1,2 \tag{5.13}$$

and inserting this into the relation (5.9) it follows that $\eta = |\omega|^2$. So η can only be 1 and this, finally, proves the spin-statistics theorem.

Acknowledgement

One of us (D.B.) gratefully acknowledges the hospitality and support extended to him by the I.H.E.S. in Bures-sur Yvette.

References

1. Fierz, M.: Uber die relativistische Theorie kräftefreier Teilchen mit beliebigem Spin. Helv. Phys. Acta 12, 3 (1939).

   Pauli, W.: On the Connection between Spin and Statistics. Phys. Rev. 58, 716 (1940).

2. Streater, R.F. and Wightman, A.S.: P C T, Spin and Statistics and all that, New York, Amsterdam: Benjamin 1964.

3. Doplicher, S., Haag, R. and Roberts J.E.: Local Observables and Particle Statistics I. Commun. Math. Phys. 23, 199 (1971).

   Local Observables and Particle Statistics II. Commun. Math. Phys. 35, 49 (1974)

4. Doplicher, S. and Roberts, J.E.: Compact Lie Groups Associated with Endomorphisms of C*-Algebras. Preprint (1984).

5. Epstein, H.: C T P Invariance in a Theory of Local Observables. J. Math. Phys. 8, 750 (1967).

6. Buchholz, D.: The Physical State Space of Quantum Electrodynamics. Commun. Math. Phys. 85, 49 (1982).

7. Buchholz, D. and Fredenhagen, K.: Locality and the Structure of Particle States. Commun. Math. Phys. 84, 1 (1982).

8. Fredenhagen, K. and Marcu, M.: Charged States in $\mathbb{Z}_2$ Gauge Theories. Commun. Math. Phys. 92, 81 (1983).

9. Fredenhagen, K.: On the Existence of Anti Particles. Commun. Math. Phys. 79, 141 (1981).

10. Ruelle, D.: On the Asymptotic Condition in Quantum Field Theory. Helv. Phys. Acta 35, 147 (1962).

11. Iagolnitzer, D.: Microcausality, Macrocausality and the Physical Region (Micro) Analytic S-Matrix. In: Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory. Lect. Notes in Physics 126, Berlin: Springer (1980).

12. Bros, F., Messiah, A., and Stora, R.: A Problem of Analytic Completion Related to the Jost-Lehmann-Dyson Formula. J. Math. Phys. 2, 639 (1961).

13. Kobayashi, S. and Nomizù, K.: Foundations of Differential Geometry. New York, London, Sidney: F. Wiley (1969).